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SOLVABILITY OF NONLOCAL SYSTEMS RELATED TO PERIDYNAMICS

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Abstract. In this work, we study the Dirichlet problem associated with a strongly coupled system of nonlocal equations. The system of equations comes from a linearization of a model of peridynamics, a nonlocal model of elasticity. It is a nonlocal analogue of the Navier-Lamé system of classical elasticity. The leading operator is an integro-differential operator characterized by a distinctive matrix kernel which is used to couple differences of components of a vector field. The paper's main contributions are proving well-posedness of the system of equations and demonstrating optimal local Sobolev regularity of solutions. We apply Hilbert space techniques for well-posedness. The result holds for systems associated with kernels that give rise to non-symmetric bilinear forms. The regularity result holds for systems with symmetric kernels that may be supported only on a cone. For some specific kernels associated energy spaces are shown to coincide with standard fractional Sobolev spaces.

1. Introduction.

We study the Dirichlet problem associated with a nonlocal system of equations

$$\mathbb{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega; \quad \mathbf{u} = \mathbf{0} \quad \text{in } \complement\Omega. \tag{1}$$

where the matrix-valued nonlocal operator $\mathbb{L}$ is of the form

$$\mathbb{L}\mathbf{u}(\mathbf{x}) = \lim_{\varepsilon\to 0^+} \int\limits_{|\mathbf{x}-\mathbf{y}|>\varepsilon} k(\mathbf{x},\mathbf{y}) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\, \mathrm{d}\mathbf{y}, \tag{2}$$

when the limit exists. In the above, $\Omega \subset \mathbb{R}^d$ denotes an open, bounded set with a sufficiently regular boundary, and $\complement\Omega$ denotes its complement. The functions $\mathbf{u}$ and $\mathbf{f}$ are vector fields defined in their respective domain. The kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ is measurable. For given vectors $\mathbf{a} = (a_1, a_2, \cdots, a_d)$, and $\mathbf{b} = (b_1, b_2, \cdots, b_d)$, the tensor $\mathbf{a} \otimes \mathbf{b}$ is the rank-one matrix with $a_i b_j$ as its ij^{th} entry. From the very

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definition of the nonlocal operator $\mathbb{L}$, it is clear that (1) is a strongly coupled system of equations.

The goal of this paper is twofold. First, we formulate a variational problem for (1), the resolution of which provides solutions to (1). We treat more general kernels than those covered in the literature. For given data $\mathbf{f}$ in an appropriate class, we describe a notion of solution and demonstrate existence of vector-valued solutions $\mathbf{u}:\mathbb{R}^d\to\mathbb{R}^d$ to the nonlocal coupled system (1). The second goal is to prove some results related to the optimal regularity of solutions. This will be carried out for a specific class of kernels.

The motivation to study the above system of equations comes from applications. Indeed, the system (1) is the equilibrium equation in linearized bond-based peridynamics, a nonlocal continuum model that has received a lot of attention in recent years [34, 35, 36]. To describe the model, a body occupying $\Omega\subset\mathbb{R}^d$ has undergone the deformation that maps a material point $\mathbf{x}\in\Omega$ to $\mathbf{x}+\mathbf{u}(\mathbf{x})$ in a deformed domain. In this case, the vector field $\mathbf{u}$ represents the displacement field. The peridynamic model treats the body as a complex mass-spring system. Any two material points $\mathbf{y}$ and $\mathbf{x}$ are assumed to be interacting through a bond vector $\boldsymbol{\xi}=\mathbf{y}-\mathbf{x}$. Under the uniform small strain theory [35], the strain of the bond $\mathbf{y}-\mathbf{x}$ is given by the nonlocal linearized strain

$$\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})=(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}.$$

A portion of this strain contributes to the volume changing component of the deformation and the remaining is the shape changing component. According to the linearized bond-based peridynamic model [35] the balance of forces is given by a system of equations that has the same form as (1) for some appropriate kernel k. The kernel k contains properties of the modeled material and represents the strength and extent of interactions between material points $\mathbf{x}$ and $\mathbf{y}$. The kernel k may depend on $\mathbf{x},\mathbf{y}$, their relative position $\mathbf{y}-\mathbf{x}$ or, in the case of homogeneous materials, only on their relative distance $|\mathbf{y}-\mathbf{x}|$. For general k, the equation may model heterogeneous and anisotropic materials. The operator $\mathbb{L}\mathbf{u}$ is then the linearized internal force density function due to the deformation $\mathbf{x}\mapsto\mathbf{x}+\mathbf{u}(\mathbf{x})$ and is a weighted average of the linearized strain function associated with the displacement $\mathbf{u}$ [26, 35]. Indeed, rewriting (2) in terms of the nonlocal strain $\mathcal{D}(\mathbf{u})$ we get $\mathbb{L}\mathbf{u}(\mathbf{x})=\lim\limits_{\varepsilon\to0^+}\int\limits_{|\mathbf{x}-\mathbf{y}|>\varepsilon}k(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}$, whenever it exists.

The usage of the "projected" difference of $\mathbf{u}$, $\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})$, in $\mathbb{L}$ makes the operator distinct from other nonlocal operators that use the full difference $\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})$. To see this distinction, it suffices to note that for smooth vector fields

$$\frac{\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}=\frac{(\mathbf{y}-\mathbf{x})^{\intercal}\big(\varepsilon(\mathbf{u})(\mathbf{x})\big)(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^2}+o(|\mathbf{y}-\mathbf{x}|)$$

whereas $\frac{\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}=\nabla\mathbf{u}(\mathbf{x})\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}+o(|\mathbf{y}-\mathbf{x}|)$, where we have used the notation $\varepsilon(\mathbf{u})(\mathbf{x})$ to represent the symmetric part of the gradient matrix $\frac{1}{2}(\nabla\mathbf{u}(\mathbf{x})+\nabla\mathbf{u}(\mathbf{x})^{\intercal})$, commonly called the strain tensor. The action $\{\}^{\intercal}$ denotes the transpose. A consequence of this is that the nonlocal system (1) can be seen as a nonlocal analogue of the Dirichlet problem corresponding to the strongly coupled system of partial differential equations

$$\mathbf{div}\,\mathfrak{C}(\mathbf{x})\varepsilon(\mathbf{u})(\mathbf{x})=\mathbf{f}\quad\text{in }\Omega;\qquad\mathbf{u}=\mathbf{0}\quad\text{in }\partial\Omega\,,$$

where $\mathfrak{C}(\mathbf{x})$ is a fourth–order tensor of bounded coefficients, which is not necessarily uniformly elliptic but rather satisfies the weaker Legendre-Hadamard condition. Systems of partial differential equations of the above type that are commonly used in the theory of linearized elasticity are well studied, see [17].

Our study of the nonlocal system (1) begins with a mathematically rigorous understanding of the operator $\mathbb{L}$. The focus is to find a large class of kernels k that may not be symmetric ($k(\mathbf{x},\mathbf{y}) \neq k(\mathbf{y},\mathbf{x})$), may have singularity along the diagonal $\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d : \mathbf{x} = \mathbf{y}\}$ or degeneracy on some directions such that both the operator $\mathbb{L}$ and associated system of equations (1) make sense. Notice that even for smooth functions the limit in (2) does not exist in general unless we put a condition on k. As with partial differential equations in divergence form with measurable coefficients, we study variational solutions based on quadratic forms. We use use Hilbert space techniques to study the Dirichlet problem (1). Applicability of harmonic analysis tools is also possible when the system of equations is posed over the entire domain $\mathbb{R}^d$.

To describe some of our results, following [14, 32] let us introduce a decomposition of $k(\mathbf{x},\mathbf{y})$ in terms of its symmetric part k_s and its anti-symmetric part k_a. They are given by

$$k_s(\mathbf{x},\mathbf{y}) = \frac{1}{2}(k(\mathbf{x},\mathbf{y}) + k(\mathbf{y},\mathbf{x})), \quad k_a(\mathbf{x},\mathbf{y}) = \frac{1}{2}(k(\mathbf{x},\mathbf{y}) - k(\mathbf{y},\mathbf{x})).$$

Throughout the paper we consider kernels whose symmetric part has locally integral second moment, i.e., we assume

$$\mathbf{x} \mapsto \int\limits_{\mathbb{R}^d} \min\{1, |\mathbf{x}-\mathbf{y}|^2\} k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} \in L^1_{loc}(\mathbb{R}^d). \tag{3}$$

We also define the function space of vector fields

$$S(\mathbb{R}^d;k) = \left\{ \mathbf{v} \in L^2(\mathbb{R}^d;\mathbb{R}^d) : \mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y}) k_s^{1/2}(\mathbf{x},\mathbf{y}) \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\}.$$

The mapping $[\mathbf{u},\mathbf{v}]_{H(\mathbb{R}^d;k)} := \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$ defines a bilinear form on $S(\mathbb{R}^d;k)$. One can easily show that the function $\|\cdot\|_{S(\mathbb{R}^d;k)}$, defined via the relation

$$\|\mathbf{v}\|^2_{S(\mathbb{R}^d;k)} = \|\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) (\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y}))^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x},$$

serves as a norm for $S(\mathbb{R}^d;k)$. Moreover, adapting the argument used in the proof of [14, Lemma 2.3], we can actually show that $S(\mathbb{R}^d;k)$ is a separable Hilbert space with inner product $(\cdot,\cdot)_{L^2} + [\cdot,\cdot]_{S(\mathbb{R}^d;k)}$. See also similar results in [12, 26, 25]. We denote the dual space of $S(\mathbb{R}^d;k)$ by $S^*(\mathbb{R}^d;k)$.

Roughly speaking, we show the following results: for those kernels k whose antisymmetric part is small relative to the symmetric part (e.g. the function $\mathbf{x} \mapsto \int_{\mathbb{R}^d} \frac{(k_a(\mathbf{x},\mathbf{y}))^2}{k_s(\mathbf{x},\mathbf{y})}\,\mathrm{d}\mathbf{y}$ is uniformly bounded for any $\mathbf{u} \in S(\mathbb{R}^d;k)$), the limit in (2) exists in the weak-* topology of the dual space $S^*(\mathbb{R}^d;k)$, and therefore $\mathbb{L}\mathbf{u} \in S^*(\mathbb{R}^d;k)$. This interpretation of the operator allows us to define a generalized or weak notion of solution to the system of equations in (1). The well-posedness of the problem is demonstrated via the application of the Lax-Milgram theorem. To this end, we introduce a bilinear form on the space $S(\mathbb{R}^d;k) \times S(\mathbb{R}^d;k)$ that is compatible with the system (1), and by imposing additional conditions on k we show that this form is continuous and coercive on appropriate subspaces. Systems of the type (1) have been studied extensively in the literature, cf. in [19, 11, 10, 13, 25]. Our results complement the well-posedness result in the above cited papers. Indeed, our work deals with kernels that give rise to non-symmetric bilinear forms while earlier works are based on kernels associated to symmetric bilinear forms. As we will see in the next section clearly, the non-symmetric bilinear forms we study account for the the presence of lower order terms that may involve "lower order fractional" derivatives, while the results in [25] deal with linear problems with lower order terms that involve the unknown function without any derivatives.

Let us comment on the case where the vector fields are scalar. In this case, the quadratic form under consideration becomes a regular Dirichlet form in the sense of [15]. For this reason there is an associated strong Markov jump process, which can be used to study the Dirichlet problem. In the particular case of translation invariant operators, i.e., when $k(\mathbf{x}, \mathbf{y})$ depends only on $(\mathbf{x} - \mathbf{y})$, the process has stationary independent increments and is called a Lévy process. The potential theory of Markov jump processes including fine properties of heat kernels has been developed in great detail in recent years. It can be shown that our notion of a variational solution coincides with the probabilistic notion of harmonicity [4, 23] if the source term $\mathbf{f}$ vanishes. For the theory of nonlocal non-symmetric Dirichlet forms we refer to [20, 16, 32]. In the case of scalar fields, the variational approach to the Dirichlet problem has been used by several authors [33, 14, 29]. Note that we only comment on nonlocal operators in bounded domains which are related to quadratic forms. For a survey of results on nonlocal Dirichlet problem in the non-variational context, see [28].

Our study of the nonlocal system (1) for general kernels follows the variational approach taken in [14] adapting it to the system of equations. This adaptation is not trivial because of the structure of the operator. For instance, one can easily check that the seminorm $[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k)}$ vanishes over a class of affine maps of the type $\mathbf{u}(\mathbf{x}) = \mathbb{B}\mathbf{x} + \mathbf{c}$, where $\mathbb{B}$ is a skew-symmetric matrix. When proving coercivity of the form over a subspace, one has to find a mechanism to remove this large class of maps, as opposed to constants in the case of equations. We will see that we need to use fractional Poincaré-Korn-type inequalities for the system in contrast to the standard fractional Poincaré inequality for problems involving scalar fields.

Let us mention that the system arising in (1) is related to the Euler-Lagrange system generated by fractional harmonic maps. Those systems were studied first in [7] for the half-Laplacian and then extended to more general situations [30, 6, 31, 27, 8]. In these works, the systems arise as Euler-Lagrange equations for critical points of functionals like $\|(-\Delta)^{\frac{s}{2}}\mathbf{u}\|_{L^p}$ for $\mathbf{u} \in \dot{H}^{s,p}(\mathbb{R}^d; \mathcal{M})$ where $M \subset \mathbb{R}^N$ is a smooth closed manifold. Obviously, these systems are nonlinear in general, which makes the regularity theory very challenging. However, the systems generated by harmonic maps do not possess a strong coupling in the main part of the operator as in (1).

