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Compactness and Löwenheim-Skolem theorems in extensions of first-order logic

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Abstract

Lindström's theorem characterizes first-order logic as the most expressive among those that satisfy the countable Compactness and downward Löwenheim-Skolem theorems. Given the importance of this results in model theory, Lindström's theorem justifies, to some extent, the privileged position of first-order logic in contemporary mathematics. Even though Lindström's theorem gives a negative answer to the problem of finding a proper extension of first-order logic satisfying the same model-theoretical properties, the study of these extensions has been of great importance during the second half of the XX. century: logicians were trying to find systems that kept a balance between expressive power and rich model-theoretical properties. The goal of this essay is to prove Lindström's theorem, along with its prerequisites, and to give weaker versions of the Compactness and Löwenheim-Skolem theorems for the logic $\mathfrak{L}(Q_1)$ (first-order logic with the quantifier "there exist uncountably many"), which we present as an example of extended logic with good model-theoretical properties.

Resumen

El teorema de Lindström caracteriza la lógica de primer orden como la más expresiva entre las que cumplen los teoremas de Compacidad numerable y Löwenheim-Skolem descendiente. Dada la importancia de ambos resultados en teoría de modelos, el teorema de Lindström justifica, en cierta medida, el lugar privilegiado que otorgamos a la lógica de primer orden en las matemáticas contemporáneas. Aunque el teorema de Lindström ofrece una respuesta negativa al problema de encontrar extensiones propias que cumplan las mismas propiedades que la lógica de primer orden, el estudio de estas extensiones ha tenido una importancia considerable en la segunda mitad del siglo XX, donde se ha tratado de encontrar sistemas lógicos que mantuvieran cierto equilibrio entre propiedades semánticas y poder expresivo. El objetivo de este trabajo es demostrar el teorema de Lindström, incluyendo sus prerrequisitos, y dar versiones débiles de los teoremas de Compacidad y Löwenheim-Skolem para la lógica $\mathfrak{L}(Q_1)$ (la lógica de primer orden junto con el cuantificador "existe una cantidad no numerable"), que presentamos como ejemplo de extensión de la lógica de primer orden con buenas propiedades semánticas.

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Azkenik, lan hau aita eta amari eskaini nahi diet, urte luze hauetan nirekin batera poz eta tristurak gainditu eta nigan konfidantza osoa izateagatik. Badira uneak non sakonki pentsatutako hitz sorta batek ezin duen bakarraren esangura ordezkatu. Milesker.

Contents

Introduction

Gizonen lana jakintza dugu: ezagutuz aldatzea. [...] ezaren gudaz baietza sortuz, ukazioa legetzat hartuz beti aurrera joatea.

—Mikel Laboa, *Izarren hautsa*

Los problemas lógicos no son solo problemas de forma; se trata de cuestiones ontológicas, cosmológicas y antropológicas. [...] toda forma se presenta, en principio, como un contenido que primero se trabaja, se elabora y solo luego deviene forma, aunque, al final del proceso, la forma parezca anterior al contenido. ¿No es así como surge la concepción griega del cosmos elaborada por Aristóteles en tanto que concepción del orden y de la necesidad, transcrita en el proceso de deducción formal?

—Henri Lefebvre, *Lógica formal, lógica dialéctica*

Model theory is a branch of mathematical logic dedicated to the study of the relation between a formal language and its interpretations. In particular, what is known as *classical* model theory deals with the interaction of first-order logic and mathematical structures. The work of authors such as Löwenheim, Skolem, Gödel and Malcev provided the basic results for the subject in the first half of the XX. century, but it was not clearly visible as a separate area of research until the early 1950's (in fact, A. Tarski coined the term *theory of models* in 1954). Since then, model theory has proven to be a rich and highly technical subject with a wide range of applications in fields such as set theory, algebra and analysis.

Model theory often gives information about the limitations of our formal system. For example, the Downward Löwenheim-Skolem theorem (1920) states that if a countable set of sentences has an infinite model (that is, a model with infinite cardinality in which those sentences are true), then it has a countable model. This means that we cannot distinguish between countable and uncountable cardinalities in first-order logic. The Compactness theorem (Gödel, 1930) (which states that a set of sentences is satisfiable if and only if every finite subset is) provides similar results, and lies in the basis of the construction of non-standard models of arithmetic, giving another interesting example of the limitations of first-order logic: we cannot characterize natural numbers up to isomorphism within it.

The working mathematician exceeds the limits of first-order logic every now and then, and since mathematical logic should have a saying in how a mathematician reasons, it was a natural step to study extensions of first-order logic that were expressive enough to formalize that reasoning. The first of such extensions was second-order logic, where one can quantify over subsets of the universe rather than just over elements of it. Due to all its expressive power, second-order logic fails to satisfy the Compactness and Löwenheim-Skolem theorems, which had slowly become two of the most widely used results in model theory. This was not an exception. It soon became clear that there is a trade-off between the expresive power and the model-theoretical properties of a logical system, and the task of finding versions of the Compactness and Löwenheim-Skolem theorems for extensions of first-order logic became a real challenge.

By the 1960s, a variety of extensions had been studied, but none of them satisfied those central theorems simultaneously. In his 1969 paper [10], the swedish logician Per Lindström[2] provided an explanation to this fact with a striking result: what currently is known as Lindström's first theorem states that every logic with more expressive power than first-order logic will fail to satisfy either the Compactness theorem or the Löwenheim-Skolem theorem. More precisely, first-order logic is the strongest logic satisfying the following properties:[3]

(1) (Countable Compactness) A countable set of sentences has a model if and only if every finite subset of it has a model.

(2) (Löwenheim Property) If a sentence has an infinite model, it has a countable model.

Giving such a characterization presupposes an understanding of what a logic is. For this purpose Lindström introduced the notion of *abstract logic*, which considered a logical system to be a pair consisting of a map sending languages to sentences and a satisfaction relation between structures and sentences. This definition, with the corresponding technicalities and closure properties, is general enough to include first-order logic and its main extensions within it.

Since the Compactness theorem (and its close relative, the Completeness theorem, which states that a set of sentences is consistent if and only if it has a model) are unconditionally needed for any strong formal language, one could think that Lindström's theorem upholds first-order logic as the only possible logic (as some logicians and philosophers have argued during the XX. century). However, the importance of the Löwenheim-Skolem theorem is more controversial:

> «The theorem of Lindström shows simultaneously the high stability of the first order logic as well as its limitations. It gives a general concept of logic and asserts that logics which apparently extend the first order logic all end up being the same as it, provided they satisfy the Löwenheim theorem (i.e. a sentence has a countable model if it has a model at all) and they either have the compactness property or are formally axiomatizable. When we are interested

[2]A brief summary of Per Lindström's life and work can be found in [15].

[3]Note that the properties listed above are weaker than the full Compactness and Löwenheim-Skolem theorems, thus making the result stronger.

in set theory or classical analysis, the Löwenheim theorem is usually taken as a sort of defect (often thought to be inevitable) of the first order logic. Therefore, what is established is not that first order logic is the only possible logic but rather that it is the only possible logic when we in a sense deny reality to the concept of uncountability and require (what seems to be a less debatable condition) that logical proofs be formally checkable (viz. the requirement of axiomatizability or compactness).» *H. Wang (1974), From mathematics to philosophy.*

By the time Lindström published his result, there had already been enough success in the model theory of infinitary logics and logics with cardinality quantifiers to justify their study even at the cost of losing important model theoretical tools. In fact, a completely new field of model theory was born with the goal of studying the relations between different logical systems and their properties: Abstract model theory.

Abstract model theory was grounded in previous work of Mostowski (who in his 1957 paper [11] introduced a new way of extending first-order logic by adding cardinality quantifiers), Tarski and his students (who in the late 1950s studied infinitary languages) and Lindström. Its motivations are elocuently suggested by Barwise in [2]:

> «Studying only the model theory of first-order logic would be analogous to the study of real analysis never knowing of any but the polynomial functions: core concepts like countinuity, differentiability, analyticity and their relations would remain at best vaguely perceived. It is only the study of more general functions that one sees the importance of these notions, and their different roles, even for the simplest case.»

The analogy is quite precise. For example, the difference between full compactness and countable compactness is not very significative in first-order applications to common mathematical structures because their language is usually countable. In general, however, countable compactness is much weaker. The task of finding a fully compact logic properly extending first-order logic was carried out by Shelah in his 1975 paper [12].

Appart from studying properties of logics, abstract model theory also attempted to give Lindström-style characterizations for them, but the project turned out to be considerably difficult and only minor results were archieved.

The goal of this essay is to present two of the main results in abstract model theory: the above mentioned Linström's First Theorem and Keisler's Completeness theorem for the logic $\mathfrak{L}(Q_1)$ which has the countable Compactness theorem for $\mathfrak{L}(Q_1)$ as a consequence. We also give a weak version of the downward Löwenheim-Skolem theorem (down to $\aleph_1$) for $\mathfrak{L}(Q_1)$. The logic $\mathfrak{L}(Q_1)$ is the logic we obtain by adding the quantifier Q_1 interpreted as "there are uncountably many" to first-order logic.

It is a natural technique in abstract model theory to *approximate* a logic as much as we can by first-order logic so that all our knowledge of first-order model theory is available. Keisler's proof is an elegant example of such technique, and this fact forces us to study *classical* model theory first. For this reason, the present essay also serves as an introduction to some foundational results and techniques in model theory, such as the back-and-forth method, Fraissé's theorem, elementary extensions or the omitting-types theorem.

In accordance with its objectives, the essay is structured into two clearly distinguished parts. The first of them covers chapters 3 and 4 and culminates with the proof of Lindström's first theorem. Chapter 3 introduces the key notions of partial and finite isomorphisms and the back-and-forth method in order to prove Fraissé's theorem, which states that two structures are elementarily equivalent if and only if they are finitely isomorphic. Chapter 4 begins with the definition of logical system, which is very similar to Lindström's idea of *abstract logic*, and concludes with the proof of Lindström's first theorem. A brief summary of the role played by the Compactness and Löwenheim-Skolem theorems is given at the end. In the second part, that is, chapters 5 and 6, the main result is Keisler's completeness theorem for $\mathfrak{L}(Q_1)$. Chapter 5 deals with the first-order prerequisites for its proof, namely, elementary extensions (especially the elementary chain theorem) and the omitting types theorem. Chapter 6 introduces the basic semantics of the language $\mathfrak{L}(Q_1)$. The key notion of weak model is highlighted because of its role in the proof of the Completeness theorem, and since the Completeness theorem needs a deductive calculus, we give a concrete set of axioms. Finally, the Completness theorem is proved using elementary chains of weak models that omit certain sets of sentences.

The essay begins with a brief review of the basic definitions in model theory, and both the Compactness and Löwenheim-Skolem theorems are proved. Chapter 2 is intended to have a quick glance at the basic examples of extended logics, thus giving a motivation for the incoming chapters and providing concrete examples of how expressive power and model-theoretical properties interact.

We mention the sources used in the beginning of each chapter, but the fundamental bibliographical sources have been [5], [8], [4] and [2]. The remaining references have been used as secondary bibliography, and the papers mentioned, althought carefully read, are just referenced for historical reasons. Needless to say, the present essay merely aspires to be a systematic exposition and a synthesis of the different sources we use, but, as far as possible, we have tried to give an original perspective to the topics treated.

Chapter 1

Preliminaries

In this chapter we introduce all the fundamental definitions and results that are recquiered afterwards. Good sourcebooks for the topics treated here are [4], [5] and [7].

A language S is a set of relation, function and constant symbols. We use the letters $R, S, T...$, $f, g, h...$ and $c, d, e...$ for them, respectively. We also use $x_1, ..., x_n, y, z...$ to denote variables. Given a language, the terms are particular string of symbols:

1. A variable is a term.
2. A constant symbol is a term.
3. If f is a n-ary function symbol and $t_1, \ldots, t_n$ are terms, $f(t_1, ..., t_n)$ is a term.

The atomic formulas are defined in a similar way:

1. If t_1, t_2 are terms, $t_1 = t_2$ is atomic.
2. If R is a n-ary relation symbol and $t_1, ..., t_n$ are terms, then $Rt_1, ..., t_n$ is atomic.

Finally, in first-order logic with equality, $\mathfrak{L}_{\omega\omega}$, the formulas of the language S constitute the least set $L^S_{\omega\omega}$ such that every atomic formula belongs to $L^S_{\omega\omega}$ and whenever $\varphi, \psi \in L^S_{\omega\omega}$ and v is a variable, $\varphi \wedge \psi$, $\neg\varphi$ and $\exists v\varphi$ all belong to $L^S_{\omega\omega}$. Note that we have not mentioned $\forall$, $\rightarrow$, $\vee$ and $\leftrightarrow$ in our definition. They can obviously be defined in terms of $\wedge$, $\exists$ and $\neg$ following the standard derivations. We are going to use this kind of reasoning when carrying out an inductive argument.

We now define by induction the function SF, which assigns to each formula the set of its subformulas:

Definition 1.1. *(a)* $SF(\varphi) = \{\varphi\}$ *if* φ *is atomic.*

(b) $SF(\neg\varphi) = \{\neg\varphi\} \cup SF(\varphi)$.

(c) $SF(\varphi * \psi) = \{\varphi * \psi\} \cup SF(\varphi) \cup SF(\psi)$ *for* $* = \wedge, \vee, \rightarrow, \leftrightarrow$.

(d) $SF(\forall x\varphi) = \{\forall x\varphi\} \cup SF(\varphi)$.

(e) $SF(\exists x\varphi) = \{\exists x\varphi\} \cup SF(\varphi)$.

We use the notation $|A|$ to represent the cardinality of the set A. It is easy to prove, using basic properties of cardinal numbers, that $|L^S_{\omega\omega}| = \omega + |S|$.

We now define the central notion of structure (or model).

Definition 1.2. *An structure $\mathfrak{A}$ in $L^S_{\omega\omega}$ consists on a nonempty set (a universe) A where to each n-ary relation symbol $R \in S$ corresponds a n-ary relation $R^{\mathfrak{A}} \subseteq A^n$, to each n-ary function symbol $f \in S$ corresponds a function $f^{\mathfrak{A}} : A^n \to A$ and to every constant symbol $c \in S$ corresponds a distinguished element $c^{\mathfrak{A}} \in A$. This correspondence is given by a interpretation γ mapping symbols of S to their corresponding relation, function or element. Thus a S-model is a pair $\mathfrak{A} = (A, \gamma)$ composed by a universe, A, and the interpretation of S in that universe.*

Remark 1.3. We usualy write $R^{\mathfrak{A}}$ instead of $\gamma(R)$ for the interpretations of the symbols, so we may omit γ when convenient. If the language is finite, say

$$S = \{R_1, ..., R_n, f_1, ..., f_m, c_1, ..., c_s\},$$

we can write the model as

$$\mathfrak{A} = (A, R_1^{\mathfrak{A}}, ..., R_n^{\mathfrak{A}}, f_1^{\mathfrak{A}}, ..., f_m^{\mathfrak{A}}, c_1^{\mathfrak{A}}, ..., c_s^{\mathfrak{A}}).$$

When the model is know, we may also omit the superindexes. For example, the natural numbers with the usual order relation are $(\omega, <)$. To simplify the notation, we assume that the universes of the structures $\mathfrak{A}$ and $\mathfrak{B}$ are A and B respectively.

Definition 1.4. *Let $S \subseteq S'$ be languages and $\mathfrak{A}$ and $\mathfrak{B}$ S and S'-structures respectively. If $A = B$ and they interpret the symbols of S in the same way, then we say that $\mathfrak{B}$ is an expansion of $\mathfrak{A}$. Conversely, we say that $\mathfrak{A}$ is the restriction of $\mathfrak{B}$ to S (written $\mathfrak{A} = \mathfrak{B}|_S$).*

We say that a variable x in a formula φ is bounded if it is under the scope of an existential or universal quantifier. Otherwise we say it is free in φ. A formula without free variables is a sentence. For example, in the formula $\forall x_0(x_0 = x_1)$, x_0 is bounded and x_1 is free. We use $\varphi(x_1, ...x_n)$ to denote that the free variables of φ are among $x_1, ..., x_n$.

Definition 1.5. *Let $\mathfrak{A}$ be an S-structure. An interpretation of the variables in $\mathfrak{A}$ is a function $\pi : VAR \to A$ where VAR is the set of all variables and π assigns an element of A to each one. The denotation of a term t in $\mathfrak{A}$ under an interpretation π is $t^{\mathfrak{A}}[\pi] \in A$ defined as follows:*

1. *$x^{\mathfrak{A}}[\pi] = \pi(x)$ for every variable x.*
2. *$c^{\mathfrak{A}}[\pi] = c^{\mathfrak{A}}$ for every constant $c \in S$.*
3. *$f(t_1, ..., t_n)^{\mathfrak{A}}[\pi] = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\pi], ..., t_n^{\mathfrak{A}}[\pi])$, for every n-ary function symbol f and terms $t_1, ..., t_n$.*

We now define the central notion of satisfaction, which gives meaning to the idea of a formula being true in a structure.

Definition 1.6. *Given an S-structure $\mathfrak{A}$ and a formula $\varphi \in L^S_{\omega\omega}$, we define $\mathfrak{A} \models \varphi[\pi]$ (where π is an interpretation of the variables in $\mathfrak{A}$) by induction on formulas:*

(a) $\mathfrak{A} \models t_1 = t_2[\pi]$ *if and only if* $t_1^{\mathfrak{A}}[\pi] = t_2^{\mathfrak{A}}[\pi]$.

(b) $\mathfrak{A} \models Rt_1, ..., t_m[\pi]$ *if and only if* $(t_1^{\mathfrak{A}}[\pi], ..., t_m^{\mathfrak{A}}[\pi]) \in R^{\mathfrak{A}}$.

(c) $\mathfrak{A} \models \neg\varphi[\pi]$ *if and only if* $\mathfrak{A} \not\models \varphi[\pi]$.

(d) $\mathfrak{A} \models \varphi \wedge \psi[\pi]$ *if and only if* $\mathfrak{A} \models \varphi[\pi]$ *and* $\mathfrak{A} \models \psi[\pi]$. *There are analogue clauses for* $\varphi \vee \psi$ *and* $\varphi \to \psi$.

(e) $\mathfrak{A} \models \exists x\varphi[\pi]$ *if and only if* $\mathfrak{A} \models \varphi[\pi_x^a]$ *for some* $a \in A$, *where* π_x^a *is the interpretation in which* $\pi_x^a(x) = a$ *and* $\pi_x^a = \pi$ *for the rest of variables.*

(f) $\mathfrak{A} \models \forall x\varphi[\pi]$ *if and only if* $\mathfrak{A} \models \varphi[\pi_x^a]$ *for all* $a \in A$.

