On Spectral Theory of Compatible Random Inflation Systems

Dissertation zur Erlangung des akademischen Grades eines
Doktors der Mathematik (Dr. math.)

vorgelegt von

Timo Spindeler

Fakultät für Mathematik
Universität Bielefeld

Dezember 2017

Gedruckt auf alterungsbeständigem Papier °°ISO 9706.

Contents

Introduction

General motivation

Crystals appear in many different areas of theoretical and applied science. They are often characterised by their lattice structure. Speaking in geometric terms, this means that small patterns are repeated in a periodic fashion. One of the most prominent examples is silicon, which is used in semiconductors. On the other hand, there are rubies (used in lasers), diamonds (used in industrial saws), quartzes and sapphires (used in watches) and many more.

Since last century, X-ray diffraction has become an important tool for the analysis of the inner structure of crystals by detecting long-range order through the appearance of sharp reflection spots, called Bragg peaks, in the diffraction image. The Bragg peaks do not display the symmetry of the lattice structure of the crystal itself but rather the symmetry of the dual lattice, which is underlying the crystal structure. For a long time, it was assumed that these point patterns are *the* characterising property of crystals. However, this changed when quasycrystals were discovered.

The study of quasicrystals started in 1982, when the materials scientist Dan Shechtman inspected an $\mathtt{Al}_{86}\mathtt{Mn}_{14}$ alloy with an electron microscope in diffraction mode. It is a well-known fact that the diffraction pattern of two- and three-dimensional crystals may only feature crystallographic point symmetries with d-fold rotational symmetry, where $d \in \{1, 2, 3, 4, 6\}$ [Cox61, Sec. 4.5]. To his astonishment, he observed a diffraction pattern with sharp Bragg peaks with tenfold rotational symmetry. From that he deduced the presence of long-range order beyond the realm of perfect crystals; the quasicrystals were discovered.

First, he was criticised blisteringly by some of his colleagues. It took Shechtman quite a while to convince colleagues of his discovery until the result was finally published in 1984 [SBGC84]. Later, Ishimasa, Nissen and Fukano [INF85] found empirical evidence for the existence of quasicystal structures, and, finally, in 2010 the Icosahedrite [BSYL11] was accepted by the *International Mineralogical Association*. Shechtman was awarded the Wolf Prize in Physics in 1999 and the Nobel Prize in Chemistry in 2011 for his astounding discovery.

Mathematical motivation

It does not surprise that mathematicians are interested in the study of (quasi)crystals, too. Their main goal is to solve the so called *inverse problem* which states:

> Given a diffraction pattern, can I (uniquely) determine the corresponding (quasi)crystal?

In order to do so, the first step is to model the (quasi)crystals mathematically. There are two successful approaches to generate such point configurations that feature long-range internal order. The first one creates a tiling, which is a partition $\mathcal{T} = \{T_i\}_{i \in I}$ of $\mathbb{R}^d$ such that each T_i is closed and $T_i^\circ \cap T_j^\circ = \varnothing$, by inflating and decomposing a finite set of prototiles. Placing atoms on the vertices of the tiles gives the desired point set. A famous examples in two dimensions is the Ammann–Beenker tiling. A finite patch is illustrated in Figure 0.1. The second one cuts through a higher-dimensional lattice and projects the resulting set into the Euclidean space. Each method has its own advantages, as we will see later, and we will discuss both in detail in the following chapters.

The second step is the mathematical analysis of the diffraction pattern of these crystals. Bombieri and Taylor were among the first to raise the question which distributions of matter diffract, i.e. show sharp spots (or Bragg peaks) in their diffraction patterns; see [BT86]. The mathematical framework for diffraction was set in the 1990s by Hof [Hof95]. Given a point set Λ, which represents the positions of the atoms in a solid, its autocorrelation measure γ is a positive and positive definite measure. The Fourier transform $\widehat{\gamma}$ of γ is called the *diffraction pattern* of Λ. If $\widehat{\gamma}$ is a pure point measure, we say that Λ is pure point diffractive. The key to understanding the structure of (quasi)crystals is the understanding of pure point diffraction. However, it turned out that different crystals can lead to the same diffraction pattern. Thus, there is no one-to-one correspondence between the crystals and the diffraction patterns. Gaining as much information about the crystal as possible is the best we can hope for.

As even this is a difficult task, we go one step back and try to determine the diffraction pattern of a given crystal. Despite many open problems (including the famous Pisot substitution conjecture), the structure of systems with pure point diffraction is rather well understood [BM04, Que10]. Due to recent progress [BBM10, BGG12], also the situation for various systems with diffraction spectra of mixed type has improved. Still, the understanding of spectra in the presence of entropy is only at its beginning and it is desirable to work out concrete examples. The objective of this thesis is the mathematical analysis of such examples in one and two dimensions.

A paradigmatic one-dimensional example was considered by Godrèche and Luck [GL89] in 1989. There, they extended the study of deterministic substitutions to random inflation tilings by mixing the two Fibonacci substitutions on the local level on the basis of a

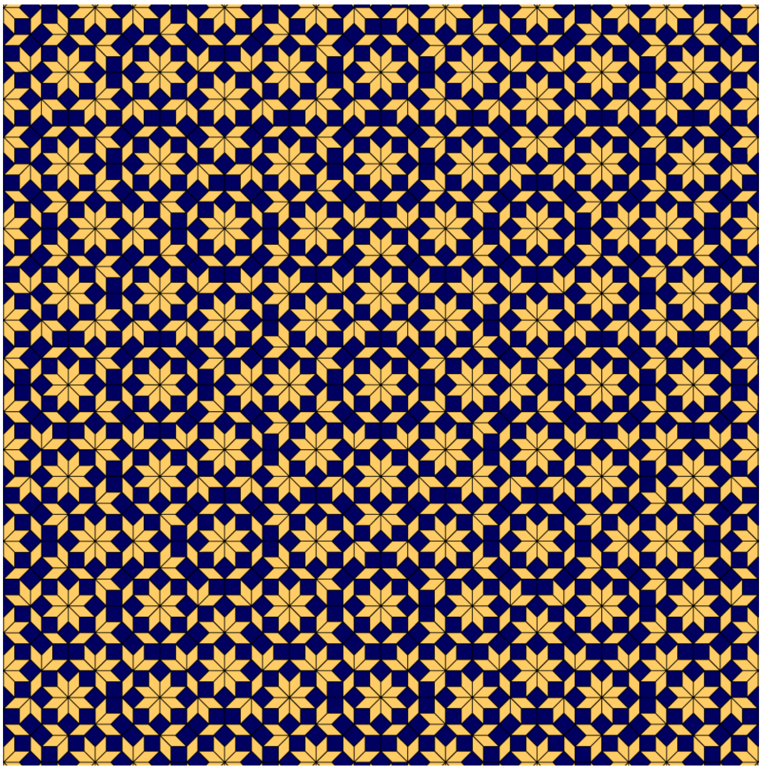

Figure 0.1: A finite patch of the Ammann–Beenker tiling. Image created by E. Harris and D. Frettlöh; image used with kind permission.

fixed probability parameter. They made heuristic suggestions for the computation of the topological entropy and the spectral type of the diffraction measure. Here, we are going to generalise this example by considering the family of random noble means substitutions. Furthermore, we study the randomised version of the period doubling chain and proceed with a two-dimensional example.

Main results of this thesis

In a deterministic setting, substitution dynamical systems are rather well understood and an extensive amount of literature has grown; see [BG13, Fog02, Lot02, Que10] and many more. For instance, given a model set with sufficiently nice window, it is well-known that the diffraction measure is pure point, and there are explicit formulas for the autocorrelation and diffraction measure [BG13, Prop. 9.8 and Thm. 9.4]. Of course, one cannot expect such optimal situations in general. Concrete results are sparse outside the realm of model sets, although there are some notable exceptions such as the Thue–Morse chain or the Rudin–Shapiro chain; the first one having a singular continuous diffraction measure, the second one having an absolutely continuous diffraction measure. Other concrete examples, concerning point processes, can be found in [BBM10] and [BKM15].

Recently, Moll started the mathematical analysis of the random noble means substitution in his Phd thesis [Mo13]. He obtained a formula for the autocorrelation measure γ_Λ for ν_m-almost all random noble means sets Λ using an ergodic measure ν_m, which acts on the continuous stochastic hull of the random noble means substitution. From this he derived the decomposition (in the special case $m = 1$ of the random Fibonacci chain)

$$\widehat{\gamma_\Lambda} = (\widehat{\gamma_\Lambda})_\ominus + (\widehat{\gamma_\Lambda})_{\mathrm{pp}} + \phi\boldsymbol{\lambda},$$

where $\phi\boldsymbol{\lambda}$ is an absolutely continuous measure (with respect to the Lebesgue measure $\boldsymbol{\lambda}$) with the known density function ϕ, $(\widehat{\gamma_\Lambda})_{\mathrm{pp}}$ is the pure point part and $(\widehat{\gamma_\Lambda})_\ominus$ is an unknown continuous measure, which he conjectured to be the null measure, in line with the implicit claims of [GL89]. This results holds almost surely with respect to the patch frequency measure.

The main result of this thesis is the exact decomposition of the diffraction measure of the random noble means substitution for arbitrary m. This includes:

1. Proving that indeed $(\widehat{\gamma_\Lambda})_\ominus \equiv 0$ and thus showing that $\widehat{\gamma_\Lambda}$ is of mixed spectral type, consisting of a pure point part and an absolutely continuous component.

2. Finding an explicit formula for the pure point part together with the determination of the set of Bragg peaks.

3. Finding an explicit formula for the absolutely continuous part.

Moreover, we will obtain the same kind of results for the random period doubling substitution and a two-dimensional example.

Acknowledgements

First and foremost, I wish to thank my supervisor Prof. Dr. Michael Baake for suggesting the topic of this thesis to me and for his persistent encouragement and motivating support.

Furthermore, I thank all former and present members of our research group: Enrico Paolo Bugarin, Dirk Frettlöh, Franz Gähler, Christian Huck, Tobias Jakobi, Chrizaldy Neil Manibo, Markus Moll, Johan Nilsson, Eden Delight Provido, Dan Rust, Venta Terauds and Peter Zeiner.

For all the non-mathematical support, I want to thank the secretary of our research group Britta Heitbreder.

Last but not least, I thank the German Research Foundation (DFG) for the financial support within the CRC 701 and the CRC 1283.

Abgrenzung des eigenen Beitrags gemäß §10(2) der Promotionsordnung

Den Inhalt der Kapitel 3, 4 und 6 hat der Autor dieser Dissertation in einer Arbeit [BSS17] gemeinsam mit seinem Betreuer, Michael Baake, und Nicolae Strungaru veröffentlicht. Diese Arbeit wurde bei der Zeitschrift Indagationes Mathematicae zur Veröffentlichung in einer Sonderausgabe zum workshop 'Aperiodic Patterns in Crystals, Numbers and Symbols' vom 19. bis 23. Juni 2017 eingereicht. Kapitel 6 entstand in Zusammenarbeit mit den anderen beiden Autoren von [BSS17], wobei alle Autoren im selben Maß zum Kapitel beigetragen haben.

1 Basics

The first chapter of this thesis is devoted to the introduction of the basic concepts of mathematical diffraction theory. After discussing the necessary methods which are used for the mathematical modeling of quasicrystals, such as point sets in $\mathbb{R}^d$, Perron–Frobenius theorem, substitution rules and model sets (Sections 1.1-1.3), we will collect tools from measure theory, functional analysis and harmonic analysis in Section 1.4 that will help us to analyse and determine the diffraction spectra of the point configurations in question. All parts of this chapter are a review of well-known material and can be found, for instance, in [BG13], [Mo13] and the references given in the text below.

1.1 Preliminaries

Before we start, let us fix some notations that will be used throughout the entire thesis.

1.1.1 Notation

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the positive integers, the integers, the rationals, the reals and the complex numbers. Additionally, we will use the notations $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := \{x \in \mathbb{R} \,|\, x > 0\}$ and $\mathbb{R}_{\geq 0} := \mathbb{R}_+ \cup \{0\}$, while the empty set is denoted by $\varnothing$.

The symbol $\dot{\cup}$ denotes the disjoint union, i.e. $A \,\dot{\cup}\, B = A \cup B$ with $A \cap B = \varnothing$ for any two sets A and B. The volume of A, with respect to the d-dimensional Lebesgue measure $\boldsymbol{\lambda}$, is referred to as $\mathrm{vol}(A)$. The interior of A is written as A°, $\overline{A}$ is its closure and ∂A its boundary (the corresponding topology will either be clear from the context or stated otherwise). The open ball of radius r with centre $x \in \mathbb{R}^d$ is denoted by $B_r(x)$. Also, we will use the notation $B_r := B_r(0)$.

The standard scalar product in $\mathbb{R}^d$ is denoted by

$$xy := \langle x \,|\, y \rangle := \sum_{i=1}^{d} x_i \, y_i$$

for $x, y \in \mathbb{R}^d$.

The ring of square $(d \times d)$-matrices over a commutative ring R is indicated by $\mathrm{Mat}(d, R)$. For $M \in \mathrm{Mat}(d, \mathbb{R})$, by $M \geq 0$, $M > 0$ or $M \gg 0$ we mean that M consists of non-negative entries only, non-negative entries with at least one being positive or of solely positive entries. The same notation will be used for elements in $\mathbb{R}^d$, too. We refer to the eigenvalue spectrum of M as $\sigma(M)$ and to the spectral radius of M as $\rho(M)$.

We denote

$$
\begin{aligned}
C(X) &:= \{f : X \to \mathbb{C} \mid f \text{ is continuous}\},\\
C_{\mathrm{B}}(X) &:= \{f \in C(X) \mid f \text{ is bounded}\},\\
C_{\mathrm{U}}(X) &:= \{f \in C_{\mathrm{B}}(X) \mid f \text{ is uniformly continuous}\},\\
C_0(X) &:= \{f \in C_{\mathrm{U}}(X) \mid f \text{ vanishes at } \infty\},\\
C_{\mathrm{c}}(X) &:= \{f \in C_0(X) \mid f \text{ has compact support}\},\\
L^p(X,\mu) &:= \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_{L^p(X,\mu)} < \infty\},
\end{aligned}
$$

where $\|f\|_{L^p(X,\mu)} := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p}$.

For $U, V \subseteq \mathbb{R}^d$, we define $U \pm V := \{u \pm v \,|\, u \in U,\ v \in V\}$, which is called the *Minkowski sum (difference)* of U and V.

1.1.2 Point sets

As mentioned in the introduction, we are interested in the study of diffraction by crystals and quasicrystals. The latter can be modeled by point sets in $\mathbb{R}^d$ with certain properties.

Here, a set consisting of one element is called a *singleton set*, and countable unions of singleton sets are called *point sets*. In particular, every finite set is a point set. However (for mathematical reasons), we will be interested in infinite point sets only, although experimentalists work, of course, with finite samples. A point set Λ is said to be *discrete* if each element $x \in \Lambda$ has an open neighbourhood $U = U(x) \subseteq \mathbb{R}^d$ such that $U \cap \Lambda = \{x\}$. The point set Λ is called *uniformly discrete* if there is an open neighbourhood U of $0 \in \mathbb{R}^d$ such that $(x + U) \cap (y + U) = \varnothing$ for all distinct $x, y \in \Lambda$. Moreover, Λ is *relatively dense* if there is an $R > 0$ such that $\mathbb{R}^d = \Lambda + \overline{B_R}$.

Definition 1.1. A point set $\Lambda \subseteq \mathbb{R}^d$ is a *Delone set* if it is uniformly discrete and relatively dense.

These two properties seem to be natural when dealing with (quasi)crystals. This is because uniform discreteness implies the existence of a minimum distance between distinct points, while relatively denseness implies that the distance between neighbouring points cannot become arbitrarily large. Thus, Delone sets are reasonable mathematical models for (quasi)crystals. Still, sometimes it is desirable to work with sets of the following type.

Definition 1.2. A point set $\Lambda \subseteq \mathbb{R}^d$ is a *Meyer set* if Λ is relatively dense and $\Lambda - \Lambda$ is uniformly discrete.

Notice that every Meyer set is a Delone set. At first sight, it might not be intuitive why we want the difference $\Lambda - \Lambda$ to have nice properties. Later however, this will turn out to be very useful; see the definition of the autocorrelation measure.

Let $\Lambda \subseteq \mathbb{R}^d$ be a discrete point set. For $x \in \mathbb{R}^d$ and $r > 0$, we call any set S of the form $B_r(x) \cap \Lambda$ a *patch* of Λ. The point set Λ is *repetitive* if for any patch S there is an $R > 0$ such that $B_R(y)$ contains at least one translate of S for any $y \in \mathbb{R}^d$. Furthermore, Λ has *finite local complexity* (or is an FLC set) if the set $\{\overline{B_R(x)} \cap \Lambda \,|\, x \in \Lambda\}$, for any $r > 0$, contains only finitely many non-empty patches up to translations. A point set Λ is *locally finite* if $K \cap \Lambda$ is a finite set, for any compact set $K \subseteq \mathbb{R}^d$. It is not difficult to see that a Meyer set is a locally finite FLC set.

Last but not least, a subset $\Gamma \subseteq \mathbb{R}^d$ is called a *point lattice* or simply a *lattice* in $\mathbb{R}^d$ if there are d vectors $b_1, \ldots, b_d$ such that

$$\Gamma = \mathbb{Z}b_1 \oplus \ldots \oplus \mathbb{Z}b_d := \left\{ \sum_{i=1}^{d} m_i b_i \,|\, m_i \in \mathbb{Z} \right\},$$

together with the requirement that its $\mathbb{R}$-span is $\mathbb{R}^d$, meaning that $\langle \Gamma \rangle_{\mathbb{R}} = \mathbb{R}^d$. The set $\{b_1, \ldots, b_d\}$ is then called a *basis* of the lattice Γ.

1.1.3 Perron–Frobenius theory

Soon, we will have to deal with non-negative matrices. In particular, primitive matrices will turn out to be very useful. Some of their most powerful properties are listed in the Perron–Frobenius theorem. In order to state this theorem, we need the following definition.

Definition 1.3. A non-negative matrix $M \in \mathrm{Mat}(d, \mathbb{R})$ is called *primitive* if there exists a positive integer $k \in \mathbb{N}$ such that $M^k \gg 0$.

These properties have powerful consequences for the eigenvalues and eigenvectors of such matrices.

Theorem 1.4. *(Perron–Frobenius, [Sen06, Que10]) If $M \in \mathrm{Mat}(d, \mathbb{R})$ is a non-negative primitive matrix, there exists a simple eigenvalue λ_{PF} of M with the following properties:*

1. $\lambda_{\mathrm{PF}} \in \mathbb{R}_+$.
2. $\lambda_{\mathrm{PF}} = \rho(M)$ *and* $\lambda_{\mathrm{PF}} > |\lambda|$ *for any* $\lambda \in \sigma(M) \setminus \{\lambda_{\mathrm{PF}}\}$.
3. *With λ_{PF} can be associated a left eigenvector $\mathbf{L}$ and a right eigenvector $\mathbf{R}$, both consisting of strictly positive entries. Those two eigenvectors are unique up to scalar multiplication.* □

From now on, the eigenvalue and the eigenvectors stated in the theorem above are called the *PF eigenvalue* and the *PF eigenvectors*.

For general background, applications and various corollaries of Theorem 1.4 we refer to [Mey00, Ch. 8].

1.2 Symbolic dynamics

Now, after we motivated what kind of sets we want to work with, we need methods to construct such structures. The first one uses methods from symbolic dynamics and is called inflation rule of a substitution. An explicit example will be given in Section 2.1.2.

1.2.1 Letters and words

Fix $n \in \mathbb{N}$ and consider the set $\mathcal{A} := \{a_i \,|\, 1 \leq i \leq n\}$, which consists of finitely many distinct elements. The set $\mathcal{A}$ is called an *alphabet* and its elements are *letters*. A *(finite) word* w is a (finite) *concatenation* $w = a_{i_1} \dots a_{i_\ell}$ of letters of $\mathcal{A}$, and we let $|w| = \ell$ denote the *length* of w. The empty word is denoted by the symbol ϵ and we define $|\epsilon| := 0$. Additionally, $|w|_{a_i} = \ell_i$ denotes the number of times ℓ_i that the letter a_i appears in the word w. The number ℓ_i is called the *occurrence number* of a_i in w. By a *subword* v of $w = w_0 \dots w_{\ell-1}$ we mean any word of the form $w_i w_{i+1} \dots w_j$, where $0 \leq i \leq j \leq \ell - 1$, and we write $v \lhd w$. If we want to emphasise the precise location of a subword, we write $w_{[i,j]} := w_i w_{i+1} \dots w_{j-1} w_j \lhd w$ for $0 \leq i \leq j \leq \ell - 1$, and we set $w_{[i,j]} := \epsilon$ in the case $i > j$. The set $\mathcal{A}^\ell := \{a_{j_1} a_{j_2} \dots a_{j_\ell} \,|\, 1 \leq j_1, j_2, \dots, j_\ell \leq n\}$ is the set of all words of length ℓ; in particular, we have $\mathcal{A}^0 := \{\epsilon\}$. Furthermore,

$$\mathcal{A}^* := \bigcup_{\ell \geq 0} \mathcal{A}^\ell$$

consists of all (finite) words on the alphabet $\mathcal{A}$ and the empty word.

Let $S \subseteq \mathcal{A}^*$. The set $\mathcal{W}_\ell(S)$ denotes the set of all subwords of length ℓ of elements in S and we call $\mathcal{W}_\ell(S)$ the ℓth factor set of S. Similarly, $\mathcal{W}(S)$ is the set of all subwords of elements in S.

It is also possible to consider infinite sequences of letters in $\mathcal{A}$; we call a sequence $w = (w_i)_{i \in \mathbb{N}_0} \in \mathcal{A}^{\mathbb{N}_0}$ a *semi-infinite word*. Analogously, we denote by

$$w = \dots w_{-3} w_{-2} w_{-1} | w_0 w_1 w_2 \dots = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}}$$

a *bi-infinite word*. The vertical bar indicates the reference point. We let

$$\mathcal{A}^{\mathbb{Z}} := \{(w_i)_{i \in \mathbb{Z}} \,|\, w_i \in \mathcal{A}\}$$

denote the set of bi-infinite sequences in $\mathcal{A}$, which is often called the *full n-shift*. Most of the concepts which were introduced above for finite words carry over to the setting of infinite and bi-infinite words. We endow $\mathcal{A}^{\mathbb{N}_0}$ and $\mathcal{A}^{\mathbb{Z}}$ with the product topology. Therefore, by an application of Tychonoff's theorem [Lan93, Thm. 3.12] the spaces $\mathcal{A}^{\mathbb{N}_0}$ and $\mathcal{A}^{\mathbb{Z}}$ are compact, since the finite alphabet $\mathcal{A}$ is equipped with the discrete topology and, hence, compact. It is well-known that the class $\mathfrak{Z}(\mathcal{A}^{\mathbb{Z}})$ of *cylinder sets*

$$\mathcal{Z}_k(v) := \{w \in \mathcal{A}^{\mathbb{Z}} \mid w_{[k,k+\ell-1]} = v\}, \quad k \in \mathbb{Z},\, v \in \mathcal{A}^{\ell} \tag{1.1}$$

forms an open, closed and countable basis for the topology of $\mathcal{A}^{\mathbb{Z}}$, which generates the Borel σ-algebra $\mathfrak{B}$ of $\mathcal{A}^{\mathbb{Z}}$. The subclass

$$\mathfrak{Z}_0(\mathcal{A}^{\mathbb{Z}}) := \{\mathcal{Z}_k(v) \in \mathfrak{Z}(\mathcal{A}^{\mathbb{Z}}) \mid 1 - |v| \leq k \leq 0\} \cup \mathcal{A}^{\mathbb{Z}}$$

generates $\mathfrak{B}$, too, and it forms a semi-algebra on $\mathcal{A}^{\mathbb{Z}}$. We refer to [Bil12, Sec. 2], [LM95, Ch. 6] and [Que10, Ch. 4] for general background. Two elements $v, w \in \mathcal{A}^{\mathbb{Z}}$ are close in this topology if they agree on a large region around the index 0. Therefore, the topology is called the *local topology*.

1.2.2 Substitution rules

Consider the *free group* $\mathfrak{F}_n := \langle a_1, \ldots, a_n \rangle$ generated by the letters of $\mathcal{A}$. This means that $\mathfrak{F}_n$ consists of all possible finite words in the letters a_i (including ϵ) and their formal inverses a_i^{-1}, up to the equivalence relation $a_i a_i^{-1} = \epsilon = a_i^{-1} a_i$. In this situation, the concatenation of words is the multiplication, and ϵ is the neutral element.

Definition 1.5. A *general substitution rule*, or just a *substitution* for short, on a finite alphabet $\mathcal{A}$ is an endomorphism of the corresponding free group $\mathfrak{F}_n$.

The endomorphism property means that

$$\vartheta(vw) = \vartheta(v)\vartheta(w) \quad \text{ and } \quad \vartheta(v^{-1}) = \big(\vartheta(v)\big)^{-1}$$

for any $v, w \in \mathfrak{F}_n$. This ensures that a substitution is completely characterised by the images of the letters in $\mathcal{A}$ under ϑ and we will restrict to these simple cases for the definition of all studied substitutions. In what follows, we are mainly interested in substitutions ϑ where the images $\vartheta(a_i)$ contain no negative powers of the letters a_i.

A useful tool in the study of substitutions is the *substitution matrix*, which is defined by

$$M_{\vartheta} := \big(|\vartheta(a_j)|_{a_i}\big)_{1 \leq i,j \leq n} \in \mathrm{Mat}(n, \mathbb{Z}).$$

It is easy to check that $M_{\vartheta \circ \sigma} = M_{\vartheta} M_{\sigma}$ for $\vartheta, \sigma \in \mathrm{End}(\mathcal{A}^*)$. However, note that different substitutions can share the same substitution matrix, as we will see in Chapter 2 below. Nevertheless, many interesting properties of a substitution can be derived from the corresponding substitution matrix.

Definition 1.6. A substitution ϑ on $\mathcal{A}$ is called *irreducible* if for each pair (i, j) with $1 \leq i, j \leq n$, there is a power $k \in \mathbb{N}$ such that a_i is a subword of $\vartheta^k(a_j)$. The substitution ϑ is *primitive* if there is a $k \in \mathbb{N}$ such that a_i is a subword of $\vartheta^k(a_j)$ for all $1 \leq i, j \leq n$.

The definitions of Section 1.1.3 immediately imply the following result.

Lemma 1.7. *A substitution rule ϑ is irreducible (or primitive) if and only if its substitution matrix M_ϑ is an irreducible (or primitive) matrix.* $\square$

As a consequence of the primitivity of a substitution ϑ we obtain $\lim_{k\to\infty} |\vartheta^k(a)| = \infty$ for any letter $a \in \mathcal{A}$, as long as $n \geq 2$. This follows from the fact that there is (by definition) a number $j \in \mathbb{N}$ such that $\vartheta^j(a)$ contains all letters $a_i \in \mathcal{A}$, which means $|\vartheta^j(a)| \geq n$. Consequently, we have $|\vartheta^{ij}(a)| \geq n^i$ for all $i \in \mathbb{N}$. In fact, we can say even more about the sequence $(\vartheta^k(a))_{k\in\mathbb{N}}$. For the next result, we need the mapping ϕ, which is defined by

$$\phi : \mathcal{A}^* \to \mathbb{Z}^n, \quad w \mapsto (|w|_{a_1}, \dots, |w|_{a_n})^\mathsf{T},$$

where x^T denotes the transpose of a vector $x \in \mathbb{R}^d$.

Proposition 1.8. *([Que10, Prop. 5.8.]) Let $a \in \mathcal{A}$ and let ϑ be a primitive substitution. Then,*

$$\lim_{k\to\infty} \frac{1}{|\vartheta^k(a)|}\, \phi(\vartheta^k(a)) = \mathbf{R},$$

where $\mathbf{R}$ is the positive right Perron–Frobenius eigenvector of M_ϑ satisfying $\|\mathbf{R}\|_1 = 1$. $\square$

By this proposition, we can interpret the entries of $\mathbf{R}$ as the frequencies of all letters in $\mathcal{A}$ in an infinite word $\lim_{k\to\infty} \vartheta^k(a)$, where the limit is taken in the local topology.

Definition 1.9. A word $w \in \mathcal{A}^*$ is *legal* with respect to a substitution ϑ (or ϑ-legal) if there is some $k \in \mathbb{N}$ and some $x \in \mathcal{A}$ such that $w \lhd \vartheta^k(x)$.

Now, let $w^{(0)} := w_{-1}|w_0 \in \mathcal{A}^2$ be ϑ-legal. Consider the sequence $\big(\vartheta^k(w^0)\big)_{k\in\mathbb{N}}$. Then, the bi-infinite word $w \in \mathcal{A}^\mathbb{Z}$ with the property

$$\lim_{k\to\infty} \vartheta^k(w^{(0)}) = w = \vartheta(w) \tag{1.2}$$

is called a bi-infinite *fixed point* of ϑ with legal seed $w^{(0)}$. If a substitution ϑ is primitive, we can always guarantee the existence of such a fixed point.

Lemma 1.10. *([BG13, Lem. 4.3]) If ϑ is a primitive substitution on $\mathcal{A}^*$ for $n \geq 2$, there exists some $k \in \mathbb{N}$ and some $w \in \mathcal{A}^\mathbb{Z}$ with w being a fixed point of ϑ^k.* $\square$

Later, see Section 2.1.2, a geometric interpretation of fixed points of primitive substitution rules will result in a model for (one-dimensional) quasicrystals.

1.2.3 The discrete hull

Definition 1.11. The *shift map* $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ on the full shift is defined by

$$S((w_k)_{k\in\mathbb{Z}}) := (w_{k+1})_{k\in\mathbb{Z}}.$$

This map is invertible. The inverse mapping satisfies $S^{-1}((w_k)_{k\in\mathbb{Z}}) = (w_{k-1})_{k\in\mathbb{Z}}$. Moreover, S is a homeomorphism. Notice that $S : \mathcal{A}^{\mathbb{N}_0} \to \mathcal{A}^{\mathbb{N}_0}$ is neither invertible nor one-to-one; see [Fog02, Ch. 1].

Definition 1.12. Let ϑ be a primitive substitution and $w \in \mathcal{A}^{\mathbb{Z}}$ be a fixed point of ϑ^k for some $k \in \mathbb{N}$. The set

$$\mathbb{X}_\vartheta := \overline{\{S^j w \mid j \in \mathbb{Z}\}} \subseteq \mathcal{A}^{\mathbb{Z}}$$

is called the *two-sided discrete hull* of the substitution ϑ, where the closure is taken in the local topology.

Remember that such a fixed point exists due to Lemma 1.10. The two-sided discrete hull is shift-invariant by definition. More generally, we call any closed and shift-invariant subset $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ a *two-sided subshift*, and it is not difficult to see that $S|_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}$ is a homeomorphism.

Remark 1.13. It is possible to construct the hull $\mathbb{X}_\vartheta$ from a semi-infinite fixed point of ϑ; see [BG13, Rem. 4.1]. This will lead to technical simplifications in Chapter 3. ◇

Two different fixed points $v, w \in \mathcal{A}^{\mathbb{Z}}$ of ϑ^k define the same two-sided discrete hull $\mathbb{X}_\vartheta$; compare [BG13, Lem. 4.2 and Prop. 4.2]. Thus, $\mathbb{X}_\vartheta$ is uniquely defined [BG13, Thm. 4.1]. The two-sided discrete hull can also be described using legal words.

Lemma 1.14. *([Fog02, Lem. 1.1.2]) For any two sequences $v, w \in \mathcal{A}^{\mathbb{Z}}$, the following statements are equivalent:*

(a) $v \in \overline{\{S^k w \mid k \in \mathbb{Z}\}}$.

(b) There exists a sequence $(r_\ell)_{\ell\in\mathbb{N}}$ such that $v_0 \dots v_\ell = w_{r_\ell} \dots w_{r_\ell+\ell}$ for every $\ell \geq 0$.

(c) $\mathcal{W}_\ell(\{v\}) \subseteq \mathcal{W}_\ell(\{w\})$ for all $\ell \in \mathbb{N}$. □

Remark 1.15. Let $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ be any subshift. The class of cylinder sets $\mathfrak{Z}(\mathbb{X})$ is given by

$$\mathfrak{Z}(\mathbb{X}) := \{\mathcal{Z} \cap \mathbb{X} \mid \mathcal{Z} \in \mathfrak{Z}(\mathcal{A}^{\mathbb{Z}})\}.$$

◇

Next, we want to characterise different hulls. A bi-infinite word $w \in \mathcal{A}^{\mathbb{Z}}$ is called *periodic* if there is some $k \in \mathbb{Z} \setminus \{0\}$, such that $S^k w = w$ and *non-periodic*, when there is no k with that property. A primitive substitution ϑ is called *aperiodic* if $\mathbb{X}_\vartheta$ contains no periodic element. The following result turns out to be useful in the case of the (random) noble means substitution.

Theorem 1.16. *([BG13, Thm. 4.6]) Let ϑ be a primitive substitution with substitution matrix M_ϑ. If the PF eigenvalue of M_ϑ is irrational, then ϑ is aperiodic.* □

The substitutions which we will consider have even more properties. An algebraic integer $\alpha > 1$ is called a *Pisot–Vijayaraghavan number*, or a *PV number* for short, if its algebraic conjugates $\alpha_1, \ldots, \alpha_k$ satisfy $|\alpha_i| < 1$. If α is a PV number and an algebraic unit, we call α a *PV unit*. A substitution ϑ is called a *Pisot substitution* if M_ϑ has a largest and simple eigenvalue $\lambda > 1$ and all other eigenvalues λ' of M_ϑ satisfy $0 < |\lambda'| < 1$. We refer to [Sin06] for more information about Pisot substitutions.

Also, it would be useful to have a sufficient condition for two substitutions to have the same two-sided discrete hull. This can be achieved as follows. Let ϑ be a substitution on the alphabet $\mathcal{A}$ and fix an element $v \in \mathfrak{F}_n$. We denote by $\vartheta_v(a) := v\vartheta(a)v^{-1}$, for any $a \in \mathcal{A}$, the conjugate substitution to ϑ corresponding to v. Here, we call a substitution *non-negative* if the images of all a_i consist of letters with exclusively non-negative exponents. For an arbitrary $w \in \mathcal{F}_n$, the definition of ϑ_v and the homomorphism property imply $\vartheta_v(w) = v\vartheta(w)v^{-1}$ because we have $v^{-1}v = \epsilon$ in $\mathfrak{F}_n$, wherefore it is enough to test conjugacy on all $a_i \in \mathcal{A}^*$.

