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STOCHASTIC RESPONSE ANALYSIS OF NONLINEAR STRUCTURES VIA PDEM

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Abstract

Stochastic response analysis of nonlinear multi-degree-of-freedom systems is still a challenging problem. Incorporating the quasi-symmetric point method into the probability density evolution method (PDEM) provides a feasible approach. The symmetry and sparseness of the quasi-symmetric point set guarantee the global accuracy of cubature, but they also lead to the overlapping of projections and thus in some cases the applicability is limited. In the present paper, the effects of rotational transform of the quasi-symmetric point sets are investigated. It is shown that when the appropriate rotational transform is conducted, the performance of the point sets will be improved considerably. Numerical examples on the probability density evolution analysis of nonlinear random-parameter structures subjected to strong earthquake input via the rotational point sets are carried out. Problems to be further studied are discussed.

1 Introduction

Stochastic nonlinear response analysis of multi-degree-of-freedom (MDOF) systems is not only a critical issue in engineering fields such as civil and offshore engineering, but also a challenging problem in stochastic dynamics (Schuëller, 1997). Traditionally, stochastic dynamics are divided, somewhat artificially, into two separate branches, i.e. stochastic structural analysis and random vibration analysis, owing to the different treatment to apparent different sources of the randomness (Li and Chen, 2009). The randomness in the excitations is taken into account in the random vibration theory, whereas in stochastic structural analysis the uncertainties involved in the structural parameters are tackled. Correspondingly, over the last half century, a variety of approaches have been developed and investigated extensively in both branches. For instance, a variety of methods are developed in random vibration, including the moment equation, the spectral analysis, the FPK equation and so on (Lutes and Sarkani, 2004). Meanwhile, in stochastic structural analysis, a number of methods have also been put forward, such as the Monte Carlo simulation, the random perturbation technique and the orthogonal polynomial expansion (Li, 1996). However, none of these methods works well in the stochastic response analysis of general MDOF nonlinear structures (Schuëller, 1997). Although breakthroughs were made in some areas of stochastic dynamics (Zhu, 2006; Jin *et al.*, 2011; Chen *et al.*, 2011), huge challenges are still encountered in the analysis of nonlinear stochastic structures with large degree-of-freedom in civil and offshore engineering.

In the past few years, the probability density evolution method (PDEM) has been proposed and developed based on the principle of preservation of probability (Li and Chen, 2009). By PDEM, the instantaneous probability density of the structural response can be obtained and the randomness in structural parameters and excitations can be handled in a unified way. Important advances in PDEM has been made, including stochastic nonlinear response analysis and reliability evaluation of MDOF systems and optimal control of stochastic systems (Li and Chen, 2010; Li *et al.*, 2012b). In PDEM, selecting representative points in the probability-assigned space is required in the point-evolution-based numerical solving method (Li *et al.*, 2012b). Researches show that the quasi-symmetric point method (Q-SPM) has relatively high accuracy and efficiency in moderate dimensions (Xu *et al.*, 2012). However, because of the overlapping of projections, the quasi-symmetric point method has some deficiencies in certain circumstances. By performing appropriate rotation, the applicability of the quasi-symmetric points can be improved considerably, which is the topic of the present paper.

2 Fundamentals of the Probability Density Evolution Method (PDEM)

Without loss of generality, consider the equation of motion of a MDOF nonlinear structure subjected to stochastic excitations

$$\mathbf{M}\ddot{\mathbf{X}}+\mathbf{G}(\mathbf{X},\dot{\mathbf{X}})=\mathbf{\Gamma}\boldsymbol{\xi}(t) \tag{1}$$

where $\mathbf{X}$, $\dot{\mathbf{X}}$ and $\ddot{\mathbf{X}}$ are the n by 1 vectors of displacement, velocity and acceleration, respectively. $\mathbf{G}(\mathbf{X},\dot{\mathbf{X}})$ is the sum of restoring force vector and the damping force vector. $\boldsymbol{\xi}(t)$ is the r-dimensional random loading vector and $\mathbf{\Gamma}$ is the n by r loading influence matrix. If the excitation is stochastic ground acceleration, then $\mathbf{\Gamma}\boldsymbol{\xi}(t)=-\mathbf{M}\mathbf{1}\ddot{X}_{\mathrm{g}}(t)$, $\mathbf{1}=(1,1,\cdots,1)^{\mathrm{T}}$. Here $\ddot{\mathbf{X}}$, $\dot{\mathbf{X}}$, $\mathbf{X}$ are accelerations, velocities and displacements of the structure relative to ground.