When $\mathfrak{A} \models \varphi[\pi]$ we say that $\mathfrak{A}$ satisfies φ with the interpretation π.

A set of formulas $\Sigma \subset L_{\omega\omega}^S$ is said to be satisfiable if there is an S-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \sigma[\pi]$ for some interpretation π and every $\sigma \in \Sigma$. When Σ is a set of sentences, we say that $\mathfrak{A}$ is a model of Σ. In that sense, we define $Mod_S(\Sigma)$ as the class of all the S-structures that are a model of Σ.

Now we can also speak about consequence relation between formulas: let $\Sigma \subseteq L_{\omega\omega}^S$ and $\varphi \in L_{\omega\omega}^S$. We say that $\Sigma \models \varphi$ when for every S-structure $\mathfrak{A}$ and every interpretation π, if $\mathfrak{A} \models \Sigma[\pi]$ then $\mathfrak{A} \models \varphi[\pi]$.

If $t = t(x_1, ..., x_n)$, $t^{\mathfrak{A}}[\pi]$ is determined by the value of π in $x_1, ..., x_n$, so any two interpretations that coincide in $x_1, ..., x_n$ will result in the same $t^{\mathfrak{A}}[\pi]$. In these cases we write $t^{\mathfrak{A}}[a_1, ..., a_n]$ to indicate that we use any denotation that sends x_i to a_i for $i \leq n$. The same happens with formulas: if $\varphi = \varphi(x_1, ..., x_n)$, then $\mathfrak{A} \models \varphi[a_1, ..., a_n]$ will mean $\mathfrak{A} \models \varphi[\pi]$ with any interpretation sending x_i to a_i for $i \leq n$. For sentences we can just write $\mathfrak{A} \models \varphi$.

In the following we define the central notion of elementary equivalence:

Definition 1.7. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *S-structures. We say that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are elementarily equivalent (and write* $\mathfrak{A} \equiv \mathfrak{B}$*) if for every sentence* $\varphi \in L_{\omega\omega}^S$, $\mathfrak{A} \models \varphi$ *iff* $\mathfrak{B} \models \varphi$.

We now turn to results regarding homomorphisms between structures.

Definition 1.8. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be S-structures. A map* $F : A \to B$ *is a homomorphism if the following conditions hold:*

(a) $F(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ *for every constant* $c \in S$.

(b) $F(f^{\mathfrak{A}}(a_1, ..., a_n)) = f^{\mathfrak{A}}(F(a_1), ..., F(a_n))$ *for every n-ary function symbol* $f \in S$ *and every* $a_1, ..., a_n \in A$.

(c) $(a_1, ..., a_n) \in R^{\mathfrak{A}}$ *implies* $(F(a_1), ..., F(a_n)) \in R^{\mathfrak{B}}$ *for every n-ary relation symbol* $R \in S$ *and every* $a_1, ..., a_n \in A$.

This definition generalizes the idea of morphism in any algebraic structure. When F is a bijection we have an isomorphism and we write $F : \mathfrak{A} \cong \mathfrak{B}$.

The following lemma shows that isomorphic structures cannot be distinguished by means of first order sentences.

Lemma 1.9. *If $F : \mathfrak{A} \cong \mathfrak{B}$, then for $\varphi(x_1, ..., x_n) \in L^S_{\omega\omega}$ and $a_1, ..., a_n \in A$, $\mathfrak{A} \models \varphi[a_1, ..., a_n]$ iff $\mathfrak{B} \models \varphi[F(a_1), ..., F(a_n)]$.*

Proof. The proof is a rutinary but tedious induction over formulas, it can be found, for example, in [5] III. 5.2. □

From the particular case of sentences in the previous lemma, we infer the following:

Corollary 1.10. *If $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$.*

We have seen that isomorphic structures satisfy the same sentences. Conversely, one could ask whether structures in which the same sentences hold are isomorphic. A counterexample is easy to find: Let $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ where $<$ is the usual order relation. They satisfy the same $L^{\{<\}}_{\omega\omega}$ sentences but they are trivially not isomorphic.

The program of giving an algebraic characterization of elementary equivalence goes back to Tarski. In the Princeton University Bicentennial Conference on Problems of Mathematics of 1946 (see [13]) he stated:

> *Algebras exist which are not isomorphic, but which cannot be distinguished by their arithmetic properties; it would be desirable to construct a theory of arithmetic equivalence of algebras as deep as the notions of isomorphism, etc. now in use.*

The task of finding an algebraic criterion for elementary equivalence, where isomorphisms failed, was first carried out by Fraïssé in 1955 with the characterization we show in chapter 3, and Ehrenfeucht independently discovered it in game form in 1961. Since them, plenty of work has been done in this direction (see [13], footnote 46).

Definition 1.11. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be S-structures. $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ (written $\mathfrak{A} \subseteq \mathfrak{B}$) if*

1. *$A \subseteq B$.*
2. *$c^{\mathfrak{A}} = c^{\mathfrak{B}}$ for every $c \in S$.*
3. *For every n-ary function symbol $f \in S$, $f^{\mathfrak{A}}$ is the restriction of $f^{\mathfrak{B}}$ to A^n.*
4. *For every n-ary relation symbol $R \in S$, $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$.*

In this case we also say that $\mathfrak{B}$ is an extension of $\mathfrak{A}$. For example, $(\mathbb{Z}, +, 0)$ is an extension of $(\omega, +, 0)$ (observe that $(\omega, +, 0)$ is a substructure of $(\mathbb{Z}, +, 0)$ but not a subgroup).

If $\mathfrak{A} \subseteq \mathfrak{B}$ then A is S-closed in $\mathfrak{B}$, that is, A is nonempty, for n-ary $f \in S$, $a_1, ..., a_n \in A$ implies $f^{\mathfrak{B}}(a_1, ..., a_n) \in A$ and $c^{\mathfrak{B}} \in A$ for $c \in S$.

Conversely, every S-closed subset $X \subseteq B$ corresponds to the domain of exactly one substructure of $\mathfrak{B}$ determined by the conditions in the definition above. We denote this substructure by $[X]^{\mathfrak{B}}$, the substructure of $\mathfrak{B}$ generated by X.

The following two lemmas give the relation between satisfaction and substructures:

Lemma 1.12. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be S-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. If π is an interpretation in $\mathfrak{A}$, then for every quantifier-free formula $\varphi \in L^S_{\omega\omega}$ we have $\mathfrak{A} \models \varphi[\pi]$ iff $\mathfrak{B} \models \varphi[\pi]$.*

Proof. The result is an easy induction, it can be found in [3], Proposition 4.3 □

We say that a formula $\varphi(y_1, ..., y_n)$ is universal if it is of the form

$$\forall x_1 ... \forall x_m \psi(x_1, ..., x_m, y_1, ..., y_n),$$

where ψ is quantifier-free.

Lemma 1.13. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be S-structures with $\mathfrak{A} \subseteq \mathfrak{B}$ and $\varphi(x_1, ..., x_n) \in L^S_{\omega\omega}$ a universal formula. Then for all $a_1, ..., a_n \in A$,*

$$\textit{if } \mathfrak{B} \models \varphi[a_1, ..., a_n] \textit{ then } \mathfrak{A} \models \varphi[a_1, ..., a_n].$$

Proof. The result follows from the previous lemma: If $\mathfrak{B} \models \forall x_1 ... \forall x_m \psi(x_1, ..., x_m, a_1, ..., a_n)$, then for all $b_1, ..., b_m \in B$ (and since $A \subseteq B$, for all $b_1, ..., b_m \in A$), $\mathfrak{B} \models \psi[b_1, ..., b_m, a_1, ..., a_n]$, and by Lemma 1.12, $\mathfrak{A} \models \forall x_1 ... \forall x_m \psi[x_1, ..., x_m, a_1, ..., a_n]$. □

1.1. Compactness and Löwenheim-Skolem theorems

As we said in the introduction, model theory has a lot to do with the limitations of our language. In the case of first-order logic, we already mentioned that we cannot distinguish between countable and uncountable cardinalities due to the downward Löwenheim-Skolem theorem. In the words of Wilfrid Hodges:

> «Ravens, so we read, can only count up to seven. They can't tell the difference between two numbers greater than or equal to eight. First-order logic is much the same as ravens, except that the cutoff point is rather hight: it's ω instead of 8.» *Wilfrid Hodges, Model theory*

An explanation for this fact is given by the Compactness and Löwenheim-Skolem theorems, which play a determinant role in the semantics of first-order logic and are widely applied to the study of mathematical structures. Not only are these theorems important, but, as Lindström proved, they characterize first-order logic. For these reasons they are considered the two basic results in the model theory of first-order logic. We also apply them to prove the existence of non-standard models of arithmetic.

The classical approach to the Compactness theorem is to have it as a corollary of the Completeness theorem. Model theorists tend to prefere direct proofs independent from the deductive calculus, for instance, using the set theoretical tools of filters and ultrafilters. The proof we give here follows the classical proof method of Henkin but using the shortcut in Lemma 1.16 to avoid Completeness theorem (see [3]).

Definition 1.14. *Let Σ be a set of $L^S_{\omega\omega}$ sentences. We say that Σ is finitely satisfiable if every finite subset of Σ is satisfiable.*

Obviously, the reciprocal is always true: if a set of sentences is satisfiable then every finite subset is satisfiable too.

Definition 1.15. *Let $C \subseteq S$ be a set of constants. We say that Σ has witnesses in C when for every formula $\varphi(x) \in L^S_{\omega\omega}$ such that $\exists x\varphi(x) \in \Sigma$ there is $c \in C$ with $\varphi(c) \in \Sigma$.*

The idea of Henkin's proof of the Compactness Theorem is to construct a model of Σ whose universe is C.

Lemma 1.16. *Let S be a language and C a set of constants such that $S \cap C = \emptyset$ and $|C| = |S| + \omega$. Then there is a set of sentences $\Delta \subseteq L^{S\cup C}_{\omega\omega}$ such that:*

(a) If Σ is a finitely satisfiable set of $L^S_{\omega\omega}$-sentences, then $\Sigma \cup \Delta$ is finitely satisfiable.

(b) If $\Gamma \supseteq \Delta$ is a set of sentences in $S \cup C$ closed under modus ponens (that is, $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ imply that $\psi \in \Gamma$), then Γ has witnesses in C.

Proof. Let $\kappa = |C| = |S| + \omega$ and $C = \{c_i \mid i < \kappa\}$. Since every sentence has a finite number of symbols, we can give an enumeration, $(\varphi_i \mid i < \kappa)$, of the formulas with a single free variable in $L^{S\cup C}_{\omega\omega}$, so that for every $i < \kappa$, $\varphi_i = \varphi_i(x_i)$ is a $S \cup \{c_j \mid j < i\}$-formula.

We now define $\Delta = \{(\exists x_i \varphi(x_i) \to \varphi_i(c_i)) \mid i < \kappa\}$. It is clear that this Δ already satisfies (b). We turn to the proof of (a).

Consider a finitely satisfiable set of S-sentences Σ. For each $\alpha \leq \kappa$, let $\Delta_\alpha = \{(\exists x_i \varphi(x_i) \to \varphi_i(c_i)) \mid i < \alpha\}$. We will prove by induction that for every $\alpha \leq \kappa$ $\Sigma \cup \Delta_\alpha$ is finitely satisfiable.

When $\alpha = 0$ or the result is trivial, and using induction hypothesis we also know that the result holds when α is a limit cardinal. Now suppose $\Sigma \cup \Delta_\alpha$ is finitely satisfiable. We know that $\Sigma \cup \Delta_{\alpha+1} = \Sigma \cup \Delta \cup \{\exists x_\alpha \varphi_\alpha(x_\alpha) \to \varphi_\alpha(c_\alpha)\}$, and since $\Sigma \cup \Delta_\alpha$ and $\exists x_\alpha \varphi_\alpha(x_\alpha)$ are $S \cup \{c_i \mid i < \alpha\}$-sentences, in every model of $\Sigma \cup \Delta$ we can always interpret c_α in a way that satisfies the conditional, so the result is proved.

□

Lemma 1.17 (Lindenbaum)**.** *Let Σ be a finitely satisfiable set of sentences. Then Σ can be extended to a maximal set with respect to finite satisfiability.*

Proof. A simple application of Zorn's Lemma. □

Lemma 1.18. *Let Σ be a maximal set of sentences with respect to finite satisfiability with witnesses in $C \subseteq S$. Then Σ has model $\mathfrak{A}$ with $A = \{c^{\mathfrak{A}} \mid c \in C\}$.*

Proof. Consider the set T of terms without variables. We define the following relation in T:

$$t_1 \sim t_2 \text{ iff } t_1 = t_2 \in \Sigma.$$

Since Σ is maximal, $\sim$ is en equivalence relation in T and the following hold:

(a) If f is an n-ary function symbol and $t_i \sim t'_i$ for $i < n$, then $f(t_1, ..., t_n) \sim f(t'_1, ..., t'_n)$.

(b) If R is an n-ary relation symbol, $Rt_1, ..., t_n \in \Sigma$ and $t_i \sim t'_i$ for $i < n$, then $Rt'_1, ..., t'_n \in \Sigma$.

We will now construct a model $\mathfrak{A}$ whose universe is $A = {}^{T}\!/\!_{\sim} = \{[t]_\sim \,|\, t \in T\}$, the equivalence classes of the relation $\sim$ in T. The interpretations of the symbols in it are the following:

1. For each $c \in S$, $c^{\mathfrak{A}} = [c]_\sim$.
2. For every n-ary function symbol $f \in S$ and arbitrary $t_1, ..., t_n \in T$, $f^{\mathfrak{A}}([t_1]_\sim, ..., [t_n]_\sim) = [f(t_1, ..., t_n)]_\sim$.
3. For every n-ary relation symbol $R \in S$ and arbitrary $t_1, ..., t_n \in T$, $([t_1]_\sim, ..., [t_n]_\sim) \in R^{\mathfrak{A}}$ iff $Rt_1, ..., t_n \in \Sigma$.

Let $a \in A$. Then there is $t \in T$ such that $a = [t]_\sim$ and since $\Sigma \models \exists xx = t$, $\exists xx = t \in \Sigma$. As Σ has witnesses in C, there is $c \in C$ with $c = t \in \Sigma$, whence $c \sim t$. This means that $a = [c]_\sim = c^{\mathfrak{A}}$, thus $A = \{c^{\mathfrak{A}} \,|\, c \in C\}$.

Now we will prove by induction that $\mathfrak{A}$ is a model of Σ. It is easy to see that for each $t \in T$, $t^{\mathfrak{A}} = [t]_\sim$ and using (b) we can prove that for each atomic sentence $\varphi \in L^S_{\omega\omega}$, $\mathfrak{A} \models \varphi$ iff $\varphi \in \Sigma$. For $\neg\varphi$ and $\varphi \vee \psi$, the maximality of Σ gives the same result, so we turn to consider the case $\exists x\varphi(x)$.

Since $A = \{c^{\mathfrak{A}} \,|\, c \in C\}$, $\mathfrak{A} \models \exists x\varphi(x)$ if and only if there is $c \in C$ such that $\mathfrak{A} \models \varphi(c)$. As C is a set of witnesses for Σ, $\exists x\varphi(x) \in \Sigma$ iff there is $c \in C$ such that $\varphi(c) \in \Sigma$, and applying the induction hypothesis to $\varphi(c)$ we have the desired result.

□

We finally prove the Compactness Theorem for $\mathfrak{L}_{\omega\omega}$:

Theorem 1.19 (Compactness Theorem). *Let S be a language and Σ a finitely satisfiable set of sentences in $L^S_{\omega\omega}$, then Σ is satisfiable (in a model of cardinality $\leq |S| + \omega$).*

Proof. Let $\kappa = |S| + \omega$ and consider C a set of constants with $C \cap S = \emptyset$ and $|C| = \kappa$. Let Δ be the set of sentences given by Lemma 1.16. Then $\Sigma \cup \Delta$ will be finitely satisfiable. Consider the extension $\Sigma \cup \Delta \subseteq \Gamma$ given by Lemma 1.17. By Lemma 1.18, Γ has a model $\mathfrak{A}$ with $A = \{c^{\mathfrak{A}} \,|\, c \in C\}$, hence $|\mathfrak{A}| \leq \kappa$. Then $\mathfrak{A}|_S$ satisfies Σ in the language S. □

If one reexamines the precedent proof, we can obtain the following result:

Theorem 1.20 (Downward Löwenheim-Skolem). *Let Σ be a satisfiable set of S-sentences. Then Σ has a model $\mathfrak{A}$ of cardinality $|A| \leq |L^S_{\omega\omega}| = |S| + \omega$.*

In particular, if Σ is countable, then it will only depend on countably many symbols, $S_0 \subseteq S$, so $\Sigma \subset L^{S_0}_{\omega\omega}$ and hence has a model of cardinality $\leq \omega$. The Downward Löwenheim-Skolem theorem is usually stated for this particular case.

Although it will not be used in the future, we prove, for the sake of completeness, the Upward Löwenheim-Skolem theorem:

Theorem 1.21 (Upward Löwenheim-Skolem). *Let Σ be a set of satisfiable S-sentences with an infinite model. Then for each cardinal $\kappa \geq |S| + \omega$, there is a model of Σ with cardinality κ.*

Proof. Let $\kappa \geq |S| + \omega$ and consider $C = \{c_i \,|\, i < \kappa\}$ a new set of constant symbols such that $C \cap S = \emptyset$ and $c_i \neq c_j$ for $i \neq j$. Let $\Gamma = \Sigma \cup \{\neg c_i \neq c_j \,|\, i < j < \kappa\}$. Since Σ has an infinite model, Γ is finitely satisfiable (giving the proper interpretation to the new constants in the old model), and by the compactness theorem, Γ has a model $\mathfrak{A}$ of cardinality $\leq |S \cup C| + \omega \leq \kappa$. But since $\mathfrak{A} \models \neg c_i = c_j$ for every $i \neq j < \kappa$, the cardinality of A has to be exactly κ. Then $\mathfrak{A}|_S$ is a model of Σ with cardinality κ in the language S. $\square$

The existence of non-standard models of arithmetic is an interesting consequence of the results in this section. Let $\mathfrak{N} = (\omega, \sigma, 0)$ be the standard structure of the natural numbers. A non standard model of arithmetic is an structure which satisfies $Th(\mathfrak{N}) = \{\varphi \in L^S_{\omega\omega} \,|\, \mathfrak{N} \models \varphi\}$ but which is not isomorphic to $\mathfrak{N}$.