Proposition 1.17. *([BG13, Prop. 4.6]) Let ϑ be a primitive substitution on the finite alphabet $\mathcal{A}$, and let v be a finite word such that ϑ_v is a non-negative substitution as well. Then, ϑ_v is primitive and ϑ and ϑ_v define the same two-sided discrete hull.* □

1.3 Model sets

The second mathematical model for quasicrystals is obtained through a cut and project scheme. This method was introduced by Y. Meyer in [Mey72] in order to generate a class of models with interesting harmonic properties. The method was rediscovered independently by Kramer [KN84] in 1984. Kramer used this method to produce a three-dimensional icosahedral quasicrystal by projection from a 6-dimensional hypercubic lattice. It was only later that Lagarias [Lag96] and Moody [Moo97a] connected the work of Meyer, Kramer and de Bruijn [Bru81], and realised the importance of this method to long range aperiodic order.

Definition 1.18. A *cut and project scheme* (CPS) is a triple $(\mathbb{R}^d, H, \mathcal{L})$ with a (compactly generated) locally compact Abelian group H, a lattice $\mathcal{L}$ in $\mathbb{R}^d \times H$ and the two natural projections

$$\pi_1 : \mathbb{R}^d \times H \to \mathbb{R}^d \quad \text{and} \quad \pi_{\text{int}} : \mathbb{R}^d \times H \to H,$$

subject to the conditions that $\pi_1|_{\mathcal{L}}$ is injective and that $\pi_{\text{int}}(\mathcal{L})$ is dense in H. The group $\mathbb{R}^d$ is called the *physical space* and H the *internal space*.

We write $L = \pi_1(\mathcal{L})$. Since, for a given CPS, π_1 is then a bijection between $\mathcal{L}$ and L, there is a well-defined mapping

$$\star : L \longrightarrow H, \quad x \mapsto x^\star := \pi_{\text{int}}\big((\pi_1|_{\mathcal{L}})^{-1}(x)\big),$$

where $(\pi_1|_{\mathcal{L}})^{-1}(x)$ is the unique point in the set $\mathcal{L} \cap \pi_1^{-1}(x)$. This mapping is called the *star map* of the CPS. The $\star$-image of L is denoted by $L^\star$. Furthermore, $\mathcal{L}$ can be viewed as a diagonal embedding of L, i.e.

$$\mathcal{L} = \{(x, x^\star) \,|\, x \in L\}.$$

The setting of a general CPS is conveniently summarised in the following diagram.

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi_1\;} & \mathbb{R}^d \times H & \xrightarrow{\;\pi_{\text{int}}\;} & H \\
\cup & & \cup & & \cup \ \text{dense} \\
\pi_1(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^\star
\end{array}$$

Let us briefly mention that there is a matching CPS that involves the dual lattice $\mathcal{L}^*$, and the dual group $\widehat{\mathbb{R}^d \times H} \simeq \widehat{\mathbb{R}^d} \times \widehat{H}$. This is known as the *dual CPS*; see [Moo97a, Sec. 5].

For a given CPS $(\mathbb{R}^d, H, \mathcal{L})$ and a (general) set $W \subseteq H$,

$$\curlywedge(W) := \{x \in L \,|\, x^\star \in W\}$$

denotes a projection set within the CPS. The set W is called its *window*. Let us now expand on the most important situation.

Definition 1.19. Let $(\mathbb{R}^d, H, \mathcal{L})$ be a CPS. If $W \subseteq H$ is a relatively compact set with non-empty interior, the projection set $\curlywedge(W)$, or any translate $t + \curlywedge(W)$ with $t \in \mathbb{R}^d$, is called a *model set*. A model set is called *regular* when $\mu_H(\partial W) = 0$, where μ_H is the Haar measure of H. If $L^\star \cap \partial W = \varnothing$, the model set is called *generic*. If the window W is not in a generic position (meaning that $L^\star \cap \partial W \neq \varnothing$), the corresponding model set is called *singular*.

Proposition 1.20. *([BG13, Prop. 7.5.]) Let $(\mathbb{R}^d, H, \mathcal{L})$ be a CPS and consider a projection set of the form $\Lambda = t + \curlywedge(W)$ with $t \in \mathbb{R}^d$ and window $W \subseteq H$. If W is relatively compact, Λ is FLC and thus uniformly discrete; if $W^\circ \neq \varnothing$, then Λ is relatively dense. If Λ is a model set, it is also a Meyer set.* □

Theorem 1.21. *([Moo97a, Thm. 9.1(i)]) A relatively dense point set $P \subseteq \mathbb{R}^d$ is Meyer if and only if P is a subset of a model set.* □

1.4 Measures and diffraction

In this section, we collect basic tools from functional analysis and harmonic analysis, which we will need to study the diffraction of some random substitutions. We denote by

$$\mathcal{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \mid \forall\, \alpha, \beta \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty\}$$

the *Schwartz space*. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then, the *convolution* of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)\, g(x - y)\ \mathrm{d}y.$$

and the *Fourier transform* of $f \in \mathcal{S}(\mathbb{R}^d)$ is

$$\mathcal{F}[f](k) := \widehat{f}(k) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} kx}\ f(x)\ \mathrm{d}x.$$

The same definitions apply in the case of $f, g \in L^1(\mathbb{R}^d)$. However, the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is (in general) not integrable, whereas $f \in \mathcal{S}(\mathbb{R}^d)$ implies $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$. The dual space $\mathcal{S}'(\mathbb{R}^d)$ of $\mathcal{S}(\mathbb{R}^d)$ is called the space of *tempered distributions*. The Fourier transform of a tempered distribution T is defined by $\widehat{T}(f) := T(\widehat{f})$ for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Example 1.22. One of the best-known tempered distributions is the *Dirac* distribution

$$\delta_x : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}, \quad f \mapsto \delta_x(f) := f(x).$$

Its Fourier transform is given by

$$\widehat{\delta_x}(f) = \delta_x(\widehat{f}) = \widehat{f}(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} xy}\ f(y)\, \mathrm{d}y.$$

One often finds the notion $\widehat{\delta_x} = \mathrm{e}^{-2\pi \mathrm{i} xy}$, where one identifies the functional $T(f) = \int_{\mathbb{R}^d} g(x)\, f(x)\ \mathrm{d}x$ with the function g.

1.4.1 Measures and linear functionals

In this thesis, we will only deal with regular Borel measures on a separable metrisable locally compact space X. Such measures can be introduced via linear functionals [Die70, Ch. XIII].

A complex-valued measure on X is a linear functional $\mu : C_{\mathrm{c}}(X) \to \mathbb{C}$ with the following property: for each compact $K \subseteq X$, there is a real constant $c_K > 0$ such that

$$|\mu(f)| \leq c_K \|f\|_\infty$$

for all $f \in C_{\mathrm{c}}(X)$ with support in K, where $\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$ is the supremum norm of f. Let $\mathcal{M}(X)$ denote the space of complex-valued measures. Following [Die70, Ch. XIII.4], we endow $\mathcal{M}(X)$ with the *vague (= weak-$*$-) topology*, which means that a sequence $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely to a measure μ if the sequence $(\mu_n(f))_{n\in\mathbb{N}}$ converges to $\mu(f)$ in $\mathbb{C}$ for all $f \in C_{\mathrm{c}}(X)$.

The *conjugate* $\overline{\mu}$ of a measure $\mu \in \mathcal{M}(X)$ is defined by the map $f \mapsto \overline{\mu(\bar{f})}$ for any $f \in C_{\mathrm{c}}(X)$. A measure μ is called *real* if $\mu = \overline{\mu}$ and a real measure μ is called *positive* if $\mu(f) \geq 0$ for all $f \geq 0$. We denote by $\mathcal{M}^+_{\mathbb{R}}(X)$ the set of real and positive measures on X. A measure $\mu \in \mathcal{M}(X)$ is called *finite* if $|\mu|(X)$ is finite, where $|\mu|$ denotes the *total variation measure* of μ. The latter is the smallest positive measure such that $|\mu(f)| \leq |\mu|(f)$ for all non-negative $f \in C_{\mathrm{c}}(X)$.

Theorem 1.23. *([RS80, Thm. IV.18]) Let X be a locally compact space. A positive linear functional ℓ on $C_c(X)$ is of the form*

$$\ell(f) = \int f \; d\mu$$

for some regular Borel measure μ. □

An application of the Riesz–Markov representation theorem (Thm. 1.23), together with the decomposition of μ into its real/imaginary and positive/negative parts, yields a one-to-one correspondence between regular Borel measures on X and the measures defined by the approach via linear functionals [Die70, Ch. XIII.2 and Ch. XIII.3]. These two different points of view are connected via $\mu(1_B) = \mu(B)$, where 1_B denotes the characteristic function

$$1_B : X \to \{0, 1\}, \quad 1_B(x) := \begin{cases} 1, & x \in B, \\ 0, & x \notin B. \end{cases}$$

Note that the function 1_B is not continuous, but due to the regularity of μ there are suitable approximations with continuous functions from above and from below; see [Die70, Ch. XIII.7] for background information.

Definition 1.24. A measure $\mu \in \mathcal{M}(X)$ is called *translation bounded* if

$$\sup_{x\in X} |\mu|(x+K) < \infty$$

for any compact $K \subseteq X$.

The subspace of $\mathcal{M}(X)$ that consists of all translation bounded measures is denoted by $\mathcal{M}^\infty(X)$. Moreover, $\mu \in \mathcal{M}(X)$ is called *positive definite* if $\mu(f * \widetilde{f}) \geq 0$ for all $f \in C_\mathrm{c}(X)$, where $\widetilde{f}(x) := \overline{f(-x)}$ is the *reflection* of f. Analogously, we define $\widetilde{\mu}(f) := \overline{\mu(\widetilde{f})}$.

Now, let us restrict for the moment to the case $X = \mathbb{R}^d$. For any $\mu \in \mathcal{M}^+_\mathbb{R}(\mathbb{R}^d)$, the set

$$P_\mu := \{x \mid \mu(\{x\}) \neq 0\}$$

is called the set of *pure points* of μ and, for any Borel set B, we define

$$\mu_\mathrm{pp}(B) := \sum_{x\in B\cap P_\mu} \mu(\{x\}) = \mu(B \cap P_\mu),$$

as the *pure point part* of μ. Thus, we call a measure $\mu \in \mathcal{M}^+_\mathbb{R}(\mathbb{R}^d)$ *pure point* if

$$\mu(B) = \sum_{x\in B} \mu(\{x\})$$

for any Borel set B. On th other hand, we define

$$\mu_\mathrm{c} := \mu - \mu_\mathrm{pp},$$

which means $\mu_\mathrm{c}(\{x\}) = 0$ for all $x \in \mathbb{R}^d$. In this case we say that μ_c has no pure points.

A measure μ is called *absolutely continuous* with respect to Lebesgue measure if there is a locally integrable function f such that $\mu = f\,\boldsymbol{\lambda}$, which means

$$\mu(g) = \int_{\mathbb{R}^d} g \,\mathrm{d}\mu = \int_{\mathbb{R}^d} gf \,\mathrm{d}\boldsymbol{\lambda} = \boldsymbol{\lambda}(gf),$$

for any $g \in C_\mathrm{c}(\mathbb{R}^d)$. Here, f is called the *Radon–Nikodym density* of μ relative to $\boldsymbol{\lambda}$.

A positive measure μ is called *singular* relative to Lebesgue measure if $\mu(B) = 0$ for some measurable set $B \subseteq \mathbb{R}^d$ with $\boldsymbol{\lambda}(\mathbb{R}^d \setminus B) = 0$. A measure that is singular relative to $\boldsymbol{\lambda}$ without having any pure points is called *singular continuous*.

Theorem 1.25. *([RS80, Thm. I.13 and Thm. I.14]). Any positive, regular Borel measure $\mu \in \mathcal{M}^+_\mathbb{R}(\mathbb{R}^d)$ has a unique decomposition*

$$\mu = \mu_\mathrm{pp} + \mu_\mathrm{ac} + \mu_\mathrm{sc},$$

where μ_pp is pure point, μ_ac is absolutely continuous and μ_sc is singular continuous with respect to Lebesgue measure. □

If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is a finite measure, its Fourier transform (or Fourier–Stieltjes transform) can directly be defined as

$$\widehat{\mu}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} kx} \, \mathrm{d}\mu(x),$$

which is a bounded and uniformly continuous function on $\mathbb{R}^d$; see [Rud62, Thm. 1.3.3(a)]. Seen as the Radon–Nikodym density (relative to the Lebesgue measure), $\widehat{\mu}$ coincides with the Fourier transform of μ in the distributional sense, i.e. $\widehat{\mu}(f) = \mu(\widehat{f})$.

Proposition 1.26. *([BF75, Sec. I.4]) If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is positive definite, its Fourier transform $\widehat{\mu}$ exists, and is a translation bounded positive measure on $\mathbb{R}^d$.* □

If μ and ν are finite measures on $\mathbb{R}^d$, their *convolution* $\mu * \nu$ is defined by

$$(\mu * \nu)(g) := \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x+y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y)$$

for any $g \in C_{\mathrm{c}}(\mathbb{R}^d)$. This again is a finite measure, and thus certainly Fourier transformable. In fact, the convolution $\mu * \nu$ can also be defined when μ is finite and ν translation bounded; see [BF75, Prop. 1.13]. Moreover, one has the following result.

Proposition 1.27. *([BF75]) Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ with μ finite and ν translation bounded. Then, the convolution $\mu * \nu$ exists and is a translation bounded measure.*

If $\widehat{\nu}$ is not only a tempered distribution, but also a measure, one has the convolution identity $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$. The latter is again a measure, which is absolutely continuous relative to $\widehat{\nu}$. □

1.4.2 Autocorrelation and diffraction measure

We have to extend the concept in the setting of infinite measures. We define the *Eberlein convolution* of $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ as

$$\mu \circledast \nu := \lim_{r \to \infty} \frac{\mu_r * \nu_r}{\mathrm{vol}(B_r)},$$

where μ_r and ν_r are the restrictions of μ and ν to $\overline{B_r}$. This limit need not exist, but if μ and ν are translation bounded, there is at least one accumulation point [BG13, Prop. 9.1]. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a measure. Since μ_r has compact support (and is a finite measure), the convolution

$$\gamma_\mu^{(r)} := \frac{\mu_r * \widetilde{\mu_r}}{\mathrm{vol}(B_r)}$$

is well-defined, and positive definite by construction. Every accumulation point of the family $\{\gamma_\mu^{(r)} \mid r > 0\}$ in the vague topology, as $r \to \infty$, is called an *autocorrelation measure* of μ, and as such it is a positive definite measure by construction. If only one point of accumulation exists, the autocorrelation measure

$$\gamma_\mu := \lim_{r\to\infty} \gamma_\mu^{(r)} = \mu \circledast \widetilde{\mu}$$

is well-defined. Moreover, we have the following result.

Proposition 1.28. *([BG13, Prop. 9.1]) If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is translation bounded, the corresponding family $\{\gamma_\mu^{(r)} \mid r > 0\}$ of approximating autocorrelations is precompact in the vague topology. Any accumulation point of this family, of which there is at least one, is translation bounded.* □

Now, diffraction of X-rays by matter results from scattering by the individual atoms and interference between the scattering waves. We omit the underlying derivation, and refer to [Hec14, Ch. 10] and [Höf01] for introductions.

Definition 1.29. Let $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ be such that the autocorrelation γ_μ exists. The Fourier transform $\widehat{\gamma_\mu}$ is then called the *diffraction measure* of μ.

The autocorrelation (which is assumed to exist) is a positive definite measure by construction, and thus Fourier transformable by Proposition 1.26.

Remark 1.30. By Theorem 1.25, we can decompose the diffraction measure $\widehat{\gamma}$ into its pure point, absolutely continuous and singular continuous part:

$$\widehat{\gamma_\mu} = (\widehat{\gamma_\mu})_{\mathrm{pp}} + (\widehat{\gamma_\mu})_{\mathrm{ac}} + (\widehat{\gamma_\mu})_{\mathrm{sc}}.$$

Speaking in terms of kinematic diffraction, the diffraction measure is decomposed into its 'Bragg peak' part and its 'diffuse' part. ◇

If the point set is given by a cut and project scheme, the autocorrelation and diffraction measure can be computed explicitly. Note that the autocorrelation measure of a point set Λ is the autocorrelation measure of δ_Λ.

Proposition 1.31. *([BG13, Prop. 9.8]) Consider the general CPS $(\mathbb{R}^d, H, \mathcal{L})$, and let $\Lambda = \curlywedge(W)$ be a regular model set for it, with a compact window $W = \overline{W^\circ}$. The autocorrelation measure γ_Λ of Λ exists and is a positive and positive definite, translation bounded, pure point measure. It is explicitly given by*

$$\gamma_\Lambda = \sum_{z\in\Lambda-\Lambda} \eta(z)\, \delta_z,$$

with the autocorrelation coefficients

$$\eta(z) = \mathrm{dens}(\Lambda) \frac{\mu_H\big(W \cap (z^\star + W)\big)}{\mu_H(W)} = \frac{\mathrm{dens}(\Lambda)}{\mu_H(W)} \int_H 1_W(y) 1_{z^\star+W}(y) \; \mathrm{d}\mu_{\mathrm{H}}(\mathrm{y})\,.$$

□

Theorem 1.32. *([BG13, Thm. 9.4]) Let* $\Lambda = \curlywedge(W)$ *be a regular model set for the CPS* $(\mathbb{R}^d, H, \mathcal{L})$ *with compact window* $W = \overline{W^\circ}$ *and autocorrelation* γ_Λ*. The diffraction measure* $\widehat{\gamma_\Lambda}$ *is a positive and positive definite, translation bounded measure. It is explicitly given by*

$$\widehat{\gamma_\Lambda} = \sum_{k \in L^{\circledast}} I(k) \delta_k \,,$$

where the diffraction intensities are $I(k) = |A(k)|^2$ *with the amplitudes*

$$A(k) = \frac{\mathrm{dens}(\Lambda)}{\mu_H(W)} \widehat{1_W}(-k^\star) = \frac{\mathrm{dens}(\Lambda)}{\mu_H(W)} \int_W \langle k^\star, y \rangle \; \mathrm{d}\mu_{\mathrm{H}}(\mathrm{y}) \,.$$

□

In Euclidean model sets, the amplitudes can be computed as follows.

Proposition 1.33. *([BG13, Prop. 9.9]) Consider a regular model set* $\Lambda = \curlywedge(W)$ *for the Euclidean CPS* $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$*, with compact window* $W = \overline{W^\circ} \subseteq \mathbb{R}^m$ *and Fourier module* $L^{\circledast} = \pi(\mathcal{L}^*) \subseteq \mathbb{R}^d$*. Then, one has*

$$\frac{1}{\mathrm{vol}(B_r)} \sum_{x \in \Lambda_r} \mathrm{e}^{-2\pi \mathrm{i} k x} \xrightarrow{r \to \infty} \begin{cases} A(k), & k \in L^{\circledast}, \\ 0, & \textit{otherwise}, \end{cases}$$

where $\Lambda_r = \Lambda \cap \overline{B_r}$ *and* $A(k)$ *is the amplitude of Theorem 1.32 for the internal space* $H = \mathbb{R}^m$*.* □

Remark 1.34. Proposition 1.33 is a special case of the Bombieri–Taylor conjecture. In their paper [BT86] Bombieri and Taylor claim that, for some one-dimensional examples, the set of Bragg peak positions is exactly the set of those k that satisfy

$$\lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \sum_{x \in \Lambda \cap B_n} \mathrm{e}^{-2\pi \mathrm{i} k x} \neq 0.$$

This claim became known as the Bombieri–Taylor conjecture. It has been proven in various degrees of generality; see for example [Len09]. ◇

2 Random noble means substitutions

It was already mentioned in the introduction that we are interested in the diffraction of systems with positive entropy. Now that we have collected the necessary tools, we can discuss the first one-dimensional example. This family of examples was first studied in [GL89], and treated mathematically in [Mo13] many years later.

As this is a stochastic generalisation, we will recollect the important facts about the diffraction of the deterministic noble means substitutions in Section 2.1; see [Mo13, Sec. 6.2]. After that, we will generalise the study of deterministic substitutions to random substitutions in Section 2.2; compare [Mo13, Ch. 2]. After these preparations, we will be ready to determine the diffraction measure of a typical random noble means set in Chapter 3.

2.1 Noble means substitutions

Consider the alphabet $\mathcal{A}_2 = \{a, b\}$. For fixed $m \in \mathbb{N}$ and $0 \leq i \leq m$, we define the substitution $\zeta_{m,i} : \mathcal{A}_2^* \to \mathcal{A}_2^*$ by

$$\zeta_{m,i} : \begin{cases} a \mapsto a^i b a^{m-i}, \\ b \mapsto a. \end{cases}$$

Let

$$\mathcal{N} := \mathcal{N}_m := \{\zeta_{m,i} \mid 0 \leq i \leq m\}$$

denote the family of *noble means substitutions (NMS)*. The most prominent and best examined member of the NMS family is the Fibonacci substitution $\zeta_{1,1} : b \mapsto a \mapsto ab$ together with its variant $\zeta_{1,0} : b \mapsto a \mapsto ba$, compare [BG13, Fog02, Sin06].

2.1.1 Elementary properties of the NMS

As mentioned in Chapter 1, it is possible to derive certain properties of a substitution from the corresponding substitution matrix. The substitution matrix of $\zeta_{m,i}$ is given by

$$M_m := M_{\zeta_{m,i}} := \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix},$$

which is independent of i. The matrix M_m is primitive (so is each $\zeta_{m,i}$), hence there is a power $k \in \mathbb{N}$ such that $\zeta_{m,i}^k$ admits the construction of a bi-infinite fixed point by Lemma 1.10. In the Fibonacci case with legal seed $a|a$, we obtain

$$\ldots abaababaabaababaababa|abaababaabaababaababa\ldots$$

as a fixed point of $\zeta_{1,1}^2$. The eigenvalues of M_m are given by

$$\lambda_m := \frac{m+\sqrt{m^2+4}}{2} \quad \text{and} \quad \lambda_m' := \frac{m-\sqrt{m^2+4}}{2},$$

which are the roots of the characteristic polynomial $P_m(x) = x^2 - mx - 1$. This identifies λ_m as the PF eigenvalue of M_m. The corresponding eigenvectors are

$$\mathbf{L} = \mathbf{L}_m = (\lambda_m, 1) \quad \text{and} \quad \mathbf{R} = \mathbf{R}_m = \left(\frac{\lambda_m}{\lambda_m+1}, \frac{1}{\lambda_m+1}\right)^{\mathsf{T}}.$$

Here, the right eigenvector is normalised such that its entries coincide with the relative frequencies of the letters a and b, see [Que10, Sec. 5.3.2].

Additionally, as λ_m is irrational, Theorem 1.16 implies the aperiodicity of each $\zeta_{m,i}$. On the other hand, it is easy to verify that λ_m is a PV unit, which makes each $\zeta_{m,i}$ a Pisot substitution. Also notice that the leading eigenvalue gives rise to the name noble mean substitutions, because their continued fraction expansion reads $\lambda_1 = [1; 1, 1, 1, \ldots]$ (the golden mean), $\lambda_2 = [2; 2, 2, 2, \ldots]$ (the silver mean) and so on.

Another interesting property of the NMS family is the fact all members of $\mathcal{N}_m$ define the same two-sided discrete hull, which is a direct consequence of Proposition 1.17 by setting $\vartheta := \zeta_{m,i}$ and $v := a^{i-j}$. Hence, for fixed $m \in \mathbb{N}$, we define *the* corresponding noble means hull by

$$\mathbb{X}_m' := \mathbb{X}_{\zeta_{m,0}} = \ldots = \mathbb{X}_{\zeta_{m,m}}. \tag{2.1}$$

Setting $\mathcal{D}_{m,i}'$ (and $\mathcal{D}_{m,i,\ell}'$) as the set of $\zeta_{m,i}$-legal words (of length ℓ) and applying Lemma 1.14, we obtain

$$\mathbb{X}_{\zeta_{m,i}} = \{w \in \mathcal{A}_2^{\mathbb{Z}} \mid \mathcal{W}(\{w\}) \subseteq \mathcal{D}_{m,i}'\}. \tag{2.2}$$

Hence, we do not need to distinguish the sets of $\zeta_{m,i}$-legal words for different i, and we denote by $\mathcal{D}_m'$ (and $\mathcal{D}_{m,\ell}'$) *the* set of all $\zeta_{m,i}$-legal words (of length ℓ).

Moreover, the noble means hull has an important structure. As $\zeta_{m,i}$ is a primitive, aperiodic substitution, it follows from [BG13, Lem. 4.4 and Prop. 4.5] that $\mathbb{X}_m'$ is a Cantor set, i.e. a metrisable, compact, perfect and totally disconnected topological space, which is uncountable.

If we sum up the properties we have collected so far, we obtain:

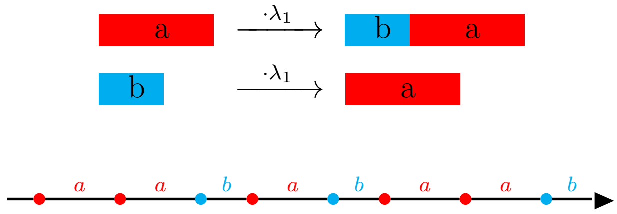

Figure 1.1: The action of the inflation rule $\zeta_{1,0}$ is shown above. The left endpoints of the resulting intervals are chosen as control points below.

Proposition 2.1. *For fixed $m \in \mathbb{N}$, each member of $\mathcal{N}_m$ is a primitive and aperiodic Pisot substitution. The two-sided discrete hulls $\mathbb{X}_{m,i}$ are uncountable Cantor sets, which coincide for $0 \leq i \leq m$.* □

2.1.2 Noble means sets

Our actual target is the study of diffraction measures. For this reason, we are less interested in the symbolic hull $\mathbb{X}'_m$, but rather in its geometric counterpart, which we will denote by $\mathbb{Y}'_m$. For this purpose, we make use of the primitivity of M_m and Theorem 1.4 to construct point sets in $\mathbb{R}$ using a fixed point of $\zeta^k_{m,i}$, which we know to exist by Lemma 1.10.

In the first step, we assign a compact interval I_a of length λ_m to the letter a and a compact interval I_b of length 1 to the letter b. In this case, the substitution $\zeta_{m,i}$ can be interpreted as an *inflation rule* for the two prototiles I_a and I_b in the following way. The prototiles are inflated by the PF eigenvalue λ_m and then dissected into copies of the original prototiles, according to the substitution rule; compare Figure 1.1. The iteration of this process leads to a tiling of the real line with prototiles I_a and I_b. We choose the left endpoints of each tile as control points to generate attributed point sets in $\mathbb{R}$.

Such point sets are called *geometric realisations*. A sketch of the geometric realisation of the Fibonacci substitution $\zeta_{1,0}$ is shown in Figure 1.1. Here, we denote geometric realisations of fixed points of $\zeta^2_{m,i}$ as *generating noble means sets* and refer to them as $\Lambda_{m,i}$. These sets can be decomposed as $\Lambda_{m,i} = \Lambda^{(a)}_{m,i} \dot{\cup} \Lambda^{(b)}_{m,i}$ with obvious meaning. Now, the *geometric* or *continuous hull* $\mathbb{Y}'_{m,i}$ is defined as

$$\mathbb{Y}'_{m,i} := \overline{\{t + \Lambda_{m,i} \,|\, t \in \mathbb{R}\}}.$$

Remark 2.2. Due to the choice of the lengths of I_a and I_b, we have

$$\Lambda_{m,i} \subseteq \mathbb{Z}[\lambda_m] = \{k + n\lambda_m \mid k, n \in \mathbb{Z}\} \subseteq \mathbb{R}$$

for any $0 \leq i \leq m$. Moreover, the *point density* of $\Lambda_{m,i}$ is given by

$$\operatorname{dens}(\Lambda_{m,i}) = \frac{1}{\langle \mathbf{L} \,|\, \mathbf{R} \rangle} = \frac{1-\lambda_m'}{\sqrt{m^2+4}},$$

since the relative frequencies of the letters a and b are encoded in $\mathbf{R}$, and the lengths of the intervals are encoded in $\mathbf{L}$; see [BG13, Sec. 2.1] for general background concerning the point density of point sets. $\diamond$

Next, we want to show that the noble means sets, i.e. the elements of $\mathbb{Y}_m'$, can be constructed within the same cut and project scheme as (translates of) model sets. This will enable us, via Theorem 1.32, to explicitly calculate the diffraction measure of each element of $\mathbb{Y}_m'$. We already know that that the physical space in Definition 1.18 is $\mathbb{R}$ and even more, we have just seen above that $\Lambda_{m,i} \subseteq \mathbb{Z}[\lambda_m]$ with $\mathbb{Z}[\lambda_m]$ being dense in $\mathbb{R}$. Now, we define the non-trivial field automorphism on the quadratic field $\mathbb{Q}(\sqrt{m^2+4})$ that is given by

$$' : \mathbb{Q}(\sqrt{m^2+4}) \to \mathbb{Q}(\sqrt{m^2+4}), \quad x + y\sqrt{m^2+4} \mapsto x - y\sqrt{m^2+4}$$

as the star map, that is $x^\star := x'$. The diagonal (Minkowski) embedding of $\mathbb{Z}[\lambda_m]$ is the lattice

$$\mathcal{L}_m := \{(x, x^\star) \mid x \in \mathbb{Z}[\lambda_m]\} \subseteq \mathbb{R} \times \mathbb{R},$$

which leads to $\mathbb{R}$ as internal space. Here, the projection of $\mathcal{L}_m$ into the physical space is also dense in $\mathbb{R}$ as $\pi_1(\mathcal{L}_m) = \mathbb{Z}[\lambda_m]$. It is not difficult to verify that

$$\mathcal{L}_m = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2,$$

where the lattice base is given by $b_1 := (1,1)^\mathsf{T}$ and $b_2 := (\lambda_m, \lambda_m')^\mathsf{T}$. Additionally, we note the dual lattice of $\mathcal{L}_m$ which is given by

$$\mathcal{L}_m^* := \{y \in \mathbb{R}^2 \mid \langle x \,|\, y \rangle \in \mathbb{Z} \text{ for all } x \in \mathcal{L}_m\} = \left\langle \frac{1}{\sqrt{m^2+4}} \begin{pmatrix} -\lambda_m' \\ \lambda_m \end{pmatrix}, \frac{1}{\sqrt{m^2+4}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{Z}}.$$

The cut and project scheme for the noble means sets can be compactly presented as follows

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\ \pi_1\ } & \mathbb{R} \times \mathbb{R} & \xrightarrow{\ \pi_{\mathrm{int}}\ } & \mathbb{R} \\
\text{dense} \ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \ \text{dense} \\
\mathbb{Z}[\lambda_m] & \xleftarrow{\ 1-1\ } & \mathcal{L}_m & \xrightarrow{\ 1-1\ } & \mathbb{Z}[\lambda_m] \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\quad \star \qquad\quad} & & L^\star
\end{array} \tag{2.3}$$

Now, the application of $\zeta_{m,i}$ to the noble means sets $\Lambda_{m,i}$ leads to the equations

$$\begin{aligned}
\Lambda^{(a)}_{m,i} &= \left\{ \dot{\bigcup_{0\le j\le i-1}} \lambda_m \Lambda^{(a)}_{m,i} + j\lambda_m \right\} \cup \lambda_m \Lambda^{(b)}_{m,i} \cup \left\{ \dot{\bigcup_{i\le j\le m-1}} \lambda_m \Lambda^{(a)}_{m,i} + j\lambda_m + 1 \right\}, \\
\Lambda^{(b)}_{m,i} &= \lambda_m \Lambda^{(a)}_{m,i} + i\lambda_m .
\end{aligned}$$

Under algebraic conjugation followed by taking the closure, one obtains a new set of equations for the closed sets $\Gamma^{(a)}_{m,i} := \overline{\left(\Lambda^{(a)}_{m,i}\right)^{\star}}$ and $\Gamma^{(b)}_{m,i} := \overline{\left(\Lambda^{(b)}_{m,i}\right)^{\star}}$,

$$\begin{aligned}
\Gamma^{(a)}_{m,i} &= \left\{ \bigcup_{j=0}^{i-1} \lambda'_m \Gamma^{(a)}_{m,i} + j\lambda'_m \right\} \cup \lambda'_m \Gamma^{(b)}_{m,i} \cup \left\{ \bigcup_{j=i}^{m-1} \lambda'_m \Gamma^{(a)}_{m,i} + j\lambda'_m + 1 \right\}, \\
\Gamma^{(b)}_{m,i} &= \lambda'_m \Gamma^{(a)}_{m,i} + i\lambda'_m ,
\end{aligned} \tag{2.4}$$

where $|\lambda'_m| < 1$, which is the PV property of λ_m. This new set of equations constitutes a coupled iterated function system (IFS) that is a contraction in the Hausdorff metric, with contraction constant λ'_m; see [Hut81, BM00a, Wic91] for general background. In this setting, one needs to work with compact sets, which was the reason for taking closures when deriving Eq. (2.4).

Proposition 2.3. *([Mo13, Sec. 5.3]) For an arbitrary but fixed $m \in \mathbb{N}$ and $0 < i < m$, the windows for the noble means sets $\Lambda_{m,i}$, within the cut and project scheme (2.3), are given by the compact intervals*

$$W_{m,i} = W^{(a)}_{m,i} \cup W^{(b)}_{m,i} = i\tau_m + [\lambda'_m, 1],$$

where $W^{(a)}_{m,i} = i\tau_m + [0,1]$ and $W^{(b)}_{m,i} = i\tau_m + [\lambda'_m, 0]$, with $\tau_m := -\frac{1}{m}(\lambda'_m + 1)$.

