# Valuation monotonicity, fairness and stability in assignment problems

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**Abstract**: In this paper, we investigate the possibility of having stable rules for two-sided markets with transferable utility, that satisfy some valuation monotonicity and fairness axioms. Valuation fairness requires that changing the valuation of a buyer for the object of a seller leads to equal changes in the payoffs of this buyer and seller. This is satisfied by the Shapley value, but is incompatible with stability. A main goal in this paper is to weaken valuation fairness in such a way that it is compatible with stability. It turns out that requiring equal changes only for buyers and sellers that are matched to each other before as well as after the change, is compatible with stability. In fact, we show that the only stable rule that satisfies weak valuation fairness is the well-known fair division rule which is obtained as the average of the buyers-optimal and the sellers-optimal payoff vectors. Our second goal is to characterize these two extreme rules by valuation monotonicity axioms. We show that the buyers-optimal (respectively sellers-optimal) stable rule is char- acterized as the only stable rule that satisfies buyer-valuation monotonicity which requires that a buyer cannot be better off by weakly decreasing his/her valuations for all objects, as long as he is assigned the same object as before (respectively object-valuation antimonotonicity which requires that a buyer cannot be worse off when all buyers weakly decrease their valuations for the object that is assigned to this specific buyer, as long as this buyer is assigned the same object as before). Finally, adding a consistency axiom, the two optimal rules are characterized in the general domain of allocation rules for two-sided assignment markets with a variable population.

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# 1 Introduction

An allocation problem consists of a finite set of agents (that we call buyers) and a finite set of objects that are to be allocated among the buyers. We assume that each object belongs to a different seller and also that each buyer is willing to acquire at most one object. Buyers have valuations over each available object and utility is quasi-linear in money. Once these valuations are reported, an allocation rule determines an outcome: who gets what and at what price. Notice that, once an object is allocated to a buyer, this buyer transfers part of its utility to the seller by means of the price paid. Hence, equivalently, an allocation rule determines a matching between buyers and sellers and the resulting payoff for each agent.

We focus on those allocation rules that produce (pairwise) stable outcomes, that is, outcomes that are not blocked by any pair of a buyer and a seller who are not matched one to another but would be better off if they were. The aim is to provide axiomatic characterizations of some outstanding stable rules.

With such buyer-seller allocation problem with transferable utility, Shapley & Shubik (1972) associates a coalitional game, the *assignment game*. These authors show that the core coincides with the set of stable payoff vectors, and a consequence of its lattice structure is the existence of two optimal stable payoff vectors: one that is optimal for all buyers and the other optimal for all sellers.

In these buyer-seller markets, the buyers-optimal stable rule produces the minimum competitive prices and coincides with the Vickrey multi-item auction. Hence it is of relevance both in theory and practice.

Our first motivation comes also from Kojima & Manea (2010). There, in the companion two-sided matching model where each agent has a preference list on the agents on the opposite side of the market and monetary transfers are not allowed, the allocation rule that is optimal for one side of the market (a deferred acceptance rule) is characterized as the only stable rule that satisfies weak Maskin monotonicity.[1]

Similarly, we prove that in our buyer-seller market, among the set of stable rules, each one of the two optimal stable rules is characterized by a monotonicity property: the sellers-optimal stable rule is the only stable rule that satisfies *object-valuation antimonotonicity* and the buyers-optimal stable rule is the only stable rule that satisfies *buyer-valuation monotonicity*. Object-valuation antimonotonicity requires that when all buyers decrease their valuation of a given object but this does not change which buyer receives that object, then the rule should not make this buyer worse off. Buyer-valuation monotonicity requires that when one buyer decreases his valuations of all objects and this fact does not change which object he receives, then the rule should not make this buyer better off.

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[1]We do not define formally weak Maskin monotonicity, since it refers to a model different to ours. Roughly speaking, an allocation rule $\varphi$ satisfies weak Maskin monotonicity if whenever agents change from a preference profile $R$ to $R'$ in such a way that, for any agent $i$, any object preferred to $\varphi_i(R)$ under $R'_i$ is also preferred under $R_i$, then it holds that $\varphi_i(R')$ is preferred to $\varphi_i(R)$ under $R'_i$.

A valuation property that treats the buyers and sellers in a more balanced way is *valuation fairness* being an equal treatment axiom which requires that, if a buyer's valuation for the object of one of the sellers changes, this has the same effect on the payoffs of this buyer and the seller that owns the object.[2] It is shown in van den Brink & Pintér (2015) that the only rule that satisfies *submarket efficiency* (which requires that every submarket allocates its own surplus among the buyers and sellers in the submarket) and valuation fairness is the Shapley value (Shapley, 1953) being a well-known single-valued solution for coalitional games. However, for assignment games, the Shapley value may not assign a stable payoff vector. Since the core obviously satisfies submarket efficiency, this implies that core stability is incompatible with valuation fairness.

A main goal in this paper is to weaken valuation fairness in a way that it becomes compatible with core stability. We introduce a weaker form of valuation fairness, called *weak valuation fairness* that requires that when a buyer modifies his valuation of the object he is assigned by the rule, and after this change this object still remains assigned to the same buyer, then the payoffs to this buyer and to the seller that owns the object change by the same amount. We show that weak valuation fairness is compatible with core stability. More specific, we show that the only stable rule that satisfies weak valuation fairness is the *fair division rule* (Thompson, 1981), being a well-known stable rule where the payoff to an agent is the average between his/her maximum and minimum stable payoffs. This fair division rule is known to be *pairwise monotonic*, see Núñez & Rafels (2002), which requires that if one buyer weakly decreases his valuation of an object, neither the buyer nor the owner of the object can be better off. However, this monotonicity property is not enough to characterize the fair division rule among the stable rules since, among others, the two optimal stable rules are also pairwise monotonic.[3]

Our third and final goal is to look for axiomatizations of the two optimal stable rules on the general domain of allocation rules for buyer-seller markets, that is, without restriction to the subclass of stable rules. We can do this if we allow for a variable population. For variable populations, consistency properties, that consider the effect on payoffs when the population varies, have played a role in the axiomatization of solutions for two-sided assignment markets. For instance, the core that can be seen as a set-valued allocation rule, has been axiomatized using consistency axioms, see e.g., Sasaki (1995) and Toda (2005). The model considered in Toda (2005) allows for individual reservation values of the agents, which represent the payoffs to those agents that remain unmatched. Roughly speaking, the consistency property used in this axiomatization requires that for each solution outcome, the same outcome should be recommended for each subgame that results when some players leave with what they have received. A single-valued stable allocation rule for the Shapley and Shubik assignment market that has been characterized using some consistency, is the nucleolus that is characterized in Llerena *et al.* (2015) by means of two axioms: *derived consistency* and *symmetry of maximum complaints of the two sides*. The second property requires that the payoff to the most disadvantaged buyer equals the payoff to the most disadvantaged seller. In this setting of a variable population, we prove that the buyers optimal stable rule is the only

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[2]This is based on the fairness axiom for communication graph games of Myerson (1977).

[3]Among non-stable rules, also the Shapley value satisfies pairwise monotonicity.

one that satisfies *derived consistency* and *buyer-valuation monotonicity*, while the sellers optimal stable rule is the only one that satisfies *derived consistency* and *object-valuation antimonotonicity*. Here, consistency is not required with respect to the subgame but with respect to the derived game introduced in Owen (1992).

The paper is organized as follows. The buyer-seller assignment model is introduced in Section 2, axiomatizations of the buyers-optimal and sellers-optimal stable rules among the set of all stable allocation rules are provided in Section 3, while Section 4 contains the characterization of the fair division rule. Section 5 provides axiomatizations of the buyers and sellers optimal rules on the general class of allocation rules.