In this paper, we also obtain local regularity results for variational solutions of the system (1) corresponding to a special class of kernels. For this aspect of our study, we concentrate on translation invariant operators with kernels of the form $k(\mathbf{x}, \mathbf{y}) = k(\mathbf{x} - \mathbf{y})$ that are even and comparable with the standard kernel of fractional order. We allow this comparability to hold true in any double cone Λ

with apex at the origin, i.e.

$$k(\mathbf{x}-\mathbf{y}) \asymp \frac{1}{|\mathbf{y}-\mathbf{x}|^{d+2s}}, \quad s\in(0,1), \quad \mathbf{x}-\mathbf{y}\in\Lambda.$$

For these types of kernels we show that the Hilbert space $S(\mathbb{R}^d;k)$ is equivalent to the standard fractional Sobolev space

$$H^s(\mathbb{R}^d;\mathbb{R}^d) := \left\{ \mathbf{u}\in L^2(\mathbb{R}^d;\mathbb{R}^d) \,:\, |\mathbf{u}|^2_{H^s} := \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} < \infty \right\}.$$

Such an equivalence will be proved using the Fourier transform. See [24, 12] for related results. For such kernels we show that actually the operator $\mathbb{L}: H^{2s}(\mathbb{R}^d;\mathbb{R}^d)\to L^2(\mathbb{R}^d;\mathbb{R}^d)$ is continuous. More generally, for any $p\in(1,\infty)$, if we define the non-homogeneous potential space

$$\mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d) = \{\mathbf{u}\in L^p(\mathbb{R}^d;\mathbb{R}^d) : (-\Delta)^s\mathbf{u}\in L^p(\mathbb{R}^d;\mathbb{R}^d)\}$$

where the fractional Laplacian $(-\Delta)^s$ is acting on each component, then it can be shown that the nonlocal matrix-valued operator $\mathbb{L}:\mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d)\to L^p(\mathbb{R}^d;\mathbb{R}^d)$ is continuous. Most importantly, we show in this paper that for any $2\le p\le \frac{2d}{d-2s}$ and $\mathbf{f}\in L^p(\Omega;\mathbb{R}^d)$, the unique variational solution $\mathbf{u}\in H^s(\mathbb{R}^d;\mathbb{R}^d)$ to the zero Dirichlet problem

$$\mathbb{L}\mathbf{u}=\mathbf{f} \quad \text{in } \Omega, \quad \mathbf{u}=0 \quad \text{in } \complement\Omega. \tag{4}$$

belongs to $\mathcal{L}^{2s,p}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$. We say $\mathbf{u}\in\mathcal{L}^{2s,p}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ if $\mathbf{u}\eta\in\mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d)$ for any $\eta\in C^\infty_c(\mathbb{R}^d)$. For nonlocal equations, results of the above type have been proved in [2, 22, 18]. We follow an approach that is used in [2, 1], where a similar but more general result is proved for the Dirichlet problem for the fractional Laplacian equation when the right hand side comes from L^p for any $1<p<\infty$. In the case of vector fields, we could not cover all ranges of p but only with the additional assumption that the weak solution $\mathbf{u}\in L^p$. In the scalar case such an assumption is not necessary since it can be proven that a solution to the Dirichlet problem of the fractional Laplacian with right hand side in L^p must also be in L^p, see [2, Lemma 2.5]. A similar Calderón-Zygmund type estimate for solutions is also proved in [22, Theorem 16]. Unfortunately we are unable to extend their proof to the vector-valued case because the argument in [2] relies on a monotonicity property of an associated semigroup and the result in [22] uses a Moser-type argument where a nonlinear function of the solution is used as a test function. Neither of these arguments can be applied for systems.

The organization of the paper is as follows: In Section 2 we introduce additional notation, provide some auxiliary results, and show well-posedness of the Dirichlet problem (1) using Hilbert space methods. We present sufficient conditions that imply the validity of fractional Poincaré-Korn-type estimates for a larger class of kernels. We also provide examples of kernels for which the theorem is applicable. For a smaller class of kernels we also link the energy space $S(\mathbb{R}^d;k)$ with classical Sobolev spaces. In Section 3 we prove higher-order interior regularity of solutions to the Dirichlet problem corresponding to a particular class of kernels.

2. **Variational formulation.** In this section we set up the variational approach to solve the system (1).

2.1. **Notations and definitions.** Throughout the paper we will be using the following function spaces and their associated norms. We assume that $D \subset \mathbb{R}^d$ is an open subset, and $\complement D$ denotes its complement. We begin with the function spaces

$$L^p_D(\mathbb{R}^d) = \{\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d) \,:\, \mathbf{u} = \mathbf{0} \text{ a.e. on } \complement D\}$$

which collects L^p functions defined over $\mathbb{R}^d$ that vanish outside of D. We also use the notation $S_D(\mathbb{R}^d;k)$ to denote the space of functions in $S(\mathbb{R}^d;k)$ that vanish outside of D:

$$S_D(\mathbb{R}^d;k) = \{\mathbf{u} \in S(\mathbb{R}^d;k) : \mathbf{u} = \mathbf{0} \text{ a.e. on } \complement D\}.$$

It is not difficult to show that $S_D(\mathbb{R}^d;k)$ is a closed subset of $S(\mathbb{R}^d;k)$ and that from the definition, $\left(S_D(\mathbb{R}^d;k), \|\cdot\|_{S(\mathbb{R}^d;k)}\right) \hookrightarrow \left(S(\mathbb{R}^d;k), \|\cdot\|_{S(\mathbb{R}^d;k)}\right)$. We denote the dual space of $S_D(\mathbb{R}^d;k)$ by $S^*_D(\mathbb{R}^d;k)$.

To set up a variational problem, we will make necessary preparations. To begin with, we introduce a bilinear form that will be used to define a generalized notion of a solution to the nonlocal systems of equations.

Definition 2.1. Given two measurable functions $\mathbf{u}$ and $\mathbf{v}$, we define

$$\begin{aligned}\mathcal{F}^k(\mathbf{u},\mathbf{v}) = \frac{1}{2}\iint_{\mathbb{R}^d\mathbb{R}^d} & k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ &+ \iint_{\mathbb{R}^d\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x},\end{aligned}$$

whenever the integrals exist.

We notice that the form is not necessarily symmetric. We aim to find conditions on k that allow us to have good control on the quadratic functional $\mathcal{F}^k(\mathbf{u},\mathbf{u})$ for $\mathbf{u}$ in the function space $S(\mathbb{R}^d;k)$. To that end, following [14, 32] let us assume that there exists a symmetric kernel $\tilde{k}$ and constants $A_1 \geq 1$, $A_2 \geq 1$ such that for all $\mathbf{x} \in \mathbb{R}^d$, the measure $|\{\mathbf{y} \in \mathbb{R}^d : k_a^2(\mathbf{x},\mathbf{y}) \neq 0 \text{ and } \tilde{k}(\mathbf{x},\mathbf{y}) = 0\}| = 0$, and

$$\iint_{\mathbb{R}^d\,\mathbb{R}^d} \tilde{k}(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \leq A_1\|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)} \tag{5}$$

for all $\mathbf{u} \in S(\mathbb{R}^d;k)$, and that

$$\sup_{\mathbf{x}\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{k_a^2(\mathbf{x},\mathbf{y})}{\tilde{k}(\mathbf{x},\mathbf{y})}\,\mathrm{d}\mathbf{y} \leq A_2. \tag{6}$$

Note that we can choose $\tilde{k} = k_s$, see [32] where it is used for scalar equations. The next lemma describes the proper definition of $\mathcal{F}^k(\mathbf{u},\mathbf{v})$ and its continuity on $S(\mathbb{R}^d;k)$. It also clarifies in what sense the operator (2) is defined.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be open and assume that k satisfies (3) and (5)-(6). For $n \in \mathbb{N}$, define the subset $D_n = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, |\mathbf{x}-\mathbf{y}| > 1/n\}$ and let*

$$\mathbb{L}_n\mathbf{u}(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|>1/n} k(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}\mathbf{y},$$

$$\mathcal{F}^k_n(\mathbf{u},\mathbf{v}) = \iint_{D_n} k(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.$$

Then we have that

i) $(\mathbb{L}_n\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)} = \mathcal{F}_n^k(\mathbf{u},\mathbf{v})$ *and* $\lim_{n\to\infty}\mathcal{F}_n^k(\mathbf{u},\mathbf{v}) = \mathcal{F}^k(\mathbf{u},\mathbf{v})$ *for all* $\mathbf{u},\mathbf{v}\in C_c^\infty(\Omega)$.

ii) Moreover, $\mathcal{F}^k : S(\mathbb{R}^d;k)\times S(\mathbb{R}^d;k)\to\mathbb{R}$ *is continuous, and thus is continuous on* $S_\Omega(\mathbb{R}^d;k)\times S_\Omega(\mathbb{R}^d;k)$.

Proof. We begin by noticing that if $\mathbf{u}\in C_c^\infty(\mathbb{R}^d)$, the expression $\mathbb{L}_n\mathbf{u}(\mathbf{x})$ is finite for almost all $\mathbf{x}\in\mathbb{R}^d$. This follows from the fact that for almost all $(\mathbf{x},\mathbf{y})\in\mathbb{R}^{d\times d}$, $k(\mathbf{x},\mathbf{y})\le k_s(\mathbf{x},\mathbf{y})$, assumption (3), and that the integration is over D_n. Similarly, for $\mathbf{u},\mathbf{v}\in C_c^\infty(\mathbb{R}^d)$, $\mathcal{F}_n^k(\mathbf{u},\mathbf{v})$ is finite as well.

Now, for $\mathbf{u},\mathbf{v}\in C_c^\infty(\mathbb{R}^d)$ we have by Fubini's theorem that

$$
\begin{aligned}
&(\mathbb{L}_n\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)}\\
&= \int_{\mathbb{R}^d}\left[\int_{\complement B(\mathbf{x},1/n)} k(\mathbf{x},\mathbf{y})\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\,\mathrm{d}\mathbf{y}\right]\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&= \iint_{D_n} k(\mathbf{x},\mathbf{y})\left((\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Split the last integral using the decomposition of k into k_s and k_a, and interchange $\mathbf{x}$ and $\mathbf{y}$ to obtain that

$$
\begin{aligned}
(\mathbb{L}_n\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)} =& \frac{1}{2}\iint_{D_n} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&+\iint_{D_n} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.
\end{aligned}
\tag{7}
$$

We will be using the Lebesgue dominated convergence theorem to pass to the limit in both term in (7). To pass to the limit in the first term we use the function $(\mathbf{x},\mathbf{y})\mapsto k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})$ as a majorant. It is integrable and by the Cauchy-Schwarz inequality,

$$
\iint_{\mathbb{R}^d\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\,\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\le[\mathbf{u},\mathbf{u}]_{H(\mathbb{R}^d;k)}[\mathbf{v},\mathbf{v}]_{H(\mathbb{R}^d;k)}<\infty \tag{8}
$$

due to (3), since $\mathbf{u},\mathbf{v}\in S(\mathbb{R}^d,k)$. We next bound the integrand in the second term in (7) as follows. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$, using Young's inequality we have that

$$
\begin{aligned}
&k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\\
&\le |k_a(\mathbf{x},\mathbf{y})|\tilde{k}^{-1/2}(\mathbf{x},\mathbf{y})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|\,|\mathbf{v}(\mathbf{x})|\tilde{k}^{1/2}(\mathbf{x},\mathbf{y})\\
&\le \frac{1}{2}\left(\mathbf{v}(\mathbf{x})^2\frac{k_a^2(\mathbf{x},\mathbf{y})}{\tilde{k}(\mathbf{x},\mathbf{y})}+\tilde{k}(\mathbf{x},\mathbf{y})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2\right),
\end{aligned}
$$

where assumption (5)-(6) guarantees that both functions in the right hand side are integrable in the product space $\mathbb{R}^d\times\mathbb{R}^d$. It is now clear that

$$
\lim_{n\to\infty}(\mathbb{L}_n\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)}=\mathcal{F}^k(\mathbf{u},\mathbf{v}).
$$

To prove the continuity of the bilinear form $\mathcal{F}^k : S(\mathbb{R}^d;k)\times S(\mathbb{R}^d;k)\to\mathbb{R}^d$ we estimate the two terms of $\mathcal{F}^k$ separately. As has been shown in (8), the first term of $\mathcal{F}^k(\mathbf{u},\mathbf{v})$ cannot exceed $\frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}[\mathbf{v},\mathbf{v}]_{S(\mathbb{R}^d;k)}$. To estimate the second term, we use (5)-(6) with $A=\max\{A_1,A_2\}$ and the Cauchy-Schwarz inequality to obtain that

$$\begin{aligned}
&\iint_{\mathbb{R}^d\,\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\leq \iint_{\mathbb{R}^d\,\mathbb{R}^d} |k_a(\mathbf{x},\mathbf{y})|\tilde{k}^{-1/2}(\mathbf{x},\mathbf{y})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})||\mathbf{v}(\mathbf{x})|\tilde{k}^{1/2}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\leq \left(\int_{\mathbb{R}^d}\mathbf{v}(\mathbf{x})^2\int_{\mathbb{R}^d}\frac{k_a^2(\mathbf{x},\mathbf{y})}{\tilde{k}(\mathbf{x},\mathbf{y})}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\right)^{1/2}\left(\iint_{\mathbb{R}^d\mathbb{R}^d}\tilde{k}(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\right)^{1/2}\\
&\leq A\,\|\mathbf{v}\|_{L^2(\mathbb{R}^d)}\,\|\mathbf{u}\|_{S(\mathbb{R}^d;k)}.
\end{aligned}$$

Combining the above estimates we have that

$$\begin{aligned}
|\mathcal{F}^k(\mathbf{u},\mathbf{v})| &\leq \frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}[\mathbf{v},\mathbf{v}]_{S(\mathbb{R}^d;k)} + A\,\|\mathbf{u}\|_{S(\mathbb{R}^d;k)}\,\|\mathbf{v}\|_{L^2(\mathbb{R}^d)}\\
&\leq C\,\|\mathbf{u}\|_{S(\mathbb{R}^d;k)}\,\|\mathbf{v}\|_{S(\mathbb{R}^d;k)}\,,
\end{aligned} \tag{9}$$

proving that $\mathcal{F}^k$ is indeed a continuous bilinear form on the space $S(\mathbb{R}^d;k)$. □

Remark 2.1. A discussion on the nature of the "limiting operator" $\mathbb{L}=\lim_{n\to\infty}\mathbb{L}_n$ is in order. First, in the event that the *kernel* $k(\mathbf{x},\mathbf{y})$ *is integrable in the sense that if for every* $\mathbf{x}\in\mathbb{R}^d$, $\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}<\infty$ *and the function* $\mathbf{x}\mapsto\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\in L^1_{loc}(\mathbb{R}^d)$, then for any $\mathbf{u}\in S(\mathbb{R}^d;k)$ and for each $n\in\mathbb{N}$, the value $\mathbb{L}_n\mathbf{u}(\mathbf{x})$ is finite for almost all $\mathbf{x}\in\mathbb{R}^d$ and for almost all $\mathbf{x}\in\mathbb{R}^d$ we have

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{L}_n\mathbf{u}(\mathbf{x}) &= \mathbb{L}\mathbf{u}(\mathbf{x})\\
&= \int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\\
&= \int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}.
\end{aligned}$$

Moreover, the above proposition implies that the sequence $\{\mathbb{L}_n\mathbf{u}\}$ is bounded in the dual space of $S(\mathbb{R}^d;k)$, and converges in the weak-* topology to $\mathcal{F}^k(\mathbf{u},\cdot)$. In this case, since for any $\mathbf{v}\in C_c^\infty(\mathbb{R}^d)$ one can verify using Fubini's theorem that

$$(\mathbb{L}\mathbf{u},\mathbf{v})_{L^2}=\mathcal{F}^k(\mathbf{u},\mathbf{v})$$

and thus we can identify $\mathcal{F}^k(\mathbf{u},\cdot)$ with the measurable vector field $\mathbb{L}\mathbf{u}$.

