

COMBINING RANGE AND DIRECTION FOR IMPROVED LOCALIZATION

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ABSTRACT

Self-localization of nodes in a sensor network is typically achieved using either range or direction measurements; in this paper, we show that a constructive combination of both improves the estimation. We propose two localization algorithms that make use of the differences between the sensors' coordinates, or *edge vectors*; these can be calculated from measured distances and angles. Our first method improves the existing edge-multidimensional scaling algorithm (E-MDS) by introducing additional constraints that enforce geometric consistency between the edge vectors. On the other hand, our second method decomposes the edge vectors onto 1-dimensional spaces and introduces the concept of *coordinate difference matrices* (CDMs) to independently regularize each projection. This solution is optimal when Gaussian noise is added to the edge vectors. We demonstrate in numerical simulations that both algorithms outperform state-of-the-art solutions.

Index Terms—Range and direction measurements, sensor arrays, calibration, node localization, measurement uncertainty.

1. INTRODUCTION

Consider a sensor network, where nodes can measure distances and angles between each other. A natural question that arises is the recovery of the sensors' locations given a set of such measurements. This problem is central to many applications, including indoor localization, autonomous vehicles, or intelligent warehouses.

If only distances are given, the problem is well studied; for instance, the theory of Euclidean distance matrices (EDMs) provides both a detailed description of fundamental limits and a plethora of algorithms to localize the sensors [1, 2, 3, 4, 5]. There have also been a number of interesting studies for the case of angle-only measurements [6, 7, 8], although the theory is not as mature as the range-only case.

However, setups leveraging both distance and angle measurements did not attract as much attention. This is surprising given the fact that a multimodal approach could provide a significant improvement in accuracy and robustness.

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Common solutions in localization assume that we have a single node, which we would like to estimate using measurements from a number of anchors at fixed known positions. This problem is known as lateration [9] when using distance measurements, and angulation [10] with angle measurements.

In contrast, many applications require to jointly localize all nodes of the network. For example, in signal processing, sensor arrays are often employed to measure physical phenomena. This includes wireless sensor networks measuring weather conditions [11], ultrasonic sensors detecting breast cancer in ultrasound tomography [12], and room geometry estimation from a microphone array [13]. Furthermore, in the field of acoustics, the most accurate solutions to common problems such as source localization, source separation and noise reduction rely on microphone arrays with precisely known microphone locations [14]. Therefore, accurate localization of all sensors in the network is critical for many tasks.

In this paper, we consider algorithms for multimodal localization of all nodes in a sensor network. Since the distance and angle measurements are often of a fundamentally different nature, it is not obvious how to combine them into a precise mathematical framework [15]. We propose two new methods that achieve state-of-the-art performance, each outperforming the other in a different noise regime.

Our first method—*constrained edge-kernel*—is in a similar spirit to the well-known multidimensional scaling (MDS) algorithm [3]. It is an extension of [16], with additional constraints added to the edge vectors to enforce geometric consistency.

The key components of our second proposed method are entitled *coordinate difference matrices* (CDMs). CDMs have elegant yet simple properties that we can leverage to find a closed-form solution to reconstruct the sensor locations. This solution can easily cope with multiple and missing entries by introducing weights for each measurement. The two proposed methods are complementary in the sense that the constrained edge-kernel performs better in the case of high angle noise, while the CDM-based approach is better at handling more noise in distances.

2. PROBLEM STATEMENT

Consider a set of points in a 2-dimensional space¹ with coordinates denoted by the vector $\mathbf{x}_n \in \mathbb{R}^2$ for $n = 1, \dots, N$. Stacking up all the N vectors in a matrix yields the coordinate matrix $\mathbf{X} \in \mathbb{R}^{N \times 2}$.

¹In this paper, we derive all the results in 2D. The generalization to 3D is mathematically straightforward for all the described algorithms. However, their performance might substantially change in dimensions higher than 2, which we aim to address in future work.