# 2 The model

Consider a market situation with two disjoint finite sets of agents: buyers and sellers.[4] The set of buyers is denoted by $B$ and the set of sellers is denoted by $S$. In this market, each seller owns one and only one indivisible object on sale. On the other side of the market, each buyer is interested in acquiring at most one object. Moreover, each buyer $i \in B$ has a non-negative valuation in terms of money, $a_{ij} \in \mathbb{R}_+$, for the object owned by the seller $j \in S$. We assume for the moment that the reservation price of each seller for the object she owns is zero. Hence, for each buyer-seller pair $(i, j) \in B \times S$, the valuation $a_{ij} \in \mathbb{R}_+$ is the joint potential monetary gain for agents $i$ and $j$ if they trade. We denote by $a_i = (a_{ij})_{j \in S}$ a vector of valuations reported by buyer $i$ and by $\mathcal{A}^S$ the set of all possible valuation vectors of each buyer $i$. A valuation profile is $a = (a_i)_{i \in B} \in \mathcal{A}^{B \times S}$ where $\mathcal{A}^{B \times S} = \mathcal{A}^S \times ... \times \mathcal{A}^S$. With some abuse of notation, for any non-empty coalition of buyers $T \subseteq B$, $a_T$ and $a_{-T}$ stand for $(a_i)_{i \in T}$ and $(a_i)_{i \in B \setminus T}$, respectively. $\mathcal{A}^{T \times S}$ is the set of all valuation profiles for buyers in $T \subseteq B$. Hence, a buyer-seller market is denoted by a triplet $(B, S, a)$.

Given a non-empty set of buyers $B' \subseteq B$ and a non-empty set of sellers $S' \subseteq S$, a *matching* of the set of buyers $B'$ to the set of sellers $S'$ is denoted by $\mu$ and it consists of a subset of $B' \times S'$ such that each agent in $B' \cup S'$ appears in at most one pair of $\mu$. We denote by $\mathcal{M}(B', S')$ the set of all such matchings. Given a matching $\mu \in \mathcal{M}(B, S)$, denote by $S_\mu$, the set of sellers matched to some buyer under $\mu$, *i.e.*, $S_\mu = \{j \in S \mid \text{there is some } i \in B \text{ such that } (i, j) \in \mu\}$. Similarly, $B_\mu$ denotes the set $\{i \in B \mid \text{there is some } j \in S \text{ such that } (i, j) \in \mu\}$.

Now, we focus on the notion of outcome of a market. An outcome consists of a payoff for each agent and a matching of $B$ to $S$.

**Definition 2.1.** *Consider a buyer-seller market $(B, S, a)$. A payoff vector $(u, v) \in \mathbb{R}^B \times \mathbb{R}^S$, is called a feasible payoff vector for $(B, S, a)$ if there exists a matching $\mu \in \mathcal{M}(B, S)$ such that*

1. $u_i + v_j = a_{ij}$ *for all $(i, j) \in \mu$,*

2. $u_i = 0$ *for all $i \in B \setminus B_\mu$,*

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[4]This model could be interpreted in alternative ways, for example, as a labour market in which the set of agents are firms and workers.

*3. $v_j = 0$ for all $j \in S \setminus S_\mu$.*

*In that case, we say that $(u, v; \mu)$ is a feasible outcome for $(B, S, a)$ and $\mu$ is compatible with $(u, v)$.*

An outcome of a market then, specifies which agents are bilaterally trading. We interpret an outcome of a particular market as follows. If a buyer $i$ is matched to some seller $j$, this buyer is paying the price $v_j$ to the seller for the object she has on sale. Thus, point *1* in Definition 2.1, states that $u_i$ is the surplus of buyer $i$ when he acquires the object owned by seller $j$ and pays $v_j$. It is natural to require that if agents are not trading, their payoff is zero. Notice that transfers between nonmatched pairs are not allowed.

**Definition 2.2.** *Consider a buyer-seller market $(B, S, a)$. A matching $\mu \in \mathcal{M}(B, S)$ is optimal for $(B, S, a)$ if*

$$\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij} \quad \text{for all } \mu' \in \mathcal{M}(B, S).$$

In the sequel, we introduce the notion of stable outcomes in order to capture the idea of stable transactions among the agents.

**Definition 2.3.** *Consider a buyer-seller market $(B, S, a)$. A feasible outcome $(u, v; \mu)$ for $(B, S, a)$ is stable for $(B, S, a)$ if*

$$(i) \quad u_i \geq 0 \text{ for all } i \in B \text{ and } v_j \geq 0 \text{ for all } j \in S$$
$$(ii) \quad u_i + v_j \geq a_{ij} \text{ for all } (i, j) \in B \times S.$$

*In that case, we say that $(u, v)$ is a stable payoff vector for $(B, S, a)$.*

If an outcome $(u, v; \mu)$ is not stable in a market $(B, S, a)$, then there is a buyer $i \in B$ and a seller $j \in S$ who are not matched to each other at $\mu$ and $u_i + v_j < a_{ij}$. Then, we say that this pair of agents $(i, j) \in B \times S$ can *block* $(u, v; \mu)$. The underlying idea is that each of them can obtain a greater payoff by leaving his/her current situation and splitting their potential gain $a_{ij}$ only among themselves. It follows easily, see for instance Roth & Sotomayor (1990), that if $(u, v; \mu)$ is a stable outcome for a market $(B, S, a)$ then

$$\mu \in \mathcal{M}(B, S) \text{ is an optimal matching for } (B, S, a) \tag{1}$$

Now, let us introduce a coalitional game with transferable utility (TU-game)[5] associated with this two-sided market. Consider a buyer-seller market $(B, S, a)$. The *assignment game* $(B \cup S, w_a)$ associated with $(B, S, a)$ is a TU-game and the worth of any coalition of agents $T \subseteq B \cup S$ is given as follows. First, since we need at least one buyer and at least one seller to create a trade surplus, every coalition that contains only buyers or only sellers has zero worth, i.e. $w_a(T) = 0$ if $T \cap B = \emptyset$ or $T \cap S = \emptyset$. Otherwise, if a coalition contains at least one buyer and at least one seller, then the

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[5] A coalitional game with transferable utility $(N, w)$ is a pair formed by a finite set of players $N$ and a characteristic function $w$ that assigns a real number $w(T)$ to each coalition $T \subseteq N$, with $w(\emptyset) = 0$.

worth of the coalition is the maximal trade surplus that can be generated by matching buyer-seller pairs, i.e.

$$w_a(T) = \max_{\mu \in \mathcal{M}(T \cap B, T \cap S)} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} \text{ if } T \cap B \neq \emptyset \text{ and } T \cap S \neq \emptyset. \tag{2}$$

Within TU-games, a prominent solution concept that captures a notion of stability is the core. More precisely, the core of a TU-game $(N, w)$ is the set $C(w) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w(N), \sum_{i \in S} x_i \geq w(S) \text{ for all } S \subset N\}$ consisting of all payoff vectors such that every coalition of agents earns at least the maximal surplus the agents in the coalition can generate among themselves. It is shown in Shapley & Shubik (1972) that the core of any assignment game is always non-empty and it coincides with the set of stable payoff vectors. With respect to the structure of stable outcomes, Shapley & Shubik (1972) shows that

if $(u, v; \mu)$ and $(u', v'; \mu')$ are stable outcomes for $(B, S, a)$,
then $(u, v; \mu')$ and $(u', v'; \mu)$ are both stable for $(B, S, a)$. $\qquad$ (3)

The set of stable payoff vectors endowed with a partial order $\geq_B$, which is the usual order on the set of buyers' payoff vectors, has a complete lattice structure (dual to the lattice associated with order $\geq_S$). As an immediate consequence, there is a unique buyers-optimal stable payoff, denoted by $(\overline{u}, \underline{v}) \in \mathbb{R}^B \times \mathbb{R}^S$ such that $\overline{u} \geq u$ and $v \geq \underline{v}$ for every other stable payoff vector $(u, v) \in \mathbb{R}^B \times \mathbb{R}^S$. And similarly there is a unique sellers-optimal stable payoff $(\underline{u}, \overline{v})$ satisfying $\underline{u} \leq u$ and $v \leq \overline{v}$ for every other stable payoff vector $(u, v) \in \mathbb{R}^B \times \mathbb{R}^S$. There is a simple expression for the maximum stable payoff for any agent in the market, see for instance Roth & Sotomayor (1990). Consider a market $(B, S, a)$ and let $k$ be a seller, then her maximum stable payoff in $(B, S, a)$ is given by

$$\overline{v}_k = \max_{\mu \in \mathcal{M}(B,S)} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} - \max_{\mu \in \mathcal{M}(B,S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\}, \tag{4}$$

and the maximum stable payoff $\overline{u}_k$ for any buyer $k \in B$ is obtained analogously.