More generally, for any kernel satisfying (5)-(6) and (3), and for any $\mathbf{u}\in S(\mathbb{R}^d;k)$ one may replace the L^2 inner product by the duality pairing to define the sequence of functionals $\mathbb{L}_n u$ defined by

$$\begin{aligned}
\langle\mathbb{L}_n\mathbf{u},\mathbf{v}\rangle := &\frac{1}{2}\iint_{D_n} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&+\iint_{D_n} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x},
\end{aligned}$$

for $\mathbf{v} \in S(\mathbb{R}^d, k)$. The proposition proved above shows that $\{\mathbb{L}_n \mathbf{u}\}$ is bounded in the dual space of $S(\mathbb{R}^d, k)$ and converges in the weak-* topology to $\mathcal{F}^k(\mathbf{u}, \cdot)$. For $\mathbf{u} \in C_c^\infty(\mathbb{R}^n)$, the limiting functional $\mathcal{F}^k(\mathbf{u}, \cdot)$ can be identified with the function

$$\mathbb{L}\mathbf{u}(\mathbf{x}) = P.V. \int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\, \mathrm{d}\mathbf{y} + \int_{\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\, \mathrm{d}\mathbf{y}.$$

In the event that k is not integrable and not necessarily symmetric, the second term in the above expression corresponds to a term with "lower order derivatives"; see [14] for a detailed discussion.

2.2. **The Dirichlet problem of system of nonlocal equations.** In this subsection we use the bilinear form introduced earlier to define a variational solution to the Dirichlet problem of the nonlocal system of equations.

2.2.1. *Zero Dirichlet data.*

Definition 2.2. Assume that k satisfies both (3) and (5)-(6). Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let $\mathbf{f} \in S^*_\Omega(\mathbb{R}^d; k)$. We say that $\mathbf{u} \in S_\Omega(\mathbb{R}^d; k)$ is a solution of

$$\mathbb{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \complement\Omega, \tag{D_0}$$

if

$$\mathcal{F}^k(\mathbf{u}, \boldsymbol{\varphi}) = (\mathbf{f}, \boldsymbol{\varphi})_{L^2(\mathbb{R}^d)} \quad \text{for all } \boldsymbol{\varphi} \in S_\Omega(\mathbb{R}^d; k). \tag{10}$$

The main result of this section is the following well-posedness of the Dirichlet problem (D_0).

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let k satisfy (3) and (5)-(6). Assume further that*

i) there exists $C_P \geq 1$ such that for all $\mathbf{u} \in L^2_\Omega(\mathbb{R}^d)$,

$$\|\mathbf{u}\|^2_{L^2(\Omega)} \leq C_P \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x}, \quad \text{and} \tag{PK}$$

ii) for every $\epsilon > 0$, there exists $C_\epsilon \geq 0$ such that

$$C_\epsilon = \sup_{\mathbf{x}\in\Omega} \int_{\complement B(\mathbf{x},\varepsilon)} |k_a(\mathbf{x},\mathbf{y})|\, \mathrm{d}\mathbf{y} < \infty, \quad \text{and} \tag{11}$$

iii)

$$\inf_{\mathbf{x}\in\mathbb{R}^d} \liminf_{\varepsilon\to 0^+} \int_{\complement B(\mathbf{x},\varepsilon)} k_a(\mathbf{x},\mathbf{y}) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right) \mathrm{d}\mathbf{y} \geq 0 \tag{12}$$

in the sense of quadratic forms.

Then corresponding to any $\mathbf{f} \in S^*_\Omega(\mathbb{R}^d; k)$ there exists a unique solution $\mathbf{u} \in S_\Omega(\mathbb{R}^d; k)$ to (D_0). Moreover, there exists a constant $c > 0$ such that

$$[\mathbf{u}, \mathbf{u}]_{S_\Omega(\mathbb{R}^d;k)} \leq c\|\mathbf{f}\|_{S^*_\Omega(\mathbb{R}^d;k)}.$$

Remark 2.2. Condition (PK) in the theorem is called a Poincaré-Korn inequality. In the theorem it appears as an assumption that restricts the choice of the kernel k. Later, we provide sufficient conditions that guarantee the validity of (PK) for a class of kernels. Conditions (11)-(12) should be treated as cancellation conditions on the antisymmetric part of the kernel. Indeed, condition (11) is an integrability requirement on k_a away from the diagonal which allows us to apply Fubini's theorem and use other properties of the integral. Condition (12) on the other hand says that the term in the energy $\mathcal{F}^k[\mathbf{u},\mathbf{u}]$ involving the antisymmetric part $k_a(\mathbf{x},\mathbf{y})$ should not be "too negative." This condition can be relaxed slightly, but verifying it may be a challenge. A relaxed condition is given in [14, Remark 3.3]. See also nonlocal variational problems that involve sign changing kernels in a different sense in [25].

Proof of Theorem 2.2. We use the Lax-Milgram theorem to prove the result. Conditions (11)-(12) will be used to show that $\mathcal{F}^k[\mathbf{u},\mathbf{u}]$ is positive semidefinite, while (PK) implies positive definiteness of the energy. We show step-by-step that all the assumptions in the Lax-Milgram theorem are satisfied. We begin by noting that as in Proposition 2.1 the conditions (3), (5)-(6) imply that the bilinear form $\mathcal{F}^k$ is a continuous form on $S(\mathbb{R}^d;k)$. Next, we will show that $\mathcal{F}^k$ is coercive on the closed subspace $S_\Omega(\mathbb{R}^d;k)$ of $S(\mathbb{R}^d;k)$. We begin by showing that

$$\mathcal{F}^k(\mathbf{u},\mathbf{u}) \geq \frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)} \quad \text{for all } \mathbf{u}\in S_\Omega(\mathbb{R}^d;k). \tag{13}$$

For any $\varepsilon>0$, and for any $\mathbf{u}\in S_\Omega(\mathbb{R}^d;k)$ we have that

$$\begin{aligned}
&\iint\limits_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&=\frac{1}{2}\iint\limits_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left((\mathbf{u}(\mathbf{x})+\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&=\frac{1}{2}\iint\limits_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \left(\left(\mathbf{u}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2-\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x},
\end{aligned}$$

where we have used the anti-symmetry of k_a in the first equality. We use the integrability assumption (11) in the last integral to apply Fubini's theorem to obtain that

$$\begin{aligned}
&\iint\limits_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&=\iint\limits_{|\mathbf{x}-y|>\varepsilon} \left(\mathbf{u}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2 k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&=\int\limits_{\mathbb{R}^d}\mathbf{u}(\mathbf{x})^{\intercal}\left[\int\limits_{\complement B(\mathbf{x},\varepsilon)} k_a(\mathbf{x},\mathbf{y})\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\right]\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

We then conclude from (12) that

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad=\lim_{\varepsilon\to 0}\iint\limits_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}
\end{aligned}$$

$$= \int_{\mathbb{R}^d} \mathbf{u}(\mathbf{x})^\mathsf{T} \left[\int_{\complement B(\mathbf{x},\varepsilon)} k_a(\mathbf{x},\mathbf{y}) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}\mathbf{y} \right] \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \geq 0.$$

Hence, from the definition of the bilinear form we have that

$$\mathcal{F}^k(\mathbf{u},\mathbf{u}) \geq \frac{1}{2} \iint_{\mathbb{R}^d\, \mathbb{R}^d} (\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2 k_s(\mathbf{x},\mathbf{y}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} \tag{14}$$

and (13) is proved. Therefore, by the Poincaré-Korn inequality (PK) and (13),

$$\mathcal{F}^k(\mathbf{u},\mathbf{u}) \geq \frac{1}{4C_P} \|\mathbf{u}\|^2_{L^2(\Omega)} + \frac{1}{4} [\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)} \geq \frac{1}{4C_P} \|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)}.$$

Finally, the Lax-Milgram theorem implies that there exists a unique $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$ such that

$$\mathcal{F}^k(\mathbf{u},\boldsymbol{\varphi}) = \langle \mathbf{f}, \varphi \rangle \qquad \text{for all } \boldsymbol{\varphi} \in S_\Omega(\mathbb{R}^d;k).$$

□

A Sufficient Condition for the Poincaré-Korn Inequality. We emphasize that the Poincaré-Korn inequality (PK) is an assumption in Theorem 2.2. Here we present a theorem that gives sufficient conditions on the kernel k for the validity of the Poincaré-Korn inequality. Given $\mathcal{I}$ an open subset of the unit sphere $\mathbb{S}^{d-1}$ such that the Hausdorff measure $\mathcal{H}^{d-1}(\mathcal{I}) > 0$, we call the set Λ defined as

$$\Lambda = \left\{ \mathbf{h} \in \mathbb{R}^d \setminus \{\mathbf{0}\} : \frac{\boldsymbol{h}}{|\mathbf{h}|} \in \mathcal{I} \cup (-\mathcal{I}) \right\}$$

a double cone with apex at the origin. Note that for any such cone $\Lambda = -\Lambda$. Define $\Lambda_{B_r} := \Lambda \cap B_r(\mathbf{0})$, a part of a double cone with apex at the origin in $B_r(\mathbf{0})$.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Assume that there is an even, nonnegative function $\rho \in L^1(\mathbb{R}^d)$ satisfying the following conditions:*

- *i) There exists $\delta_0 > 0$ and a cone Λ with apex at the origin such that $\Lambda_{B_{\delta_0}} \subset \{\rho > 0\}$.*
- *ii) There exists $c_0 > 0$ such that for all $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$*

$$\begin{aligned} &\iint_{\mathbb{R}^d\, \mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} \\ &\qquad \geq c_0 \iint_{\mathbb{R}^d\, \mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x}\,. \end{aligned} \tag{15}$$

Then, there exists $C_P = C_P(\Omega, c_0, \rho, \delta_0, \Lambda) > 0$ such that for all $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$

$$\|\mathbf{u}\|_{L^2(\mathbb{R}^d)} \leq C_P \iint_{\mathbb{R}^d\, \mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x}. \tag{16}$$

In the next subsection we give a number of examples of kernels that satisfy the hypothesis of the proposition. We need the following lemma which generalizes [38, Proposition 1.2] and [12, Lemma 2.2] that give a characterization of infinitesimal rigid motions.

Lemma 2.4. *Suppose that* $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^d$ *is a measurable vector-valued function such that for some fixed* $\delta_0 > 0$*, and* $\mathcal{J} \subset \mathcal{S}^{d-1}$*, an open subset of the unit sphere* $\mathbb{S}^{d-1}$ *with* $\mathcal{H}^{d-1}(\mathcal{J}) > 0$*, it holds that for almost every* $\mathbf{x} \in \mathbb{R}^d$*,*

$$(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot (\mathbf{x} - \mathbf{y}) = 0 \quad \textit{for almost every } \mathbf{y} \in \left\{ \mathbf{y} \in B_{\delta_0}(\mathbf{x}) : \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \in \mathcal{J} \right\}.$$

Then $\mathbf{u}$ *is an affine map of the form* $\mathbf{u}(\mathbf{x}) = \mathbb{A}\mathbf{x} + \mathbf{b}$*, almost everywhere, where* $\mathbb{A}$ *is a constant skew symmetric matrix (*$\mathbb{A}^\intercal = -\mathbb{A}$*), and* $\mathbf{b} \in \mathbb{R}^d$*.*

Proof. For $\mathbf{x} \in \mathbb{R}^d$, define $\Gamma(\mathbf{x}) = \left\{ \mathbf{y} \in B_{\delta_0}(\mathbf{x}) : \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \in \mathcal{J} \right\}$. For each $\mathbf{x}$, the set $\Gamma(\mathbf{x})$ is an open set and is in fact the intersection of the ball $B_{\delta_0}(\mathbf{x})$ with the cone whose directions lie in $\mathcal{J}$ with apex at $\mathbf{x}$. Let $\{\mathbf{e}_i\}_{i=1}^d$ denote a basis for $\mathbb{R}^d$ contained in $\mathcal{J}$; such a basis exists because $\mathcal{J}$ is nontrivial. Then since the Lebesgue integral is continuous with respect to translations, there exists a $\delta_1 > 0$ such that the function

$$\delta \mapsto \int\limits_{\mathbb{R}^d} \left(\prod_{i=1}^d \chi_{\Gamma(\delta \mathbf{e}_i)} \right) \chi_{\Gamma(0)} \, \mathrm{d}\mathbf{x}$$

is positive. For $\mathbf{x} \in \mathbb{R}^d$, set $\tilde{\Gamma}(\mathbf{x}) := \left(\bigcap_{i=1}^d \Gamma(x + \delta_1 \mathbf{e}_i) \right) \cap \Gamma(\mathbf{x})$. By the discussion above, $\tilde{\Gamma}(\mathbf{x})$ is an open set of positive measure.

