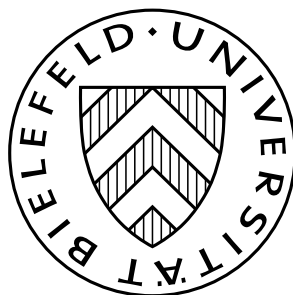**Working Papers**

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Existence of Financial Equilibria in Continuous Time with Potentially Complete Markets

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Abstract

We prove that in smooth Markovian continuous–time economies with potentially complete asset markets, Radner equilibria with endogenously complete markets exist.

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Introduction

The hallmark of economics is still the general theory of competitive markets as expressed masterfully in the work of Arrow and Debreu. While this theory can be considered as complete, its extension to competitive markets under uncertainty in continuous time remains still imperfect. In discrete models, it is well known that for potentially complete markets of real assets, one generically has a Radner equilibrium with endogenously generated

complete markets that implement the efficient allocation of the corresponding Arrow–Debreu equilibrium, see Magill and Shafer (1985) or Magill and Quinzii (1998), Theorem 25.7.

Anderson and Raimondo (2008) prove a version of this theorem for specific continuous–time economies where endowments and dividends are smooth functions of Brownian motion and time, and agents have time–separable expected utility functions. They establish their result with the help of non-standard analysis, an intriguing approach to analysis and stochastics via mathematical logic that allows, e.g., to work with infinitely large and infinitesimally small numbers, and to identify Brownian motion with a random walk of infinite length and infinitesimally small time steps. We believe that such an important theorem deserves a standard proof – we provide it here.

At the same time, we extend the result to more general classes of state variables. Many finance models nowadays rely on more general diffusions; prominent examples include the stochastic volatility models (Heston93 (1993)) , where the volatility of the risky asset is a mean–reverting process, term structure models like Vasicek (1977) or more generally affine term structure models as in Duffie, Pan, and Singleton (2000). It is thus important to have sound equilibrium foundations for such models as well.

The paper is set up as follows. The next section describes a smooth continuous–time Markov economy where all relevant functions are analytic on the open interior of their domain. In this paper, the term "analytic" (=real analytic) refers to infinitely differentiable functions that can be written locally as an infinite power series[1]. Then, we formulate our main theorem on existence of a Radner equilibrium with endogenously dynamically complete markets. The proof is split in several steps. We first recall Dana's (1993) result on existence of an Arrow–Debreu equilibrium and show that in our setup, allocation and prices are analytic functions of time and the state variable. The natural candidates for security prices are the expected present values of future dividends. We show that these can also be expressed as analytic functions of time and the state variable if natural assumptions on the coefficients of the diffusion are satisfied. On the one hand, if one has a closed–form version of the state variable's transition density, the result holds true. This is straightforward to check in the case of Brownian motion, or mean–reverting diffusions, e.g. From an abstract point of view, it is better to have conditions on the primitive of the model that ensure such a nice transition density. We state sufficient conditions on the drift and dispersion coefficients of our state variable for such a result.

The analyticity of security prices allows us to extend the local indepen-

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[1]Our reference is Krantz and Parks (2002).

dence assumption on terminal dividends to security prices, proving dynamic completeness, as in Anderson and Raimondo (2008). The implementation of the Arrow–Debreu equilibrium as a Radner equilibrium is then standard.

1 A Diffusion Exchange Economy with Potentially Complete Asset Markets

In this section, we set up an exchange economy in continuous time where the relevant information is generated by a diffusion $X = (X_t)_{t \in [0,T]}$ with values in $\mathbb{R}^K$. It is well known that one needs at least $K + 1$ financial assets to span a dynamically complete market. We thus assume that this necessary condition is satisfied. The market is thus *potentially complete*. Below, we show that in sufficiently smooth economies a Radner equilibrium with dynamically complete markets exists.

