LOCAL WELL-POSEDNESS AND BLOW-UP FOR THE HALF GINZBURG-LANDAU-KURAMOTO EQUATION WITH ROUGH COEFFICIENTS AND POTENTIAL

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Abstract. We study the Cauchy problem for the half Ginzburg-Landau-Kuramoto (hGLK) equation with the second order elliptic operator having rough coefficients and potential type perturbation. The blow-up of solutions for hGLK equation with non-positive nonlinearity is shown by an ODE argument. The key tools in the proof are appropriate commutator estimates and the essential self-adjointness of the symmetric uniformly elliptic operator with rough metric and potential type perturbation.

Contents

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1. Introduction

In this paper, we study the Cauchy problem for the focusing half Ginzburg-Landau-Kuramoto (hGLK) type equation

$$i\partial_t u + \mathcal{D}_{A,V} u = i|u|^{p-1}u, \qquad p > 1. \tag{1.1}$$

Here $\mathcal{D}_{A,V}$ is the fractional Hamiltonian (see [16] for a more general choice of the fractional powers of the Laplacian)

$$\mathcal{D}_{A,V} = \mathcal{H}_{A,V}^{1/2},$$

where

$$\mathcal{H}_{A,V}\,|_{C_c^\infty(\mathbb{R}^n)} = -\Delta_{A,V} = -\nabla \cdot A\nabla + V = -\sum_{j,k=1}^{n} \partial_j (A_{j,k}(x)\partial_k) + V$$

is a self-adjoint non-negative operator with a real-valued potential, such that the positive Hermite matrix A and the potential V satisfy appropriate assumptions given below. The fractional power of $\mathcal{H}_{A,V}$ is defined by spectral analysis. For details, see Definition 6 below. Beside the other ones, it is worth mentioning that A is supposed to ensure that $\mathcal{H}_{A,0}$ is an elliptic second order operator in divergence form. Furthermore, *focusing* stands for the "+" sign in front of the nonlinearity in (1.1).

We recall that the classical Ginzburg-Landau equation is instead typically associated with the standard Laplacian as Hamiltonian (see [24] for a recent review and references on this classical subject).

The idea to replace the Laplace operator in the Hamiltonian of some quantum mechanical models by its fractional powers was initiated in [16] and has been intensively studied in the last decade (see [22], for instance, for motivations to take the square root of the Laplacian and for an overview of the results in this context).

The half Ginzburg-Landau-Kuramoto equation (1.1), which is the main subject of this paper, is closely connected with the Kuramoto model (see [15], [1]) and the idea (proposed in [16] and [22]) to use the square root of the Laplacian in the definition of the Hamiltonian.

In order to define $\mathcal{D}_{A,V}$, we need to prove that $-\Delta_{A,V}$ has a self-adjoint extension, where we regard the domain of $-\Delta_{A,V}$ as $C_c^\infty(\mathbb{R}^n)$. One can find a self-adjoint extension for $-\Delta_{A,V}$ with rough coefficients A and rough potential V by using the Friedrichs type extension under the non-negativity assumption (see [4, Theorem 1.2.7]). Recall that the domain of Friedrichs type extension can be defined as the set of all $f \in H^1(\mathbb{R}^n)$, such that there exists $g \in L^2(\mathbb{R}^n)$ satisfying

$$-\Delta_{A,V} f = g \tag{1.2}$$

in distributional sense. On the other hand, since the argument of Friedrichs type extension does not guarantee the uniqueness of self-adjoint extensions, in order to clarify the definition of fractional power of $\mathcal{H}_{A,V}$, we also need to show the uniqueness of self-adjoint extensions of $-\Delta_{A,V}$. In this case, we say that the operator $-\Delta_{A,V}$ is essentially self-adjoint (the problem is referred to as *quantum completeness*, too). Some sufficient conditions for the essential self-adjointness for general symmetric operators on manifolds have been discussed in [3], for instance. In this paper, we give a detailed proof of the essential self-adjointness of $-\Delta_{A,V}$ (see the Subsection 1.2 below for the precise hypothesis).

We started the study of this model in [6], where local and global well-posedness were discussed for the *defocusing* ("$-$" sign in front of the nonlinearity) equation

$$i\partial_t u + (-\Delta)^{1/2} u = -i|u|^{p-1}u$$

in space dimensions $n = 1, 2, 3$. The blow-up result for the focusing equation

$$i\partial_t u + (-\Delta)^{1/2} u = i|u|^{p-1}u$$

is obtained instead in [5] for $n = 1$. In [5], the proof of the blow-up result uses the following simple commutator estimates:

$$\|[(-\Delta)^{1/2}, f]g\|_{L^2} \leqslant C\|f\|_{\mathrm{Lip}}\|g\|_{L^2},$$

where f is a Lipschitz function with corresponding norm $\|f\|_{\mathrm{Lip}}$. In order to show the blow-up of solutions to (1.1), we shall prove the following estimates

$$\|[f, \mathcal{D}_{A,0}]g\|_{L^2} \leqslant C\|f\|_{\dot{B}^1_{\infty,1}}\|g\|_{L^2}, \tag{1.3}$$

$$\|[f, \mathcal{D}_{A,V}]g\|_{L^2} \leqslant C\|f\|_{B^1_{\infty,1}}\|g\|_{L^2}, \tag{1.4}$$

where $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ are the standard inhomogeneous and homogeneous Besov spaces on $\mathbb{R}^n$, respectively. Since

$$B^1_{\infty,1} \cup \dot{B}^1_{\infty,1} \subsetneq \mathrm{Lip},$$

it would be natural to pose the question if the estimates (1.3) and (1.4) are optimal for the case of rough coefficients; but this is not our goal, hence we do not investigate this question, as well as the question if the commutator

$$[\mathcal{D}_{A,V}, \langle x\rangle]$$

is a bounded operator in L^2. However, by replacing $\langle x\rangle$ by $\langle x\rangle^a$, our aim shall be to check that the commutator

$$[\mathcal{D}_{A,V}, \langle x\rangle^a]$$

is an L^2-bounded operator for any $a \in (1/2, 1)$ and this shall be a sufficient tool to obtain our blow-up result at least for $n = 1$.

1.1. **Notations.** We collect here some notations used along the paper. Given two quantities A and B, we denote $A \lesssim B$ ($A \gtrsim B$, respectively) if there exists a positive constant C such that $A \leqslant CB$ ($A \geqslant CB$, respectively). We also denote $A \sim B$ if $A \lesssim B \lesssim A$. Given two operators $\mathcal{M}$ and $\mathcal{N}$, the commutator between them is defined as the operator $[\mathcal{M}, \mathcal{N}] = \mathcal{M}\mathcal{N} - \mathcal{N}\mathcal{M}$. For $1 \leqslant p \leqslant \infty$, the $L^p = L^p(\mathbb{R}^n; \mathbb{C})$ are the classical Lebesgue spaces endowed with norm $\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}$ if $p \neq \infty$ or $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|$ for $p = \infty$. Given an interval $I \subset \mathbb{R}$, bounded or unbounded, we define by $L^p(I; X)$ the Bochner space of vector-valued functions $f : I \to X$ endowed with the norm $\left(\int_I \|f(s)\|_X^p \, dx\right)^{1/p}$ for $1 \leqslant p < \infty$, with similar modification as above for $p = \infty$. If $f : I \to X$ is a continuous function up to the m^{th}-order of derivatives, we write $f \in C^m(I; X)$. For any $s \in \mathbb{R}$, we set $H^s = H^s(\mathbb{R}^n; \mathbb{C}) := (1 - \Delta)^{-s/2} L^2$ and its homogeneous version $\dot{H}^s = \dot{H}^s(\mathbb{R}^n; \mathbb{C}) := (-\Delta)^{-s/2} L^2$. For a pair of functions in L^2, the inner product $\langle f, g \rangle = \langle f, g \rangle_{L^2}$ is classically defined as $\langle f, g \rangle = \int_{\mathbb{R}^n} f \bar{g} \, dx$, being $\bar{z}$, the usual complex conjugate to $z \in \mathbb{C}$. For $x \in \mathbb{R}^n$ instead, $\langle x \rangle := \sqrt{1 + |x|^2}$. The space $W^{1,\infty} = W^{1,\infty}(\mathbb{R}^n)$ is the space of Lipschitz functions. The operator $\mathfrak{F} f(\xi) = \hat{f}(\xi)$ is the standard Fourier transform, $\mathfrak{F}^{-1}$ being its inverse. For $s \in \mathbb{R}$ and $0 < p, q \leqslant \infty$, $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ is the homogeneous Besov space of functions having finite $\|\cdot\|_{\dot{B}^s_{p,q}}$-norm, the last defined as

