Mean field approach to breakdown phenomena in disordered systems

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(Dated: June 12, 2018)

Breakdown phenomena can be understood as a phase transition that occur in athermal systems constituted of many interacting elements governed by the interplay between an external driving force and disorder. Representative examples are magnets driven by a magnetic field out of equilibrium and ruptures, in a stress driven fiber bundle. In this work we analyze critical behaviors of both cases within the mean field approach. The magnetic case has been compared with the Weiss model for a ferromagnet with no disorder at finite temperature. Finally, in the case of the fiber bundle model we show the existence of certain conditions for criticality to occur.

I. INTRODUCTION

In our everyday life we confront with systems which present a dynamical equilibrium that, at some point, can be disturbed, giving rise to irreversible breach of such stability. A bridge, for example, needs to support tensile and shear stresses of vehicles that drive over. If many of them pass trough at the same time, the strength may be enough to disassemble and break the bridge, involving an irreversible breakdown of the system.

Another simpler example is the case of a pulled rope made of fibers. Some will be stronger and others weaker, considering it in a random way. As the pulling strength increases, every fiber will be more stressed until it overcomes the threshold of the weaker ones and breaks them. Stress will now be redistributed in all ropes that remain entire. This phenomena can conduct the system towards a sudden breakdown at some value of the applied force.

We can also imagine a ferromagnetic system subjected to an external applied field. As we increase the field from a large negative value towards a large positive value, the magnetization of the material will change slowly until we approach a certain value of the external field. At this point, the magnetization will undergo an abrupt change that can be interpreted as a breaking of our system. In some aspects this phenomenon is similar to the case of the fiber bundle model, but not entirely, as the breakdown of the fibers is totally irreversible. Actually, the change of magnetization can be reverted, adding some hysteresis which measures the degree of irreversibility [1].

Many more examples like the ones that have been considered can be thought, including earthquakes [2], neuronal cultures [3] and compressing of nanoporous materials [4] among others. All of these, however, present similarities. In all cases, a heterogeneous system driven by an external force undergoes a breaking when a threshold is overcome. For the detailed study of this phenomena, we postulate the existence of a local random distribution of any kind of disorder. For example, it is easy to see that, in the case of a pulled rope made of fibers, we can introduce a random distribution of the thresholds at which each fiber breaks. The same can be done in a magnetic case, where we can postulate the existence of a random local field at every volume element of the magnet.

We will consider only athermic systems, meaning that the temperature will not play an important role. Then, thermal fluctuations will not be considered, as we will just focus on the effects of an effective field with a random distribution. We will begin the discussion of the magnetic case. We will introduce a simple model and solve it numerically within the mean field approximation. Finally, we will consider the fiber bundle case and show that the system can display criticality for certain distributions of disorder.

II. RANDOM FIELD IN A MAGNET

We will consider the general case of a ferromagnetic material, modelized as a spin cubic lattice, in a mean field approximation. At every point, where spins (taking ± 1 values) are located, we will define a random field, $h(\vec{r})$. We apply an external magnetic field, H, varying from $-\infty$ to $+\infty$. As we said before, the dynamics will be assumed athermal, and interactions are just given by the external magnetic field, the random field and the exchange coupling between spins, J. With all these ingredients, we can assume the hamiltonian that describes our system as the Random Field Ising Model [5]

$$\mathcal{H} = -J\sum_{i,j} s_i s_j - \sum_i h(r_i)\, s_i - H\sum_i s_i \ . \qquad (1)$$

Focusing on certain spin s_j, defining the magnetization per particle $M \equiv 1/N\sum_i s_i$ and $J' \equiv Jz$ as an effective coupling constant (z is the coordination number) the distributions of spin flips will be determined as follows

$$\begin{aligned} s_j > 0 \quad &\rightarrow \quad h(\vec{r}) > -J'M - H \\ s_j < 0 \quad &\rightarrow \quad h(\vec{r}) < -J'M - H \ . \end{aligned} \qquad (2)$$