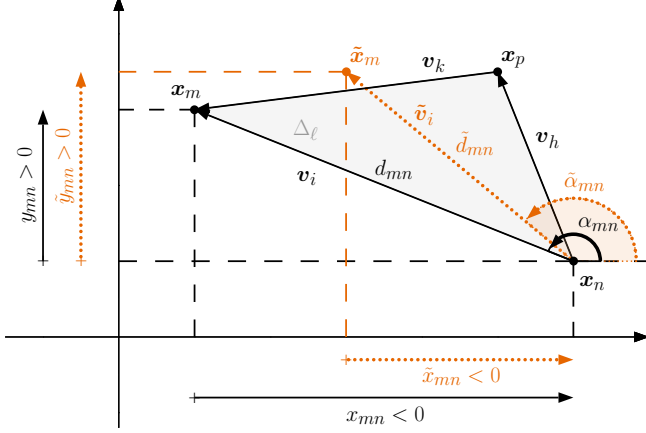


Fig. 1. Localization setup in 2D. The measured distance \tilde{d}_{mn} and the directed angle $\tilde{\alpha}_{mn}$ can be mutually expressed with the vector $\tilde{\mathbf{v}}_{mn}$ or, equivalently, with the signed projections on the x - and y -axis: \tilde{x}_{mn} and \tilde{y}_{mn} . Triangle Δ_ℓ is such that $\mathbf{v}_i - \mathbf{v}_h = \mathbf{v}_k$.

We define the edge vector \mathbf{v}_{mn} corresponding to the pair of points $(\mathbf{x}_n, \mathbf{x}_m)$ by

$$\mathbf{v}_{mn} = \mathbf{x}_m - \mathbf{x}_n. \quad (1)$$

The length of the edge vector is the distance between the points, denoted by

$$d_{mn} = \|\mathbf{x}_n - \mathbf{x}_m\|_2, \quad (2)$$

while \tilde{d}_{mn} refers to its noisy measurement. The orientation of the vector is defined with a *directed angle*, meaning that it lies between 0 and 2π with respect to some common reference direction (in this case the x -axis). This assumption is realistic because devices are often equipped with a compass, providing their absolute orientation. Therefore, we can use any off-the-shelf method [17] to measure the directions of arrival and then convert them into directed angles. We denote the noiseless and noisy measured directed angles by α_{mn} and $\tilde{\alpha}_{mn}$, respectively, as depicted in Fig. 1. Then, the coordinates of edge vectors relate to distances and angles as

$$(v_{mn})_x = d_{mn} \cos \alpha_{mn} \quad \text{and} \quad (v_{mn})_y = d_{mn} \sin \alpha_{mn}. \quad (3)$$

Our goal is to estimate the points $\hat{\mathbf{X}}$, given noisy measurements of distances \tilde{d}_{mn} and angles $\tilde{\alpha}_{mn}$.

3. EDGE-KERNEL BASED SOLUTIONS

Before describing our constrained edge-kernel approach, we first summarize the original edge-kernel method [16].

3.1. The Edge-Kernel Formulation

In [16], Macagnano *et al.* propose a new MDS formulation, referred to as edge multidimensional scaling (E-MDS), which allows a unified processing of angle and range information. In the E-MDS-based formulation, the entities are the vectors pointing from one point to another, while the measure of dissimilarity is their inner product.