1.1 The State Variables

Let W be a K–dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by W augmented by the null sets. We assume that the relevant economic information can be described by the state of a diffusion process X with values in $\mathbb{R}^K$ given by

$$X_0 = x, dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad (1)$$

for an initial state $x \in \mathbb{R}^K$ and measurable functions

$$b : \mathbb{R}^K \to \mathbb{R}^K$$

and

$$\sigma : \mathbb{R}^K \to \mathbb{R}^{K \times K}$$

that are called the drift and dispersion function, resp. We let

$$a(x) := \sigma(x)\sigma(x)^T$$

be the diffusion matrix.

Assumption 1   *1. b and σ are Lipschitz–continuous: there exist $L, M > 0$ such that for all $x, y \in \mathbb{R}^K$*

$$\|b(x) - b(y)\| \leq L \|x - y\|, \|\sigma(x) - \sigma(y)\| \leq L \|x - y\|$$

2. *The diffusion matrix satisfies the uniform ellipticity condition*

$$\|x \cdot a(x)x\| \geq \epsilon \|x\|^2 \tag{2}$$

for some $\epsilon > 0$.

Part 1 of the assumption ensures that the stochastic differential equation has a unique strong solution and so our state variable is well–defined. The uniform ellipticity condition (2) ensures that there is enough volatility in every state and the diffusion does not degenerate to a locally deterministic process; in particular, it ensures that the distribution of X has full support, see Stroock and Varadhan (1972).

We shall need below that our candidate security prices are analytic functions of time and the state variable. We present two ways to establish this analyticity. An easy approach is just to impose analyticity for the transition density of the state variable.

Assumption 2 *The Markov process X has a transition density $P[X_{s+t} \in dy|X_s = x] = p(t, x, y)\, dy$ for a continuous function*

$$p : (0, T] \times \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}_+$$

that is analytic on $(0, T) \times \mathbb{R}^K \times \mathbb{R}^K$. Moreover, the transition density p is bounded on $(\eta, T] \times \mathbb{R}^K \times \mathbb{R}^K$ for all $\eta > 0$.

In a number of applications, the transition density is explicitly known and it it then easy to verify the above assumption.

Example 3   *1. If $X = W$, the transition density is explicitly given by*

$$p(t, x, y) = \phi(t, x, y) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\|y - x\|^2}{2t}\right),$$

which remains bounded for times $t \geq \eta > 0$ and is analytic on $(0, T) \times \mathbb{R}^K \times \mathbb{R}^K$ as a composition of analytic functions.

2. *An important stationary state variable is the Ornstein–Uhlenbeck process with $b(x) = a - bx$ and $\sigma(x) = \sigma x$ for constants a and $b, \sigma \neq 0$. It has normal transition density*

$$p(t, x, y) = \phi\left(e^{-bt}(x - \frac{a}{b}), \frac{\sigma^2}{2b}\left(1 - e^{-2bt}\right)\right),$$

which satisfies again Assumption 2.

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On the other hand, we find it important to have also a quite general result at hand where the above assumption is a consequence of assumptions on the drift and dispersion coefficients.

Assumption 4 *b and σ as well as its derivatives are bounded, Hölder–continuous, and analytic functions.*

In fact, we have

Lemma 5 *Assumption 4 implies Assumption 2.*

The proof of this lemma is in the appendix.

1.2 Commodities and Agents

There is one physical commodity in the economy. Our agents consume a flow $(c_t)_{0\leq t<T}$ and a lump-sum c_T of that commodity at terminal time T. We introduce the measure $\nu = dt \otimes \delta_T$, the product of the Lebesgue measure on $[0,T]$ and the Dirac measure on $\{T\}$. This allows us to model the consumption plans succinctly as one process $c = (c_t)_{0\leq t\leq T}$ in the following way. The commodity space $\mathcal{X}$ consists of p–integrable consumption rate processes and a p–integrable terminal lump sum consumption for some $p \geq 1$,

$$\mathcal{X} = L^p\left(\Omega \times [0,T], \mathcal{O}, P \otimes \nu\right).$$

The consumption set is the positive cone $\mathcal{X}_+$. We will use occasionally the dual space of $\mathcal{X}$ that we shall call the price space

$$\Psi = L^q\left(\Omega \times [0,T], \mathcal{O}, P \otimes \nu\right)$$

for q with $1/q + 1/p = 1$.