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We assume a distribution probability density $p\left(h\right)$. Then the magnetization of the entire system is given by

$$M = 1 - 2P\left(-J'M - H\right) \ , \tag{3}$$

where $P(h) = \int_{-\infty}^{h} p(h)dh$ represents the fraction of spins pointing down given an applied external field. Therefore, it is easy to see that when we take large positive values of H, $P(-J'M - H)$ becomes zero, meaning that all spins are orientated upwards. In this case the magnetizaion per particle is $M = 1$. On the other hand, when H becomes large and negative, $M = -1$. At this point we can proceed to obtain numerically M-H curves for given distributions of the local fields. The usual situation consists of assuming that random fields are gaussian distributed.

1. *Gaussian distribution*

We assume that $p\left(h\right) = 1/\sqrt{2\pi\sigma^2}e^{h^2/2\sigma^2}$. Note that local fields are distributed with zero mean and standard deviation σ. The latest represents a measure of the amount of disorder. In the model, dimensionality is not relevant (as in any mean field). We will consider for the numerical simulations a 3D simple cubic system. Computing then a $100 \times 100 \times 100$ spins system, defining h_i in every lattice point and applying H we can measure the field evolution of the magnetization per particle as it is shown in Fig. 1. It is remarkable to check three different cases of $M\left(H\right)$ curves depending on the value of σ related to the critical value $\sigma_c = J'\sqrt{2/\pi}$. We obtain the latest solving Eq. (3) and analyzing the derivative in M vs. H when it goes to zero. Results can be summarized as follows [6].

- For $\sigma > \sigma_c$, random fields are distributed around the zero mean, but values spread out considerably. This broad gaussian distribution allows the system to go from one magnetization to another smoothly, without experiencing a sudden abrupt change. Thus, it can be understood also as if the system had more possibilities to reach a point where the external field permits the flip of a small number of spins without diverging or breaking.

- For $\sigma = \sigma_c$ the gaussian distribution is such that susceptibility ($\chi = \partial M/\partial H$) has a divergence at $H = 0$. Later we will analyze this case in more detail, obtaining the critical exponents that characterize the behavior of the system.

- If $\sigma < \sigma_c$, the values of the random fields are distributed very close to the zero mean. This makes that the changes of spin are concentrated in a small region of the external field, provoking a sudden large amount of spin flips at once, and a breakdown in $M\left(H\right)$. Moreover, for a given value of σ, when going from a negative external field to a positive one, the breakdown will occur at $H = +H_{\rm co}$. If we turn the external field back to a negative one, the breakdown will happen at $H = -H_{\rm co}$. A hysteresis cycle of the magnetization is then defined.

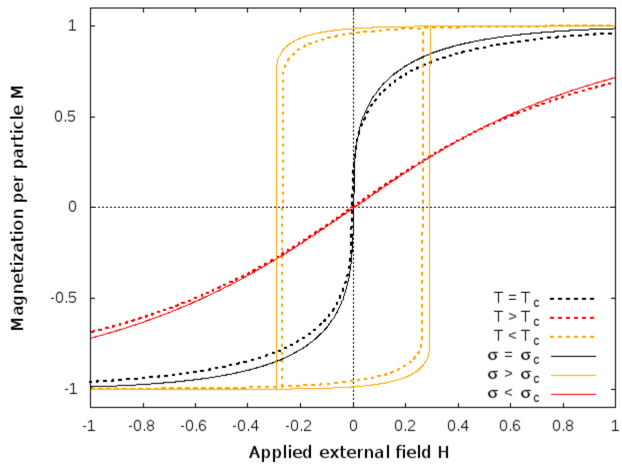

FIG. 1: M-H curves obtained with a gaussian random local field (continuous lines) and a temperature (dashed curves), as a factor of the applied external field. Note that we only expect to see the hysteresis cycle for values of the disorder parameter and temperatures below the critical point. Otherwise, magnetization will always be zero if there is no external applied field. It is also interesting to see how at the critical point, susceptibility diverges at $H = 0$, and then, as the disorder parameter decreases the coercive field, $H_{\rm co}$, increases.