The stochastic process can be represented by random functions (Li *et al.*, 2012a). All the random variables involved in the system could be denoted by $\mathbf{\Theta}=(\Theta_1,\Theta_2,\cdots,\Theta_s)$ where s is the total number of the random variables. Eq. (1) can thus be rewritten into

$$\mathbf{M}(\mathbf{\Theta})\ddot{\mathbf{X}}+\mathbf{G}(\mathbf{\Theta},\mathbf{X},\dot{\mathbf{X}})=\mathbf{F}(\mathbf{\Theta},t) \tag{2}$$

For a well-posed dynamical problem, the solution of Eq. (2) exists, is unique and must be a function of $\mathbf{\Theta}$. Generally speaking, any physical quantities (e.g. the displacement or the stress and strain at some points of the structure) take the form

$$\mathbf{Z}(t)=\mathbf{H}(\mathbf{\Theta},t),\ \dot{\mathbf{Z}}(t)=\mathbf{h}(\mathbf{\Theta},t) \tag{3}$$

In the stochastic dynamical system (3), all the randomness comes from $\mathbf{\Theta}$. Therefore the augmented system $(\mathbf{Z}(t),\mathbf{\Theta})$ is a probability preserved system. According to the principle of preservation of probability viewed from the random event description and the physical equation (3), we have the generalized density evolution equations (GDEE) as follows (Li and Chen, 2009; Li *et al.*, 2012b)

$$\frac{\partial p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)}{\partial t}+\sum_{j=1}^{m}\dot{Z}_j(\boldsymbol{\theta},t)\frac{\partial p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)}{\partial z_j}=0 \tag{4}$$

where $p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)$ is the joint PDF of $(\mathbf{Z}(t),\mathbf{\Theta})$. The initial condition of Eq. (4) is generally

$$p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t_0)=\delta(\mathbf{z}-\mathbf{z}_0)p_{\Theta}(\boldsymbol{\theta}) \tag{5}$$

in which $\mathbf{z}_0$ is the deterministic initial value of $\mathbf{z}$. Then the PDF of $\mathbf{Z}(t)$ could be obtained eventually by solving Eq. (4) under the condition (5) and taking the following integral

$$p_{\mathbf{Z}}(\mathbf{z},t)=\int_{\Omega_\Theta}p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)d\boldsymbol{\theta} \tag{6}$$

Compared with the traditional probability density evolution equations e.g. the FPK equation of which the dimension must be identical to the dimension of the original state equation, it is remarkable that the dimension of a GDEE is free from the dimension of the original system. And usually one or two dimension is already adequate, which makes the solving process much easier. What's more, the GDEE is not confined to the stochastic dynamic system which makes it possible to extend the usage of probability density evolution theory (e.g. construct equivalent extreme-value event to evaluate the extreme-value distribution (Chen and Li, 2007)).

Point evolution and ensemble evolution are two approaches to solve GDEE numerically. It is shown that the point evolution method is easier and has sufficient accuracy for the probability density of stochastic response (Li *et al.*, 2012b). There are several steps in the point-evolution-based numerical method. The first step is to select a set of representative points in the distribution domain and calculate the corresponding assigned probabilities. Then, for each specified point, the physical equation should be solved to get the time rate of the physical quantities $\dot{Z}_j(\boldsymbol{\theta},t)$. After that the GDEE is solved under the initial condition by the finite difference method with TVD scheme to acquire the numerical solution. Finally, the probability density function can be got by summing up all the results acquired before.