Consider $S = \{\sigma, 0\}$ and the standard structure of the natural numbers $\mathfrak{N} = (\omega, \sigma, 0)$ where σ is the usual successor function.. We can expand the language with a new constant c and consider the set of sentences in $L^{S\cup\{c\}}_{\omega\omega}$, $\Gamma = Th(\mathfrak{N}) \cup \{c \neq 0\} \cup \{c \neq \sigma 0\} \cup \{c \neq \sigma\sigma 0\}$.... Obviously, every finite subset of the previous set is satisfiable, because we can take a model which is the same as $\mathfrak{N}$ and interprets c as $\sigma^{\mathfrak{N}} n$ where n is the biggest natural such that $c \neq n$ appears in the subset. (Recall that we use n as an abbreviation for $(\sigma \underbrace{\ldots}_{\text{n times}} \sigma 0)^{\mathfrak{N}}$). Hence every finite subset of Γ is satisfiable and by the Compactness theorem, Γ itself has a model $\mathfrak{M}$. Since $Th(\mathfrak{N}) \subset \Gamma$, $\mathfrak{M} \models Th(\mathfrak{N})$, thus we have a non-standard model of arithmetic:

Suppose $f : \mathfrak{N} \to \mathfrak{M}$ is an isomorphism. Then there is $n \in \mathfrak{N}$ such that $f(n) = c^{\mathfrak{M}}$. But in that case $f(\sigma^{\mathfrak{N}}(n-1)) = \sigma^{\mathfrak{N}}(f(n-1)) = c^{\mathfrak{M}}$, in contradiction with the sentence $c \neq \sigma(f(n-1)) \in \Gamma$.

Another way of obtaining a non-standard model of arithmetic is to use the upward Löwenheim-Skolem theorem: since $Th(\mathfrak{N})$ is satisifable and infinite, it has models of every cardinality $\kappa > \omega$, and those models are trivially not isomorphic to $\mathfrak{N}$.

As we have just seen, we cannot characterize natural numbers up to isomorphism in first-order logic. This seems a drawback of our formal language, but experience has shown that non-standard models are of great mathematical interest. In fact, non-standard analysis can be built with similar techniques, blurring the line between desired and undesired results in logic and suggesting that we should not try to avoid those limitations of the formal method but to learn as much as possible from them.

Chapter 2

Extensions of first-order logic

In the following we present three basic examples of logical systems which extend first-order logic in some way. For now, the very notions of logical system or extension of a logic will remain vague, but a proper definition will be given in Chapter 4. The basic idea is that a logical system is just bunch of rules to build sentences and formulas out of symbols and a satisfaction relation that tells whether a sentence is true in a model or not.

Extensions of first-order logic started to gain popularity in the mid-1950s with the works of Mostowski, Henkin, Tarski and many others who tried to obtain logical systems with more expresive power than $\mathfrak{L}_{\omega\omega}$, in many occasions for applications in algebra. [1]

The task of finding an extension satisfying the same basic model-theoretical properties as $\mathfrak{L}_{\omega\omega}$ soon became very difficult. The systems we briefly survey here are intended to give a glance of how these systems are valuable for mathematical purposes and how they fail to satisfy Compactness and (downward) Löwenheim-Skolem theorems, hence giving a motivation for Lindström's (first) theorem. The main references for this chapter are [5] and [2].

2.1. Second-order logic

Second-order logic differs from first-order logic in the fact that in the former one can quantify over second-order objects such as subsets of the domain of a structure or n-ary relations and functions. In the following we present the basic syntax and semantics of the second-order language $\mathfrak{L}_{II}$.

Let S be a language. In adition to the symbols of $L^S_{\omega\omega}$, L^S_{II} will contain, for each $n \geq 1$, countable many n-ary relational variables $X^n_0, X^n_1, \ldots$.

[1]"The continuous development of infinitary logic that began in 1954 was part of a broader effort during the mid-1950s to extend first-order logic to logics that hopefully retained valuable features of first-order logic but had greater expressive power (e.g. weak second-order logic, generalized quantifiers). Foremost among those features was completeness (every valid sentence is provable) and strong completeness (every consistent set of sentences has a model). Yet the work of Henkin and Tarski (and their students Karp and Scott) in the mid-1950s was motivated at first by algebraic concerns and especially by Boolean algebras, sometimes with additional operators (e.g. cylindric algebras)." *Moore G.H. (1997) The Prehistory of Infinitary Logic: 1885-1955. pg. 105.*

We define the set L_{II}^S of second-order S-formulas to be the set generated with the syntax for first-order formulas extended by the following rules:

(a) If X is an n-ary relational variable and $t_1, ..., t_n$ are S-terms, then $Xt_1, ..., t_n$ is and S-formula.

(b) If φ is an S-formula and X is a relational variable, then $\exists X\varphi$ is an S-formula.

We now extend the satisfaction relation from $\mathfrak{L}_{\omega\omega}$ by taking into account (a) and (b): If $\mathfrak{A}$ is an S-structure, γ a second-order assignment in $\mathfrak{A}$ (a map sending each variable x to an element of A and each n-ary relational variable X^n to an n-ary relation on A) and $\mathfrak{I} = (\mathfrak{A}, \gamma)$, then we set:

(a) $\mathfrak{I} \models Xt_1, ..., t_n$ iff $(t_1^{\mathfrak{A}}[\gamma], ..., t_n^{\mathfrak{A}}[\gamma]) \in \gamma(X)$.

(b) For n-ary X, $\mathfrak{I} \models \exists X\varphi$ iff there is $C \subseteq A^n$ such that $\mathfrak{I}(\frac{C}{X}) \models \varphi$, where $(\frac{C}{X})$ entails the substitution of C in every free occurence of X in φ.

We use $\mathfrak{L}_{II}$ to denote second-order logic. Note that we can also use and quantify function variables, since we can make them relational taking their graph.

Remark 2.1. Arithmetic can be properly stated in second-order logic: let $S = \{\sigma, 0\}$ where σ is a unary function and 0 a constant. Then we have the following L_{II}^S-axioms (these are the classical Peano axioms):

(a) $\forall x \neg \sigma x = 0$,

(b) $\forall x \forall y(\sigma x = \sigma y \to x = y)$,

(c) $\forall X((X0 \wedge \forall x(Xx \to X\sigma x)) \to \forall y Xy)$.

It is a theorem by Dedekind that every two Peano structures (structures satisfying Peano's axioms) are isomorphic, so we have that, unlike in first-order logic, we can characterize natural numbers up to isomorphisms. Note that the fact that we can write down the induction axiom (c), prevents us from constructing a non-standard model using the Compactness theorem as we did in the previous chapter.

Another advantage of $\mathfrak{L}_{II}$ is that we can characterize finiteness with it. Let g be an unary function variable and $\varphi_{\text{fin}} := \forall g(\forall x \forall y(gx = gy \to x = y) \to \forall x \exists y x = gy)$. $\mathfrak{A} \models \varphi_{\text{fin}}$ iff every injective function from A to A is surjective, that is, $\mathfrak{A} \models \varphi_{\text{fin}}$ iff A is finite.

This kind of expressive power regarding infinity usually suggests that the Compactness theorems fail:

Theorem 2.2. *The Compactness theorem does not hold for $\mathfrak{L}_{II}$.*

Proof. Consider the sentences $\varphi_{\geq n} = \exists x_1 ... \exists x_n(x_1 \neq x_2 \wedge ... \wedge x_1 \neq x_n \wedge ... \wedge x_{n-1} \neq x_n)$ for each $n \in \omega$. $\mathfrak{A} \models \varphi_{\geq n}$ if and only if A has more than n elements. Then the following set of sentences works as a counterexample for the theorem:

$$\{\varphi_{\text{fin}}\} \cup \{\varphi_{\geq n} \mid n \geq 2\}.$$

□

Using a similar argument, we can prove the following:

Theorem 2.3. *The Löwenheim-Skolem theorem does not hold for $\mathfrak{L}_{\omega\omega}$.*

Proof. We give a sentence φ_{unc} such that for all structures $\mathfrak{A}$, $\mathfrak{A} \models \varphi_{\text{unc}}$ iff A is uncountable. Then φ_{unc} will be satisfiable but will not have any countable model.

We first define a formula $\psi_{\text{fin}}(X)$ with unary X such that $(\mathfrak{A}, \gamma) \models \psi_{\text{fin}}(X)$ iff $\gamma(X)$ is finite: $\psi_{\text{fin}}(X) :=$

$$\forall g(\forall x \forall y((Xx \wedge Xy \wedge Xgx \wedge Xgy \wedge gx = gy) \rightarrow x = y) \rightarrow \forall x(Xx \rightarrow \exists y(Xy \wedge x = gy))).$$

Note that $\psi_{\text{fin}}(X)$ is the relativization of φ_{fin} to X. We know that a set A is at most countable iff there is a total order on A such that every element has only finitely many predecessors, so we define, using a binary relation variable Y, $\varphi_{\leq ctbl} := \exists Y(\forall x \neg Yxx \wedge \forall x \forall y \forall z((Yxy \wedge Yyz) \rightarrow Yxz) \wedge \forall x \forall y(Yxy \vee x = y \vee Yyx) \wedge \forall x \exists X(\psi_{\text{fin}}(X) \wedge \forall y(Xy \leftrightarrow Yyx)))$. Then we have $\mathfrak{A} \models \varphi_{\leq \text{ctbl}}$ iff A is at most countable and hence we can set $\varphi_{\text{unc}} := \neg \varphi_{\leq \text{ctbl}}$. ☐

As we have seen, second-order logic sacrifices many good model theoretical properties in favour of great expressive power, but the lack of any kind of Compactness theorem makes it difficult to handle for mathematical purposes. For these reasons logicians usually study weaker versions of second-order logic where some analogue of the Compactness theorem holds.

2.2. Infinitary logic

Another natural way of extending first-order logic is to allow formulas of infinite lenght, in particular, infinite disjunctions and conjunctions. These logics are called infinitary logics. We now present the system $\mathfrak{L}_{\omega_1\omega}$ where only countable conjunctions and disjunctions are allowed.

Let S be a language. We add the symbol $\bigvee$ for infinite (countable) disjunctions. We define the set $L^S_{\omega_1\omega}$ to be the set generated by the rules of first-order formulas extended with the following:

- If Φ is an at most countable set of S-formulas, then $\bigvee \Phi$ is an S-formula.

We define satisfaction relation for $\bigvee$ with the following clause: $\mathfrak{A} \models \bigvee \Phi$ if and only if $\mathfrak{A} \models \varphi$ for some $\varphi \in \Phi$.

Remark 2.4.
- Obviously, the infinite conjunction can be defined in the natural way: $\bigwedge \Phi := \neg \bigvee \neg \Phi$, where $\neg \Phi := \{\neg \varphi \mid \varphi \in \Phi\}$.
- As in second order logic, we can characterize natural numbers up to isomorphism with the first two Peano Axioms in Remark 2.1 and the sentence

$$\forall x \bigvee \{x = \underbrace{\sigma \cdots \sigma}_{\text{n-times}} 0 \mid n \geq 0\}.$$

Theorem 2.5. *The Compactness theorem does not hold for $\mathfrak{L}_{\omega_1\omega}$*

Proof. Consider the sentence $\psi_{\text{fin}} := \bigvee\{\neg\varphi_{\geq n} \mid n \geq 2\}$. The set of sentences $\{\psi_{\text{fin}}\} \cup \{\varphi_{\geq n} \mid n \geq 2\}$ is a counterexample. □

The lack of a strict analogue for the Compactness theorem makes the model theory of infinitary logics weaker, but one can still transfer results from first-order logic by taking into account the set theoretical issues derived from the fact that we are dealing with infinitely long formulas. See, for example, the Barwise Compactness Theorem for $\mathfrak{L}_{\omega_1\omega}$ in [9].

The definition of the set of subformulas of $\varphi \in L_{\omega_1\omega}$ is obtained adding the clause $SF(\bigvee\Phi) := \{\bigvee\Phi\} \cup \bigcup_{\psi\in\Phi} SF(\psi)$ to the first-order ones.

Lemma 2.6. *Let S be at most countable, φ and $L^S_{\omega_1\omega}$-sentence and $\mathfrak{B}$ an S-structure such that $\mathfrak{B} \models \varphi$. Then there is an at most countable substructure $\mathfrak{A} \subseteq \mathfrak{B}$ with $\mathfrak{A} \models \varphi$.*

Proof. We define a sequence $A_0, A_1, ...$ of at most countable subsets of B so that for $m \in \omega$,

(a) $A_0 \supseteq \{c^{\mathfrak{B}} \mid c \in S\}$.

(b) $A_m \subseteq A_{m+1}$.

(c) For $\psi(x_1, ..., x_n, x) \in SF(\varphi)$ or $\psi = (fx_1 \ldots x_n = x)$ (with n-ary $f \in S$) and $a_1, ..., a_n \in A_m$, if $\mathfrak{B} \models \exists x\psi[a_1, ..., a_n]$ then there is an $a \in A_{m+1}$ such that $\mathfrak{B} \models \psi[a_1, ..., a_n, a]$.

Suppose A_m is already defined and is at most countable. In order to construct A_{m+1}, for every formula $\psi(x_1, ..., x_n, x) \in SF(\varphi)$ or which has the form $fx_1 \ldots x_n = x$ (with n-ary $f \in S$) and $a_1, ..., a_n \in A_m$ with $\mathfrak{B} \models \exists x\psi[a_1, ..., a_n]$ we choose $b \in B$ such that $\mathfrak{B} \models \psi[a_1, ..., a_n, b]$. Let A'_m the set of b's chosen this way. Since $SF(\varphi)$ and A_m are countable so is A'_m. Then $A_{m+1} = A_m \cup A'_m$. Now we set $A := \bigcup_{m\in\omega} A_m$. We have:

Obviously A is (at most) countable and S-closed, so A is the domain of an at most countable substructure $\mathfrak{A}$ of $\mathfrak{B}$. Now it is straightforward to prove by induction that for all $\psi(x_1, ..., x_n) \in SF(\varphi)$ and $a_1, ..., a_n \in A$, $\mathfrak{A} \models \psi[a_1, ..., a_n]$ iff $\mathfrak{B} \models \psi[a_1, ..., a_n]$. Then $\mathfrak{A} \models \varphi$. □

Since for every $\mathfrak{L}_{\omega_1\omega}$-sentence there is an at most countable S such that $\varphi \in L^S_{\omega_1\omega}$, we already have Löwenheim-Skolem theorem for $\mathfrak{L}_{\omega_1\omega}$:

Theorem 2.7. *Every satisfiable $\mathfrak{L}_{\omega_1\omega}$-sentence (or countable set of sentences) has an at most countable model.*

2.3. Cardinality quantifiers

Cardinality quantifiers were first suggested by Mostowski [11] in 1957 in the framework of his more general concept of *generalized quantifiers*, and they significantly expand the expressive power of the language. We focus on the most basic example: the logic $\mathfrak{L}(Q_0)$, where $Q_0x\varphi$ means "there are infinitely x satisfying φ".

Let α be an ordinal and consider the first-order logic $\mathfrak{L}_{\omega\omega}$ with identity. We form the logic $\mathfrak{L}(Q_\alpha)$ by adding to $\mathfrak{L}_{\omega\omega}$ a new quantifier $Q_\alpha x$ which will be interpreted as "there are $\aleph_\alpha$ many x"[2]. Then we add the rule

- If $\varphi(x)$ is a formula then so is $Q_\alpha x\varphi$.

The satisfaction relation is expanded in the natural way. Let $\varphi(x, x_1..., x_n)$ be a $L^S(Q_\alpha)$-formula and $a_1, ..., a_n \in A$. Then

$$\mathfrak{A} \models Q_\alpha x\varphi[a_1, ..., a_n] \text{ iff } |\{b \in A \mid \mathfrak{A} \models \varphi[b, a_1, ..., a_n]\}| \geq \aleph_\alpha.$$

Now, we turn to the particular system $\mathfrak{L}(Q_0)$. This logic builds in the finite/infinite distinction missing from first-order logic. Consider the set

$$\Gamma = \{\neg Q_0 xx = x\} \cup \{\varphi_{\geq n} \mid n \geq 2\}.$$

It is clear that every finite subset is satisfiable, but Γ itself is not, hence the Compactness theorem does not hold in $\mathfrak{L}(Q_0)$.

However, Löwenheim-Skolem theorem does hold: we just have to use the argument of Lemma 2.6 and make a similar construction taking countably many elements into A_{m+1} whenever we have $\mathfrak{B} \models Q_0 x\varphi[a_1, ..., a_n]$, with $a_1, ..., a_n \in A_m$.

Furthermore, the structure of natural numbers can also be axiomatized in this logic, the only difference with respect to previous examples is that we have to expand our language to include the usual order relation.

Indeed, the following axioms, along with (a) and (b) in Remark 2.1, characterize $(\omega, \sigma, 0, <)$ up to isomorphism:

- $\forall x \neg Q_0 y(y < x)$
- $\forall x \forall y(x < y \leftrightarrow \sigma(x) < \sigma(y))$
- $\forall x(x \neq 0 \rightarrow 0 < x)$
- $\forall x \forall y(x < y \leftrightarrow (\sigma(x) = y \vee \sigma(x) < y))$
- "$<$ is a linear order without last element".

The incompactness of $\mathfrak{L}(Q_0)$ is a great disadvantage, and it turns the system quite useless for mathematical purposes. For that reason, model theorists have headed their attention to better behaving extensions such as $\mathfrak{L}(Q_1)$ which we present in Chapter 6. $\mathfrak{L}(Q_0)$ remains as an example of a *natural* logic which turns out to be unsatisfactory.

[2]In the standard representation of infinite cardinals, $\aleph_\alpha$ represents the α-th infinite cardinal. For example, $\aleph_1$ is the first uncountable cardinal.

Chapter 3

An algebraic characterization of elementary equivalence

We now introduce partial isomorphisms and the back-and-forth method, a well stablished technique in model theory that constitutes the basis for the algebraic characterization of elementary equivalence we seek. The main result of the chapter is, therefore, Fraïsse's theorem, which states that for two structures $\mathfrak{A}$ and $\mathfrak{B}$, being elementarily equivalent is tantamount to being isomorphic in a sense we introduce in 3.1. This result provides us with a new way to disprove the elementary equivalence between two structures without any reference to first-order language. This approach will be very useful in the proof of Lindström's theorem.

In the first subsection we present all the theoretical machinery we need for Fraïssé's theorem, including the key notions of two structures being finitely (partially) isomorphic, and in 3.2 we give a complete proof.

The entire chapter follows the spirit of [5]. For a classical discussion of the back-and-forth method and some of its consequences, see [1].