There are $i = 1, \ldots, I$ agents with time–separable expected utility preferences of the form

$$U^i(c) = \mathbb{E}\int_0^T u^i\left(t, c_t\right)\nu(dt)$$

for a period utility function

$$u^i : [0,T] \times \mathbb{R}_+ \to \mathbb{R}.$$

Assumption 6 *The period utility functions u^i are continuous on $[0,T]\times\mathbb{R}_+$ and analytic on $(0,T)\times\mathbb{R}_{++}$. They are differentiably strictly increasing and differentiably strictly concave in consumption on $[0,T]\times\mathbb{R}_{++}$, i.e.*

$$\frac{\partial u^i}{\partial c}(t,c) > 0, \frac{\partial^2 u^i}{\partial c^2}(t,c) > 0.$$

They satisfy the Inada conditions

$$\lim_{c\downarrow 0} \frac{\partial u^i}{\partial c}(t,c) = \infty$$

and

$$\lim_{c\to\infty} \frac{\partial u^i}{\partial c}(t,c) = 0$$

uniformly in $t \in [0,T]$.

Each agent comes with a $P \otimes \nu$–strictly positive entitlement[2] $e^i \in \mathcal{X}_+$ that can be written as a function of the state variables:

$$e^i_t = e^i(t, X_t)$$

for continuous functions $e^i : [0,T] \times \mathbb{R}^K \to \mathbb{R}$, $i = 1, \ldots, I$.

Assumption 7 *The functions e^i are analytic on $(0,T) \times \mathbb{R}^K$.*

1.3 The Financial Market

There are $K+1$ financial assets. These are *real* assets in the sense that they pay dividends in terms of the underlying physical commodity. The assets' dividends can be written as

$$A^k_t = g^k(t, X_t), t \in [0,T]$$

for continuous functions $g^k : [0,T] \times \mathbb{R}^K \to \mathbb{R}_+$, $k = 0, \ldots, K$. As for consumption processes, we interpret dividends as a flow on $[0,T)$ plus a lump sum payment at time T.

Assumption 8 *The dividends belong to the consumption set, $A^k \in \mathcal{X}_+$. The functions g^k are analytic on $(0,T) \times \mathbb{R}^K$. Asset 0 is a real zero–coupon bond with maturity T, i.e. it has no intermediate dividends, i.e. $A^0_t = 0$ for $t < T$.[3]*

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[2]We use the word "entitlement" here to distinguish it from the total initial endowment used below which is the sum of the entitlement and the dividends of assets initially owned by the agent.

[3]We can also work with intermediate dividends. In that case, an additional small detour is necessary in order to construct a suitable numéraire asset. As this part is not at the heart of the present analysis, we do not present this generalization here. The argument is available from the authors.

Agent i owns initially $n_k^i \geq 0$ shares of asset k. Without any trade, the agent is thus endowed with his individual endowment

$$\varepsilon_t^i = e_t^i + n^i \cdot A_t \,.$$

We denote by $N_k = \sum_{i=1}^{I} n_k^i$ the total number of shares in asset k. The aggregate endowment of agents is then

$$\varepsilon_t = \sum_{i=1}^{I} e_t^i + \sum_{k=0}^{K} N_k A_t^k = \sum_{i=1}^{I} \varepsilon_t^i \,.$$

A consumption price process is a positive Itô process ψ. A (cum–dividend) security price for asset k is a nonnegative Itô process $S^k = \left(S_t^k\right)_{0 \leq t \leq T}$. We interpret S^k as the nominal price of the asset k. We denote by

$$G_t^k = S_t^k + \int_{[0,t)} A_s^k \psi_s \nu(ds), \qquad (0 \leq t \leq T)$$

the (nominal) gain process for asset k. Note that by no arbitrage we must have $S_T^k = A_T^k$ at maturity.