3 Quasi-symmetric Point Method (Q-SPM) and Rotational Q-SPM

3.1 Quasi-symmetric Point Method (Q-SPM)

For Gaussian weighted high-dimensional integrals as

$$I(f)=\frac{1}{(2\pi)^{d/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{\mathbf{x}^{\mathrm{T}}\mathbf{x}}f(\mathbf{x})dx_1\cdots dx_d \tag{7}$$

Victoir constructed some cubature formulae (Victoir, 2004), called the quasi-symmetric point method (Q-SPM), by the invariant theory and orthogonal arrays with the 5th degree of algebraic accuracy. One class of the integral points is given by

$$\mathbf{x}_0=(hr,\cdots,0),\ \mathbf{x}_1=(hs,\cdots,hs) \tag{8}$$

$$r^2=(d+2)/2,\ \ s^2=(d+2)/(d-2) \tag{9}$$

Here d is the dimension and h is the permutation of ±1. The sums of the weights read

$$\omega_0=8d/(d+2)^2\,,\ \omega_1=(d-2)^2/(d+2)^2 \tag{10}$$

The sum of the weights is 1 and all weights are positive, which is a crucial property for the application of Q-SPM to PDEM.

Apparently, these integral points can be employed as the representative points in PDEM and the weights can be adopted as the assigned probabilities if all the basic variables are Gaussian. This is a reasonable treatment considering the resemblance between Eqs. (6) and (7). However, after detailed studies, an essential difference between PDEM and cubature is exposed which can be illustrated by the concept of the rank of an integral (Xu *et al.*, 2012). The rank of an integral is the highest degree of the polynomials with respect to the weights in the cubature formulae. It can be easily shown that: (i) the rank of a common cubature formula is 1; (ii) the rank of the computation of second order moments about origin is 1 whereas the rank of the computation of second order central moments is 2; and (iii) the integral (6) is essentially a rank-∞ integral. One of the necessary conditions for the cubature formulae of an integral whose rank is 2 or above is that the sum of squares of the weights should be no more than 1. Since the sum of weights is 1, it is required that the absolute value of the weights should never exceed 1. All the weights are required to be nonnegative for the higher rank integral, which explains the reason why the Q-SPM is well behaved in PDEM, whereas other kinds of cubature points with negative weights might behave badly.

3.2 The Problem Encountered by Q-SPM

Because of the symmetry and sparseness of the quasi-symmetric point sets, the marginal probability density could not be reflected sufficiently by these points. Thus the application of Q-SPM is limited in some cases. For instance, it is impossible to obtain the extreme value distribution or the marginal distribution of the basic random variables via the Quasi-symmetric points.

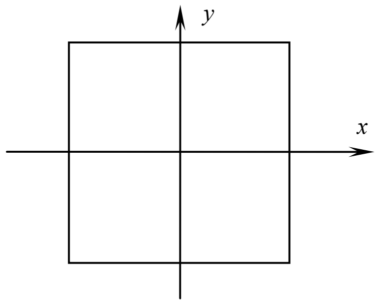

Fig.1 Quasi-symmetric points

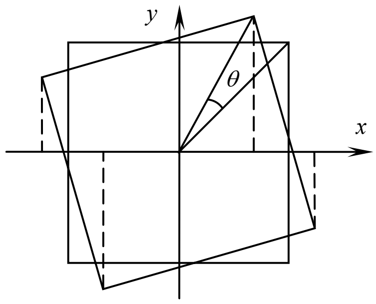

Fig. 2 Rotation of the quasi-symmetric points

It can be shown from Eq. (8) that there are two sets for the quasi-symmetric points. One set is located on the coordinate axis and the other set includes the vertex of hyper-cubes which can be illustrated by the vertexes of the square in Fig. 1. The overlapping of the projection of the four points in the x-axis and y-axis can be easily observed, so does the extreme value of these points in both axis. Such overlapping is more severe in the higher-dimensional space and would result in very few effective points for the constructed extreme-value event.

3.3 The Rotation of the Quasi-symmetric Points

To resolve the matter of overlapping, a natural idea is to rotate these quasi-symmetric points (Fig.2). Rotation does not change the qualitative property of the integral function because rotation is a linear transformation. For instance, if the integral function is rational, performing rotation does not change the degree of the polynomials in the numerator and denominator. What's more, the Gaussian weight function has rotational symmetry. Thus the accuracy of the integral is expected to decrease little and the problem about overlapping is expected to be solved by rotation. The corresponding equation to Fig.2 is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{11}$$

Fig. 2 shows that the rotation can solve the problem of overlapping of 2 dimensions effectively. This thought can apparently be extended to multi-dimensional condition. In Euclidean geometry, a rotation has three properties. First, a rotation is an example of an isometry. Second, a rotation leaves at least one point fixed. Third, a rotation do not change left- or right-handed ordering which, however, is actually unimportant for the transformation of the quasi-symmetric points.