When $i \in \{0, m\}$, the windows are given by

$$W^{(a|a)}_{m,0} = [\lambda'_m, 1[\,, \qquad W^{(a|b)}_{m,0} =]\lambda'_m, 1]\,,$$

$$W^{(a|a)}_{m,m} =]-1, -\lambda'_m]\,, \qquad W^{(b|a)}_{m,m} = [-1, -\lambda'_m[\,,$$

distinguished according to the legal two-letter seeds. □

A direct consequence of this proposition and Theorem 1.21 is the following result.

Corollary 2.4. *([Mo13, Cor. 5.20]) For an arbitrary but fixed $m \in \mathbb{N}$, and for every $0 \le i \le m$, the generating noble means sets $\Lambda_{m,i}$ (and therefore all noble means sets) are Meyer sets.* □

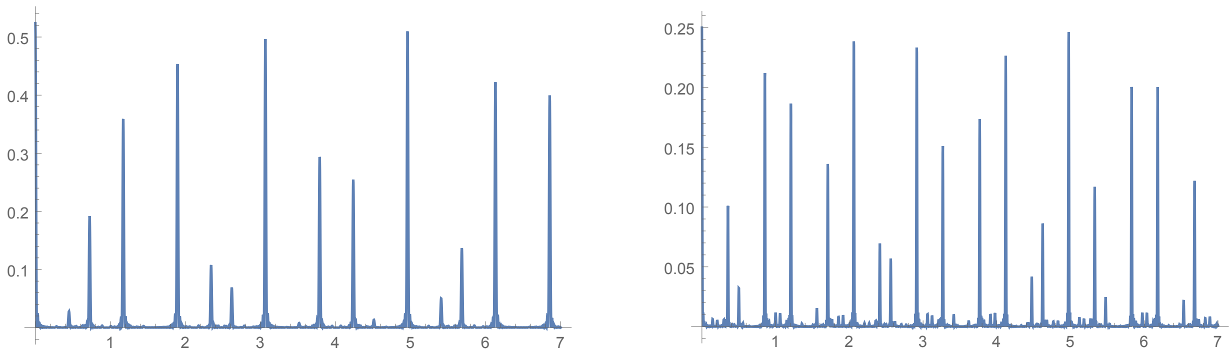

Figure 2.1: The Bragg peaks of the deterministic substitution $\zeta_{1,i}$ (left) and $\zeta_{2,i}$ (right) for $i = 0, 1, 2$.

2.1.3 Diffraction of the noble means sets

To complete this section, we will determine the diffraction measures of the noble means sets. We already know that each element of $\mathbb{Y}'_m$ can be viewed as a (translate of a) regular Euclidean model set, wherefore we can make use of Theorem 1.32 (compare [BG13, Rem. 9.14] for the singular cases $i \in \{0, m\}$). As we already know that

$$L^{\circledast} = \pi_1(\mathcal{L}^*_m) = \frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2+4}}, \tag{2.5}$$

the only thing left to do is the computation of the amplitudes $A(k)$. A short calculation shows that

$$A(k) = \mathrm{dens}(\Lambda_{m,i})\; \mathrm{e}^{\pi \mathrm{i} k^\star (\lambda'_m+1)(1-2i/m)}\; \mathrm{sinc}\left(\pi k^\star (1-\lambda'_m)\right);$$

see [Mo13, Cor. 6.10]. Hence, we conclude as follows.

Proposition 2.5. *For an arbitrary but fixed $m \in \mathbb{N}$, the diffraction measure of an arbitrary element of $\mathbb{Y}'_m$ is a positive and positive definite, translation bounded pure point measure. It is explicitly given by*

$$\widehat{\gamma} = \sum_{k \in L^{\circledast}} I(k)\, \delta_k,$$

where

$$I(k) = \left(\frac{1-\lambda'_m}{\sqrt{m^2+4}} \cdot \mathrm{sinc}\left(\pi k^\star (1-\lambda'_m)\right)\right)^2 .$$

□

Let us extend this result a little further. Consider a generating noble means set $\Lambda_{m,i} = \Lambda_a \,\dot{\cup}\, \Lambda_b$. The amplitudes can be decomposed into $A(k) = A_{\Lambda_a}(k) + A_{\Lambda_b}(k)$, where

$$A_{\Lambda_a}(k) = \mathrm{dens}(\mathcal{L}_m) \cdot \widehat{1}_{W^{(a)}_{m,i}}(-k^\star) = \frac{1}{\sqrt{m^2+4}}\; \mathrm{e}^{\pi \mathrm{i} k^\star (1-2i(\lambda'_m+1)/m)}\; \mathrm{sinc}(\pi k^\star),$$

$$A_{\Lambda_b}(k) = \mathrm{dens}(\mathcal{L}_m) \cdot \widehat{1}_{W^{(b)}_{m,i}}(-k^\star) = \frac{1}{\sqrt{m^2+4}}\,|\lambda'_m|\ \mathrm{e}^{\pi \mathrm{i} k^\star(\lambda'_m - 2i(\lambda'_m+1)/m)}\ \mathrm{sinc}(\pi\lambda'_m k^\star).$$

Consequently, when we consider the Dirac comb $\omega = u_a\delta_{\Lambda_a} + u_b\delta_{\Lambda_b}$ with weights $u_a, u_b \in \mathbb{C}$, the corresponding intensities read

$$I(k) = |u_a A_{\Lambda_a}(k) + u_b A_{\Lambda_b}(k)|^2.$$

This result can be generalised further via deformed model sets, see [BD00]. Here, we look at the modified hull $\mathbb{Y}'_{m,\alpha}$ that emerges from $\mathbb{X}'_m$ by taking a and b type intervals of lengths

$$\ell_a = \lambda_m + \alpha\lambda'_m \quad \text{ and } \quad \ell_b = 1 + \alpha,$$

respectively, where $\alpha \in]-1, \lambda_m^2[$ is a real parameter. One should note that the average tile length, and hence also the point density, does not change for different choices of α. By [BD00], we obtain the following corollary of Proposition 2.5.

Corollary 2.6. *Consider the dynamical system $(\mathbb{Y}'_{m,\alpha}, \mathbb{R})$ with parameter α as above. Select any element $\Lambda = \Lambda_a \dot\cup \Lambda_b$ from $\mathbb{Y}'_{m,\alpha}$. Then, the Dirac comb $\omega = u_a\delta_{\Lambda_a} + u_b\delta_{\Lambda_b}$, with complex weights $u_a, u_b \in \mathbb{C}$, is pure point diffractive, with diffraction measure*

$$\widehat{\gamma} = \sum_{k \in L^{\circledast}} I(k)\,\delta_k,$$

where $I(k) = |u_a A_{\Lambda_a} + u_b A_{\Lambda_b}|^2$ with

$$A_{\Lambda_a}(k) = \frac{1}{\sqrt{m^2+4}}\ \mathrm{e}^{\pi\mathrm{i}(1-2i(\lambda'_m+1)/m)(k^\star - \alpha k)}\ \mathrm{sinc}\big(\pi(k^\star - \alpha k)\big),$$

$$A_{\Lambda_b}(k) = \frac{1}{\sqrt{m^2+4}}\,|\lambda'_m|\ \mathrm{e}^{\pi\mathrm{i}(\lambda'_m - 2i(\lambda'_m+1)/m)(k^\star - \alpha k)}\ \mathrm{sinc}\big(\pi\lambda'_m(k^\star - \alpha k)\big).$$

□

2.2 A stochastic generalisation

It is a well-known fact that the topological dynamical system $(\mathbb{X}'_m, \mathbb{Z})$ has zero entropy [BG13, Prop. 4.13]. This can also be derived from [BLR07]. In this section, we generalise the noble means substitutions $\zeta_{m,i}$ to obtain a new topological dynamical system $(\mathbb{X}_m, \mathbb{Z})$ with positive entropy. This approach, which we will call *local mixtures of substitutions* in the following, was first considered in [GL89, Sec. 5.1]. In the same paper, one also finds heuristic considerations concerning the computation of topological entropy and the spectral type of the diffraction measure. We begin this section with the introduction of random substitutions; compare [Mo13, Sec. 2.2].

2.2.1 Random substitutions

Definition 2.7. A substitution $\vartheta : \mathcal{A}^* \to \mathcal{A}^*$ is called *stochastic* or a *random substitution* if there are $k_1, \dots, k_n \in \mathbb{N}$ and probability vectors

$$\left\{ \boldsymbol{p}_i = (p_{i1}, \dots, p_{ik_i}) \mid \boldsymbol{p}_i \in [0,1]^{k_i} \text{ and } \sum_{j=1}^{k_i} p_{ij} = 1,\ 1 \leq i \leq n \right\},$$

such that

$$\vartheta : a_i \mapsto \begin{cases} w^{(i,1)}, & \text{with probability } p_{i1}, \\ \quad\vdots & \quad\quad\vdots \\ w^{(i,k_i)}, & \text{with probability } p_{ik_i}, \end{cases}$$

for $1 \leq i \leq n$ where each $w^{(i,j)} \in \mathcal{A}^*$. If $p_{ij} \neq 0$ for all values of i, j then we say that ϑ is non-degenerate; otherwise, we say that ϑ is degenerate. Moreover, the *substitution matrix* is defined by

$$M_\vartheta := \left(\sum_{q=1}^{k_j} p_{jq} |w^{(j,q)}|_{a_i} \right)_{1 \leq i,j \leq n} \in \mathrm{Mat}(n, \mathbb{R}_{\geq 0}).$$

Remark 2.8. Since ϑ is applied to each letter of a given word w, $\vartheta(a_i)$ and $\vartheta(w)$ have to be considered as random variables with finitely many realisations. This means that M_ϑ is actually the expectation of the substitution matrix, the latter also viewed as a random variable. We suppress such extensions, as we do not need them for our situation. $\diamond$

In the stochastic situation we need to modify the notion of the subword relation. For any $v, w \in A^*$, by $v \blacktriangleleft w$ we mean that v is a subword of at least one image of w under ϑ^k for some $k \in \mathbb{N}$. Similarly, $v \mathrel{\dot{=}} \vartheta^k(w)$ means that there is at least one image of w under ϑ^k that coincides with v.

Definition 2.9. A random substitution $\vartheta : \mathcal{A}^* \to \mathcal{A}^*$ is *irreducible* if for each pair (i, j) with $1 \leq i, j \leq n$, there is a power $k \in \mathbb{N}$ such that $a_i \blacktriangleleft \vartheta^k(a_j)$. The substitution ϑ is *primitive* if there is a $k \in \mathbb{N}$ such that $a_i \blacktriangleleft \vartheta^k(a_j)$ for all $1 \leq i, j \leq n$.

Remark 2.10. As in the deterministic case, a non-degenerate random substitution ϑ is irreducible/ primitive if and only if M_ϑ is an irreducible/primitive matrix. This follows immediately from Definition 2.9 and the definitions of Section 1.1.3. Note that degenerate random substitutions can be irreducible/primitive and have non-irreducible/non-primitive substitution matrix. $\diamond$

Also, in the stochastic setting, we agree on a slightly modified notion of legality of words.

Definition 2.11. Let $\vartheta : \mathcal{A}^* \to \mathcal{A}^*$ be a random substitution. A word w is called *legal* (or *ϑ-legal*) if there is a $k \in \mathbb{N}$ and $a_i \in \mathcal{A}$ such that $w \blacktriangleleft \vartheta^k(a_i)$.

2.2.2 The family of random noble means substitutions

Now, let us have a look at a stochastic generalisation of the noble means family. Let $m \in \mathbb{N}$ and $\boldsymbol{p}_m = (p_0, \ldots, p_m)$ be a probability vector that are both assumed to be fixed. The random substitution $\zeta_m : \mathcal{A}_2^* \to \mathcal{A}_2^*$ is defined by

$$\zeta_m : \begin{cases} a \mapsto \begin{cases} \zeta_{m,0}(a), & \text{with probability } p_0, \\ \vdots & \vdots \\ \zeta_{m,m}(a), & \text{with probability } p_m, \end{cases} \\ b \mapsto a, \end{cases}$$

and the one-parameter family $\mathcal{R} = \{\zeta_m\}_{m\in\mathbb{N}}$ is called the family of *random noble means substitutions (RNMS)*. We refer to the p_j as the *choosing probabilities* and call $\zeta_m(w)$ for any $w \in A_2^*$ an image of w under ζ_m. Let us usually assume that $\boldsymbol{p}_m \gg 0$ unless specified otherwise.

The substitution matrix in the sense of Definition 2.7 is given by

$$M_m := \begin{pmatrix} \sum_{j=0}^m p_j |\zeta_{m,j}(a)|_a & 1 \\ \sum_{j=0}^m p_j |\zeta_{m,j}(a)|_b & 0 \end{pmatrix} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix},$$

which is the same as in the deterministic case[1].

Lemma 2.12. *For all $m \in \mathbb{N}$, the substitution ζ_m is primitive.*

Proof. This is an immediate consequence of Definition 2.9. □

For $\ell \geq 0$, we define the ζ_m-dictionary

$$\mathcal{D}_m := \{w \in \mathcal{A}_2^* \mid w \text{ is } \zeta_m - \text{legal}\} \quad \text{and} \quad \mathcal{D}_{m,\ell} := \{w \in \mathcal{D}_m \mid |w| = \ell\}.$$

Every word $w \stackrel{\bullet}{=} \zeta_m^k(b)$, for some $k \in \mathbb{N}_0$, is called an *exact substitution word*. We define for any $k > 1$ the set of exact substitution words (of order k) as

$$\mathcal{G}_{m,k} := \{w \in \mathcal{A}_2^* \mid w \stackrel{\bullet}{=} \zeta_m^{k-1}(b)\}.$$

Obviously, all subwords of legal words are legal words again, and non-empty legal words are mapped to non-empty legal words by ζ_m because if $w \in \mathcal{D}_m$, there is a $k \in \mathbb{N}$ with $w \blacktriangleleft \zeta_m^k(b)$. Applying ζ_m once more, immediately leads to $\zeta_m(w) \blacktriangleleft \zeta_m^{k+1}(b)$ which is the legality of $\zeta_m(w)$.

Usually, the two-sided discrete hull of primitive substitutions is defined via fixed points. This is no longer possible in the stochastic setting, since there is no direct analogue of a fixed point. Consequently, we have to modify the definition of the hull of a primitive random substitution.

[1] Note that M_m is not only the expectation value, but also the substitution matrix of every individual substitution $\zeta_{m,i}$.

Definition 2.13. For an arbitrary but fixed $m \in \mathbb{N}$, define

$$X_m := \left\{ w \in \mathcal{A}_2^{\mathbb{Z}} \mid w \text{ is an accumulation point of } \left(\zeta_m^k(a|a)\right)_{k\in\mathbb{N}_0} \right\},$$

where 'accumulation' point is meant in the sense of one for any of the possible realisations of the random substitution sequence. The *two-sided discrete stochastic hull* $\mathbb{X}_m$ is defined as the smallest closed and shift-invariant subset of $\mathcal{A}_2^{\mathbb{Z}}$ with $X_m \subseteq \mathbb{X}_m$. Elements of X_m are called *generating random noble means words*.

By definition, $\mathbb{X}_m$ is a closed subset of the compact Hausdorff space $\mathcal{A}_2^{\mathbb{Z}}$, which is also metrisable. Hence, the two-sided discrete stochastic hull is compact, wherefore each sequence $\left(\zeta_m^k(a|a)\right)_{k\in\mathbb{N}}$ possesses at least one accumulation point, $\mathbb{X}_m$ is non-empty.

Let us state some further properties of $\mathbb{X}_m$, compare Eq. (2.2).

Proposition 2.14. *([Mo13, Prop. 2.22]) For any $m \in \mathbb{N}$, we have*

1. $\mathbb{X}_m = \{w \in \mathcal{A}_2^{\mathbb{Z}} \mid \mathcal{W}(\{w\}) \subseteq \mathcal{D}_m\}$,
2. $\mathbb{X}_m$ *is invariant under* ζ_m,
3. $\mathbb{X}'_m \subsetneqq \mathbb{X}_m$.

□

Remark 2.15. It follows from Eq. (2.1) that the global mixture of the substitutions in $\mathcal{N}_m$ does not enlarge the two-sided discrete hull, whereas we have just seen (Proposition 2.14) that the local mixture leads to a larger hull, since $\mathbb{X}'_m \subsetneqq \mathbb{X}_m$. ◇

2.2.3 Random noble means sets

Like in Section 2.1.2, we want to generate geometric realisations in $\mathbb{R}$ using the random substitution ζ_m. For this, we consider the geometric realisations of accumulation points of the sequence $\left(\zeta_m^k(a|a)\right)_{k\in\mathbb{N}_0}$, which we call *generating random noble means sets*. Any instance of these is referred to as Λ_m.

Remark 2.16. Again, we find that

$$\Lambda_m \subseteq \mathbb{Z}[\lambda_m]$$

and that the point density is obviously given by

$$\mathrm{dens}(\Lambda_m) = \frac{1-\lambda'_m}{\sqrt{m^2+4}};$$

compare Remark 2.2. ◇

Now, it is natural to ask whether the generating random noble means sets can be described as model sets with compact windows. Then, we would again benefit from Theorem 1.32. It turns out that this is not the case, since the entropy of regular model sets is 0, and the entropy of $(\mathbb{X}_m, S)$ is positive, see [Mo13, Ch. 3]. Still, one can obtain the following result.

Proposition 2.17. *Let Λ_m be any of the generating random Fibonacci sets from the previous remark. Then, one has $\Lambda_m \subsetneqq \curlywedge(W_m)$ with covering window $W_m := [\lambda'_m - 1, 1 - \lambda'_m]$, where $W_m = W_m^{(a)} \cup W_m^{(b)}$ with $W_m^{(a)} = [-1, 1-\lambda'_m]$, $W_m^{(b)} = [\lambda'_m - 1, -\lambda'_m]$ and*

$$\mathrm{dens}(\curlywedge(W_m)) = 2 \cdot \frac{1-\lambda'_m}{\sqrt{m^2+4}} = 2\,\mathrm{dens}(\Lambda_m).$$

Proof. Due to [Mo13, Prop. 5.21], it suffices to prove the claimed equalities. The first equality follows from the well-known formula $\mathrm{dens}(\curlywedge(W)) = \mathrm{dens}(\mathcal{L})\,\mathrm{vol}(W)$ and the fact that $\mathrm{dens}(\mathcal{L}_m) = \frac{1}{\sqrt{m^2+4}}$ and $\mathrm{vol}(W_m) = 2\,(1-\lambda'_m)$. The second equality is a direct consequence of the previous remark. □

Definition 2.18. For an arbitrary but fixed $m \in \mathbb{N}$ and $0 \le i \le m$, we define

$$\mathbb{Y}_{m,i} := \overline{\{t + \Lambda_{m,i} \mid t \in \mathbb{R}\}} \tag{2.6}$$

as the *continuous hull* of the inflation rule $\zeta_{m,i}$, where $\Lambda_{m,i}$ are the noble means sets from Section 2.1.2.

Note that the closure in Eq. (2.6) is taken with respect to the local topology. Here, two FLC point sets M and N are close if, after a small translation, they agree on a large interval. That is, if

$$M \cap \left]-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right[= (-t+N) \cap \left]-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right[$$

for some $t \in \left]-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right[$.

Definition 2.19. For any $m \in \mathbb{N}$, let

$$Y_m := \{\Lambda_m \mid \Lambda_m \text{ is generating random noble means set}\}$$

and define the *continuous stochastic hull* $\mathbb{Y}_m$ of the inflation rule ζ_m as the smallest closed and translation-invariant subset $\mathfrak{D}(\mathbb{R})$ (the set of Delone subsets of $\mathbb{R}$) with $Y_m \subseteq \mathbb{Y}_m$. The elements of Y_m are called *random noble means sets*.

At the end of this section, we can formulate the following consequence of Proposition 2.17 and Theorem 1.21.

Theorem 2.20. *([Mo13, Thm. 5.25]) Each random noble means set $\Lambda \in \mathbb{Y}_m$ is a Meyer set.* □

3 Diffraction of the RNMS

In this chapter, finally, we will study the diffraction measure of the random noble means sets, which means to determine the average over the diffraction measures of the elements of $\mathbb{Y}_m$ with respect to an invariant measure on it. The latter is chosen as the patch frequency measure ν_m. Note that ν_m is a completely natural choice, and is both translation invariant and ergodic; see [Mo13, Ch. 4] and [Goh17].

In order to do so, we will explain the approach discussed in [GL89] in Section 3.1. To understand the splitting of the diffraction measure, which is one of the main ideas in the approach of Godrèche and Luck, we introduce weakly and strongly almost periodic measures in Section 3.2. Eventually, we give explicit formulas for the continuous and pure point part of the diffraction measure in Sections 3.3 and 3.4.

Before we start with explicit calculations, let us find out how much information we can gather from abstract theory. It was shown in [Mo13, Ch. 3] that, for fixed $m \in \mathbb{N}$, the substitution ζ_m has positive entropy. Considering recent developments [BLR07] and taking the following result into account, we expect to find a diffraction spectrum of mixed type in the RNMS cases.

Proposition 3.1. *([Str05, Prop. 3.12]) Let P be a Meyer set and suppose that its autocorrelation γ_P exists. Then, the set of Bragg peaks lies relatively dense. Moreover, if P is not pure point diffractive, it has a relatively dense support for the continuous spectrum as well.* □

3.1 An approach via concatenation

Let us be more concrete. Also in the stochastic situation, the diffraction measure can be achieved as follows. We consider the Dirac combs

$$\delta_{\Lambda_m} := \sum_{x \in \Lambda_m} \delta_x \quad \text{ and } \quad \delta_{\Lambda_{m,n}} := \sum_{x \in \Lambda_{m,n}} \delta_x$$

with $\Lambda_{m,n} := \Lambda_m \cap B_n$ for any random noble means set $\Lambda_m \in \mathbb{Y}_m$. The diffraction measure is given by

$$\widehat{\gamma_{\Lambda_m}} = \lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \Big| \sum_{x \in \Lambda_{m,n}} \mathrm{e}^{-2\pi \mathrm{i} k x} \Big|^2 = \lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \big| X_n(k) \big|^2,$$

where $X_n(k) := \sum_{x\in\Lambda_{m,n}} \mathrm{e}^{-2\pi \mathrm{i} kx}$ and the limit is taken in the vague topology. It exists, and is the same, for ν_m-almost all $\Lambda \in \mathbb{Y}_m$, both as a consequence of the ergodic theorem. In general, it is difficult to compute such exponential sums. Here, we apply an idea from [GL89]. Using (a generalisation of) the corresponding concatenation rule (see [GL89, Eq. 5.9]), we obtain:

$$\mathcal{X}_n(k) := \sum_{r=0}^{j-1} \mathrm{e}^{-2\pi \mathrm{i} r \lambda_m^{n-1} k} \; \mathcal{X}_{n-1}^{(r)}(k) + \mathrm{e}^{-2\pi \mathrm{i} j \lambda_m^{n-1} k} \; \mathcal{X}_{n-2}(k) + \sum_{r=j}^{m-1} \mathrm{e}^{-2\pi \mathrm{i} (\lambda_m^{n-2} + r\lambda_m^{n-1}) k} \; \mathcal{X}_{n-1}^{(r)}(k) \tag{3.1}$$

with probability p_j for $0 \leq j \leq m$ and $\mathcal{X}_0(k) := 1$ and $\mathcal{X}_1(k) := 1$. In fact, this sequence is a subsequence of $\big(X_n(k)\big)_{n\in\mathbb{N}_0}$ that corresponds to exact substitution words.

Remark 3.2. One should remark that the initial conditions $\mathcal{X}_0$ and $\mathcal{X}_1$ differ from the ones stated in [GL89]. This is due to the fact that we chose the left endpoints of the intervals I_a and I_b in Section 2.1.2. However, this will not affect the final result. ◇

Note the following. It does not matter that the random variables $\mathcal{X}_n$ are defined via concatenation and not via the substitution rule. Due to [Mo13, Lem. 2.29] these two approaches lead to the same exact substitution words with the same probabilities.

Together with the ergodicity of ν_m, we obtain

$$\widehat{\gamma_{\Lambda_m}} = \mathbb{E}\big(\widehat{\gamma_{\Lambda_m}}\big) = \lim_{n\to\infty} \frac{1}{\lambda_m^n} \mathbb{E}(|\mathcal{X}_n|^2) = \lim_{n\to\infty} \frac{1}{\lambda_m^n} \left|\mathbb{E}(\mathcal{X}_n)\right|^2 + \lim_{n\to\infty} \frac{1}{\lambda_m^n} \operatorname{Var}(\mathcal{X}_n), \tag{3.2}$$

for ν_m-almost all Λ_m, as long as the two limits on the right hand side exist.

3.2 Weakly almost periodic measures

There is a reason why we decomposed the diffraction measure as shown in Eq. (3.2). To be able to understand this, we need some preparation. For further details, see [Str05, BG17].

Definition 3.3. A measure μ on $\mathbb{R}^d$ is called *Fourier transformable* if there exists a measure $\widehat{\mu}$ on $\mathbb{R}^d$, called the Fourier transform of μ, such that

$$\widehat{\mu}(g) = \mu(\widehat{g}),$$

for all $g \in \mathcal{K}_2(\mathbb{R}^d) := \operatorname{span}\{f * g \mid f, g \in C_\mathrm{c}(\mathbb{R}^d)\}$.

If μ is also translation bounded, the Fourier transform as a measure and tempered distribution coincide. The basic properties of Fourier transformable measures can be found in [BF75, Ch. 1] or [AL90, Chs. 10-11].

Definition 3.4. The map $\mu \mapsto \{\mu * f\}_{\{f\in C_c(\mathbb{R}^d)\}}$ is an embedding of $\mathcal{M}^\infty(\mathbb{R}^d)$ in the space $[C_U(\mathbb{R}^d)]^{C_c(\mathbb{R}^d)}$. Giving $[C_U(\mathbb{R}^d)]^{C_c(\mathbb{R}^d)}$ the usual product topology, the induced topology on $\mathcal{M}^\infty$ is called the *product topology*. We will also refer to this topology as the *strong topology*. The *weak topology* is defined by the dual space of $\mathcal{M}^\infty(\mathbb{R}^d)$.

Definition 3.5. Let μ be a translation bounded measure on $\mathbb{R}^d$. Let $D_\mu := \{\delta_x * \mu\}_{x\in\mathbb{R}^d}$ and C_μ be its closed convex hull. We say that μ is *amenable* if C_μ contains exactly one scalar multiple μ_0 of $\boldsymbol{\lambda}$. In this case we write

$$\mu_0 = M(\mu)\,\boldsymbol{\lambda},$$

and we call $M(\mu)$ the *mean* of μ.

We say that $f \in C_U(\mathbb{R}^d)$ is *amenable* if C_f, the closed convex hull of $D_f := \{\delta_x * f\}_{x\in\mathbb{R}^d}$, contains exactly one constant function.

Remark 3.6. One can prove that $f \in C_U(\mathbb{R}^d)$ is amenable if and only if the measure $f\,d\boldsymbol{\lambda}$ is amenable. In this case we define: $M(f) = M(f\,\boldsymbol{\lambda})$. $\diamond$

Definition 3.7. The function $f \in C_U(\mathbb{R}^d)$ is called *strongly almost periodic* if C_f is compact in the strong topology. f is called *weakly almost periodic* if C_f is compact in the weak topology. Last, f is called *null weakly almost periodic* if f is weakly almost periodic and $M(|f|) = 0$.

We denote by $\mathrm{SAP}(\mathbb{R}^d)$, $\mathrm{WAP}(\mathbb{R}^d)$ and $\mathrm{WAP}_0(\mathbb{R}^d)$ the spaces of *strongly, weakly, and null weakly almost periodic functions on* $\mathbb{R}^d$, respectively.

These definitions extend to translation bounded measures.

Definition 3.8. A measure $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ is called *strongly almost periodic* if C_μ is compact in the product topology and μ is called *weakly almost periodic* if C_μ is compact in the weak topology. We denote by $\mathcal{SAP}(\mathbb{R}^d)$ and $\mathcal{WAP}(\mathbb{R}^d)$ the spaces of strongly and respectively weakly almost periodic measures on $\mathbb{R}^d$. A translation bounded measure μ on $\mathbb{R}^d$ is called *null weakly almost periodic* if and only if for each $g \in C_c(\mathbb{R}^d)$, the function $g * \mu$ is a null weakly almost periodic function. The corresponding space of measures is denoted by $\mathcal{WAP}_0(\mathbb{R}^d)$.

For these properties we can talk about a duality between measures and functions.

Proposition 3.9. *([AL90, Cor. 5.4 and Cor. 5.5]) If P is the property of being strongly, weakly, null weakly almost periodic or amenable, then the following statements are true:*

(a) $f \in C_U(\mathbb{R}^d)$ has property P if and only if $f\,\boldsymbol{\lambda} \in \mathcal{M}^\infty(\mathbb{R}^d)$ has property P.

(b) $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ has property P if and only if $f*\mu$ has property P for every $f \in C_c(\mathbb{R}^d)$.

□

Concerning the diffraction, there is an important decomposition of weakly almost periodic measures.

Proposition 3.10. *([BG17, Thm 4.10.10]) Let* $\mu \in \mathcal{WAP}(\mathbb{R}^d)$*. Then,* μ *can be written uniquely in the form*

$$\mu = \mu_s + \mu_0,$$

with $\mu_s \in \mathcal{SAP}(\mathbb{R}^d)$ *and* $\mu_0 \in \mathcal{WAP}_0(\mathbb{R}^d)$*.* □

The decomposition from Proposition 3.10 is called the *Eberlein decomposition* of μ. The connection to the diffraction is the following proposition.

Proposition 3.11. *([BG17, Thm. 4.10.12]) Let* μ *be a Fourier transformable and translation bounded measure on* $\mathbb{R}^d$*. Then,* μ_s *and* μ_0 *are Fourier transformable, and one has*

$$(\widehat{\mu})_{\mathrm{pp}} = (\widehat{\mu_s}) \quad \textit{and} \quad (\widehat{\mu})_{\mathrm{c}} = (\widehat{\mu_0}).$$

□

It will turn out that

$$\lim_{n\to\infty} \frac{1}{\lambda_m^n} \left|\mathbb{E}(\mathcal{X}_n)\right|^2 = \left(\widehat{\gamma_\Lambda}\right)_{\mathrm{pp}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_m^n} \operatorname{Var}(\mathcal{X}_n) = \left(\widehat{\gamma_\Lambda}\right)_{\mathrm{c}},$$

for ν_m-almost all $\Lambda \in \mathbb{Y}_m$.

3.3 The continuous part of the diffraction measure

We proceed with the derivation $\lim_{n\to\infty} \lambda_m^{-n} \operatorname{Var}(\mathcal{X}_n)$; for more details see [Mo13, Sec. 6.2.3]. Here, we restrict (for the moment) to the case $m = 1$. The general case can be handled in the same way, but is technically more involved. To this end, we derive recursion formulas for $\mathbb{E}(\mathcal{X}_n(k))$ and $\operatorname{Var}(\mathcal{X}_n(k))$. For the sake of readability, we introduce the following abbreviations:

$$\mathbb{E}_n := \mathbb{E}(\mathcal{X}_n(k)) \quad \text{and} \quad \mathbb{V}_n := \operatorname{Var}(\mathcal{X}_n(k)),$$

for any $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Using the definition of $\mathcal{X}_n$, it is immediate that, for $n \geq 2$, we have

$$\mathbb{E}_n = (p_1 + p_0\, \mathrm{e}^{-2\pi\mathrm{i}k\lambda_1^{n-2}})\mathbb{E}_{n-1} + (p_0 + p_1\, \mathrm{e}^{-2\pi\mathrm{i}k\lambda_1^{n-1}})\mathbb{E}_{n-2}, \tag{3.3}$$

where $\mathbb{E}_0 = 1$ and $\mathbb{E}_1 = 1$. Applying Eq. (3.3) for any $n \geq 2$, we find

$$\mathbb{V}_n = \mathbb{V}_{n-1} + \mathbb{V}_{n-2} + 2p_0p_1\Psi_n,$$

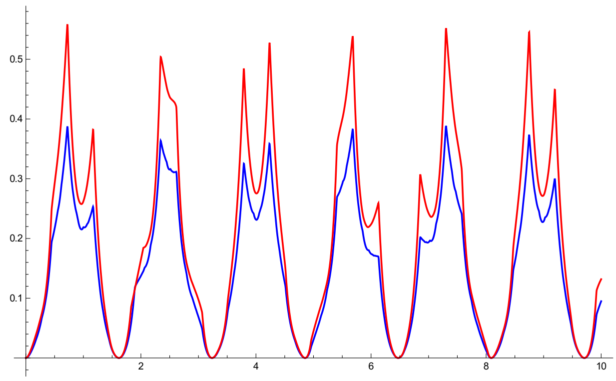

Figure 3.1: The density function ϕ is shown with probability vector $\boldsymbol{p}_1 = (0.5, 0.5)$ (red) respectively $\boldsymbol{p}_1 = (0.8, 0.2)$ (blue).

where $\mathbb{V}_0 = \mathbb{V}_1 = 0$ and

$$\Psi_n := \Psi_n(k) := \frac{1}{2}\left|(1 - \mathrm{e}^{-2\pi \mathrm{i} k \lambda_1^{n-2}})\mathbb{E}_{n-1} - (1 - \mathrm{e}^{-2\pi \mathrm{i} k \lambda_1^{n-1}})\mathbb{E}_{n-2}\right|^2,$$

for any $n \geq 2$. One can obtain the following result.