Now fix $\mathbf{x}_0 \in \mathbb{R}^d$ (up to a set of measure zero). Then by the main assumption in the lemma, for almost every $\mathbf{x} \in \tilde{\Gamma}(\mathbf{x}_0)$ we have

$$((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)) \cdot (\mathbf{x} - \mathbf{x}_0)) = 0 \tag{17}$$

and

$$((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0 + \delta_1 \mathbf{e}_i)) \cdot (\mathbf{x} - \mathbf{x}_0 - \delta_1 \mathbf{e}_i)) = 0\,. \tag{18}$$

Therefore, adding and subtracting $\mathbf{u}(\mathbf{x}_0)$ in the first argument of (18) and $\mathbf{x}_0$ in the second and using (17) we see that

$$(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)) \cdot \delta_1 \mathbf{e}_i = -(\mathbf{u}(\mathbf{x}_0 + \delta_1 \mathbf{e}_i) - \mathbf{u}(\mathbf{x}_0)) \cdot (\mathbf{x} - \mathbf{x}_0).$$

So,

$$\mathbf{u}(\mathbf{x}) \cdot \mathbf{e}_i = \frac{1}{\delta_1} \left((\mathbf{u}(\mathbf{x}_0 + \delta_1 \mathbf{e}_i) - \mathbf{u}(\mathbf{x}_0)) \cdot (\mathbf{x} - \mathbf{x}_0) \right) + \mathbf{u}(\mathbf{x}_0) \cdot \mathbf{e}_i$$

for every $\mathbf{x} \in \tilde{\Gamma}(\mathbf{x}_0)$ and every i, which is clearly a linear map. Then, letting $\mathbb{E} = [\mathbf{e}_i]$ be the matrix of basis vectors, and $\mathbf{u} = (u_1, u_2, \cdots, u_d)$, we have that

$$u_i(\mathbf{x}) = \left(\mathbb{E}^{-1} (\mathbb{E}\mathbf{u}) \right)_i = \sum_j e_{ij}^{-1} (\mathbf{e}_j \cdot \mathbf{u}(\mathbf{x}))$$

which, being a sum of of linear maps, is still linear. We conclude that for almost all $\mathbf{x} \in \tilde{\Gamma}(\mathbf{x}_0)$ the vector field $\mathbf{u}$ is of the form $\mathbb{A}(\mathbf{x}_0)\mathbf{x} + \mathbf{b}(\mathbf{x}_0)$, where $\mathbb{A}$ is matrix with constant entries (depending possibly on $\mathbf{x}_0$) and $\mathbf{b}$ is a constant vector (also depending on $\mathbf{x}_0$) in $\mathbb{R}^d$.

Next given any two points in $\mathbb{R}^d$, outside of a set of measure zero, we connect them by finitely many sets of the form $\tilde{\Gamma}(\mathbf{x})$, i.e. for any two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in $\mathbb{R}^d$ there exists a finite subcover of $(\tilde{\Gamma}(\mathbf{x}))_{\mathbf{x} \in \mathbb{R}^d}$, denoted $(\tilde{\Gamma}(\mathbf{x}_k))_{k=1}^N$, such that $\tilde{\Gamma}(\mathbf{x}_k) \cap \tilde{\Gamma}(\mathbf{x}_{k+1}) \neq \emptyset$ and $\mathbf{x}_0 \in \tilde{\Gamma}(\mathbf{x}_0)$, $x_1 \in \tilde{\Gamma}(\mathbf{x}_N)$. This is possible, since the line segment connecting $\mathbf{x}_0$ and $\mathbf{x}_1$ is compact. Therefore the $\mathbf{u}$ given above is the same in neighboring intersecting open sets and so $\mathbf{u} = \mathbb{A}\mathbf{x} + \mathbf{b}$ on $\mathbb{R}^d$ where $\mathbb{A}$, $\mathbf{b}$ are now constants. Again from the main assumption, the matrix $\mathbb{A}$ must be skew symmetric. $\square$

Corollary 2.4.1. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Assume that there is a nonnegative even function $\rho \in L^1(\mathbb{R}^d)$ satisfying the following:*

There exist $\delta_0 > 0$ and a symmetric cone Λ with vertex at the origin such that $\Lambda \cap B_{\delta_0}(0) \subset \operatorname{supp} \rho$.

Suppose that $\mathbf{u} \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ satisfies

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} = 0.$$

Then $\mathbf{u} = \mathbf{0}$ almost everywhere.

Proof. Since the integrand is nonnegative, we see that for almost every $\mathbf{x} \in \mathbb{R}^d$,

$$(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot (\mathbf{x}-\mathbf{y}) = 0$$

for almost every $\mathbf{y} \in \operatorname{supp} \rho + \mathbf{x} := \{\mathbf{z} \,:\, \mathbf{z} - \mathbf{x} \in \operatorname{supp} \rho\}$. By assumption and by Lemma 2.4, $\mathbf{u}$ is an affine map. But since $\mathbf{u} \in L^2(\mathbb{R}^d)$, it follows that $\mathbf{u}$ must be the zero vector field. □

Now, we are ready to prove the sufficiency for the Poincaré-Korn inequality. The proof follows the argument presented in the proof of [26, Proposition 2] that applies to the case when ρ is radial.

Proof of Lemma 2.3. Without loss of generality we assume that ρ has compact support of positive measure. (else replace ρ by $\rho\chi_{B(0,r)}$). Then ρ satisfies (3). To prove the lemma, it suffices to show that there exists a constant $C > 0$ such that for all $\mathbf{u} \in L^2_\Omega(\mathbb{R}^d)$,

$$\|\mathbf{u}\|_{L^2} \leq C[\mathbf{u}, \mathbf{u}]_{H(\mathbb{R}^d;\rho)}.$$

Suppose to the contrary; that there exists $\{\mathbf{u}_n\} \subset S_\Omega(\mathbb{R}^d;\rho)$ such that $\forall\, n \in \mathbb{N}$ $\|\mathbf{u}_n\|_{L^2(\mathbb{R}^d)} = 1$ and $[\mathbf{u}_n, \mathbf{u}_n]_{S(\mathbb{R}^d;\rho)} \to 0$ as $n \to \infty$. Let $\mathbf{u}$ be the weak $L^2(\mathbb{R}^d)$ limit of $\{\mathbf{u}_n\}$. We first show that $\mathbf{u} = \mathbf{0}$. Note that because of the properties of ρ the operator

$$\mathbb{L}_\rho \mathbf{u}(\mathbf{x}) := \int\limits_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\,\mathrm{d}\mathbf{y}$$

is a bounded linear map from $L^2(\mathbb{R}^d;\mathbb{R}^d)$ to $L^2(\mathbb{R}^d;\mathbb{R}^d)$. Let $\boldsymbol{\varphi} \in C_c^\infty(\mathbb{R}^d)$. Then, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
&[\mathcal{F}^\rho(\mathbf{u}_n, \boldsymbol{\varphi})]^2 \\
&= \left[\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left((\mathbf{u}_n(\mathbf{x}) - \mathbf{u}_n(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{y}-\mathbf{x}|} \right) \left(\boldsymbol{\varphi}(\mathbf{x}) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{y}-\mathbf{x}|} \right) \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \right]^2 \\
&\leq \left(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left((\mathbf{u}_n(\mathbf{x}) - \mathbf{u}_n(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{y}-\mathbf{x}|} \right) \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \right) \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) |\boldsymbol{\varphi}(\mathbf{x})|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&= \|\rho\|_{L^1(\mathbb{R}^d)} [\mathbf{u}_n, \mathbf{u}_n]_{S(\mathbb{R}^d;\rho)} \|\boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

and the last term approaches 0 as $n \to \infty$. Now, since ρ is symmetric, $\mathcal{F}^\rho$ is symmetric. Thus,

$$(\mathbb{L}_\rho \mathbf{u}_n, \boldsymbol{\varphi})_{L^2} = \mathcal{F}^\rho(\mathbf{u}_n, \boldsymbol{\varphi}) = \mathcal{F}^\rho(\boldsymbol{\varphi}, \mathbf{u}_n) = (\mathbf{u}_n, \mathbb{L}_\rho \boldsymbol{\varphi})_{L^2} \quad \forall\, n \in \mathbb{N}.$$

Since $\mathbb{L}_\rho \boldsymbol{\varphi} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, it follows that $\mathcal{F}^\rho(\boldsymbol{\varphi},\mathbf{u}_n) \to \mathcal{F}^\rho(\boldsymbol{\varphi},\mathbf{u})$ as $n\to\infty$. Therefore, for all $\boldsymbol{\varphi} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, $(\mathbb{L}_\rho\varphi,\mathbf{u})_{L^2} = (\mathbb{L}_\rho\mathbf{u},\boldsymbol{\varphi})_{L^2} = 0$. Thus $\mathbb{L}_\rho\mathbf{u} = \mathbf{0}$ a.e. As a consequence, since ρ is even and assumption $ii)$

$$
\begin{aligned}
&(\mathbb{L}_\rho\mathbf{u},\mathbf{u})_{L^2} = \mathcal{F}^\rho(\mathbf{u},\mathbf{u})\\
&= \frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;\rho)} = \frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y})\left((\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2 \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} = 0.
\end{aligned}
$$

We can now apply Corollary 2.4.1 to conclude that $\mathbf{u}\equiv\mathbf{0}$ on $\mathbb{R}^d$.

Next we show that in fact, up to a subsequence, $\mathbf{u}_n \to 0$ strongly in L^2, and arrive at our contradiction. To show this it suffices to demonstrate that $\|\mathbf{u}_n\|_{L^2(\Omega)} \to 0$ as $n\to\infty$. Define $\mathbb{K}(\boldsymbol{\xi}) = \rho(\boldsymbol{\xi})\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$. Note that $\mathbb{K}\in L^1(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$ since $\rho\in L^1\cap L^\infty(\mathbb{R}^d)$. Then, define $\mathbb{K}*\mathbf{u}(\mathbf{x})$ and $\mathbb{B}$ as

$$
(\mathbb{K}*\mathbf{u}(\mathbf{x}))_i = \sum_{j=1}^d \int\limits_{\mathbb{R}^d} (\mathbb{K}(\mathbf{x}-\mathbf{y}))_{ij}\mathbf{u}(\mathbf{y})_j\,\mathrm{d}\mathbf{y}, \qquad \mathbb{B} = \int\limits_{\mathbb{R}^d}\mathbb{K}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}.
$$

Both quantities converge absolutely and are well-defined. Further, $\mathbb{L}_\rho\mathbf{u}(\mathbf{x}) = \mathbb{K}*\mathbf{u}(\mathbf{x}) - \mathbb{B}\mathbf{u}(\mathbf{x})$. Note that $\mathbb{B}$ is a positive definite constant matrix, which follows from the fact that $\Phi(v) = \mathbf{v}^\intercal\mathbb{B}\mathbf{v} = \int\limits_{\mathbb{R}^d}\rho(\boldsymbol{\xi})\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\mathbf{v}\right|^2\mathrm{d}\boldsymbol{\xi}$ is a continuous and positive function on the unit sphere $\mathbb{S}^{d-1}$. From an above estimate, we have that

$$
(\mathbb{L}_\rho\mathbf{u}_n,\mathbf{u}_n)_{L^2} \le \|\rho\|_{L^1(\mathbb{R}^d)}\,[\mathbf{u}_n,\mathbf{u}_n]_{S(\mathbb{R}^d;\rho)}\,\|\mathbf{u}_n\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } n\to\infty.
$$

Since $\mathbf{u}_n \rightharpoonup \mathbf{0}$ weakly in $L^2(\mathbb{R}^d)$, by compactness of the convolution operator [3, Corollary 4.28] we have that

$$
\mathbb{K}*\mathbf{u}_n(\mathbf{x}) \to \mathbf{0} \text{ strongly in } L^2(\Omega;\mathbb{R}^d).
$$

Therefore, since $\mathbb{B}\ge\gamma\mathbb{I}$ in the sense of quadratic forms, we have that

$$
\begin{aligned}
\gamma\lim_{n\to\infty}\int\limits_{\mathbb{R}^d}|\mathbf{u}_n|^2\,\mathrm{d}\mathbf{x} &\le \lim_{n\to\infty}(\mathbb{B}\mathbf{u}_n,\mathbf{u}_n)_{L^2(\mathbb{R}^d)}\\
&= \lim_{n\to\infty}(\mathbb{B}\mathbf{u}_n,\mathbf{u}_n)_{L^2(\mathbb{R}^d)} + \lim_{n\to\infty}(\mathbb{K}*\mathbf{u}_n,\mathbf{u}_n)_{L^2(\Omega)}\\
&= \lim_{n\to\infty}(\mathbb{L}_\rho\mathbf{u}_n,\mathbf{u}_n)_{L^2(\Omega)} = 0,
\end{aligned}
$$

which completes the proof. □

2.2.2. *Examples of kernels.* There are several examples of kernels that satisfy all the conditions of the theorem; a number of them are discussed in detail in [14] in connection with the solvability of the Dirichlet problem associated to nonlocal equations. For some of these examples, the verification of (PK) is nontrivial. We list several examples of nontrivial kernels, for which we can verify all the conditions. This shows that the nonlocal Dirichlet problem for the corresponding system of equations is well-posed.

Example 1: Suppose that $\rho(\boldsymbol{\xi})$ is a nonnegative, even, integrable function in $\mathbb{R}^d$. Define now

$$
k(\mathbf{x},\mathbf{y}) = \rho(\mathbf{x}-\mathbf{y})\chi_{\Lambda_{B_1}}(\mathbf{x}-\mathbf{y}),
$$

where Λ_{B_1} is as defined before Proposition 2.3. Since K is symmetric, we only need to verify the Poincaré-Korn inequality (PK). But this is a consequence of Lemma

2.3. See also [26, Proposition 2] for a similar result that is valid for radial kernels. Note that for these types of kernels the space $S(\mathbb{R}^d;k)$ is just $L^2(\mathbb{R}^d;\mathbb{R}^d)$.