3.1. Partial isomorphisms and back-and-forth

Cantor was the first to understand that partial isomorphisms can be extended to an isomorphism provided the structures in which they are defined are countable. Using this idea, he proved that any two countable and dense linear orderings without endpoints are isomorphic, which in an abstract version, can be stated as Lemma 3.4(d). The technique he used is now called back-and-forth method, and is widely used in model theory. We start by defining partial isomorphisms:

Definition 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be S-structures and let p be a map such that $dom(p) \subseteq A$ and $rg(p) \subseteq B$. Then p is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ if and only if p has the following properties:*

1. *p is injective.*

2. *p is a morphism in the following sense:*

   a) *For n-ary $P \in S$ and $a_1, ..., a_n \in dom(p)$, $P^{\mathfrak{A}} a_1 \ldots a_n$ iff $P^{\mathfrak{B}} p(a_1)...p(a_n)$.*

   b) *For n-ary $f \in S$ and $a_1, ..., a_n, a \in dom(p)$, $f^{\mathfrak{A}}(a_1, ..., a_n) = a$ iff $f^{\mathfrak{B}}(p(a_1)...p(a_n)) = p(a)$.*

   c) *For $c \in S$ and $a \in dom(p)$, $c^{\mathfrak{A}} = a$ iff $c^{\mathfrak{B}} = p(a)$.*

We write $Part(\mathfrak{A}, \mathfrak{B})$ for the set of all partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

Definition 3.2. *$\mathfrak{A}$ and $\mathfrak{B}$ are finitely isomorphic ($\mathfrak{A} \cong_f \mathfrak{B}$) if and only if there is a sequence $(I_n)_{n\in\omega}$ such that:*

(a) Every I_n is a nonempty set of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

(b) (Forth-property) For every $p \in I_{n+1}$ and $a \in A$, there is $q \in I_n$ such that $p \subseteq q$ and $a \in dom(q)$.

(c) (Back-property) For every $p \in I_{n+1}$ and $b \in B$, there is $q \in I_n$ such that $p \subseteq q$ and $b \in rg(q)$.

We can informally understand (b) and (c) as follows: partial isomorphisms in I_n can be extendend n times and these extensions lie in $I_{n-1}, ..., I_0$.

Definition 3.3. *$\mathfrak{A}$ and $\mathfrak{B}$ are partially isomorphic ($\mathfrak{A} \cong_p \mathfrak{B}$) if and only if there is a set I such that:*

(a) I is a nonempty set of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

(b) (Forth-property) For every $p \in I$ and $a \in A$, there is $q \in I$ such that $p \subseteq q$ and $a \in dom(q)$.

(c) (Back-property) For every $p \in I$ and $b \in B$, there is $q \in I$ such that $p \subseteq q$ and $b \in rg(q)$.

The following lemma lists the basic relations between all the notions of isomorphism we have described.

Lemma 3.4. *(a) If $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \cong_p \mathfrak{B}$.*

(b) If $\mathfrak{A} \cong_p \mathfrak{B}$, then $\mathfrak{A} \cong_f \mathfrak{B}$.

(c) If $\mathfrak{A} \cong_f \mathfrak{B}$ and A is finite, then $\mathfrak{A} \cong \mathfrak{B}$.

(d) If $\mathfrak{A} \cong_p \mathfrak{B}$ and A and B are countable, then $\mathfrak{A} \cong \mathfrak{B}$.

Proof. (a) If $\pi : \mathfrak{A} \cong \mathfrak{B}$, then $I : \mathfrak{A} \cong_p \mathfrak{B}$ for $I = \{\pi\}$.

(b) If $I : \mathfrak{A} \cong_p \mathfrak{B}$, then $(I)_{n\in\omega} : \mathfrak{A} \cong_f \mathfrak{B}$.

(c) If $(I_n)_{n\in\omega} : \mathfrak{A} \cong_f \mathfrak{B}$ and A has exactly r elements, $A = \{a_1, ..., a_r\}$, we choose $p \in I_{r+1}$. If we apply the forth property r times, we obtain $q \in I_1$ such that $a_1, ..., a_r \in dom(q) = A$. If $rg(q) \neq B$, we take $b \in B \backslash rg(q)$ and by the back-property there would be a proper extension q' of q in I_0 with $b \in rg(q')$. Since $dom(q) = A$, this is not possible. Therefore $q : \mathfrak{A} \cong \mathfrak{B}$.

(d) Suppose $I : \mathfrak{A} \cong_p \mathfrak{B}$, $A = \{a_i\}_{i\in\omega}$ and $B = \{b_j\}_{j\in\omega}$. Taking an arbitrary $p_0 \in I$, we can repeatedly apply the back-and-forth properties obtaining extensions $p_1, p_2, ... \in I$ such that $a_1 \in dom(p_1), b_1 \in rg(p_2), a_2 \in dom(p_3), b_2 \in rg(p_4), ...$ and so on. That is, a sequence $(p_n)_{n\in\omega}$ of partial isomorphisms in I that for all n satisfy:

(i) $p_n \subseteq p_{n+1}$.

(ii) If $n = 2r - 1$, then $a_r \in dom(p_n)$.

(iii) If $n = 2r$, then $b_r \in rg(p_n)$.

By (i), $p := \bigcup_{n\in\omega} p_n$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. By (ii) and (iii), $dom(p) = A$ and $rg(p) = B$, thus $p : \mathfrak{A} \cong \mathfrak{B}$.

□

3.2. Fraïssé's theorem

In this section, **S will be a finite relational language**. We can make any set relational (i.e. having only relation symbols) by replacing functions and constants by their graphs.

We start with a easy but useful result that gives a first conection between partial isomorphisms and satisfaction relation (To simplify the notation, in this section L^S_r will be the subset of $L^S_{\omega\omega}$ composed by the formulas with r free variables $v_0, ..., v_{r-1}$).

Lemma 3.5. *For $a_0, ..., a_{r-1} \in A$ and $b_0, ..., b_{r-1} \in B$, the following statements are equivalent:*

- *$p(a_i) := b_i$ for $i < r$ determines a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.*
- *For every atomic formula $\psi \in L^S_r$, $\mathfrak{A} \models \psi[a_0, ..., a_{r-1}]$ iff $\mathfrak{B} \models \psi[b_0, ..., b_{r-1}]$.*

Proof. A straightforward induction. It can be found in XII. 1.2(c) in [5]. □

Remark 3.6. It is easy to check that, in general, a partial isomorphism does not preserve the validity of formulas with quantifiers. For example, take $S = \{<\}$ and let p the partial isomorphism from $(\mathbb{R}, <)$ to $(\mathbb{Z}, <)$ with $dom(p) = \{2, 3\}$ and $p(2) = 3$, $p(3) = 4$. Then

$$(\mathbb{R}, <) \models \exists v_2(v_0 < v_2 \wedge v_2 < v_1)[2, 3],$$

but

$$\text{not } (\mathbb{Z}, <) \models \exists v_2(v_0 < v_2 \wedge v_2 < v_1)[p(2), p(3)].$$

However, there is still some subtle relation between partial isomorphisms and the validity of formulas: following with the previous example, for any two $a < b$ the validity of $(\mathbb{Z}, <) \models \exists v_2(v_0 < v_2 \wedge v_2 < v_1)[p(a), p(b)]$ is equivalent to the existence of a partial isomorphism q which extends p and has (for example) $\frac{a+b}{2}$ in its domain. Clearly, in the particular case with $dom(p) = \{2, 3\}$ and $p(2) = 3$, $p(3) = 4$, such extension does not exist.

This remark indicates the key idea for the algebraic characterization of elementary equivalence: if the truth of formulas with quantifiers is preserved under partial isomorphisms, provided that these admit certain extensions, then elementary equivalence of structures amounts to the existence of these extensions. Properly formalized, this is the main result of the section:

Theorem 3.7 (Fraïssé). *For any two S-structures $\mathfrak{A}$ and $\mathfrak{B}$,*

$$\mathfrak{A} \equiv \mathfrak{B} \text{ iff } \mathfrak{A} \cong_f \mathfrak{B}.$$

We devote the rest of the section to prove this theorem. First we define the quantifier rank of a formula as the maximum number of nested quantifiers occurring in it:

Definition 3.8. *For each $\varphi \in L^S_{\omega\omega}$ we recursively define his quantifier rank as follows:*

(a) $qr(\varphi) = 0$ *if φ is atomic.*

(b) $qr(\neg\varphi) = qr(\varphi)$.

(c) $qr(\varphi \vee \psi) = max\{qr(\varphi), qr(\psi)\}$.

(d) $qr(\exists x\varphi) = qr(\varphi) + 1$.

Informally, the following lemma says that partial isomorphisms in I_n preserve truth value of formulas with quantifier rank $\leq n$.

Lemma 3.9. *Let $(I_n)_{n\in\omega} : \mathfrak{A} \cong_f \mathfrak{B}$. Then for every $\varphi \in L^S_r$ with $qr(\varphi) \leq n$, any $p \in I_n$ and any $a_0, ..., a_{r-1} \in dom(p)$, $\mathfrak{A} \models \varphi[a_0, ..., a_{r-1}]$ if and only if $\mathfrak{B} \models \varphi[p(a_0), ..., p(a_{r-1})]$.*

Proof. We prove it by induction on formulas:

(a) For atomic formulas the result follows from Lemma 3.5.

(b) If $\varphi = \neg\psi$, then $\mathfrak{A} \models \varphi[a_0, ..., a_{r-1}]$ iff not $\mathfrak{A} \models \psi[a_0, ..., a_{r-1}]$, which is equivalent, by induction hypothesis, to not $\mathfrak{B} \models \psi[p(a_0), ..., p(a_{r-1})]$ iff $\mathfrak{B} \models \varphi[p(a_0), ..., p(a_{r-1})]$.

(c) For $\varphi = \varphi_0 \vee \varphi_1$ the proof is analogous.

(d) If $\varphi = \exists x\psi$, we may assume $x = v_r$. Because $qr(\varphi) = qr(\exists x\psi) \leq n$, we have $qr(\psi) \leq n-1$. Now, $\mathfrak{A} \models \varphi[a_0, ..., a_{r-1}]$ iff there is $a \in A$ such that $\mathfrak{A} \models \psi[a_0, ..., a_{r-1}, a]$, and by induction hypothesis and the forth property of p, there is $q \in I_{n-1}$ such that $p \subseteq q$, $a \in dom(q)$ and $\mathfrak{B} \models \psi[p(a_0), ..., p(a_{r-1}), q(a)]$, and taking $b = q(a)$ we have that there is $b \in B$ such that $\mathfrak{B} \models \psi[p(a_0), ..., p(a_{r-1}), b]$, which means $\mathfrak{B} \models \varphi[p(a_0), ..., p(a_{r-1})]$. Now, using the back property and induction hypothesis, the reciprocal is completely symmetric.

□

Corollary 3.10. *If $\mathfrak{A} \cong_f \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$.*

Proof. Just use Lemma 3.9 with $r = 0$, $n = qr(\varphi)$ and any $p \in I_n$. $\square$

To prove the reciprocal, for any $b^r := b_0, ..., b_{r-1} \in B$ we introduce certain formulas $\varphi^0_{\mathfrak{B},b^r} \in L^S_r$ that will describe the isomorphism type of the substructure $[b_0, ..., b_{r-1}]^{\mathfrak{B}}$. For $n > 0$, $\varphi^n_{\mathfrak{B},b^r}$ will indicate to which isomorphism types b^r can be extended in $\mathfrak{B}$ by adding n elements.

Our goal will be to build the formulas so that $\mathfrak{B} \models \varphi^n_{\mathfrak{B},b^r}[b^r]$ for all n, and if $\mathfrak{A} \models \varphi^n_{\mathfrak{B},b^r}[a^r]$ for $a^r \in A$ then the map $a_i \mapsto b_i$ for $i < r$ is a partial isomorphism that can be extended back-and-forth n times.

Definition 3.11. *We set* $\boldsymbol{\phi}_r := \{\varphi \in L^S_r \mid \varphi$ *is atomic or negated atomic*$\}$. *By recursion in n and for all r and* $b^r \in \mathfrak{B}$, *we define:*

$$\varphi^0_{\mathfrak{B},b^r} := \bigwedge\{\varphi \in \boldsymbol{\phi}_r \mid \mathfrak{B} \models \varphi[b^r]\},$$

$$\varphi^{n+1}_{\mathfrak{B},b^r} := \forall v_r \bigvee\{\varphi^n_{\mathfrak{B},b^r b} \mid b \in B\} \wedge \bigwedge\{\exists v_r \varphi^n_{\mathfrak{B},b^r b} \mid b \in B\}.$$

Remark 3.12. Since S is finite and only contains relation symbols, $\boldsymbol{\phi}_r$ is finite and $\boldsymbol{\phi}_0 = \emptyset$. By induction on n, we easily obtain that the set $\{\varphi^n_{\mathfrak{B},b^r} \mid b^r \in B\}$ is finite and the conjunctions and disjunctions occurring in Definition 3.11 are finite, hence $\varphi^n_{\mathfrak{B},b^r}$ are first-order formulas.

It is also easy to check, using induction over n again, that $qr(\varphi^n_{\mathfrak{B},b^r}) = n$ and $\mathfrak{B} \models \varphi^n_{\mathfrak{B},b^r}[b^r]$ for all $n \in \omega$.

Lemma 3.13. *If* $\mathfrak{A} \models \varphi^n_{\mathfrak{B},b^r}[a^r]$, *then* $a^r \mapsto b^r \in Part(\mathfrak{A}, \mathfrak{B})$.

Proof. The case $n = 0$ is a consequence of Lemma 3.5. Let $\mathfrak{A} \models \varphi^{n+1}_{\mathfrak{B},b^r}[a^r]$. We choose an arbitrary $a \in A$. Since $\mathfrak{A} \models \forall v_r \bigvee\{\varphi^n_{\mathfrak{B},b^r b} \mid b \in B\}[a^r]$, there is some $b \in B$ such that $\mathfrak{A} \models \varphi^n_{\mathfrak{B},b^r b}[a^r a]$. By induction hypothesis, $a^r a \mapsto b^r b \in Part(\mathfrak{A}, \mathfrak{B})$, hence $a^r \mapsto b^r \in Part(\mathfrak{A}, \mathfrak{B})$. $\square$

For $n \in \omega$ we set $J_n := \{a^r \mapsto b^r \mid r \in \omega, a^r \in A, b^r \in B \text{ and } \mathfrak{A} \models \varphi^n_{\mathfrak{B},b^r}[a^r]\}$. Putting everything together:

Lemma 3.14. *(a)* $J_n \subseteq Part(\mathfrak{A}, \mathfrak{B})$.

(b) If $n > 0$ *and* $\mathfrak{A} \models \varphi^n_{\mathfrak{B}} (= \varphi^n_{\mathfrak{B},\emptyset})$, *then* $\emptyset \in J_n$, *hence* $J_n \neq \emptyset$.

(c) $(J_n)_{n\in\omega}$ *has the back-and-forth properties.*

Proof. (a) Follows inmediately from Lemma 3.13.

(b) Is a direct consequence of the definition of J_n.

(c) Let $p = a^r \mapsto b^r \in J_{n+1}$ and $a \in A$. Then $\mathfrak{A} \models \varphi^{n+1}_{\mathfrak{B},b^r}[a^r]$, in particular, $\mathfrak{A} \models \forall v_r \bigvee\{\varphi^n_{\mathfrak{B},b^r b} \mid b \in B\}[a^r]$. Therefore, there is $b \in B$ such that $\mathfrak{A} \models \varphi^n_{\mathfrak{B},b^r b}[a^r a]$. Then, by Lemma 3.13, $a^r a \mapsto b^r b \in J_n$ which extends p (forth-property). Since also $\mathfrak{A} \models \bigwedge\{\exists v_r \varphi^n_{\mathfrak{B},b^r b} \mid b \in B\}[a^r]$, for each $b \in B$ there is $a \in A$ such that $\mathfrak{A} \models \varphi^n_{\mathfrak{B},b^r b}[a^r a]$ and, again by Lemma 3.13, $a^r a \mapsto b^r b \in J_n$ (back-property).

$\square$

We can finally prove the left-to-right direction of Fraïssé's theorem: If $\mathfrak{A} \equiv \mathfrak{B}$, since for $n > 1$, $\mathfrak{B} \models \varphi^n_{\mathfrak{B}}$, we have $\mathfrak{A} \models \varphi^n_{\mathfrak{B}}$, and by the previous lemma we get $(J_n)_{n\in\omega} : \mathfrak{A} \cong_f \mathfrak{B}$.

In the proof of Lindström's theorem we will not need Fraïssé's theorem in its full extension, so we now state a weaker version of it which is going to be used in the next chapter.

Definition 3.15. *Two structures are said to be m-isomorphic ($\mathfrak{A} \cong_m \mathfrak{B}$) if there is a sequence $I_0, ..., I_m$ of nonempty sets of partial isomporphisms from $\mathfrak{A}$ to $\mathfrak{B}$ with the back-and-forth properties.*

By Lemma 3.9 we know that the partial isomorphisms in I_m preserve the validity of formulas with quantifier rank $\leq m$, so if we write $\mathfrak{A} \equiv_m \mathfrak{B}$ in case $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of quantifier rank $\leq m$, by the Fraïssé's theorem we know that $\mathfrak{A} \cong_m \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_m \mathfrak{B}$.

It is very common to present these theorems in a game theoretical form so that the ins and outs of the relation between elementary equivalence and partial isomorphisms are easier to handle. These games are known as Ehrenfeucht-Fraïssé games or back-and-forth games, and play an important role in theoretical computer science, since they are also valid in finite structures. For a game-theoretical approach to the results in this section we refer to [14].

Chapter 4

Lindström's theorem

In this chapter we prove one of the main results of this work: Lindström's first theorem. Informally, this states that first-order logic is the most expresive among those that still satisfy the Compactness and Löwenheim-Skolem theorems. We first formalize the notion of a logic "being more expresive than" other and in section 4.2 we prove Lindström's theorem using the results from Chapter 3. The whole chapter follows [5] and [2].

4.1. Logical systems

In the following we give an abstract definition of logical system that collects properties shared by all the logics we have considered. As we mentioned before, the idea goes back to Lindström's definition of *abstract logic* or *generalized first-order logic* in [10].

Definition 4.1. *A logical system $\mathfrak{L}$ consists of a function L and a binary relation $\models_{\mathfrak{L}}$. L maps every language S to the set $L(S)$ of S-sentences of $\mathfrak{L}$ and $\models_{\mathfrak{L}}$ is a relation between structures and sentences of $\mathfrak{L}$. The following properties hold:*

(a) If $S_0 \subseteq S_1$, then $L(S_0) \subseteq L(S_1)$.