A portfolio process is a predictable process θ with values in $\mathbb{R}^{K+1}$ that is G–integrable, i.e. the stochastic integrals $\int_0^t \theta_u^k dG_u^k$ are well–defined. The value of such a portfolio is $V_t = \theta \cdot S$.

We call a portfolio admissible (without reference to an agent) if its value process is bounded below by a martingale. This admissibility condition rules out doubling strategies[4].

A portfolio is admissible for agent i if its present value plus the present value of the agent's endowment is nonnegative, or

$$V_t + \mathbb{E}\left[\left. \int_{t+}^{T} e_s^i \psi_s \nu(ds) \right| \mathcal{F}_t \right] \geq 0 \,.$$

Note that this implies $V_T \geq 0$ for the terminal value of the portfolio.

A portfolio θ finances a consumption plan $c \in \mathcal{X}_+$ for agent i if θ is admissible for agent i and the intertemporal budget constraint is satisfied for the associated value process V:

$$V_t = n^i \cdot S_0 + \int_0^t \theta_u dG_u + \int_0^t \left(e_u^i - c_u\right) \psi_u \nu(du) \,.$$

---

[4]Anderson and Raimondo use a martingale condition to rule out such strategies. This requires to impose a martingale condition on potential security prices. As this martingale property is a consequence of equilibrium, we prefer not to impose this assumption ex ante. Nevertheless, either way works here.

We then call the portfolio/consumption pair (θ, c) i–feasible. More generally, we say that a portfolio θ finances a net consumption plan $z \in \mathcal{X}$ if its value process satisfies

$$V_t = V_0 + \int_0^t \theta_u dG_u + \int_0^t \left(e_u^i - c_u \right) \psi_u \nu(du).$$

A Radner equilibrium consists of asset prices S, a consumption price ψ, portfolios θ^i and consumption plans $c^i \in \mathcal{X}_+$ for each agent i such that θ^i is admissible for agent i and finances c^i, c^i maximizes agent i's utility over all such i–feasible portfolio/consumption pairs, and markets clear, i.e. $\sum_{i=1}^I c^i = e$ and $\sum_{i=1}^I \theta^i = N$.

Our way to a Radner equilibrium with dynamically complete markets will lead over the intermediate step of an Arrow–Debreu equilibrium. For the existence of such an equilibrium, the following assumption is, in general, necessary[5]:

Assumption 9 *For each agent, the marginal utility of his endowment belongs to the price space Ψ:*

$$\frac{\partial}{\partial c} u^i(t, \varepsilon_t^i) \in \Psi.$$

If the assets are linearly dependent, there is no hope to span a dynamically complete market. To exclude this, we follow Anderson and Raimondo (2008) and impose a full rank condition on terminal payoffs:

Assumption 10 *On a nonempty open set $V \subset \mathbb{R}^K$, the dividend of the zero–th asset is strictly positive at maturity,*

$$g^0(T, x) > 0, \qquad (x \in V).$$

---

[5]Assumption 9 cannot be weakened in general. Assume that there is only one agent. Then, to establish a no-trade equilibrium in the Arrow-Debreu sense, it is necessary to find a price $\psi \in L$ that separates the endowment e from the set $G = \{d \in L; U(d) \geq U(e)\}$ of consumption streams preferred to e. The only candidate in a smooth model like this one for such a price process is the marginal felicity $\frac{\partial}{\partial c} u(t, e_t)$. If it is not square-integrable, then there exists no equilibrium. For more on the necessity of Assumption 9, the reader may consult the overview of Mas-Colell and Zame (1991), especially Example 6.5, and the paper of Araujo and Monteiro (1991), where it is shown that an equilibrium does generically not exist if one does not have a condition on the integrability of marginal felicities like Assumption 9.