In the following, we will introduce allocation rules for buyer-seller markets with the objective of focusing on those that always produce a stable outcome.

**Definition 2.4.** *Fix a set $B$ of buyers and a set $S$ of sellers. An allocation rule $\varphi$ consists of maps $(u, v; \mu)$ from valuation profiles to feasible outcomes. That is, for each $a \in \mathcal{A}^{B \times S}$, $\varphi(a) \equiv (u(a), v(a); \mu(a))$ is a feasible outcome for $(B, S, a)$.*

Now, let us introduce stable rules as a subclass of allocation rules. A stable rule is an allocation rule that always selects stable outcomes.

**Definition 2.5.** *Fix a set $B$ of buyers and a set $S$ of sellers. An allocation rule $\varphi \equiv (u, v; \mu)$ is a stable rule if for each valuation profile $a \in \mathcal{A}^{B \times S}$, $\varphi(a) = (u(a), v(a); \mu(a))$ is a stable outcome for $(B, S, a)$.*

In the next section, we introduce some monotonicity properties in order to study some outstanding stable rules: the buyers-optimal stable rule, that for each valuation matrix selects the buyers-optimal stable payoff vector together with a compatible matching, and the sellers-optimal stable rule that selects the sellers-optimal payoff vector and a matching compatible with this vector. Notice that for each of these two rules, the associated payoff vector is uniquely determined but the compatible matching may not be unique.

# 3 Axiomatization of optimal stable rules

In this section, we introduce axioms to obtain characterizations of the optimal stable rules. The first axiom, *object-valuation antimonotonicity*, reflects the behaviour of an allocation rule when all buyers weakly decrease their valuation for a single object.

Suppose that in a market $(B, S, a)$, buyer $t \in B$ is acquiring the object owned by seller $k \in S$. Now, assume that every buyer in the market decreases his willingness to pay for the object owned by this seller $k$. This could be explained by a loss of the quality of the good. Then, if after this decrease in valuation, the rule again assigns object $k$ to buyer $t$, the payoff of buyer $t$ will not decrease. Although buyer $t$'s willingness to pay for the good owned by seller $k$ might also decrease, compared to the other buyer's decrease in valuations, the valuation of buyer $t$ is still high enough to guarantee that he is matched to seller $k$. Then, the lower valuations for the good owned by seller $k$ brings buyer $t$ in a stronger position against seller $k$.

**Definition 3.1.** *Fix a set $B$ of buyers and a set $S$ of sellers. An allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **object-valuation antimonotonicity (OVA)** if for all $a, a' \in \mathcal{A}^{B \times S}$ for which there is a $k \in S$ such that $a'_{ij} = a_{ij}$ for all $(i, j) \in B \times (S \setminus \{k\})$ and $a'_{ik} \leq a_{ik}$ for all $i \in B$,*

$$(t, k) \in \mu(a') \cap \mu(a) \Rightarrow u_t(a') \geq u_t(a).$$

Notice that this is a weak form of antimonotonicity since it only applies if the object is assigned again to the same buyer. It turns out that among the stable rules, the sellers-optimal stable rule is the only one satisfying object-valuation antimonotonicity.

**Theorem 3.2.** *Fix a set $B$ of buyers, a set $S$ of sellers and let $\varphi \equiv (u, v; \mu)$ be a stable rule. For each $a \in \mathcal{A}^{B \times S}$, the payoff vector $(u(a), v(a))$ is the sellers-optimal stable payoff vector for $(B, S, a)$ if and only if $\varphi$ satisfies OVA.*

*Proof.* The "if" part. We prove that if a stable rule $\varphi \equiv (u, v; \mu)$ satisfies OVA, then for each valuation profile $a \in \mathcal{A}^{B \times S}$, the payoff vector $(u(a), v(a))$ is the sellers-optimal stable payoff vector for the market $(B, S, a)$. Assume by way of contradiction that for some valuation profile $a \in \mathcal{A}^{B \times S}$ and for some seller $k \in S$, we have that $v_k(a) < \overline{v}_k$ where $\overline{v}_k$ is seller $k$'s maximum stable payoff in $(B, S, a)$. Since $\varphi$ is a stable rule and because of the lattice structure of the set of stable payoff vectors, we have that $0 \leq \underline{v}_k \leq v_k(a) < \overline{v}_k$. Notice that $\mu(a)$ is an optimal matching for $(B, S, a)$, see (1). Moreover, because of (3), $(\underline{u}, \overline{v}; \mu(a))$ is a stable outcome for $(B, S, a)$. Since $\overline{v}_k > 0$,

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then feasibility implies that $k$ is matched by $\mu(a)$. Let $t \in B$ be the buyer such that $(t, k) \in \mu(a)$.

Now, define the following valuation profile $a' \in \mathcal{A}^{B \times S}$ by $a'_{ij} = a_{ij}$ for all $(i, j) \in B \times (S \setminus \{k\})$, $a'_{ik} = 0$ for all $i \in B \setminus \{t\}$ and $a_{tk} - \overline{v}_k < a'_{tk} < a_{tk} - v_k(a)$. Notice that $0 \leq a'_{tk} < a_{tk}$. Now, we prove the following claim:

*Claim:* $(t, k) \in \overline{\mu}$ for all $\overline{\mu} \in \mathcal{M}(B, S)$ that is optimal for $(B, S, a')$.

First, define the following set of matchings $\mathcal{M}^- \subseteq \mathcal{M}(B, S)$ by $\mathcal{M}^- = \{\mu \in \mathcal{M}(B, S) \,|\, (t, k) \notin \mu\}$. Notice that $\mu(a) \in \mathcal{M}(B, S) \setminus \mathcal{M}^-$ where $\mu(a)$ is the matching given by the rule $\varphi$ considered at the beginning of this proof. We see that $\sum_{(i,j) \in \mu(a)} a'_{ij} > \sum_{(i,j) \in \mu'} a'_{ij}$ for all $\mu' \in \mathcal{M}^-$. Indeed,

$$
\sum_{(i,j) \in \mu(a)} a'_{ij} = \sum_{(i,j) \in \mu(a) \setminus \{(t,k)\}} a_{ij} + a'_{tk} > \sum_{(i,j) \in \mu(a)} a_{ij} - \overline{v}_k
$$

$$
= \sum_{(i,j) \in \mu(a)} a_{ij} - \left( \sum_{(i,j) \in \mu(a)} a_{ij} - \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} \right)
$$

$$
= \max_{\mu \in \mathcal{M}^-} \left\{ \sum_{(i,j) \in \mu} a'_{ij} \right\}, \tag{5}
$$

where the first inequality follows from $a'_{tk} > a_{tk} - \overline{v}_k$, the second equality comes from expression (4) for the maximum stable payoff and the third one from definition of $a'$. Therefore, as a consequence of expression (5), we can guarantee that, if $\overline{\mu} \in \mathcal{M}(B, S)$ is an optimal matching for $(B, S, a')$, then it must hold $(t, k) \in \overline{\mu}$ and the claim is proved.