(b) If $\mathfrak{A} \models_{\mathfrak{L}} \varphi$, then $\mathfrak{A}$ is an S-structure for some language S and $\varphi \in L(S)$.

(c) (Isomorphism property) If $\mathfrak{A} \models_{\mathfrak{L}} \varphi$ and $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{B} \models_{\mathfrak{L}} \varphi$.

(d) (Reduct property) If $S_0 \subseteq S_1$, $\varphi \in L(S_0)$ and $\mathfrak{A}$ is an S_1-structure, then $\mathfrak{A} \models_{\mathfrak{L}} \varphi$ iff $\mathfrak{A}|_{S_0} \models_{\mathfrak{L}} \varphi$.

If $\mathfrak{L}$ is a logical system and $\varphi \in L(S)$, we can define

$$Mod^S_{\mathfrak{L}}(\varphi) := \{\mathfrak{A} \mid \mathfrak{A} \text{ is an S-structure and } \mathfrak{A} \models_{\mathfrak{L}} \varphi\}.$$

This suggests the following definition for the relative expresive power of logical systems:

Definition 4.2. *Let $\mathfrak{L}$ and $\mathfrak{L}^*$ be logical systems.*

(a) Let S be a language, $\varphi \in L(S)$ and $\psi \in L^*(S)$. The sentences φ and ψ are said to be logically equivalent iff $Mod^S_{\mathfrak{L}}(\varphi) = Mod^S_{\mathfrak{L}^*}(\psi)$.

(b) *$\mathfrak{L}^*$ is at least as strong as $\mathfrak{L}$ ($\mathfrak{L} \leq \mathfrak{L}^*$) iff for every S and every $\varphi \in L(S)$ there is a $\psi \in L^*(S)$ such that φ and ψ are logically equivalent.*

(c) *$\mathfrak{L}$ and $\mathfrak{L}^*$ are equally strong ($\mathfrak{L} \sim \mathfrak{L}^*$) iff $\mathfrak{L} \leq \mathfrak{L}^*$ and $\mathfrak{L}^* \leq \mathfrak{L}$.*

We can now define some general semantic notions in an analogous way to $\mathfrak{L}_{\omega\omega}$. For example, $\varphi \in L(S)$ is said to be satisfiable if $Mod^S_{\mathfrak{L}} \neq \emptyset$. Also, if $\Phi \subseteq L(S)$ then $\Phi \models_{\mathfrak{L}} \varphi$ iff every model of Φ is a model of φ.

Definition 4.3. *A logical system if said to be regular if it satisfies the following three properties:*

- *$Boole(\mathfrak{L})$ ($\mathfrak{L}$ contains Boolean connectives):*

  (a) *Given S and $\varphi \in L(S)$, there is $\psi \in L(S)$ such that for every S-structure $\mathfrak{A}$, $\mathfrak{A} \models_{\mathfrak{L}} \psi$ iff not $\mathfrak{A} \models_{\mathfrak{L}} \varphi$.*

  (b) *Given S and $\varphi, \psi \in L(S)$, there is $\chi \in L(S)$ such that for every S-structure $\mathfrak{A}$, $\mathfrak{A} \models_{\mathfrak{L}} \chi$ iff ($\mathfrak{A} \models_{\mathfrak{L}} \varphi$ or $\mathfrak{A} \models_{\mathfrak{L}} \psi$).*

- *$Rel(\mathfrak{L})$ ($\mathfrak{L}$ permits relativization):*

  For S, $\varphi \in L(S)$ and unary U, there is $\psi \in L(S \cup \{U\})$ such that $(\mathfrak{A}, U^A) \models_{\mathfrak{L}} \psi$ iff $[U^A]^{\mathfrak{A}} \models_{\mathfrak{L}} \varphi$ for all S-structures $\mathfrak{A}$ and all S-closed susbsets U^A of A.

- *$Repl(\mathfrak{L})$ ($\mathfrak{L}$ permits replacement of function symbols and constants by relation symbols). As we have already done in previous chapter, the most common way to replace functions and constants by relations is to define their graph as a new relation.*

It is easy to prove that first-order logic is regular. The same arguments in the case of $\mathfrak{L}_{\omega\omega}$ are valid, with minimal changes, to prove that all the logical systems considered here are regular too.

For a regular logical system such that $\mathfrak{L}_{\omega\omega} \leq \mathfrak{L}$, let φ^* be a $L(S)$-sentence equivalent for the first-order S-sentence φ. We can define $\Phi^* = \{\varphi^* \mid \varphi \in \Phi\}$.

We introduce abbreviations for the properties we are going to consider in Lindström's theorem:

- **LöSko**($\mathfrak{L}$) (Löwenheim-Skolem Theorem holds for $\mathfrak{L}$): If $\varphi \in L(S)$ is satisfiable, then there is a model of φ whose domain is at most countable.

- **Comp**($\mathfrak{L}$) (The Compactness Theorem holds for $\mathfrak{L}$): if $\Phi \subset L(S)$ and every finite subset of Φ is satisfiable, then Φ is satisfiable.

The following lemma is an easy but helpful consequence of $Comp(\mathfrak{L})$:

Lemma 4.4. *Suppose $Comp(\mathfrak{L})$ and let $\Phi \cup \{\varphi\} \subseteq L(S)$ and $\Phi \models_{\mathfrak{L}} \varphi$. Then there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models_{\mathfrak{L}} \varphi$.*

Proof. Choose $\neg\varphi$ by $Boole(\mathfrak{L})$. Then $\Phi \cup \{\neg\varphi\}$ is not satisfiable, and by $Comp(\mathfrak{L})$ there is a finite subset $\Phi_0 \subseteq \Phi$ so that $\Phi_0 \cup \{\neg\varphi\}$ is not satisfiable, i.e., $\Phi_0 \models_{\mathfrak{L}} \varphi$. □

4.2. Lindström's theorem

In the following, let $\mathfrak{L}$ be a regular logical system with $\mathfrak{L}_{\omega\omega} \leq \mathfrak{L}$.

Lemma 4.5. *Suppose $Comp(\mathfrak{L})$ and $\psi \in L(S)$. Then there is a finite subset $S_0 \subset S$ such that for all S-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$ then $\mathfrak{A} \models_{\mathfrak{L}} \psi$ iff $\mathfrak{B} \models_{\mathfrak{L}} \psi$.*

Proof. Let Φ be the set of the following $S \cup \{U, V, f\}$-sentences:

- $\exists x Ux$, $\exists x Vx$, $\forall x(Ux \to Vfx)$, $\forall y(Vy \to \exists x(Ux \land fx = y))$,
- $\forall x \forall y((Ux \land Uy \land fx = fy) \to x = y)$ and, for every $R_n \in S$,
- $\forall x_1 ... \forall x_n((Ux_1 \land ... \land Ux_n) \to (Rx_1 \dots x_n \leftrightarrow Rfx_1 \dots fx_n))$.

Φ says that f is an isomorphism between the substructure induced by U and the one induced by V. Then $\Phi^* \models_{\mathfrak{L}} \psi^U \leftrightarrow \psi^V$:

In fact, if $\mathfrak{A}$ is an S-structure and $(\mathfrak{A}, U^A, V^A, f^A) \models_{\mathfrak{L}} \Phi^*$, U^A and V^A are nonempty and $f^A|_{U^A}$ is an isomorphism from $[U^A]^{\mathfrak{A}}$ to $[V^A]^{\mathfrak{A}}$. Then $[U^A]^{\mathfrak{A}} \models_{\mathfrak{L}} \psi$ iff $[V^A]^{\mathfrak{A}} \models_{\mathfrak{L}} \psi$, and by $Rel(\mathfrak{L})$, $(\mathfrak{A}, U^A) \models_{\mathfrak{L}} \psi^U$ iff $(\mathfrak{A}, V^A) \models_{\mathfrak{L}} \psi^V$. Finally, by the reduct property and $Boole(\mathfrak{L})$, $(\mathfrak{A}, U^A, V^A, f^A) \models_{\mathfrak{L}} \psi^U \leftrightarrow \psi^V$.

Now, by $Comp(\mathfrak{L})$ there is a finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0^* \models \psi^U \leftrightarrow \psi^V$ and we may choose a finite $S_0 \subseteq S$ such that Φ_0 consists of S_0-sentences.

Suppose $\pi : \mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$. We can assume, without loss of generality that $A \cap B = \emptyset$ (if $A \cap B \neq \emptyset$ just take an isomorphic copy of B). We define over $C := A \cup B$ an $S \cup \{U, V, f\}$-structure $(\mathfrak{C}, U^C, V^C, f^C)$:

- $R^C := R^A \cup R^B$, for $R \in S$.
- $U^C = A$, $V^C = B$.
- f^C such that $f^C|_{U^C} = \pi$.

Then $(\mathfrak{C}, U^C, V^C, f^C) \models_{\mathfrak{L}} \Phi_0^*$, thus $(\mathfrak{C}, U^C, V^C, f^C) \models_{\mathfrak{L}} \psi^U \leftrightarrow \psi^V$, and by $Rel(\mathfrak{L})$, $\mathfrak{A} \models_{\mathfrak{L}} \psi$ iff $\mathfrak{B} \models_{\mathfrak{L}} \psi$. □

From now on, **S will be a relational language** and ψ an $L(S)$-sentence not logically equivalent to any first-order sentence. The following lemma says that there are structures $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathfrak{L}} \psi$ and $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$, which are identical with respect to $\mathfrak{L}_{\omega\omega}$.

Lemma 4.6. *Let S be a relational language and $\psi \in L(S)$ a sentence which is not equivalent to any first-order sentence. Then for every finite $S_0 \subseteq S$ and every $m \in \omega$, there are S-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that:*

$$\mathfrak{A}|_{S_0} \cong_m \mathfrak{B}|_{S_0}, \mathfrak{A} \models_{\mathfrak{L}} \psi \text{ and } \mathfrak{B} \models_{\mathfrak{L}} \neg\psi.$$

Proof. Let $S_0 \subseteq S$ be finite and $m \geq 1$. We set, following Definition 3.11,

$$\varphi := \bigvee \{\varphi^m_{\mathfrak{A}|_{S_0}} \,|\, \mathfrak{A} \text{ is an S-structure and } \mathfrak{A} \models_{\mathfrak{L}} \psi\}.$$

As we saw in the previous chapter, φ is a first-order sentence. Obviously $\psi \to \varphi^*$ is valid, and since ψ is not logically equivalent to φ (hence not to φ^*), there is an S-structure $\mathfrak{B}$ such that $\mathfrak{B} \models_{\mathfrak{L}} \varphi^*$ and $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$. Now, $\mathfrak{B} \models \varphi$ means that there is an S-structure $\mathfrak{A}$ such that $\mathfrak{A} \models_{\mathfrak{L}} \psi$ and $\mathfrak{B} \models \varphi^m_{\mathfrak{A}|_{S_0}}$. Therefore, we finally have $\mathfrak{A}|_{S_0} \cong_m \mathfrak{B}|_{S_0}$. □

In the proof os Lindström's Theorem we are going to use the fact that this result can be formulated in $\mathfrak{L}$. With that goal in mind, we turn to such a formulation.

For $m \in \omega$ we choose $\psi, \mathfrak{A}, \mathfrak{B}, S_0$ and $(I_n)_{n \leq m}$ such that $(I_n)_{n\leq m} : \mathfrak{A}|_{S_0} \cong_m \mathfrak{B}|_{S_0}$, $\mathfrak{A} \models_{\mathfrak{L}} \psi$ and $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$. Let $S^+ = S \cup \{c, f, P, U, V, W, <, I, G\}$ where c is a constant, f is a unary function, P, U, V, W are unary relations, $<, I$ are binary relations and G is a ternary relation. We are going to define an S^+-structure $\mathfrak{C}$ that contains $\mathfrak{A}$ and $\mathfrak{B}$ and will allow to describe the m-isomorphism property by including the partial isomorphisms from I_n as elements of its domain:

- $C = A \cup B \cup \{0, ..., m\} \cup \bigcup_{n \leq m} I_n$.
- $U^C = A$ and $[U^C]^{\mathfrak{C}|_S} = \mathfrak{A}$, $V^C = B$ and $[V^C]^{\mathfrak{C}|_S} = \mathfrak{B}$.
- $W^C = \{0, ..., m\}$ and $<^C$ is the natural order in W^C. Furthermore, $c^C = m$ and $f|_{W^C}$ is the predecessor function in W^C, i.e, $f^C(n+1) = n$, for all $n < m$ and $f^C(0) = 0$.
- $P^C = \bigcup_{n \leq m} I_n$.
- $I^C np$ iff $n \leq m$ and $p \in I_n$.
- $G^C pab$ iff $P^C p$, $a \in dom(p)$ and $p(a) = b$.

$\mathfrak{C}$ is then a model of the conjunction, χ, of the following finite set of sentences of $L(S^+)$, which yields the desired formulation of the m-isomorphism property:

(I) $\forall p(Pp \to \forall x \forall y(Gpxy \to (Ux \wedge Vy)))$.

(II) $\forall p(Pp \to \forall x \forall x' \forall y \forall y'((Gpxy \wedge Gpx'y') \to (x = x' \leftrightarrow y = y')))$.

(III) For every n-ary $R \in S_0$, $\forall p(Pp \to \forall x_1 ... \forall x_n \forall y_1 ... \forall y_n((Gpx_1y_1 \wedge ... \wedge Gpx_ny_n) \to (Rx_1...x_n \leftrightarrow Ry_1...y_n)))$.

(IV) The axioms of partial orderings.

(V) $\forall x(Wx \leftrightarrow (x = c \vee \exists y(y < x \vee x < y))) \wedge \forall x(Wx \to (x < c \vee x = c))$.

(VI) $\forall x(\exists y(y < x) \to (fx < x \wedge \neg\exists z(fx < z \wedge z < x)))$.

(VII) $\forall x(Wx \to \exists p(Pp \wedge Ixp))$.

(VIII) $\forall x \forall p \forall u((fx < x \wedge Ixp \wedge Uu) \to \exists q \exists v(Ifxq \wedge Gquv \wedge \forall x' \forall y'(Gpx'y' \to Gqx'y')))$.

(IX) $\forall x \forall p \forall v((fx < x \wedge Ixp \wedge Vu) \to \exists q \exists u(Ifxq \wedge Gquv \wedge \forall x' \forall y'(Gpx'y' \to Gqx'y')))$.

(X) $\exists x Ux \wedge \exists y Vy \wedge \psi^U \wedge (\neg\psi)^V$.

Remark 4.7.

- The reader can easily check the meaning of the above sentences in the structure $\mathfrak{C}$. As an example, (i) and (ii) say that P is a set of partial isomorphisms and (viii) and (ix) say that P has the back-and-forth properties.
- χ does not depend on m.
- We use first-order sentences as an intuitive notation for corresponding $L(S)$-sentences.
- For every $m \in \omega$ there is a model $\mathfrak{C}$ of χ in which the domain W^C of $<^C$ consists exactly of $m + 1$ elements.

The next important lemma paves the way towards Lindström's theorem and shows the role of LöSko($\mathfrak{L}$) in it.

Lemma 4.8. *Assume LöSko($\mathfrak{L}$). Then (a) or (b) holds:*

(a) There are S-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models_{\mathfrak{L}} \psi$, $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$.

(b) In all models $\mathfrak{D}$ of χ, W^D is finite.

Proof. We divide the proof in two parts:

First we show that if the S^+-structure $\mathfrak{D}$ is a model of χ in which the domain W^D of $<^D$ is infinite, then U^D and V^D are universes of S-structures $\mathfrak{A} := [U^D]^{\mathfrak{D}|_S}$ and $\mathfrak{B} := [V^D]^{\mathfrak{D}|_S}$ such that $\mathfrak{A} \models_{\mathfrak{L}} \psi, \mathfrak{B} \models_{\mathfrak{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong_p \mathfrak{B}|_{S_0}$.

Indeed: Since $\mathfrak{D}$ satisfies the sentence (x), $U^D, V^D \neq \emptyset$ and since S is relational, U^D and V^D are domains of S-structures. Again by (x), $\mathfrak{D} \models_{\mathfrak{L}} \psi^U$, $\mathfrak{D} \models_{\mathfrak{L}} (\neg\psi)^V$ and therefore $\mathfrak{A} \models_{\mathfrak{L}} \psi$, $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$. From (i), (ii) and (iii) we know that every $p \in P^D$ corresponds via G^D to a partial isomorphism from $\mathfrak{A}|_{S_0}$ to $\mathfrak{B}|_{S_0}$. We build the set I in the following way: Since W^D is infinite and c^D is the last element of $<^D$, $<^D$ has an infinite descending chain: ... $<^D (f^2c)^D <^D (fc)^D <^D c^D$. We set $I := \{p \mid \text{there is an } n \text{ with } I^D(f^n c)^D p\}$. By (vii) we know that $I \neq \emptyset$, and by (viii) and (ix) that I has the back-and-forth properties, hence the first claim is proved.

Now, suppose that (b) does not hold. So there is a model of χ in which W is infinite. We may assume that the domain of the model is countable:

Let $\mathfrak{D}$ a model of χ with W^D infinite. Let Q be a new unary symbol and let θ be the $L(S^+ \cup \{Q\})$-sentence: $\theta = Qc \wedge \forall x(Qx \rightarrow (fx < x \wedge Qfx))$, with $Q^D := \{(f^n c)^D \mid n \in \omega\}$. Then $(\mathfrak{D}, Q^D) \models_{\mathfrak{L}} \chi \wedge \theta$, and by LöSko($\mathfrak{L}$) there exists a countable model $(\mathfrak{E}, Q^E)$ of $\chi \wedge \theta$ (hence a model of χ) with infinite W^E.

So, using LöSko($\mathfrak{L}$) we may assume the domain of $\mathfrak{D}$ to be countable. Then, by the first part, $\mathfrak{A}|_{S_0} \cong_p \mathfrak{B}|_{S_0}$. Since both models are countable, they are isomorphic by Lemma 3.4(d) and (a) is satisfied.