To simplify the notation and allow the construction of a matrix whose elements depend on two edge vectors i and j , we first re-index

the edges \mathbf{v}_{mn} from (1) to \mathbf{v}_i with $i = (m-1)N - \frac{m(m+1)}{2} + n$, where $m = 1, \dots, N-1$, $n = m+1, \dots, N$, and $i = 1, \dots, E$, $E = \frac{N(N-1)}{2}$. If we consider a matrix consisting of elements \mathbf{v}_{mn} , this corresponds to flattening the elements of its upper triangle in a row-wise order. Then, we define the edge-kernel $\mathbf{K}_E \in \mathbb{R}^{E \times E}$ of a fully connected point set with E edges and the element (i, j) as

$$\begin{aligned} (\mathbf{K}_E)_{ij} &= \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{x}_m - \mathbf{x}_n, \mathbf{x}_q - \mathbf{x}_p \rangle \\ &= d_{mn} d_{pq} \cos(\phi_{ij}), \end{aligned} \quad (4)$$

where d_{mn} is defined in (2), and ϕ_{ij} is the inner angle between vectors \mathbf{v}_i and \mathbf{v}_j . In matrix form, (4) simplifies to

$$\mathbf{K}_E = \mathbf{V} \mathbf{V}^\top = \mathbf{d}_E \mathbf{d}_E^\top \circ \boldsymbol{\Omega}, \quad (5)$$

where $\mathbf{V} \in \mathbb{R}^{E \times 2}$ is the matrix of the edges, $\mathbf{d}_E \in \mathbb{R}^E$ is the vector of the edge lengths and \circ represents the Hadamard (entrywise) product. The elements Ω_{ij} of the angle matrix $\boldsymbol{\Omega}$ contain the cosines of the inner angles between \mathbf{v}_i and \mathbf{v}_j . From (5), it is easy to verify that the rank of the edge kernel is at most 2. Given noisy angle and distance measurements, the authors in [16] propose to reconstruct a denoised kernel by imposing this low-rank constraint and positive-semidefiniteness:

$$\begin{aligned} \hat{\mathbf{V}} &= \arg \min_{\mathbf{V} \in \mathbb{R}^{E \times 2}} \|\tilde{\mathbf{K}}_E - \mathbf{V} \mathbf{V}^\top\|_F \\ &= \text{diag}(\tilde{\mathbf{d}}_E) \cdot [\mathbf{U}_{\tilde{\boldsymbol{\Omega}}}]_{1:E, 1:2} \cdot [\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\Omega}}}]_{1:2, 1:2}^{\frac{1}{2}}, \end{aligned} \quad (6)$$

where $\tilde{\mathbf{K}}_E = \tilde{\mathbf{d}}_E \tilde{\mathbf{d}}_E^\top \circ \tilde{\boldsymbol{\Omega}}$ is the measured edge kernel and $\mathbf{U}_{\tilde{\boldsymbol{\Omega}}}$ and $\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\Omega}}}$ are the matrices of eigenvectors and eigenvalues of $\tilde{\boldsymbol{\Omega}}$, respectively. Recovering the point set \mathbf{X} from $\hat{\mathbf{V}}$ is straightforward once we set the translation of \mathbf{X} by fixing one point. It consists of solving a sparse linear system of equations defined in [16].

3.2. Constrained Edge-Kernel

The above solution outperforms previous approaches, which include distance-only techniques [18] as well as methods combining both distances and angles [18, 19]. However, it neglects certain geometric constraints, which means the recovered vectors are not guaranteed to be geometrically consistent. Indeed, imposing the kernel structure ignores triangle equalities of the edge vectors. Therefore, we propose a new method, which we call *constrained edge-kernel*, to address this limitation.

To better exploit geometric information for denoising, we introduce the novel notion of a *triangle constraints matrix* $\mathbf{M} \in \mathbb{R}^{C \times E}$. This matrix incorporates the mutual dependencies of edge vectors forming a triangle, with $M_{\ell i} = 1$, $M_{\ell k} = M_{\ell j} = -1$, assuming that the vectors \mathbf{v}_i , \mathbf{v}_j and \mathbf{v}_k form the triangle Δ_ℓ (see Fig. 1). The number of constraints C equals to the number of triangles in the point set, $C = \binom{N}{3}$. The problem given in (6) can then be written as the rank-constrained optimization,