Now, since $\varphi$ is a stable rule, if we apply $\varphi$ to $a' \in \mathcal{A}^{B \times S}$ we know that $\mu(a')$ is optimal for $(B, S, a')$. Hence, as a consequence of the previous claim, we have that $(t, k) \in \mu(a) \cap \mu(a')$ and by the axiom OVA, we obtain

$$
u_t(a') \geq u_t(a).
$$

Making use of the fact that $\varphi$ is a stable rule and stable outcomes are feasible outcomes, we have

$$
u_t(a') = a'_{tk} - v_k(a') \quad \text{and} \quad u_t(a) = a_{tk} - v_k(a).
$$

Hence, we obtain $a'_{tk} - v_k(a') = u_t(a') \geq u_t(a) = a_{tk} - v_k(a)$, which implies that $v_k(a') \leq a'_{tk} + v_k(a) - a_{tk} < a_{tk} - v_k(a) + v_k(a) - a_{tk} = 0$. This contradicts the fact that $\varphi$ selects a stable outcome because we should have $v_k(a') \geq 0$ by condition (i) in Definition 2.3. We conclude that our assumption that $v_k(a) < \overline{v}_k$ for some $k \in S$ does not hold, and thus for each valuation profile $a \in \mathcal{A}^{B \times S}$, the payoff vector $(u(a), v(a))$ is the sellers-optimal stable payoff vector for the market $(B, S, a)$.

The "only if" part. We prove that if $\varphi_* \equiv (\underline{u}, \overline{v}; \mu_*)$ is such that for each $a \in \mathcal{A}^{B \times S}$, the payoff vector $(\underline{u}(a), \overline{v}(a))$ is the sellers-optimal stable payoff vector for $(B, S, a)$,

then $\varphi_*$ satisfies OVA. Let $a, a' \in \mathcal{A}^{B \times S}$ and $k \in S$ be such that $a'_{ij} = a_{ij}$ for all $(i, j) \in B \times (S \setminus \{k\})$ and $a'_{ik} \leq a_{ik}$ for all $i \in B$. Moreover, assume that $(t, k) \in \mu_*(a') \cap \mu_*(a)$. By expression (4) of the maximum stable payoff, we have

$$
\begin{aligned}
\underline{u}_t(a) &= a_{tk} - \overline{v}_k(a) = a_{tk} - \sum_{(i,j) \in \mu_*(a)} a_{ij} + \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} \\
&= \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} - \max_{\mu \in \mathcal{M}(B \setminus \{t\}, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} \\
&= \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a'_{ij} \right\} - \max_{\mu \in \mathcal{M}(B \setminus \{t\}, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a'_{ij} \right\} \\
&= a'_{tk} - \sum_{(i,j) \in \mu_*(a')} a'_{ij} + \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a'_{ij} \right\} = a'_{tk} - \overline{v}_k(a') = \underline{u}_t(a'),
\end{aligned}
$$
(6)

where the first equality comes from the feasibility of stable outcomes, the third one because

$$
\mu^*(a) \setminus \{(t, k)\} \in \operatorname{argmax}_{\mu \in \mathcal{M}(B \setminus \{t\}, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\},
$$

and the fifth and seventh follow from a similar reasoning with respect to $a'$. Therefore $\varphi_*$ satisfies OVA and this completes the proof. $\qquad\square$

The reader will easily see that we can obtain a sort of "dual" axiomatization for the buyers-optimal stable rule. To this end, we introduce the axiom of *buyer-valuation monotonicity* which refers to the behavior of an allocation rule when a single buyer weakly decreases his valuations of all objects.

**Definition 3.3.** *Fix a set $B$ of buyers and a set $S$ of sellers. An allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **buyer-valuation monotonicity (BVM)** if for all $a, a' \in \mathcal{A}^{B \times S}$ for which there is a $t \in B$ such that $a'_{ij} = a_{ij}$ for all $(i, j) \in (B \setminus \{t\}) \times S$ and $a'_{tj} \leq a_{tj}$ for all $j \in S$,*

$$
(t, k) \in \mu(a') \cap \mu(a) \Rightarrow u_t(a') \leq u_t(a).
$$

Buyer-valuation monotonicity states that a buyer cannot be better off by weakly decreasing his valuations for all objects, as long as this buyer is assigned the same object in both situations. Then, an immediate consequence of Theorem 3.2, by simply interchanging the roles of buyers and sellers in its proof, is the following axiomatization of the buyers-optimal stable rule.

**Theorem 3.4.** *Fix a set $B$ of buyers, a set $S$ of sellers and let $\varphi \equiv (u, v; \mu)$ be a stable rule. For each $a \in \mathcal{A}^{B \times S}$, the payoff vector $(u(a), v(a))$ is the buyers-optimal stable payoff vector for every buyer-seller market $(B, S, a)$ if and only if $\varphi$ satisfies BVM.*

The above monotonicity axioms, OVA and BVM, refer to how payoffs of the rule react to the change of some valuation. This is different from manipulability axioms where some agents may report false valuations in search of a higher surplus. The following definition states a notion of manipulability by a group of buyers.

**Definition 3.5.** *An allocation rule $\varphi \equiv (u, v; \mu)$ is manipulable by a non-empty group of buyers $B' \subseteq B$ at $a \in \mathcal{A}^{B \times S}$ if there is a profile $a'_{B'} \in \mathcal{A}^{B' \times S}$ such that for each $i \in B'$, the following two conditions hold:*

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1. *there is $j \in S$ such that $(i, j) \in \mu(a'_{B'}, a_{-B'})$,*

2. *$a_{ij} - v_j(a'_{B'}, a_{-B'}) > u_i(a)$.*

**Definition 3.6.** *Fix a set $B$ of buyers and a set $S$ of sellers. A stable rule $\varphi \equiv (u, v; \mu)$ is **buyers strategy-proof (BSP)** if it is not manipulable by any group of buyers $B' \subseteq B$ at any $a \in \mathcal{A}^{B \times S}$.*

It is known, from Pérez-Castrillo & Sotomayor (2017), that the property of buyers strategy-proofness also characterizes the buyers-optimal competitive-equilibrium rule among the class of all competitive-equilibrium rules. Since in the Shapley & Shubik (1972) assignment game the set of stable payoff vectors coincides with the set of competitive equilibrium payoff vectors, the following characterization of the buyers-optimal stable rule in terms of buyers strategy-proofness is straightforward.

**Theorem 3.7.** *(Pérez-Castrillo & Sotomayor, 2017) For any set of buyers $B$ and any set of sellers $S$, let $\varphi \equiv (u, v; \mu)$ be a stable rule. The vector $(u(a), v(a))$ is the buyers-optimal stable payoff vector for any valuation profile $a \in \mathcal{A}^{B \times S}$ if and only if $\varphi$ is BSP.*

Buyer-valuation monotonicity seems a rather weak axiom, but it turns out to be strong enough to characterize the buyers-optimal rule among the stable rules. On the other hand, buyers strategy-proofness, requiring non manipulability by groups of buyers, seems rather strong, but it is weak enough for existence of a stable rule that satisfies this property.

For more general preferences, not necessarily quasi-linear, and allowing for budget constraints, Miyake (1998) also proves that the buyers-optimal auction is the only rule that is (individually) strategy-proof among those rules that select a stable allocation. The fact that the buyers-optimal stable rule satisfies buyers strategy-proofness can also be deduced from a result in Demange & Gale (1985) for a related model with more general preferences.

# 4 Axiomatization of the fair division rule

Apart from the two optimal stable rules analyzed until now, another outstanding stable rule for assignment markets is the rule that produces the *fair division point* (Thompson, 1981). Given an assignment market $(B, S, a)$, the payoff vector of the fair division rule $\varphi^\tau \equiv (u^\tau, v^\tau; \mu)$ satisfies $u^\tau(a) = \frac{1}{2}(\overline{u}(a) + \underline{u}(a))$ and $v^\tau(a) = \frac{1}{2}(\overline{v}(a) + \underline{v}(a))$. The stability of the fair division rule follows straightforwardly since its payoff vector is the midpoint of two stable payoff vectors. Núñez & Rafels (2002) proves that the fair division rule of an assignment market $(B, S, a)$ equals the $\tau$-value (Tijs, 1981) of the related assignment game $(B \cup S, w_a)$ and that this rule satisfies pairwise monotonicity.