Proposition 3.12. *([Mo13, Prop. 6.18 and Cor. 6.19]) For any $n \in \mathbb{N}_0$, consider the function*

$$\phi_n : \mathbb{R} \to \mathbb{R}_+, \quad k \mapsto \frac{1}{\lambda_1^n} \operatorname{Var}(\mathcal{X}_n(k)).$$

On $\mathbb{R}$, the sequence $(\phi_n)_{n \in \mathbb{N}_0}$ converges uniformly to the continuous function

$$\phi : \mathbb{R} \to \mathbb{R}_+, \quad k \mapsto \frac{2 p_0 p_1 \lambda_1}{\sqrt{5}} \sum_{i=2}^{\infty} \frac{\Psi_i(k)}{\lambda_1^i}.$$

Moreover, the roots of ϕ are precisely the roots of Ψ_2, and they are given by all integer multiples of λ_1. $\square$

It was also implicitly shown in [Mo13, Cor. 6.19] that

$$\Psi_n(k) = \Psi_2(k) \cdot \prod_{j=1}^{n-2} \left| p_0 + p_1 \,\mathrm{e}^{-2\pi \mathrm{i} \lambda_1^j k} \right|^2.$$

Thus, we obtain

Corollary 3.13. *The measure $\lim_{n\to\infty} \frac{1}{\lambda_1^n} \operatorname{Var}(\mathcal{X}_n)$ is absolutely continuous with respect to $\boldsymbol{\lambda}$. Its density function is*

$$\phi(k) = \frac{\lambda_1}{\sqrt{5}} \sum_{n=2}^{\infty} \frac{2 p_0 p_1 \big(1 - \cos(2\pi \lambda_1' k)\big)}{\lambda_1^n} \prod_{j=1}^{n-2} \left| p_0 + p_1 \,\mathrm{e}^{-2\pi \mathrm{i} \lambda_1^j k} \right|^2.$$

$\square$

Since $\prod_{j=1}^{n-2} \left| p_0 + p_1 \, \mathrm{e}^{-2\pi \mathrm{i} \lambda_1^j k} \right|^2 \leq 1$ and $1 - \cos(2\pi \lambda_1' k) \leq 2$, we obtain

$$\phi(k) \leq \frac{4 p_0 p_1 \lambda_1}{\sqrt{5}} \sum_{n=2}^{\infty} \frac{1}{\lambda_1^n} = \frac{4 p_0 p_1 \lambda_1}{\sqrt{5}}$$

for every $k \in \mathbb{R}$.

Remark 3.14. Note that Proposition 3.12 is consistent with what we know about the deterministic cases $\Lambda_{1,0}$ and $\Lambda_{1,1}$: Proposition 2.5 implies that the corresponding diffraction measure is pure point. Setting $p_0 = 0$ or $p_1 = 0$ in the representation of ϕ from Proposition 3.12 indeed shows that $\phi \equiv 0$, which means that the absolutely continuous part vanishes here. ◇

Let us turn our attention at the end of this section to the general case $m \in \mathbb{N}$. Exactly as before, it can be shown that that $\lim_{n\to\infty} \frac{1}{\lambda_m^n} \operatorname{Var}(\mathcal{X}_n)$ is an absolutely continuous measure. Its density function ϕ can again be represented by

$$\phi(k) = \frac{\lambda_m}{\sqrt{m^2+4}} \sum_{n=2}^{\infty} \frac{\psi_m^{(n)}(k)}{\lambda_m^n},$$

where $\psi_m^{(n)}$ are uniformly bounded, continuous functions on $\mathbb{R}$.

3.4 The pure point part of the diffraction measure

Now, let us focus on the limit $\lim_{n\to\infty} \frac{1}{\lambda_m^n} \left| \mathbb{E}(\mathcal{X}_n) \right|^2$. Let $\mathcal{M}_n$ denote the finite random Dirac comb such that $\widehat{\mathcal{M}_n} = \mathcal{X}_n$. Thus, we obtain

$$\mathcal{M}_n = \sum_{r=0}^{j-1} \left(\delta_{r\lambda_m^{n-1}} * \mathcal{M}_{n-1}^{(r)} \right) + \delta_{j\lambda_m^{n-1}} * \mathcal{M}_{n-2} + \sum_{r=j}^{m-1} \left(\delta_{\lambda_m^{n-2} + r\lambda_m^{n-1}} * \mathcal{M}_{n-1}^{(r)} \right),$$

with probability p_j for $0 \leq j \leq m$ and $\mathcal{M}_0 = \mathcal{M}_1 = \delta_0$; compare (3.1). By definition, we have $|\zeta_m(a)|_a = m$ and $|\zeta_m(a)|_b = 1$ for every realisation of $\zeta_m(a)$. Thus, the geometric length of $\zeta_m(w)$ is independent of the realisation for every legal word w. Consequently, it is not difficult to verify the following fact.

Fact 3.15. *For any $n \in \mathbb{N}$, all realisations of the random Dirac comb M_n have support in the finite point set $⅄(W_m) \cap [0, \lambda_m^n[$.* □

Here, we can identify $\mathcal{M}_n$ with a random variable with values in $\Omega := \{0,1\}^{\Lambda_{\geq 0}}$, where $\Lambda_{\geq 0} := \{x \in ⅄(W_m) \mid x \geq 0\}$. In this case, we obtain sequences $(m_x)_{x \in \Lambda_{\geq 0}}$ with $m_x \in \{0,1\}$. We denote by θ_n the corresponding (discrete) probability distribution on Ω.

Lemma 3.16. *Fix $m \in \mathbb{N}$ and let $\boldsymbol{p}_m \gg 0$ be a fixed probability vector. Then, the sequence $(\theta_n)_{n\in\mathbb{N}}$ of probability measures on Ω is weakly convergent.*

Proof. By standard arguments from probability theory, it suffices to show the convergence of $(\theta_n(Z))_{n\in\mathbb{N}}$ for every cylinder set Z that is specified at a finite set of positions in $\Lambda_{\geq 0}$. A simple inclusion-exclusion argument shows that this is equivalent to the convergence of $\theta_n(\{m_x = 1 \text{ for } x \in F\})$ for any finite set $F \subseteq \Lambda_{\geq 0}$. Define

$$g^{(n)}(x) := \theta_n(\{m_x = 1\}) \tag{3.4}$$

and write $g^{(n)}(x) = g_a^{(n)}(x) + g_b^{(n)}(x)$, with obvious meaning. Then, $g^{(n)}(x)$ is the occupation probability of position x under θ_n, split into those for type a and b. By an application of the random inflation rule ζ_m, one obtains

$$\begin{aligned} g_a^{(n+1)}(x) &= \sum_{\ell=0}^{m} p_\ell \left[\sum_{j=0}^{\ell-1} g_a^{(n)}(\mathfrak{f}_j^{-1}(x)) + \sum_{j=\ell}^{m-1} g_a^{(n)}(\mathfrak{g}_j^{-1}(x)) + g_b^{(n)}(\mathfrak{f}_0^{-1}(x)) \right], \\ g_b^{(n+1)}(x) &= \sum_{\ell=0}^{m} p_\ell \, g_a^{(n)}(\mathfrak{f}_\ell^{-1}(x)), \end{aligned} \tag{3.5}$$

together with the initial condition $g_\alpha^{(0)}(x) = \delta_{\alpha,a}\,\delta_{x,0}$, for $\alpha \in \{a, b\}$, and $g^{(n)}(x) = 0$ for all $x \notin \Lambda_{\geq 0}$, where $\mathfrak{f}_j(x) = \lambda_m x + j\lambda_m$ and $\mathfrak{g}_j(x) = \lambda_m x + j\lambda_m + 1$ are the functions of the iterated function system $\mathcal{F}$ in [Mo13, Sec. 5.3.2]. Considering $x = 0$, we obtain

$$\begin{pmatrix} g_a^{(n+1)}(0) \\ g_b^{(n+1)}(0) \end{pmatrix} = \begin{pmatrix} 1 - p_0 & 1 \\ p_0 & 0 \end{pmatrix} \begin{pmatrix} g_a^{(n)}(0) \\ g_b^{(n)}(0) \end{pmatrix}. \tag{3.6}$$

Since $\boldsymbol{p}_m \gg 0$, the matrix on the right hand side is primitive. Moreover it is the transpose of a Markov matrix, which means that its Perron–Frobenius eigenvalue equals 1. This implies the convergence of the sequence $(g_\alpha^{(n)}(0))_{n\in\mathbb{N}}$. Next, we write $\Lambda_{\geq 0} = \{0 = x_0 < x_1 < x_2 < \ldots\}$. The convergence of $(g_\alpha^{(n)}(x_i))_{n\in\mathbb{N}}$ follows inductively, because every function value $g_\alpha^{(n)}(x_i)$ is determined by function values $g_\alpha^{(n)}(x_j)$ with $j < i$. This establishes the convergence of the marginals for all cylinder sets that are specified at a single location.

A similar argument, based on the propagation of prefix probabilities, also works for the cylinder sets specified on a finite set of positions. Here, given any finite set $\varnothing \neq F \subseteq \Lambda_{\geq 0}$, one chooses an integer n such that the geometric realisation (as a patch) of any legal word of length n is longer than the largest element of F. We consider this as a collection of prefix patches. Taking all possible inflations of these patches and sorting them according to the same prefix collection, we derive a transition matrix for the prefix collection under one inflation step, which is a Markov matrix by construction. When $\boldsymbol{p}_m \gg 0$, this matrix is primitive. This follows immediately from the primitivity of the substitution. Now, we equip the prefix patches with initial probabilities, for instance via a set of exact inflation patches of sufficient size (we know that this is possible because we only look at legal words of length n). An iteration of the primitive Markov matrix gives a converging sequence of prefix probability vectors (independent of the initial probabilities), with the limit being independent of the initial choice. This implies the convergence of $\theta_n(m_x = 1 \text{ for } x \in F)$ as $n \to \infty$, and our claim follows. $\square$

Let us denote by g_a and g_b the pointwise limits of $\big(g_a^{(n)}\big)_{n\in\mathbb{N}}$ respectively $\big(g_b^{(n)}\big)_{n\in\mathbb{N}}$. From Eq. (3.5) we obtain the relations

$$\begin{aligned} g_a(x) &= \sum_{\ell=0}^{m} p_\ell \left[\sum_{j=0}^{\ell-1} g_a(\mathfrak{f}_j^{-1}(x)) + \sum_{j=\ell}^{m-1} g_a(\mathfrak{g}_j^{-1}(x)) + g_b(\mathfrak{f}_0^{-1}(x)) \right], \\ g_b(x) &= \sum_{\ell=0}^{m} p_\ell \, g_a(\mathfrak{f}_\ell^{-1}(x)), \end{aligned} \tag{3.7}$$

together with $g_a(x) = g_b(x) = 0$ for all $x \notin \Lambda_{\geq 0}$.

Proposition 3.17. *Let $\boldsymbol{p}_m \gg 0$. Then, the space of solutions of the renormalisation relations from Eq.* (3.7) *is one-dimensional. Moreover, the support of the function $g^{(n)}$ from Eq.* (3.4) *is a subset of $\Lambda^{(n)} := \Lambda \cap [0, \lambda_m^n[$, for every $n \in \mathbb{N}$.*

Proof. Consider Eq. (3.6). When we let n tend to infinity, we find

$$\begin{pmatrix} g_a(0) \\ g_b(0) \end{pmatrix} = \begin{pmatrix} 1-p_0 & 1 \\ p_0 & 0 \end{pmatrix} \begin{pmatrix} g_a(0) \\ g_b(0) \end{pmatrix}.$$

This is an eigenvector equation with eigenvalue 1. The corresponding eigenspace is given by $\{t \cdot (1, p_0)^{\mathsf{T}} \mid t \in \mathbb{R}\}$. Since all other values $g_\alpha(x)$ can be determined recursively, the first claim follows.

The second claim follows recursively from Eqs. (3.5) and (3.6). $\square$

To determine this solution explicitly, we consider the measure θ on Ω, which is the weak limit of the sequence $(\theta_n)_{n\in\mathbb{N}}$. Lemma 3.16 implies the existence of a random variable $\mathcal{M}$ with law θ, which is the limit (in distribution) of the sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$. Moreover, we obtain

$$\mathbb{E}(\mathcal{M}) = \lim_{n\to\infty} \mathbb{E}(\mathcal{M}_n) = \sum_{x\in\Lambda_{\geq 0}} \theta(\{m_x = 1\})\, \delta_x,$$

and additionally $\theta(\{m_x = 1\}) = g_a(x) + g_b(x) = g(x)$. Making use of the CPS we can represent the expectation values $\mathbb{E}(\mathcal{M}_n)$ and $\mathbb{E}(\mathcal{M})$ as

$$\mathbb{E}(\mathcal{M}_n) = \sum_{x\in\Lambda_{\geq 0}} h^{(n)}(x^\star)\, \delta_x \quad \text{and} \quad \mathbb{E}(\mathcal{M}) = \sum_{x\in\Lambda_{\geq 0}} h(x^\star)\, \delta_x,$$

where $h^{(n)} = h_a^{(n)} + h_b^{(n)}$. Now, define

$$\mu_a^{(n)} := \frac{c}{F_{m,n}} \sum_{x \in \curlywedge(W)} h_a^{(n)}(x^\star)\, \delta_{x^\star} \quad \text{and} \quad \mu_b^{(n)} := \frac{c}{F_{m,n}} \sum_{x \in \curlywedge(W)} h_b^{(n)}(x^\star)\, \delta_{x^\star},$$

where $F_{m,n}$ is recursively defined by $F_{m,n} = m\, F_{m,n-1} + F_{m,n-2}$ and $F_{m,0} = F_{m,1} = 1$ for fixed $m \in \mathbb{N}$ and every $n \in \mathbb{N}$, and $c > 0$ is a constant, which will be determined later. This means that $F_{m,n}$ is the number of tiles in the patch that underlies $\mathcal{M}_n$. The measures $\mu_a^{(n)}$ and $\mu_b^{(n)}$ are finite, pure point measure measures that are supported in $W \subseteq H = \mathbb{R}$ for every $n \in \mathbb{N}$. Now, consider the functions $f_j(x) = \lambda'_m x + j\lambda'_m$ and $g_j(x) = \lambda'_m x + j\lambda'_m + 1$, which are precisely the contractions of the iterated function system $\mathcal{F}^\star$, see Eq. (2.4) and [Mo13, Sec. 5.3.2]. A computation similar to [Mo13, Eq. 6.28] leads to

$$\begin{aligned} \mu_a^{(n+1)} &= \frac{F_{m,n}}{F_{m,n+1}} \sum_{q=0}^{m} p_q \left[\sum_{j=0}^{q-1} f_j.\mu_a^{(n)} + \sum_{j=q}^{m-1} g_j.\mu_a^{(n)} + f_0.\mu_b^{(n)} \right], \\ \mu_b^{(n+1)} &= \frac{F_{m,n}}{F_{m,n+1}} \sum_{q=0}^{m} p_q\, f_q.\mu_a^{(n)}, \end{aligned} \tag{3.8}$$

with initial condition $\mu_a^{(1)} = \delta_0$ and $\mu_b^{(1)} = 0$.

Lemma 3.18. *The sequences $\big(\mu_a^{(n)}\big)_{n\in\mathbb{N}}$ and $\big(\mu_b^{(n)}\big)_{n\in\mathbb{N}}$ of finite measures are weakly converging. The limit measures μ_a and μ_b are compactly supported and satisfy the system of rescaling equations*

$$\begin{aligned} \mu_a &= |\lambda'_m| \sum_{q=0}^{m} p_q \left[\sum_{j=0}^{q-1} f_j.\mu_a + \sum_{j=q}^{m-1} g_j.\mu_a + f_0.\mu_b \right], \\ \mu_b &= |\lambda'_m| \sum_{q=0}^{m} p_q\, f_q.\mu_a. \end{aligned} \tag{3.9}$$

Proof. The claimed convergence follows by an application of Levy's continuity theorem (the characteristic functions will be given explicitly below). Since the measure $\mu_a^{(n)}$ and $\mu_b^{(n)}$ are compactly supported in W, so are μ_a and μ_b. The limit measures satisfy the system of rescaling equations because of Eq. (3.8) and the fact that $\lim_{n\to\infty} \frac{F_{m,n}}{F_{m,n+1}} = |\lambda'_m|$. $\square$

Next, we wish to obtain an explicit formula for μ_a and μ_b. Applying the Fourier transform in Eq. (3.9), we get

$$\widehat{\mu_a}(k) = |\lambda'_m| \sum_{q=0}^{m} p_q \left[\left(\sum_{j=0}^{q-1} \mathrm{e}^{-2\pi \mathrm{i} j \lambda'_m k} + \sum_{j=q}^{m-1} \mathrm{e}^{-2\pi \mathrm{i}(j\lambda'_m + 1)k} \right) \widehat{\mu_a}(\lambda'_m k) + \widehat{\mu_b}(\lambda'_m k) \right], \tag{3.10}$$

$$\widehat{\mu_b}(k) = |\lambda'_m| \left(\sum_{q=0}^{m} p_q\, \mathrm{e}^{-2\pi \mathrm{i} q \lambda'_m k} \right) \widehat{\mu_a}(\lambda'_m k). \tag{3.11}$$

We claim that a solution of this system (which is unique up to a constant multiple; compare Proposition 3.17) is given by

$$
\begin{aligned}
\widehat{\mu_a}(k) &= \widetilde{c}\, \mathrm{e}^{-\pi \mathrm{i} k}\, \mathrm{sinc}(\pi k) \prod_{j\geq 1} \left(\sum_{q=0}^{m} p_q\, \mathrm{e}^{-2\pi \mathrm{i} q (\lambda'_m)^j k} \right), \\
\widehat{\mu_b}(k) &= \widetilde{c}\, |\lambda'_m|\, \mathrm{e}^{-\pi \mathrm{i} \lambda'_m k}\, \mathrm{sinc}(\pi \lambda'_m k) \prod_{j\geq 1} \left(\sum_{q=0}^{m} p_q\, \mathrm{e}^{-2\pi \mathrm{i} q (\lambda'_m)^j k} \right).
\end{aligned}
\tag{3.12}
$$

To see this, set $P_\ell(k) := \prod_{j\geq \ell} \left(\sum_{q=0}^{m} p_q\, \mathrm{e}^{-2\pi \mathrm{i} q (\lambda'_m)^j k} \right)$. Let us start with Eq. (3.10).

$$
\begin{aligned}
&|\lambda'_m| \sum_{q=0}^{m} p_q \left[\left(\sum_{j=0}^{q-1} \mathrm{e}^{-2\pi \mathrm{i} j \lambda'_m k} + \sum_{j=q}^{m-1} \mathrm{e}^{-2\pi \mathrm{i} (j\lambda'_m + 1) k} \right) \widehat{\mu_a}(\lambda'_m k) + \widehat{\mu_b}(\lambda'_m k) \right] \\
&\quad = \widetilde{c}\, P_2(k)\, \mathrm{e}^{-\pi \mathrm{i} k} \sum_{q=0}^{m} p_q \Bigg[\sum_{j=0}^{q-1} |\lambda'_m|\, \mathrm{e}^{-\pi \mathrm{i} k (2j\lambda'_m + \lambda'_m - 1)}\, \mathrm{sinc}(\pi \lambda'_m k) \\
&\qquad\qquad + \sum_{j=q}^{m-1} |\lambda'_m|\, \mathrm{e}^{-\pi \mathrm{i} k (2j\lambda'_m + \lambda'_m + 1)}\, \mathrm{sinc}(\pi \lambda'_m k) \\
&\qquad\qquad + (\lambda'_m)^2\, \mathrm{e}^{-\pi \mathrm{i} k ((\lambda'_m)^2 - 1)}\, \mathrm{sinc}(\pi (\lambda'_m)^2 k) \Bigg] \\
&\quad = \widetilde{c}\, P_2(k)\, \mathrm{e}^{-\pi \mathrm{i} k} \sum_{q=0}^{m} p_q \Bigg[\sum_{j=0}^{q-1} \widehat{1}_{[(j+1)\lambda'_m - \frac{1}{2},\, j\lambda'_m - \frac{1}{2}]}(k) + \sum_{j=q}^{m-1} \widehat{1}_{[(j+1)\lambda'_m + \frac{1}{2},\, j\lambda'_m + \frac{1}{2}]}(k) \\
&\qquad\qquad + \widehat{1}_{[-\frac{1}{2},\, (\lambda'_m)^2 - \frac{1}{2}]}(k) \Bigg] \\
&\quad = \widetilde{c}\, P_2(k)\, \mathrm{e}^{-\pi \mathrm{i} k} \sum_{q=0}^{m} p_q \widehat{1}_{[q\lambda'_m - \frac{1}{2},\, q\lambda'_m + \frac{1}{2}]}(k) \\
&\quad = \widetilde{c}\, P_2(k)\, \mathrm{e}^{-\pi \mathrm{i} k} \sum_{q=0}^{m} p_q\, \mathrm{e}^{-2\pi \mathrm{i} q \lambda'_m k}\, \mathrm{sinc}(\pi k) \\
&\quad = \widehat{\mu_a}(k),
\end{aligned}
$$

where we used the fact that $(\lambda'_m)^2 = m\lambda'_m + 1$. Now, we focus on Eq. (3.11).

$$
\begin{aligned}
|\lambda'_m| \left(\sum_{q=0}^{m} p_q\, \mathrm{e}^{-2\pi \mathrm{i} q \lambda'_m k} \right) \widehat{\mu_a}(\lambda'_m k) &= \widetilde{c}\, |\lambda'_m| \left(\sum_{q=0}^{m} p_q\, \mathrm{e}^{-2\pi \mathrm{i} q \lambda'_m k} \right) \mathrm{e}^{-\pi \mathrm{i} \lambda'_m k}\, \mathrm{sinc}(\pi \lambda'_m k) P_2(k) \\
&= \widehat{\mu_b}(k).
\end{aligned}
$$

Hence, the claimed solutions are indeed solutions. Later, it will become clear that it makes sense to choose $\widetilde{c} = 1$.

Remark 3.19. It is not completely obvious that the infinite product

$$\prod_{j\geq 1}\left(\sum_{n=0}^{m} p_n\,\mathrm{e}^{-2\pi\mathrm{i}n(\lambda_m')^j k}\right)$$

from Eq. (3.12) converges. Remember that an infinite product of complex numbers $(a_n)_{n\in\mathbb{N}}$ converges if and only if the series $\sum_{n=1}^{\infty}\log(a_n)$ converges. But this is true for the numbers $a_j=\sum_{n=0}^{m} p_n\,\mathrm{e}^{-2\pi\mathrm{i}n(\lambda_m')^j k}$ because $\big((\lambda_m')^j\big)_{j\in\mathbb{N}}$ converges to 0 exponentially fast. $\diamond$

As $\widehat{\mu_a},\widehat{\mu_b}\in L^2(\mathbb{R})$, it follows from [Wol03, Thm. 3.12] that μ_a and μ_b are absolutely continuous with respect to the Lebesgue measure $\boldsymbol{\lambda}$, i.e. $\mu_a=h_a\boldsymbol{\lambda}$ and $\mu_b=h_b\boldsymbol{\lambda}$, where $h_a,h_b\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ and

$$\widehat{h_a}=\widehat{\mu_a}\quad\text{and}\quad\widehat{h_b}=\widehat{\mu_b};$$

compare [AL90, Thm. 2.2] and [Rud62, Sec. 1.3.4]. The measures μ_a and μ_b are compactly supported. Consequently, the same is true for h_a and h_b. In fact, we are going to show that there are continuous representatives in the equivalence class of h_a and h_b, when $\boldsymbol{p}_m\gg 0$. To do so, consider the measure

$$\mu:=\mathop{\Large\ast}_{j=1}^{\infty}\Big(\sum_{q=0}^{m} p_q\delta_{q(\lambda_m')^j}\Big)=\mathop{\Large\ast}_{j=1}^{\infty}\mu_j,\tag{3.13}$$

where $\mu_j=\sum_{q=0}^{m} p_q\delta_{q(\lambda_m')^j}$.

Lemma 3.20. *The infinite convolution product for the measure μ from Eq. (3.13) is absolutely convergent [2] to a probability measure in the weak topology. Moreover, it converges to the same limit in any order of the terms.*

Proof. We apply [JW35, Thm. 6]. Therefore, we have to show that

$$\sum_{j=1}^{\infty}|c(\mu_j)|<\infty\quad\text{and}\quad\sum_{j=1}^{\infty}M_2(\mu_j)<\infty,$$

where

$$c(\mu_j):=\int_{\mathbb{R}} x\,\mathrm{d}\mu_j(x)\quad\text{and}\quad M_2(\mu_j):=\int_{\mathbb{R}} x^2\,\mathrm{d}\mu_j(x).$$

We have

$$c(\mu_j)=\int_{\mathbb{R}} x\,\mathrm{d}\mu_j(x)=\sum_{q=0}^{m} p_q\cdot q\cdot(\lambda_m')^j=c_{m,\boldsymbol{p}}\cdot(\lambda_m')^j,$$

[2] An infinite convolution of probability measures is called *absolutely convergent* if it is convergent in any order of the terms, and the limit is also independent of the order of terms.

where $c_{m,\boldsymbol{p}} := \sum_{q=0}^{m} p_q q$. As $|\lambda'_m| < 1$, the series $\sum_{j=1}^{\infty} |c(\mu_j)|$ is convergent. Similarly, one can see that the series $\sum_{j=1}^{\infty} M_2(\mu_j)$ is convergent, too. Hence, μ is absolutely convergent. Since all measures μ_j are probability measures, so is the weak limit μ, and the claim follows. $\square$

Consequently, we can write

$$\mu = \mu_i * \nu_i, \qquad \text{with} \quad \nu_i := \mathop{\Huge *}_{j \neq i} \mu_j$$

for all $i \in \mathbb{N}$.

To prove the continuity of h_a and h_b, we will proceed in two steps. First, we will show that μ is a continuous measure in the sense that $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. Afterwards, we will show that the convolution of an indicator function of a compact interval with a continuous measure is a continuous function, which together will imply the continuity of h_a and h_b.

Lemma 3.21. *When $\boldsymbol{p}_m \gg 0$, the measure μ from Eq.* (3.13) *is a continuous measure.*

Proof. This proof is a generalisation of the special case $m = 1$ with $\boldsymbol{p} = (\frac{1}{2}, \frac{1}{2})$, which can be found in [JW35, Sec. 6]. Consider the sequence of probability measures $(\mu_j)_{j\in\mathbb{N}}$ defined in Eq. (3.13). Now, suppose there is an element $x \in \mathbb{R}$ such that $\mu(\{x\}) > 0$. Notice that

$$\mu(\{x\}) = \sum_{n=0}^{m} p_n \, \nu_j\big(\{x - n(\lambda'_m)^j\}\big) \tag{3.14}$$

because

$$\mu(\{x\}) = \big(\mu_j * \nu_j\big)(\{x\}) = \Big(\sum_{n=0}^{m} p_n \delta_{n(\lambda'_m)^j} * \nu_j\Big)(\{x\}) = \sum_{n=0}^{m} p_n \, \nu_j\big(\{x - n(\lambda'_m)^j\}\big).$$

Analogously, we obtain

$$\mu(\{x - (\lambda'_m)^j\}) = \sum_{n=0}^{m} p_n \, \nu_j\big(\{x - (n+1) \cdot (\lambda'_m)^j\}\big) \tag{3.15}$$

and

$$\mu(\{x + (\lambda'_m)^j\}) = \sum_{n=0}^{m} p_n \, \nu_j\big(\{x - (n-1) \cdot (\lambda'_m)^j\}\big). \tag{3.16}$$

Eqs. (3.14), (3.15) and (3.16) imply

$$\Big(\prod_{\kappa=0}^{m} p_\kappa\Big) \cdot \mu(\{x\}) \leq \mu(\{x - (\lambda'_m)^j\}) + \mu(\{x + (\lambda'_m)^j\}) \tag{3.17}$$

because (setting $p_{-2} = p_{-1} = p_{m+1} = p_{m+2} = 0$)

$$\begin{aligned}
\mu(\{x - (\lambda'_m)^j\}) + \mu(\{x + (\lambda'_m)^j\}) &= \sum_{n=-1}^{m+1} (p_{n+1} + p_{n-1}) \cdot \nu_j(\{x - n(\lambda'_m)^j\}) \\
&\geq \sum_{n=-1}^{m+1} \max\{p_{n+1}, p_{n-1}\} \cdot \nu_j(\{x - n(\lambda'_m)^j\}) \\
&\geq \sum_{n=0}^{m} \Big(\prod_{\kappa=0}^{m} p_\kappa\Big) p_n \cdot \nu_j(\{x - n(\lambda'_m)^j\}) \\
&= \Big(\prod_{\kappa=0}^{m} p_\kappa\Big) \cdot \mu(\{x\}) > 0,
\end{aligned}$$

where the penultimate step holds because $p_k \geq p_\ell \cdot \prod_{\kappa=0}^{m} p_\kappa > 0$ is true for any k, ℓ. Assume now that all $p_\kappa > 0$ and choose $r \in \mathbb{N}$ with $\Big(\prod_{\kappa=0}^{m} p_\kappa\Big) \cdot \mu(\{x\}) > \frac{1}{r}$ and $j_1 < j_2 < \ldots < j_r$. We obtain

$$\begin{aligned}
1 &\geq \mu\left(\dot{\bigcup_{1\leq q\leq r}} \Big\{x - (\lambda'_m)^{j_q}\Big\} \dot\cup \Big\{x + (\lambda'_m)^{j_q}\Big\}\right) \\
&= \sum_{q=1}^{r} \left(\mu\Big(\Big\{x - (\lambda'_m)^{j_q}\Big\}\Big) + \mu\Big(\Big\{x + (\lambda'_m)^{j_q}\Big\}\Big)\right) \\
&\overset{(3.17)}{\geq} \sum_{q=1}^{r} \Big(\prod_{\kappa=0}^{m} p_\kappa\Big) \cdot \mu(\{x\}) > r \cdot \frac{1}{r} = 1.
\end{aligned}$$

This contradiction shows that μ is continuous. □

Remark 3.22. One can even show that μ is purely singular continuous; see [JW35, Thm. 35]. But as we only need the continuity of μ, we leave it like that. ◇

Remark 3.23. Let us comment on the case when $p_\kappa = 0$ for some $0 \leq \kappa \leq m$. Using

$$\prod_{\substack{0\leq\kappa\leq m,\\ p_\kappa\neq 0}} p_\kappa \quad \text{instead of} \quad \prod_{\kappa=0}^{m} p_\kappa$$

in the above proof, we can also deal with incomplete mixtures (with essentially the same proof) as long as

$$\max\{p_{n+1}, p_{n-1}\} > 0 \quad \text{for all } 0 \leq n \leq m \text{ with } p_n > 0.$$

◇

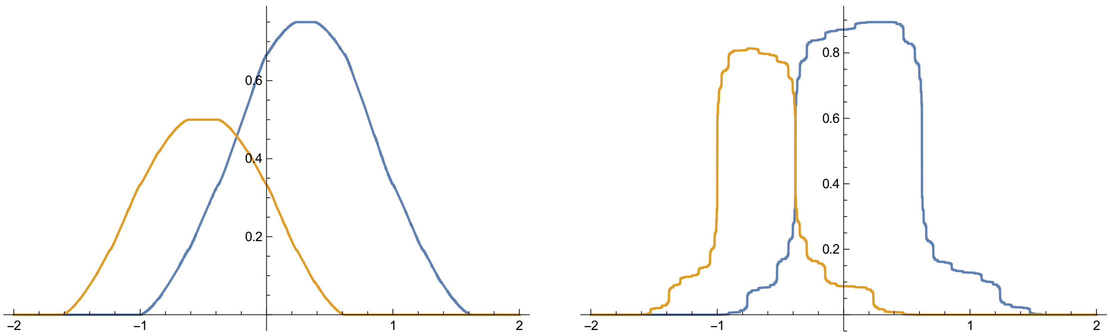

Figure 3.2: Left: The functions $h_a(x)$ (blue) and $h_b(x)$ (yellow) are shown with $m = 1$ and probability vector $\boldsymbol{p}_1 = (0.5, 0.5)$.
Right: The functions $h_a(x)$ (blue) and $h_b(x)$ (yellow) are shown with $m = 1$ and probability vector $\boldsymbol{p}_1 = (0.1, 0.9)$

To proceed, we need the following result.

Lemma 3.24. *Let $a, b \in \mathbb{R}$ with $a < b$, and let ν be a finite, regular Borel measure on $\mathbb{R}$ that is continuous. Now, let J be any of the intervals $[a, b]$, (a, b), $[a, b)$ or $(a, b]$. Then, the function $1_J * \nu$ is continuous on $\mathbb{R}$.*

Proof. First, let $J = [a, b]$. Fix $x \in \mathbb{R}$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \searrow x$. Then,

$$\begin{aligned}|f * \nu(x) - f * \nu(x_n)| &= \Big| \int_{\mathbb{R}} \big(1_{[a,b]}(x - y) - 1_{[a,b]}(x_n - y)\big) \,\mathrm{d}\nu(y)\Big| \\ &= \Big|\nu([x - b, x_n - b]) - \nu([x - a, x_n - a])\Big| \\ &\to 0,\end{aligned}$$

when $n \to \infty$ because $\nu(\{x - b\}) = \nu(\{x - a\}) = 0$. The same holds when $x_n \nearrow x$. Thus, $f * \nu \in C(\mathbb{R})$.

Due to the assumed continuity of ν, the same type of argument applies to half-open or open intervals as well. □

Now, we are able to prove the continuity of h_a and h_b.

Proposition 3.25. *If $\boldsymbol{p}_m \gg 0$, the Radon–Nikodym densities h_a and h_b are continuous functions with compact support.*

Proof. The convolution theorem implies $\widehat{\mu} = \prod_{j\geq 1} \Big(\sum_{q=0}^{m} p_q \, \mathrm{e}^{-2\pi \mathrm{i} q (\lambda_m')^j k} \Big)$, where μ is the measure from Eq. (3.13). Then, it follows from Eq. (3.12) that

$$h_a(x) = \Big(1_{[0,1[} * \mu\Big)(x) \quad \text{and} \quad h_b(x) = \Big(1_{[\lambda_m',0[} * \mu\Big)(x). \tag{3.18}$$

The claim now follows from Lemma 3.21 and Lemma 3.24. Due to Lemma 3.18, the functions h_a and h_b are compactly supported. □

Remark 3.26. Note that Eq. (3.18) also holds when $p_j = 1$ for some j. In that case, the functions h_a and h_b are indicator functions supported on some interval inside W. Consequently, they are no longer continuous on W. $\diamond$

In what follows, we need to compare the function values of $h_a(x^\star)$ and $g_a(x)$ respectively $h_b(x^\star)$ and $g_b(x)$.