We have seen how the value of σ plays a role similar to an effective temperature. Also, as it can be seen in Fig. 1, the M-H hysteresis cycle shows a behavior very similar to the one of a magnet without a disorder at a finite temperature. We can now compare this similarity considering a thermal mean field model for a ferromagnet based on the Weiss theory.

2. *Thermal case*

We will consider the Weiss theory of ferromagnetism, which states that the equation of state of a ferromagnetic system is

$$M = \tanh\left[\frac{sg\mu_B}{2k_BT}\left(H + J'M\right)\right] \ . \tag{4}$$

Here we consider only systems with spin $s = 1/2$ and $g \simeq 2$, being the Landé factor. Using Eq. (3) and $P\left(h\right)$ defined before we can find the distribution of an effective thermal disorder that lead to Eq. (4). The distribution is, $p\left(h\right) \sim \text{sech}^2\left(h\mu_B/2k_BT\right)$. Knowing that, we compute a 3D system with the same conditions as we did before. The results are shown in Fig. 1, with dashed lines, where it can be observed the similarities of the behaviors in the three cases of disorder mentioned before with the gaussian distribution of random fields. Here the defined parameter, T, (which is measured in units of K) plays a role of disorder similar to σ in the case of the athermal Random Field Ising Model. The critical value of the disorder variable in this case is $T_C = J'\mu_B/2k_B$. Both quantities, σ and T, can be treated as effective temperatures.

3. Free energy and critical behavior

We will begin taking Eq. (3) considering that random fields are gaussian distributed. We obtain

$$M = \mathrm{erf}\left(\frac{H + J'M}{\sqrt{2}\sigma}\right) \quad (5)$$

being $\mathrm{erf}\,(x)$ the error function. This expression can be understood as an evolution equation (or equation of state) of the magnetization per particle. Inverting the error function and taking a series expansion of the left hand side term the result is [7],

$$H = \sqrt{\frac{\pi}{2}}\left(\sigma - \sqrt{\frac{2}{\pi}}J'\right)M + \frac{\sqrt{2\pi^3}}{24}\sigma M^3 + \cdots \quad (6)$$

Assuming that σ is an effective temperature, we can identify the critical value $\sigma_c = J'\sqrt{2/\pi}$. By integration of the thermodynamic relation $M = \partial F/\partial H$ we obtain the effective energy

$$F_{eff} = \sqrt{\frac{\pi}{2}}\left(\sigma - \sigma_c\right)\frac{M^2}{2} + \frac{\sqrt{2\pi^3}}{24}\sigma\frac{3}{4}M^4 + \cdots \quad (7)$$

It can be written as a power series of the order parameter, which in this case would be the magnetization per particle. From Eq. (7) we can obtain the critical exponents for the model. For this, we will consider as we have already mentioned the parameter σ as the effective temperature. Then it is clear to see that the behavior of the order parameter close to the critical point will be $M \sim (\sigma - \sigma_c)^{1/2}$, being then $\beta = 1/2$. We can also see how susceptibility behaves in our system, obtaining $\chi \sim (\sigma - \sigma_c)^{-1}$, being $\gamma = 1$. Moreover, taking the dependence of the susceptibility in the external field, at the critical point, we see $\chi \sim H^{-2/3}$. Magnetization near the critical point, according to Eq. (6), goes as $H \sim M^3$, with $\delta = 3$. All these critical exponents we have obtained are, as expected, those of the mean field approximation theory.