Consider the rotation of a point $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ in n-dimensional space. The distance between this point to the origin is given as

$$\|\mathbf{p}\|^2 = \mathbf{p}^{\mathrm{T}}\mathbf{p} \tag{12}$$

In the view of analytic geometry, the transformation of point in space can be expressed as the product of a matrix and a vector. Let $\mathbf{Q}$ be the matrix, due to the property of isometry we have

$$\|\mathbf{p}\|^2 = \mathbf{p}^{\mathrm{T}}\mathbf{p} = \mathbf{p}^{\mathrm{T}}\mathbf{I}\mathbf{p} = (\mathbf{Q}\mathbf{p})^{\mathrm{T}}(\mathbf{Q}\mathbf{p}) = \mathbf{p}^{\mathrm{T}}(\mathbf{Q}^{\mathrm{T}}\mathbf{Q})\mathbf{p} \tag{13}$$

For Eq. (13) is required to hold for any $\mathbf{p}$, $\mathbf{Q}$ is an orthogonal matrix. It can be further proved that the $n \times n$ rotation matrix can be parameterized by $n(n-1)/2$ plan rotations within which each can be seen as a Givens rotation (William *et al.*, 1986). For instance, rotating a vector $\mathbf{p}$ in the (i, j) plane of θ radians counterclockwise can be given as

$$\mathbf{p}' = \mathbf{G}(i, j, \theta)\mathbf{p} \tag{14}$$

where

$$
\mathbf{G}(i,j,\theta)=\begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cos\theta & \cdots & -\sin\theta & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \sin\theta & \cdots & \cos\theta & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{15}
$$

Therefore, any rotational transformation of the point in the space can be expressed as

$$
\mathbf{p}'=\sum_{i=1}^{n-1}\sum_{j=i}^{n}\mathbf{G}(i,j,\theta_{ij})\mathbf{p} \tag{16}
$$

It is a natural thought to require the projections on the axis to scatter as uniformly as possible to resolve the problem of overlapping of the projections. When the objective index for the rotation is determined, the optimal result can be obtained by running over through Eq. (16). However, as the dimension grows, the computational efforts grow exponentially. The intelligent algorithm is preferred in this situation. We use the genetic algorithm (Wang *et al.*, 2007) in the present paper. The angles of rotation are chosen as individuals and the rotational index is used to determine the calibration function.

4 Illustrative Examples

Consider a 9-DOF nonlinear structure as shown in Fig 3. The masses and the lateral inter-story stiffness are all taken as independent normal random variables. The mean values of the masses from bottom to top are 3.442, 3.056, 3.056, 2.739, 2.739, 2.739, 2.739, 2.739 and 2.692 ($\times 10^5$kg) and all the coefficients of variation are 0.15. The mean values of the lateral inter-story stiffness are 2.94, 3.26, 3.24, 3.20, 3.20, 3.84, 3.70, 1.92 and 1.78 ($\times 10^8$kN/m) and their coefficients of variation are all 0.1. The Rayleigh damping is adopted such that $\mathbf{C}=a\mathbf{M}+b\mathbf{K}$, where $\mathbf{C},\mathbf{M}$ and $\mathbf{K}$ are the damping, mass and stiffness matrices, respectively, and $a=0.2643\mathrm{s}^{-1}$, $b=0.0071\mathrm{s}$. In the case where nonlinearity is involved in the inter-story restoring forces, the extended Bouc-Wen model is adopted. In this model, totally 13 parameters are included (Ma *et al*, 2004), taking the values $A=1$, $n=1$, $q=0.25$, $p=1000$, $\lambda=0.5$, $\phi=0.05$, $d_\phi=5$, $d_v=2000$, $d_\eta=2000$ and $\zeta=0.99$. β and γ in the Bouc-Wen model are also regarded as independent random variables and the mean values are 30 and 10, respectively. The coefficients of variation of them are both 0.2. Totally 20 basic random variables are involved. The El Centro accelerogram in E-W direction is adopted as the ground motion and 552 representative points are adopted employing Q-SPM. A typical sample of the restoring force v.s. inter-drift in Fig. 4 shows that the system exhibits strong nonlinearity.