The functions $h^k : x \mapsto \frac{g^k(T,x)}{g^0(T,x)}$ are continuously differentiable on V for $k = 1, \ldots, K$ and the Jacobian matrix

$$Dh(x) = \begin{pmatrix} \frac{\partial h^1(T,x)}{\partial x_1} & \cdots & \frac{\partial h^1(T,x)}{\partial x_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial h^K(T,x)}{\partial x_1} & \cdots & \frac{\partial h^K(T,x)}{\partial x_K} \end{pmatrix}$$

has full rank on V.

2 Existence of Radner Equilibrium with Dynamically Complete Markets

We are now in the position to state our main result. We call the market given by the asset prices S, dividends A, and consumption price ψ dynamically complete if every net consumption plan $z \in \mathcal{X}$ can be financed by an admissible portfolio θ in the sense that its value process satisfies

$$V_t = V_0 + \int_0^t \theta_u dG_u + \int_0^t z_u \psi_u \nu(du) \,.$$

Theorem 11 *There exists a Radner equilibrium $\left(S, \psi, (\theta^i, c^i)_{i=1,\ldots,I} \right)$ with a dynamically complete market (S, A, ψ); the prices and dividends are linked by the present value relation*

$$S_t^k = \mathbb{E}\left[\left. \int_t^T A_s^k \psi_s \, \nu(ds) \right| \mathcal{F}_t \right] \,. \tag{3}$$

The proof of this theorem runs as follows. In a first step, we establish the existence of an Arrow–Debreu equilibrium. In the current time–additive setup, this is a result by Dana (2002). We extend her result by showing that in our smooth economy the equilibrium consumption price ψ and the allocation $(c^i)_{i=1,\ldots,I}$ are analytic functions of time and the state variable. It is well known that one can implement the Arrow–Debreu equilibrium as a Radner equilibrium if one has dynamically complete markets. With *nominal* assets, this is more or less trivial (see Duffie and Huang (1985) and Huang (1987)). Here, our assets pay real dividends, and the completeness depends on the endogenous consumption price ψ and cannot be assumed exogenously.

The natural candidates for our asset prices are, of course, the present values of their future dividends as in (3). We have to show dynamic completeness then. We do this by proving that the (local) linear independence

of the dividends at maturity T carries over to the volatility matrix of asset prices. This yields dynamic completeness. This step needs the intermediate mathematical result that our candidate security prices are analytic functions of time and state variable.

The implementation of the Arrow–Debreu equilibrium as a Radner equilibrium is then standard.

2.1 Existence of an Analytic Arrow–Debreu Equilibrium

We quickly recall the notions of classical General Equilibrium Theory. An allocation is an element $(c^i)_{i=1,\dots,I} \in \mathcal{X}_+^I$. Is is feasible if we have $\sum_{i=1}^I c^i \leq \varepsilon$. A price is a nonnegative, optional process $\psi \in \mathcal{X}_+$. It defines a continuous linear price functional $\Psi(c) = \mathbb{E} \int_0^T c_t \psi_t \, \nu(dt)$ on $\mathcal{X}$.

An Arrow–Debreu equilibrium consists of a feasible allocation $(c^i)_{i=1,\dots,I}$ and a price ψ such that c^i is budget–feasible and optimal for all agents $i = 1,\dots,I$, i.e. $\Psi(c^i) \leq \Psi(\varepsilon^i)$, and for all consumption plans $c \in \mathcal{X}_+$ the relation $U^i(c) > U^i(c^i)$ implies $\Psi(c) > \Psi(\varepsilon^i)$.

Existence and uniqueness of Arrow–Debreu equilibria in our separable setting have been clarified by Dana (1993). We recall her existence result and show the additional refinement that equilibrium price and consumption plans are analytic functions of time and the state variable on $(0,T) \times \mathbb{R}^K$.