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{sjq} \|P_j f\|_{L^p}^q\right)^{1/q}$$

with obvious modifications for $p, q = \infty$. The non-homogeneous version $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$ is induced by the norm

$$\|f\|_{B^s_{p,q}} = \|Qf\|_{L^p} + \left(\sum_{j \in \mathbb{N}} 2^{sjq} \|P_j f\|_{L^p}^q\right)^{1/q}.$$

Here the Littlewood-Paley projectors P_j are defined by means of a radial cut-off function $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ and the dyadic functions $\varphi_j(\xi) = \chi_0(2^{-j}\xi) - \chi_0(2^{-j+1}\xi)$ yielding to the partition of the unity $\chi_0(\xi) + \sum_{j \geqslant 1} \varphi_j(\xi) = 1$, for any $\xi \in \mathbb{R}^n$. Hence the projectors are given by $Qf := \mathfrak{F}^{-1}(\chi_0 \mathfrak{F} f)$ and $P_j f := \mathfrak{F}^{-1}(\varphi_j \mathcal{F} f)$. The Lorentz space $L^{\beta,\infty}$ is given by

$$L^{\beta,\infty} = \{f \,:\, \|f\|_{L^{\beta,\infty}}^\beta = \sup_{t > 0} t^\beta |\{|f| > t\}| < \infty\}.$$

For $1 \leqslant p \leqslant \infty$, p' is the conjugate index defined by $1/p + 1/p' = 1$.

1.2. **Assumptions and the main results.** We give now the precise assumptions that we make on the structure of our Hamiltonian $-\Delta_{A,V}$

and the main results contained in the paper. We start with the hypotheses on $A = A(x)$, which is a Hermitian matrix-valued function. We assume:

A1. Uniform ellipticity of A: There exist two positive constants C_1 and C_2 satisfying

$$C_1|\xi|^2 \leqslant \sum_{j,k=1}^{n} A_{j,k}(x)\xi_j\bar{\xi}_k \leqslant C_2|\xi|^2, \qquad \forall\, \xi \in \mathbb{C}^n, \quad \forall\, x \in \mathbb{R}^n; \tag{1.5}$$

A2. Regularity of the coefficients: A is in the Lipschitz class of matrix-valued functions, namely

$$A_{j,k} \in W^{1,\infty}(\mathbb{R}^n) \qquad j,k \in \{1, \cdots, n\};$$

A3. Boundedness: The multiplication operator

$$f \mapsto \; ((-\Delta)^{1/4}A_{j,k})f$$

maps $\dot{H}^{1/2}$ into L^2, namely

$$\max_{j,k} \|((-\Delta)^{1/4}A_{j,k})f\|_{L^2} \leqslant C\|(-\Delta)^{1/4}f\|_{L^2}, \qquad \forall\, f \in \dot{H}^{1/2}. \tag{1.6}$$

Let us turn our attention to the potential perturbation $V = V(x)$. It is a real-valued function satisfying the following conditions:

H1. Boundedness of the potential:

$$V \in L^{q,\infty}(\mathbb{R}^n) + L^{\infty}(\mathbb{R}^n)$$

for some q with $q > \max\{2, n/2\}$;

H2. Non-negativity of the Hamiltonian $-\Delta_{A,V}$: There exists $\theta \in (0,1)$ such that

$$\theta\langle A\nabla f, \nabla f\rangle + \langle Vf, f\rangle \geqslant 0, \qquad \forall\, f \in C_0^{\infty}(\mathbb{R}^n).$$

Though for the moment it is not our aim to weaken the non-negativity assumption in H2, it is worth mentioning that this hypothesis could be relaxed, at least in the case $n = 1$. For example for $A = 1$, perturbations of the Laplacian which belongs to the Miura class should imply positivity of such Hamiltonians (see [11]). The non-negativity assumption is needed to guarantee that the square root of the operator is well defined.

First we state the result on the self-adjoint extension of the operator $-\Delta_{A,V}$. This theorem is crucial for the local well-posedness theory below and for the commutator estimates we are going to prove.

Theorem 1. *Assume the assumptions* A1, A2, H1 *and* H2 *are satisfied. Then the operator $-\Delta_{A,V}$ is essentially self-adjoint, i.e. there exists a unique self-adjoint extension $\mathcal{H}_{A,V}$ of this operator with domain*

$$D(\mathcal{H}_{A,V}) = H^2(\mathbb{R}^n).$$

The key point in our blow-up result shall be instead the following commutator estimate.

Proposition 2. *Assume the conditions* A1, A2, A3 *and* H1, H2 *are satisfied. We have the two commutator estimates in two cases below.*

Case 1. *Let $f \in B^1_{\infty,1}$. If V also belongs to $L^{q,\infty}$ for $q > \max\{2, n\}$,*

$$\|[f, \mathcal{D}_{A,V}]g\|_{L^2} \leqslant C \left(\|f\|_{\dot{B}^1_{\infty,1}} + \|V\|_{L^{q,\infty}} \|f\|_{L^\infty} \right) \|g\|_{L^2}. \tag{1.7}$$

Case 2. *Suppose $n \geqslant 3$ and $f \in \dot{B}^1_{\infty,1}$. If V also belongs to $L^{n/2,\infty}$ then,*

$$\|[f, \mathcal{D}_{A,V}]g\|_{L^2(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{B}^1_{\infty,1}(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}. \tag{1.8}$$

Next we turn to the local well-posedness of (1.1).

Theorem 3. *Let $n = 1, 2, 3$. Assume that the conditions* A1, A2, H1 *and* H2 *are satisfied. Then for any $u_0 \in H^s$ with $s = 1$ if $n = 1$ or $s = 2$ if $n = 2, 3$, there exists a positive time $T > 0$ and a solution $u \in C([0, T); H^s)$ to (1.1).*

The next result is the finite time blow-up result for solutions to (1.1) in one space dimension.

Theorem 4. *Let $n = 1$. Assume the conditions* A1, A2, A3 *and* H2 *are satisfied and $V \in L^{q,\infty}$ for some q with $q > 2$.*

Case 1. *Let $u_0 \in L^2$ and $w \in B^1_{\infty,1}$ satisfy $1/w \in L^\infty \cap L^2$ and the following estimate:*

$$\|wu_0\|^2_{L^2} \geqslant C^{\frac{2}{p-1}} \|1/w\|^{\frac{2}{p-1}}_{L^\infty} \|w\|^{\frac{2}{p-1}}_{B^1_{\infty,1}} \|1/w\|^2_{L^2}. \tag{1.9}$$

If there exists a solution $u \in C([0, T_{\max}); L^2 \cap L^{p+1})$, then the maximal time of existence is finite: $T_{\max} < \infty$.

Case 2. *Suppose that $V \equiv 0$. Let $1 < p < 3$ and let $u_0 \in L^2 \backslash \{0\}$. If there exists a solution $u \in C([0, T_{\max}); L^2 \cap L^{p+1})$, then the maximal time of existence is finite: $T_{\max} < \infty$.*

Remark 1.1. *Here the condition $p = 3$ corresponds to the critical exponent $p_F = 1 + 2/n$ defined also in a multidimensional framework. See the results in* [5].

2. Self-Adjointness of $-\Delta_{A,V}$

The proof of Theorem 1 can be reduced to the proof that $-\Delta_{A,0}$ is essentially self-adjoint. Indeed, if $-\Delta_{A,0}$ has unique self-adjoint extension $\mathcal{H}_{A,0}$ with domain

$$D(\mathcal{H}_{A,0}) = H^2(\mathbb{R}^n),$$

then we can use the estimate (A.1) of Lemma A.1 in combination with the KLMN lemma (see [21, Theorem X.17]) and deduce that $\mathcal{H}_A + V$ is an essentially self-adjoint operator with domain $H^2(\mathbb{R}^n)$.