$$\begin{aligned} \hat{\mathbf{K}}_E &= \arg \min_{\mathbf{K}} \|\tilde{\mathbf{K}}_E - \mathbf{K}\|_F \\ &\text{subject to} \quad \mathbf{K} \in \mathcal{C} \text{ and } \mathbf{K} \in \mathcal{R}, \end{aligned} \quad (7)$$

where we introduce the feasibility sets

$$\mathcal{R} = \left\{ \mathbf{X} \in \mathbb{R}^{E \times E} : \mathbf{X} \succeq 0, \text{rank}(\mathbf{X}) = 2 \right\}, \quad (8)$$

$$\mathcal{C} = \left\{ \mathbf{X} \in \mathbb{R}^{E \times E} : \mathbf{M} \mathbf{X} = \mathbf{0} \right\}. \quad (9)$$

Because of the rank constraint, Problem (7) is non-convex and, furthermore, it has no analytic solution. In such situations, it is typical to relax the non-convex constraint.

To do this, we implement the widely used nuclear norm relaxation from [20]. Since the solution is constrained to be positive-semi-definite, its nuclear norm is equal to its trace, resulting in the following relaxation:

$$\begin{aligned} \hat{\mathbf{K}}_E = \arg \min_{\mathbf{K}} \quad & \|\tilde{\mathbf{K}}_E - \mathbf{K}\|_F + \lambda \text{trace}(\mathbf{K}) \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}, \mathbf{K} \succeq 0, \end{aligned} \quad (10)$$

where λ is a regularization parameter, which controls to what extent the solution satisfies the rank property and the linear constraints, respectively.

As an alternative to relaxation, we can apply the lift-and-project method from [21]. It is an iterative method which consists of projecting the estimate at iteration k ($\hat{\mathbf{K}}_{E,k}$) onto the feasible sets \mathcal{C} and \mathcal{R} in an alternating fashion. The optimal projection onto the non-convex set \mathcal{R} is equivalent to the E-MDS solution given by (6). The orthogonal projection of $\hat{\mathbf{K}}_E$ onto the convex set \mathcal{C} is given by

$$\hat{\mathbf{K}}_{E,k+1} = \left(\mathbf{I} - \mathbf{M}^\top (\mathbf{M}\mathbf{M}^\top)^{-1} \mathbf{M} \right) \hat{\mathbf{K}}_{E,k}. \quad (11)$$

Experimentally, we found that this second approach shows excellent convergence rate in simulations (convergence is always achieved in at most two iterations), and no parameter tuning is required, so we rely on this approach for the presented results.

4. COORDINATE DIFFERENCE MATRICES

Consider a set of 1-dimensional points whose locations are collected in the vector $\mathbf{s} \in \mathbb{R}^N$. We can compute their pairwise differences and arrange them in a coordinate difference matrix, \mathbf{S} :

$$\mathbf{S} \stackrel{\text{def}}{=} \mathbf{s}\mathbf{1}^\top - \mathbf{1}\mathbf{s}^\top, \quad (12)$$

where $\mathbf{1}$ is the all-ones vector. In the following, we propose an algorithm that estimates the points \mathbf{s} from their pairwise differences \mathbf{S} , but first we show how one can relate the 2D multimodal localization problem to CDMs.

We assume to measure distances d_{mn} and angles α_{mn} , from which we can compute the edge vectors \mathbf{v}_{mn} by (3). The x - and y -coordinates of the edge vectors are then used to create two independent coordinate difference matrices \mathbf{S}^x and \mathbf{S}^y , such that $S_{mn}^x = (v_{mn})_x$ and $S_{mn}^y = (v_{mn})_y$. Observe that \mathbf{S}^x (\mathbf{S}^y) is a valid CDM, as its entries are pairwise differences of 1-dimensional points obtained by taking only x - (y -) coordinates of points in \mathbf{X} .