Example 2: More generally, if $\mathcal{C}=\left\{\mathbf{h}\in B_1(\mathbf{0}):\frac{\boldsymbol{h}}{|\mathbf{h}|}\in\mathcal{J}\right\}$, and $\mathcal{J}$ is an open subset of the unit sphere $\mathbb{S}^{d-1}$ with Hausdorff measure $\mathcal{H}^{d-1}(\mathcal{J})>0$, then $k(\mathbf{x},\mathbf{y})=\rho(\mathbf{x}-\mathbf{y})\chi_{\mathcal{C}}(\mathbf{x}-\mathbf{y})$ satisfies all the conditions of the theorem. In this case the kernel is not symmetric. However, its symmetric part is $k_s(\mathbf{x},\mathbf{y})=\rho(\mathbf{x}-\mathbf{y})\chi_{\mathcal{C}\cup(-\mathcal{C})}(\mathbf{x}-\mathbf{y})$, and the union $\mathcal{C}\cup(-\mathcal{C})$ is now a double cone with apex at the origin. The antisymmetric part k_a is given by $k_a(\mathbf{x},\mathbf{y})=\frac{1}{2}\left(\rho(\mathbf{x}-\mathbf{y})\chi_{\mathcal{C}}(\mathbf{x}-\mathbf{y})-\rho(\mathbf{x}-\mathbf{y})\chi_{(-\mathcal{C})}(\mathbf{x}-\mathbf{y})\right)$ and satisfies both conditions (11) and (12) as can easily be seen.

Before we give other examples let us first prove a lemma that helps us compare function spaces. The result is an improvement of [12, Lemma 2.12], where the same result is shown for radial kernels that are supported on $\Lambda_{B_r}=B_r$.

Lemma 2.5 (**Fractional Korn inequality**). *Let $s\in(0,1)$ and let $m(\boldsymbol{\xi})$ be an even function defined on $B_r(\mathbf{0})$ with the property that $0<\alpha_1\le m(\boldsymbol{\xi})\le\alpha_2<\infty$ for some positive constants α_1 and α_2. For a given Λ a double cone with apex at the origin and a given $r>0$ define the kernel*

$$k_r(\mathbf{x},\mathbf{y})=\frac{m(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_{\Lambda_{B_r}}(\mathbf{x}-\mathbf{y}).$$

Then the function space $S(\mathbb{R}^d;k_r)$ is precisely $H^s(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, there exists a function $\beta(r)$ with the property that $\beta(r)\to0$ as $r\to\infty$ and positive constants C_1, C_2 such that

$$C_1[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k_r)}\le|\mathbf{u}|^2_{H^s}\le C_2[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k_r)}+C_2\beta(r)\|\mathbf{u}\|^2_{L^2}. \tag{19}$$

If Λ_{B_r} is replaced by Λ, then β can be taken to be the zero function. The constants C_1, C_2 and the function β depend on α_i, Λ, d and s.

Proof. We prove the lemma using the Fourier transform. First let us introduce the following modification

$$\tilde{m}(\boldsymbol{\xi})=\begin{cases}m(\boldsymbol{\xi}) & \boldsymbol{\xi}\in\Lambda_{B_r}\\ \alpha_1 & \boldsymbol{\xi}\in\Lambda\cap\complement B_r(\mathbf{0}).\end{cases}$$

Then $\tilde{m}$ is even, and $\tilde{m}(\boldsymbol{\xi})\ge\alpha_1$ for all $\boldsymbol{\xi}\in\Lambda$. Now, for $\mathbf{u}\in S(\mathbb{R}^d;k_r)$

$$\begin{aligned}
&[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k_r)}+\iint_{\mathbb{R}^d\,\mathbb{R}^d}\alpha_1\frac{\left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}}\chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad=\iint_{\mathbb{R}^d\,\mathbb{R}^d}\tilde{m}(\mathbf{x}-\mathbf{y})\frac{\left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}}\chi_{\Lambda}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad=\iint_{\mathbb{R}^d\,\mathbb{R}^d}\tilde{m}(\mathbf{x}-\mathbf{y})\frac{\left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}}\chi_{\Lambda}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad=\int_{\Lambda}\frac{\tilde{m}(\mathbf{h})}{|\mathbf{h}|^{\mathbf{d+2s}}}\|\tau_{\mathbf{h}}\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\mathbf{h}
\end{aligned}$$

where $\tau_{\mathbf{h}}\mathbf{u}(\mathbf{x}) = (\mathbf{u}(\mathbf{x}+\mathbf{h}) - \mathbf{u}(\mathbf{x})) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}$. Note that the Fourier transform of $\tau_{\mathbf{h}}\mathbf{u}(\mathbf{x})$ is given by

$$\mathcal{F}(\tau_{\mathbf{h}}\mathbf{u})(\xi) = (e^{\imath 2\pi \boldsymbol{\xi}\cdot\mathbf{h}} - 1)\mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}.$$

Using Parseval's identity and after a simple calculation we see that

$$\|\tau_{\mathbf{h}}\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} = 2\int_{\mathbb{R}^d} \left|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 (1 - \cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))\,\mathrm{d}\boldsymbol{\xi}.$$

Plugging the last expression in the above semi-norm and interchanging the integral we get that

$$\begin{aligned}
&\int_{\Lambda} \frac{\tilde{m}(\mathbf{h})}{|\mathbf{h}|^{\mathbf{d+2s}}} \|\tau_{\mathbf{h}}\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\mathbf{h} \\
&= 2\int_{\mathbb{R}^d} \left[\int_{\Lambda} \frac{\tilde{m}(\mathbf{h})}{|\mathbf{h}|^{\mathbf{d+2s}}} \left|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 (1 - \cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))\,\mathrm{d}\mathbf{h}\right] \mathrm{d}\boldsymbol{\xi} \\
&\geq 2\alpha_1 \int_{\mathbb{R}^d} \left[\int_{\Lambda} \frac{(1 - \cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))}{|\mathbf{h}|^{\mathbf{d+2s}}} \left|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h}\right] \mathrm{d}\boldsymbol{\xi} \\
&= 2\alpha_1 \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} \left[\int_{\Lambda} \frac{(1 - \cos(2\pi\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\mathbf{h}))}{|\mathbf{h}|^{\mathbf{d+2s}}} \left|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h}\right] \mathrm{d}\boldsymbol{\xi},
\end{aligned}$$

where in the last step we have made a change of variables $\mathbf{h} \mapsto |\boldsymbol{\xi}|\mathbf{h}$ and used the fact that Λ remains invariant under scaling. Notice that the last inequality can be written as

$$\begin{aligned}
[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k_r)} &+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \alpha_1 \frac{\left|(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \cdot \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}} \chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&\geq 2\alpha_1 \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} |\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|^2 \Psi\left(\frac{\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})}{|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \mathrm{d}\boldsymbol{\xi},
\end{aligned}$$

where $\Psi(\boldsymbol{\nu},\boldsymbol{\eta}) = \int_{\Lambda} \frac{(1-\cos(2\pi\boldsymbol{\nu}\cdot\mathbf{h}))}{|\mathbf{h}|^{\mathbf{d+2s}}} \left|\boldsymbol{\eta}\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h}$, and maps $\Psi : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to [0,\infty)$. It is not difficult to see that Ψ is a continuous positive function on the compact set $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$, and therefore has a positive minimum, Ψ_{min}. As a consequence we have

$$\begin{aligned}
2\alpha_1\Psi_{min}|\mathbf{u}|^2_{H^s} &= 2\alpha_1\Psi_{min} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} |\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi} \\
&\leq [\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k_r)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \alpha_1 \frac{\left|(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \cdot \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}} \chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Next, we estimate the second term on the right hand side of the above inequality. Again using the Fourier transform we have that

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \alpha_1 \frac{\left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}}\chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&\quad = 2\alpha_1 \int\limits_{\mathbb{R}^d}\int\limits_{\Lambda\cap\complement B_r(\mathbf{0})} \frac{(1-\cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))}{|\mathbf{h}|^{\mathbf{d+2s}}}\left|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h}\,\mathrm{d}\boldsymbol{\xi} \\
&\quad \le 2\alpha_1\beta(r)\int\limits_{\mathbb{R}^d}|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi},
\end{aligned}$$

where $\beta(r) = \int_{\Lambda\cap\complement B_r(\mathbf{0})}\frac{\mathrm{d}\mathbf{h}}{|\mathbf{h}|^{d+2s}} \to 0$, as $r\to\infty$ and depends only on d, s, and Λ. We conclude that there exists a constant $C>0$ such that for every $\mathbf{u}\in S(\mathbb{R}^d;k_r)$ we have $|\mathbf{u}|^2_{H^s} \le C[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k_r)} + C\beta(r)\|\mathbf{u}\|^2_{L^2}$. The bound

$$[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k_r)} \le 2\alpha_2\Psi_{max}|\mathbf{u}|^2_{H^s}$$

can be proved in a similar fashion. □

Let us now continue discussing examples of kernels that may satisfy our well-posedness result.

Example 3: Let k_r be as in Lemma 2.5. Since the kernel is symmetric, to check the applicability of Theorem 2.2 for this kernel, we need to verify only the Poincaré-Korn inequality. But this follows from Lemma 2.3 by taking $\rho(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 k_r(\boldsymbol{\xi})$, for any $r>0$. By the above lemma, the space $S(\mathbb{R}^d;k_r)$ is in fact $H^s(\mathbb{R}^d;\mathbb{R}^d)$.

Example 4: Another nontrivial non-symmetric kernel given in [14] is the following. For $s\in(0,1)$, fix $\alpha\in\left(0,\frac{s}{2}\right)$. Let Λ be a double cone with apex at the origin. Given the cone $\mathcal{C} = \left\{\mathbf{h}\in B_1(\mathbf{0}) : \frac{\boldsymbol{h}}{|\mathbf{h}|}\in\mathcal{J}\right\}$, where $\mathcal{J}$ is a nontrivial open subset of the unit sphere $\mathbb{S}^{d-1}$ such that $-\mathcal{J}\neq\mathcal{J}$, let us consider the kernel

$$k(\mathbf{x},\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_\Lambda(\mathbf{y}-\mathbf{x}) + \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{\mathcal{C}}(\mathbf{y}-\mathbf{x}).$$

Then the symmetric and antisymmetric part of k are given by

$$\begin{aligned}
k_s(\mathbf{x},\mathbf{y}) &= \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_\Lambda(\mathbf{y}-\mathbf{x}) + \frac{1}{2}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{\mathcal{C}\cup(-\mathcal{C})}(\mathbf{y}-\mathbf{x}) \\
k_a(\mathbf{x},\mathbf{y}) &= \frac{1}{2}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{\mathcal{C}}(\mathbf{y}-\mathbf{x}) - \frac{1}{2}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{(-\mathcal{C})}(\mathbf{y}-\mathbf{x}).
\end{aligned}$$

Conditions (3) and (11)-(12) can be shown as in [14]. Let us show (5)-(6) with $\tilde{k}(\mathbf{x},\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}$. Again, (6) is shown in [14] where the constant A_2 depends on $\mathcal{C}$ and $s-2\alpha$, but to show (5) we use the fact that $k_s(\mathbf{x},\mathbf{y}) \ge \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_\Lambda(\mathbf{y}-\mathbf{x})$. Indeed, using Lemma 2.5 and the remark following it, there exist constants $c_1, c_2>0$

such that

$$\begin{aligned}&\iint_{\mathbb{R}^d\,\mathbb{R}^d} \tilde{k}(\mathbf{x},\mathbf{y})\left((\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\right)^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\ &\leq c_1|\mathbf{u}|^2_{H^s}\\ &\leq c_2\iint_{\mathbb{R}^d\,\mathbb{R}^d}\frac{\chi_\Lambda(\mathbf{y}-\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\ &\leq 2c_2[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}\,.\end{aligned}$$

The Poincaré-Korn inequality (PK) now follows from the standard Fractional Poincaré inequality, because the function space $S(\mathbb{R}^d;k)$ coincides with $H^s(\mathbb{R}^d;\mathbb{R}^d)$, and because by Lemma 2.5

$$|\mathbf{u}|^2_{H^s}\leq C[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}.$$

2.2.3. *Variants of the Dirichlet problem.* As indicated earlier in the proof of Theorem 2.2, conditions (11)-(12) on the kernel k are used to show the positive semi-definiteness of the bilinear form on $S(\mathbb{R}^d;k)$. There are however kernels for which either these conditions are not true or difficult to verify. For this class of kernels, well-posedness of the Dirichlet problem corresponding to the addition of a positive multiple of the identity operator can be obtained.