Therefore, it remains to verify that $-\Delta_{A,0}$ is essentially self-adjoint. This is done below in Proposition 5 and this yields to Theorem 1. Firstly, we recall sufficient equivalent conditions guaranteeing the self-adjointness property of an operator.

Lemma 2.1. [21, Theorem X.26] *Assume the operator $-\Delta_{A,V}$ is non-negative (in sense of quadratic form acting on C_0^∞ functions). Then the following conditions are equivalent:*

i) $-\Delta_{A,V}$ *is essentially self-adjoint;*

ii) *the kernel of the adjoint operator satisfies*

$$\operatorname{Ker}(-\Delta_{A,V}+1)^* = \{0\};$$

iii) *the range of $(-\Delta_{A,V}+1)$ is dense in $L^2(\mathbb{R}^n)$:*

$$\overline{[\operatorname{Ran}(-\Delta_{A,V}+1)]} = L^2(\mathbb{R}^n). \tag{2.1}$$

Next, we recall some fundamental operator calculus.

Lemma 2.2. *For f smooth enough,*

$$[-\Delta_{A,V}, f] = (\nabla f)\cdot A\nabla + \nabla\cdot A(\nabla f). \tag{2.2}$$

Proof. For completeness, we shall sketch the proof. The relation (2.2) follows directly from the simple commutator rule

$$\begin{aligned}[B_1B_2, f] &= B_1B_2f - fB_1B_2 - B_1fB_2 + B_1fB_2\\ &= B_1[B_2,f] + [B_1,f]B_2.\end{aligned}$$

□

Lemma 2.3. *Let $\mathcal{A}$ be non-negative self-adjoint operator. Then*

$$[(\lambda+\mathcal{A})^{-1}, f] = (\lambda+\mathcal{A})^{-1}[\mathcal{A}, f](\lambda+\mathcal{A})^{-1}.$$

Proof. For completeness, we shall sketch the proof. Noting the identity

$$\begin{aligned}0 &= [(\lambda+\mathcal{A})(\lambda+\mathcal{A})^{-1}, f]\\ &= (\lambda+\mathcal{A})[(\lambda+\mathcal{A})^{-1}, f] + [\mathcal{A}, f](\lambda+\Delta_{A,V})^{-1}\end{aligned}$$

and applying the resolvent $(\lambda+\mathcal{A})^{-1}$ from the left, we obtain the assertion. □

We can now give the following:

Proposition 5. *Assume the assumptions* A1 *and* A2 *are satisfied. Then the operator $-\Delta_{A,0}$ is essentially self-adjoint, i.e. there exists a unique self-adjoint extension $\mathcal{H}_A$ of this operator with domain*

$$D(\mathcal{H}_A) = H^2(\mathbb{R}^n).$$

Proof. We show that the closure $\overline{(-\Delta_{A,0})}$ is self-adjoint. Lemma A.2 in the Appendix A below, implies that

$$D(\overline{(-\Delta_{A,0})}) = H^2(\mathbb{R}^n).$$

Thanks to symmetry and regularity of A, there exists at least one self-adjoint extension of $-\Delta_{A,0}$. Indeed, since $-\Delta_{A,0}$ is symmetric and $A \in W^{1,2}_{loc}(\mathbb{R}^n)$, the quadratic form

$$Q(f) = \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} A_{jk}(x)\partial_{x_j} f(x)\overline{\partial_{x_k} f(x)}\, dx, \quad D(Q) = H^1(\mathbb{R}^n)$$

is closable and possesses a self-adjoint operator $\mathcal{H}_A$ satisfying

$$Q(f) = \langle \mathcal{H}_A f, f\rangle$$

for any $f \in D(\mathcal{H}_A) \subset H^1(\mathbb{R}^n)$ (see [4, Theorem 1.2.5]). We recall that any self-adjoint extension of $-\Delta_{A,0}$ is also an extension of $\overline{(-\Delta_{A,0})}$ and so is $\mathcal{H}_A$.

Now we show that the self-adjointness of $\mathcal{H}_A$ implies that also $\overline{(-\Delta_{A,0})}$ is self-adjoint. For this purpose, we shall check the equivalent assertion (2.1) in Lemma 2.1. Let $h \in L^2$ satisfy

$$h \perp \mathrm{Ran}(-\Delta_{A,0} + 1), \tag{2.3}$$

namely h is orthogonal to the range of $(-\Delta_{A,0}+1)$. Our goal is to show that $h = 0$. We define the Yosida type approximation of Laplacian

$$\rho_j = j(j - \Delta)^{-1}$$

with $j \geqslant 1$. We show that

$$\overline{(-\Delta_{A,0})}(\rho_j f) \overset{L^2}{\rightharpoonup} \mathcal{H}_A f, \quad \forall f \in D(\mathcal{H}_A).$$

We remark that, for any $j \geqslant 1$ and $f \in L^2$, $\rho_j f \in H^2$. For any $b \in L^2$,

$$\begin{aligned}
\langle \overline{(-\Delta_{A,0})}\rho_j f, b\rangle &= \lim_{k\to\infty} \langle \overline{(-\Delta_{A,0})}\rho_j f, \rho_k b\rangle \\
&= \lim_{k\to\infty} \langle f, \rho_j \overline{(-\Delta_{A,0})}\rho_k b\rangle \\
&= \lim_{k\to\infty} (\langle f, \overline{(-\Delta_{A,0})}\rho_j \rho_k b\rangle + R_{j,k}),
\end{aligned}$$

where $R_{j,k} = [\rho_j, \overline{(-\Delta_{A,0})}]\rho_k$. Since for $g \in H^2(\mathbb{R}^n)$, $\overline{(-\Delta_{A,0})}g = \nabla \cdot A\nabla g$ in the distributional sense and ρ_j commutes with ∇, by Lemma 2.2,

$$\begin{aligned}
[\rho_j, \overline{(-\Delta_{A,0})}] &= j\nabla \cdot [(j - \Delta)^{-1}, A]\nabla \\
&= j(j - \Delta)^{-1}\nabla \cdot [-\Delta, A]\nabla(j - \Delta)^{-1},
\end{aligned}$$

where

$$[\Delta, A]_{j,k} = (\nabla A_{j,k}) \cdot \nabla + \nabla \cdot (\nabla A_{j,k}).$$

Therefore

$$
\begin{aligned}
|R_{j,k}| &= j|\langle \nabla f, (j-\Delta)^{-1}[A,\Delta]\nabla (j-\Delta)^{-1}\rho_k b\rangle| \\
&= \sum_{m_1,m_2,m_3} j|\langle \partial_{m_1} f, (j-\Delta)^{-1}(\partial_{m_2}A_{m_1,m_3})\rho_k \partial_{m_2}\partial_{m_3}(j-\Delta)^{-1}b\rangle| \\
&+ \sum_{m_1,m_2,m_3} j|\langle \partial_{m_1} f, (j-\Delta)^{-1}\partial_{m_2}(\partial_{m_2}A_{m_1,m_3})\rho_k \partial_{m_3}(j-\Delta)^{-1}b\rangle| \\
&\leqslant n^3\|\nabla A\|_{L^\infty}\|\rho_j\nabla f\|_{L^2}\|\rho_k \nabla\otimes\nabla(j-\Delta)^{-1}b\|_{L^2} \\
&+ n^3\|\nabla A\|_{L^\infty}\|j^{1/2}(j-\Delta)^{-1}\nabla\otimes\nabla f\|_{L^2}\|\rho_k j^{1/2}(j-\Delta)^{-1}\nabla b\|_{L^2}.
\end{aligned}
$$