Theorem 12 *There exists an Arrow–Debreu equilibrium $\left(\psi, (c^i)_{i=1,\dots,I}\right)$ such that*

$$\psi_t = \psi(t, X_t)$$
$$c_t^i = c^i(t, X_t)$$

for some continuous functions

$$\psi, c^i : [0,T] \times \mathbb{R}^K \to \mathbb{R}_+$$

that are analytic on $(0,T) \times \mathbb{R}^K$.

PROOF : By Dana (1993), there exists an equilibrium $(\psi, (c^i))$ with $\psi > 0$ $P \otimes \nu$–a.s. and the allocation (c^i) is the solution of the social planner problem

$$\max_{c \in \mathcal{X}_+^I, \sum c^i \leq \varepsilon} \sum \lambda^i U^i(c^i)$$

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for some $\lambda^i > 0^6$.

As we have separable utility functions, the social planner's problem can be solved point– and state–wise; we thus look at the real–valued problem

$$v(t,x) := \max_{\substack{\sum_{i=1}^I x^i = x \\ x^i \geq 0, i=1,\dots,I}} \sum_{i=1}^I \lambda^i u^i(t,x^i).$$

By Assumption 6, the unique solution of the above real–valued maximization problem is characterized by the equations

$$\lambda^i \frac{\partial u^i}{\partial c}\left(t, x^i\right) = \mu \qquad (4)$$

$$\sum_{i=1}^I x^i = x \qquad (5)$$

for some Lagrange parameter $\mu > 0$. By Dana (1993), Proposition 2.1, the solution of the above equations is given by continuous functions $x^i, \mu :$ $[0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ of (t,x). By the Analytic Implicit Function Theorem and Assumption 6, these are even analytic on $(0,T) \times (0,\infty)$ (see also Anderson and Raimondo (2008), page 881). By Dana (1993), we have $c_t^i = x^i(t, \varepsilon_t)$ and $\psi_t = \mu(t, \varepsilon_t)$. As aggregate endowment is a function of time and state variable that is continuous on $[0,T] \times \mathbb{R}_+$ and analytic on $(0,T) \times \mathbb{R}_+$ (Assumptions 7 and 8), the result follows. $\qquad \square$

2.2   Analytic Security Prices

The natural candidates for security prices are, of course, the present values of their future dividends, or (3). It is of essential importance for our development that these expectations are themselves analytic functions of time and state variable jointly.

Theorem 13 *Define S by (3). Under either Assumption 2 or 4, there exist continuous functions $s : [0,T] \times \mathbb{R}^K \to \mathbb{R}_+$ that are analytic on $(0,T) \times \mathbb{R}^K$ and*

$$S_t = s(t, X_t).$$

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[6]$\lambda^i = 0$ is not possible. This is already implicit in Dana's proof. Here is another argument based on our Assumption 9. For, if, say, $\lambda^1 = 0$, then $c^1 = 0$ (by Negishi). By the strict monotonicity of utility functions, $c^1 = 0$ is an equilibrium demand only if wealth is zero, i.e. $E \int_0^T \psi_t \varepsilon_t^1 \nu(dt) = 0$. But by Assumption 9 and the Inada assumption, $\varepsilon^1 > 0$ $P \otimes \nu$–a.s. Hence $E \int_0^T \psi_t \varepsilon_t^1 \nu(dt) > 0$, a contradiction.

The first derivatives with respect to x, $\frac{\partial s}{\partial x_l}$ are continuous on $[0,T] \times \mathbb{R}^K$ and we have

$$\lim_{t \uparrow T} \frac{\partial s}{\partial x_l}(t,x) = \frac{\partial s}{\partial x_l}(T,x) = \frac{\partial g}{\partial x_l}(T,x)$$

PROOF :