Proposition 2.6. *Let $\Omega\subset\mathbb{R}^d$ be open, bounded. Let k satisfy (3) and (5)-(6). Assume also that (PK) holds. Then there exists $\beta_0>0$ such that for any $\beta>\beta_0$ and any $\mathbf{f}\in S^*_\Omega(\mathbb{R}^d;k)$, there exists a unique solution $\mathbf{u}\in S_\Omega(\mathbb{R}^d;k)$ to*

$$\begin{cases}\mathbb{L}\mathbf{u}+\beta\mathbf{u}=\mathbf{f} & \text{in } \Omega\,,\\ \mathbf{u}=\mathbf{0} & \text{on } \complement\Omega\,.\end{cases}\tag{20}$$

Moreover, there exists a constant $c>0$ independent of $\mathbf{f}$ such that

$$[\mathbf{u},\mathbf{u}]_{S_\Omega(\mathbb{R}^d;k)}\leq c\|\mathbf{f}\|_{S^*_\Omega(\mathbb{R}^d;k)}.$$

Proof. The proof follows from standard arguments once Gårding-type estimates are established. To that end, we show that there is a constant $\gamma=\gamma(A_1,A_2)>0$ such that

$$\mathcal{F}^k(\mathbf{u},\mathbf{u})\geq\frac{1}{4}\|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)}-\gamma\|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\quad\text{for all }\mathbf{u}\in S(\mathbb{R}^d;k).\tag{21}$$

To prove this, let $\mathbf{u}\in S(\mathbb{R}^d;k)$. From (5)-(6) and by Young's inequality,

$$\begin{aligned}&\mathcal{F}^k(\mathbf{u},\mathbf{u})\\ &\geq\frac{1}{2}\iint_{\mathbb{R}^d\,\mathbb{R}^d}k_s(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}-\iint_{\mathbb{R}^d\,\mathbb{R}^d}k_a(\mathbf{x},\mathbf{y})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|\left|\mathbf{u}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right|\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ &\geq\frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}-\iint_{\mathbb{R}^d\,\mathbb{R}^d}|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|\tilde{k}^{1/2}(\mathbf{x},\mathbf{y})|\mathbf{u}(\mathbf{x})|k_a(\mathbf{x},\mathbf{y})\tilde{k}^{-1/2}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ &\geq\frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}-\iint_{\mathbb{R}^d\,\mathbb{R}^d}\left(\varepsilon|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2\tilde{k}(\mathbf{x},\mathbf{y})+\frac{1}{4\varepsilon}|\mathbf{u}(\mathbf{x})|^2k_a^2(\mathbf{x},\mathbf{y})\tilde{k}^{-1}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ &\geq\frac{1}{4}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}-C(\epsilon)\|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\\ &\geq\frac{1}{4}\|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)}-\gamma\|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

if ε is chosen sufficiently small such that $A_1\epsilon < 1/4$ and then $\gamma = \gamma(A_1, A_2)$ chosen sufficiently large. □

We next discuss an example of a nontrivial kernel that satisfies all the conditions of the proposition. The example is taken from [32, 16, 14] and discussed in detail there. For two given positive numbers $0 < \alpha_1 \le \alpha_2 < 2$, let $\alpha : \mathbb{R}^d \to [\alpha_1, \alpha_2]$ be a continuous function, with its modulus of continuity $\omega[\alpha]$ satisfying $\int_0^1 \frac{(\omega[\alpha](r)\ln(r))^2}{r^{1+\alpha_2}}\mathrm{d}r < \infty$. We introduce the non-symmetric kernel

$$k(\mathbf{x},\mathbf{y}) = \frac{b(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{d+\alpha(\mathbf{x})}},$$

where $b(\mathbf{x})$ is a continuous function bounded from below and above by positive numbers and satisfying the inequality $|b(\mathbf{x}) - b(\mathbf{y})| \le c|\alpha(\mathbf{x}) - \alpha(\mathbf{y})|$ for some $c > 0$ provided $|\mathbf{x}-\mathbf{y}| < 1$. To see if Proposition 2.6 applies to this kernel, we need to verify (3), (5)-(6) and (PK). It has been shown in [32] that this kernel satisfies (3) and (5)-(6), with $\tilde{k}$ taken to be the symmetric part k_s of k. What remains is the show the Poincaré-Korn inequality (PK) holds for k. But this follows from Lemma 2.3 and the fact that $k_s(\mathbf{x},\mathbf{y}) \ge \frac{b_{min}}{|\mathbf{x}-\mathbf{y}|^{d+\alpha_1}}$ when $|\mathbf{x}-\mathbf{y}| < 1$.

We remark that in [14] for the kernel $k'(\mathbf{x},\mathbf{y}) = \chi_{B_R(\mathbf{0})}(\mathbf{y}-\mathbf{x})k(\mathbf{x},\mathbf{y})$ with $1 \ll R$, the Dirichlet problem for scalar equations is shown to be well-posed even for $\beta = 0$, see [14, Theorem 4.4]. This was proved using the Fredholm Alternative theorem via the application of the weak maximum principle that is used to prove uniqueness of the solution to the Dirichlet problem with zero right-hand side. Following the argument in [14], one can write a Fredholm Alternative theorem for the Dirichlet problem (D_0) of the system of nonlocal equations. However, since we are dealing with a system of equations a maximum principle is not applicable and we are unable to show uniqueness of the solution of the linear system of equations (D_0). The uniqueness of the zero solution (D_0) corresponding to $\mathbf{f} = 0$ under the assumption of Proposition 2.6 or even the stronger assumption on k given in [14, Theorem 4.4] remains an open problem.

We end this section by noting that well-posedness of the Dirichlet problem with nonzero complementary data can also be proved. To that end, again following the set up in [14], let us introduce the function space of vector fields $\mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$V(D;k) = \left\{ \mathbf{v}|_D \in L^2(D;\mathbb{R}^d) \,:\, (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} k_s^{1/2}(\mathbf{x},\mathbf{y}) \in L^2(D\times\mathbb{R}^d) \right\}.$$

The mapping $[\mathbf{u},\mathbf{v}]_{V(D;k)}$ is given by

$$[\mathbf{u},\mathbf{v}]_{V(D;k)} := \iint\limits_{D\,\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) \left((\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$$

defines a bilinear form on $V(D;k)$. In the event that $D = \mathbb{R}^d$, it is clear that $V(\mathbb{R}^d;k) = S(\mathbb{R}^d;k)$. For a given $\mathbf{g} \in V(\Omega;k)$, we say $\mathbf{u} \in V(\Omega;k)$ is called a solution of

$$\mathbb{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{g} \quad \text{on } \complement\Omega, \tag{D}$$

if $\mathbf{u} - \mathbf{g} \in S_\Omega(\mathbb{R}^d;k)$ and (10) holds.

We now state the well-posedness of the Dirichlet Problem. We omit the proof here as it can be done following the argument in [14].

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let k be a kernel that satisfies* (3), (11)-(12)*, and* (PK)*. Assume further that there exists a $\tilde{k}$ such that for all $\mathbf{u} \in V(\Omega; k)$*

$$\iint_{\Omega\,\mathbb{R}^d} \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \tilde{k}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \leq A_1 \iint_{\Omega\,\mathbb{R}^d} \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \tag{22}$$

and such that (6) *holds for this $\tilde{k}$. Then (D) has a unique solution $\mathbf{u} \in V(\Omega; k)$, with*

$$[\mathbf{u}, \mathbf{u}]_{V(\Omega;k)} \leq C \left(\|f\|^2_{S^*_\Omega(\mathbb{R}^d;k)} + [g, g]_{V(\Omega;k)} \right), \tag{23}$$

where $C = C(C_P, A_1, A_2) > 0$.

Remark 2.3. Condition (22) obviously holds if one chooses $\tilde{k}(\mathbf{x},\mathbf{y}) = k_s(\mathbf{x},\mathbf{y})$. The integration allows for more flexibility here, see [14] for examples. Note that Theorem 2.7 opens up an interesting question concerning data on $\complement\Omega$. The result requires $\mathbf{g} \in V(\Omega; k)$, i.e., the data is given in all of $\mathbb{R}^d$. This condition is similar to the condition $g \in H^1(\Omega)$ when searching for a solution v solving some partial differential equation of second order in Ω with $v - g \in H^1_0(\Omega)$. From the point of view of applications it is desirable to prescribe g only on the complement $\complement\Omega$ and to have some extension theorem. Such results are nowadays standard for classical Sobolev function spaces. They put into relation the trace space $H^{\frac{1}{2}}(\Omega)$ with $H^1(\Omega)$. For spaces characterized by derivatives of fractional order, a similar relation has been addressed in [21].

3. Interior regularity of solutions.

3.1. Setup and main results.

We now turn to the question of regularity of solutions. We want to answer the following question: if the data $\mathbf{f}$ are in $L^p(\Omega; \mathbb{R}^d)$, what is the optimal space for the weak solution $\mathbf{u}$ of the Dirichlet problem of the system of nonlocal equations (D_0)? From the existence result proved in the previous section, if $\mathbf{f} \in S^*_\Omega(\mathbb{R}^d; k)$ then $\mathbf{u} \in S_\Omega(\mathbb{R}^d; k)$, which is the largest space to which the solution can belong. This space is, in general, not the optimal space. Moreover, for general kernels k there is no good characterization of the space or other finer subspaces in which the solution may live. With this in mind, in this section we give a partial result concerning regularity of solutions. The result applies to systems of equations with leading operator $\mathbb{L}$ defined using an even function comparable with the fractional kernel. To be precise, let $s \in (0, 1)$ and $m(\boldsymbol{\xi})$ be an even function with the property that $0 < \alpha_1 \leq m(\boldsymbol{\xi}) \leq \alpha_2 < \infty$ for some positive constants α_1 and α_2. For a given Λ, a double cone with apex at the origin, and $0 < r \leq \infty$ we consider translation-invariant kernels that may be supported on Λ:

$$k_r(\mathbf{x}-\mathbf{y}) = \frac{m(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \chi_{\Lambda_{B_r}}(\mathbf{x}-\mathbf{y}). \tag{24}$$

For kernels of this form we have shown in Lemma 2.5 that $S(\mathbb{R}^d; k) = H^s(\mathbb{R}^d; \mathbb{R}^d)$. Thus, $S_\Omega(\mathbb{R}^d; k) = L^2_\Omega(\mathbb{R}^d; \mathbb{R}^d) \cap S(\mathbb{R}^d; k) = L^2_\Omega(\mathbb{R}^d; \mathbb{R}^d) \cap H^s(\mathbb{R}^d; \mathbb{R}^d)$. We denote this set by $H^s_\Omega(\mathbb{R}^d; \mathbb{R}^d)$. We also denote

$$H^{2s}_{loc}(\Omega; \mathbb{R}^d) = \{\mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \eta\mathbf{u} \in H^{2s}(\mathbb{R}^d; \mathbb{R}^d), \quad \forall \eta \in C^\infty_c(\Omega)\}.$$

We also need the following potential spaces. For $\mathbf{u} \in \mathcal{S}'$, the space of tempered distributions, then

$$\mathcal{L}^{s,p}(\mathbb{R}^d) := \{\mathbf{u} \in \mathcal{S}' : \mathcal{F}^{-1}[(1+|\boldsymbol{\xi}|^2)^{s/2}\mathcal{F}(\mathbf{u})] \in L^p(\mathbb{R}^d;\mathbb{R}^d)\},$$

where $1 < p < \infty$ and $s \geq 0$. The space is equivalent to $\{\mathbf{u} \in \mathcal{S}' : (-\Delta)^{s/2}\mathbf{u} \in L^p\}$, where $(-\Delta)^{s/2}$ is the standard fractional Laplacian operator applied componentwise. We also denote

$$\mathcal{L}^{s,p}_{loc}(\Omega;\mathbb{R}^d) := \{\mathbf{u} \in L^p(\Omega;\mathbb{R}^d) : \eta\mathbf{u} \in \mathcal{L}^{s,p}(\mathbb{R}^d;\mathbb{R}^d), \quad \forall \eta \in C_c^\infty(\Omega)\}.$$

When $p = 2$, $\mathcal{L}^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d;\mathbb{R}^d)$.

The following theorem contains one of the results of this section.

Theorem 3.1. *Assume that k_r has the form (24). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $\mathbf{f} \in L^2_\Omega(\mathbb{R}^d;\mathbb{R}^d)$ and $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$ be the unique weak solution to the system*

$$\begin{cases} \mathbb{L}\mathbf{u} = \mathbf{f} & \text{on } \Omega \\ \mathbf{u} = 0 & \text{on } \complement\Omega. \end{cases} \tag{25}$$

Then $\mathbf{u} \in H^{2s}_{loc}(\Omega;\mathbb{R}^d)$. Moreover, for any $\eta \in C_c^\infty(\Omega)$, there exists a constant $C > 0$ depending on η such that

$$|\eta\mathbf{u}|_{H^{2s}} \leq C\|\mathbf{f}\|_{L^2}.$$

Our second regularity result corresponds to the case when $\mathbf{f} \in L^p_\Omega(\mathbb{R}^d;\mathbb{R}^d)$ for $p \geq 2$. For this result, we only study $\mathbb{L}$ corresponding to $m = 1$ and $\Lambda = \mathbb{R}^d$. That is, k is the standard fractional kernel given by $k(x,y) = |x-y|^{-d-2s}$. To separate this special operator from generic ones, we introduce the notation $(-\mathring{\boldsymbol{\Delta}})^s$ to denote the matrix operator. That is,

$$(-\mathring{\boldsymbol{\Delta}})^s\mathbf{u} = P.V.\int_{\mathbb{R}^d} \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right) \frac{\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\, d\mathbf{y}.$$

Note that, here and above, we omit a normalizing constant depending on s and d in the intro-differential representation of the fractional Laplace-type operator $(-\mathring{\boldsymbol{\Delta}})^s$. We do not study the limit cases $s \nearrow 1$ or $s \searrow 0$.

Theorem 3.2. *Let $p \in [2,\infty)$, $\mathbf{f} \in L^p_\Omega(\mathbb{R}^d)$ and $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$ be the unique weak solution to the Dirichlet problem (25) with $\mathbb{L}$ replaced by $(-\mathring{\boldsymbol{\Delta}})^s$. Then $\mathbf{u} \in \mathcal{L}^{2s,p}_{loc}(\Omega)$ provided*

a) $2 \leq p \leq 2^{*_s}$, *where* $2^{*_s} := \frac{2d}{d-2s}$;
or
b) $p > 2^{*_s}$ *and* $\mathbf{u} \in L^p(\Omega;\mathbb{R}^d)$.

As we described in the introduction, to prove Theorem 3.1 and Theorem 3.2 we follow an argument used in [2], where a similar but more general result is proved for the Dirichlet problem for the fractional Laplacian equation when the right-hand side comes from L^p for any $1 < p < \infty$. The argument relies on an optimal regularity result for weak solutions of the same system posed on the entire space. Multiplying the weak solution of the Dirichlet problem by a cutoff function, the product becomes a weak solution of a system of equations posed on $\mathbb{R}^d$ with a perturbed right-hand side. The task is then to show that the perturbed force term lives in the same space as the original right-hand side function. In implementing the strategy of [2] to our case, although the cutoff function argument remains the same, we have to demonstrate the optimal regularity result for weak solutions of the strongly coupled system in $\mathbb{R}^d$. For strong solutions of nonlocal equations defined on $\mathbb{R}^d$, optimal regularity is obtained in [9].

We should mention that, for the scalar case, the result [2, Theorem 1.4] does not require $\mathbf{u}$ be in $L^p(\Omega)$ for large p as we do in Theorem 3.2. Roughly speaking, they prove that a solution to the Dirichlet problem of the fractional Laplacian with right-hand side in L^p must also be in L^p, see [2, Lemma 2.5]. A similar Calderón-Zygmund type estimate is also proved in [22, Theorem 16]. Unfortunately we are unable to extend their proof to the vector-valued case because the argument in [2] relies on a monotonicity property of an associated semigroup and in the result in [22] uses a Moser-type argument where a function that is a power of the solution is used as a test function. Neither of these arguments can be applied for systems.