Lemma 3.27. *If $\boldsymbol{p}_m \gg 0$, one has $g_\alpha(x) = h_\alpha(x^\star)$ for $\alpha \in \{a, b\}$ and all $x \in \Lambda_{\geq 0}$.*

Proof. When $\boldsymbol{p}_m \gg 0$, the functions h_α are continuous, and satisfy the recursions

$$\begin{aligned} h_a(x^\star) &= \sum_{\ell=0}^{m} p_\ell \left[\sum_{j=0}^{\ell-1} h_a((\mathfrak{f}_j^{-1}(x))^\star) + \sum_{j=\ell}^{m-1} h_a((\mathfrak{g}_j^{-1}(x))^\star) + h_b((\mathfrak{f}_0^{-1}(x))^\star) \right], \\ h_b(x^\star) &= \sum_{\ell=0}^{m} p_\ell \, h_a((\mathfrak{f}_\ell^{-1}(x))^\star), \end{aligned} \tag{3.19}$$

which follows from Lemma 3.18, rewritten in terms of the densities. Since $h_a(0^\star) = \frac{1}{1+p_0}$, we obtain the same type of renormalisation equation as in Eq. (3.7). Now, the claim follows from Proposition 3.17. $\square$

Before we come back to the measure $\lim_{n\to\infty} \frac{1}{\lambda_m^n} \big| \mathbb{E}(\mathcal{X}_n) \big|^2$, we need the following lemma.

Lemma 3.28. *If $\boldsymbol{p}_m \gg 0$, we have*

$$\lim_{n\to\infty} \max_{x \in \Lambda^{(n)}} |h^{(n)}(x^\star) - h(x^\star)| = 0.$$

Proof. For simplicity, we assume $m = 1$. The general case $m \in \mathbb{N}$ can be treated analogously. First, we notice that Eq. (3.3) implies

$$g^{(n+1)}(x) = p_1 \, g^{(n)}(x) + p_0 \, g^{(n)}(x - \lambda_1^{n-1}) + p_0 \, g^{(n-1)}(x) + p_1 \, g^{(n-1)}(x - \lambda_1^n).$$

Consequently, we have

$$g^{(n+1)}(x) = \begin{cases} p_1 \, g^{(n)}(x) + p_0 \, g^{(n-1)}(x), & x \in \Lambda^{(n-1)}, \\ p_1 \, g^{(n)}(x) + p_0 \, g^{(n)}(x - \lambda_1^{n-1}), & x \in \Lambda^{(n)} \setminus \Lambda^{(n-1)}, \\ p_0 \, g^{(n)}(x - \lambda_1^{n-1}) + p_1 \, g^{(n-1)}(x - \lambda_1^n), & x \in \Lambda^{(n+1)} \setminus \Lambda^{(n)}. \end{cases} \tag{3.20}$$

Soon, we will need the following result.

Claim A. Let $r \in \mathbb{N}$, and let $x \in \Lambda^{(r)} \setminus \Lambda^{(r-1)}$. Then,

$$g^{(n)}(x) = \frac{p_0 \, g^{(r)}(x) + g^{(r+1)}(x)}{1 + p_0} - \frac{g^{(r+1)}(x) - g^{(r)}(x)}{1 + p_0} \, (-p_0)^{n-r}$$

for all $n \geq r$. In particular, we obtain

$$g^{(n)}(x) - g(x) = -\frac{g^{(r+1)}(x) - g^{(r)}(x)}{1+p_0}\,(-p_0)^{n-r}.$$

Proof of Claim A. We deduce from Eq. (3.20) that

$$g^{(n+1)}(x) = p_1\, g^{(n)}(x) + p_0\, g^{(n-1)}(x)$$

for all $n \geq r+1$. Setting $a_n := g^{n+r}(x)$ we get for all $n \geq 0$ the second order recurrence relation

$$a_{n+2} = p_1 a_{n+1} + p_0 a_n.$$

The characteristic equation

$$x^2 = p_1 x + p_0$$

has the solutions $x = 1$ and $x = -p_0$. Hence, the general solution is given by

$$a_n = C_1 + C_2(-p_0)^n.$$

From a_0 and a_1 one immediately obtains

$$C_1 = \frac{p_0 a_0 + a_1}{1+p_0} \qquad \text{and} \qquad C_2 = \frac{a_1 - a_0}{1+p_0},$$

and the claim follows. $\boxdot$

Let us return to the proof of the lemma. Since the function h is uniformly continuous, there is a null sequence $(\widetilde{a_n})_{n\in\mathbb{N}}$ such that

$$\max_{x\in\Lambda} |h(x^\star) - h(x^\star - (\lambda_1^\star)^{n-1})| = \widetilde{a_{n-1}}. \tag{3.21}$$

Now, choose a null sequence $(a_n)_{n\in\mathbb{N}}$ with the following properties:

(i) $\widetilde{a_n} \leq a_n$ for all $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = 1$.

To finish the proof, we will now show by induction that there is a number $n_0 \in \mathbb{N}$ such that

$$\max_{x\in\Lambda^{(n)}} |h^{(n)}(x^\star) - h(x^\star)| \leq \sqrt{a_n} \qquad \text{for all } n \geq n_0.$$

The claim is obvious for $n = n_0$ (which will be chosen later). One only has to take a sequence $(a_n)_{n\in\mathbb{N}}$ such that a_{n_0} is large enough. Next, fix $n \geq n_0$, and assume that the claim is true for $n_0, \ldots, n-1$. First, we observe that

$$\begin{aligned}\max_{x\in\Lambda^{(n)}} |h^{(n)}(x^\star) - h(x^\star)| &\leq \max\Big\{ \max_{x\in\Lambda^{(n-1)}} |h^{(n)}(x^\star) - h(x^\star)|,\ \max_{x\in\Lambda^{(n)}\setminus\Lambda^{(n-1)}} |h^{(n)}(x^\star) - h(x^\star)| \Big\} \\ &=: \max\{I_1,\, I_2\}.\end{aligned}$$

We first focus on I_1. Using the induction hypothesis and Claim A, we obtain

$$\max_{x\in\Lambda^{(n-1)}} |h^{(n)}(x^\star) - h(x^\star)| \le p_0 \sqrt{a_{n-1}}.$$

The right hand side is bounded from above by $\sqrt{a_n}$ if and only if $p_0 \le \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}}$. Since $p_0 < 1$ and $\lim_{n\to\infty} \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}} = 1$ because of (ii), there is a number $n_1 \in \mathbb{N}$ such that $p_0 \le \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}}$ for all $n \ge n_1$. Consequently, the claim is true in the case I_1.

Now, let us turn our attention to I_2. Let $x \in \Lambda^{(n)} \setminus \Lambda^{(n-1)}$. Then, Claim A and Eq. (3.20) imply

$$\begin{aligned}
|h^{(n)}(x^\star) - h(x^\star)| &= \frac{1}{1+p_0}\, |h^{(n+1)}(x^\star) - h^{(n)}(x^\star)| \\
&= \frac{1}{1+p_0}\, |p_1\, h^{(n)}(x^\star) + p\, h^{(n)}(x^\star - (\lambda_1^\star)^{n-1}) - h^{(n)}(x^\star)| \\
&= \frac{p_0}{1+p_0}\, |h^{(n)}(x^\star) - h^{(n)}(x^\star - (\lambda_1^\star)^{n-1})| \\
&\le \frac{p_0}{1+p_0} \Big(|h^{(n)}(x^\star) - h(x^\star)| + |h(x^\star) - h(x^\star - (\lambda_1^\star)^{n-1})| \\
&\qquad + |h(x^\star - (\lambda_1^\star)^{n-1}) - h^{(n)}(x^\star - (\lambda_1^\star)^{n-1})| \Big).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&|h^{(n)}(x^\star) - h(x^\star)| \\
&\qquad \le p_0\, |h(x^\star) - h(x^\star - (\lambda_1^\star)^{n-1})| + p_0\, |h(x^\star - (\lambda_1^\star)^{n-1}) - h^{(n)}(x^\star - (\lambda_1^\star)^{n-1})|.
\end{aligned}$$

Together with Eq. (3.21) and the induction hypothesis, this implies

$$|h^{(n)}(x^\star) - h(x^\star)| \le p_0\, a_{n-1} + p_0^2 \sqrt{a_{n-1}}.$$

The right hand side is bounded from above by $\sqrt{a_n}$ if and only if

$$p_0 \sqrt{a_{n-1}} + p_0^2 \le \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}}.$$

Since $p_0^2 < 1$, $\lim_{n\to\infty} \sqrt{a_n} = 0$ and $\lim_{n\to\infty} \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}} = 1$, there is a number $n_2 \in \mathbb{N}$ such that

$$p_0 \sqrt{a_{n-1}} + p_0^2 \le \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}} \qquad \text{for all } n \ge n_2.$$

By induction, the claim follows in the case I_2.

Finally, if we set $n_0 := \max\{n_1, n_2\}$, the lemma is proven. □

To continue, we would like to rewrite our measure $\lim_{n\to\infty} \frac{1}{\lambda_m^n} \left|\mathbb{E}(\mathcal{X}_n)\right|^2$ in the following way:

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\lambda_m^n} \left|\mathbb{E}(\mathcal{X}_n)\right|^2 &= \lim_{n\to\infty} \frac{1}{\lambda_m^n} \left|\widehat{\mathbb{E}(\mathcal{M}_n)}\right|^2 = \lim_{n\to\infty} \frac{1}{\lambda_m^n} \mathcal{F}[\mathbb{E}(\mathcal{M}_n) * \widetilde{\mathbb{E}(\mathcal{M}_n)}] \\ &= \mathcal{F}[\mathbb{E}(\mathcal{M}) \circledast \widetilde{\mathbb{E}(\mathcal{M})}]. \end{aligned} \tag{3.22}$$

By construction, $\mathbb{E}(\mathcal{M}) = \lim_{n\to\infty} \mathbb{E}(\mathcal{M}_n)$ is the weighted Dirac comb on the model set $人([\lambda'_m - 1,\, 1 - \lambda'_m])$, where the weight of some element $x \in 人([\lambda'_m - 1,\, 1 - \lambda'_m])$ is the probability that x is an element of Λ_m, i.e.

$$\mathbb{E}(\mathcal{M}) = \sum_{x \in 人([\lambda'_m - 1,\, 1 - \lambda'_m])} h(x^\star)\, \delta_x, \tag{3.23}$$

where $h := h_a + h_b$. The last step in Eq. (3.22) needs clarification, since $\mathbb{E}(\mathcal{M})|_{\Lambda^{(n)}} \neq \mathbb{E}(\mathcal{M}_n)$. To verify Eq. (3.22), it suffices to show that

$$\lim_{n\to\infty} \left| \left(\frac{\mathbb{E}(\mathcal{M}_n) * \widetilde{\mathbb{E}(\mathcal{M}_n)}}{\lambda_m^n} - \frac{\mathbb{E}(\mathcal{M})|_{\Lambda^{(n)}} * \widetilde{\mathbb{E}(\mathcal{M})|_{\Lambda^{(n)}}}}{\lambda_m^n} \right)(f) \right| = 0$$

for all $f \in C_{\mathrm{c}}(\mathbb{R})$. But this is true because, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\left| \left(\frac{\mathbb{E}(\mathcal{M}_n) * \widetilde{\mathbb{E}(\mathcal{M}_n)}}{\lambda_m^n} - \frac{\mathbb{E}(\mathcal{M})|_{\Lambda^{(n)}} * \widetilde{\mathbb{E}(\mathcal{M})|_{\Lambda^{(n)}}}}{\lambda_m^n} \right)(f) \right| \\ &\quad = \frac{1}{\lambda_m^n} \left| \sum_{x\in\Lambda^{(n)}} \sum_{y\in\Lambda^{(n)}} h^{(n)}(x^\star) h^{(n)}(y^\star) \delta_{x-y}(f) - \sum_{x\in\Lambda^{(n)}} \sum_{y\in\Lambda^{(n)}} h(x^\star) h(y^\star) \delta_{x-y}(f) \right| \\ &\quad \le \frac{\|f\|_\infty}{\lambda_m^n} \sum_{x\in\Lambda^{(n)}} \sum_{\substack{y\in\Lambda^{(n)},\\ x-y\in\operatorname{supp}(f)}} |h^{(n)}(x^\star) h^{(n)}(y^\star) - h(x^\star) h(y^\star)| \\ &\quad \le \frac{\|f\|_\infty}{\lambda_m^n} \sum_{x\in\Lambda^{(n)}} \sum_{\substack{y\in\Lambda^{(n)},\\ x-y\in\operatorname{supp}(f)}} \Big(|h^{(n)}(x^\star) - h(x^\star)| + |h^{(n)}(y^\star) - h(y^\star)| \Big) \\ &\quad \overset{\text{Lem. 3.28}}{\le} 2\varepsilon \cdot \frac{\|f\|_\infty}{\lambda_m^n} \sum_{x\in\Lambda^{(n)}} \sum_{\substack{y\in\Lambda^{(n)},\\ x-y\in\operatorname{supp}(f)}} 1 \\ &\quad \le 4\varepsilon \|f\|_\infty c_f \operatorname{dens}(\Lambda). \end{aligned}$$

Here, the last step follows from the fact that

$$\sum_{\substack{y\in\Lambda^{(n)},\\ x-y\in\operatorname{supp}(f)}} 1 \le c_f \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_m^n} \sum_{x\in\Lambda^{(n)}} 1 = \operatorname{dens}(\Lambda),$$

for some constant $c_f > 0$, which only depends on f, because the support of f is compact and Λ is a Meyer set.

Now, as we already know that $h_a, h_b \in C_{\mathrm{c}}(\mathbb{R})$, we can apply the following result.

Theorem 3.29. *([BG13, Thm. 9.5], [Str14, Lem. 3.12]). Let*

$$\omega_h = \sum_{x\in\Lambda} h(x^\star)\,\delta_x$$

be a weighted Dirac comb for a regular model set $\Lambda = \curlywedge(W)$ with CPS $(\mathbb{R}^d, H, \mathcal{L})$ and compact window $W = \overline{W^\circ}$, with a function $h : H \to \mathbb{C}$ which is continuous on W and vanishes on its complement. Then, $\omega_h \in \mathcal{SAP}(\mathbb{R}^d)$ has the positive, translation bounded, pure point diffraction measure

$$\widehat{\gamma} = \sum_{k\in L^{\circledast}} |A(k)|^2\,\delta_k \quad \textit{with} \quad A(k) = \mathrm{dens}(\mathcal{L})\,\widehat{h}(-k^\star),$$

where $L^{\circledast} = \pi(\mathcal{L}^\star)$ is the Fourier module. □

It follows that $\lim_{n\to\infty} \frac{1}{\lambda_m^n}\big|\mathbb{E}(\mathcal{X}_n)\big|^2$ is a pure point measure with

$$\begin{aligned} A(k) &= \mathrm{dens}(\mathcal{L})\,\widehat{h}(-k^\star) \\ &= \mathrm{dens}(\Lambda_m)\,\mathrm{e}^{\pi\mathrm{i}(1+\lambda_m')k^\star}\,\mathrm{sinc}\big(\pi(1-\lambda_m')k^\star\big)\prod_{j=1}^{\infty}\Big(\sum_{n=0}^{m} p_n\,\mathrm{e}^{2\pi\mathrm{i}n(\lambda_m')^j k^\star}\Big), \end{aligned}$$

hence $\lim_{n\to\infty} \frac{1}{\lambda_m^n}\big|\mathbb{E}(\mathcal{X}_n)\big|^2 = \sum_{k\in L^{\circledast}} I(k)\,\delta_k$ with

$$I(k) = \Big(\mathrm{dens}(\Lambda_m)\,\mathrm{sinc}\big(\pi(1-\lambda_m')k^\star\big)\Big)^2 \cdot \prod_{j=1}^{\infty}\Big|\sum_{n=0}^{m} p_n\,\mathrm{e}^{2\pi\mathrm{i}n(\lambda_m')^j k^\star}\Big|^2. \tag{3.24}$$

Let us summarise the results of this chapter.

Theorem 3.30. *Let $\Lambda_m \in Y_m$ be a generating random noble means set. Then, its diffraction measure $\widehat{\gamma_{\Lambda_m}}$ consists of an absolutely continuous part and a pure point part. More precisely:*

$$\widehat{\gamma_{\Lambda_m}} = \sum_{k\in\frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2+4}}} I(k)\,\delta_k + \phi\boldsymbol{\lambda},$$

where $I(k)$ is given in Eq. (3.24) *and ϕ is the Radon–Nikodym density from Corollary 3.13.*

It should be noted that it is not entirely obvious that the supporting set of the Bragg peaks is still $\frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2+4}}$ in the stochastic situation. This follows from an explicit calculation using exponential sums; see [Spi17].

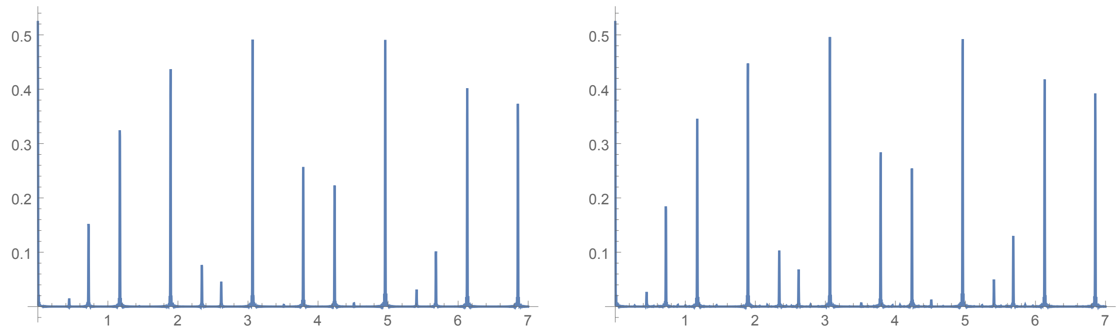

Figure 3.3: The diffraction intensities of the random substitution ζ_1 with $\boldsymbol{p}_1 = (0.5, 0.5)$ (left) and $\boldsymbol{p}_1 = (0.2, 0.8)$ (right).

Remark 3.31. The triangle inequality and $\sum_{n=0}^{m} p_n = 1$ imply that

$$\prod_{j=1}^{\infty} \left| \sum_{n=0}^{m} p_n \, \mathrm{e}^{2\pi \mathrm{i} n (\lambda_m')^j k^\star} \right|^2 \leq 1$$

for all $k \in \frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2+4}}$. Equality holds if and only if $p_n = 1$ for one $n \in \{0, \ldots, m\}$, which coincides with the deterministic cases. This is consistent with earlier results. On the one hand, we had

$$I(k) = \Big(\operatorname{dens}(\Lambda) \ \operatorname{sinc}(\pi(1 - \lambda_m')k^\star) \Big)^2$$

in the deterministic situation, see Proposition 2.5, which is equivalent to the fact that the infinite product equals one. On the other hand, there is an absolutely continuous part in the random case, which explains the fact that the intensities are strictly smaller in this situation (except for $k = 0$). $\Diamond$

3.5 The deformed random Fibonacci chain

The next question, which naturally arises, is whether or not one can generalise this situation even further. Due to the Pisot nature of λ_1, one can consider a modified hull $\mathbb{Y}_\alpha'$ that emerges from $\mathbb{X}_1'$ by taking a and b type intervals of lengths

$$\ell_a \; = \; \tau + \alpha(1 - \tau) \quad \text{and} \quad \ell_b \; = \; 1 + \alpha, \tag{3.25}$$

respectively, where $\alpha \in \,]-1, \tau + 1[\,$ is a real parameter. This choice is made such that the average tile length, and hence also the density of left endpoints, is the same for $\mathbb{Y}_1' := \mathbb{Y}_{1,1} = \mathbb{Y}_{1,0}$ and $\mathbb{Y}_\alpha'$. Since the elements of $\mathbb{Y}_\alpha'$ are always considered as tilings with two distinct prototiles, even if they have the same length (as happens for $\alpha = \tau^{-2}$), the dynamical systems $(\mathbb{Y}'_1, \mathbb{R})$ and $(\mathbb{Y}'_\alpha, \mathbb{R})$ are topologically conjugate in this setting. This follows from the description of the elements of $\mathbb{Y}'_\alpha$ as *deformed model sets*, see [BD00, BL05], and is in line with the general analysis of [CS03]. In terms of the diffraction, and in complete analogy to [BG13, Ex. 9.9], the result is the following.

Proposition 3.32. *Let $\Lambda' \in \mathbb{Y}'_\alpha$ with $\Lambda' = \Lambda'_a \dot{\cup} \Lambda'_b$. Consider the Dirac comb*

$$\omega' = u_a \delta_{\Lambda'_a} + u_b \delta_{\Lambda'_b}$$

with fixed weights $u_a, u_b \in \mathbb{C}$. Then, ω' is pure point diffractive with diffraction measure

$$\widehat{\gamma'} = \sum_{k \in \frac{\mathbb{Z}[\lambda_1]}{\sqrt{5}}} I'(k)\, \delta_k,$$

where $I'(k) = |u_a A_{\Lambda'_a}(k) + u_b A_{\Lambda'_b}(k)|^2$ with

$$A_{\Lambda'_a}(k) = \frac{1}{\sqrt{5}} \int_{\tau-2}^{\tau-1} \mathrm{e}^{2\pi \mathrm{i}(k^\star - \alpha k)y} \, \mathrm{d}y = \frac{1}{\sqrt{5}} \, \mathrm{e}^{\pi \mathrm{i}(2\lambda_1 - 3)(k^\star - \alpha k)} \, \mathrm{sinc}\big(\pi(k^\star - \alpha k)\big)$$

and

$$A_{\Lambda'_b}(k) = \frac{1}{\sqrt{5}} \int_{-1}^{\tau-2} \mathrm{e}^{2\pi \mathrm{i}(k^\star - \alpha k)y} \, \mathrm{d}y = \frac{1}{\sqrt{5}} \, \mathrm{e}^{\pi \mathrm{i}(\lambda_1 - 3)(k^\star - \alpha k)} \, \mathrm{sinc}\big(\pi(\lambda_1 - 1)(k^\star - \alpha k)\big).$$

□

Now, if we consider the random Fibonacci chain and tiles with the length ratio given in Eq. (3.25), we obtain, with an analogous proof as in case of the tile lengths 1 and λ_1, the following result.

Theorem 3.33. *Let $\Lambda \in \mathbb{Y}_\alpha$ be a typical realisation and $\boldsymbol{p}_1 = (p_0, p_1)$ a fixed probability vector. Then, the diffraction measure $\widehat{\gamma_\Lambda}$ is of mixed type, consisting of a pure point part and an absolutely continuous part. The diffraction intensities are explicitly given by*

$$I(k) = \left(\frac{\lambda_1}{\sqrt{5}} \, \mathrm{sinc}\big(\pi \lambda_1 (k^\star - \alpha k)\big) \right)^2 \prod_{j=1}^{\infty} \left| p_0 + p_1 \, \mathrm{e}^{2\pi \mathrm{i}(\lambda'_1)^j (k^\star - \alpha k)} \right|^2.$$

□

4 The random period doubling substitution

Undoubtedly, the Fibonacci substitution is one of the most prominent and best examined substitutions. However, it is not immediately clear how to extend this substitution to two two-dimensional substitutions that give rise to the same hull. Therefore, we are going to study a second one-dimensional substitution, namely the period doubling substitution, which is a substitution of constant length. As we will see in the next chapter, it is not difficult to extend such substitutions to higher dimensions.

4.1 The deterministic period doubling substitution

Once again, let us consider the binary alphabet $\mathcal{A}_2 = \{a, b\}$. The period doubling substitution $\rho_{\text{pd}} : \mathcal{A}_2^* \to \mathcal{A}_2^*$ is defined by

$$\rho_{\text{pd}} : \begin{cases} a \mapsto ab, \\ b \mapsto aa. \end{cases}$$

In contrast to the Fibonacci example, this substitution is of constant length, i.e. every element of $\mathcal{A}_2$ is mapped to a word with the same number of letters. Furthermore, we will need the substitution $\rho'_{\text{pd}} : \mathcal{A}_2^* \to \mathcal{A}_2^*$ defined by

$$\rho'_{\text{pd}} : \begin{cases} a \mapsto ba, \\ b \mapsto aa. \end{cases}$$

Applying Prop. 1.17 with $v := a^{-1}$, we find that ρ_{pd} and ρ'_{pd} are conjugate. Hence, they define the same discrete hull, which we will denote by $\mathbb{X}_{\text{pd}}$.

4.1.1 Elementary properties of ρ_{pd}

In the first section, we are going to summarise some results concerning the deterministic period doubling substitution; see [BG13, Sects. 4.5.1 and 9.4.4]. As before, we are able to gain some information about the period doubling substitution by studying its substitution matrix. This matrix is given by

$$M_{\mathrm{pd}} := \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

This matrix is primitive, which means that there is a power $k \in \mathbb{N}$ such that ρ_{pd}^k admits the construction of a bi-infinite fixed point by Lemma 1.10. Starting from the legal seed $a|a$, an iteration of ρ_{pd} leads to the fixed point [3]

$$\dots abaaababababaaabaa|abaaababababaaabaa\dots$$

One can show that this sequence is aperiodic, see [BG13, Prop. 4.7]. The eigenvalues of M_{pd} are 2 and -1, which are the roots of the characteristic polynomial $P(x) = x^2 - x - 2$. Hence, it follows that ρ_{pd} is a primitive and aperiodic Pisot substitution with leading PF eigenvalue $\lambda_{\mathrm{PF}} = 2$. The left and right PF eigenvectors are

$$\mathbf{L}_{\rho_{\mathrm{pd}}} = (1,1) \quad \text{and} \quad \mathbf{R}_{\rho_{\mathrm{pd}}} = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)^{\mathsf{T}}.$$

As described in Section 2.1.2, the left PF eigenvector can be used to construct a point set in $\mathbb{R}$. This time, the letters a and b are identified with compact intervals I_a and I_b, whose length is in both cases 1. The left endpoints of these intervals lead to the point set

$$\Lambda = \Lambda_a \,\dot\cup\, \Lambda_b = \mathbb{Z}. \tag{4.1}$$

The geometric fixed point equation for ρ_{pd}^2 implies the identities

$$\begin{aligned} \Lambda_a &= 4\Lambda_a \,\dot\cup\, (4\Lambda_a + 2) \,\dot\cup\, (4\Lambda_a + 3) \,\dot\cup\, 4\Lambda_b \,\dot\cup\, (4\Lambda_b + 2), \\ \Lambda_b &= (4\Lambda_a + 1) \,\dot\cup\, (4\Lambda_b + 1) \,\dot\cup\, (4\Lambda_b + 3), \end{aligned} \tag{4.2}$$

which can be deduced from $\rho_{\mathrm{pd}}^2(a) = abaa$ and $\rho_{\mathrm{pd}}^2(b) = abab$. Combining Eqs. (4.1) and (4.2), we get

$$\Lambda_a = 2\mathbb{Z} \,\dot\cup\, (4\Lambda_a + 3) \quad \text{and} \quad \Lambda_b = (4\mathbb{Z} + 1) \,\dot\cup\, (4\Lambda_b + 3).$$

By iteration, this leads to the solutions

$$\begin{aligned} \Lambda_a &= \dot{\bigcup_{i \ge 0}} \big(2 \cdot 4^i\mathbb{Z} + (4^i - 1)\big) \,\dot\cup\, \{-1\}, \\ \Lambda_b &= \dot{\bigcup_{i \ge 1}} \big(4^i\mathbb{Z} + (2 \cdot 4^{i-1} - 1)\big), \end{aligned} \tag{4.3}$$

[3] A different choice of the legal seed might lead to another fixed point.

where the singleton set $\{-1\}$ emerges from the iteration only as a limit in the topology of the 2-adic numbers.

Indeed, these point sets can, again, be constructed via a cut and project scheme. For this purpose, we choose $H = \mathbb{Z}_2$ as (locally) compact Abelian group to obtain a CPS $(\mathbb{R}, \mathbb{Z}_2, \mathcal{L})$ with lattice

$$\mathcal{L} = \big\{(x, \iota(x)) \mid x \in \mathbb{Z}\big\} \subseteq \mathbb{R} \times \mathbb{Z}_2,$$

where $\iota : \mathbb{Z} \hookrightarrow \mathbb{Z}_2$ is the canonical embedding and also the $\star$-map in this case. Because of

$$\overline{\Lambda_a}^2 \cup \overline{\Lambda_b}^2 = \mathbb{Z}_2 \quad \text{and} \quad \overline{\Lambda_a}^2 \cap \overline{\Lambda_b}^2 = \{-1\},$$

we find

$$\Lambda_a = \curlywedge(W_a) \quad \text{and} \quad \Lambda_b = \curlywedge(W_b)$$

with $W_a = \overline{\Lambda_a}^2$ and $W_b = \overline{\Lambda_b}^2 \setminus \{-1\}$.

4.1.2 Diffraction of the period doubling chain

Due to the last section, an application of Thm. 1.32 shows that the diffraction measure of ρ_{pd} is pure point and that it is explicitly given by

$$\widehat{\gamma_\Lambda} = \sum_{k \in L^{\circledast}} |u_a A(k) + u_b A_b(k)|^2 \, \delta_k,$$

where $L^{\circledast} = \mathbb{Z}\big[\frac{1}{2}\big] = \big\{\frac{m}{2^r} \;\big|\; (r = 0,\, m \in \mathbb{Z}) \text{ or } (r \geq 1,\, m \text{ odd})\big\}$, $u_a, u_b \in \mathbb{C}$, and

$$A(k) = \frac{\operatorname{dens}(\Lambda_a)}{\mu_{\mathbb{Z}_2}(W_a)} \int_{W_a} \langle k^\star, y\rangle \, \mathrm{d}\mu_{\mathbb{Z}_2}(y) \quad \text{and} \quad B(k) = \frac{\operatorname{dens}(\Lambda_b)}{\mu_{\mathbb{Z}_2}(W_b)} \int_{W_b} \langle k^\star, y\rangle \, \mathrm{d}\mu_{\mathbb{Z}_2}(y).$$

However, this requires some techniques from 2-adic analysis. Alternatively, one can make use of the limit-periodic structure of the period doubling chain as described in [BG13, Sec. 9.4.4]. Either way, one obtains

$$A(k) = \frac{2}{3}\,\frac{(-1)^r}{2^r}\,\mathrm{e}^{2\pi\mathrm{i}k} \quad \text{and} \quad B(k) = \delta_{r,0} - A(k)$$

for any $k = \frac{m}{2^r} \in L^{\circledast}$. Hence, the diffraction intensities for $k \in L^{\circledast}$ can be calculated as

$$I(k) \;=\; \begin{cases} \frac{1}{9 \cdot 4^{r-1}}\,|u_a - u_b|^2, & r \geqslant 1, \\ \frac{1}{9}\,|2u_a + u_b|^2, & r = 0. \end{cases}$$

4.2 The randomised case

Now, let us focus on the local mixture of the two substitutions ρ_{pd} and ρ'_{pd}, i.e. we consider the random substitution $\rho : \mathcal{A}_2^* \to \mathcal{A}_2^*$ given by

$$\rho : \begin{cases} a \mapsto \begin{cases} ab, & \text{with probability } p, \\ ba, & \text{with probability } q, \end{cases} \\ b \mapsto aa, \end{cases}$$

where $p, q \in [0,1]$ with $p + q = 1$. This substitution is called *random period doubling substitution*. The substitution matrix (in the sense of Definition 2.7) is given by

$$M_\rho := \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Again, the substitution matrices of the deterministic and the stochastic substitutions coincide[4], and ρ is primitive. Therefore, we have $\boldsymbol{L}_\rho = \boldsymbol{L}_{\rho_{\mathrm{pd}}}$ and $\boldsymbol{R}_\rho = \boldsymbol{R}_{\rho_{\mathrm{pd}}}$.

As before, we define the two-sided stochastic hull $\mathbb{X}_\rho$ as the smallest closed and shift-invariant subset of $\mathcal{A}_2^{\mathbb{Z}}$ with $X_\rho \subseteq \mathbb{X}_\rho$, where

$$X_\rho := \left\{ w \in \mathcal{A}_2^{\mathbb{Z}} \mid w \text{ is an accumulation point of } \left(\rho^k(a|a)\right)_{k \in \mathbb{N}_0} \right\}.$$

Each geometric realisation of elements of $\mathbb{X}_\rho$ is called a *generating random period doubling set* and is referred to as Λ. These realisations are constructed as described in Sections 2.1.2 and 2.2.3. An analogue of Proposition 2.14 and Remark 2.15 holds for the random period doubling substitution as follows.

Proposition 4.1. *We have*

1. $\mathbb{X}_\rho = \{w \in \mathcal{A}_2^{\mathbb{Z}} \mid \mathcal{W}(\{w\}) \subseteq \mathcal{D}_\rho\}$,
2. $\mathbb{X}_\rho$ *is invariant under* ρ,
3. $\mathbb{X}_{pd} \subsetneq \mathbb{X}_\rho$. □

4.3 Diffraction of the random period doubling chain

It was shown in [BSS17] that the system $(\mathbb{X}_\rho, \mathbb{Z})$ has positive entropy $s > 0$. More precisely, one calculates $s = \frac{2}{3}\log(2)$. Therefore, due to [BLR07] and Proposition 3.1, we expect the random period doubling substitution to have a diffraction measure of mixed type.

[4]Again, this matrix is not only the expectation value, but also the substitution matrix of every individual substitution.