□

We can now summarize all the information we have in the following result:

Lemma 4.9 (Main Lemma). *Let $\mathfrak{L}$ be a regular logical system with $\mathfrak{L}_{\omega\omega} \leq \mathfrak{L}$ and LöSko($\mathfrak{L}$). Let S be a relational language and $\psi \in L(S)$ not logically equivalent to any first-order sentence. Then (a) or (b) holds:*

(a) *For all finite subsets $S_0 \subseteq S$ there are S-structures $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \models_{\mathfrak{L}} \psi$, $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$.*

(b) *For a unary relation symbol W and a suitable $S \cup \{W\} \subseteq S^+$ with finite $S^+ \setminus S$, there is an $L(S^+)$-sentence χ such that:*

  (I) *In every model $\mathfrak{C}$ of χ, W^C is finite and nonempty.*

  (II) *For every $m \geq 1$ there is a model $\mathfrak{C}$ of χ in which W^C has exactly m elements.*

Finally, we show:

Theorem 4.10 (Lindström's First Theorem)**.** *For a regular logical system $\mathfrak{L}$ with $\mathfrak{L}_{\omega\omega} \leq \mathfrak{L}$, the following is true:*

$$\textit{If LöSko}(\mathfrak{L}) \textit{ and Comp}(\mathfrak{L}), \textit{ then } \mathfrak{L}_{\omega\omega} \sim \mathfrak{L}.$$

Proof. Assume that there is $\psi \in L(S)$ not logically equivalent to any first-order sentence. We can assume S is relational. Since $Comp(\mathfrak{L})$ holds, the meaning of ψ depends only on finitely many symbols, so we can choose a finite $S_0 \subseteq S$ such that for all S-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$.

Then Lemma 4.9(a) is not satisfied and 4.9(b) must hold: there is a $L(S^+)$-sentence χ which satisfies 4.9(b)(i) and 4.9(b)(ii). But this contradicts $Comp(\mathfrak{L})$: By 4.9(b)(i), $\{\chi\} \cup \{$"W contains at least n elements"$\,|\, n \in \omega\}$ is not satisfiable but by 4.9(b)(ii), every finite subset has a model. The contradiction arises by the assumption that such ψ exists, so the theorem is proved. □

The technicalities of the proof we have just developed can easily distract us from understanding the role of the Compactness and Löwenheim-Skolem theorems in it. To clarify it, we describe the main idea of the proof:

Starting with the assumption that ψ is a $L(S)$-sentence which is not logically equivalent to any first-order sentence, for any $m \geq 1$ we obtain structures $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathfrak{L}} \psi$, $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$ and $\mathfrak{A} \cong_m \mathfrak{B}$. By Comp$(\mathfrak{L})$ we get structures $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathfrak{L}} \psi$, $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$ and $\mathfrak{A} \cong_p \mathfrak{B}$. Now LöSko$(\mathfrak{L})$ allows to find countable structures which satisfy the previous relations, therefore, by Lemma 3.4(d), $\mathfrak{A} \models_{\mathfrak{L}} \psi$, $\mathfrak{B} \models_{\mathfrak{L}} \neg\psi$ and $\mathfrak{A} \cong \mathfrak{B}$, a contradiction.

One could point out that the name of Lindström's *first* theorem suggests that there is at least a *second* one. In fact there are a handful of characterizations of first-order logic which are equivalent to Lindström's first theorem grouped under the generic name of *Lindström's theorems*. For a discussion of different characterizations of first-order logic see chapter III in [2].

Chapter 5

Elementary extensions and the omitting types theorem

In this chapter we introduce two basic techniques of model theory with a wide range of applications. Our goal is to use them in the next chapter, and with that instrumental purpose in mind, we just develop the necessary tools to prove the two main theorems. For a extense exposition of the topics treated here see [4].

5.1. Elementary extensions

Definition 5.1 (Elementary extension)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures of the language S. We say that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ (or that $\mathfrak{B}$ is a elementary extension of $\mathfrak{A}$) if $\mathfrak{A} \subseteq \mathfrak{B}$ and for every formula $\phi(x_1, ..., x_n)$ in $L^S_{\omega\omega}$ and every n-tuple $a_1, ..., a_n \in A$, $\mathfrak{A} \models \phi[a_1, ..., a_n]$ if and only if $\mathfrak{B} \models \phi[a_1, ..., a_n]$. In this case we write $\mathfrak{A} \preceq \mathfrak{B}$.*

Remark 5.2. The following points give some basic properties of elementary extensions:

(1) $\mathfrak{A} \preceq \mathfrak{A}$.

(2) If $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathfrak{B} \preceq \mathfrak{C}$ then $\mathfrak{A} \preceq \mathfrak{C}$.

(3) If $\mathfrak{A} \preceq \mathfrak{C}$, $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{A} \preceq \mathfrak{B}$.

Remark 5.3. Obviously, if $\mathfrak{A} \preceq \mathfrak{B}$, we have $\mathfrak{A} \equiv \mathfrak{B}$, but not viceversa:

- Let $\mathfrak{B} = \langle \omega, < \rangle$ and $\mathfrak{A} = \langle \omega \setminus \{0\}, < \rangle$. Since they are isomorphic, $\mathfrak{A} \equiv \mathfrak{B}$, but when $\phi(y) = \forall x (y < x \vee y = x)$, we have $\mathfrak{A} \models \phi[1]$ and $\mathfrak{B} \not\models \phi[1]$, which means $\mathfrak{A} \not\preceq \mathfrak{B}$.

The following theorem gives a necessary and sufficient condition for a substructure to be elementary.

Theorem 5.4 (Vaught-Tarski test)**.** *Let $\mathfrak{B}$ be a structure in the language S and $\mathfrak{A} \subseteq \mathfrak{B}$. The following are equivalent:*

(1) $\mathfrak{A} \preceq \mathfrak{B}$.

(2) For every $a_2, ..., a_n \in A$ *and every formula* $\phi(x_1, ..., x_n) \in L^S_{\omega\omega}$*, if* $\mathfrak{B} \models \exists x_1 \phi[a_2, ..., a_n]$*, then there is* $a_1 \in A$ *such that* $\mathfrak{B} \models \phi[a_1, ..., a_n]$.

Proof. The first implication is trivial and we can turn to prove the reciprocal by induction on ϕ.

The induction is easy to carry out for atomic formulas and for sentential connectives. Now assume $\phi(x_2, ..., x_n) = \exists x_1 \varphi(x_1, ..., x_n)$. Given $a_2, ..., a_n \in A$, if $\mathfrak{A} \models \exists x_1 \varphi[a_2, ..., a_n]$, then there is $a_1 \in A$ such that $\mathfrak{A} \models \varphi[a_1, ..., a_n]$. By hypothesis, $\mathfrak{B} \models \varphi[a_1, ..., a_n]$, whence $\mathfrak{B} \models \exists x_1 \varphi[a_2, ..., a_n]$. On the other hand, if $\mathfrak{B} \models \exists x_1 \varphi[a_2, ..., a_n]$, by assumption, there is $a_1 \in A$ such that $\mathfrak{B} \models \varphi[a_1, .., a_n]$. Finally, by induction, $\mathfrak{A} \models \varphi[a_1, ..., a_n]$ and $\mathfrak{A} \models \exists x_1 \varphi[a_2, ..., a_n]$.

□

We now introduce the definition of elementary chain of models. The idea behind is that we can build a new model considering the union of the models in the chain.

Definition 5.5. *Let* α *be an ordinal. A chain of models of lenght* α *is a sequence* $(\mathfrak{A}_i, i < \alpha)$ *of models where* $\mathfrak{A}_i \subseteq \mathfrak{A}_{i+1}$ *for every* $i < \alpha$*. The chain is elementary if* $\mathfrak{A}_i \preceq \mathfrak{A}_j$ *for every* $i \leq j < \alpha$.

Remark 5.6. We can build the union of the chain, $\mathfrak{A} = \bigcup_{i<\alpha} \mathfrak{A}_i$, in the following way:

(a) $A = \bigcup_{i<\alpha} A_i$.

(b) If $c \in S$, then $c^{\mathfrak{A}} = c^{\mathfrak{A}_i}$, where $i < \alpha$ is arbitrary.

(c) If $R \in S$ is a relation symbol, $R^{\mathfrak{A}} = \bigcup_{i<\alpha} R^{\mathfrak{A}_i}$.

(d) If $f \in S$ is an n-ary function symbol and $a_1, ..., a_n \in A$, $f^{\mathfrak{A}}(a_1, ..., a_n) = f^{\mathfrak{A}_i}(a_1, ..., a_n)$, where $i < \alpha$ and $a_1, ..., a_n \in A_i$.

The following lemma gives the inspiration for the next theorem. The proof is trivial by the construction of $\mathfrak{A}$.

Lemma 5.7. *Let* $(\mathfrak{A}_i, i < \alpha)$ *be a chain.* $\mathfrak{A} = \bigcup_{i<\alpha} \mathfrak{A}_i$ *is a model such that* $\mathfrak{A}_j \subseteq \mathfrak{A}$*, for every* $j < \alpha$.

We are now ready to prove the main result in this section, a classical result in model theory attributed to Tarski. It will prove to be very useful in the next chapter.

Theorem 5.8. *If* $(\mathfrak{A}_i, i < \alpha)$ *is an elementary chain,* $\mathfrak{A}_\beta \preceq \bigcup_{i<\alpha} \mathfrak{A}_i$*, for every* $\beta < \alpha$.

Proof. Let $\mathfrak{A} = \bigcup_{i<\alpha} \mathfrak{A}_i$. We prove the following for all formulas $\varphi(x_1, ..., x_n) \in L$, and for every $\beta < \alpha$, $a_1, ..., a_n \in A_\beta$:

$$\mathfrak{A}_\beta \models \varphi[a_1, ..., a_n] \text{ if and only if } \mathfrak{A} \models \varphi[a_1, ..., a_n].$$

The case of atomic formulas and sentential connectives is an easy routine. Assume $\varphi = \exists x_1 \psi$ and $a_2, ..., a_n \in A_\beta$. If $\mathfrak{A}_\beta \models \psi[a_2, ..., a_n]$, then there is $a_1 \in A_\beta$ such that $A_\beta \models \psi[a_1, ..., a_n]$. By induction hypothesis, $\mathfrak{A} \models \psi[a_1, ..., a_n]$, hence $\mathfrak{A} \models \exists x_1 \psi[a_2, ..., a_n]$.

On the other hand, $\mathfrak{A} \models \exists x_1 \psi[a_2, ..., a_n]$ means that there is $a_1 \in A$ such that $\mathfrak{A} \models \psi[a_1, ..., a_n]$. Then for some $\gamma < \alpha$, $a_1 \in A_\gamma$ and $a_1, ..., a_n \in A_\gamma$. By induction, $\mathfrak{A}_\gamma \models \psi[a_1, ..., a_n]$, hence $\mathfrak{A}_\gamma \models \exists x_1 \psi[a_2, ..., a_n]$. If $\gamma \leq \beta$ we are done, so we may assume $\beta \leq \gamma$. Since $\mathfrak{A}_\beta \preceq \mathfrak{A}_\gamma$, we have, by definition, $\mathfrak{A}_\beta \models \exists x_1 \psi[a_2, ..., a_n]$. □

5.2. Omitting types

Recall that a theory $T \subseteq L^S_{\omega\omega}$ is a satisfiable set of sentences that is closed under consequence. A theory is complete if for every S-sentence φ we have $\varphi \in T$ or $\neg\varphi \in T$.

Definition 5.9. *Let $\mathfrak{A}$ be a model, $a_1, ..., a_n \in A$ and $\Sigma = \{\sigma_i(x_1, ..., x_n) \in L^S_{\omega\omega} \mid i \in I\}$. We say that $a_1, ..., a_n$ realizes Σ in $\mathfrak{A}$ if and only if for every $\sigma_i \in \Sigma$, $\mathfrak{A} \models \sigma_i[a_1, ..., a_n]$.*

Definition 5.10. *We say that $\mathfrak{A}$ omits Σ if and only if $\mathfrak{A}$ does not realize Σ, that is, if and only if there are not $a_1, ..., a_n \in A$ such that $\mathfrak{A} \models \sigma_i[a_1, ..., a_n]$ for every $\sigma_i \in \Sigma$.*

Remark 5.11. If Σ is finite, it is easy to determine whether $\mathfrak{A}$ omits Σ: $\mathfrak{A}$ realizes $\Sigma = \{\sigma_1, ..., \sigma_m\}$ if and only if $\mathfrak{A} \models \phi := \exists x_1 \ldots x_n(\sigma_1 \wedge \ldots \wedge \sigma_m)$.

Definition 5.12. *We say that a formula $\sigma(x_1, ..., x_n) \in L^S_{\omega\omega}$ is consistent with a theory T if and only if there is a model of T which realizes σ. More generaly, $\Sigma = \{\sigma_i(x_1, ..., x_n) \in L^S_{\omega\omega} \mid i \in I\}$ is consistent with T if and only if T has a model which realizes Σ.*

Definition 5.13. *Let $\Sigma(x_1, ..., x_n)$ be a set of formulas from the language S. We say that Σ is isolated in T if and only if there is $\phi(x_1, ..., x_n) \in L^S_{\omega\omega}$ such that:*

1. *ϕ is consistent with T.*
2. *$T \models \phi \to \sigma$ for every $\sigma \in \Sigma$.*

If such ϕ does not exist, we say that Σ is not isolated in T.

Remark 5.14. It is easy to check that if Σ is not isolated, for every $\phi \in L^S_{\omega\omega}$ consistent with T there is $\sigma \in \Sigma$ such that $\phi \wedge \neg\sigma$ is consistent with T.

The following result is the converse of the omitting types theorem, with the extra condition that T has to be a complete theory.

Proposition 5.15. *Let T a complete theory of L and $\Sigma(x_1, ..., x_n)$ a set of formulas. iF T has a model which omits Σ, then Σ is not isolated in T.*

Proof. We will set that if Σ is isolated in T, then every model of T realizes Σ. Let $\phi(x_1, ..., x_n) \in L^S_{\omega\omega}$ a formula consistent with T such that $T \models \phi \to \sigma$, for every $\sigma \in \Sigma$. Let $\mathfrak{A}$ a model of T. Since T is complete, $T \models \exists x_1 \ldots x_n \phi$, i. e., there are $a_1, ..., a_n \in A$ such that $\mathfrak{A} \models \phi[a_1, ..., a_n]$, and by propositional logic, $\mathfrak{A} \models \sigma[a_1, ..., a_n]$ for every $\sigma \in \Sigma$. This clearly means that $a_1, ..., a_n$ realize Σ in $\mathfrak{A}$. □

We now formulate the omitting types theorem in the case of one free variable. Both the result and the proof are the same for the n variable case, but the notation becomes needlessly complicated.

Theorem 5.16 (Omitting types theorem)**.** *Let T be a consistent and countable theory of $L^S_{\omega\omega}$ and $\Sigma = \Sigma(x)$ a set of formulas. If Σ is not isolated in T, then T has a countable model which omits Σ.*

Proof. Assume that Σ is not isolated in T. Let $C = \{c_i \mid i < \omega\}$ be a countable set of new constants and consider $S' = S \cup C$. S' is still countable and we can arrange the sentences of $L^{S'}_{\omega\omega}$: ϕ_0, ϕ_1,... Now are going to build an increasing sequence of theories $T = T_0 \subseteq T_1 \subseteq T_2 \subseteq$ ... such that:

(1) Every T_m is consistent and a finite extension of T made up by sentences of $L^{S\cup\{c_i \mid i<m\}}_{\omega\omega}$.

(2) $\phi_m \in T_{m+1}$ or $\neg\phi_m \in T_{m+1}$ for every $m \in \omega$.

(3) If $\phi_m = \exists x\psi(x)$ and $\phi_m \in T_{m+1}$, then $\psi(c_m) \wedge \neg\sigma(c_m) \in T_{m+1}$, for some $\sigma \in \Sigma$.

First we construct the sequence satisfying (1)-(3). Assume that we already have T_n. If ϕ_n is not consistent with T_n, $\neg\phi_n$ will. If T_n is consistent with ϕ_n (or $\neg\phi_n$) and ϕ_n does not start with an existential quantifier, $T_{n+1} = T_n \cup \{\phi_n\}$ (or $T_{n+1} = T_n \cup \{\neg\phi_n\}$). In the case $\phi_n = \exists x\psi(x)$ is consistent with T_n, let φ be the conjunction of the sentences from $T_n \setminus T$. Let $\chi(c_0, ...c_{n-1}, x) = \varphi \wedge \psi(x)$. Then $\exists x_0 \dots x_{n-1}\chi(x_0, \dots, x_{n-1}, x)$ is consistent with T.

Since Σ is not isolated in T, there is $\sigma \in \Sigma$ such that $\exists x_0 \dots x_{n-1}\chi(x_0, ...x_{n-1}, x) \wedge \neg\sigma(x)$ is consistent with T. And in conclusion, we can ensure that $\chi(c_0, ...c_{n-1}, c_n) \wedge \neg\sigma(c_n)$ is also consistent with T, guaranteeing that $T_{n+1} = T_n \cup \{\psi(c_n) \wedge \neg\sigma(c_n)\}$ is consistent.

Once the sequence is constructed, we can consider $T_\omega = \bigcup_{n<\omega} T_n$. By (1) and (2) we know that T_ω is a maximal consistent theory in $L^{S'}_{\omega\omega}$. Let $\mathfrak{B}' = (\mathfrak{B}, C^{\mathfrak{B}})$ be a countable model of T_ω and consider $\mathfrak{A}' = (\mathfrak{A}, C^{\mathfrak{B}})$, the submodel of $\mathfrak{B}'$ generated by $c_0^{\mathfrak{B}'}, c_1^{\mathfrak{B}'}$,.... By (3) we know that T_ω has witnesses in C, so by Lemma 1.18 $A = \{c_i^{\mathfrak{B}'}\}_{i<\omega}$. Clearly $\mathfrak{A}'$ is a model of T_ω, hence $\mathfrak{A}$ is a countable model of T.

Now we are ready to prove that $\mathfrak{A}$ omits Σ. Assume that $a \in A$ is a realization of Σ. In that case, $a = c_n^{\mathfrak{B}'}$ for some $n < \omega$. Consider the sentence $\phi_m = \exists x x = c_n$. Since the sentence is consistent with T_m, there is $\sigma \in \Sigma$ such that $(c_m = c_n) \wedge \neg\sigma(c_m) \in T_{m+1}$ by (3), whence $T_\omega \models \neg\sigma(c_n)$. This means that $\mathfrak{A} \models \neg\sigma[a]$ in contradiction with the assumption that a realizes Σ. □

For a topological proof of Theorem 5.16, see [3].