Let the securities price process S be defined by (3), so that — in light of Theorem 12 and the assumptions on the dividends — we have

$$S_t^k = \mathbb{E}\left[\int_t^T g^k(s, X_s)\, \psi(s, X_s)\, \nu(ds) \,\middle|\, \mathcal{F}_t \right],$$

hence by an application of the Fubini–Tonelli theorem,

$$S_t^k = \int_t^T \int_{\mathbb{R}^K} g^k(s,y)\, \psi(s,y) p(s-t, X_t, y)\; dy\, ds$$
$$+ \int_{\mathbb{R}^K} g^k(T,y)\, \psi(T,y) p(T-t, X_t, y)\; dy.$$

Since g is analytic by Assumption 8 and ψ is jointly analytic by Theorem 12, the joint analyticity of p is sufficient for the analyticity of s (as integrals of analytic functions are again analytic, see Proposition 2.2.3 of Krantz and Parks (2002) and compare the analogous reasoning at the end of the proof of Theorem B.4 in Anderson and Raimondo (2008)). This, however, is Assumption 2 or the content of Lemma 5.

The continuous differentiability of s with respect to the second argument x follows from Theorem 10.3 on p. 143 of Friedman (1969). □

2.3 Dynamically Complete Markets

Theorem 14 *The market (S, A, ψ) is dynamically complete.*

PROOF :   By Assumption 10, $\psi > 0$ and the fact that X has full support, we have $S_t^0 > 0$ a.s. Hence, we can take asset 0 as a numéraire. Define

$$R_t^k = \frac{S_t^k}{S_t^0}.$$

By Theorem 13, $R_t^k = r^k(t, X_t)$ for continuous functions $r^k : [0,T] \times \mathbb{R}^K \to \mathbb{R}_+$ that are analytic on $(0,T) \times \mathbb{R}^K$, $k = 1, \ldots, K$.

After this change of numéraire, we have a riskless asset (with interest rate 0, of course) and K risky assets, as many as indepedent Brownian motions.

The asset market is dynamically complete if the volatility matrix is a.s. invertible (see, e.g., Karatzas and Shreve (1998), Theorem 1.6.6)[7]. By Itô's lemma, the volatility matrix is given by $I(t,x)Dr(t,x)\sigma(t,x)$ where Dr is the Jacobian matrix of r and I the triangular matrix

$$I(t,x) = \begin{pmatrix} \frac{1}{r_1(t,x)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{r_K(t,x)} \end{pmatrix}.$$

Now suppose that the volatility matrix has determinant 0 on a set of positive Lebesgue measure. By analyticity and Theorem B.3 in Anderson and Raimondo (2008), we conclude that the determinant vanishes everywhere on $(0,T) \times \mathbb{R}^K$. As Dr, r, and σ are continuous on $[0,T]$, it then follows that

$$\det I(T,x)Dr(T,x)\sigma(T,x) = 0\,.$$

(For Dr and r, this is Theorem 13.) As σ has full rank by Assumption 1 and $I(T,x)$ is triangular, we conclude that

$$\det Dr(T,x) = 0\,.$$

But $r(T,x) = g(T,x)/g^0(T,x) = h(x)$, so

$$\det Dr(T,x) \neq 0$$

on a set of positive measure by Assumption 10. This contradiction shows that the volatility matrix is invertible a.s. We conclude that the market (S,A,ψ) is dynamically complete.

$\square$

With dynamically complete asset markets, it is a standard argument to show that the Arrow–Debreu equilibrium can be implemented as a Radner equilibrium. The basic argument is as in Duffie and Huang (1985), translated to our more complex setting, see also Dana and Jeanblanc (2003), Theorem 7.1.10 (apply this theorem to the asset market with asset 0 as numéraire).

A   Appendix: Analytic Transition Densities

We provide here the proof of Lemma 5.

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[7]To apply this result, we check quickly that the asset market is also standard in the sense of Karatzas and Shreve (1998): by construction ((3)), the gain processes are martingales; hence, our market is arbitrage–free. As our state–price deflator ψ is in Ψ, also the martingale condition in Karatzas and Kou (1998) is satisfied.