Now, we proceed as described in Sections 3.1, 3.3 and 3.4. Once again, we make use of the idea of Godrèche and Luck. This time, our random variables satisfy

$$\mathcal{X}_n(k) = \begin{cases} \mathcal{X}_{n-1}(k) + \mathrm{e}^{-2\pi \mathrm{i} k\cdot 2^{n-1}}\, \mathcal{X}_{n-2}(k) + \mathrm{e}^{-2\pi \mathrm{i} k\cdot 3\cdot 2^{n-2}}\, \mathcal{X}'_{n-2}(k), & \text{with probability } p, \\ \mathcal{X}_{n-2}(k) + \mathrm{e}^{-2\pi \mathrm{i} k\cdot 2^{n-2}}\, \mathcal{X}'_{n-2}(k) + \mathrm{e}^{-2\pi \mathrm{i} k\cdot 2^{n-1}}\, \mathcal{X}_{n-1}(k), & \text{with probability } q, \end{cases} \tag{4.4}$$

together with $\mathcal{X}_0(k) = u_a$ and

$$\mathcal{X}_1(k) = \begin{cases} u_a + u_b\, \mathrm{e}^{-2\pi \mathrm{i} k}, & \text{with probability } p, \\ u_b + u_a\, \mathrm{e}^{-2\pi \mathrm{i} k}, & \text{with probability } q, \end{cases}$$

where, $u_a, u_b \in \mathbb{C}$ are fixed weights. The expected value $\mathbb{E}_n := \mathbb{E}(\mathcal{X}_n)$ of $\mathcal{X}_n(k)$ is given by

$$\mathbb{E}_n = (p + q\,\mathrm{e}^{-2^n \pi \mathrm{i} k}) \cdot \mathbb{E}_{n-1} + (q + q\,\mathrm{e}^{-2^{n-1}\pi \mathrm{i} k} + p\,\mathrm{e}^{-2^n \pi \mathrm{i} k} + p\,\mathrm{e}^{-3\cdot 2^{n-1}\pi \mathrm{i} k}) \cdot \mathbb{E}_{n-2} \tag{4.5}$$

together with $\mathbb{E}_0(k) = u_a$ and $\mathbb{E}_1(k) = (pu_a + qu_b) + (pu_b + qu_a)\,\mathrm{e}^{-2\pi \mathrm{i} k}$. Moreover, we obtain a recurrence relation for $\mathbb{V}_n := \mathrm{Var}(\mathcal{X}_n)$:

$$\mathbb{V}_n = \mathbb{V}_{n-1} + 2\mathbb{V}_{n-2} + 2pq\Psi_n, \quad n \geq 2,$$

with $\mathbb{V}_0 \equiv 0$ and $\mathbb{V}_1(k) = 2pq|u_a - u_b|^2\big(1 - \cos(2\pi k)\big)$, where the functions Ψ_n are defined by

$$\Psi_n(k) := \frac{1}{2}\Big|(1 - \mathrm{e}^{-2\pi \mathrm{i} k 2^{n-1}})\mathbb{E}_{n-1} - (1 - \mathrm{e}^{-2\pi \mathrm{i} k 2^{n-1}} + \mathrm{e}^{-2\pi \mathrm{i} k 2^{n-2}} - \mathrm{e}^{2\pi \mathrm{i} k 3\cdot 2^{n-2}})\mathbb{E}_{n-2}\Big|^2. \tag{4.6}$$

As before, we decompose the diffraction measure

$$\widehat{\gamma_\Lambda} = \lim_{n\to\infty} \frac{1}{2^n}\, |\mathbb{E}(\mathcal{X}_n)|^2 + \lim_{n\to\infty} \frac{1}{2^n}\, \mathrm{Var}(\mathcal{X}_n) =: \widehat{\gamma_1} + \widehat{\gamma_2}.$$

4.3.1 The continuous part of the diffraction measure

We want to show in the same way as in Section 3.3 that the vague limit $\lim_{n\to\infty} \frac{\mathbb{V}_n}{2^n}\boldsymbol{\lambda}$ is an absolutely continuous measure. First, we need the following lemma.

Lemma 4.2. *We have*

$$\Psi_n(k) = |u_a - u_b|^2\big(1 - \cos(2^n \pi k)\big) \prod_{j=1}^{n-1} \Big| q + p\,\mathrm{e}^{-2^j \pi \mathrm{i} k}\Big|^2$$

In particular, this implies $\|\Psi_n\|_\infty \leq 2|u_a - u_b|^2$ *for all* $n \geq 2$.

Proof. This can be proved by induction using Eqs. (4.5) and (4.6). □

This leads to

Proposition 4.3. *For every $n \in \mathbb{N}_0$, consider the function*

$$\phi_n : \mathbb{R} \to \mathbb{R}_+, \quad k \mapsto \frac{1}{2^n} \operatorname{Var}(\mathcal{X}_n(k)).$$

On $\mathbb{R}$, the sequence $(\phi_n)_{n\in\mathbb{N}_0}$ converges uniformly to the continuous function

$$\phi : \mathbb{R} \to \mathbb{R}_+, \quad k \mapsto \frac{1}{3}\mathbb{V}_1 + \frac{4pq}{3} \sum_{i=2}^{\infty} \frac{\Psi_i(k)}{2^i}.$$

Proof. Consider the sequence $a_n = \frac{1}{3}\left(2^n - (-1)^n\right)$. One can easily show via induction that

$$\mathbb{V}_n = a_n \mathbb{V}_1 + 2pq \sum_{j=2}^{n} a_{n+1-j} \Psi_j,$$

which implies

$$\frac{\mathbb{V}_n}{2^n} = \frac{1}{3}\frac{2^n - (-1)^n}{2^n}\mathbb{V}_1 + 2pq \sum_{j=2}^{n} \frac{1}{3}\frac{2^{n+1-j} - (-1)^{n+1-j}}{2^n}\Psi_j.$$

Now, it is not difficult to see that $\frac{\mathbb{V}_n(k)}{2^n}$ converges uniformly to $\phi(k)$, which finishes the proof. □

Thus, we obtain

Corollary 4.4. *The measure $\lim_{n\to\infty} \frac{1}{2^n} \operatorname{Var}(\mathcal{X}_n)$ is absolutely continuous with respect to $\boldsymbol{\lambda}$. Its Radon–Nikodym density is given by*

$$\phi(k) = \frac{2}{3}|u_a - u_b|^2 \sum_{n=1}^{\infty} \frac{2pq\bigl(1 - \cos(2^n\pi k)\bigr)}{2^n} \prod_{j=1}^{n-1} \left| q + p\,\mathrm{e}^{-2^j\pi\mathrm{i}k} \right|^2.$$

□

Remark 4.5. Proposition 4.3 is consistent with the results from Section 4.1.2. We saw that, in the deterministic case, the diffraction measure of the period doubling chain is pure point. Setting $\boldsymbol{p} = (1,0)$ or $\boldsymbol{p} = (0,1)$ gives $\phi \equiv 0$. This confirms that there is no absolutely continuous component in these cases. ◇

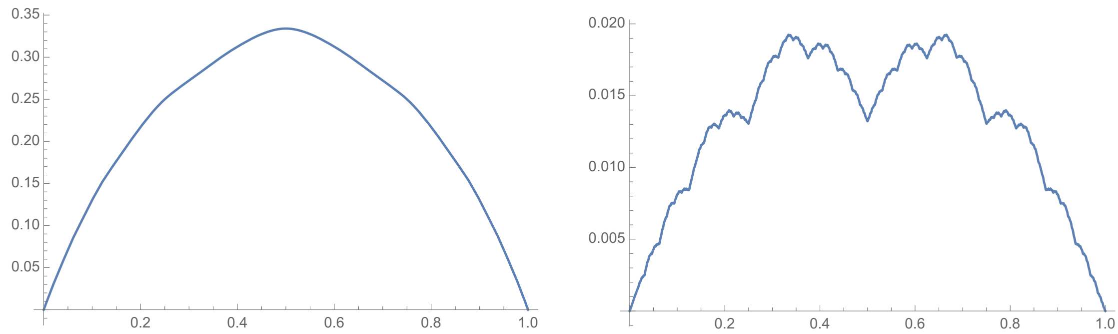

Figure 4.1: The Radon–Nikodym density ϕ of Corollary 4.4, for $\boldsymbol{p} = (0.5, 0.5)$ (left) and $\boldsymbol{p} = (0.01, 0.99)$ (right).

4.3.2 The pure point part of the diffraction measure

Similar to Eq. (3.2), we want to rewrite the measure $\lim_{n\to\infty} \frac{1}{2^n} \left|\widehat{\mathbb{E}(\mathcal{M}_n)}\right|^2$ as:

$$\lim_{n\to\infty} \frac{1}{2^n} \left|\widehat{\mathbb{E}(\mathcal{M}_n)}\right|^2 = \lim_{n\to\infty} \frac{1}{2^n} \mathcal{F}[\mathbb{E}(\mathcal{M}_n) * \widetilde{\mathbb{E}(\mathcal{M}_n)}] = \mathcal{F}[\mathbb{E}(\mathcal{M}) \circledast \widetilde{\mathbb{E}(\mathcal{M})}], \tag{4.7}$$

where $\mathcal{M}_n$ is the measure with $\widehat{\mathcal{M}_n} = \mathcal{X}_n$. By construction, $\mathbb{E}(\mathcal{M}) = \lim_{n\to\infty} \mathbb{E}(\mathcal{M}_n)$ is the weighted Dirac comb on $\mathbb{Z}$, where the weight at $x \in \mathbb{Z}$ is given by

$$u_a \mathbb{P}(\text{type at } x \text{ is } a) + u_b \mathbb{P}(\text{type at } x \text{ is } b).$$

Consequently, we get

$$\mathbb{E}(\mathcal{M}) \;=\; u_b \delta_{\mathbb{Z}} + (u_a - u_b) \sum_{x\in\mathbb{Z}} a_x \, \delta_x, \tag{4.8}$$

where a_x is the probability that at position x is an a. Thus, we can restrict our attention to the case $u_a = 1$ and $u_b = 0$, which leads to

$$\mathbb{E}(\mathcal{M}) \;= \sum_{x\in\mathbb{Z}} a_x \, \delta_x.$$

Remark 4.6. Since $\mathbb{E}(\mathcal{M})|_{\Lambda^{(n)}} \neq \mathbb{E}(\mathcal{M}_n)$, we need to justify the last step in Eq. (4.7). It suffices to show that

$$\lim_{n\to\infty} \max_{x\in\Lambda^{(n)}} |a_x^{(n)} - a_x| = 0,$$

where $a_x^{(n)}$ is the probability that at position x is an a in the n-th step, and $\Lambda^{(n)} = \Lambda \cap [0, 2^n[$; see page 46. $\diamond$

Lemma 4.7. *If $p, q > 0$, we have*

$$\lim_{n\to\infty} \max_{x\in\Lambda^{(n)}} |a_x^{(n)} - a_x| = 0.$$

Proof. The action of ρ implies the following recursion relations for all $x, n \in \mathbb{N}_0$:

$$a_{2x+1}^{(n)} = 1 - pa_x^{(n-1)}, \qquad a_{2x}^{(n)} = 1 - qa_x^{(n-1)} \tag{4.9}$$

and

$$a_{2x+1} = 1 - pa_x, \qquad a_{2x} = 1 - qa_x. \tag{4.10}$$

Let $r := \max\{p, q\}$. Next, we are going to show that

$$\max_{x \in \Lambda^{(n)}} |a_x^{(n)} - a_x| \leq r^n,$$

which proves the lemma. The claim is obviously true for $n = 0$. Now, fix $n \in \mathbb{N}$, and assume that the claim is true for $n - 1$. Then, Eqs. (4.9) and (4.10) imply

$$\begin{aligned}
\max_{x \in \Lambda^{(n)}} |a_x^{(n)} - a_x| &= \max\Big\{ \max_{\substack{x \in \Lambda^{(n)}, \\ x \text{ even}}} |a_x^{(n)} - a_x|, \max_{\substack{x \in \Lambda^{(n)}, \\ x \text{ odd}}} |a_x^{(n)} - a_x| \Big\} \\
&= \max\Big\{ q \max_{y \in \Lambda^{(n-1)}} |a_y^{(n-1)} - a_y|, p \max_{y \in \Lambda^{(n-1)}} |a_y^{(n-1)} - a_y| \Big\} \\
&\leq r \cdot r^{n-1} = r^n.
\end{aligned}$$

Hence, the claim is also true for n. $\square$

As before, in order to apply Theorem 3.29, the next aim is to find a function $g \in C_{\mathrm{c}}(\mathbb{Z}_2, \mathbb{R})$ such that $a_x = g(x^\star)$. We will see that it is enough to consider the dense subset $\mathbb{N}_0 \subseteq \mathbb{Z}_2$. In this case, we have $g(x^\star) = g(x)$ because the $\star$-map is the identity on $\mathbb{Z}$. It is not difficult to see that

$$g(0) = \frac{1}{1+q}.$$

Furthermore, due to the action of ρ, we obtain for $n \geq 0$:

$$g(2n+1) = 1 - pg(n) \quad \text{and} \quad g(2n) = 1 - qg(n).$$

We will prove that the mapping $g : (\mathbb{N}_0, |\cdot|_2) \to (\mathbb{R}, |\cdot|)$ defined as above is uniformly continuous. For that, we need the following two lemmas.

Lemma 4.8. *(a) For $m, j \in \mathbb{N}_0$, we have:*

$$g(m2^j) = g(0) + \left(-\frac{1}{1+q} + g(m) \right) \cdot (-q)^j.$$

(b) For all $k \in \mathbb{N}_0$, we have:

$$g(m2^j + k) = g(k) + x_j \quad \textit{with} \quad |x_j| \leq (\max\{p, q\})^j$$

for all $m, j \in \mathbb{N}_0$.

Proof. (a) We obtain

$$g(m2^j) = g(2 \cdot m2^{j-1}) = 1 - qg(m2^{j-1}) = 1 - q(1 - qg(m2^{j-2})) = 1 - q + q^2 g(m2^{j-2})$$

and inductively

$$g(m2^j) = \sum_{k=0}^{\ell-1} (-q)^k + (-q)^\ell g(m2^{j-\ell})$$

for every $0 \leq \ell \leq j$. Hence,

$$\begin{aligned} g(m2^j) &= \frac{1-(-q)^j}{1+q} + (-q)^j g(m) = \frac{1}{1+q} + (-q)^j \cdot \left(-\frac{1}{1+q} + g(m) \right) \\ &= g(0) + \left(-\frac{1}{1+q} + g(m) \right) \cdot (-q)^j. \end{aligned}$$

(b) Proof by induction. For $k = 0$, the claim follows from (a) because $\frac{1}{1+q}$ and g are both bounded by 1. Now, let the assertion be true for $1, \ldots, k$, where $k = 2r$ is even (the case k odd can be handled analogously). Then,

$$\begin{aligned} g(m2^j + k + 1) = g(2(m2^{j-1} + r) + 1) &= 1 - pg(m2^{j-1} + \underbrace{r}_{\leq k}) \\ &= 1 - p(g(r) + x_{j-1}) = 1 - pg(r) - px_{j-1} \\ &= g(k+1) - px_{j-1}. \end{aligned}$$

□

Finally, we can prove the uniform continuity of g. Let $\varepsilon > 0$. Choose $j \in \mathbb{N}$ such that $(\max\{p,q\})^j < \varepsilon$ and $\delta := 2^{-j}$. Then, we have $|x - y|_2 \leq \delta$ if and only if $x \equiv y \mod 2^j$. Hence, for all $i \geq j$, we obtain by Lemma 4.8

$$\begin{aligned} |g(x) - g(y)| = |g(x) - g(x + m2^i)| &= |x_i| \leq (\max\{p,q\})^i \\ &\leq (\max\{p,q\})^j < \varepsilon. \end{aligned}$$

As $\mathbb{N} \subseteq \mathbb{Z}_2$ is dense and g is uniformly continuous, g can be extended to a uniformly continuous function $G : \mathbb{Z}_2 \to \mathbb{R}$, see [Die70, (3.15.6)].

Now, Theorem 3.29 implies that $\widehat{\gamma_1}$ is a pure point measure.

Corollary 4.9. *The measure $\widehat{\gamma_1}$ is pure point.*

Next, we can apply [Hof95, Thm. 3.2] and [Len09, Thm. 5, Cor. 5] to obtain

$$\widehat{\gamma_\Lambda}(\{k\}) = \lim_{n\to\infty} \frac{1}{4^n} \left|\mathbb{E}(\mathcal{X}_n(k))\right|^2. \tag{4.11}$$

Remember that every $k \in L^{\circledast}$ can be written in the form $k = \frac{m}{2^r}$. If $n \geq r+2$, then

$$\mathbb{E}_n \;=\; \mathbb{E}_{n-1} + 2\mathbb{E}_{n-2}$$

by Eq. (4.5). With the initial conditions $\mathbb{E}_r$ and $\mathbb{E}_{r+1}$, this recurrence relation has the unique solution

$$\mathbb{E}_n \;=\; \frac{1}{3}\left(2^{n-r}\cdot(\mathbb{E}_r + \mathbb{E}_{r+1}) + (-1)^{n-r}\cdot(2\mathbb{E}_r - \mathbb{E}_{r+1})\right).$$

This and Eq. (4.11) imply

$$\widehat{\gamma_\Lambda}\left(\left\{\tfrac{m}{2^r}\right\}\right) \;=\; \tfrac{1}{9\cdot 4^r}\left|E_r\left(\tfrac{m}{2^r}\right) + E_{r+1}\left(\tfrac{m}{2^r}\right)\right|^2.$$

Applying Eq. (4.5), it is not difficult to see that $E_{r+1}\left(\frac{m}{2^r}\right) = 2\cdot E_r\left(\frac{m}{2^r}\right)$. Hence, we have

$$\widehat{\gamma_\Lambda}\left(\left\{\tfrac{m}{2^r}\right\}\right) \;=\; \tfrac{1}{9\cdot 4^{r-1}}\left|E_r\left(\tfrac{m}{2^r}\right)\right|^2.$$

Moreover, we obtain from Eq. (4.5) inductively

$$\begin{aligned}
E_r\left(\frac{m}{2^r}\right) &= \left(E_{r-1}\left(\frac{m}{2^r}\right) - \left(1+\mathrm{e}^{-2^{r-1}\pi\mathrm{i}\frac{m}{2^r}}\right)E_{r-2}\left(\frac{m}{2^r}\right)\right)\cdot\left(-q - p\,\mathrm{e}^{-2^r\pi\mathrm{i}\frac{m}{2^r}}\right)\\
&= \left(E_1\left(\frac{m}{2^r}\right) - \left(1+\mathrm{e}^{-2\pi\mathrm{i}\frac{m}{2^r}}\right)E_0\left(\frac{m}{2^r}\right)\right)\cdot\prod_{j=2}^{r}\left(-q - p\,\mathrm{e}^{-2^j\pi\mathrm{i}\frac{m}{2^r}}\right)\\
&= \prod_{j=1}^{r}\left(-q - p\,\mathrm{e}^{-2^j\pi\mathrm{i}\frac{m}{2^r}}\right).
\end{aligned} \tag{4.12}$$

Finally, this gives

$$\begin{aligned}
\widehat{\gamma_\Lambda}\left(\left\{\tfrac{m}{2^r}\right\}\right) &= \tfrac{1}{9\cdot 4^{r-1}}\prod_{j=1}^{r}\left|q + p\,\mathrm{e}^{-2^j\pi\mathrm{i}\frac{m}{2^r}}\right|^2\\
&= \widehat{\gamma_{\Lambda,\mathrm{det}}}\left(\left\{\tfrac{m}{2^r}\right\}\right)\cdot\prod_{j=1}^{r}\left|q + p\,\mathrm{e}^{-2^j\pi\mathrm{i}\frac{m}{2^r}}\right|^2.
\end{aligned} \tag{4.13}$$

Let us summarise the results of this chapter.

Theorem 4.10. *The diffraction measure $\widehat{\gamma_\Lambda}$ of the random period doubling substitution consists of an absolutely continuous part and a pure point part. More precisely:*

$$\widehat{\gamma_\Lambda} = \sum_{k\in L^{\circledast}} I(k)\,\delta_k \;+\; \phi\boldsymbol{\lambda},$$

where $I(k)$ is given in Eq. (4.13) and ϕ is the density function from Corollary 4.4.

Remark 4.11. As in the case of the random Fibonacci substitution, we find that the diffraction intensities in the stochastic situation are given by the deterministic ones multiplied by a function that depends on the probabilities p and q. One might wonder why we obtain a finite and not an infinite product in the diffraction formula. However, we have

$$\prod_{j=1}^{r} \left| q + p\, \mathrm{e}^{-2^j \pi \mathrm{i} \frac{m}{2^r}} \right|^2 = \prod_{j=1}^{\infty} \left| q + p\, \mathrm{e}^{-2^j \pi \mathrm{i} \frac{m}{2^r}} \right|^2 .$$

Therefore, we obtain again that the diffraction in the stochastic setting is given by the deterministic one multiplied with an infinite product which arises from the Bernoulli process of the local mixture. $\diamond$

5 An example in two dimensions

In this chapter, we extend the study of the previous chapter by considering two block substitutions of constant length. It is not uncommon that things become much more complicated when dealing with higher dimensions. However, we will see that we can prove the analogue of Theorem 4.10 in exactly the same way.

5.1 The deterministic block substitution

Consider the the block substitution σ_0 defined by

$$a \mapsto \begin{smallmatrix} a & b \\ a & a \end{smallmatrix}, \qquad b \mapsto \begin{smallmatrix} a & a \\ a & a \end{smallmatrix}$$

with substitution matrix

$$M_{\sigma_0} := \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

This matrix is primitive. Starting from the legal seed $\begin{smallmatrix} \mathrm{a} & \mathrm{a} \\ \mathrm{a} & \mathrm{a} \end{smallmatrix}$, an iteration of σ_0^2 leads to the fixed point (finite patch)

$$\begin{array}{cccc|cccc}
\mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{a} \\
\mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} \\
\mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} \\
\mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} \\
\hline
\mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{a} \\
\mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} \\
\mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} & \mathrm{a} & \mathrm{b} \\
\mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a}
\end{array}$$

One can show that this sequence is aperiodic, see [BG13, Prop. 4.7]. The eigenvalues of M_{σ_0} are 4 and -1 with corresponding left and right PF eigenvectors $\mathbf{L}_{\sigma_0} = (1,1)$ and $\mathbf{R}_{\sigma_0} = \left(\frac{4}{5}, \frac{1}{5}\right)^{\mathsf{T}}$. Hence, it follows that σ_0 is a primitive and aperiodic Pisot substitution with leading PF eigenvalue $\lambda_{\mathrm{PF}} = 4$.

Once again, the left PF eigenvector can be used to construct a point set; this time in $\mathbb{R}^2$. Here, we identify the letter a with a square $\square$ and the letter b with a square $\blacksquare$. In both cases, the length of the sides of the squares is 1. Additionally, we choose the left bottom edge of each square as reference points. This leads to the point set

$$\Lambda = \Lambda_a \,\dot\cup\, \Lambda_b = \mathbb{Z}^2. \tag{5.1}$$

The geometric fixed point equation for σ_0^2 implies the identities

$$\begin{aligned}
\Lambda_a \;=\;& 4\Lambda_a \,\dot\cup\, (4\Lambda_a + e_1) \,\dot\cup\, (4\Lambda_a + 2e_1) \,\dot\cup\, (4\Lambda_a + 3e_1) \,\dot\cup\, (4\Lambda_a + e_2) \,\dot\cup\, (4\Lambda_a + 2e_1 + e_2)\\
&\dot\cup\, (4\Lambda_a + 2e_2) \,\dot\cup\, (4\Lambda_a + e_1 + 2e_2) \,\dot\cup\, (4\Lambda_a + 2e_1 + 2e_2) \,\dot\cup\, (4\Lambda_a + 3e_1 + 2e_2)\\
&\dot\cup\, (4\Lambda_a + 3e_2) \,\dot\cup\, (4\Lambda_a + 2e_1 + 3e_2) \,\dot\cup\, (4\Lambda_a + 3e_1 + 3e_2) \,\dot\cup\, 4\Lambda_b \,\dot\cup\, (4\Lambda_b + e_1)\\
&\dot\cup\, (4\Lambda_b + 2e_1) \,\dot\cup\, (4\Lambda_b + 3e_1) \,\dot\cup\, (4\Lambda_b + e_2) \,\dot\cup\, (4\Lambda_b + 2e_1 + e_2) \,\dot\cup\, (4\Lambda_b + 2e_2)\\
&\dot\cup\, (4\Lambda_b + e_1 + 2e_2) \,\dot\cup\, (4\Lambda_b + 2e_1 + 2e_2) \,\dot\cup\, (4\Lambda_b + 3e_1 + 2e_2) \,\dot\cup\, (4\Lambda_b + 3e_2)\\
&\dot\cup\, (4\Lambda_b + 2e_1 + 3e_2),\\
\Lambda_b \;=\;& (4\Lambda_a + e_1 + e_2) \,\dot\cup\, (4\Lambda_a + 3e_1 + e_2) \,\dot\cup\, (4\Lambda_a + e_1 + 3e_2) \,\dot\cup\, (4\Lambda_b + e_1 + e_2)\\
&\dot\cup\, (4\Lambda_b + 3e_1 + e_2) \,\dot\cup\, (4\Lambda_b + e_1 + 3e_2) \,\dot\cup\, (4\Lambda_b + 3e_1 + 3e_2).
\end{aligned} \tag{5.2}$$

Combining Eqs. (5.1) and (5.2), we get

$$\begin{aligned}
\Lambda_a =\;& 4\mathbb{Z}^2 \,\dot\cup\, (4\mathbb{Z}^2 + e_1) \,\dot\cup\, (4\mathbb{Z}^2 + 2e_1) \,\dot\cup\, (4\mathbb{Z}^2 + 3e_1) \,\dot\cup\, (4\mathbb{Z}^2 + e_2) \,\dot\cup\, (4\mathbb{Z}^2 + 2e_1 + e_2)\\
&\dot\cup\, (4\mathbb{Z}^2 + 2e_2) \,\dot\cup\, (4\mathbb{Z}^2 + e_1 + 2e_2) \,\dot\cup\, (4\mathbb{Z}^2 + 2e_1 + 2e_2) \,\dot\cup\, (4\mathbb{Z}^2 + 3e_1 + 2e_2)\\
&\dot\cup\, (4\mathbb{Z}^2 + 3e_2) \,\dot\cup\, (4\mathbb{Z}^2 + 2e_1 + 3e_2) \,\dot\cup\, (4\Lambda_a + 3e_1 + 3e_2)\\
\Lambda_b =\;& (4\mathbb{Z}^2 + e_1 + e_2) \,\dot\cup\, (4\mathbb{Z}^2 + 3e_1 + e_2) \,\dot\cup\, (4\mathbb{Z}^2 + e_1 + 3e_2) \,\dot\cup\, (4\Lambda_b + 3e_1 + 3e_2)
\end{aligned}$$

By iteration, this leads to the solutions (with $c_i^{(j)} := j \cdot 4^i - 1$)

$$\begin{aligned}
\Lambda_a \;=\;& \dot{\bigcup_{i\geq 1}} \Big[\big(4^i\mathbb{Z}^2 + c_{i-1}^{(1)}\, e_1 + c_{i-1}^{(1)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(2)}\, e_1 + c_{i-1}^{(1)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(3)}\, e_1 + c_{i-1}^{(1)}\, e_2\big)\\
&\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i}^{(1)}\, e_1 + c_{i-1}^{(1)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(1)}\, e_1 + c_{i-1}^{(2)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(3)}\, e_1 + c_{i-1}^{(2)}\, e_2\big)\\
&\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(1)}\, e_1 + c_{i-1}^{(3)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(2)}\, e_1 + c_{i-1}^{(3)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(3)}\, e_1 + c_{i-1}^{(3)}\, e_2\big)\\
&\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i}^{(1)}\, e_1 + c_{i-1}^{(3)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(1)}\, e_1 + c_{i}^{(1)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(3)}\, e_1 + c_{i}^{(1)}\, e_2\big)\Big],\\
&\dot\cup\, \{(-1,-1)\}\\
\Lambda_b \;=\;& \dot{\bigcup_{i\geq 1}} \Big[\big(4^i\mathbb{Z}^2 + c_{i-1}^{(2)}\, e_1 + c_{i-1}^{(2)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i-1}^{(2)}\, e_1 + c_{i}^{(1)}\, e_2\big) \,\dot\cup\, \big(4^i\mathbb{Z}^2 + c_{i}^{(1)}\, e_1 + c_{i-1}^{(2)}\, e_2\big)\Big].
\end{aligned}$$

Remark 5.1. In general, equations like Eq. (5.2) have several solutions, which distinguish themselves by a set of measure zero. $\diamond$

If we choose $H = \mathbb{Z}_2^2$ as (locally) compact Abelian group, these point sets can be obtained as model sets within the CPS $(\mathbb{R}^2, \mathbb{Z}_2^2, \mathcal{L})$ with lattice

$$\mathcal{L} = \big\{(x, \iota(x)) \mid x \in \mathbb{Z}^2\big\} \subseteq \mathbb{R}^2 \times \mathbb{Z}_2^2,$$

where $\iota : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}_2^2$ is the canonical embedding and also the $\star$-map.

Therefore, the diffraction measure of σ_0 is pure point and is explicitly given by

$$\widehat{\gamma_\Lambda} = \sum_{k \in L^{\circledast}} |u_a A(k) + u_b A_b(k)|^2 \, \delta_k,$$

where $L^{\circledast} = \mathbb{Z}\big[\frac{1}{2}\big] \times \mathbb{Z}\big[\frac{1}{2}\big]$, $u_a, u_b \in \mathbb{C}$. Again, one can make use of the limit-periodic structure of the block substitution as described in [BG13, Sec. 9.4.4]. One obtains

$$A(k) = \frac{4}{5} \frac{(-1)^r}{4^r} \, \mathrm{e}^{2\pi \mathrm{i}(k_1 + k_2)} \qquad \text{and} \qquad B(k) = \delta_{r,0} - A(k)$$

for any $(k_1, k_2) = \big(\frac{m_1}{2^{r_1}}, \frac{m_2}{2^{r_2}}\big) \in L^{\circledast}$. Consequently, the diffraction intensities for $k \in L^{\circledast}$ can be stated as

$$I(k) \; = \; \begin{cases} \frac{1}{25 \cdot 16^{r-1}} \, |u_a - u_b|^2, & r \geqslant 1, \\ \frac{1}{25} \, |4u_a + u_b|^2, & r = 0. \end{cases}$$

5.2 The randomised case

Let $\boldsymbol{p} = (p_0, p_1, p_2, p_3) \gg 0$ be a fixed probability vector. Consider the random substitution

$$\sigma : \begin{cases} a \mapsto \begin{cases} w_0(a), & \text{with probability } p_0, \\ w_1(a), & \text{with probability } p_1, \\ w_2(a), & \text{with probability } p_2 \\ w_3(a), & \text{with probability } p_3, \end{cases} \\ b \mapsto w_4(a), \end{cases}$$

where

$$w_0(a) \; = \; \begin{smallmatrix} b\,a \\ a\,a \end{smallmatrix}, \quad w_1(a) \; = \; \begin{smallmatrix} a\,b \\ a\,a \end{smallmatrix}, \quad w_2(a) \; = \; \begin{smallmatrix} a\,a \\ a\,b \end{smallmatrix}, \quad w_3(a) \; = \; \begin{smallmatrix} a\,a \\ b\,a \end{smallmatrix}, \quad w_4(a) \; = \; \begin{smallmatrix} a\,a \\ a\,a \end{smallmatrix}$$

with corresponding substitution matrix (in the sense of Definition 2.7)

$$M_\sigma := \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

Once again, the substitution matrices of the deterministic and the stochastic substitutions coincide, and σ is primitive. The eigenvalues of this matrix are given by 4 and -1, and the PF eigenvectors are given by $\mathbf{L}_\sigma = \mathbf{L}_{\sigma_0}$ and $\mathbf{R}_\sigma = \mathbf{R}_{\sigma_0}$. Consequently, σ is a primitive Pisot substitution.

As before, the stochastic hull $\mathbb{X}_\rho$ is defined as the smallest closed and shift-invariant subset of $\mathcal{A}_2^{\mathbb{Z}^2}$ with $X_\sigma \subseteq \mathbb{X}_\sigma$, where

$$X_\sigma := \Big\{ w \in \mathcal{A}_2^{\mathbb{Z}^2} \mid w \text{ is an accumulation point of } \big(\sigma^k(\begin{smallmatrix} a\,a \\ a\,a \end{smallmatrix})\big)_{k \in \mathbb{N}_0} \Big\}.$$

The geometric realisations of these elements are constructed as usual.

Lemma 5.2. *If $\boldsymbol{p} \gg 0$, the topological entropy of the dynamical system $(X_\sigma, \mathbb{Z})$ is positive; more precisely, we have $h_{top} = \frac{8}{15}\log(2)$.*

Proof. Compare the proof of Lemma 6.1 in [BSS17]. Just note that there are exactly $2^{\frac{1}{15}(8\cdot 4^n - 3\cdot(-1)^n - 5)}$ different realisations of $\sigma^n(a)$ and

$$\lim_{n\to\infty} \frac{\log\left(2^{\frac{1}{15}(8\cdot 4^n - 3\cdot(-1)^n - 5)}\right)}{4^n} = \frac{8}{15}\log(2).$$

□

Consequently, due to [BLR07] and Proposition 3.1, we expect the diffraction measure of the random block substitution to be a of mixed type.