The internal energy also matches perfectly with the mean field theory. Considering an effective entropy as $S_{\mathrm{eff}} = -\partial F_{\mathrm{eff}}/\partial\sigma$, the internal energy $U_{\mathrm{eff}} = F_{\mathrm{eff}} + \sigma S_{\mathrm{eff}}$ is given by

$$U_{\mathrm{eff}} = -\frac{1}{2}M^2 J' \ . \quad (8)$$

It is also useful to take a look at the coercive field and to understand when the system will undergo the breakdown in magnetization. In Eq. (5) we see on the left side the magnetization of the system, and on the right side the amount of spins (in magnetization per particle units) that remains pointing in a given direction (up, +, or down, −). Depending on the value of σ we can see the three cases we mentioned before. Now we focus in $\sigma < \sigma_c$, happening only when the breakdown transition of the magnetization occurs. The process is clear in Fig. 2, where both sides of Eq. (5) are represented. Beginning from a large negative value of the applied field, the equality shows how both sides of Eq. (5) cross at $M = -1$. Increasing H we reach a point where both sides still agree at one point which is also tangent. It is right after that point when the system breaks and the magnetization turns abruptly to $M \simeq +1$. The coercive field is determined from the tangent condition explained in the caption of Fig. 2. The result is

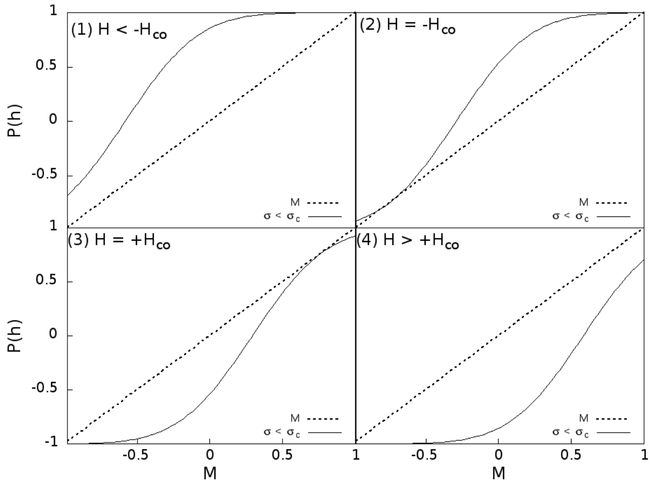

FIG. 2: As we decrease the external field, the equation of evolution of our system evolves as these figures, having $\sigma < \sigma_c$. For high values of H, there is only one intersection of both lines, being at $M = +1$ (all spins are flipped up). Decreasing the external field, at $H = +H_{\mathrm{co}}$ in (3), the intersection disappears, leaving the system evolve to the rightmost one, happening then the breakdown transition of the magnetization. Due to the symmetry, the same transition will occur when H is increased from large negative values at the same H_{co}.

$$H_{co} = \sqrt{2}\sigma\sqrt{\ln\left(\frac{\sigma_c}{\sigma}\right)} - J'\mathrm{erf}\left(\sqrt{\ln\left(\frac{\sigma_c}{\sigma}\right)}\right) \ . \quad (9)$$

This equation agrees with the results obtained from numerical computations (see Fig. 3).

III. FIBER BUNDLE MODEL

Imagine a large number of fibers, N_0, pulled by an external force, F, which is shared among all democratically. Each one withstands a strength F/N, being N

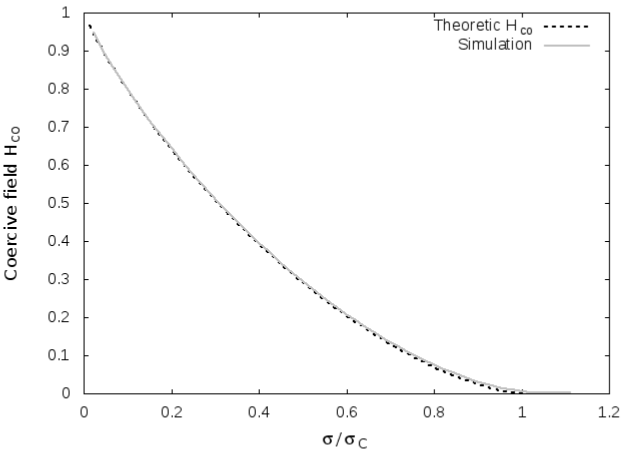

FIG. 3: Values obtained of coercive field with the numerical simulation of a 3D cubic system with 10^6 spins (grey line) compared with the values obtained by 9 (dashed line). Here can be observed how the computations match the model for low values of σ, but approaching to the critical point, some discrepancies appear, due to numerical errors.