Chapter 6

Logic with the quantifier "there exist uncountably many"

As we have mentioned, the idea of adding quantifiers to first-order logic goes back to Mostowski [11]. For obvious mathematical reasons, cardinality quantifiers were a natural way of doing it, and Fuhrken [6] and Vaught [16] were the first to study compactness and completeness for the logic $\mathfrak{L}(Q_1)$, obtained by adjoining the quantifier Q_1 (there exists uncountably many) to first-order logic. The first systematical study of $\mathfrak{L}(Q_1)$ was due to Keisler [8], whose paper is the basis for our exposition. The main result in this chapter is the Completeness theorem for $\mathfrak{L}(Q_1)$, that has countable Compactness as a corollary. As a proper extension of first-order logic, $\mathfrak{L}(Q_1)$ will not satisfy both the countable Compactness and the Löwenheim-Skolem theorems, but we will show that it is very close to satisfying them, and in that sense $\mathfrak{L}(Q_1)$ will be the best behaving extension studied here.

6.1. The language L(Q)

From now on we will consider the first-order logic $\mathfrak{L}_{\omega\omega}$ with identity. We form the logical system $\mathfrak{L}(Q_1)$ by adding to $\mathfrak{L}_{\omega\omega}$ a new quantifier Q_1x which will be interpreted as "there are uncountably many x". Then, the set of formulas of $L^S(Q_1)$ is the least set Φ which contains all the atomic formulas of $\mathfrak{L}_{\omega\omega}$ and satisfies the following: If $\varphi, \psi \in \Phi$ and y is a variable, then $\varphi \wedge \psi$, $\neg\varphi$, $\exists y\varphi$ and $Qy\varphi \in \Phi$. From now on we follow **Keisler's notation, writing Q instead of Q_1, L for the language and $L(Q)$ for the formulas in the language L.**

Our way of dealing with $L(Q)$ will be to reduce it to first-order logic in some manner we may apply first-order model theory to obtain the desired results. This approach will rely on weak models as they were introduced by Keisler:

Definition 6.1. *A* weak model *for $L(Q)$ is a pair $(\mathfrak{A}, q)$ such that $\mathfrak{A}$ is a first-order structure and q is a set of subsets of A. Given a n-tuple $a_1, ..., a_n \in A$ and the formula $\varphi(v_1, ..., v_n) \in L(Q)$, the satisfaction relation is defined in the usual way by induction on the complexity of φ, and the*

Qx clause in the definition is:

$$(\mathfrak{A}, q) \models Qv_m\varphi[a_1, ..., a_n] \textit{ iff } \{b \in A \mid (\mathfrak{A}, q) \models \varphi[a_1, ..., a_{m-1}, b, a_{m+1}, ..., a_n]\} \in q$$

for $m \leq n$.

In a weak model the validity of Qx depends on some subsets of the universe, and when those subsets are exactly those we expect, we have an standard model:

Definition 6.2. $\mathfrak{A}$ *is an* standard model *of* $L(Q)$ *if it is a weak model* $(\mathfrak{A}, q)$ *where* q *is the set of all uncountable subsets of* A.

Thus an standard model interprets Qx appropiately as "there are uncountably many x".

While the downward Löwenheim-Skolem theorem fails in $L(Q)$ (consider the sentence $Qx(x = x)$), a modified version of it holds:

Theorem 6.3 (Downward Löwenheim-Skolem theorem for $\leq \omega_1$)**.** *If* $\mathfrak{A}$ *is an standard model of* $L(Q)$*, then there exists* $\mathfrak{B} \prec_Q \mathfrak{A}$ *(where* $\prec_Q$ *means that* $\mathfrak{B}$ *is an elementary substructure of* $\mathfrak{A}$ *in* $L(Q)$*) such that* $|B| \leq \omega_1$.

Proof. Assuming $|A| > \omega_1$ (otherwise the argument is done), the usual proof of the downward Löwenheim-Skolem theorem can be easily modified to provide ω_1 witnesses to each $Qx\varphi(x)$ instead of only one, see for example Lemma 2.6. □

Compactness theorem for $\mathfrak{L}(Q)$ is proved here following the classical method: first prove completeness and then derive compactness from it. This means that we have to give a deductive calculus for $L(Q)$, that is, a set of axioms and rules of inference, and prove the Completeness theorem with it. We now introduce axioms for $L(Q)$. This will be schemes of formulas which are obviously true in all standard models:

Axiom 0 All the axiom schemes for $\mathfrak{L}_{\omega\omega}$.

Axiom 1 $\neg Qx(x = y \vee x = z)$;

Axiom 2 $\forall x(\varphi \to \psi) \to (Qx\varphi \to Qx\psi)$;

Axiom 3 $Qx\varphi(x) \leftrightarrow Qy\varphi(y)$;

Axiom 4 $Qy\exists x\varphi \to (\exists xQy\varphi \vee Qx\exists y\varphi)$;

The intuitive content of Axiom 1 is: "every set of power ≤ 2 is countable". The intuitive content of Axiom 2 is: "every set with an uncountable subset is uncountable". Axiom 4 says "If $\bigcup_{x \in X} a_x$ is uncountable then either some a_x is uncountable or X is uncountable". Vaught and Fuhrken proposed this set of axioms and conjectured that $L(Q)$ was complete with them, but did not give a proof.

We hope the reader is now convinced that the following lemma is straightforward:

Lemma 6.4. *Every model* $\mathfrak{A}$ *of* $\mathfrak{L}_{\omega\omega}$ *is a standard model of the axioms of* $L(Q)$*.*

It is easy to find weak models which do not satisfy the axioms of $L(Q)$. However, there are weak models (other than the standard models) which do satisfy the axioms. For example, if q is the set of all infinite subsets of A, $(\mathfrak{A}, q)$ is a weak model of all the axioms of $L(Q)$.

The rules of inference if $L(Q)$ are the same as for $\mathfrak{L}_{\omega\omega}$:

Modus ponens: From φ and $\varphi \to \psi$ we infer ψ.

Generalization: From φ we infer $\forall x \varphi$.

The following four lemmas give $L(Q)$ versions of basic proof-theoretical properties of first-order logic. Their first-order proofs can easily be modified for this case, so we enunciate them without proof.

Lemma 6.5. *Let $(\mathfrak{A}, q)$ be a weak model of all the axioms of $L(Q)$. If $\vdash \varphi$, then $(\mathfrak{A}, q) \models \varphi$. If $\Sigma \vdash \varphi$ and $(\mathfrak{A}, q)$ is a weak model of Σ, then $(\mathfrak{A}, q) \models \varphi$.*

Lemma 6.6. *If Σ is a set of sentences of $L(Q)$ and Σ has a standard model, then Σ is consistent in $L(Q)$. If $\mathfrak{A}$ is a standard model of Σ and $\Sigma \vdash \varphi$, then φ holds in $\mathfrak{A}$.*

Lemma 6.7 (Deduction theorem)**.** *Let Σ be a set of sentences of $L(Q)$ and φ a sentence of $L(Q)$. If $\Sigma \cup \{\varphi\} \vdash \psi$ then $\Sigma \vdash \varphi \to \psi$.*

Lemma 6.8. *Let Σ be a maximal consistent set of sentences of $L(Q)$. Then for all sentences φ, ψ of $L(Q)$ we have*

$$\neg\varphi \in \Sigma \textit{ iff } \varphi \notin \Sigma$$

and

$$\varphi \wedge \psi \in \Sigma \textit{ iff } \varphi \in \Sigma \textit{ and } \psi \in \Sigma.$$

The following lemma shows that certain formulas are provable in $L(Q)$. They will be used during the proof of the Completeness theorem.

Lemma 6.9. *Let φ, ψ be formulas, and x, y distinct variables, of $L(Q)$.*

(i) $\vdash Qx\varphi \to \exists x\varphi$.

(ii) $\vdash \exists x Qy\varphi \to Qy\exists\varphi$.

(iii) *If x does not occur free in φ, then* $\vdash Qx(\varphi \wedge \psi) \to \varphi \wedge Qx\psi$.

(iv) $\vdash Qx(\varphi \wedge \psi) \leftrightarrow Qx\varphi \vee Qx\psi$.

(v) $\vdash Qx\varphi \wedge \neg Qx\psi \to Qx(\varphi \wedge \neg\psi)$.

Proof. (I) Let y, z no occuring in φ. By propositional logic,

$$\vdash \neg\exists x\varphi \to \forall x(\varphi \to (x = y \vee x = z)).$$

By Axiom 2 and propositional logic,

$$\vdash \neg\exists x\varphi \to (Qx\varphi \to Qx(x = y \vee x = z)).$$

Now using Axiom 1, $\vdash \neg\exists x\varphi \to \neg Qx\varphi$ and (i) follows form propositional logic.

The rest of the formulas are derived in a similar way, but the proofs would be too long for this section. They can be found in pages 9-12 of [8]. □

6.2. Completeness theorem for L(Q)

From now in L will be a countable language. In this section we shall prove the following:

Theorem 6.10 (Completenes theorem for L(Q))**.** *Let Σ be a set of sentences of $L(Q)$, then Σ has a standard model if and only if Σ is consistent in $L(Q)$.*

One direction is elementary and is stated in Lemma 6.6 above. We are going to prove the hard direction: If Σ is consistent in $L(Q)$ then Σ has a standard model. In a first stage, we will use Henkin's method of constructing models to show that Σ has a weak model in which certain types can be omitted. Then we will construct a standard model from an elementary chain of weak models.

Definition 6.11. *Let Γ be a set of sentences of $L(Q)$ and C a set of constant symbols of S. C is said to be a set of witnesses for Γ iff for every sentence of the form $\exists x\varphi(x)$ there is $c \in C$ such that $\Gamma \vdash \exists x\varphi(x) \to \varphi(c)$.*

Lemma 6.12. *Let Γ be a maximal consistent set of sentences of $L(Q)$ and let C be a set of witnesses for Γ. Then Γ has a weak model $(\mathfrak{A}, q)$ such that every element of A is the interpretation of some $c \in C$.*

Proof. Let Γ_0 be the set of all first-order sentences in Γ. Then by Lemma 6.8, Γ_0 is a maximal consistent set in $\mathfrak{L}_{\omega\omega}$. Moreover, C is a set of witnesses for Γ_0, so it follows from Henkin's proof of the completeness theorem that Γ_0 has a model $\mathfrak{A}$ such that every element of A is the interpretation of some $c \in C$. If $\bar{c}$ is the interpretation of $c \in C$, $A = \{\bar{c} \mid c \in C\}$. Our goal now is to make $\mathfrak{A}$ into a weak model $(\mathfrak{A}, q)$ of Γ. For each formula $\varphi(x)$ of $L(Q)$ with only one free variable x, let

$$S_\varphi = \{\bar{c} \mid c \in C \text{ and } \Gamma \vdash \varphi(c)\}.$$

We then define

$$q = \{S_\varphi \mid \varphi \text{ has only one free variable, say x, and } \Gamma \vdash Qx\varphi\}.$$

Obviously q is a set of subsets of A, and we shall show by induction on φ that for all sentences φ of $L(Q)$,

(1) $(\mathfrak{A}, q) \models \varphi$ iff $\Gamma \vdash \varphi$.

Henkin's proof of the Completeness theorem gives the result for first-order sentences, so we just have to consider the case $\varphi = Qx\psi(x)$. By induction hypothesis we can assume that (1) holds for all sentences $\psi(c)$, $c \in C$. Then

(2) $S_\psi = \{\bar{c} \,|\, \Gamma \vdash \psi(c)\} = \{\bar{c} \,|\, (\mathfrak{A}, q) \models \psi(c)\} = \{\bar{c} \,|\, (\mathfrak{A}, q) \models \psi[c]\}$.

If $\Gamma \vdash Qx\psi(x)$, then $S_\psi \in q$ by definition, so $(\mathfrak{A}, q) \models Qx\psi(x)$ by (2). Lets turn to the reciprocal: Assume $(\mathfrak{A}, q) \models Qx\psi(x)$. Then $S_\psi \in q$ by (2). From the definition of q, there is a formula $\theta(y)$ such that $S_\psi = S_\theta$ and $\Gamma \vdash Qy\theta(y)$. By (2), $\bar{c} \in S_\psi$ implies $\Gamma \vdash \psi(c)$, for all $c \in C$, and the same occurs for θ. It follows from this that

$$\Gamma \vdash \psi(c) \text{ iff } \Gamma \vdash \theta(c), \text{ for all } c \in C.$$

Thus $\Gamma \vdash \psi(c) \leftrightarrow \theta(c)$, for all $c \in C$. Let u be a variable occurring in neither ψ nor θ. Since C is a set of witnesses for Γ, $\Gamma \vdash \forall u(\psi(u) \leftrightarrow \theta(u))$. Using Axiom 2 we have $\Gamma \vdash Qu\psi(u) \leftrightarrow Qu\theta(u)$, and finally, by Axiom 3 and propositional logic, $\Gamma \vdash Qx\psi(x) \leftrightarrow Qy\theta(y)$. Since $\Gamma \vdash Qy\theta(y)$, we have $\Gamma \vdash Qx\psi(x)$. This completes the proof of (1), and it follows that $(\mathfrak{A}, q)$ is a model of Γ.

□

We can now use techniques inherited from first-order logic to prove that every consistent set of sentences of $L(Q)$ has a weak model:

Theorem 6.13 (Weak completeness theorem)**.** *Let Σ be a set of sentences of $L(Q)$. Then Σ is consistent if and only if Σ has a countable weak model in which all the axioms of $L(Q)$ hold.*

Proof. The proof is essentially the same as Henkin's proof of the completeness theorem for $\mathfrak{L}_{\omega\omega}$. First assume Σ is consistent. We can enlarge the languange L to L^* by adding a countable set C of new individual constants. Σ is still consistent in $L^*(Q)$, and by the method of Henkin, Σ can be extended to a maximal consistent set of sentences Γ of $L^*(Q)$ such that C is a set of witnesses for Γ. By the previous lemma Γ has a weak model $(\mathfrak{A}^*, q)$ in which every element is an interpretation of some $c \in C$. Since C is countable, A^* is countable too. Since Γ is maximal consistent in $L^*(Q)$, all axioms of $L^*(Q)$ belong to Γ and hold in $(\mathfrak{A}^*, q)$. Let $\mathfrak{A}$ be the reduct of $\mathfrak{A}^*$ to L. Then $(\mathfrak{A}, q)$ is the requiered countable weak model of Σ satisfying all axioms of $L(Q)$. The converse follows from Lemma 6.5. □

Lemma 6.14. *Let Γ be a consistent set of sentences of $L(Q)$, and for each $n < \omega$ let $\Sigma_n(x_n)$ be a set of formulas of $L(Q)$. Assume that for every $n < \omega$ and every formula $\varphi(x_n)$ of $L(Q)$, if $\exists x_n \varphi(x_n)$ is consistent with Γ then there exists $\sigma \in \Sigma_n$ such that $\exists x_n(\varphi \wedge \neg\sigma)$ is consistent with Γ. Then Γ has a countable weak model which omits each Σ_n.*

This lemma is the generalization to $L(Q)$ of the first-order omitting types theorem, and the proof is essentially the same. We next introduce elementary chains of weak models.

Definition 6.15. *$(\mathfrak{B}, r)$ is said to be an elementary extension of $(\mathfrak{A}, q)$, $(\mathfrak{A}, q) \preceq (\mathfrak{B}, r)$, iff $A \subseteq B$ and for all formulas $\varphi(x_1, ..., x_n)$ of $L(Q)$ and all $a_1, ..., a_n \in A$ we have*

$$(\mathfrak{A}, q) \models \varphi[a_1, ..., a_n] \text{ iff } (\mathfrak{B}, r) \models \varphi[a_1, ..., a_n].$$

A sequence $(\mathfrak{A}_\alpha, q_\alpha)$, $\alpha < \gamma$, of weak models is said to be an elementary chain iff we have $(\mathfrak{A}_\alpha, q_\alpha) \prec (\mathfrak{A}_\beta, q_\beta)$ for all $\alpha < \beta < \gamma$.

The union of such an elementary chain is the weak model $(\mathfrak{A}, q) = \cup_{\alpha<\gamma}(\mathfrak{A}_\alpha, q_\alpha)$ such that $\mathfrak{A} = \cup_{\alpha<\gamma}\mathfrak{A}_\alpha$ and $q = \{S \subseteq A \mid \text{ For some } \beta < \gamma, \beta \leq \alpha < \gamma \text{ implies } S \cap A_\alpha \in q_\alpha\}$.

As the reader can see, the construction of the union of an elementary chain of weak models is identical to the first-order case, with the exception of q, which is buit as the set of all $S \subseteq A$ such that $A_\alpha \cap S$ is eventually in q_α.

The following lemma extends Tarski's theorem to $L(Q)$:

Lemma 6.16. *Let $(\mathfrak{A}_\alpha, q_\alpha)$, $\alpha < \gamma$, be an elementary chain and let $(\mathfrak{A}, q)$ be the union. Then for all $\alpha < \gamma$, $(\mathfrak{A}_\alpha, q_\alpha) \prec (\mathfrak{A}, q)$.*

Proof. We will show, by induction on the complexity of $\varphi(x_1, ..., x_n)$, that

(1) For all $\alpha < \gamma$ and all $a_1, ..., a_n \in A_\alpha$,

$$(\mathfrak{A}_\alpha, q_\alpha) \models \varphi[a_1, ..., a_n] \text{ iff } (\mathfrak{A}, q) \models \varphi[a_1, ..., a_n].$$

The induction steps for first-order formulas are the same as in the proof of Theorem 5.8. We turn to the case $\varphi = Qx\psi$. Assume $\alpha < \gamma$ and $a_1, ..., a_n \in A_\alpha$. We define $S = \{a \in A \mid (\mathfrak{A}, q) \models \psi[a, a_1, ..., a_n]\}$.

Let $(\mathfrak{A}, q) \models Qx\psi[a_1, ..., a_n]$. Then $S \in q$, and for some β, $\alpha \leq \beta < \gamma$ and $S \cap A_\beta \in q_\beta$. Since, by induction hypothesis, (1) holds for ψ, $S \cap A_\beta = \{a \in A_\beta \mid (\mathfrak{A}_\beta, q_\beta) \models \psi[a, a_1, ..., a_n]\}$. Whence $(\mathfrak{A}_\beta, q_\beta) \models Qx\psi[a_1, ..., a_n]$, and since $(\mathfrak{A}_\alpha, q_\alpha) \prec (\mathfrak{A}_\beta, q_\beta)$, $(\mathfrak{A}_\alpha, q_\alpha) \models Qx\psi[a_1, ..., a_n]$.