5.3 Diffraction of the random block substitution

Now, we proceed exactly as we have done before. This time, our random variable reads

$$\mathcal{X}_n(k_1,k_2) = \begin{cases} F_0(k_1,k_2), & \text{with probability } p_0, \\ F_1(k_1,k_2) & \text{with probability } p_1, \\ F_2(k_1,k_2) & \text{with probability } p_2, \\ F_3(k_1,k_2) & \text{with probability } p_3, \end{cases} \tag{5.3}$$

together with $\mathcal{X}_0(k) = u_a$ and

$$\mathcal{X}_1(k) = \begin{cases} u_a + u_a\,\mathrm{e}^{-2\pi\mathrm{i}k_1} + u_b\,\mathrm{e}^{-2\pi\mathrm{i}k_2} + u_a\,\mathrm{e}^{-2\pi\mathrm{i}(k_1+k_2)}, & \text{with probability } p_0, \\ u_a + u_a\,\mathrm{e}^{-2\pi\mathrm{i}k_1} + u_a\,\mathrm{e}^{-2\pi\mathrm{i}k_2} + u_b\,\mathrm{e}^{-2\pi\mathrm{i}(k_1+k_2)}, & \text{with probability } p_1, \\ u_a + u_b\,\mathrm{e}^{-2\pi\mathrm{i}k_1} + u_a\,\mathrm{e}^{-2\pi\mathrm{i}k_2} + u_a\,\mathrm{e}^{-2\pi\mathrm{i}(k_1+k_2)}, & \text{with probability } p_2, \\ u_b + u_a\,\mathrm{e}^{-2\pi\mathrm{i}k_1} + u_a\,\mathrm{e}^{-2\pi\mathrm{i}k_2} + u_a\,\mathrm{e}^{-2\pi\mathrm{i}(k_1+k_2)}, & \text{with probability } p_3, \end{cases}$$

where $u_a, u_b \in \mathbb{C}$ are fixed weights and

$$\begin{aligned} F_0(k_1,k_2) := {} & \mathcal{X}^{(1)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}k_1}\,\mathcal{X}^{(2)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)}\,\mathcal{X}^{(3)}_{n-1}(k_1,k_2) \\ & + \mathrm{e}^{-2^n\pi\mathrm{i}k_2}\,\mathcal{X}^{(1)}_{n-2}(k_1,k_2) + \mathrm{e}^{-3\cdot 2^{n-1}\pi\mathrm{i}k_2}\,\mathcal{X}^{(2)}_{n-2}(k_1,k_2) \\ & + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}(k_1+2k_2)}\,\mathcal{X}^{(3)}_{n-2}(k_1,k_2) + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}(k_1+3k_2)}\,\mathcal{X}^{(4)}_{n-2}(k_1,k_2), \end{aligned}$$

$$\begin{aligned} F_1(k_1,k_2) := {} & \mathcal{X}^{(1)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}k_1}\,\mathcal{X}^{(2)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}k_2}\,\mathcal{X}^{(3)}_{n-1}(k_1,k_2) \\ & + \mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)}\,\mathcal{X}^{(1)}_{n-2}(k_1,k_2) + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}(2k_1+3k_2)}\,\mathcal{X}^{(2)}_{n-2}(k_1,k_2) \\ & + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}(3k_1+2k_2)}\,\mathcal{X}^{(3)}_{n-2}(k_1,k_2) + \mathrm{e}^{-3\cdot 2^{n-1}\pi\mathrm{i}(k_1+k_2)}\,\mathcal{X}^{(4)}_{n-2}(k_1,k_2), \end{aligned}$$

$$
\begin{aligned}
F_2(k_1,k_2) := {} & \mathcal{X}^{(1)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}k_2}\,\mathcal{X}^{(2)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)}\,\mathcal{X}^{(3)}_{n-1}(k_1,k_2)\\
& + \mathrm{e}^{-2^n\pi\mathrm{i}k_1}\,\mathcal{X}^{(1)}_{n-2}(k_1,k_2) + \mathrm{e}^{-3\cdot 2^{n-1}\pi\mathrm{i}k_1}\,\mathcal{X}^{(2)}_{n-2}(k_1,k_2)\\
& + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}(2k_1+k_2)}\,\mathcal{X}^{(3)}_{n-2}(k_1,k_2) + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}(3k_1+k_2)}\,\mathcal{X}^{(4)}_{n-2}(k_1,k_2),\\
F_3(k_1,k_2) := {} & \mathrm{e}^{-2^n\pi\mathrm{i}k_1}\,\mathcal{X}^{(1)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}k_2}\,\mathcal{X}^{(2)}_{n-1}(k_1,k_2) + \mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)}\,\mathcal{X}^{(3)}_{n-1}(k_1,k_2)\\
& + \mathcal{X}^{(1)}_{n-2}(k_1,k_2) + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_1}\,\mathcal{X}^{(2)}_{n-2}(k_1,k_2)\\
& + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_2}\,\mathcal{X}^{(3)}_{n-2}(k_1,k_2) + \mathrm{e}^{-2^{n-1}\pi\mathrm{i}(k_1+k_2)}\,\mathcal{X}^{(4)}_{n-2}(k_1,k_2).
\end{aligned}
$$

The expected value $\mathbb{E}_n := \mathbb{E}(\mathcal{X}_n)$ of $\mathcal{X}_n(k)$ is given by

$$
\mathbb{E}_n(k_1,k_2) = G(k_1,k_2)\mathbb{E}_{n-1}(k_1,k_2) + H(k_1,k_2)\mathbb{E}_{n-2}(k_1,k_2), \tag{5.4}
$$

where

$$
\begin{aligned}
G(k_1,k_2) := {} & p_0(1+\mathrm{e}^{-2^n\pi\mathrm{i}k_1}+\mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)}) + p_1(1+\mathrm{e}^{-2^n\pi\mathrm{i}k_1}+\mathrm{e}^{-2^n\pi\mathrm{i}k_2})\\
& + p_2(1+\mathrm{e}^{-2^n\pi\mathrm{i}k_2}+\mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)}) + p_3(\mathrm{e}^{-2^n\pi\mathrm{i}k_1}+\mathrm{e}^{-2^n\pi\mathrm{i}k_2}+\mathrm{e}^{-2^n\pi\mathrm{i}k_1}),\\
H(k_1,k_2) := {} & \big(p_0\,\mathrm{e}^{-2^n\pi\mathrm{i}k_2} + p_1\,\mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)} + p_2\,\mathrm{e}^{-2^n\pi\mathrm{i}k_1} + p_3\big)\big(1+\mathrm{e}^{-2^n\pi\mathrm{i}k_1}\big)\big(1+\mathrm{e}^{-2^n\pi\mathrm{i}k_2}\big)
\end{aligned}
$$

Moreover, we obtain a recurrence relation for $\mathbb{V}_n := \mathrm{Var}(\mathcal{X}_n)$, $n \geq 2$:

$$
\mathbb{V}_n = 3\mathbb{V}_{n-1} + 4\mathbb{V}_{n-1} + 2\Big((p_0p_1+p_2p_3)\Psi^{(1)}_n + p_0p_2\Psi^{(2)}_n + (p_0p_3+p_1p_2)\Psi^{(3)}_n + p_1p_3\Psi^{(4)}_n\Big),
$$

with $\mathbb{V}_0 \equiv 0$ and $\mathbb{V}_1(k) = 2pq|u_a-u_b|^2\big(1-\cos(2\pi k)\big)$, where the functions $\Psi^{(\nu)}_n$, for $\nu \in \{1,2,3,4\}$, are defined by

$$
\begin{aligned}
\Psi^{(1)}_n &:= \frac{1}{2}\Big|1-\mathrm{e}^{-2^n\pi\mathrm{i}k_1}\Big|^2\Big|\mathbb{E}_{n-1} - (1+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_1}+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_2}-\mathrm{e}^{-2^{n-1}\pi\mathrm{i}(k_1+k_2)})\mathbb{E}_{n-2}\Big|^2,\\
\Psi^{(2)}_n &:= \frac{1}{2}\Big|1-\mathrm{e}^{-2^n\pi\mathrm{i}(k_1-k_2)}\Big|^2\Big|\mathbb{E}_{n-1} - (1+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_1}+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_2}-\mathrm{e}^{-2^{n-1}\pi\mathrm{i}(k_1+k_2)})\mathbb{E}_{n-2}\Big|^2,\\
\Psi^{(3)}_n &:= \frac{1}{2}\Big|1-\mathrm{e}^{-2^n\pi\mathrm{i}k_2}\Big|^2\Big|\mathbb{E}_{n-1} - (1+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_1}+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_2}-\mathrm{e}^{-2^{n-1}\pi\mathrm{i}(k_1+k_2)})\mathbb{E}_{n-2}\Big|^2,\\
\Psi^{(4)}_n &:= \frac{1}{2}\Big|1-\mathrm{e}^{-2^n\pi\mathrm{i}(k_1+k_2)}\Big|^2\Big|\mathbb{E}_{n-1} - (1+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_1}+\mathrm{e}^{-2^{n-1}\pi\mathrm{i}k_2}-\mathrm{e}^{-2^{n-1}\pi\mathrm{i}(k_1+k_2)})\mathbb{E}_{n-2}\Big|^2.
\end{aligned}
$$

Moreover, we decompose the diffraction measure as before

$$
\widehat{\gamma_\Lambda} = \mathbb{E}(\widehat{\gamma_\Lambda}) = \lim_{n\to\infty}\frac{1}{4^n}\left|\mathbb{E}(\mathcal{X}_n)\right|^2 + \lim_{n\to\infty}\frac{1}{4^n}\,\mathrm{Var}(\mathcal{X}_n) =: \widehat{\gamma}_1 + \widehat{\gamma}_2
$$

for ν_m-almost all $\Lambda \in \mathbb{Y}_m$.

5.3.1 The continuous part of the diffraction measure

Let us discuss $\widehat{\gamma_2}$ first. To do so, we need the next lemma, and use the following abbreviations

$$k^{(1)} := k_1, \qquad k^{(2)} := k_1 - k_2, \qquad k^{(3)} := k_2, \qquad k^{(4)} := k_1 + k_2.$$

Lemma 5.3. *We have*

$$\Psi_n^{(\nu)}(k_1,k_2) = |u_a - u_b|^2 \big(1 - \cos(2^n \pi k^{(\nu)})\big) \prod_{j=1}^{n-1} \Big| p_3 + p_2\, \mathrm{e}^{-2^j \pi \mathrm{i} k_1} + p_0\, \mathrm{e}^{-2^j \pi \mathrm{i} k_2} + p_1\, \mathrm{e}^{-2^j \pi \mathrm{i}(k_1+k_2)} \Big|^2 .$$

In particular, this implies $\|\Psi_n^{(\nu)}\|_\infty \leq 2|u_a - u_b|^2$ for all $n \geq 2$.

Proof. This lemma is proved in the same way as Lemma 4.2. □

This leads to the following result.

Proposition 5.4. *Consider the function*

$$\phi_n : \mathbb{R} \to \mathbb{R}_+, \quad k \mapsto \frac{1}{4^n} \operatorname{Var}(\mathcal{X}_n(k)).$$

On $\mathbb{R}$, the sequence $(\phi_n)_{n \in \mathbb{N}_0}$ converges uniformly to the continuous function

$$\phi : \mathbb{R} \to \mathbb{R}_+, \quad k \mapsto \frac{1}{5}\mathbb{V}_1 + \frac{8}{5} \sum_{\nu=1}^{4} p^{(\nu)} \sum_{i=2}^{\infty} \frac{\Psi_i^{(\nu)}(k)}{4^i},$$

where

$$p^{(1)} := p_0 p_1 + p_2 p_3, \quad p^{(2)} := p_0 p_2, \quad p^{(3)} := p_0 p_3 + p_1 p_2, \quad p^{(4)} := p_1 p_3.$$

The measure $\lim_{n\to\infty} \frac{1}{4^n} \operatorname{Var}(\mathcal{X}_n)$ is absolutely continuous with respect to $\boldsymbol{\lambda}$. Its density function is

$$\phi(k_1,k_2) = \frac{4}{5}|u_a - u_b|^2 \sum_{n=1}^{\infty} \frac{f_n(k_1,k_2)}{4^n} \prod_{j=1}^{n-1} \Big| p_3 + p_2\, \mathrm{e}^{-2^j \pi \mathrm{i} k_1} + p_0\, \mathrm{e}^{-2^j \pi \mathrm{i} k_2} + p_1\, \mathrm{e}^{-2^j \pi \mathrm{i}(k_1+k_2)} \Big|^2 ,$$

where $f_n(k_1,k_2) := \sum_{\nu=1}^{4} 2p^{(\nu)}\big(1 - \cos(2^n \pi k^{(\nu)})\big)$ with $k^{(\nu)}$ as above. □

Remark 5.5. If $p_i = 1$ for some i, we have $\phi \equiv 0$. This means that the diffraction measure of the block substitution is pure point. Thus, Proposition 5.4 is consistent with the results from Section 5.1. ◇

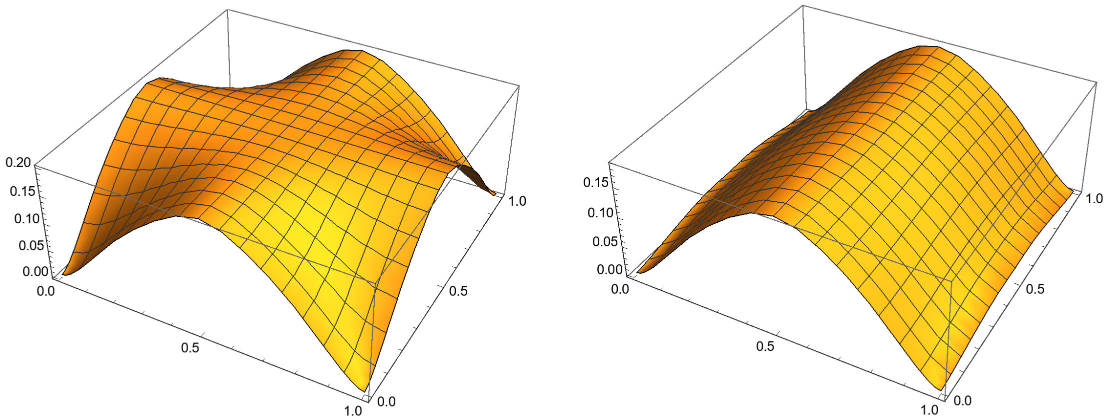

Figure 5.1: The Radon–Nikodym density ϕ of Proposition 5.4 for $\boldsymbol{p} = (0.25, 0.25, 0.25, 0.25)$ (left) and $\boldsymbol{p} = (0.01, 0.02, 0.35, 0.62)$ (right).

5.3.2 The pure point part of the diffraction measure

Next, we focus on $\widehat{\gamma}_1$. We have

$$\widehat{\gamma}_1 = \lim_{n\to\infty} \frac{1}{4^n} \big|\widehat{\mathbb{E}(m_n)}\big|^2 = \lim_{n\to\infty} \frac{1}{4^n} \mathcal{F}[\mathbb{E}(m_n) * \widetilde{\mathbb{E}(m_n)}] = \mathcal{F}[\mathbb{E}(m) \circledast \widetilde{\mathbb{E}(m)}],$$

where m_n is the measure with $\widehat{m_n} = \mathcal{X}_n$. The last identity holds due to a similar argument presented in Remark 4.6 and Lemma 4.7. By construction, $\mathbb{E}(m) = \lim_{n\to\infty} \mathbb{E}(m_n)$ is the weighted Dirac comb on $\mathbb{Z}^2$, where the weight at $x \in \mathbb{Z}^2$ is given by

$$u_a \mathbb{P}(\text{type at } x \text{ is } a) + u_b \mathbb{P}(\text{type at } x \text{ is } b) =: u_a a_x + u_b b_x,$$

so that

$$\mathbb{E}(m) \;=\; u_b \delta_{\mathbb{Z}^2} + (u_a - u_b) \sum_{x\in\mathbb{Z}^2} a_x\, \delta_x. \tag{5.5}$$

Thus, without loss of generality, we consider the case $u_a = 1$ and $u_b = 0$, which leads to

$$\mathbb{E}(m) \;=\; \sum_{x\in\mathbb{Z}^2} a_x\, \delta_x.$$

Now, Theorem 3.29 implies that $\widehat{\gamma}_1$ is pure point if we can find a function $g \in C_{\mathrm{c}}(\mathbb{Z}_2^2, \mathbb{R})$ such that $a_x = g(x^\star)$. On the dense subset $\mathbb{N}_0^2 \subseteq \mathbb{Z}_2^2$, we have $g(x^\star) = g(x)$ because the $\star$-map is the identity on $\mathbb{Z}^2$. One easily proves

$$g(0,0) = \frac{1}{1+p_3}$$

and, due to the action of σ, we obtain for $n, m \geq 0$:

$$g(2n, 2m) = 1 - p_3\, g(n,m), \qquad g(2n, 2m+1) = 1 - p_2\, g(n,m),$$

$$g(2n+1,2m+1) = 1 - p_1\, g(n,m), \qquad g(2n+1,2m) = 1 - p_0\, g(n,m).$$

We will prove that the mapping $g : (\mathbb{N}_0^2, |\cdot|_2) \to (\mathbb{R}, |\cdot|)$ defined as above is uniformly continuous. For that, we need the following lemma.

Lemma 5.6. *For all $k, n \in \mathbb{N}_0$ we have:*

$$g(n + m_1 2^{j_1}, k + m_2 2^{j_2}) = g(n,k) + x_{j_1,j_2} \quad \text{with } |x_{j_1,j_2}| \leq 2 \max_{j\in\{j_1,j_2\}} \max_{i\in\{0,\ldots,3\}} p_i^j$$

for all $m_1, m_2, j_1, j_2 \in \mathbb{N}_0$.

Proof. This lemma can be proved as shown in the proof of Lemma 4.8. □

Now, the uniform continuity of g follows the same lines as described after Lemma 4.8, and Theorem 3.29 implies that $\widehat{\gamma_1}$ is a pure point measure.

Next, we can apply [Hof95, Thm. 3.2] and [Len09, Thm. 5, Cor. 5] to obtain

$$\widehat{\gamma_\Lambda}(\{k\}) = \lim_{n\to\infty} \frac{1}{16^n} \big|\mathbb{E}(\mathcal{X}_n(k))\big|^2. \tag{5.6}$$

Every $k \in L^{\circledast}$ can be written in the form $k = \left(\frac{m_1}{2^{r_1}}, \frac{m_1}{2^{r_1}}\right)$. If $n \geq r+2$ (with $r := \max\{r_1, r_2\}$), then

$$\mathbb{E}_n = 3\,\mathbb{E}_{n-1} + 4\,\mathbb{E}_{n-2}.$$

With the initial conditions E_r and E_{r+1}, this recurrence relation has the unique solution

$$\mathbb{E}_n = \frac{1}{5}\Big(4^{n-r}\cdot(E_r + E_{r+1}) + (-1)^{n-r}\cdot(4E_r - E_{r+1})\Big).$$

Combining this and Eq. (5.6), we obtain

$$\widehat{\gamma_\Lambda}\left(\left\{\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right)\right\}\right) = \tfrac{1}{25\cdot 16^r}\left|\mathbb{E}_r\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right) + \mathbb{E}_{r+1}\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right)\right|^2.$$

Applying Eq. (5.4), it is not difficult to see that $\mathbb{E}_{r+1}\left(\frac{m_1}{2^{r_1}}, \frac{m_2}{2^{r_2}}\right) = 3\cdot\mathbb{E}_r\left(\frac{m_1}{2^{r_1}}, \frac{m_2}{2^{r_2}}\right)$. Hence, we have

$$\widehat{\gamma_\Lambda}\left(\left\{\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right)\right\}\right) = \tfrac{1}{25\cdot 16^{r-1}}\left|\mathbb{E}_r\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right)\right|^2.$$

Moreover, if we apply Eq. (5.4) repeatedly, we get (compare Eq. (4.12))

$$\mathbb{E}_r\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right) = \prod_{j=1}^{r}\left(-p_3 - p_2\,\mathrm{e}^{-2^j\pi\mathrm{i}k_1} - p_0\,\mathrm{e}^{-2^j\pi\mathrm{i}k_2} - p_1\,\mathrm{e}^{-2^j\pi\mathrm{i}(k_1+k_2)}\right).$$

Finally, we obtain

$$\widehat{\gamma_\Lambda}\left(\left\{\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right)\right\}\right) = \widehat{\gamma_{\Lambda,\mathrm{det}}}\left(\left\{\left(\tfrac{m_1}{2^{r_1}}, \tfrac{m_2}{2^{r_2}}\right)\right\}\right)\cdot\prod_{j=1}^{r}\left|p_3 + p_2\,\mathrm{e}^{-2^j\pi\mathrm{i}k_1} + p_0\,\mathrm{e}^{-2^j\pi\mathrm{i}k_2} + p_1\,\mathrm{e}^{-2^j\pi\mathrm{i}(k_1+k_2)}\right|^2. \tag{5.7}$$

Let us summarise the results of this chapter as follows.

Theorem 5.7. *The diffraction measure $\widehat{\gamma_\Lambda}$ of the random block substitution consists of an absolutely continuous part and a pure point part. More precisely:*

$$\widehat{\gamma_\Lambda} = \sum_{k \in L^{\circledast}} I(k)\, \delta_k \ + \ \phi \boldsymbol{\lambda},$$

where $I(k)$ is given in Eq. (5.7) *and ϕ is the density function from Proposition 5.4.*

Remark 5.8. Once again, we find that the diffraction intensities in the stochastic situation are given by the deterministic ones multiplied by some function that depends on the probabilities p_i. ◇

The results from Sections 3, 4 and 5 suggest that the methods used for the computation of the diffraction measure (for example the decomposition from Eq. (3.2)) can be applied to a larger class of random substitutions. To this end, let us call a finite set of deterministic substitutions $\vartheta_1, \ldots, \vartheta_n$ *compatible* if they define the same hull and can be described

(i) via an inflation rule and

(ii) within the same cut and project scheme.

Question: Let ϑ be a primitive random substitution which is a local mixture of compatible Pisot substitutions, such that $h_{\text{top}}(\mathbb{X}_\vartheta) > 0$. Is it then true that the diffraction spectrum is of mixed type, consisting of a pure point part and an absolutely continuous part, where Eq. (3.2) gives the correct decomposition of $\widehat{\gamma}$?

This is not obvious because many of the computations rely on the explicit structure of the inflation rule and the CPS.

6 The topological point spectrum

Up to now, we have only studied the diffraction spectrum of the random substitutions under consideration. In the deterministic setting, it is known that the diffraction spectrum (to be more precise, the supporting set of the Bragg peaks) coincides with the dynamical spectrum, since the system is pure point [BLvE13]. However, we obtained a diffraction spectrum of mixed type in the stochastic setting. Therefore, we cannot immediately obtain the dynamical spectrum from the diffraction spectrum.

In Section 6.1, we collect some basic facts concerning the measure-theoretic and topological spectrum of a dynamical system; see [Wal00]. After that, we study these two spectra in the case of the random noble means substitution and the random period doubling substitution. Sections 6.2 and 6.3 closely follow [BSS17, Sec. 7].

6.1 Kronecker factor versus maximal equicontinuous factor

Let us start with a measure-theoretic dynamical system $(X, \mathfrak{A}, T, \mu)$. Here, X denotes a compact space, $\mathfrak{A}$ a σ-algebra on X, μ a probability measure on X and $T : X \to X$ a measure-preserving function, which means that T is measurable and $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathfrak{A}$. Now, the function T induces the mapping (the so-called *Koopman operator*)

$$U_T : L^2(X, \mu) \to L^2(X, \mu), \quad (U_T f)(x) := f(Tx).$$

If T is invertible, the mapping U_T is a unitary operator on the Hilbert space $L^2(X, \mu)$, with the usual inner product

$$\langle f \,|\, g \rangle = \int_X \overline{f(x)}\, g(x) \; \mathrm{d}\mu(x).$$

We refer to the spectrum of U_T as the *dynamical spectrum* of $(X, \mathfrak{A}, T, \mu)$.

Definition 6.1. Suppose $(X_i, \mathfrak{A}_i, \mu_i)$ is a probability space and $T_i : X_i \to X_i$ is a measure-preserving function, $i \in \{1, 2\}$. We say $(X_2, \mathfrak{A}_2, T_2, \mu_2)$ is a *measure-theoretic factor* of $(X_1, \mathfrak{A}_1, T_1, \mu_1)$ if there exist $A_i \in \mathfrak{A}_i$ with $\mu_i(A_i) = 1$ and $T_i A_i \subseteq A_i$, and there exists a *factor map* $\Phi : A_1 \to A_2$ with

$$\Phi \circ T_1 = T_2 \circ \Phi \quad \text{ and } \quad \mu_1(\Phi^{-1}A) = \mu_2(A) \;\text{ for all } A \in \mathfrak{A}_2.$$

Some of the basic properties of the eigenvalues and eigenfunctions of U_T are collected in the next proposition.

Proposition 6.2. *([Wal00, Thm. 3.1]) Let T be a measure-preserving function of a probability space $(X, \mathfrak{A}, \mu)$, and suppose the system is ergodic. Then, the following statements are true.*

(i) If f is an eigenfunction of U_T with corresponding eigenvalue λ, then $|\lambda| = 1$, and $|f|$ is constant almost everywhere.

(ii) Eigenfunctions corresponding to different eigenvalues are orthogonal.

(iii) If f and g are both eigenfunctions corresponding to the eigenvalue λ, then $f = cg$ almost everywhere for some constant c.

(iv) The eigenvalues of T form a subgroup of the unit circle $\mathbb{S}$. □

We will now introduce an important measure-theoretic factor [HK06, Ch. 11]. Let $(X, \mathfrak{A}, T, \mu)$ be an ergodic measure-theoretic dynamical system. A closed subspace S of $L^2(X, \mu)$ is called a *unitary $*$-algebra* if S is invariant under complex conjugation, bounded functions are dense in S and the product of any two bounded functions from S is again in S. In this case, the characteristic functions generate S and S defines a measurable partition ξ of X in the following way.

Proposition 6.3. *([HK06, Ch. 11, Prop. 1.2]) Any unitary $*$-subalgebra consists of all functions in $L^2(X, \mu)$ which are constant mod 0 on elements of a measurable partition. If a unitary $*$-subalgebra is U_T-invariant, then the corresponding measurable partition is T-invariant and defines a factor of $(X, \mathfrak{A}, T, \mu)$.* □

The complex conjugate of an eigenfunction of U_T is also an eigenfunction with complex conjugate eigenvalue. Since the system is ergodic, Proposition 6.2 implies that eigenfunctions have constant absolute value. If $U_T f = \lambda f$, then

$$U_T(f\bar{f}) = U_T(f) \cdot U_T(\bar{f}) = \lambda\bar{\lambda} f\bar{f} = f\bar{f}.$$

Hence, $f\bar{f} = |f|^2$ is constant. Furthermore, both the eigenfunctions and the eigenvalues form a group invariant under complex conjugation. Consequently, linear combinations of eigenfunctions form a $*$-algebra, and hence their L^2-closure is an invariant unitary $*$-subalgebra of $L^2(X, \mu)$, which we will denote by $\mathcal{K}(T)$. Thus, by Proposition 6.3, $\mathcal{K}(T)$ determines a factor of $(X, \mathfrak{A}, T, \mu)$, called the *Kronecker factor* of $(X, \mathfrak{A}, T, \mu)$. It is the maximal factor with pure point spectrum [Que10, Wal00]. The Kronecker factor is never the empty set, since the constant function (and every constant multiple of it) is always an eigenfunction with eigenvalue 1. Now, two extremal cases can arise. If the Kronecker factor only consists of the constant functions, the dynamical system is said to have *trivial spectrum*. On the other hand, if the set of eigenfunctions generates the whole space $L^2(X, \mu)$, we say that $(X, \mathfrak{A}, T, \mu)$ has *pure point dynamical spectrum* (sometimes also called *discrete dynamical spectrum*).

Sometimes, several of the eigenfunctions can be chosen to be continuous. In this case, the corresponding eigenvalues are called *topological eigenvalues*, and we call the set of all topological eigenvalues the *topological point spectrum* of the dynamical system $(X, \mathfrak{A}, T, \mu)$.

Example 6.4. Consider the dynamical system $(\mathbb{X}_{\zeta_{m,i}}, S, \mu_m)$ that arises from a deterministic noble means substitution. Since the system is pure point and the diffraction spectrum equals $\frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2+4}}$, we find that every $\lambda \in \frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2+4}}$ is an eigenvalue of U_S. Moreover, it is well-known that all eigenvalues are topological eigenvalues, which means that the topological point spectrum is $\frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2+4}}$, too. This is a special situation. In general, the eigenfunctions of U_T (except the constant one) are merely measurable.

Next, let (X, T) be a topological dynamical system, where X is a compact metric space and $T : X \to X$ is a homeomorphism.

Definition 6.5. Let (X, T) and (Y, R) be two topological dynamical systems.

(1) A *morphism* $\Phi : (X, T) \to (Y, R)$ is a continuous map $\Phi : X \to Y$ satisfying $\Phi \circ T = R \circ \Phi$.

(2) If Φ is bijective, we say that Φ is a *conjugacy* and that (X, T) and (Y, R) are *conjugate*.

(3) If Φ is surjective, we say that Φ is a *factor map* and (Y, R) is a *factor* of (X, T) or that (X, T) is an *extension* of (Y, R).

(4) If Φ is injective, we say that (X, T) is a *subsystem* of (Y, R).

We will consider an important class of factors.

Definition 6.6. A topological dynamical system (X, T) is called *equicontinuous* if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in X : \; d(x, y) < \delta \implies \forall n \in \mathbb{N} : \; d(T^n x, T^n y) < \varepsilon.$$

Moreover, we say that (Y, R) is an *equicontinuous factor* of (X, T) if (Y, R) is a factor of (X, T), and (Y, R) is equicontinuous.

One can show that any topological dynamical system possesses a *maximal equicontinuous factor* $(X_{\max}, T_{\max})$ in the sense that every equicontinuous factor of (X, T) is a factor of $(X_{\max}, T_{\max})$. Let us look at an important case of general interest.

Example 6.7. Consider a regular model set $\Lambda = \curlywedge(W)$ of a CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$. It was shown in [BLM07] that Λ gives rise to a strictly ergodic dynamical system which is almost everywhere one to one over its maximal equicontinuous factor. This factor is given by the 2-torus $\mathbb{T} = (\mathbb{R} \times \mathbb{R})/\mathcal{L}$ together with the canonical $\mathbb{R}$-action

$$(u, v) + \mathcal{L} \mapsto (u + t, v) + \mathcal{L}$$

for all $t \in \mathbb{R}$ and $(u, v) \in \mathbb{R} \times H$, which turns $\mathbb{T}$ into a topological dynamical system and also into a measure-theoretic dynamical system (with the corresponding Haar measure).

6.2 The topological point spectrum of the random noble means chain

In this section, we will study the topological point spectrum and the Kronecker factor of the random noble means chain.

6.2.1 Generic elements of the stochastic hull

Let us look one more time at the dynamical system $(\mathbb{Y}_m, \mathbb{R}, \nu_m)$. Consider the set

$$Y := \{\Lambda \mid \Lambda \text{ is the one-sided geometric realisation of an accumulation point of } \big(\zeta_m^n(a)\big)_{n\in\mathbb{N}}\}.$$

Remember that we identify the letter a with the left endpoint of an interval of length λ_m and b with the left endpoint of an interval of length 1.

Now, we are going to play the 'chaos game' [Pal92, Sec. 4.2]. In order to do so, we need to introduce the following notation. By (α, x), we mean that at position x there is the left endpoint of an interval I_α with $\alpha \in \{a, b\}$. An application of ζ_m leads to the two mappings $(b, x) \mapsto (a, \lambda_m x)$ and

$$(a,x) \mapsto \begin{cases} \{(b, \lambda_m x), (a, \lambda_m x + 1), (a, \lambda_m x + \lambda_m + 1), \ldots, (a, \lambda_m x + (m-1)\lambda_m + 1)\}, & p_0, \\ \{(a, \lambda_m x), (b, \lambda_m x + \lambda_m), (a, \lambda_m x + \lambda_m + 1), \ldots, (a, \lambda_m x + (m-1)\lambda_m + 1)\}, & p_1, \\ \qquad\vdots & \\ \{(a, \lambda_m x), (a, \lambda_m x + \lambda_m), (a, \lambda_m x + 2\lambda_m), \ldots, (b, \lambda_m x + m\lambda_m)\}, & p_m. \end{cases}$$

Let Λ_α denote the set of all type-α positions of all possible realisations. Then, the sets $\overline{\Lambda_\alpha^\star}$ coincide with W_m^α from Proposition 2.17. Consequently, Λ_α is a subset of the regular model set $\curlywedge(W_m^{(\alpha)})$. To find out to what extent these windows are determined by a single realisation, we apply the so-called 'chaos game' [Pal92, Sec. 4.2] and Elton's ergodic theorem [Elt87]; see also [Bar89, Thm. 10].

Consider the single-point iteration in internal space, as defined by $p(0) = (a, 0)$ together with $p(n+1) = \Theta(p(n))$ for $n \geq 0$, where Θ is a random mapping in internal space, defined by $(b, y) \mapsto (a, \lambda'_m y)$ and

$$(a,y) \mapsto \begin{cases} (b, \lambda'_m y), & \text{with probability } q_0, \\ (b, \lambda'_m y + \lambda'_m), & \text{with probability } q_1, \\ \qquad \vdots & \qquad \vdots \\ (b, \lambda'_m y + m\lambda'_m), & \text{with probability } q_m, \\ (a, \lambda'_m y), & \text{with probability } q_{m+1}, \\ (a, \lambda'_m y + \lambda'_m), & \text{with probability } q_{m+2}, \\ \qquad \vdots & \qquad \vdots \\ (a, \lambda'_m y + m\lambda'_m), & \text{with probability } q_{2m-1}, \\ (a, \lambda'_m y + 1), & \text{with probability } q_{2m}, \\ (a, \lambda'_m y + \lambda'_m + 1), & \text{with probability } q_{2m+1}, \\ \qquad \vdots & \qquad \vdots \\ (a, \lambda'_m y + (m-1)\lambda'_m), & \text{with probability } q_{3m}, \end{cases}$$

where $q_i > 0$ and $\sum_{i=0}^{3m} q_i = 1$. Now, Elton's theorem (more precisely [Elt87, Cor. 2]) asserts that, almost surely, the corresponding (infinite) random point sequences lie dense in the attractor of the iterated function system, as long as $\boldsymbol{p}_m \gg 0$. This observation immediately establishes the following result.

Proposition 6.8. *For almost every realisation of the one-sided random noble means inflation tiling, the lift of the positions of type α via the $\star$-map lies dense in W_m^{α}. In particular, the lift of all left endpoints together is a dense subset of W_m.* □

The corresponding result applies to one-sided tilings that extend to the left, for instance when starting from $(b, -1)$ or from $(a, -\lambda_m)$. Since the intersection of two sets of full measure is again a set of full measure, one obtains the following consequence.

Corollary 6.9. *Almost every realisation of the two-sided random noble means inflation tiling that emerges from one of the legal seeds $a|a$, $a|b$, $b|a$ or $b|b$ completely determines the window of the covering model set, as in Proposition* 6.8. □

We will call an element of the stochastic hull whose realisation has a dense lift in the window (as described in Corollary 6.9) *generic*. Obviously, not all elements are generic, since the corresponding windows in deterministic cases are strictly smaller than W_m, compare Proposition 2.3 and Corollary 2.4. Let Y_0 be the set of all generic elements.