Recall that for $g \in L^2$

$$j^{1/2}\nabla(j-\Delta)^{-1}g \to 0 \qquad \text{in} \quad L^2$$

and

$$\nabla\otimes\nabla(j-\Delta)^{-1}g \to 0 \qquad \text{in} \quad L^2,$$

where

$$(\nabla\otimes\nabla)_{m,\ell} = \partial_m\partial_\ell.$$

Since $f \in H^1$, we get

$$
\begin{aligned}
&n^{-3}\limsup_{j\to\infty}\limsup_{k\to\infty}|R_{j,k}| \\
&\leqslant \limsup_{j\to\infty}\|\nabla A\|_{L^\infty}\|\rho_j\nabla f\|_{L^2}\|\nabla\otimes\nabla(j-\Delta)^{-1}b\|_{L^2} \\
&+ \limsup_{j\to\infty}\|\nabla A\|_{L^\infty}\|j^{1/2}(j-\Delta)^{-1}\nabla\otimes\nabla f\|_{L^2}\|j^{1/2}(j-\Delta)^{-1}\nabla b\|_{L^2} \\
&\leqslant \limsup_{j\to\infty}\|\nabla A\|_{L^\infty}\|\nabla f\|_{L^2}\|\nabla\otimes\nabla(j-\Delta)^{-1}b\|_{L^2} \\
&+ \limsup_{j\to\infty}\|\nabla A\|_{L^\infty}\|\nabla f\|_{L^2}\|j^{1/2}(j-\Delta)^{-1}\nabla b\|_{L^2} = 0.
\end{aligned}
$$

Moreover, as $j \to \infty$,

$$\lim_{k\to\infty}\langle f, \overline{(-\Delta_{A,0})}\rho_j\rho_k b\rangle = \lim_{k\to\infty}\langle \rho_j\mathcal{H}_A f, \rho_k b\rangle = \langle \rho_j\mathcal{H}_A f, b\rangle \to \langle \mathcal{H}_A f, b\rangle.$$

Hence, if h satisfies (2.3), then for any $f \in D(\mathcal{H}_A)$,

$$\langle(\mathcal{H}_A+1)f, h\rangle = \lim_{j\to\infty}\langle((-\Delta_{A,0})+1)\rho_j f, h\rangle = 0.$$

Therefore, $h \perp \operatorname{Ran}(\mathcal{H}_A+1)$ and the self-adjointness of $\mathcal{H}_A$ implies $h = 0$. □

3. Local well-posedness of (1.1)

This section is devoted to the proof of the local well-posedness for the Cauchy problem associated with the model (1.1), where $u_0(x) = u(0,x)$ is considered as initial datum. More precisely, we give now a proof of Theorem 3.

At first, we give the definition of $\mathcal{D}_{A,V}$. We use a functional calculus for the fractional powers of self-adjoint operators based on the integral representation below (see (4.7) in [7], for example).

Definition 6. *Let $\mathcal{A}$ be a non-negative self-adjoint operator. For $0 < s < 2$,*

$$\mathcal{A}^{s/2} = C_0(s) \int_0^\infty \lambda^{s/2-1} \mathcal{A}(\lambda + \mathcal{A})^{-1} d\lambda \tag{3.1}$$

where

$$C_0(s) = \left(\int_0^\infty \lambda^{s/2-1} (\lambda + 1)^{-1} d\lambda \right)^{-1} = \frac{\sin\left(s\frac{\pi}{2}\right)}{\pi}.$$

We remark that here the formula

$$x^{s/2} = \frac{\sin\left(s\frac{\pi}{2}\right)}{\pi} \int_0^{+\infty} t^{s/2-1} \frac{x}{t+x} \, dt, \qquad x \geqslant 0\,, \quad s \in (0,2)\,. \tag{3.2}$$

plays a critical role.

Now we can conclude this section by proving Theorem 3.

Proof of Theorem 3. We rewrite (1.1) in the integral form by means of its Duhamel's formulation

$$u(t) = e^{it\mathcal{D}_{A,V}} u_0 + \int_0^t e^{i(t-\tau)\mathcal{D}_{A,V}} |u(\tau)|^{p-1} u(\tau) \, d\tau. \tag{3.3}$$

Here $e^{it\mathcal{D}_{A,V}}$ stands for the propagator associated with linear hGLK equation, namely (1.1) with trivial RHS. Briefly speaking, $e^{it\mathcal{D}_{A,V}} f$ solves the linear hGLK with f as initial datum. By Lemma A.2, $e^{it\mathcal{D}_{A,V}}$ is a uniformly bounded operator on H^s for $n = 1$ and $s = 1$ or for $n = 2$, 3 and $s = 2$. A standard fixed point argument implies that (3.3) has a solutions in $C([0,T); H^1)$ if $n = 1$ and in $C([0,T); H^2)$ if $n = 2, 3$. $\square$

4. Commutator Estimates

In this section, we assume A1, A2, A3, H1, and H2.

4.1. **Preliminary.**

The following representation is essential for our approach to study commutator estimates.

Lemma 4.1.

$$\begin{aligned}
&\left\langle g, [(\mathcal{H}_{A,V})^{s/2}, f] h \right\rangle \\
&= -C_0(s) \int_0^\infty \lambda^{s/2} \langle (\lambda + \mathcal{H}_{A,V})^{-1} g, [\mathcal{H}_{A,V}, f] (\lambda + \mathcal{H}_{A,V})^{-1} h \rangle \, d\lambda.
\end{aligned}$$

Proof. By (3.1), we have

$$\begin{aligned}
&\langle g, [\mathcal{H}_{A,V}^{s/2}, f]h\rangle \\
&= C_0(s) \int_0^\infty \lambda^{s/2-1} \langle g, [\mathcal{H}_{A,V}(\lambda + \mathcal{H}_{A,V})^{-1}, f]h\rangle \, d\lambda \\
&= C_0(s) \left(\int_0^\infty \lambda^{s/2} \langle g, [(\lambda + \mathcal{H}_{A,V})^{-1}, f]h\rangle \, d\lambda \right).
\end{aligned}$$

Therefore Lemma 2.3 implies Lemma 4.1. □

Lemma 4.2. *Let $\mathcal{A}$ be a non-negative self-adjoint operator. For $\sigma > \frac{1}{4}$,*

$$\|(\cdot)^{\sigma-3/4} \mathcal{A}^{1/4} (\cdot + \mathcal{A})^{-\sigma} f\|_{L^2((0,\infty);L^2)} \leqslant \left(\int_0^\infty \frac{\lambda^{2\sigma-3/2}}{(\lambda+1)^{2\sigma}} \, d\lambda \right)^{1/2} \|f\|_{L^2}.$$

Proof. Using the spectral measures E_μ for $\mathcal{A}$ (see [20, Theorem VII.7], for instance), we can write

$$\begin{aligned}
&\|(\cdot)^{\sigma-3/4} \mathcal{A}^{1/4} (\cdot + \mathcal{A})^{-\sigma} f\|_{L^2((0,\infty);L^2)}^2 \\
&\leqslant \int_0^\infty \int_0^\infty \frac{\lambda^{2\sigma-3/2} \mu^{1/2}}{(\lambda+\mu)^{2\sigma}} \, d\|E_\mu(f)\|_{L^2}^2 \, d\lambda \\
&= \int_0^\infty \frac{\lambda^{2\sigma-3/2}}{(\lambda+1)^{2\sigma}} \, d\lambda \, \|f\|_{L^2}^2.
\end{aligned}$$

□

In the next lemma, we recall the well-known result that the function $t \to t^s$, $s \in [0, 1]$, is operator monotone on the set of bounded operators in a Hilbert space. One can see [18] for the original matrix-valued version of the statement, [10, Proposition 4.2.8] for the case $s = 1/2$ and [19] for a short proof of the general case. See also [9, 12].

Lemma 4.3 ([18], [19], [10])**.** *Let $(\mathcal{A}_1, D(\mathcal{A}_1))$ and $(\mathcal{A}_2, D(\mathcal{A}_2))$ be two positive self-adjoint operators on L^2 satisfying $D(\mathcal{A}_2) \subset D(\mathcal{A}_1)$ and*

$$\langle f, \mathcal{A}_1 f\rangle \leqslant \langle f, \mathcal{A}_2 f\rangle.$$

Then

$$\langle f, \mathcal{A}_1^s f\rangle \leqslant \langle f, \mathcal{A}_2^s f\rangle \tag{4.1}$$

for $0 < s \leqslant 1$. Moreover, if $\mathcal{A}_1$ is invertible, so is $\mathcal{A}_2$, and

$$\langle f, \mathcal{A}_2^{-1} f\rangle \leqslant \langle f, \mathcal{A}_1^{-1} f\rangle.$$

4.2. **Fractional Leibniz Rules.** Here we collect some useful Leibniz rules for fractional power of the classical Laplace operator.