**Definition 4.1.** *Fix a set $B$ of buyers and a set $S$ of sellers. An allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **pairwise monotonicity (PM)** if for all $a, a' \in \mathcal{A}^{B \times S}$ such that there is a buyer-seller pair $(t, k) \in B \times S$, $a'_{ij} = a_{ij}$ for all $(i, j) \in B \times S \setminus \{(t, k)\}$ and $a'_{tk} \leq a_{tk}$, it holds that*

$$u_t(a') \leq u_t(a) \text{ and } v_k(a') \leq v_k(a).$$

Pairwise monotonicity states that if only one buyer weakly decreases his valuation for an object, then nor this buyer neither the owner of the object can be better off. In fact, in the aforementioned paper, the pairwise monotonicity of the fair division rule is obtained as a consequence of the pairwise monotonicity of the two optimal stable rules.

It is shown in van den Brink & Pintér (2015) that also the Shapley value is pairwise monotonic on the domain of assignment games. Moreover, these authors show that the Shapley value is the only solution for assignment games that satisfies *submarket efficiency* and *valuation fairness*.

Valuation fairness requires that when a single valuation is modified, both the buyer and the seller that owns the object see their payoff changed by the same amount.

**Definition 4.2.** *Fix a set $B$ of buyers and a set $S$ of sellers. An allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **valuation fairness (VF)** if for all $a, a' \in \mathcal{A}^{B \times S}$ such that there is a buyer-seller pair $(t, k) \in B \times S$, $a'_{ij} = a_{ij}$ for all $(i, j) \in (B \setminus \{t\}) \times (S \setminus \{k\})$ and $a'_{tk} \leq a_{tk}$, it holds that*

$$u_t(a') - u_t(a) = v_k(a') - v_k(a).$$

Given a buyer-seller market $(B, S, a)$, a submarket is determined by a subset of buyers $B' \subseteq B$ and a subset of sellers $S' \subseteq S$ such that every buyer in $B'$ has value zero for the objects that belong to sellers in $S \setminus S'$, and buyers in $B \setminus B'$ have value zero for the objects that belong to sellers in $S'$. A payoff vector for market $(B, S, a)$ is submarket efficient if, for every submarket, the total payoff to agents in the submarket equals the worth $w_a(B' \cup S')$ of the submarket.

Notice that no stable rule satisfies VF, since, by the definition of the core, they all are submarket efficient. Hence, if a stable rule also satisfied VF, then, by van den Brink & Pintér (2015), its payoff vector should coincide with the Shapley value which contradicts that the rule is stable.

However, from our Theorem 3.2 and Theorem 3.4, we deduce that the fair division rule satisfies a weaker form of valuation fairness which, similar as OVA and BVM, requires equal changes only for matched pairs.

**Definition 4.3.** *Fix a set $B$ of buyers and a set $S$ of sellers. An allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **weak valuation fairness (WVF)** if for all $a, a' \in \mathcal{A}^{B \times S}$ such that there is a buyer-seller pair $(t, k) \in B \times S$, $a'_{ij} = a_{ij}$ for all $(i, j) \in (B \setminus \{t\}) \times (S \setminus \{k\})$ and $a'_{tk} \leq a_{tk}$, it holds that*

$$(t, k) \in \mu(a) \cap \mu(a') \Rightarrow u_t(a') - u_t(a) = v_k(a') - v_k(a).$$

**Proposition 4.4.** *The fair division rule satisfies WVF in assignment markets with set of buyers $B$ and set of sellers $S$.*

*Proof.* Let $\varphi^\tau \equiv (u^\tau, v^\tau, \mu)$ be the fair division rule. Fix a set $B$ of buyers and a set $S$ of sellers, and $a, a' \in \mathcal{A}^{B \times S}$ such that $a'_{ij} = a_{ij}$ for all $(i, j) \in (B \setminus \{t\}) \times (S \setminus \{k\})$ and $a'_{tk} \leq a_{tk}$. Assume $(t, k) \in \mu(a) \cap \mu(a')$. In the proof of Theorem 3.2, showing that the sellers-optimal stable rule satisfies OVA, we get $\underline{u}_t(a') = \underline{u}_t(a)$ (see (6)) which, with the fact that stable outcomes are feasible, implies that $\overline{v}_k(a) - \overline{v}_k(a') = a_{tk} - a'_{tk}$. Similarly, from the proof that the buyers-optimal stable rule satisfies BVM, we obtain

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$\underline{v}_k(a') = \underline{v}_k(a)$ and hence $\bar{u}_t(a) - \bar{u}_t(a') = a_{tk} - a'_{tk}$. By definition of the fair division rule $\varphi^\tau$ we have

$$u_t^\tau(a') - u_t^\tau(a) = v_k^\tau(a') - v_k^\tau(a) = \frac{1}{2}(a'_{tk} - a_{tk}). \tag{7}$$

$\square$

Notice from (7) that when a single valuation is increased (or decreased), the payoff of the fair division rule to the two related agents increases (or decreases) by the same amount.

Moreover, the next theorem shows that on the domain of stable rules on buyer-seller markets with a fixed set of agents, WVF characterizes the fair division rule.

**Theorem 4.5.** *Fix a set of buyers $B$ and a set of sellers $S$, and let $\varphi \equiv (u, v; \mu)$ be a stable rule. The payoff vector $(u(a), v(a))$ is the fair division point of $(B, S, a)$ for each $a \in \mathcal{A}^{B \times S}$ if and only if $\varphi$ satisfies WVF.*

*Proof.* Let $\varphi \equiv (u, v; \mu)$ be a stable rule that satisfies WVF and denote by $(u^\tau(a), v^\tau(a))$ the fair division point for the buyer-seller market $(B, S, a)$, for any $a \in \mathcal{A}^{B \times S}$. Assume that $\varphi$ is not the fair division rule. Then, without loss of generality we can assume there is a $k \in S$ and $a^0 \in \mathcal{A}^{B \times S}$ such that

$$0 \leq v_k(a^0) < v_k^\tau(a^0). \tag{8}$$

Let us write $\varepsilon = v_k^\tau(a^0) - v_k(a^0) > 0$. Notice that $v_k^\tau(a^0) > 0$ implies that $k$ is matched by $\mu(a^0)$. Let $t \in B$ be such that $(t, k) \in \mu(a^0)$.

Starting with $a^0$, define recursively for $r = 1, 2, ...,$ a sequence of valuation profiles $a^r \in \mathcal{A}^{B \times S}$, as long as $v_k(a^{r-1}) < v_k^\tau(a^{r-1})$, by

$$\begin{aligned}
a_{ij}^r &= a_{ij}^{r-1} && \text{for all } (i, j) \in B \times (S \setminus \{k\}), \\
a_{ik}^r &= 0 && \text{for all } i \in B \setminus \{t\}, \text{ and} \\
a_{tk}^r &= a_{tk}^{r-1} - v_k^\tau(a^{r-1}). && 
\end{aligned} \tag{9}$$

So, in all these valuation profiles, all buyers except $t$ have valuation zero for the object owned by seller $k$, while buyer $t$'s valuation for the object owned by seller $k$ is updated by subtracting the fair division payoff of the seller from the valuation in the previous valuation profile. All other valuations stay the same.

Notice that, for all $r \geq 1$, $a_{tk}^r \geq 0$ since the fair division rule is a stable rule and as a consequence $a_{tk}^r = a_{tk}^{r-1} - v_k^\tau(a^{r-1}) = u_t^\tau(a^{r-1}) \geq 0$. Moreover,

$$a_{tk}^r = a_{tk}^{r-1} - v_k^\tau(a^{r-1}) < a_{tk}^{r-1} - v_k(a^{r-1}) \tag{10}$$

since we continue as long as $v_k(a^{r-1}) < v_k^\tau(a^{r-1})$.

We first prove the following claim.