3.2. **Interior H^{2s} regularity for the Dirichlet problem of the system of equations.** Now we turn to the main point. We recall that for a given $\mathbf{f} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, we say that $\mathbf{u} \in S(\mathbb{R}^d;k)$ is a weak solution to $\mathbb{L}\mathbf{u} = \mathbf{f}$ in $\mathbb{R}^d$ if for any $\boldsymbol{\psi} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, we have

$$\langle \mathbb{L}\mathbf{u}, \boldsymbol{\psi}\rangle := \iint_{\mathbb{R}^d\,\mathbb{R}^d} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\boldsymbol{\psi})(\mathbf{x},\mathbf{y})k(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{26}$$

The following lemma gives an optimal regularity result for weak solutions of the system.

Lemma 3.3. *Assume that k_r has the form (24). Suppose that $\mathbf{f} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. Let $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{R}^d)$ be a weak solution to the system of nonlocal equations*

$$\mathbb{L}\mathbf{u} = \mathbf{f}, \quad \text{in } \mathbb{R}^d.$$

Then $\mathbf{u} \in H^{2s}(\mathbb{R}^d;\mathbb{R}^d)$, and $|\mathbf{u}|_{H^{2s}} \le c\,(\|\mathbf{f}\|_{L^2} + \|\mathbf{u}\|_{L^2})$ for some constant c depending only on r, s, d, and Λ.

Remark 3.1. Note that [5] establishes very similar regularity results.

Proof. Let $\boldsymbol{\psi} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Then iterating the integral in (26) and changing variables we have that

$$\int_{\mathbb{R}^d} k_r(\mathbf{h}) \int_{\mathbb{R}^d} \left((\mathbf{u}(\mathbf{x}+\mathbf{h}) - \mathbf{u}(\mathbf{x}))\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right)\left((\boldsymbol{\psi}(\mathbf{x}+\mathbf{h}) - \boldsymbol{\psi}(\mathbf{x}))\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{h}$$
$$= \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

We apply the Fourier transform and Plancherel theorem to rewrite the above integral in the frequency space as

$$\int_{\mathbb{R}^d} (\mathbb{M}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}))\cdot\hat{\boldsymbol{\psi}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^d} \hat{\mathbf{f}}(\boldsymbol{\xi})\cdot\hat{\boldsymbol{\psi}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

where $\mathbb{M}_r(\boldsymbol{\xi})$ is the matrix of Fourier symbols given by

$$\mathbb{M}_r(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} k_r(\mathbf{h})(e^{\imath 2\pi\boldsymbol{\xi}\cdot\mathbf{h}} - 1)^2\frac{\mathbf{h}}{|\mathbf{h}|}\otimes\frac{\mathbf{h}}{|\mathbf{h}|}\,\mathrm{d}\mathbf{h}.$$

We now use the density of the Fourier transform of $C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ in $L^2(\mathbb{R}^d;\mathbb{R}^d)$ to conclude that

$$\mathbb{M}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{f}}(\boldsymbol{\xi}), \quad \text{almost everywhere in } \mathbb{R}^d. \tag{27}$$

Let us write $k_r = k - \tilde{k}_r$ where $k(\mathbf{z}) = \frac{m(\mathbf{z})}{|\mathbf{z}|^{d+2s}}\chi_\Lambda(\mathbf{z})$. Notice that $\tilde{k}_r$ is supported outside of the ball B_r. If we denote the matrix of symbols by $\mathbb{M}$ and $\tilde{\mathbb{M}}_r$, we have that

$$\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \tilde{\mathbb{M}}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) + \hat{\mathbf{f}}(\boldsymbol{\xi}), \quad \text{almost everywhere in } \mathbb{R}^d. \tag{28}$$

To estimate the relevant norms of $\mathbf{u}$, let us first estimate the eigenvalues of the matrix $\mathbb{M}(\boldsymbol{\xi})$. To that end, for any $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$, noting the form of k we have that

$$\begin{aligned}(\mathbb{M}(\boldsymbol{\xi})\boldsymbol{\eta})\cdot\boldsymbol{\eta} &= \int_{\mathbb{R}^d} k(\mathbf{h})(e^{\imath 2\pi\boldsymbol{\xi}\cdot\mathbf{h}} - 1)^2 \left|\boldsymbol{\eta}\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h} \\ &\geq 2\alpha_1 \int_\Lambda \frac{(1-\cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))}{|\mathbf{h}|^{d+2s}} \left|\boldsymbol{\eta}\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h} \\ &\geq 2\alpha_1 \Psi_{min}|\boldsymbol{\xi}|^{2s},\end{aligned}$$

where the last inequality is from the proof of Lemma 2.5. As a consequence the eigenvalues of the matrix function $|\boldsymbol{\xi}|^{-2s}\mathbb{M}(\boldsymbol{\xi})$ are uniformly bounded from below by a positive number. We also note that since $\mathbb{M}(\boldsymbol{\xi})$ is symmetric and positive definite for each $\boldsymbol{\xi}$, the eigenvalues of the square of $\mathbb{M}(\boldsymbol{\xi})$ are precisely the squares of the eigenvalues of $\mathbb{M}(\boldsymbol{\xi})$. It then follows that for any vector $\mathbf{w}$

$$|\mathbb{M}(\boldsymbol{\xi})\mathbf{w}|^2 = \mathbb{M}(\boldsymbol{\xi})\mathbf{w}\cdot(\mathbb{M}(\boldsymbol{\xi})\mathbf{w}) = \mathbb{M}(\boldsymbol{\xi})\mathbb{M}(\boldsymbol{\xi})\mathbf{w}\cdot\mathbf{w} = |\mathbf{w}|^2 \min_\beta\{\beta(\boldsymbol{\xi})^2\},$$

where the minimum is taken over the eigenvalues $\beta(\boldsymbol{\xi})$ of $\mathbb{M}(\boldsymbol{\xi})$. We conclude that there exists a positive number α_0 that depends only on α_1, s, Λ, such that for all vectors $\boldsymbol{\xi}$ and $\mathbf{w}$ in $\mathbb{R}^d$ we have that

$$|\mathbb{M}(\boldsymbol{\xi})\mathbf{w}|^2 \geq \alpha_0(|\boldsymbol{\xi}|^{2s}|\mathbf{w}|)^2.$$

We now easily see from (27) that

$$\begin{aligned}\|(-\Delta)^s\mathbf{u}\|_{L^2(\mathbb{R}^d)} = \||\boldsymbol{\xi}|^{2s}\hat{\mathbf{u}}\|_{L^2} &\leq \alpha_0\|\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}\|_{L^2} \\ &= \alpha_0\left(\|\hat{\mathbf{f}}\|_{L^2(\mathbb{R}^d)} + +\|\tilde{\mathbb{M}}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi})\|^2_{L^2}\right) \\ &\leq \alpha_0\left(\|\hat{\mathbf{f}}\|_{L^2(\mathbb{R}^d)} + \beta(r)\|\hat{\mathbf{u}}\|^2_{L^2}\right),\end{aligned}$$

where in the last inequality $\beta(r)$ is from Lemma 2.5. Thus, since we already know that $\mathbf{u} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, we get that $\mathbf{u} \in \mathcal{L}^{2s,2}(\mathbb{R}^d)$. ☐

Proof of Theorem 3.1. Let $\Omega_1 \Subset \Omega_2 \Subset \Omega$. Let $\eta \in C_c^\infty(\Omega_2)$ be a real-valued function such that

$$\eta(\mathbf{x}) \equiv 1,\ \mathbf{x}\in\Omega_1, \quad \eta(\mathbf{x})\in[0,1],\, \mathbf{x}\in\Omega_2\setminus\Omega_1, \quad \text{and}\, \eta(\mathbf{x}) = 0,\ \mathbf{x}\in\mathbb{R}^d\setminus\Omega_2.$$

Let $\mathbf{f} \in L^2_\Omega(\mathbb{R}^d;\mathbb{R}^d)$ and let $\mathbf{u} \in H_\Omega(\mathbb{R}^d;k)$ be the unique weak solution to the Dirichlet problem (25). Notice that because of the form of k, $\mathbf{u}$ is in fact in $H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d)$. Now, it is clear that the function $\eta\mathbf{u} \in H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d)$. Using the identity

$$\mathcal{D}(\mathbf{u}\eta)(\mathbf{y},\mathbf{x}) = (\eta(\mathbf{x})-\eta(\mathbf{y}))\mathbf{u}(\mathbf{x})\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} + \eta(\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) + (\eta(\mathbf{y})-\eta(\mathbf{x}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})$$

we see that for every $\mathbf{v} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

$$\mathcal{F}^k(\eta\mathbf{u},\mathbf{v}) - \mathcal{F}^k(\mathbf{u},\eta\mathbf{v}) = ([\mathbb{L}\eta]\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)} - (I_s(\mathbf{u},\eta),\mathbf{v})_{L^2(\mathbb{R}^d)}, \tag{29}$$

where for almost all $\mathbf{x} \in \mathbb{R}^d$ the vector valued function is

$$I_s(\mathbf{u}, v)(\mathbf{x}) = \int_{\mathbb{R}^d} k(\mathbf{y}-\mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y},$$

which is finite via Hölder's inequality. In the above and hereafter we suppress the dependence of k on r. The matrix valued function $\mathbb{L}\eta(\mathbf{x})$ is given by

$$\mathbb{L}\eta(\mathbf{x}) = P.V.\int_{\mathbb{R}^d} k(\mathbf{x}-\mathbf{y})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}.$$

Let us justify the L^2 inner products in the right-hand side of (29). To that end, we introduce the vector field

$$\mathbf{g} := (\mathbb{L}\eta)\mathbf{u} - I_s(\mathbf{u}, \eta),$$

and show that $\mathbf{g} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. In fact, we also show that there exists a constant $C > 0$ independent of $\mathbf{u}$ (but depending on η) such that

$$\|\mathbf{g}\|_{L^2(\mathbb{R}^d)} \le C\,\|\mathbf{u}\|_{H^s}. \tag{30}$$

The rest of the argument is similar to that given in [2] adjusted for the system case. We include it here for clarity and completeness. We begin by noting that $\mathbb{L}\eta$ is uniformly bounded in $\mathbb{R}^d$. Indeed, using the fact that $\eta \in C_c(\mathbb{R}^d)$ and k is even, we can easily show that $\|\mathbb{L}\eta\|_\infty \le C(\|D^2\eta\|_{L^\infty} + \|\eta\|_{L^\infty}.)$ As a consequence of this and the Poincaré-Korn inequality, since $\mathbf{u} \in H^s_\Omega(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\|(\mathbb{L}\eta)\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \le \|\mathbb{L}\eta\|^2_{L^\infty(\mathbb{R}^d)}\,\|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \le C|\mathbf{u}|^2_{H^s}. \tag{31}$$

To bound the L^2 norm of $I_s(\mathbf{u}, \eta)(\mathbf{x})$, we begin by breaking the region of integration as

$$\begin{aligned}
I_s(\mathbf{u}, \eta)(\mathbf{x}) &= \int_{\Omega} k(\mathbf{y}-\mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} \\
&\quad + \int_{\mathbb{R}^d\setminus\Omega} k(\mathbf{y}-\mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} \\
&= \int_{\Omega} k(\mathbf{y}-\mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} \\
&\quad + \eta(\mathbf{x})\int_{\mathbb{R}^d\setminus\Omega} k(\mathbf{y}-\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} \\
&:= I_1(\mathbf{x}) + I_2(\mathbf{x}).
\end{aligned}$$

Let us estimate the first integral $I_1(\mathbf{x})$. Using Cauchy-Schwarz,

$$|I_1(\mathbf{x})| \le \left(\int_{\Omega} k(\mathbf{y}-\mathbf{x})|\eta(\mathbf{x}) - \eta(\mathbf{y})|^2\,\mathrm{d}\mathbf{y}\right)^{1/2}\left(\int_{\Omega} k(\mathbf{y}-\mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2\,\mathrm{d}\mathbf{y}\right)^{1/2}.$$

Since Ω is bounded, taking $R = \mathrm{diam}(\Omega)$, we see that for any $\mathbf{x} \in \Omega$, $\Omega \subset B_{2R}(\mathbf{x})$. We now use the fact that η is smooth and the kernel is comparable with the fractional kernel to obtain that a constant $C > 0$ depending on η such that for any $\mathbf{x} \in \Omega$

$$\int_{\Omega} k(\mathbf{y}-\mathbf{x})|\eta(\mathbf{x})-\eta(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y} \le \|\nabla\eta\|_{L^\infty} \int_{B_{2R}(\mathbf{x})} k(\mathbf{y}-\mathbf{x})|\mathbf{y}-\mathbf{x}|^2 \,\mathrm{d}\mathbf{y}$$
$$= \|\nabla\eta\|_{L^\infty} \int_{B_{2R}(\mathbf{0})} k(\boldsymbol{\xi})|\boldsymbol{\xi}|^2 \,\mathrm{d}\boldsymbol{\xi} \le C.$$

For $\mathbf{x} \in \complement\Omega$, we use the fact $\eta(\mathbf{x}) = 0$, and $\mathrm{supp}(\eta) \subset \Omega_2$, and that $\delta = \mathrm{dist}(\Omega_2, \partial\Omega) > 0$ to conclude that

$$\int_{\Omega} k(\mathbf{y}-\mathbf{x})|\eta(\mathbf{x})-\eta(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y} = \int_{\Omega_2} k(\mathbf{y}-\mathbf{x})|\eta(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y} \le \|\eta\|^2_{L^\infty} \int_{\{|\boldsymbol{\xi}|>\delta\}} k(\boldsymbol{\xi}) \,\mathrm{d}\boldsymbol{\xi} \le C.$$

Using the two preceding estimates, we see that there exists a positive constant $C > 0$, that depends on η such that

$$\int_{\mathbb{R}^d} |I_1(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \le C \int_{\mathbb{R}^d}\int_{\Omega} k(\mathbf{y}-\mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \le C|\mathbf{u}|^2_{H^s}. \tag{32}$$

To estimate the L^2 norm of I_2, again using Cauchy-Schwarz we get that

$$|I_2(\mathbf{x})|^2 \le |\eta(\mathbf{x})|^2 \left(\int_{\mathbb{R}^d\setminus\Omega} k(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{y}\right)\left(\int_{\mathbb{R}^d\setminus\Omega} k(\mathbf{y}-\mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2 \,\mathrm{d}\mathbf{y}\right).$$

As before, since $\eta(\mathbf{x}) = 0$, and $\mathrm{supp}(\eta) \subset \Omega_2$, and that $\delta = \mathrm{dist}(\Omega_2, \partial\Omega) > 0$, we have that the function

$$\mathbf{x} \mapsto |\eta(\mathbf{x})|^2 \int_{\mathbb{R}^d\setminus\Omega} k(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{y}$$

is bounded. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} |I_2(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} &= \int_{\Omega_2} |I_2(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\ &\le C \int_{\Omega_2}\int_{\complement\Omega} k(\mathbf{y}-\mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\ &\le C|\mathbf{u}|^2_{H^s}.\end{aligned} \tag{33}$$

Therefore, the estimate (30) of $\mathbf{g}$ follows from (32) and (33). We have shown that $\eta\mathbf{u}$ is a weak solution to the equation

$$\mathbb{L}(\eta\mathbf{u}) = \mathbf{F} \quad \text{ in } \mathbb{R}^d,$$

where $\mathbf{F} = \eta\mathbf{f} + (\mathbb{L}\eta)\mathbf{u} - I_s(\mathbf{u},\eta) \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. By Lemma 3.3, $\eta\mathbf{u} \in H^{2s,2}(\mathbb{R}^d;\mathbb{R}^d)$. Thus $\mathbf{u} \in H^{2s,2}_{loc}(\Omega)$ and the proof is complete. □

3.3. **Interior $\mathcal{L}^{2s,p}$ regularity for $p > 2$.** In this section we prove Theorem 3.2. As before we start with an optimal regularity estimate for the system of equations posed on $\mathbb{R}^d$.