Lemma 4.4 ([23, Proposition 4.1.A])**.** *Let f be a Lipschitz function. Then for any $g \in H^1$*

$$\|[(-\Delta)^{1/2}, f]g\|_{L^2} \leqslant C\|f\|_{\mathrm{Lip}}\|g\|_{L^2}.$$

Lemma 4.5 ([14, Lemma A.12])**.** *For $0 < s < 1$, there exists $C > 0$ such that*

$$\|(-\Delta)^{s/2}(fg) - f(-\Delta)^{s/2}g - g(-\Delta)^{s/2}f\|_{L^2} \leqslant C\|f\|_{L^\infty}\|(-\Delta)^{s/2}g\|_{L^2}.$$

Remark 4.1. *In* [17]*, one can find the refined estimate*

$$\|(-\Delta)^{s/2}(fg) - f(-\Delta)^{s/2}g - g(-\Delta)^{s/2}f\|_{L^2} \lesssim \|f\|_{\mathrm{BMO}}\|(-\Delta)^{s/2}g\|_{L^2}.$$

and more general estimates, but for simplicity, we use only Lemma 4.5.

In the sequel, we shall also need a generalization, obtained in [8], of the classical Kato-Ponce estimate, introduced in the seminal and well celebrated work [13]. We recall it.

Lemma 4.6 ([8, Theorem 1])**.** *Let $1/2 < r < \infty$, $1 < p_1, p_2, q_1, q_2 \leqslant \infty$ satisfying*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

For $s > \max\{0, n/r - n\}$ or $s \in 2\mathbb{N}$ (the set of positive even integers), there exists $C > 0$ such that

$$\begin{aligned}&\|(-\Delta)^{s/2}(fg)\|_{L^r}\\ &\leqslant C\|(-\Delta)^{s/2}f\|_{L^{p_1}}\|g\|_{L^{q_1}} + C\|f\|_{L^{p_2}}\|(-\Delta)^{s/2}g\|_{L^{q_2}}.\end{aligned}$$

4.3. **Key estimate for Proposition 2.** The purpose of this subsection is to show that the commutator between $\mathcal{D}_{A,V}$ and a localized weight function is realized as a bounded operator in L^2 under the following assumptions:

$$\|(\lambda + \mathcal{H}_{A,V})^{-\sigma}f\|_{L^2} \lesssim \|(\lambda - \Delta)^{-\sigma}f\|_{L^2}, \quad \forall f \in L^2,\ \lambda > 0 \tag{4.2}$$

for some $1/4 < \sigma \leqslant 1$;

$$\|\mathcal{D}_{A,V}f\|_{L^2} \lesssim \|(-\Delta)^{1/2}f\|_{L^2}, \quad \forall f \in H^1; \tag{4.3}$$

$$\|(-\Delta)^{1/4}f\|_{L^2} \lesssim \|\mathcal{H}_{A,V}^{1/4}f\|_{L^2}, \quad \forall f \in H^{1/2}; \tag{4.4}$$

for any $g, h \in H^{1/2}$

$$\begin{aligned}\langle g, \nabla(A(\nabla f)h)\rangle &\lesssim \|(-\Delta)^{1/4}\nabla f\|_{L^\infty}\|(-\Delta)^{1/4}g\|_{L^2}\|h\|_{L^2}\\ &\quad + \|\nabla f\|_{L^\infty}\|(-\Delta)^{1/4}g\|_{L^2}\|(-\Delta)^{1/4}h\|_{L^2}.\end{aligned} \tag{4.5}$$

Lemma 4.7. *Assume* A1, A2, A3, H1, *and* H2. *Let* A *and* V *satisfy the properties* (4.2), (4.3), (4.4), *and* (4.5). *Then for any* $j \in \mathbb{Z}$,

$$\|[\mathcal{D}_{A,V}, P_j f]P_{\leqslant j}h\|_{L^2} \lesssim 2^j \|P_j f\|_{L^\infty}\|h\|_{L^2}, \tag{4.6}$$

$$|\langle P_{>j}g, [\mathcal{D}_{A,V}, P_j f]P_{>j}h\rangle| \lesssim 2^j \|P_j f\|_{L^\infty}\|g\|_{L^2}\|h\|_{L^2}. \tag{4.7}$$

Proof. We first prove (4.6). The same relation given at the beginning of the proof of Lemma 2.2 and the triangular inequality gives

$$\begin{aligned}
&\|[\mathcal{D}_{A,V}, P_j f]P_{\leqslant j}h\|_{L^2} \\
&\leqslant \|\mathcal{D}_{A,V}(-\Delta)^{-1/2}[(-\Delta)^{1/2}, P_j f]P_{\leqslant j}h\|_{L^2} \\
&+ \|[\mathcal{D}_{A,V}(-\Delta)^{-1/2}, P_j f](-\Delta)^{1/2}P_{\leqslant j}h\|_{L^2}.
\end{aligned} \tag{4.8}$$

By (4.3) and Lemma 4.4, the first term on the R.H.S. of (4.8) is estimated as

$$\begin{aligned}
&\|\mathcal{D}_{A,V}(-\Delta)^{-1/2}[(-\Delta)^{1/2}, P_j f]P_{\leqslant j}h\|_{L^2} \\
&\leqslant \|[(-\Delta)^{1/2}, P_j f]P_{\leqslant j}h\|_{L^2} \lesssim 2^j \|P_j f\|_{L^\infty}\|P_{\leqslant j}h\|_{L^2},
\end{aligned}$$

where we have used the fact that

$$\|\nabla P_j f\|_{L^\infty} \lesssim 2^j \|P_j f\|_{L^\infty}.$$

By (4.3), the second term on the R.H.S. of (4.8) is estimated as

$$\begin{aligned}
&\|[\mathcal{D}_{A,V}(-\Delta)^{-1/2}, P_j f](-\Delta)^{1/2}P_{\leqslant j}h\|_{L^2} \\
&\lesssim \|P_j f\|_{L^\infty}\|(-\Delta)^{1/2}P_{\leqslant j}h\|_{L^2} \lesssim 2^{j+1}\|P_j f\|_{L^\infty}\|h\|_{L^2}.
\end{aligned}$$

We next prove (4.7). By Lemma 4.1, it is enough to show

$$\begin{aligned}
&\left|\int_0^\infty \lambda^{1/2}\big\langle(\lambda + \mathcal{H}_{A,V})^{-1}P_{>j}g, \nabla\cdot A(\nabla P_j f)(\lambda + \mathcal{H}_{A,V})^{-1}P_{>j}h\big\rangle\, d\lambda\right| \\
&\lesssim 2^j \|P_j f\|_{L^\infty}\|g\|_{L^2}\|h\|_{L^2}.
\end{aligned} \tag{4.9}$$

By (4.4) and (4.5), the L.H.S. of (4.9) is estimated by

$$\begin{aligned}
&2^j \|P_j f\|_{L^\infty}\|g\|_{L^2}\|h\|_{L^2} \\
&\lesssim 2^{3j/2}\|A\|_{L^\infty}\|P_j f\|_{L^\infty} \\
&\times \int_0^\infty \lambda^{1/2}\|\mathcal{H}_{A,V}^{1/4}(\lambda + \mathcal{H}_{A,V})^{-1}P_{>j}g\|_{L^2}\|(\lambda + \mathcal{H}_{A,V})^{-1}P_{>j}h\|_{L^2}\, d\lambda \\
&+ 2^j \|A\|_{W^{1,\infty}}\|P_j f\|_{L^\infty} \\
&\times \int_0^\infty \lambda^{1/2}\|\mathcal{H}_{A,V}^{1/4}(\lambda + \mathcal{H}_{A,V})^{-1}P_{>j}g\|_{L^2}\|\mathcal{H}_{A,V}^{1/4}(\lambda + \mathcal{H}_{A,V})^{-1}P_{>j}h\|_{L^2}\, d\lambda.
\end{aligned}$$