Now suppose $(\mathfrak{A}, q) \models \neg Qx\psi[a_1, .., a_n]$. Then $S \notin q$, so there exists β such that $\alpha \leq \beta < \gamma$ and $S \cap A_\beta \notin q_\beta$. Arguing as before, we have $(\mathfrak{A}_\alpha, q_\alpha) \models \neg Qx\psi[a_1, ..., a_n]$. This shows that $\varphi = Qx\psi$ satisfies (1). □

Given a model $\mathfrak{A}$ for $L(Q)$, we denote by $\mathfrak{A}^*$ the model obtained by adding a new constant c_a to L for each $a \in A$ and interpreting c_a in $\mathfrak{A}^*$ by a. The following technical fact is needed later:

Lemma 6.17. *Let $(\mathfrak{A}, q)$ be a weak model for $L(Q)$ and let L^* be the language of $\mathfrak{A}^*$. If $(\mathfrak{A}, q)$ satisfies all the axioms of $L(Q)$, then $(\mathfrak{A}^*, q)$ satisifies all the axioms of $L^*(Q)$.*

Proof. Let φ be any axiom of $L * (Q)$. Let $c_{a_1}, ..., c_{a_n}$ be all the constat symbols of $L * \setminus L$ which occur in φ. We form ψ by replacing each occurrence of c_{a_m} in φ by v_m (a new variable), $m \leq n$. Then ψ is an axiom of $L(Q)$, therefore ψ holds in $(\mathfrak{A}, q)$, hence $\forall v_1 .... \forall v_n \psi$ holds in $(\mathfrak{A}, q)$. It follows that $(\mathfrak{A}^*, q) \models \psi[a_1, ..., a_n]$, hence φ holds in $(\mathfrak{A}^*, q)$. □

We now prove our main lemma, which says that every countable weak model of all the axioms of $L(Q)$ has an elementary extension in which a set in q gets new elements and every definable set which is not in q stays the same. This will solve the main problem of constructing an standard model, because it will help us to keep the set of elements of $\mathfrak{A}$ satisfying $\psi(x)$ countable when $\mathfrak{A} \models \neg Qx\psi(x)$.

Lemma 6.18 (Main lemma)**.** *Let $(\mathfrak{A}, q)$ be a countable weak model in which all the axioms of $L(Q)$ hold. Let L^* be the language of $\mathfrak{A}^*$ and let $\varphi(x)$ be a formula of $L^*(Q)$ such that $(\mathfrak{A}^*, q) \models Qx\varphi(x)$. Then there is a countable elementary extension $(\mathfrak{B}, r)$ of $(\mathfrak{A}, q)$ such that:*

(I) *For some $b \in B \setminus A$, $(\mathfrak{B}^*, r) \models \varphi[b]$.*

(II) *For every formula $\varphi(y)$ of $L^*(Q)$ such that $(\mathfrak{A}^*, q) \models \neg Qy\psi(y)$, we have $\{a \in B \mid (\mathfrak{B}^*, r) \models \psi[a]\} \subseteq A$.*

Proof. We extend L^* to L' by adding a new constant c. Let Γ be the following set of sentences:

(a) All sentences of $L^*(Q)$ true in $(\mathfrak{A}^*, q)$.

(b) $\varphi(c)$.

(c) $\neg\psi(c)$ for every $\varphi(y)$ in $L^*(Q)$ such that $(\mathfrak{A}^*, q) \models \neg Qy\psi(y)$.

We first prove the following claim:

(1) A sentence $\theta(c)$ is consistent with Γ iff $(\mathfrak{A}^*, q) \models Qu(\theta(u) \wedge \varphi(u))$.

Here $\theta(c)$ comes from a formula $\theta(y)$ of L^*. To prove one direction of (1), assume that $\neg Qu(\theta(u) \wedge \varphi(u))$ holds in $(\mathfrak{A}^*, q)$. Then $\neg(\theta(c) \wedge \varphi(c))$ belongs to Γ, and by propositional logic, $\Gamma \vdash \neg\theta(c)$, whence $\theta(c)$ is not consistent with Γ.

Now assume that $\theta(c)$ is not consistent with Γ. By the deduction theorem, $\Gamma \vdash \neg\theta(c)$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \neg\theta(c)$. Let Γ_1 be the set of all sentences of type (a) and let $\neg\psi_1(c), ..., \neg\psi_n(c)$ be the sentences of type (c) which belong to Γ_0. Then

$$\Gamma_1 \cup \{\varphi(c), \neg\psi_1(c), ..., \neg\psi_n(c)\} \vdash \neg\theta(c).$$

By the deduction theorem and propositional logic, $\Gamma_1 \vdash (\varphi(c) \wedge \theta(c) \to \psi_1(c) \vee ... \vee \psi_n(c))$. Since c does not occur in Γ_1, $\Gamma_1 \vdash (\varphi(u) \wedge \theta(u) \to \psi_1(u) \vee ... \vee \psi_n(u))$ where u is a variable not occurring in the sentence on the right. By generalization we have

$$\Gamma_1 \vdash \forall u(\varphi(u) \wedge \theta(u) \to \psi_1(u) \vee ... \vee \psi_n(u)),$$

and by Axiom 2,

$$\Gamma_1 \vdash Qu(\varphi(u) \wedge \theta(u)) \to Qu(\psi_1(u) \vee ... \vee \psi_n(u)).$$

Since the sentences $\psi_i(c)$ are of type (c), we have $\Gamma_1 \vdash \neg Qu\psi_i(u)$, $1 \leq i \leq n$. Using Lemma 6.9(iv), $\Gamma_1 \vdash \neg Qu(\psi_1(u) \vee ... \vee \psi_n(u))$, thus $\Gamma_1 \vdash \neg Qu(\varphi(u) \wedge \theta(u))$. As $(\mathfrak{A}^*, q)$ is a model of Γ_1, by Lemma 6.5, $(\mathfrak{A}^*, q) \models \neg Qu(\varphi(u) \wedge \theta(u))$. This ends the proof of (1).

Lets check that Γ is consistent. To see this, consider the valid sentence $c = c$. By hypothesis, $(\mathfrak{A}^*, q) \models Qx\varphi(x)$, hence $(\mathfrak{A}^*, q) \models Qu(\varphi(u) \wedge u = u)$, and it follows from (1) that $c = c$ is consistent with Γ, so Γ itself is consistent.

Since the set A is countable, the language of $(\mathfrak{A}^*, q)$ has only countable many formulas and we can arrange the formulas of $\psi(x) \in L^*(Q)$ such that $\psi(c)$ is in Γ of type (c) in a countable list $\psi_0(y_0)$, $\psi_1(y_1), \ldots$.

For each $n < \omega$, let $\Sigma_n = \{\psi_n(y_n)\} \cup \{\neg y_n = c_a \mid a \in A\}$. Rawly, to omit Σ_n will mean that there are no "new" elements that realize $\psi_n(y_n)$ aside from those in A.

We now are going to verify that the conditions of the omitting types theorem hold for the set of sentences Γ and the sets of formulas Σ_n, $n < \omega$. Suppose $\theta(y_n, c)$ is a formula of $L'(Q)$ such that $\exists y_n \theta(y_n\, c)$ is consistent with Γ. We must show that

(2) There exists $\sigma \in \Sigma_n$ such that $\exists y_n(\theta \wedge \neg\sigma)$ is consistent with Γ.

By (1) we have $(\mathfrak{A}^*, q) \models Qu(\varphi(u) \wedge \exists y_n \theta(y_n, u))$. By Lemma 6.9(iv), we have either

(3) $$(\mathfrak{A}^*, q) \models Qu[\varphi(u) \wedge \exists y_n(\theta(y_n, u) \wedge \psi_n(y_n))] \text{ or}$$

(4) $$(\mathfrak{A}^*, q) \models Qu[\varphi(u) \wedge \exists y_n(\theta(y_n, u) \wedge \neg\psi_n(y_n))].$$

If (4) is the case, then by (1), $\exists y_n(\theta(y_n, c) \wedge \neg\psi_n(y_n))$ is consistent with Γ, and since $\psi_n(y_n) \in \Sigma_n$, (2) holds. Suppose (3) is the case. By predicate logic and Axiom 2, $(\mathfrak{A}^*, q) \models Qu\exists y_n(\varphi(u) \wedge \theta(y_n, u) \wedge \psi_n(y_n))$, and by Axiom 4 we have either

(5) $$(\mathfrak{A}^*, q) \models Qy_n \exists u(\varphi(u) \wedge \theta(y_n, u) \wedge \psi_n(y_n)) \text{ or else}$$

(6) $$(\mathfrak{A}^*, q) \models \exists y_n Qu(\varphi(u) \wedge \theta(y_n, u) \wedge \psi_n(y_n)).$$

But $(\mathfrak{A}^*, q) \models \neg Qy_n \psi_n(y_n)$, and hence by Axiom (2), (5) is impossible. Therefore (6) must hold. Thus for some $a \in A$, $(\mathfrak{A}^*, q) \models Qu(\varphi(u) \wedge \theta(c_a, u) \wedge \varphi_n(c_a))$. This means that $\theta(c_a, c) \wedge \psi_n(c_a)$ is consistent with Γ, so $\theta(c_a, c)$ is consistent with Γ too. It follows that $\exists y_n(\theta(y_n, c) \wedge \neg\neg y_n = c_a)$ is consistent with Γ, and since $\neg y_n = c_a \in \Sigma_n$, condition (2) holds.

Having verified (2) in both cases, we may apply the omitting types theorem (Lemma 5.16) and we get a countable weak model $(\mathfrak{B}', r)$ of Γ which omits each type Σ_n, $n < \omega$. We may assume without loss of generality that for each $a \in A$, c_a is interpreted in $\mathfrak{B}'$ by a. Let $\mathfrak{B}$ be the reduct of $\mathfrak{B}'$ to the language L. Since $(\mathfrak{B}', r)$ satisfies all sentences true in $(\mathfrak{A}^*, q)$, we have $(\mathfrak{A}, q) \prec (\mathfrak{B}, r)$.

If b is the interpretation of the constant c in $\mathfrak{B}'$, since $(\mathfrak{B}', r)$ satisfies $\varphi(c)$, $(\mathfrak{B}^*, r) \models \varphi[b]$. Also, by Axiom 1 we have for all $a \in A$, $(\mathfrak{A}^*, q) \models \neg Qx(x = c_a)$, so the sentences $\neg(c = c_a)$ are of type (c) and belong to Γ, hence they hold in $(\mathfrak{B}', r)$. It follows that $b \neq a$ for all $a \in A$, whence $b \in B \setminus A$. This shows that $(\mathfrak{B}, r)$ satisfies conclusion (i) of the lemma.

Now let $\psi(y)$ be any formula of $L^*(Q)$ such that $(\mathfrak{A}^*, q) \models \neg Qy\psi(y)$. Then $\psi(y)$ is $\psi_n(y_n)$ for some $n \in \omega$ and (ii) follows from the fact that $(\mathfrak{B}', r)$ omits Σ_n.

□

We can iterate the main lemma using elementary extensions to get the following stronger result:

Lemma 6.19. *Let $(\mathfrak{A}, q)$ be a countable weak model in which all the axioms of $L(Q)$ hold, and let L^* be the language of $\mathfrak{A}^*$. Then $(\mathfrak{A}, q)$ has a countable elementary extension $(\mathfrak{B}, r)$ such that for all formulas $\varphi(x)$ of $L^*(Q)$, $(\mathfrak{A}^*, q) \models Qx\varphi(x)$ if and only if there exists $b \in B \setminus A$ such that $(\mathfrak{B}^*, r) \models \varphi[b]$.*

Proof. Since the set A is countable, we may arrange all formulas of $\varphi(x)$ of $L^*(Q)$ such that $Qx\varphi(x)$ holds in $(\mathfrak{A}^*, q)$: $\varphi_0(x_0)$, $\varphi_1(x_1)$,... Then using the main lemma countably many times we construct an elementary chain $(\mathfrak{A}_n, q_n)_{i\in\omega}$ such that:

(1) $(\mathfrak{A}_0, q_0) = (\mathfrak{A}, q)$.

(2) There exists $b_n \in A_{n+1} \setminus A_n$ such that $(\mathfrak{A}^*_{n+1}, q_{n+1}) \models \varphi_n[b_n]$, $n < \omega$.

(3) For all formulas $\psi(y)$ of $L^*(Q)$ such that $(\mathfrak{A}^*_n, q_n) \models \neg Qy\psi(y)$, and all $n < \omega$, $\{a \in A_{n+1} \mid (\mathfrak{A}^*_{n+1}, q_{n+1}) \models \psi[a]\} \subseteq A_n$.

Let $(\mathfrak{B}, r) = \cup_{n<\omega}(\mathfrak{A}_n, q_n)$. Then by Lemma 6.16, $(\mathfrak{B}, r)$ is an elementary extension of each $(\mathfrak{A}_n, q_n)$ and in particular of $(\mathfrak{A}, q)$. Then it follows from (1)-(3) that $(\mathfrak{B}, r)$ satisfies the result. □

We are now ready to prove the main result in this chapter:

Proof of the Completeness Theorem (Hard direction):

Suppose Σ is a consistent set of sentences of $L(Q)$. By the weak completeness theorem, Σ has a countable weak model $(\mathfrak{A}_0, q_0)$ in which all the axioms of $L(Q)$ hold. Now, iterating Lemma 6.19 ω_1 times and using Lemma 6.16 at the limit stages, we obtain an elementary chain $(\mathfrak{A}_\alpha, q_\alpha)_{\alpha<\omega_1}$, of countable weak models such that:

(1) If α is a limit ordinal, $(\mathfrak{A}_\alpha, q_\alpha) = \cup_{\beta<\alpha}(\mathfrak{A}_\beta, q_\beta)$.

(2) For any $\alpha < \omega_1$ and any formula $\varphi(x)$ of the language of $(\mathfrak{A}^*_\alpha, q_\alpha)$,

$$(\mathfrak{A}^*_\alpha, q_\alpha) \models Qx\varphi(x) \text{ iff for some } a \in A_{\alpha+1} \setminus A_\alpha,\ (\mathfrak{A}^*_{\alpha+1}, q_{\alpha+1}) \models \varphi[a].$$

Let $\mathfrak{B} = \cup_{\alpha<\omega_1}\mathfrak{A}_\alpha$. We shall show that $\mathfrak{B}$ is a standard model of Σ.

Consider the weak model $(\mathfrak{B}, r) = \cup_{\alpha<\omega_1}(\mathfrak{A}_\alpha, q_\alpha)$. Since $(\mathfrak{A}_0, q_0) \prec (\mathfrak{B}, r)$, $(\mathfrak{B}, r)$ is a weak model of Σ. Our goal is to prove the following:

(3) For all formulas $\varphi(x_1, ..., x_n)$ of $L(Q)$ and all $b_1, ..., b_n \in B$,

$$(\mathfrak{B}, r) \models \varphi[b_1, ..., b_n] \text{ iff } \mathfrak{B} \models \varphi[b_1, ..., b_n].$$

We shall proceed by induction on φ. Since the clauses for the definition of satisfaction of first-order formulas are the same for $(\mathfrak{B}, r)$ as for $\mathfrak{B}$, the problem reduces to prove (3) for $\varphi = Qx_0\psi(x_0, ..., x_n)$.

For some $\alpha < \omega_1$, $b_1, ..., b_n \in A_\alpha$. Suppose $(\mathfrak{B}, r) \models Qx_0\psi[b_1, ..., b_n]$. Then $(\mathfrak{B}_\beta, q_\beta) \models Qx_0\psi[b_1, ..., b_n]$, $\alpha \leq \beta < \omega_1$. By (2), for every β such that $\alpha \leq \beta < \omega_1$, there exists $a_\beta \in A_{\beta+1} \setminus A_\beta$ with $(\mathfrak{A}_{\beta+q}, q_{\beta+1}) \models \psi[a_\beta, b_1, ..., b_n]$. This means that $(\mathfrak{B}, r) \models \psi[a_\beta, b_1, ..., b_n]$, an since all the a_β are distinct and there are ω_1 of them, the set $S = \{b_0 \in B \mid (\mathfrak{B}, r) \models$

$\psi[b_0, ..., b_n]\}$ has power ω_1. But since ψ satisfies (3), the set S is the same as $\{b_0 \in B \mid \mathfrak{B} \models \psi[b_0, ..., b_n]\}$, and therefore $\mathfrak{B} \models Qx_0\psi[b_1, ..., b_n]$.

Now suppose that $(\mathfrak{B}, r) \models \neg Qx_0\psi[b_1, ..., b_n]$. Hence whenever $\alpha \leq \beta < \omega_1$, $(\mathfrak{A}_\beta, q_\beta) \models \neg Qx_0\psi[b_1, ..., b_n]$. It follows from (1) and (2) that $S = \{b_0 \in B \mid (\mathfrak{B}, r) \models \psi[b_0, ..., b_n]\} \subseteq A_\alpha$. But since A_α is countable, by induction hypothesis we conclude that $\mathfrak{B} \models \neg Qx_0\psi[b_1, ..., b_n]$. This completes the proof. □

Since $\Sigma \vdash \varphi$ iff there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \varphi$ (proofs are finite), Σ is consistent iff all finite Σ_0 are consistent. Now using the Completeness theorem:

Corollary 6.20. *$L(Q)$ is countably compact.*

Compactness theorem does not hold if L has uncountably many constants: if we consider the set of sentences $\Sigma = \{\neg QxP(x)\} \cup \{P(c_\alpha) \mid \alpha < \omega_1\} \cup \{\neg c_\alpha = c_\beta \mid \alpha < \beta < \omega_1\}$, it is easy to check that every finite subset of Σ is satisfiable but Σ is not. This means that $L(Q)$ is just countably compact.

The first fully compact extension of first-order logic was given by Shelah (1975) in his paper [12], where he introduces a quantifier $Q^{cf}xy\,\varphi(x, y)$ saying "$\varphi(x, y)$ is an ordering with cofinality ω".

Conclusions

Model theorists lost interest in abstract model theory by the early 1990s. The main reason for this was that they were not successful in providing characterization theorems for the variety of extensions they were studying. Maybe because of that, Lindström's theorems remain an oddity in the universe of model theory and are often overlooked. They do, however, give very valuable information about the limitations of our formal systems and, by extension, about our mathematical practice. For that reason, they will have a permanent place in the history of mathematics. Furthermore, all the new techniques developed in the study of extended logics (of which the proof of Keisler's Completeness theorem for $\mathfrak{L}(Q_1)$ is a great example) have greatly enriched the model-theoretic toolbox. We hope that we have been able to transmit this importance to the reader.

With respect to my personal experience, the task of writing this essay has demanded a considerable mathematical maturity and I have faced some challenges for the first time: dealing with an extense bibliography and putting together all my ideas was not a smooth process. However, I am satisfied with the results, and especially with the fact that this essay has opened the doors to a branch of mathematics that was completely unknown for me one year ago. I would be pleased to continue studying more advanced results in model theory and especially its applications in algebra.

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