the number of fibers remaining entire. Note that if a fiber breaks, the shared force increases and can lead to another to break. It can be deduced that this kind of behavior holds a breakdown phenomena. This model is often called the democratic fiber bundle model. In this brief study we will focus on analyzing this model assuming a mean field approach similar to the one used in the magnetic case. We will consider that every fiber is able to support a strength threshold before it breaks. These thresholds will be assumed to follow a distribution $p\left(f\right)$.

First of all, we should define a generic equation of evolution for a generic distribution. The number of fibers remaining entire will be the original number of fibers when no strength is applied minus the number of fibers that has been broken due to the pulling tension. That is,

$$N = N_0 \left\{ 1 - P\left(\frac{F}{N}\right) \right\} \ , \qquad (10)$$

being $P\left(f\right) = \int_0^f p\left(f\right) df$ the fraction of the number of fibers broken given a force per fiber f. The distribution is chosen with a certain mean value $\mu > 0$ and a standard deviation σ.

The interesting part begins when applying the gaussian distribution to our model. This states that all thresholds have values around μ with width σ. Mathematically, if $\mu < \sigma$, it is possible to observe fibers with negative threshold. Therefore, translated to physical meaning, there exist fibers which will never break due to the fact that we will never achieve a negative external force able to create the rupture. The gaussian distribution for low values of the thresholds and wide distributions is not a suitable approximation, but we can consider it in the cases of $\mu >> \sigma$.

Although now in the latest cases we are able to describe and compute this model, with the gaussian distribution we never observe a critical behavior. Let us analyze and try to obtain the kind of distribution which involves a critical behavior. We base all of our computations in an equation of evolution, such as Eq. (3) in the magnetic case, or Eq. (10) in the fiber bundle case. Usually these take the form $m = f\left(m, h; \alpha\right)$, where m is the order parameter, h is the external perturbation and α are characteristic parameters of the distribution. The smooth breakdown associated with a critical behavior is defined when the infinitesimal variations of the order parameter are continuous, wich means that,

$$1 = \lim_{m \to m_c} \lim_{h \to h_c} \frac{df\left(m, h; \alpha\right)}{dm} \ . \qquad (11)$$

Where m_c and h_c are the values expected to be at the critical point. Note that the order parameter is defined to be zero at this point. If this relation is not acomplished, the system will have a suddent breakdown. In the fiber bundle model, Eq. (11) forces the distribution to be, in critical conditions, $p\left(f\right)_{\text{crit}} \propto 1/f^2$ (wich means that the gaussian distribution does not satisfy the condition), so there will exsist a continuous break.

Following with the discussion, if now we propose a normalized distribution such as $p\left(f\right)_{\text{crit}} = \xi\left(\alpha\right)/f^2$, being $\xi\left(\alpha\right)$ a generic function, at the critical point:

$$1 = \lim_{N \to 0} \lim_{F \to F_{\text{crit}}} \frac{N_0 F}{N^2} \frac{1}{f^2} \xi\left(\alpha\right) = \frac{N_0}{F_{\text{crit}}} \xi\left(\alpha\right) \qquad (12)$$

We can relate the external force at the critical point, $F_{\text{crit}} > 0$ with the characteristic parameters of the distribution. This means that, if the system is able to reach such external force without breaking, the later will be continuous. In this case we see how the critical behavior is subjected to the degree of disorder and other parameters of the random distribution, α, as these define the force at wich the system breaks.