Then, by Lemma 4.2, the first integral on the R.H.S. of the last inequality is estimated by

$$
\begin{aligned}
&2^{3j/2}\int_0^\infty \lambda^{1/2}\|\mathcal{H}_{A,V}^{1/4}(\lambda+\mathcal{H}_{A,V})^{-1}P_{>j}g\|_{L^2}\|(\lambda+\mathcal{H}_{A,V})^{-1}P_{>j}h\|_{L^2}\,d\lambda\\
&\lesssim 2^{3j/2}\int_0^\infty \lambda^{\sigma-1/2}\|\mathcal{H}_{A,V}^{1/4}(\lambda+\mathcal{H}_{A,V})^{-1}P_{>j}g\|_{L^2}\|(\lambda-\Delta)^{-\sigma}P_{>j}h\|_{L^2}\,d\lambda\\
&\lesssim 2^{j}\|(\cdot)^{1/4}\mathcal{H}_{A,V}^{1/4}(\cdot+\mathcal{H}_{A,V})^{-1}P_{>j}g\|_{L^2(0,\infty;L^2)}\\
&\times\|(\cdot)^{\sigma-3/4}(-\Delta)^{1/4}(\cdot-\Delta)^{-\sigma}P_{>j}h\|_{L^2(0,\infty;L^2)}\\
&\lesssim 2^j\|g\|_{L^2}\|h\|_{L^2}
\end{aligned}
$$

with $1/4<\sigma\leqslant 1$ satisfying (4.2). The second integral is also estimated by

$$
\begin{aligned}
&2^{j}\int_0^\infty \lambda^{1/2}\|\mathcal{H}_{A,V}^{1/4}(\lambda+\mathcal{H}_{A,V})^{-1}P_{>j}g\|_{L^2}\|\mathcal{H}_{A,V}^{1/4}(\lambda+\mathcal{H}_{A,V})^{-1}P_{>j}h\|_{L^2}\,d\lambda\\
&\lesssim 2^{j}\|(\cdot)^{1/4}\mathcal{H}_{A,V}^{1/4}(\cdot+\mathcal{H}_{A,V})^{-1}P_{>j}g\|_{L^2(0,\infty;L^2)}\\
&\times\|(\cdot)^{1/4}\mathcal{H}_{A,V}^{1/4}(\cdot+\mathcal{H}_{A,V})^{-1}P_{>j}h\|_{L^2(0,\infty;L^2)}\\
&\lesssim 2^j\|g\|_{L^2}\|h\|_{L^2}.
\end{aligned}
$$

□

4.4. **Proof of Proposition 2.** We are now in a position to prove Proposition 2. We treat separately the two cases.

Proof of Case 1. At first, we show that Lemma 4.7 implies (1.7) with $V\equiv 0$. (4.2), (4.3), and (4.4) follow from A1. Indeed, (1.5) implies

$$C_1\|\nabla f\|_{L^2}^2\leqslant\langle f,\mathcal{H}_{A,0}f\rangle=\|\mathcal{D}_{A,0}f\|_{L^2}^2\leqslant C_2\|\nabla f\|_{L^2}^2$$

which coincides with (4.3). Therefore, Lemma 4.3 can be applied with $\mathcal{H}_{A,0}$ and $-\Delta$. Hence, the relation (4.1), with $s=1/2$, $\mathcal{A}_1=\mathcal{H}_{A,0}$ and $\mathcal{A}_2=-C\Delta$, coincides with (4.4). Moreover (1.5) implies that one can find two constants c,C with $0<c\leqslant 1\leqslant C$ such that

$$c\langle f,(\lambda-\Delta)f\rangle\leqslant\langle f,(\lambda+\mathcal{H}_{A,0})f\rangle\leqslant C\langle f,(\lambda-\Delta)f\rangle$$

for any $f\in H^2$ and $\lambda\geqslant 0$. Then, Lemma 4.3 implies that for any $f\in L^2$

$$\langle f,(\lambda+\mathcal{H}_{A,0})^{-1}f\rangle\leqslant\langle f,c^{-1}(\lambda-\Delta)^{-1}f\rangle,$$

which coincides with (4.2) with $\sigma=1/2$.

(4.5) may be obtained by decomposing $\partial_j(A_{j,k}(\partial_k f)h)$ as follows:

$$
\begin{aligned}
\partial_j(A_{j,k}(\partial_k f)h) &= (-\Delta)^{1/4}R_j(-\Delta)^{1/4}(A_{j,k}(\partial_k f)h)\\
&= (-\Delta)^{1/4}R_jA_{j,k}(-\Delta)^{1/4}((\partial_k f)h)\\
&+ (-\Delta)^{1/4}R_j((-\Delta)^{1/4}A_{j,k})(\partial_k f)h\\
&+ (-\Delta)^{1/4}R_jB(A_{j,k},(\partial_k f)h), \qquad (4.10)
\end{aligned}
$$

where

$$\mathfrak{F}(R_j f) = \frac{\xi_j}{|\xi|}\hat{f}(\xi)$$

is, up to a complex constant, the standard Riesz transform, and

$$B(A_{j,k}, \partial_k f) := (-\Delta)^{1/4}(A_{j,k}\partial_k f) - A_{j,k}(-\Delta)^{1/4}\partial_k f - \partial_k f(-\Delta)^{1/4}A_{j,k}.$$

The first term on the R.H.S. of (4.10) is easily estimated by the Hölder inequality and Lemma 4.6. Here we recall that (1.5) implies $\|A_{j,k}\|_{L^\infty} < \infty$. The other terms are estimated similarly, since by Lemma 4.5 and (1.6), we have

$$\|B(A_{j,k}, (\partial_k f)h)\|_{L^2} \lesssim \|A\|_{L^\infty}\|(-\Delta)^{1/4}((\partial_k f)h)\|_{L^2}$$

and

$$\|((-\Delta)^{1/4}A_{j,k})(\partial_k f)h\|_{L^2} \lesssim \|(-\Delta)^{1/4}((\partial_k f)h)\|_{L^2},$$

respectively.

We now show Proposition 2 with $V \equiv 0$. Since

$$\begin{aligned}
&\langle g, [\mathcal{D}_{A,0}, f]h\rangle \\
&= \langle P_{\leqslant j}g, [\mathcal{D}_{A,0}, f]h\rangle + \langle g, [\mathcal{D}_{A,0}, f]P_{\leqslant j}h\rangle + \langle P_{>j}g, [\mathcal{D}_{A,0}, f]P_{>j}h\rangle \\
&= -\overline{\langle h, [\mathcal{D}_{A,0}, f]P_{\leqslant j}g\rangle} + \langle g, [\mathcal{D}_{A,0}, f]P_{\leqslant j}h\rangle + \langle P_{>j}g, [\mathcal{D}_{A,0}, f]P_{>j}h\rangle,
\end{aligned}$$

Lemma 4.2 implies the estimate.

We next show (1.7) with $V \not\equiv 0$. (1.7) follows from the fact that for any $g \in H^1$,

$$\|(\mathcal{D}_{A,0} - \mathcal{D}_{A,V})g\|_{L^2} \leqslant \|V\|_{L^{q,\infty}}\|g\|_{L^2}.$$

Indeed, by Lemma 4.1,

$$\begin{aligned}
&C_0(1/2)^{-1}(\mathcal{D}_{A,0} - \mathcal{D}_{A,V})g \\
&= \left(\int_0^\infty \lambda^{1/2}((\lambda + \mathcal{H}_{A,0})^{-1} - (\lambda + \mathcal{H}_{A,V})^{-1})\, d\lambda\right) g \\
&= \left(\int_0^1 \lambda^{1/2}((\lambda + \mathcal{H}_{A,0})^{-1} - (\lambda + \mathcal{H}_{A,V})^{-1})\, d\lambda\right) g \\
&+ \left(\int_1^\infty \lambda^{1/2}(\lambda + \mathcal{H}_{A,V})^{-1}V(\lambda + \mathcal{H}_{A,0})^{-1}\, d\lambda\right) g.
\end{aligned}$$