Proposition 6.10. *The topological point spectrum of* $(\mathbb{Y}_m, \mathbb{R}, \nu_m)$ *is trivial.*

Proof. Let f be a continuous eigenfunction, i.e.

$$f(t+y) = \mathrm{e}^{2\pi \mathrm{i} k t} f(y) \tag{6.1}$$

for all $t \in \mathbb{R}$ and ν_m-almost all $y \in \mathbb{Y}_m$. The ergodicity of ν_m implies that $|f|$ is constant. Fix $y \in Y_0$, and set $c = f(y)$. By construction, we have $y = y_{\mathrm{L}}|y_{\mathrm{R}}$, where y_{L} and y_{R} are two level-infinity (half-)tilings glued together at the origin. Let $y' \in Y_0 \setminus \{y\}$. This element is again of the form $y'_{\mathrm{L}}|y'_{\mathrm{R}}$. Then, we can construct a new element $y'' = y_{\mathrm{L}}|y'_{\mathrm{R}}$, which is obviously an element of Y_0, and we obtain

$$|f(y) - f(y'')| = \lim_{t\to\infty} |\,\mathrm{e}^{2\pi \mathrm{i} k t}(f(y) - f(y''))| = \lim_{t\to\infty} |f(t+y) - f(t+y'')| = 0,$$

since f is a continuous eigenfunction. Hence, we have $f(y) = f(y'')$. Analogously, one shows that $f(y') = f(y'')$, which implies $f(y) = f(y')$. Consequently, f is constant on Y_0.

Now, let y_1 be the fixed point of $\zeta_{m,m}$ starting from the legal seed $a|a$ and y_2 the fixed point of starting from the legal seed $b|a$. It is clear that $y_1, y_2 \in Y_0$. Remember that

$$W_{m,m}^{(a|a)} =]-1, -\lambda'_m] \quad \text{and} \quad W_{m,m}^{(b|a)} = [-1, -\lambda'_m[$$

by Proposition 2.3. A comparison with the windows for Λ_a and Λ_b shows that $\lambda_m + y_1 \in Y_0$ and $1 + y_2 \in Y_0$, and we conclude

$$f(y_1) = f(\lambda_m + y_1) = \mathrm{e}^{2\pi \mathrm{i} k \lambda_m} f(y_1) \quad \text{and} \quad f(y_2) = f(1+y_2) = \mathrm{e}^{2\pi \mathrm{i} k} f(y_2).$$

This immediately implies $k\lambda_m \in \mathbb{Z}$ and $k \in \mathbb{Z}$, thus, $k = 0$. Therefore, the only topological eigenvalue is 1. Since $(\mathbb{Y}_m, \mathbb{R}, \nu_m)$ is ergodic, 1 is a simple eigenvalue. This proves that the constant function is the only continuous eigenfunction of U_S. □

6.2.2 The Kronecker factor of the random noble means chain

We have seen in the previous section that every continuous eigenfunction of U_S is constant. But there are many other eigenfunctions, which are merely measurable. The objective of this section is the study of the Kronecker factor of $(\mathbb{Y}_m, \mathbb{R}, \nu_m)$.

In order to do so, we will consider the CPS $(\mathbb{R}, H, \mathcal{L})$ from (2.3) with compact window $W_m = [-\lambda_m, \lambda_m]$. We use the symbol H instead of $\mathbb{R}$ to explicitly distinguish direct and internal space in the following arguments. By Example 6.7, we know that a regular model set Λ which is obtained by such a CPS defines a strictly ergodic dynamical system that is almost everywhere one-to-one over its maximal equicontinuous factor, which is given by $\mathbb{T} = (\mathbb{R} \times H)/\mathcal{L}$. On the other hand, there is the classic *torus parametrisation*, where we assume that the singular element Λ is the union of all elements in the fibre over $(0,0) \in \mathbb{T}$.

There is the following connection. The fibre Y_0 is linked to Λ itself and hence mapped to $(0,0)$. Since almost every element in the fibre uniquely determines the window of Λ by Corollary 6.9, we can unambiguously map these elements to $(0,0)$. To extend this to a mapping from ν_m-almost every element of $\mathbb{Y}_m$ to $\mathbb{T}$, we first select a generic element $y_0 \in Y_0$. Now, for any $y \in \mathbb{Y}_m$, there is a sequence $(t_n)_{n\in\mathbb{N}}$ of translations such that

$$y = \lim_{n\to\infty} (t_n + y_0). \tag{6.2}$$

Without loss of generality, we may assume that $y \in \mathbb{Y}_0 := \{u \in \mathbb{Y}_m \,|\, 0 \in u\}$. Then, we always have $y \subseteq \mathbb{Z}[\lambda_m]$, so that all t_n from Eq. (6.2) lie in $\mathbb{Z}[\lambda_m]$ as well, and the $\star$-map is well-defined. Also, the convergence then implies that we may choose t_n such that $y \cap [-n,n] = (t_n + y_0) \cap [-n,n]$ holds, because our point sets have finite local complexity.

Lemma 6.11. *If (r,s) is a cluster point of $(t_n,0)_{n\in\mathbb{N}}$ in $\mathbb{T}$, with translations t_n from Eq. (6.2), we have $-r + y \subseteq \curlywedge(-s + W_m)$.*

Proof. Let U and V be open, relatively compact neighbourhoods of $0 \in \mathbb{R}$ and $0 \in H$, respectively, and assume $V = -V$. Then, our assumption implies that there is a subsequence $(n_j)_{j\in\mathbb{N}}$ of integers such that

$$(t_{n_j}, 0) \in (r,s) + U \times V + \mathcal{L} \tag{6.3}$$

holds for all sufficiently large j, say $j > N$. For any such j, we have $n_j > j$ and thus

$$y \cap [-j,j] = (t_{n_j} + y_0) \cap [-j,j] \subseteq t_{n_j} + y_0 \subseteq t_{n_j} + \curlywedge(W_m).$$

By (6.3), we have $(t_{n_j}, 0) = (r,s) + (u,v) + (x,x^\star)$ for some $u \in U$, $v \in V$ and $(x,x^\star) \in \mathcal{L}$, hence $t_{n_j} = r + u + x$ and $s + v + x^\star = 0$. Consequently, we have

$$y \cap [-j,j] \subseteq r + u + x + \curlywedge(W_m),$$

where $x + \curlywedge(W_m) = \curlywedge(x^\star + W_m)$ because $(x,x^\star) \in \mathcal{L}$. This implies

$$y \cap [-j,j] \subseteq r + u + \curlywedge(-s - v + W_m) \subseteq r + U + \curlywedge(-s + W_m - V) = r + U + \curlywedge(-s + W_m + V),$$

which holds for all $j > N$ and thus implies

$$y = \bigcup_{j>N} (y \cap [-j,j]) \subseteq r + U + \curlywedge(-s + W_m + V).$$

Since this holds for any open neighbourhood U of 0, and since $\curlywedge(-s+W_m+V)$ is a Delone set due to the relative compactness of V, we get

$$\bigcap_{0\in U \text{ open}} U + (r + \curlywedge(-s + W_m + V)) = r + \curlywedge(-s + W_m + V)$$

so that $y \subseteq r + \curlywedge(-s + W_m + V)$ and hence also

$$y \subseteq \bigcap_{\substack{0 \in V = -V, \\ V \text{ open}}} r + \curlywedge(-s + W_m + V)$$

together with

$$\bigcap_{\substack{0 \in V = -V, \\ V \text{ open}}} r + \curlywedge(-s + W_m + V) \supseteq r + \curlywedge(-s + W_m).$$

Now, our claim follows if we show that the last inclusion actually is an equality.

To do so, we may assume $r = 0$ without loss of generality. Let $x \in L \backslash \curlywedge(-s + W_m)$, where $L = \pi(\mathcal{L})$ from the CPS 2.3, so $x^\star \notin -s + W_m$. Then, there is an open neighbourhood V of $0 \in H$ with $V = -V$ such that $(x^\star + V) \cap (-s + W_m) = \varnothing$, which implies that $x^\star \notin -s + W_m + V$ and thus $x \notin \curlywedge(-s + W_m + V)$. Consequently, $y \in r + \curlywedge(-s + W_m)$ as claimed. $\square$

For technical reasons, we also need the following lemma.

Lemma 6.12. *If (r_1, s_1) and (r_2, s_2) are two cluster points of the sequence $(t_n, 0)_{n \in \mathbb{N}}$ as in the previous lemma, and if the element $y \in \mathbb{Y}_0$ from 6.2 is generic, one has the identity $(r_1, s_1) + \mathcal{L} = (r_2, s_2) + \mathcal{L}$.*

Proof. By the previous lemma, we have $-r_i + y \subseteq \curlywedge(-s_i + W_m)$ for $i \in \{1, 2\}$, hence also the inclusions $(-r_i + y)^\star \subseteq -s_i + W_m$ and thus $s_i + (-r_i + y)^\star \subseteq W_m$, which means that the sets $-r_i + y$ are translates of elements from Y_0. When y is generic, the window is uniquely determined, which is to say that

$$\overline{s_i + (-r_i + y)^\star} = s_i + \overline{(-r_i + y)^\star} = W_m.$$

But this implies

$$\begin{aligned} -s_2 + W_m &= \overline{(-r_2 + y)^\star} = \overline{(-r_2 + r_1 - r_1 + y)^\star} \\ &= (r_1 - r_2)^\star + \overline{(-r_1 + y)^\star} = (r_1 - r_2)^\star - s_1 + W_m. \end{aligned}$$

Since $v + W_m = W_m$ is only possible for $v = 0$, we conclude that $s_1 - s_2 = (r_1 - r_2)^\star$, which means nothing but $(r_1 - r_2, s_1 - s_2) \in \mathcal{L}$ and our claim follows. $\square$

By an application of Lemma 6.11, Lemma 6.12 and Corollary 6.9, we obtain a continuous mapping $\psi : \widetilde{\mathbb{Y}}_m \to \mathbb{T}$, where $\widetilde{\mathbb{Y}}_m \subseteq \mathbb{Y}_m$ is a subset of full measure. Then, for each character $\chi : \mathbb{T} \to \mathbb{C}$, the mapping $\chi \circ \psi$ defines an eigenfunction of $(\mathbb{Y}_m, \mathbb{R}, \nu_m)$ that is continuous on $\widetilde{\mathbb{Y}}_m$. This complements the statement of Proposition 6.10. We can now formulate the main result of this section.

Theorem 6.13. *The Kronecker factor of the dynamical system* $(\mathbb{Y}_m, \mathbb{R}, \nu_m)$ *can be identified with the maximal equicontinuous factor of the dynamical system obtained from the covering model set. It is explicitly given by* $\mathbb{T} = (\mathbb{R} \times H)/\mathcal{L}$ *within the CPS* (2.3)*, with* $H = \mathbb{R}$*.*

Proof. The mapping $\psi : \widetilde{\mathbb{Y}}_m \to \mathbb{T}$ from above is the measure-theoretic factor map onto $\mathbb{T}$. The maximality of this factor is a consequence of Theorem 3.30, as the dual group of $\mathbb{T}$ precisely is the Fourier module of the pure point spectrum, which is tantamount to saying that the mappings $\chi \circ \psi$ on $\widetilde{\mathbb{Y}}_m$ account for all eigenfunctions of our system. $\square$

6.2.3 Interpretation via disintegration

Let us consider the regular model set $\Lambda = \curlywedge(W_m)$ and the associated dynamical system $\big(\mathbb{X}(\Lambda), \mathbb{R}\big)$ [BLM07, Sec. 2]. This is a uniquely ergodic system with pure point spectrum, and it is almost everywhere one-to-one over its maximal equicontinuous factor $\mathbb{T}$; see Example 6.7. This one also acts as the Kronecker factor for our system $(\mathbb{Y}_m, \mathbb{R}, \nu_m)$, where the map is only defined for ν_m-almost every element of $\mathbb{Y}_m$ by first identifying the unique covering model set and then projecting down to the maximal equicontinuous factor. We equip the compact Abelian group $\mathbb{T}$ with its Haar measure, which is the Lebesgue measure.

Now, over every $a \in \mathbb{T}$, we have a fibre $Y_a \subseteq \mathbb{Y}_m$ together with a probability measure μ_a on it. For $a = 0$, this is just our fibre Y_0 from above. These fibre measures are compatible with the (normalised) Haar measure on $\mathbb{T}$ as needed for a disintegration formula. Therefore, for any $f \in L^1(\mathbb{Y}_m, \nu_m)$, we then have

$$\mathbb{E}(f) = \int_{\mathbb{Y}_m} f(y) \, \mathrm{d}\nu_m(y) = \int_{\mathbb{T}} \int_{Y_a} f(y) \, \mathrm{d}\mu_a(y) \, \mathrm{d}a = \int_{\mathbb{T}} \mathbb{E}(f|Y_a) \, \mathrm{d}a, \tag{6.4}$$

in line with the general theory; see [Fur81, Ch. 5.4].

6.3 The Kronecker factor of the random period doubling chain

At the end, let us devote our attention to the Kronecker factor and the maximal equicontinuous factor of the random period doubling chain. The random period doubling substitution is a substitution of constant length, which means that we can identify the discrete and the tiling picture and work with the $\mathbb{Z}$-action of the shift. Moreover, the internal space is no longer $\mathbb{R}$, but the set of 2-adic integers. As a consequence, we cannot work with compact intervals as windows any more, which makes things more complicated.

Now, consider the dynamical system $(\mathbb{X}_\rho, \mathbb{Z}, \nu)$, where ν denotes the frequency measure, which is ergodic. The hull $\mathbb{X}_\rho$ contains the special fibre X_0, which is defined as before.

Proposition 6.14. *The topological point spectrum of* $(\mathbb{X}_\rho, \mathbb{Z}, \nu)$ *is trivial.*

Proof. Let f be a continuous eigenfunction, i.e.

$$f(n+x) = \mathrm{e}^{2\pi \mathrm{i} k n} f(x)$$

for some $k \in [0,1[$, the dual group of $\mathbb{Z}$, and all $n \in \mathbb{Z}$. As in the proof of Proposition 6.10, $|f|$ is constant on $\mathbb{X}_\rho$, while f is constant on X_0.

Next, observe that $\mathbb{X}_\rho$ contains a periodic element, namely the one obtained by periodic repetition of the three-letter word aab. What is more, it is contained in the fibre X_0 in three different ways, as is apparent from

$$\ldots \underline{ba}\,\underline{ab}\,\underline{aa}\,\underline{ba}\,\underline{ab} \,|\, \underline{aa}\,\underline{ba}\,\underline{ab}\,\underline{aa}\,\underline{ba} \ldots$$

$$\ldots \underline{ab}\,\underline{aa}\,\underline{ba}\,\underline{ab}\,\underline{aa} \,|\, \underline{ba}\,\underline{ab}\,\underline{aa}\,\underline{ba}\,\underline{ab} \ldots$$

$$\ldots \underline{aa}\,\underline{ba}\,\underline{ab}\,\underline{aa}\,\underline{ba} \,|\, \underline{ab}\,\underline{aa}\,\underline{ba}\,\underline{ab}\,\underline{aa} \ldots$$

Since f takes the same value on all three, which are translates of one another, we get

$$\mathrm{e}^{2\pi \mathrm{i} k} = 1 \quad \text{with } k \in [0,1[\,,$$

which implies $k = 0$. Thus f must be a constant eigenfunction as claimed. □

The diffraction analysis in Chapter 4 suggests that the measure-theoretic point spectrum of $(\mathbb{X}_\rho, \mathbb{Z}, \nu)$ is given by $[0,1[\cap \mathbb{Z}\big[\frac{1}{2}\big]$. As before, $\mathbb{X}_\rho$ contains an open set of full measure, $\mathbb{X}'_\rho$ say, with the property that all eigenfunctions are continuous on it. The discontinuity is thus once again caused by a null set in the hull.

Remark 6.15. The discrete dynamical system $(\mathbb{X}_\rho, \mathbb{Z}, \nu)$ can be embedded into a flow, written as $(\mathbb{X}, \mathbb{R}, \nu_{\mathbb{R}})$, via a suspension with a constant roof function. Here, $\nu_{\mathbb{R}}$ denotes the standard extension of ν to an invariant probability measure on $\mathbb{X}$. This system is also ergodic [EW11, Lem. 9.24], and the topological point spectrum becomes $\mathbb{Z}$, while the measure-theoretic point spectrum is $\mathbb{Z}\big[\frac{1}{2}\big]$. The additional continuous eigenfunctions in comparison to Proposition 6.14 trivially emerge from the suspension. In terms of the approach via Fourier–Bohr coefficients, this can be seen by adding a complex weight of the form $\mathrm{e}^{2\pi \mathrm{i} n k}$ with a fixed $n \in \mathbb{Z}$, which results in a phase change for the continuous flow, but remains invisible for the discrete shift. ◇

From here, the remainder of the argument is similar to before. We get a covering two-component model set and its maximal equicontinuous factor as the Kronecker factor of $(\mathbb{X}, \mathbb{R}, \nu_{\mathbb{R}})$. Consequently, we also have the disintegration as in Eq. (6.4), which explains the nice formulas we were able to obtain in Chapter 4.

Summary and outlook

In this thesis, we investigated the local mixture of the family of noble means substitutions

$$\zeta_m : \begin{cases} a \mapsto \begin{cases} \zeta_{m,0}(a), & \text{with probability } p_0, \\ \quad\vdots & \qquad\qquad\vdots \\ \zeta_{m,m}(a), & \text{with probability } p_m, \end{cases} \\ b \mapsto a. \end{cases}$$

Each of these substitutions defines the same two-sided discrete hull. The corresponding geometric counterpart, the continuous hull, can be constructed either via the inflation rule associated to $\zeta_{m,i}$ or as a regular model set within the cut and project scheme (2.3). The latter implies that the system is pure point diffractive. These properties promised some technical simplifications.

Now, the crucial difference between the deterministic and the random situation is that the entropy of the deterministic system is zero, while we obtain positive entropy of the random system. Due to this fact, the random system cannot have pure point spectrum any more [BLR07]. This corresponds to the presence of a certain amount of disorder in systems with positive entropy.

The key observation for the exact formula of the diffraction measure of ζ_m was the decomposition of $\widehat{\gamma_{\Lambda_m}}$ into the first and second moments:

$$\widehat{\gamma_{\Lambda_m}} = \lim_{n\to\infty} \frac{1}{\lambda_m^n} \left|\mathbb{E}(\mathcal{X}_n)\right|^2 + \lim_{n\to\infty} \frac{1}{\lambda_m^n} \operatorname{Var}(\mathcal{X}_n),$$

see Eq. (3.2), which was already suggested by Godrèche and Luck in the special case $m = 1$ [GL89]. It turned out that this is the right decomposition into the pure point and continuous component not only for the family of random noble means substitutions (each of which is a primitive Pisot substitution of non-constant length with an irrational inflation multiplier), but also for the random period doubling (primitive Pisot substitution of constant length with integer inflation multiplier) and a higher-dimensional example.

The influence of the randomness is also reflected in the dynamical spectrum. While it is well-known that in the deterministic case the topological dynamical spectrum coincides with the diffraction spectrum, the topological point spectrum is trivial in the random case. The eigenfunctions are no longer continuous on the entire hull (which is true in the deterministic case), but they are continuous when restricted to a subset of full measure.

One possible extension is to consider the local mixture of substitutions, which can be described within the same cut and project scheme (and are therefore pure point diffractive), but no longer give rise to the same two-sided discrete hull. An example of such substitutions is given by

$$\eta : \begin{cases} a \mapsto \begin{cases} abb, & \text{with probabiliy } p_0, \\ bab, & \text{with probabiliy } p_1, \\ bba, & \text{with probabiliy } p_2, \end{cases} \\ b \mapsto a. \end{cases}$$

Here, the first and third substitution define the same aperiodic hull, whereas the second one is periodic. This substitution was studied in [Goh17]. It is shown that the decomposition of the diffraction measure into the first and second moments leads again to a splitting into the pure point and absolutely continuous part. However, the Radon–Nikodym density of the absolutely continuous component is no longer bounded and continuous, but merely locally integrable.

Other objects of interest are Lyapunov exponents. It is shown in [Man17] that one can rule out the existence of an absolutely continuous component in the diffraction spectrum of constant length substitutions on a binary alphabet (for example the period doubling substitution) by showing that the smallest Lyapunov exponent is strictly positive. As we have seen in this thesis, the diffraction spectrum of every random substitution under consideration contains a non-trivial absolutely continuous component. Therefore, one expects the smallest Lyapunov exponent to be zero, which would be interesting to verify.

Last but not least, another interesting field is the analysis of Schrödinger operators associated with a random substitution subshift $\mathbb{X}_\vartheta$. Schrödinger operators associated with deterministic subshifts are rather well understood [Dam07, DEG15], and the study of their spectral properties give much insight into the underlying physical processes. But many important physical phenomena are influenced by random processes, wherefore it is necessary to also consider systems with positive entropy such as random substitution subshifts.

Bibliography

[AL90] J. Gil. de Lamadrid and L. N. Argabright: *Almost periodic measures*, Memoirs of the Amer. Math. Soc., Vol 85, No. 428 (1990).

[BBM10] M. Baake, M. Birkner and R.V. Moody: Diffraction of stochastic point sets: Explicitly computable examples, *Commun. Math. Phys.* **293** (2010), 611–660; `arXiv:0803.1266`

[BGG12] M. Baake, F. Gähler and U. Grimm: Spectral and topological properties of a family of generalised Thue-Morse sequences, *J. Math. Phys.* **53** (2012), 032701 (24pp); `arXiv:1201.1423`.

[BG13] M. Baake and U. Grimm: *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge University Press, Cambridge (2013).

[BG17] M. Baake and U. Grimm (eds.): *Aperiodic Order. Vol. 2: Crystallography and Almost Periodicity*, Cambridge University Press, Cambridge (2017).

[BKM15] M. Baake, H. Kösters and R. V. Moody: Diffraction Theory of Point Processes: Systems with Clumping and Repulsion, *J. Stat. Phys.* **159** (2015), 915–936; `arXiv:1415.4255`.

[BL05] M. Baake and D. Lenz: Deformation of Delone dynamical systems and pure point diffraction, *J. Fourier Anal. Appl.* **11** (2005), 125–150; `arXiv:math.DS/0404155`.

[BLM07] M. Baake, D. Lenz and R.V. Moody: Characterization of model sets by dynamical systems, *Ergodic Th. & Dynam. Syst.* **27** (2007) 341–382; `arXiv:math/0511648`.

[BLR07] M. Baake, D. Lenz and C. Richard: Pure point diffraction implies zero entropy for Delone sets with uniform cluster frequencies, *Lett. Math. Phys.* **82** (2007), 61–77; `arXiv:0706.1677`.

[BLvE13] M. Baake, D. Lenz and A. van Enter: Dynamical versus diffractionspectrum for structures with finite local complexity, *Ergod. Th. & Dynam. Syst.* **35** (2015), 2017–2043; `arXiv:math.DS/1307.7518v2`.

[BM04] M. Baake and R.V. Moody: Weighted Dirac combs with pure point diffraction, *J. reine angew. Math. (Crelle)* **573** (2004), 61–94; `arXiv:math.MG/0203030`.

[BM00] M. Baake and R.V. Moody (eds.): *Directions in Mathematical Quasicrystals*, CRM Monograph Series, vol. 13, AMS, Providence, RI (2000).

[BM00a] M. Baake and R.V. Moody: Self-similar measures for quasicrystals, in [BM00] (2000), 1–42; `arXiv:math.MG/0008063`.

[BSS17] M. Baake, T. Spindeler and N. Strungaru: Diffraction of compatible random substitutions in one dimension, submitted.

[Bar89] M.F. Barnsley: Lecture notes on iterated function systems, in *Chaos and Fractals*, eds. R.L. Devaney and L. Keen, Proc. Symp. Appl. Math. vol. 39, AMS, Providence, RI (1989), 127–144.

[BF75] C. Berg and G. Forst: *Potential Theory on Locally Compact Abelian Groups*, Springer, Berlin (1975).

[BD00] G. Bernuau and M. Duneau: Fourier analysis of deformed model sets, in [BM00] (2000), 43–60.

[Bil12] P. Billingsley: *Probability and Measure*, annivers. ed. Wiley, New York (2012).

[BSYL11] L. Bindi, P.J. Steinhardt, N. Yao and P.J. Lu: Icosahedrite, $Al_{63}Cu_{24}Fe_{13}$, the first natural quasicrystal, *Am. Min.* **96** (2011), 928–931.

[BT86] E. Bombieri and J.E. Taylor: Which distributions of matter diffract? An initial investigation, *J. Phys. Colloque* **47** (C3) (1986), 19–28.

[Bru81] N. G. de Bruijn: Algebraic theory of Penroses nonperiodic tilings of the plane. I, II, *Nederl. Akad. Wetensch. Indag. Math.* **43** (1981), 39–52, 53–66.

[CS03] A. Clark and L. Sadun: When size matters: Subshifts and their related tiling spaces, *Ergodic Th. & Dynam. Syst.* **23** (2003) 1043–1057; `arXiv:math.DS/0201152`.

[Cox61] H.S.M. Coxeter: *Introduction to Geometry*, Wiley, New York (1961)

[Dam07] D. Damanik: Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, *Proc. Sympos. Pure Math.* **76**, Part 2, Amer. Math. Soc., Providence, RI, (2007), 505–538; `arXiv:math/0509197`.

[DEG15] D. Damanik, M. Embree and A. Gorodetski: Spectral Properties of Schrdinger Operators Arising in the Study of Quasicrystals, *Prog. Math. Phys.* **309** (2015), 307–370; `arXiv:1210.5753`.

[Die70] J. Dieudonné: *Treatise on Analysis,* vol. II, Academic Press, New York (1970).

[EW11] M. Einsiedler and T. Ward: *Ergodic Theory: With a View Towards Number Theory*, Springer, London (2011).

[Elt87] J.H. Elton: An ergodic theorem for iterated maps, *Ergodic Th. & Dynam. Syst.* **7** (1987), 481–488.

[Fog02] N.P. Fogg: *Substitutions in Dynamics, Arithmetics and Combinatorics*, Springer, Berlin (2002).

[Fur81] H. Furstenberg: *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton, NJ (1981).

[GL89] C. Godrèche and J.M. Luck: Quasyperiodicity and randomness in tilings of the plane, *J. Stat. Phys.* **55** (1989), 1–28.

[Goh17] P. Gohlke: *On a Family of Semi-Compatible Random Substitutions*, Masters Thesis, Bielefeld University (2017).

[HK06] B. Hasselblatt and A. Katok (eds.): *Handbook of Dynamical Systems*, Volume 1, Part B, (2006).

[Hec14] E. Hecht: *Optics*, 4. ed., Harlow/Essex, Pearson (2014).

[Hof95] A. Hof: On diffraction by aperiodic structures, *Commun. Math. Phys.* **169** (1995), 25–43.

[Höf01] M. Höffe: *Diffraktionstheorie stochastischer Parkettierungen*, PhD thesis, Univ. Tübingen (2001).

[Hut81] J.E. Hutchinson: Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–743.

[INF85] T. Ishimasa, H.-U. Nissen and Y. Fukano, New ordered state between crystalline and amorphous in Ni-Cr particles, *Phys. Rev. Lett.* **55** (1985), 511–513.

[JW35] B. Jessen and A. Wintner: Distribution functions and the Riemann Zeta function, *Trans. Amer. Math. Soc.* **38** (1935), 48–88.

[KN84] P. Kramer and R. Neri: On periodic and non-periodic space fillings of $\mathbb{E}^m$ obtained by projection, *Acta Cryst. A* **40** (1984), 580–587.

[Lag96] J. Lagarias: Meyer's concept of quasicrystal and quasiregular sets, *Commun. Math. Phys.* **179** (1996), 365–376.

[Lan93] S. Lang: *Real and Functional Analysis*, 3rd ed. Springer, New York (1993).

[Len09] D. Lenz: Continuity of eigenfunctions of uniquely ergodic dynamical systems and intensity of Bragg peaks, *Commun. Math. Phys.* **287** (2009), 225–258; `arXiv:math-ph/0608026`.

[LM95] D. Lind and B. Marcus: *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge (1995).

[Lot02] M. Lothaire: *Algebraic Combinatorics on Words*, Cambridge University Press, Cambridge (2002).

[Man17] N. Manibo: Lyapunov exponents for binary substitutions of constant length, *J. Math. Phys.* **58** 113504, 2017; `arXiv:1706.00451v2`.

[Mey72] Y. Meyer: *Algebraic Numbers and Harmonic Analysis*, North Holland, Amsterdam (1972).

[Mey00] C.D. Meyer: *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia (2000).

[Mo13] M. Moll: *On a Family of Random Noble Means Substitutions*, PhD thesis, Univ. Bielefeld (2013).

[Moo97] R.V. Moody (ed.): *The Mathematics of Long-Range Aperiodic Order*, NATO ASI Series C 489, Kluwer, Dordrecht (1997).

[Moo97a] R.V. Moody: Meyer sets and their duals, in [Moo97] (1997), 403–449.

[Pal92] K.J. Palmer: Bifurcations, chaos and fractals, in *Nonlinear Dynamics and Chaos*, eds. R.L. Dewar and B.I. Henry, World Scientific, Singapore (1992), 91–133.

[Que10] M. Queffélec: *Substitution Dynamical Systems - Spectral Analysis*, 2nd. ed. LNM 1294, Springer, Berlin (2010).

[RS80] M. Reed and B. Simon: *Functional Analysis*, Rev. ed., Elsevier Science, San Diego (1980).

[Rud62] W. Rudin: *Fourier Analysis on Groups*, Wiley, New York (1962).

[Sen06] E. Seneta: *Non-negative Matrices and Markov Chains*, rev. 2nd ed., Springer, New York (2006).

[SBGC84] D. Shechtman, I. Blech, D. Gratias and J.W. Cahn: Metallic phase with long-range orientational order and no translational symmetry, *Phys. Rev. Lett* **53** (1984), 1951–1954.

[Sin06] B. Sing: *Pisot Substitutions and Beyond*, PhD thesis, Univ. Bielefeld (2006).

[Spi17] T. Spindeler: Diffraction intensities of a class of binary Pisot substitutions via exponential sums, *Monatsh. Math.* **182** (2017), 143–153; `arXiv:1608.01969`.

[Str05] N. Strungaru: Almost periodic measures and long-range order in Meyer sets, *Discr. Comput. Geom.* **33** (2005), 483–505.

[Str14] N. Strungaru: On weighted Dirac combs supported inside model sets, *J. Phys. A: Math. Theor.* **47** (2014); `arXiv:1309.7947`.

[Wal00] P. Walters: *An Introduction to Ergodic Theory*, reprint, Springer, New York (2000).

[Wic91] K.R. Wicks: *Fractals and Hyperspaces*, LNM 1492, Springer, Berlin (1991).

[Wol03] T. Wolff: *Lectures on Harmonic Analysis*, University Lecture Series, vol. 29, AMS, Providence, RI (2003).

List of Symbols

$\lhd$	subword relation
$\blacktriangleleft$	randomised subword relation
$\doteq$	randomised word equality relation
1_B	characteristic function of the set B
$\mathcal{A}_n$	alphabet with n letters
$\mathcal{A}_n^\ell$	words of length ℓ over $\mathcal{A}_n$
$\mathcal{A}_n^*$	all finite words over $\mathcal{A}_n$
$\mathfrak{B}_m$	Borel-σ-algebra generated by $\mathfrak{Z}(\mathbb{X}_m)$
$C(X)$	vector space of continuous functions $X \to \mathbb{C}$
$C_B(X)$	bounded $C(X)$-functions
$C_U(X)$	uniformly continuous $C_B(X)$-functions
$C_0(X)$	$C_U(X)$-functions vanishing at infinity
$C_c(X)$	$C_0(X)$-functions with compact support
$\mathcal{D}_\rho$	set of ρ-legal words
$\mathcal{D}_m$	set of ζ_m-legal words
$\mathcal{D}_{m,\ell}$	set of ζ_m-legal words of length ℓ
$\mathcal{D}'_m$	set of $\zeta_{m,i}$-legal words
$\mathcal{D}'_{m,\ell}$	set of $\zeta_{m,i}$-legal words of length ℓ
$\mathfrak{F}_n$	free group, generated by the letters of $\mathcal{A}_n$
$\mathcal{G}_{m,k}$	set of ζ_m-exact substitution words of order k

λ_m	PF eigenvalue of M_m
λ'_m	algebraic conjugate of λ_m
$\Lambda_{m,i}$	noble means set
Λ_m	generating random noble means set
$\mathcal{L}_m$	diagonal embedding of $\mathbb{Z}[\lambda_m]$
$L^{\circledast}_m$	Fourier module $\pi_1(\mathcal{L}^*_m)$
$\mathcal{M}(X)$	space of measures on X
$\mathrm{Mat}(d, R)$	square $(d \times d)$-matrices over R
M_m	substitution matrix of $\zeta_{m,i}$ and ζ_m
M_{pd}	substitution matrix of ρ_{pd}
μ_m	patch frequency measure on $\mathbb{X}_m$
$\mathcal{N}_m$	family of noble means substitutions
ν_m	patch frequency measure on $\mathbb{Y}_m$
$\mathcal{P}(X)$	space of probability measures on X
$\mathcal{P}_T(X)$	space of T-invariant probability measures on X
$\mathbf{p}_m$	probability vector for ζ_m
$\mathcal{R}$	family of random noble means substitutions
ρ	random period doubling substitution
ρ_{pd}	period doubling substitution
ρ'_{pd}	variant of the period doubling substitution
$\mathcal{S}(\mathbb{R}^d)$	Schwartz space or space of tempered distributions
S	shift map
U_T	Koopman operator of the homeomorphism T
$\mathcal{W}(S)$	factor set of S

$\mathcal{W}_\ell(S)$	ℓth factor set of S
$\mathbb{X}_\vartheta$	two-sided discrete hull defined by ϑ
$\mathbb{X}_\rho$	two-sided discrete stochastic hull defined by ρ
$\mathbb{X}_m$	two-sided discrete stochastic hull defined by ζ_m
$\mathbb{X}_{m,i}, \mathbb{X}'_m$	two-sided discrete hull defined by $\zeta_{m,i}$
$\mathbb{X}_{\mathrm{pd}}$	two-sided discrete hull defined by ρ_{pd}
$\mathbb{Y}_m$	continuous stochastic hull of ζ_m
$\mathbb{Y}'_m$	continuous hull of $\zeta_{m,i}$
$\mathbb{Y}^{\circledcirc}_m$	punctured continuous stochastic hull of ζ_m
$\mathbb{Y}_{m,i}$	continuous hull of $\zeta_{m,i}$
ζ_m	random noble means substitution
$\zeta_{m,i}$	noble means substitution
$\mathcal{Z}_k(v)$	cylinder set of v at index k
$\mathfrak{Z}(\mathbb{X})$	cylinder sets for the product topology of the subshift $\mathbb{X} \subseteq \mathcal{A}_n^{\mathbb{Z}}$

Index