Lemma 3.4. *Let $p \in (1,\infty)$. For $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d) \cap L^2(\mathbb{R}^d;\mathbb{R}^d)$, if $\mathbf{u} \in H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d) \cap L^p(\mathbb{R}^d;\mathbb{R}^d)$ is a weak solution of*

$$(-\mathring{\mathbf{\Delta}})^s \mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{R}^d.$$

Then $\mathbf{u} \in \mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d)$, Moreover, there exists a constant $C = C(d,s) > 0$ such that

$$\|(-\mathring{\mathbf{\Delta}})^s \mathbf{u}\|_{L^p} \le C\|\mathbf{f}\|_{L^p}.$$

Proof. We proceed as in the proof of Lemma 3.3 to obtain that in the Fourier space the equation is $\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{f}}(\boldsymbol{\xi})$ almost everywhere, where $\mathbb{M}(\boldsymbol{\xi})$ is as given in (28) with k replaced by the kernel $\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}$. For the particular form of the kernel, we can explicitly compute the matrix of symbols. $\mathbb{M}(\boldsymbol{\xi})$ is given by

$$\mathbb{M}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2s}\left(\ell_1 \mathbb{I} + \ell_2 \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \otimes \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right),$$

where ℓ_1 and ℓ_2 are positive numbers given by the formula $\ell_i = \int_{\mathbb{R}^d} \frac{1-\cos(2\pi h_1)}{|\mathbf{h}|^{d+2s}} \frac{h_i^2}{|\mathbf{h}|^2}\, \mathrm{d}\mathbf{h}$, for $i = 1,2$, and $\mathbb{I}$ is the $d \times d$ identity matrix. The matrix $\ell_1\mathbb{I} + \ell_2 \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \otimes \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ is invertible for any $\boldsymbol{\xi} \ne \mathbf{0}$, with

$$\left(\ell_1 \mathbb{I} + \ell_2 \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \otimes \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)^{-1} = \left(\frac{1}{\ell_1}\mathbb{I} - \frac{\ell_2}{\ell_1(\ell_1+\ell_2)} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \otimes \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right).$$

Using this formula, we can rewrite $\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{f}}(\boldsymbol{\xi})$ as

$$|\boldsymbol{\xi}|^{2s}\hat{\mathbf{u}}(\boldsymbol{\xi}) = \left(\frac{1}{\ell_1}\mathbb{I} - \frac{\ell_2}{\ell_1(\ell_1+\ell_2)} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \otimes \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)\hat{\mathbf{f}}(\boldsymbol{\xi}).$$

The conclusion of the lemma now follows from the assumption that $\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ and for any real numbers a and b, the matrix multiplier $a\mathbb{I} + b\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \otimes \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ is a L^p-multiplier for any $1 < p < \infty$. The latter follows immediately from the L^p-boundedness of the Riesz Transforms. □

We use this regularity theorem to prove the second main result of this section. Let us begin by reviewing the standard fractional Sobolev spaces. For $p \in [1,\infty)$, Ω an open subset of $\mathbb{R}^d$ and $s \in (0,1)$, we define

$$W^{s,p}(\Omega;\mathbb{R}^d) := \left\{\mathbf{u} \in L^p(\Omega) \;:\; \iint_{\Omega\,\Omega} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} < \infty\right\}.$$

With norm $\|\mathbf{u}\|^p_{W^{s,p}} := \|\mathbf{u}\|_{L^p} + \int_\Omega\int_\Omega \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}$, it is well known that $W^{s,p}(\Omega;\mathbb{R}^d)$ is a Banach space.

If $s > 1$, then let we write $s = m + \sigma$, where m is the largest integer less than s, and define

$$W^{s,p}(\Omega;\mathbb{R}^d) = \{\mathbf{u} \in W^{m,p}(\Omega;\mathbb{R}^d) : D^\alpha \mathbf{u} \in W^{\sigma,p}(\Omega;\mathbb{R}^d), \quad |\alpha| = m\}.$$

With the norm $\|\mathbf{u}\|^p_{W^{s,p}} := \|\mathbf{u}\|_{W^{m,p}} + \sum_{|\alpha|=m} \|D^\alpha \mathbf{u}\|_{W^{\sigma,p}}$, the space $W^{s,p}(\Omega;\mathbb{R}^d)$ is known to be a Banach space.

We also need the Sobolev embedding result

$$W^{r,p}(\mathbb{R}^d;\mathbb{R}^d) \hookrightarrow W^{s,q}(\mathbb{R}^d;\mathbb{R}^d),$$

provided $0 < s \leq r$, $1 < p \leq q < \infty$, and $r - \frac{d}{p} = s - \frac{d}{q}$. If Ω is open and bounded with smooth enough boundary then

$$W^{r,p}(\Omega;\mathbb{R}^d) \hookrightarrow W^{s,q}(\Omega;\mathbb{R}^d),$$

provided $0 < s \leq r$, $1 < p \leq q < \infty$, and $r - \frac{d}{p} \geq s - \frac{d}{q}$. Let us recall the relation between the potential spaces and the fractional Sobolev spaces, [37, Chapter 5, Theorem 5]. For our purpose it suffices to recall

$$\mathcal{L}^{s,p}(\mathbb{R}^d;\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d;\mathbb{R}^d), \text{ if } p \geq 2.$$

For $p = 2$, the spaces are the same; $H^s(\mathbb{R}^d;\mathbb{R}^d) = \mathcal{L}^{s,2}(\mathbb{R}^d;\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$.

Proof of Theorem 3.2. Let $\mathbf{f} \in L^p_\Omega(\mathbb{R}^d)$. Since $p \geq 2$, and Ω is bounded, $\mathbf{f} \in L^2_\Omega(\mathbb{R}^d)$. Therefore a unique weak solution $\mathbf{u}$ in $H^s_\Omega(\mathbb{R}^d)$ exists. Let η be the cutoff function constructed in the proof of Theorem 3.1. Then we have that $\eta\mathbf{u} \in H^{2s}(\mathbb{R}^d;\mathbb{R}^d) = W^{2s,2}(\mathbb{R}^d)$. By Sobolev Embedding, $\eta\mathbf{u} \in W^{s,2^{*s}}(\mathbb{R}^d;\mathbb{R}^d)$. Now, let ω_1, ω_2 be open sets such that $\omega \Subset \omega_1 \Subset \omega_2 \Subset \Omega$. The cutoff function η was arbitrary, so therefore $\mathbf{u} \in W^{s,2^{*s}}(\omega_2;\mathbb{R}^d)$.

Part a. ($p \leq 2^{*s}$) Since ω_2 is bounded and $p \leq 2^{*s}$, again by Sobolev Embedding $\mathbf{u} \in W^{s,p}(\omega_2;\mathbb{R}^d)$. Moreover, since $\mathbf{u} \in H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d)$, we have that $\mathbf{u} \in W^{s;2}(\Omega) \hookrightarrow L^p_\Omega(\mathbb{R}^d)$. In summary, $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{R}^d) \cap W^{s,p}(\omega_2;\mathbb{R}^d) \cap L^p_\Omega(\mathbb{R}^d;\mathbb{R}^d)$. Now we proceed as in the proof of Theorem 3.1. With the same reasoning, $\eta\mathbf{u}$ is a weak solution of

$$(-\mathring{\mathbf{\Delta}})^s(\eta\mathbf{u}) = \mathbf{F} \quad \text{in } \mathbb{R}^d,$$

where $\mathbf{F} = \eta\mathbf{f} + ((-\mathring{\mathbf{\Delta}})^s\eta)\mathbf{u} - I_s(\mathbf{u},\eta)$. Let $\mathbf{g} := ((-\mathring{\mathbf{\Delta}})^s\eta)\mathbf{u} - I_s(\mathbf{u},\eta)$. We have shown already that $\mathbf{g} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. Noting that $\eta\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{R}^d) \cap L^p_\Omega(\mathbb{R}^d;\mathbb{R}^d)$, we can now apply Lemma 3.4 to conclude that $\eta\mathbf{u} \in \mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d)$ provided we successfully show $\mathbf{g} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. In fact, we demonstrate that for some constant $C > 0$ independent of $\mathbf{u}$

$$\|\mathbf{g}\|_{L^p(\mathbb{R}^d)} \leq C(\|\mathbf{u}\|_{W^{s,p}(\omega_2;\mathbb{R}^d)} + \|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^d)}). \tag{34}$$

The last estimate follows from a similar argument as in Theorem 3.1. We sketch its proof. More detail can be found in [2]. As before, the matrix function $(-\mathring{\mathbf{\Delta}})^s\eta \in L^\infty(\mathbb{R}^d)$. Thus,

$$\begin{aligned}\left\|(-\mathring{\mathbf{\Delta}})^s\eta)\mathbf{u}\right\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\frac{\eta(\mathbf{x})-\eta(y)}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathbf{u}(\mathbf{x})\right|^p \mathrm{d}\mathbf{x}\\ &\leq \left\|(-\mathring{\mathbf{\Delta}})^s\eta\right\|^p_{L^\infty(\mathbb{R}^d)}\|\mathbf{u}\|^p_{L^p(\Omega)}.\end{aligned}$$

The second term $I_s(\mathbf{u},\eta)$ can also be dealt with in the same way as in the proof of Theorem 3.1. We begin by breaking the integral as

$$\begin{aligned}I_s(\mathbf{u},\eta)(\mathbf{x}) &= \int_{\omega_1}\frac{\eta(\mathbf{x})-\eta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\mathcal{D}\mathbf{u}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} + \eta(\mathbf{x})\int_{\mathbb{R}^d\setminus\omega_1}\frac{\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}\\ &:= I_1(\mathbf{x}) + I_2(\mathbf{x}).\end{aligned}$$

We estimate $I_1(\mathbf{x})$: Using Hölder's inequality with conjugate $p' = p/(p-1)$,

$$|I_1(\mathbf{x})| \leq \int_{\omega_1}\frac{|\eta(\mathbf{x})-\eta(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\left|(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right|\mathrm{d}\mathbf{y}$$

$$\leq \left(\int_{\omega_1} \frac{|\eta(\mathbf{x})-\eta(\mathbf{y})|^{p'}}{|\mathbf{x}-\mathbf{y}|^{d+sp'}}\,\mathrm{d}\mathbf{y}\right)^{1/p'}$$
$$\times \left(\int_{\omega_1} \left|(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right|^p \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{y}\right)^{1/p},$$

from which we get that

$$\int_{\mathbb{R}^d} |I_1(\mathbf{x})|^p\,\mathrm{d}\mathbf{x} \leq C\left(\|\mathbf{u}\|^p_{W^{s,p}(\omega_2)} + \|\mathbf{u}\|^p_{L^p(\Omega)}\right). \tag{35}$$

Similarly, we also get that

$$\int_{\mathbb{R}^d} |I_2(\mathbf{x})|^p\,\mathrm{d}\mathbf{x} \leq C\,\|\mathbf{u}\|^p_{L^p(\Omega)}. \tag{36}$$

Therefore, the estimate (34) of $\mathbf{g}$ follows from (35) and (36).

Part b. ($p > 2^{*_s}$) From Part a) we have that $\mathbf{u} \in W^{2s,2^{*_s}}_{loc}(\Omega;\mathbb{R}^d)$, and so $\mathbf{u} \in W^{2s,2^{*_s}}(\omega_2;\mathbb{R}^d)$. By Sobolev Embedding we have that $\mathbf{u} \in W^{s,q_1}(\omega_2;\mathbb{R}^d)$ with $q_1 = \min\left\{p, \frac{N2^{*_s}}{N-s2^{*_s}}\right\} = \min\left\{p, \frac{2N}{N-4s}\right\}$. By assumption $\mathbf{u} \in L^{q_1}(\Omega;\mathbb{R}^d)$. With this information, we can now repeat the argument in Part a) to conclude that $\mathbf{u} \in W^{2s,q_1}_{loc}(\Omega;\mathbb{R}^d)$. Now, if $2 \leq p \leq \frac{2N}{N-4s}$, the proof is completed. Otherwise we iterate the above procedure to obtain $\mathbf{u} \in W^{2s,q_j}_{loc}(\Omega;\mathbb{R}^d)$ with $q_j = \min\left\{p, \frac{2N}{N-js}\right\}$, for all $j \geq 2$. For $p \geq 2^{*_s}$, we can now choose $j \in \mathbb{N}$ such that $2 \leq p \leq \frac{2N}{N-js}$. That completes the proof of the theorem. □

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