Now, let us consider a generic CDM defined in (12). Assume that we are given a set of differences and we would like to reconstruct the original 1-dimensional points up to translation. In the complete case, that is when all the pairwise distances between the points are available, this task is as simple as averaging the rows of \mathbf{S} . However, the majority of problems have more restrictive conditions: distances are noisy, measurements are missing or they appear in multiple occurrences. Therefore, we introduce a masking matrix \mathbf{W} with non-negative integer entries, such that W_{mn} denotes the number of measurements of the distance S_{mn} . This formulation allows multiple measurements of the same distance, which is particularly useful in case of noisy entries. We adopt the convention that $W_{mm} = 0$ for all m ; as we see later, this simplifies the notation.

A noisy and incomplete coordinate difference matrix $\tilde{\mathbf{S}}$ is then defined as $\tilde{\mathbf{S}} = (\mathbf{S} + \mathbf{Z}) \circ \mathbf{W}$, where the entries of \mathbf{Z} are independent

Algorithm 1 Points recovery in 1D using CDMs

Input: An incomplete noisy CDM $\tilde{\mathbf{S}}$ and its mask matrix \mathbf{W} .

Output: The vector of 1D points \mathbf{s} that minimizes (13).

Compute \mathbf{W}' and $\tilde{\mathbf{S}}'$ from \mathbf{W} and $\tilde{\mathbf{S}}$ by removing the first row and column.

Compute Λ' , such that $\Lambda'_m = \frac{1}{N_m}$, where $N_m = \sum_{n=1}^N W_{mn}$.

$\mathbf{d}' = \Lambda'(\tilde{\mathbf{S}}' \circ \mathbf{W}')\mathbf{1}$

$\mathbf{A}' = \mathbf{I} - \Lambda'\mathbf{W}'$

$\hat{\mathbf{s}} = (\mathbf{A}')^{-1}\mathbf{d}'$

return $[0 \quad \hat{\mathbf{s}}]$

noise realizations. In case we have multiple measurements between the points s_m and s_n , S_{mn} and Z_{mn} are the average between the measurements and the noise realizations, respectively.

Our goal now is to minimize the following cost function:

$$f(\mathbf{s}) = \sum_m \sum_n W_{mn} (s_m - s_n - \tilde{S}_{mn})^2. \quad (13)$$

By calculating the Hessian matrix, it is easy to prove that the cost function $f(\mathbf{s})$ is convex. Thus, setting the first derivative of $f(\mathbf{s})$ to zero leads to the optimal solution:

$$\begin{aligned} \frac{\partial f(\mathbf{s})}{\partial s_m} &= 2 \sum_{n=1}^N (s_m - s_n - \tilde{S}_{mn}) W_{mn} = 0. \\ \Rightarrow \hat{s}_m &= \frac{1}{N_m} \sum_{n=1}^N s_n W_{mn} + \frac{1}{N_m} \sum_{n=1}^N \tilde{S}_{mn} W_{mn}, \end{aligned} \quad (14)$$

where $N_m = \sum_{n=1}^N W_{mn}$ is the number of measurements in the m th row in \mathbf{W} . We can rewrite this result in matrix form

$$\mathbf{A}\mathbf{s} = (\mathbf{I} - \Lambda\mathbf{W})\mathbf{s} = \mathbf{d}, \quad (15)$$

with $\Lambda = \text{diag}\left(\frac{1}{N_1}, \frac{1}{N_2}, \dots, \frac{1}{N_N}\right)$ and $\mathbf{d} = \Lambda(\tilde{\mathbf{S}} \circ \mathbf{W})\mathbf{1}$.