The L^2-norm of the first integral on the R.H.S. of the last equality is shown to be bounded by the fact that for any non-negative self-adjoint operator $\mathcal{A}$

$$\|(\lambda + \mathcal{A})^{-1}g\|_{L^2} \leqslant \lambda^{-1}\|g\|_{L^2}.$$

By (1.5) and Lemma 4.3,

$$\begin{aligned}
\|V(\lambda + \mathcal{H}_{A,0})^{-1}g\|_{L^2} &\lesssim \|V\|_{L^{q,\infty}}\|(-\Delta)^{n/2q}(\lambda + \mathcal{H}_{A,0})^{-1}g\|_{L^2} \\
&\lesssim \|V\|_{L^{q,\infty}}\|\mathcal{H}_{A,0}^{n/2q}(\lambda + \mathcal{H}_{A,0})^{-1}g\|_{L^2} \\
&\lesssim \|V\|_{L^{q,\infty}}\lambda^{-1+n/2q}\|g\|_{L^2}.
\end{aligned}$$

Then, the L^2-norm of the second integral is shown to be bounded by

$$\int_1^\infty \lambda^{-3/2+n/2q}\, d\lambda < \infty.$$

□

Proof of Case 2. (1.8) follows if we are able to show that

$$-\Delta \sim -\Delta_{A,V}, \tag{4.11}$$

where the equivalence is in the sense of bilinear forms. Indeed, if (4.11) is shown, then (4.2), (4.3), and (4.4) are satisfied and therefore Lemma 4.7 implies (1.8). The relation (4.11) is proved as follows:

$$\begin{aligned}\langle f, -\Delta_{A,V} f\rangle &\geqslant (1-\theta)\langle A\nabla f, \nabla f\rangle \\ &\geqslant C_1(1-\theta)\|\nabla f\|_{L^2}^2, \\ \langle f, -\Delta_{A,V} f\rangle &\leqslant C_2\|\nabla f\|_{L^2}^2 + \||V|^{1/2} f\|_{L^2}^2 \\ &\leqslant C_2\|\nabla f\|_{L^2}^2 + C^2\||V|^{1/2}\|_{L^{n,\infty}}^2\|(-\Delta)^{1/2} f\|_{L^2}^2.\end{aligned}$$

□

5. The finite time blow-up result

Theorem 4 may be concluded be means of the following ODE argument.

Lemma 5.1. *Let $A, B > 0$ and $q > 1$. If $f \in C^1([0,T);\mathbb{R}^+)$ satisfies $f(0) > 0$ and*

$$f' + Af = Bf^q \quad \text{on } [0,T) \text{ for some } T > 0,$$

then

$$f(t) = e^{-At}\Big(f(0)^{-(q-1)} + A^{-1}Be^{-A(q-1)t} - A^{-1}B\Big)^{-\frac{1}{q-1}}.$$

Moreover, if $f(0) > A^{\frac{1}{q-1}}B^{-\frac{1}{q-1}}$, then $T < -\frac{1}{A(q-1)}\log(1 - AB^{-1}f(0)^{-q+1})$.

Proof. For completeness, we sketch the proof. Let $f = e^{-At}g$. Then

$$g' = Be^{-A(q-1)t}g^q$$

and therefore,

$$\frac{1}{1-q}\Big(g^{1-q}(t) - g^{1-q}(0)\Big) = \frac{B}{A(1-q)}(e^{-A(q-1)t} - 1).$$

The conclusion follows straightforward. □

We exploit Lemma 5.1 in the proof of Theorem 4.

Proof of Theorem 4. Case 1. Let $w \in B^1_{\infty,1}(\mathbb{R})$ be a non-negative function satisfying $1/w \in L^\infty \cap L^2$. We put $u = vw$. Then v satisfies

$$\partial_t v + \frac{i}{w}[\mathcal{D}_{A,V}, w]v = w^{p-1}|v|^{p-1}v. \tag{5.1}$$

By multiplying $\overline{v}$ on the both hand sides of (5.1), integrating the resulting equation, and taking the real part,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 \\
&\geqslant \|w^{\frac{p-1}{p+1}}v(t)\|_{L^{p+1}}^{p+1} - \|1/w\|_{L^\infty}\|[\mathcal{D}_{A,V}, w]v\|_{L^2}\|v\|_{L^2} \\
&\geqslant \|1/w\|_{L^2}^{-p+1}\|v\|_{L^2}^{p+1} - \|1/w\|_{L^\infty}\|[\mathcal{D}_{A,V}, w]v\|_{L^2}\|v\|_{L^2} \\
&\geqslant \|1/w\|_{L^2}^{-p+1}\|v\|_{L^2}^{p+1} - C\|1/w\|_{L^\infty}\|w\|_{B^1_{\infty,1}}\|v\|_{L^2}^2,
\end{aligned} \tag{5.2}$$

where we have used that

$$\|v\|_{L^2} \leqslant \|1/w^{\frac{p-1}{p+1}}\|_{L^{\frac{2(p+1)}{p-1}}}\|w^{\frac{p-1}{p+1}}v(t)\|_{L^{p+1}} \leqslant \|1/w\|_{L^2}^{\frac{p-1}{p+1}}\|w^{\frac{p-1}{p+1}}v(t)\|_{L^{p+1}}.$$

By (5.2), we apply Lemma 5.1 with

$$\begin{aligned}
A &= C\|1/w\|_{L^\infty}\|w\|_{B^1_{\infty,1}}, \\
B &= \|1/w\|_{L^2}^{-p+1}.
\end{aligned}$$

Then (1.9) implies that $\|v(t)\|_{L^2}$ is not uniformly controlled.
Case 2. We rescale $w \in \dot{B}^1_{\infty,1}$ as $w_R = w(\cdot/R)$ with $R > 0$. Then by (5.2),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 \\
&\geqslant \|1/w_R\|_{L^2}^{-p+1}\|v\|_{L^2}^{p+1} - \|1/w_R\|_{L^\infty}\|[(-\Delta_{A,V})^{1/2}, w_R]v\|_{L^2}\|v\|_{L^2} \\
&\geqslant R^{-(p-1)/2}\|1/w\|_{L^2}^{-p+1}\|v\|_{L^2}^{p+1} - CR^{-1}\|1/w\|_{L^\infty}\|w\|_{\dot{B}^1_{\infty,1}}\|v\|_{L^2}^2.
\end{aligned}$$

We apply Lemma 5.1 with

$$\begin{aligned}
A &= CR^{-1}\|1/w\|_{L^\infty}\|w\|_{B^1_{\infty,1}}, \\
B &= R^{-(p-1)/2}\|1/w\|_{L^2}^{-p+1},
\end{aligned}$$

which means $AB^{-1} \sim R^{-1+(p-1)/2}$. Therefore, if $1 < p < 3$, $AB^{-1} \to 0$ as $R \to \infty$ and this shows Theorem 4. □

Appendix A. Equivalence of Sobolev norms

We show the equivalence of the standard H^s-norms (for $s = 1, 2$) and the ones induced by the Hamiltonian $\mathcal{H}_{A,V}$. We begin with simple a priori estimates that imply the equivalence of H^1 norms.

Lemma A.1. *Assume* H2. *If* $V \in L^{q,\infty}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ *with* $q > \max\{1, n/2\}$, *then for any* $\alpha \in (0,1)$ *there exists* $C > 0$, *so that for any* $f \in C_c^\infty(\mathbb{R}^n)$,

$$\langle(-\alpha\nabla \cdot A\nabla - |V|)f, f\rangle \geqslant -C\|f\|_{L^2}^2 \tag{A.1}$$

and

$$\langle A\nabla f, \nabla f\rangle + \langle Vf, f\rangle + \|f\|_{L^2}^2 \sim \|f\|_{H^1}^2. \tag{A.2}$$

Proof. We know that uniform ellipticity assumption implies

$$\langle(-\nabla \cdot A\nabla)\, f, f\rangle \sim \|f\|_{\dot{H}^1}.$$

We need to prove the inequality

$$\int_{\mathbb{R}^n} |V||f|^2 dx \lesssim \|f\|_{\dot{H}^s}^2 \tag{A.3}$$

with $0 < s < 1$, since this estimate and the Gagliardo-Nirenberg interpolation inequality

$$\|f\|_{\dot{H}^s} \lesssim \|f\|_{\dot{H}^1}^s \|f\|_{L^2}^{1-s}$$

imply

$$\int_{\mathbb{R}^n} |V||f|^2 dx \leqslant \langle(-\alpha\nabla \cdot A\nabla)f, f\rangle + C\|f\|_{L^2}^2,$$

so we have (A.1) and (A.2).