*Claim:* For all $r \geq 1$,

    a) $(t, k) \in \bar{\mu}$ for any $\bar{\mu}$ that is optimal for $(B, S, a^r)$,
    b) $v_k^\tau(a^r) - v_k(a^r) = v_k^\tau(a^0) - v_k(a^0) = \varepsilon$, and
    c) $v_k^\tau(a^r) = \frac{1}{2^r} v_k^\tau(a^0)$

We prove the claim by induction on $r$. Consider first $r = 1$. To this end, define the following set of matchings $\mathcal{M}^- \subseteq \mathcal{M}(B, S)$ by $\mathcal{M}^- = \{\mu \in \mathcal{M}(B, S) \,|\, (t, k) \notin \mu\}$. Notice that $\mu(a^0) \in \mathcal{M}(B, S) \setminus \mathcal{M}^-$. We will see that $\sum_{(i,j) \in \mu(a^0)} a_{ij}^1 > \sum_{(i,j) \in \mu'} a_{ij}^1$ for all $\mu' \in \mathcal{M}^-$. Indeed,

$$
\sum_{(i,j) \in \mu(a^0)} a_{ij}^1 = \sum_{(i,j) \in \mu(a^0) \setminus \{(t,k)\}} a_{ij}^0 + a_{tk}^1 > \sum_{(i,j) \in \mu(a^0)} a_{ij}^0 - \overline{v}_k(a^0)
$$

$$
= \sum_{(i,j) \in \mu(a^0)} a_{ij}^0 - \left( \sum_{(i,j) \in \mu(a^0)} a_{ij}^0 - \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij}^0 \right\} \right)
$$

$$
= \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij}^1 \right\} = \max_{\mu \in \mathcal{M}^-} \left\{ \sum_{(i,j) \in \mu} a_{ij}^1 \right\}, \tag{11}
$$

where

(i) the first inequality follows from $a_{tk}^1 = a_{tk}^0 - v_k^\tau(a^0) > a_{tk}^0 - \overline{v}_k(a^0)$ which in its turn follows from $v_k^\tau(a^0) \le \overline{v}_k(a^0)$ by stability of the fair division rule, and $v_k^\tau(a^0) = \overline{v}_k(a^0)$ would imply $v_k^\tau(a^0) = \overline{v}_k(a^0) = \underline{v}_k(a^0)$ which is in contradiction with the assumption (8),

(ii) the second equality comes from expression (4) for the maximum stable payoff and

(iii) the third (and fourth) equality follows from the definition of $a_{ik}^1$ for all $i \in B \setminus \{t\}$.

Therefore, as a consequence of expression (11), we can guarantee that, if $\overline{\mu} \in \mathcal{M}(B, S)$ is an optimal matching for $(B, S, a^1)$, then it must hold that $(t, k) \in \overline{\mu}$ and (a) is proved for $r = 1$.

Now, since $\varphi$ is a stable rule, $\mu(a^1)$ is optimal for $(B, S, a^1)$ and (a) guarantees that $(t, k) \in \mu(a^0) \cap \mu(a^1)$. Then WVF of $\varphi$ implies

$$
u_t(a^0) - u_t(a^1) = v_k(a^0) - v_k(a^1)
$$

and again from stability of $\varphi$ we have $u_t(a^0) = a_{tk}^0 - v_k(a^0)$ and $u_t(a^1) = a_{tk}^1 - v_k(a^1)$, which leads to

$$
v_k(a^0) - v_k(a^1) = \frac{1}{2}(a_{tk}^0 - a_{tk}^1) = v_k^\tau(a^0) - v_k^\tau(a^1),
$$

where the second equality follows from (7). As a consequence, $v_k^\tau(a^1) - v_k(a^1) = v_k^\tau(a^0) - v_k(a^0) = \varepsilon$ and (b) is proved for $r = 1$.

Finally, by definition of $a^1$ in (9),

$$
v_k^\tau(a^1) - v_k^\tau(a^0) = \frac{1}{2}a_{tk}^1 - \frac{1}{2}a_{tk}^0 = \frac{1}{2}a_{tk}^0 - \frac{1}{2}v_k^\tau(a^0) - \frac{1}{2}a_{tk}^0 = -\frac{1}{2}v_k^\tau(a^0)
$$

which gives $v_k^\tau(a^1) = \frac{1}{2}v_k^\tau(a^0)$ and (c) is proved for $r = 1$.

To prove the claim for $r > 1$, we assume as induction hypothesis that $a^{r-1}$ satisfies (a), (b) and (c) and prove that $a^r$ also does. The proof of part (a) is analogous to the one in case $r = 1$. Now, stability of $\varphi$ guarantees that $\mu(a^r)$ is optimal for $(B, S, a^r)$ and hence $(t, k) \in \mu(a^r) \cap \mu(a^{r-1})$. Then, since $\varphi$ satisfies WVF we obtain $u_t(a^{r-1}) - u_t(a^r) =$

$v_k(a^{r-1}) - v_k(a^r)$ and again from stability of $\varphi$ we have $u_t(a^{r-1}) = a_{tk}^{r-1} - v_k(a^{r-1})$ and $u_t(a^r) = a_{tk}^r - v_k(a^r)$, which leads to

$$v_k(a^{r-1}) - v_k(a^r) = \frac{1}{2}(a_{tk}^{r-1} - a_{tk}^r) = v_k^\tau(a^{r-1}) - v_k^\tau(a^r),$$

where the second equality follows from (7). As a consequence, $v_k^\tau(a^r) - v_k(a^r) = v_k^\tau(a^{r-1}) - v_k(a^{r-1}) = \varepsilon$ and $v_k^\tau(a^r) = \frac{1}{2}v_k^\tau(a^{r-1}) = \frac{1}{2^r}v_k^\tau(a^0)$, showing (b) and (c).

This concludes the proof of the claim.

Once the claim is proved, notice that since $v_k^\tau(a^0) \geq \varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $v_k^\tau(a^m) = \frac{1}{2^m}v_k^\tau(a^0) < \varepsilon$. Then, $v_k^\tau(a^m) - \varepsilon = v_k(a^m) < 0$, which contradicts that $\varphi(a^m)$ is a stable outcome. $\qquad\square$

# 5 Axioms for the optimal stable rules on the general domain of rules and variable population

The characterization results presented in Section 3 for the two optimal stable rules are obtained on the class of assignment markets with a fixed set of agents $B \cup S$, and among the set of stable allocation rules for these markets. Next, we show that the previous results easily provide characterizations of the two optimal stable rules among the set of general allocation rules, without imposing stability. To this end, we will make use of a consistency property and hence we will allow for a variable population. Moreover, as in Owen (1992), in order to guarantee that the reduced market remains in the class of assignment markets, we enlarge this class by allowing for individual reservation values, which stand for the gain of an agent when he remains unassigned.

Let $U^b$ and $U^s$ be the universe of buyers and sellers respectively. Given a set of buyers $B \subseteq U^b$ and a set of sellers $S \subseteq U^s$, each buyer $i \in B$ has a non-negative valuation $a_{ij} \in \mathbb{R}$ for the object of seller $j \in S$, and also a reservation value $a_{i0} \geq 0$. Each seller $j \in S$ has also a reservation value $a_{0j} \geq 0$ for his own object. By introducing a fictitious agent on each side of the market, we summarize these valuations in a matrix $a = (a_{ij})_{(i,j) \in B^0 \times S^0}$, where $B^0$ and $S^0$ are the set of buyers and sellers, respectively, enlarged with the fictitious agents, and by convention $a_{00} = 0$. A triple $(B, S, a)$ as defined above is called a *buyer-seller assignment market with reservation values*.

Given a non-empty subset of buyers $B' \subseteq B$ and a non-empty subset of sellers $S' \subseteq S$, a matching is now a partition of $B' \cup S'$ in mixed pairs and singletons. With some abuse of notation we continue denoting by $\mathcal{M}(B', S')$ the set of matchings for $B'$ and $S'$. Given a market $(B, S, a)$, a matching $\mu$ is optimal if the addition of the values of the elements of partition $\mu$ is not less than that of any other matching $\mu' \in \mathcal{M}(B, S)$.