Observe that the points \mathbf{s} and their translations $\mathbf{s} + c$ lead to the same CDM \mathbf{S} :

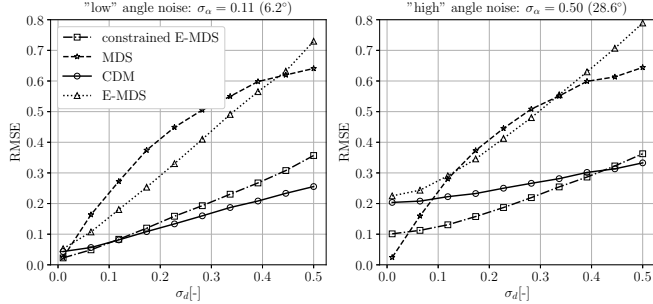
$$(\mathbf{s} + c)\mathbf{1}^\top - \mathbf{1}(\mathbf{s} + c)^\top = \mathbf{s}\mathbf{1}^\top - \mathbf{1}\mathbf{s}^\top.$$

Due to this translation invariance, $\text{rank}(\mathbf{A}) = N - 1$, hence the system of equations (15) is not invertible. To anchor the translation and obtain a full-rank matrix, we arbitrarily fix $s_1 = 0$ and remove the first entry of \mathbf{d} and \mathbf{s} as well as the first row and column of \mathbf{A} . By defining the matrices \mathbf{A}' and \mathbf{W}' analogously, (15) becomes

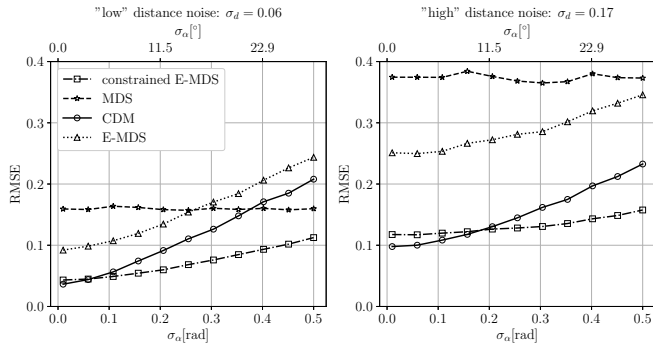
$$\mathbf{A}'\mathbf{s}' = \mathbf{d}'. \quad (16)$$

The matrix \mathbf{A}' has a particular structure and belongs to the class of the so-called *M-matrices* [22, 23]. One of their interesting properties is that they are inverse-positive; that is \mathbf{A}' is invertible and the entries of $(\mathbf{A}')^{-1}$ are non-negative. Hence, the estimated points are given by $\hat{\mathbf{s}} = (\mathbf{A}')^{-1}\mathbf{d}'$.

To summarize, Algorithm 1 formally states how we reconstruct points in 1D from noisy and incomplete CDMs. By running Algorithm 1 independently for CDMs $\tilde{\mathbf{S}}^x$ and $\tilde{\mathbf{S}}^y$, we jointly reconstruct the x - and y -coordinates of the points in \mathbf{X} . A noteworthy observation is that the solution of the CDM-based algorithm is derived in a closed-form using least squared error criterion in (14). Therefore, it is optimal for Gaussian noise added on the edge vectors.



(a) Performance vs. distance noise (σ_d) for two representative angle noise levels.



(b) Performance vs. angular noise (σ_α) for two representative distance noise levels.

Fig. 2. Simulated results of localization performance averaged over 1000 independent realizations of point sets, angle and distance noise. We compare the proposed methods (CDM and constrained E-MDS) with E-MDS and the distance-only MDS.

5. SIMULATION RESULTS

In this section, we compare both our algorithms with E-MDS, which is the state-of-the art in multimodal localization, and MDS, which provides a baseline for range-only measurements.

We run 1000 experiments with $N = 6$ points chosen uniformly at random from the unit square at each iteration. The noise is added independently to all pairwise distances and angles. This is motivated by the fact that in most real-world scenarios, the measurements of distances and angles are obtained in an independent manner, e.g. from time-of-arrival and angle-of-arrival measurements. Thus, the noise on the edge vectors is governed by independent noise on these measurements, and the angular noise is amplified with larger distances (see Fig. 1). To reproduce this behavior, we run simulations with independently randomly generated additive noise on the distances and angles between the points. The noise on distances and angles is Gaussian with zero mean and standard deviation σ_d and σ_α , respectively. Note that noise exceeding $\pm \pi$ will distort the angle noise distribution, however for the range of standard deviations chosen in these experiments, this effect is negligible.