In order to prove (A.3), we take

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{2q}, \;\; s = \frac{n}{2q}$$

and then we can write

$$\left(\int_{\mathbb{R}^n} |V||f|^2 dx\right)^{1/2} \lesssim \||V|^{1/2}\|_{L^{2q,\infty}} \|f\|_{L^{r,2}} \lesssim \|V\|_{L^{q,\infty}}^{1/2} \|f\|_{\dot{H}^s}^2$$

due to Hölder inequality in Lorentz spaces and Sobolev embedding. The requirement $0 < s < 1$ is fulfilled due to the assumption $q > n/2$. □

Lemma A.2. *Assume* A1, A2, H1, *and* H2. *Then one can find positive constants* $C_1 < C_2$ *so that for any* $f \in C_c^\infty(\mathbb{R}^n)$,

$$C_1\|f\|_{H^2} \leqslant \|-\Delta_{A,V} f\|_{L^2} + \|f\|_{L^2} \leqslant C_2\|f\|_{H^2}. \tag{A.4}$$

Proof. The right inequality of (A.4) follows directly from the representation of $\Delta_{A,V}$. Indeed

$$\Delta_{A,V} f = (\nabla A) \cdot \nabla f + \sum_{j,k=1}^n A_{j,k}(x)\partial_j\partial_k f + Vf.$$

Further we can take

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{q}, \;\; s = \frac{n}{q}$$

and then we can write

$$\|Vf\|_{L^2} \lesssim \|V\|_{L^{q,\infty}} \|f\|_{L^{r,2}} \lesssim \|V\|_{L^{q,\infty}} \|f\|_{H^s} \tag{A.5}$$

with $s \in (0,2)$ so interpolation yields the right-side estimate.

Next we show the left inequality of (A.4) with $V = 0$. By A1,

$$\begin{aligned}
&C_1 \|(-\Delta) f\|_{L^2}^2 \\
&= C_1 \int_{\mathbb{R}^n} \overline{(-\Delta) f(x)} (-\Delta) f(x)\, dx \\
&\leqslant \int_{\mathbb{R}^n} \overline{(-\Delta)^{1/2} f(x)} (-\Delta_{A,0})(-\Delta)^{1/2} f(x)\, dx \\
&= \int_{\mathbb{R}^n} \overline{(-\Delta) f(x)} (-\Delta_{A,0}) f(x)\, dx + \int_{\mathbb{R}^n} \nabla \overline{(-\Delta)^{1/2} f(x)} \cdot \mathcal{G}_A f(x)\, dx \\
&\leqslant \|(-\Delta) f\|_{L^2} (\| - \Delta_{A,0} f\|_{L^2} + \|\mathcal{G}_A f\|_{L^2}),
\end{aligned}$$

where $\mathcal{G}_A = [(-\Delta)^{1/2}, A]\nabla$ so that

$$\nabla \cdot \mathcal{G}_A f = [(-\Delta_{A,0}), (-\Delta)^{1/2}] f = \nabla [A, (-\Delta)^{1/2}] \nabla f.$$

Then, by Lemma 4.4, the Gagliardo-Nirenberg and the Young inequalities,

$$\|\mathcal{G}_A f\|_{L^2} \leqslant C \|\nabla A\|_{L^\infty} \|\nabla f\|_{L^2} \leqslant \frac{1}{2} \|(-\Delta) f\|_{L^2} + C \|f\|_{L^2},$$

which in turn implies

$$\|(-\Delta) f\|_{L^2} \leqslant C \|f\|_{L^2} + C \| - \Delta_{A,0} f\|_{L^2}.$$

This inequality and (A.5) prove the left estimate in (A.4). □

Appendix B. Estimate of the weight function

Our choice of w for the proof of the blow-up result is $w(x) = \langle x \rangle^a$ with $a \in (1/2, 1)$. The lower bound of a is required to guarantee that $1/w \in L^2(\mathbb{R})$ for Theorem 4. The upper bound of a follows from the following Proposition:

Proposition 7. *For $a < 1$,*

$$\langle \cdot \rangle^a \in \dot{B}^1_{\infty,1}.$$

Proof. We recall that $2^{-sj} P_j (-\Delta)^{s/2}$ is a bounded operator on L^∞. Therefore for $j \geqslant 0$,

$$\|P_j (-\Delta)^{1/2} \langle x \rangle^a\|_{L^\infty} \lesssim 2^{-j} \|2^j P_j (-\Delta)^{-1/2} \Delta \langle x \rangle^a\|_{L^\infty} \lesssim 2^{-j} \|\Delta \langle x \rangle^a\|_{L^\infty}$$

which implies $P_{\geqslant 0} \langle \cdot \rangle^a \in \dot{B}^1_{\infty,1}$. Moreover, for $a > 0$ since

$$\|P_j f\|_{L^\infty} \lesssim 2^{jn/p} \|f\|_{L^p}$$

and

$$|\nabla \langle x \rangle^a| \lesssim \langle x \rangle^{a-1},$$

by taking $p = \frac{2n}{1-a}$

$$\begin{aligned}\|P_j(-\Delta)^{1/2}\langle x\rangle^a\|_{L^\infty} &\lesssim 2^{jn/p}\|\nabla(-\Delta)^{-1/2}\nabla\langle x\rangle^a\|_{L^p}\\ &\lesssim 2^{j(1-a)/2}\|\langle x\rangle^{-1}\|_{L^{2n}}^{1-a}.\end{aligned}$$

Therefore

$$P_{\leqslant 0}\langle x\rangle^a \in \dot{B}^1_{\infty,1}. \tag{B.1}$$

For $a \leqslant 0$, it is easy to see (B.1). □

Remark B.1. *It is worth mentioning that the estimate above is valid in arbitrary dimension, but we can use only $n = 1$ in order to prove Theorem 4.*

Remark B.2. *The upper bound for the function a in Proposition 7 is optimal. Indeed,*

$$\langle\cdot\rangle \notin \dot{B}^1_{\infty,1}(\mathbb{R}^n)$$

for any n. In order to show this, we estimate the following equivalent norm for $\dot{B}^1_{\infty,1}(\mathbb{R}^n)$:

$$|||f|||_{\dot{B}^1_{\infty,1}(\mathbb{R}^n)} = \int_0^\infty \sup_{|y|<t}\|f(\cdot+y) - 2f(\cdot) + f(\cdot-y)\|_{L^\infty(\mathbb{R}^n)}\frac{dt}{t^2}.$$

For details, see [2, 6.3.1. Theorem]. *Then, by substituting $x = 0$, for $t \geqslant 1$,*

$$\begin{aligned}&\sup_{|y|<t}\sup_{x\in\mathbb{R}^n}|\langle x+y\rangle - 2\langle x\rangle + \langle x-y\rangle|\\ &\geqslant \sup_{|y|<t} 2(\langle y\rangle - 1)\\ &\geqslant 2\left(\left(1+\frac{t^2}{4}\right)^{1/2} - 1\right) > \frac{t}{8},\end{aligned}$$

where we have used the fact that

$$1 + \frac{t^2}{4} \geqslant 1 + \frac{t^2}{8} + \frac{t^2}{256} \geqslant \left(1 + \frac{t}{16}\right)^2.$$

Therefore,

$$\begin{aligned}|||\langle\cdot\rangle|||_{\dot{B}^1_{\infty,1}(\mathbb{R}^n)} &\geqslant \int_1^\infty \sup_{|y|<t}\|\langle\cdot+y\rangle - 2\langle\cdot\rangle + \langle\cdot-y\rangle\|_{L^\infty(\mathbb{R}^n)}\frac{dt}{t^2}\\ &\geqslant \frac{1}{8}\int_1^\infty \frac{dt}{t} = \infty.\end{aligned}$$

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