Now, the notions of feasible outcome and stable outcome follow easily for this setting.

**Definition 5.1.** *Consider a buyer-seller assignment market with reservation values* $(B, S, a)$. *A payoff vector* $(u, v) \in \mathbb{R}^B \times \mathbb{R}^S$, *is called a feasible payoff vector for* $(B, S, a)$ *if there exists a matching* $\mu \in \mathcal{M}(B, S)$ *such that*

$(i)$ $u_i + v_j = a_{ij}$ *for all* $i \in B$, $j \in S$ *such that* $\{i, j\} \in \mu$ *and*
$(ii)$ $u_k = a_{k0}$ *if* $k \in B$, $\{k\} \in \mu$, $\quad v_k = a_{0k}$ *if* $k \in S$, $\{k\} \in \mu$.

*Then we say that $(u, v; \mu)$ is a feasible outcome for $(B, S, a)$ and $\mu$ is compatible with $(u, v)$.*

**Definition 5.2.** *Consider a buyer-seller assignment market with reservation values $(B, S, a)$. A feasible outcome $(u, v; \mu)$ for $(B, S, a)$ is stable for $(B, S, a)$ if*

$$(i) \quad u_i \geq a_{i0} \text{ for all } i \in B, v_j \geq a_{0j} \text{ for all } j \in S \text{ and}$$
$$(ii) \quad u_i + v_j \geq a_{ij} \text{ for all } i \in B \text{ and } j \in S.$$

*Then we say that $(u, v)$ is a stable payoff vector for $(B, S, a)$.*

To each buyer-seller assignment market with reservation values $(B, S, a)$, we associate a TU game $(B \cup S, w_a)$ in a way similar to (2). We first define $w_a(\{i\}) = a_{i0}$ for all $i \in B$, $w_a(\{j\}) = a_{0j}$ for all $j \in S$ and $w_a(\{i, j\}) = a_{ij}$ for all $(i, j) \in B \times S$. Then, for all $\emptyset \neq T \subseteq B \cup S$,

$$w_a(T) = \max_{\mu \in \mathcal{M}(T \cap B, T \cap S)} \sum_{R \in \mu} w_a(R).$$

The core of this assignment game with reservation values coincides with the set of stable payoff vectors of $(B, S, a)$ and also has a lattice structure with an optimal stable payoff for each side of the market.

Consistency is a standard property used to analyze the behavior of allocation rules with respect to a reduction of the population. A notion of reduced market for assignment markets with reservation values is the *derived market* of Owen (1992). Consistency with respect to this reduced assignment market is used in Llerena *et al.* (2015) to characterize the nucleolus. In the derived assignment market relative to a coalition $T$, only the buyers and sellers who belong to $T$ are active, values for the objects 'that are still in the market' are the same as in the original market, and the individual reservation values are modified taking into account the possibilities to trade with agents outside the derived market.

**Definition 5.3.** *Let $(B, S, a)$ be an assignment market, $\emptyset \neq T \subseteq B \cup S$ and $z = (u, v) \in \mathbb{R}^B \times \mathbb{R}^S$. The derived assignment market relative to $T$ at $z$ is $(B \cap T, S \cap T, a^{T,z})$ where $a_{ij}^{T,z} = a_{ij}$ for all $(i, j) \in (B \cap T) \times (S \cap T)$ and*

$$(i) \quad a_{i0}^{T,z} = \max\left\{a_{i0}, \max_{j \in S \setminus T}\{a_{ij} - v_j\}\right\}, \text{ for all } i \in B \cap T,$$
$$(ii) \quad a_{0j}^{T,z} = \max\left\{a_{0j}, \max_{i \in B \setminus T}\{a_{ij} - u_i\}\right\}, \text{ for all } j \in S \cap T.$$

**Definition 5.4.** *On the domain of assignment markets with reservation values, an allocation rule $\varphi = (u, v; \mu)$ assigns to each market $(B, S, a)$ an outcome*

$$\varphi(B, S, a) \equiv (u(B, S, a), v(B, S, a); \mu(B, S, a))$$

*that is feasible for this market.*

Derived consistency means that in a derived market, the payoffs for the buyers and sellers that are still in the market do not change.

**Definition 5.5.** *On the domain of assignment markets with reservation values, an allocation rule $\varphi \equiv (u, v; \mu)$ is **derived consistent (DC)** if for all markets $(B, S, a)$ and all coalitions $\emptyset \neq T \subseteq B \cup S$,*

$$u_i(B \cap T, S \cap T, a^{T, z^{\varphi(B,S,a)}}) = u_i(B, S, a) \qquad \text{for all } i \in B \cap T \text{ and}$$
$$v_j(B \cap T, S \cap T, a^{T, z^{\varphi(B,S,a)}}) = v_j(B, S, a) \qquad \text{for all } j \in S \cap T,$$

*where $z^{\varphi(B,S,a)} = (u, v)$ if $\varphi(B, S, a) = (u, v)$.*

**Theorem 5.6.** *On the domain of assignment markets with reservation values,*

1. *the sellers-optimal stable rules are the only ones that satisfy DC and OVA,*

2. *the buyers-optimal stable rules are the only ones that satisfy DC and BVM.*

*Proof.* We sketch the proof for the buyers-optimal stable rule, since the proof for the sellers-optimal stable rule is analogous. From Proposition 2 in Llerena *et al.* (2015), on the class of assignment markets with reservation values, derived consistency implies core selection, i.e. stability of the rule. Together with Theorem 3.2 in the previous section[6], this guarantees the uniqueness of the rule's payoff vector, that is, the buyers-optimal stable rules are the only stable rules that can be DC and BVM. It only remains to prove that indeed these rules are DC.

Take a buyer-seller market $(B, S, a)$ and apply a buyers-optimal stable rule $\varphi(B, S, a) = (\overline{u}(a), \underline{v}(a); \mu(a))$, and let $z^{\varphi(B,S,a)} = (\overline{u}(a), \underline{v}(a))$ be the buyers-optimal stable payoff. The derived market at $\emptyset \neq T \subseteq B \cup S$ and $z^{\varphi(B,S,a)} = (\overline{u}(a), \underline{v}(a))$ is $(B \cap T, S \cap T, a^{T, z^{\varphi(B,S,a)}})$. We denote by $z_{|T}^{\varphi(B,S,a)} = (\overline{u}(a)_{|T}, \underline{v}(a)_{|T})$ the restriction of $z^{\varphi(B,S,a)}$ to agents in $T$. Since the core of the assignment game satisfies derived consistency, see Llerena *et al.* (2015), we have that $z_{|T}^{\varphi(B,S,a)} = (\overline{u}(a)_{|T}, \underline{v}(a)_{|T})$ is a stable payoff vector for the derived market. We show that $z_{|T}^{\varphi(B,S,a)}$ is the buyers-optimal stable payoff vector of the derived market $(B \cap T, S \cap T, a^{T, z^{\varphi(B,S,a)}})$.

Assume on the contrary that there exists a stable payoff vector $(u', v')$ of $(B \cap T, S \cap T, a^{T, z^{\varphi(B,S,a)}})$ and $i_0 \in B \cap T$ such that $u'_{i_0} > \overline{u}_{i_0}(a)$. Define then the following payoff vector: $(u'', v'') \in \mathbb{R}^B \times \mathbb{R}^S$ with

$$u''_i = u'_i \text{ for all } i \in B \cap T; \quad u''_i = \overline{u}_i(a) \text{ for all } i \in B \setminus T,$$
$$v''_j = v'_j \text{ for all } j \in S \cap T; \quad v''_j = \underline{v}_j(a) \text{ for all } j \in S \setminus T.$$

We want to see that $(u'', v'')$ is a stable payoff vector for the initial market $(B, S, a)$. Indeed, notice first, taking Definition 5.3 into account, that $u''_i \geq a_{i0}$ for all $i \in B$ and $v''_j \geq a_{0j}$ for all $j \in S$. The reader will find easily that, since $(\overline{u}(a), \underline{v}(a); \mu)$ is a stable outcome of $(B, S, a)$, then

(a) $a_{i0}^{T, z^{\varphi(B,S,a)}} = \overline{u}_i(a)$ for all $i \in B \cap T$,

(b) $a_{0j}^{T, z^{\varphi(B,S,a)}} = \underline{v}_j(a)$ for all $j \in S \cap T$ and

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[6] Theorem 3.2 is stated and proved for buyer-seller markets with null reservation values but it is straightforward to see that it also holds if we allow for non-negative reservation values.