PROOF :   By Theorem 5.4 in Chapter 6 of Friedman (1975) X possesses a transition density p, whence

$$P[X_{s+t} \in dy | X_s = x] = p\,(t, x, y)\,dy$$

for all $s \geq 0, t > 0, x \in \mathbb{R}^K$.

The proof is divided into three parts:

1. First, we verify that the transition density p is bounded as bounded away from zero, i.e. p is bounded on $[\delta, T] \times \mathbb{R}^K \times \mathbb{R}^K$ for all $\delta > 0$.

   It suffices to prove this kind of boundedness for the function Γ^*, defined by

   $$\Gamma^*(x, s; y, t) = p(t - s, x, y)$$

   for $t > s \geq 0$. By our Assumption 4, the hypotheses (A1)-(A4) in Chapter 6 of Friedman (1975) are satisfied. By Theorem 4.7 in Chapter 6 of Friedman (1975),

   $$\Gamma^*(x, s; y, t) = \Gamma(y, t; x, s)$$

   where Γ is the fundamental solution of the partial differential equation

   $$-\frac{\partial}{\partial s}u + \mathcal{L}u = 0\,.$$

   By Theorem 4.5 in Chapter 6 of Friedman (1975), we have

   $$\Gamma(y, t; x, s) \leq C \frac{1}{\sqrt{t - s}^K} \exp\left(-c\frac{\|x - y\|^2}{t - s}\right) \qquad (6)$$

   for some constants $c, C > 0$. Now take $s = 0$. As long as t is bounded away from zero, the density p is thus bounded.

   Remark: The density remains also bounded as t goes to zero and $x \neq y$, compare also Mikhailov's example. But we might have a singularity in $t = 0$, $x = y$, as for Brownian motion.

2. Next, we establish the analyticity of p in the time variable.[8] The tran-

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[8] An alternative proof for analyticity in time would go as follows: The transition density p solves the evolution equation

$$u'(t) = \mathcal{L}u\,.$$

As such, it is analytic by Theorem 2.1 in Part 3, Chapter 2 of Friedman (1969) (see also the Corollary on p. 209) if the conditions (E1)-(E3) on p. 206 or the conditions (F1)-(F4) on p. 210 of Friedman (1969) are satisfied. Given that $\mathcal{L}$ is independent of t, all we need to verify is condition (E2) or condition (F2), which requires that the resolvent of the Markov process exists in some complex sector around zero. However, this has been proven in Eq. (2.11), Theorem 1 of Yosida (1959).

sition density p solves the evolution equation

$$u'(t) = \mathcal{L}u$$

with initial condition $p(0, \cdot) = f$ for some square-integrable f. It can thus be written $p(t, x, y) = T_t f(x, y)$ for all $t > 0$, $x, y \in \mathbb{R}^K$, wherein T is the semi-group generated by the smallest closed extension of the operator $\mathcal{L}$. Theorem 2 of Yosida (1959) (whose proof depends crucially on the earlier paper Yosida (1958) and the resolvent estimate of Theorem 1 of Yosida (1959)) yields that T is strongly differentiable and for sufficiently small $t > 0$ even analytic. As in the proof of Eq. (1.3) of Yosida (1959), we can now argue that for $t_0 > 0$ the difference between $p(t_0 + h, \cdot)$ and the nth Taylor expansion of $T_{t_0+h} f$ around t_0 converges to 0 in the L^2 norm as $n \to \infty$. Since any L^2-convergent sequence has an a.e. convergent subsequence, $p(t_0+h, x, y)$ can be written as a power series around t_0 for almost every $x, y \in \mathbb{R}^K$. Using the analyticity of p in x, y (see below), one can then finally show that the resulting power series must even converge for every $x, y \in \mathbb{R}^K$.

3. Finally, note that p is analytic in (x, y) by Theorem 1.2 in Part 3, Chapter 1 of Friedman (1969) and recall that functions which are bounded and separately analytic are jointly analytic (a result of Osgood (1899)).

$\square$

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