With the purpose of showing an example, we take the gaussian distribution and the Lorentzian function,

$$p\left(f\right) = \frac{\sigma}{\pi} \frac{1}{\left(f - \mu\right)^2 + \sigma^2} \ , \qquad (13)$$

as a normalized distribution wich satisfies Eq. (11) giving the relation $F_{\text{crit}} = N_0 \sigma/\pi$ between the external critical force and the degree of disorder, without considering the mean value. It is because of the exsistence of this force that the critical behavior can appear. It can also be demonstrated that having no disorder, meaning $\sigma = 0$, the breaking will occur at $F_{\text{break}}\left(\sigma = 0\right) = N_0 \mu$. Taking that into account, the continuous breaking of the system will take place if the fibers are able to hold up the applied force until the value $F_{\text{cont}} = N_0\left(\sigma/\pi + \mu\right)$ at wich it will occur (see Fig. 4).

Though the model now is able to describe a critical behavior, the distributions still fail to describe real phenomena as it considers negative values of thresholds. Matematically, these cases are considered as unbreakable fibers.

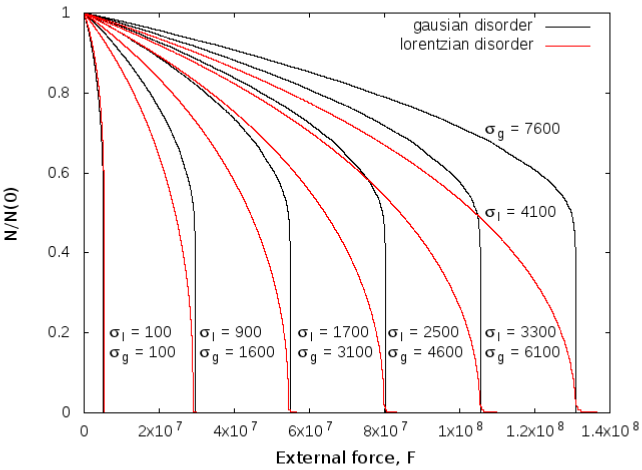

FIG. 4: In black it is shown the behavior of the fiber breaking of a system with a gaussian distributed disorder and in red with the Lotentzian distribution. Each curve corresponds to a diferent degree of disorder of the thresholds, as it is displayed in the figure. The latest reaches a certain value of σ_c between the leftmost curve and the following on the right, and every value above involve a continuous breaking of the fibers. With the gaussian distribution, on the other hand, we will never reach this critical state. The computation is done with $\mu = 100$ and $N_0 = 10^5$ values.

However, from the point of view of the numerical simulations, they are supposed as already broken fibers with zero force. In Fig. 4 though, the representation of the evolution of the number of fibers has been normalized at the number of fibers remaining at force zero, $N(0)$, (not in the total number of fibers, N_0) for an easier understanding. It is worth mention that the effective internal energy, computed as in the case of a magnet, for both distributions is $U_{\text{eff}} = N^2\mu/2$, which is the expected mean field expression.

It would be interesting in the future to study in more detail this problem. Taking into account Eq. (11), it would be useful to analyze distributions suitable to describe real cases that include the possibility of criticality.

IV. CONCLUSIONS

Modeling the athermal Random Field Ising Model in the case of a magnet using a gaussian random distributed field in a lattice, we have obtained a the critical behavior expected following the mean field approximation theory. We have also been able to obtain an equation which provides the value of the coercitive field for each level of disorder of the random fields, and the critical value σ_C for which the magnetic breakdown disappears. Comparing the latest case with a thermal model such as the Weiss theory for a ferromagnetic material, we have compared both behaviors using numerical simulations and obtained the critical value T_C. We conclude that we can treat a thermal mean field ferromagnetic modelized magnet with the Random Field Ising Model and obtain the same critical behaviors.

Finnally, we have analyzed the fiber bundle model using the same procedure as with the magnetic case, and we have seen that the distribution is restricted to several conditions to show a critical behavior. Numerical simulations of the model agree with this explanation as the gussian distributed thresholds do not display criticality. However, in the case of a Lorantzian distribution the system reaches a level of disorder in which the breaking becomes continuous.

Acknowledgements

I would like to thank Antoni Planes for supervising and helping me through this work.

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