We evaluate the performance based on the root mean squared error (RMSE) between the original and the estimated point set $\hat{\mathbf{X}}$. Fig. 2a shows the localization performance vs. distance noise level for two chosen levels of angle noise (“low” and “high”, with $\sigma_\alpha = 6.2^\circ$ and 28.6° , respectively). For low enough angle noise, it is beneficial to include angle information (left plot), since the three multimodal methods surpass the distance-only (MDS) algorithm. Further-

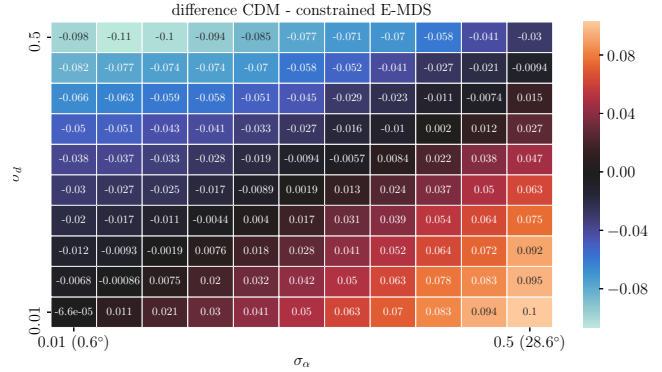


Fig. 3. Difference of RMSE of the two proposed algorithms. A negative value means that CDM is better for this level of angle/distance noise. Both σ_α and σ_d are linearly spaced between 0.01 and 0.5.

more, CDM and constrained E-MDS have a substantial advantage over E-MDS. For higher angle noise, both our proposed methods again significantly improve over E-MDS for any noise level on distances. They also demonstrate superior performance over MDS, as long as the distance noise is non-negligible.

Similarly, Fig. 2b shows the localization performance vs. angle noise level for two chosen levels of distance noise (“low” and “high”, with $\sigma_d = 0.06$ and 0.17 , respectively). The left plot shows that for low distance and angle noise, it is beneficial to use both, but the use of angle information becomes detrimental if it is significantly noisier than the measured distances. For high distance noise, angular information always significantly improves the localization result for all investigated noise levels, again with favorable performance of our methods.

All the plots in Fig. 2 show that CDM and constrained E-MDS algorithms are affected quite differently by angle and distance noise; this deserves additional investigations. To this end, we compare the RMSE of the two methods for different values of distance and angle noise. As the result in Fig. 3 suggests, CDM is more efficient on high distance noise regimes, while constrained E-MDS performs better when the angle noise is higher. Interestingly, when the two levels of noise are equal, both algorithms perform equally well.

6. CONCLUSIONS

We have proposed two algorithms for multimodal sensor localization that exploit geometric information of points in space. The first method improves on the existing edge-multidimensional algorithm by adding geometric constraints on triplets of points. In the second algorithm, we explore a different perspective of the problem based on a novel tool called CDM, which allows us to estimate the points’ coordinates independently along each dimension. Numerical simulations demonstrate that both proposed methods significantly outperform existing distance-based and multimodal localization algorithms. Our future work focuses on analyzing the dependence of our algorithms on the noise models, and obtaining a better understanding of why both algorithms perform equally well with identical distance and angle noise levels. Furthermore, we will extend their formulation to 3D and study the performance of both algorithms. We expect the CDM-based approach to benefit more from higher dimensional setups, as it exploits both the azimuth and elevation, whereas the edge kernel-based solution relies only on the smallest inner angle between two vectors.

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