The comparison of the results among the rotational quasi-symmetric points (Rot), the original quasi-symmetric points (NonRot) and the Monte Carlo simulation is shown in Fig. 5. The relative error of the standard deviation by the rotational quasi-symmetric points is almost the same as the results computed by the quasi-symmetric points. What's more, the increased workload is negligible because the rotation is independent of the structural analysis and could be conducted in advance. Rotational quasi-symmetric points would have more extensive applicability considering that there are much less overlapping of the projections in the axis. For instance, higher accuracy is expected for the evaluation of the extreme value distributions, which needs further study. Fig. 6 shows that the pattern of distribution of the typical instantaneous PDF is irregular for the nonlinear stochastic structures.

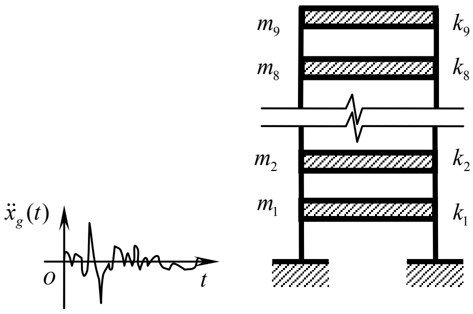

Fig.3 Shear frame structure

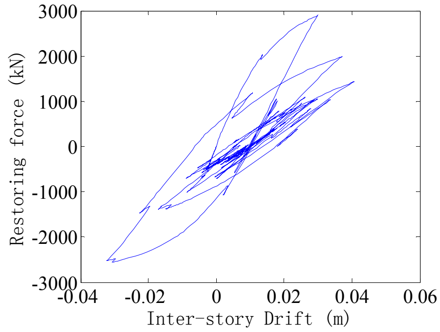

Fig.4 Typical restoring v.s. inter-story drift

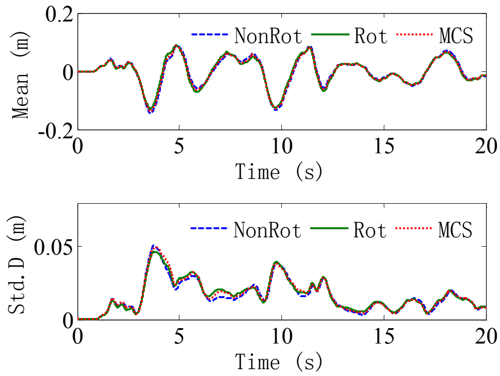

Fig.5 The mean and standard deviation

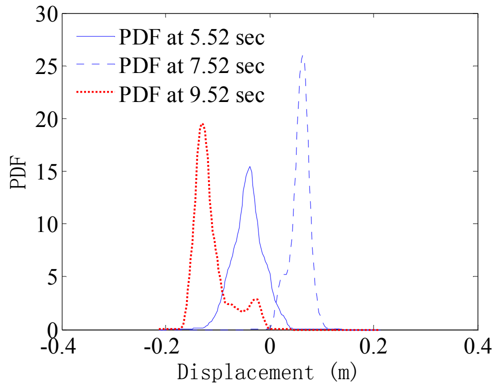

Fig.6 Typical instantaneous PDF

5 Conclusions

Incorporating the quasi-symmetric point method into the probability density evolution method could implement the stochastic analysis of nonlinear multi-degree-of-freedom systems. Rotation is introduced to resolve the problem of projection overlapping. It is shown via numerical examples that the rotation can overcome the projection overlapping of the quasi-symmetric points but add little additional workload for stochastic nonlinear response analysis. Researches based on the genetic algorithm show that the computed results do not always get better along with the improvement of objective index of the rotation. Further investigations on finding more appropriate objective index are needed.

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