**(c)** the matching $\mu' \in \mathcal{M}(B \cap T, S \cap T)$ such that $\{i, j\} \in \mu'$ if and only if $\{i, j\} \in \mu$ and $\{i, j\} \subseteq T$ is optimal for $(B \cap T, S \cap T, a^{T, z^{\varphi(B,S,a)}})$.

In a derived market at a stable payoff vector, buyers are matched to the same sellers as in the original market if both buyer and seller belong to the derived market, and buyers (respectively sellers) who originally are matched to a seller (respectively buyer) outside the market, stay unassigned in the derived market but with a modified reservation value. As a consequence, $(u'', v''; \mu)$ is a feasible outcome for $(B, S, a)$. Moreover,

**(a')** For all $(i, j) \in (B \cap T) \times (S \cap T)$ and $(i, j) \in (B \setminus T) \times (S \setminus T)$, $u_i'' + v_j'' \geq a_{ij}$.

**(b')** For all $(i, j) \in (B \cap T) \times (S \setminus T)$,

$$u_i'' + v_j'' = u_i' + \underline{v}_j(a) \geq \max_{k \in S \setminus T} \{a_{ik} - \underline{v}_k(a)\} + \underline{v}_j(a) \geq a_{ij},$$

where the first inequality follows since $u_i' \geq a_{i_0}^{T, z^{\varphi(B,S,a)}} \geq \max_{k \in S \setminus T} \{a_{ik} - \underline{v}_k(a)\}$.

**(c')** For all $(i, j) \in (B \setminus T) \times (S \cap T)$,

$$u_i'' + v_j'' = \overline{u}_i(a) + v_j' \geq \overline{u}_i(a) + \max_{k \in B \setminus T} \{a_{kj} - \overline{u}_k(a)\} \geq a_{ij},$$

where the first inequality follows similar as under (b).

Since $(u'', v'')$ is a stable allocation for $(B, S, a)$, $u_{i_0}'' = u_{i_0}' > \overline{u}_{i_0}(a)$, contradicts that $\varphi$ is the buyers-optimal stable rule. Hence, $\overline{u}_i(a) = \overline{u}_i(a^{T, z^{\varphi(B,S,a)}})$ for all $i \in B \cap T$, and the buyers-optimal stable rule $\varphi$ is derived consistent. $\square$

The axioms in the above characterizations are independent. The rule that associates each valuation profile with the nucleolus payoff vector satisfies DC but neither OVA nor BVM. In order to show that a rule satisfies OVA but not DC, tag the agents in the following way, $B = \{1, 2, 3, ...\}$ and $S = \{1', 2', 3', ...\}$. Consider now the rule that associates each valuation profile with the diagonal matching ($\{i, i'\} \in \mu(a)$ for $i \in \{1, 2, ..., n\}$ where $n = \min\{|B|, |S|\}$) and the payoff vector $u_i = 0$ for all $i \in B$, $v_j = a_{ij}$ if $j \in S$ and there exists $i \in B$ such that $\{i, j\} \in \mu(a)$, $v_j = 0$ otherwise. This rule satisfies OVA but not DC. A similar rule can be defined that satisfies BVM but not DC.

Comparing Theorems 3.2 and 3.4 on one hand, and parts 1 and 2 of Theorem 5.6 on the other hand, we see that Theorems 3.2 and 3.4 characterize the sellers-optimal and buyers-optimal rules *among the stable rules*, while Theorem 5.6 characterizes these rules *among all rules*. But this comes 'at a cost', namely that we need to allow for a variable population and we need to introduce reservation values for unassigned agents. However, it is then sufficient to use derived consistency which is a very natural property for a variable population.

Since the fair division rule is not derived consistent, we cannot provide as a corollary a straightforward axiomatization of this rule in the general class of rules (not imposing stability) for variable population, as we have done for the two optimal stable rules.

# 6 Concluding remarks

In the class of all coalitional games, Young (1985) proves that core stability is incompatible with coalitional monotonicity, which states that if the worth of only one coalition is raised, no agent in this coalition can be worse off. For assignment markets, coalitional monotonicity is not an appealing property since when one buyer raises only one of his valuations, several coalitions increase their worth. Pairwise monotonicity is a more suitable monotonicity requirement for these buyer-seller markets, and it is compatible with stability since several stable rules (such as the buyers-optimal stable rule, the sellers-optimal stable rule and the fair division rule) meet this requirement.

In this paper, we have proved that stability is also compatible with other monotonicity properties such as object-valuation antimonotonicity and buyer-valuation monotonicity. Moreover, these two monotonicity properties allow to discriminate between the two optimal stable rules: the sellers-optimal stable rule, respectively the buyers-optimal stable rule. Adding derived consistency, we obtain characterizations of the two optimal stable rules in the class of assignment markets with reservation values and variable population. Notice that derived consistency is a property that these optimal stable rules have in common with the nucleolus on the same class of markets (Llerena *et al.*, 2015).

Young (1985) also introduces *strong monotonicity* for TU-games which requires that the payoff for a player does not decrease if the game changes in such a way that none of his marginal contributions decrease. In van den Brink *et al.* (2013), Young's monotonicity is weakened by requiring the payoffs of a player only to be nondecreasing if its marginal contributions do not decrease and, moreover, also the worth of the 'grand coalition' $v(N)$ does not decrease. Casajus & Huettner (2014) shows that weakening strong monotonicity in this way in Young's axiomatizations characterizes the class of egalitarian Shapley values being all convex combinations of the Shapley value and equal division solution introduced by Joosten (1996).[7] Considering assignment games, it is interesting to observe that buyer-valuation monotonicity and pairwise monotonicity are weaker than weak monotonicity since only one buyer increasing his value for the good of one or more seller, does not decrease his marginal contributions and also does not decrease the worth of the grand coalition in an assignment game. This makes buyer-valuation monotonicity and pairwise monotonicity even nicer axioms for assignment games.

Although, in two-sided markets where one side represents an institution, it may be of social interest to choose the stable allocation rule that favors the weak side of the market (students, resident doctors or bidders in an auction), in a buyer-seller market where both sides are equally important, a more balanced payoff distribution between the two sides of the market could be more reasonable. The fair division rule is characterized as the only stable rule that satisfies weak valuation fairness, which is a weaker form of valuation fairness as used in the characterization of the Shapley value for assignment games provided in van den Brink & Pintér (2015), hence establishing some kind of connection between the fair division rule and the Shapley value in buyer-seller markets.

Another weaker version of valuation fairness is based on weak differential marginality of Casajus & Yokote (2017) which weakens the fairness property of van den Brink (1991)

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[7]This is shown for games where there are at least three players.

or the equivalent differential marginality property of Casajus (2011). Applying weak differential marginality to assignment games requires that only increasing the valuation of a buyer for the good offered by one seller, changes the payoffs of this buyer and seller in the same direction (but not necessarily by the same amount). Our weak valuation fairness is neither stronger nor weaker than this weak fairness axiom. On one hand, our axiom makes a requirement on the payoffs of a buyer and a seller only if the buyer and seller stay matched while weak differential marginality makes a requirement on the payoffs irrespective whether the buyer and seller stay matched or not. On the other hand, our axiom requires equal changes, while weak differential marginality only requires changes to be in the same direction.

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