Center for
Mathematical Economics
Working Papers

530

November 2014

Irreversible Investment under Lévy Uncertainty: an Equation for the Optimal Boundary

Giorgio Ferrari and Paavo Salminen

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

Irreversible Investment under Lévy Uncertainty: an Equation for the Optimal Boundary*

Giorgio Ferrari† Paavo Salminen ‡

November 10, 2014

Abstract. We derive a new equation for the optimal investment boundary of a general irreversible investment problem under exponential Lévy uncertainty. The problem is set as an infinite time-horizon, two-dimensional degenerate singular stochastic control problem. In line with the results recently obtained in a diffusive setting, we show that the optimal boundary is intimately linked to the unique optional solution of an appropriate Bank-El Karoui representation problem. Such a relation and the Wiener-Hopf factorization allow us to derive an integral equation for the optimal investment boundary. In case the underlying Lévy process hits any point in $\mathbb{R}$ with positive probability we show that the integral equation for the investment boundary is uniquely satisfied by the unique solution of another equation which is easier to handle. As a remarkable by-product we prove the continuity of the optimal investment boundary. The paper is concluded with explicit results for profit functions of (i) Cobb-Douglas type and (ii) CES type. In the first case the function is separable and in the second case non-separable.

Key words: free-boundary, irreversible investment, singular stochastic control, optimal stopping, Lévy process, Bank and El Karoui's representation theorem, base capacity.

MSC2010 subsject classification: 91B70, 93E20, 60G40, 60G51.

JEL classification: C02, E22, D92, G31.

1 Introduction

Investment problems under uncertainty have received increasing attention in the last years in both the economic and the mathematical literature (see, for instance, [20] for an extensive review). Several economic papers tackle the problem of a firm maximizing profits when the operating profit function depends on an exogenous stochastic shock process reflecting the changes in, e.g., technologically feasible output, demand, and macroeconomic conditions and so on (see, e.g., [1], [8], [10], and [34]), and relate irreversible investment decisions and their timing to real

*Financial support by the German Research Foundation (DFG) via grant Ri 1128-4-1 is gratefully acknowledged by the first author. Research of the second author has been supported in part by a grant from Svenska kulturfonden via Stiftelsernas professorspool, Finland.

†Center for Mathematical Economics, Bielefeld University, Germany; giorgio.ferrari@uni-bielefeld.de

‡Åbo Akademi University, Department of Natural Sciences/Mathematics and Statistics, Fänriksgatan 3 B, FIN-20500 Åbo, Finland; phsalmin@abo.fi

options (cf. [27] and [34], among others). Usually in those models profit functions are of separable type (as Cobb-Douglas) and the economic shock process is a geometric Brownian motion.

In the mathematical-economic literature, problems of continuous-time irreversible investment under uncertainty are usually modeled as concave (or convex) stochastic control problems with monotone controls (see, e.g., [14], [22], [30], [33] and [36]). In fact, due to the economic constraint that does not allow disinvestment, an irreversible investment problem under uncertainty may be seen as a so-called 'monotone follower' problem; that is, a problem in which the investment strategies are given by nondecreasing stochastic processes, whose associated random Borel measures on $\mathbb{R}_+$ may be singular with respect to the Lebesgue measure. In this setting the connection between irreversible investment under uncertainty and real options found in the economic literature (cf., e.g., [27] and [34]) may be seen as the well known connection between concave (or convex) stochastic control problems with monotone controls and certain problems in optimal stopping. This kind of connections have been firstly rigorously shown in [21], [23] and [24].

When the stochastic process X underlying the optimization is Markovian, e.g., a diffusion or a Lévy process, the optimal control policy usually consists in splitting the state space of the singular stochastic control problem into two regions by a curve, called the optimal investment boundary or the free-boundary. These regions are generally called 'action' and 'inaction' regions as it is optimal therein to turn the control on or off, respectively. Similarly in optimal stopping problems one has 'continuation' and 'stopping' regions where it is optimal to let the evolution of X continue and cease, respectively. The main feature of concave (or convex) singular stochastic control problems is that the action region of the singular stochastic control problem coincides with the stopping region of a suitably associated optimal stopping problem and the optimal policy is to keep the controlled process just inside the inaction (continuation) region, with minimal control. It is then evident that a study of the optimal stopping problem associated with a singular stochastic control problem through a characterization of its free-boundary separating the stopping and continuation regions leads to a complete understanding of the optimal control, i.e., the optimal investment policy of the firm.

In this paper we essentially consider the same problem as in [22] but now the economic shock is modeled by an exponential Lévy process rather than a regular linear diffusion. Lévy processes may exhibit heavy tails and skewness in the probability distributions commonly found in time-series from the market data. We solve the irreversible investment problem by a stochastic first order conditions approach in the spirit of [4], [7], [36], among others, and by relying on a suitable application of the Bank-El Karoui representation theorem (cf. [6], Theorem 3). As in [22] we prove that the unique optional solution of the Bank-El Karoui representation problem is closely linked to the free-boundary of the one-dimensional, infinite time-horizon, parameter-dependent optimal stopping problem naturally associated with the original singular control problem. Such a relation and the Wiener-Hopf factorization for Lévy processes enable us to derive an integral equation for the free-boundary. If the underlying Lévy process hits any point in $\mathbb{R}$ with positive probability (as α-stable Lévy processes with $\alpha \in (1, 2)$ or jump-diffusion processes which play an important role in Financial Economics) the free-boundary is then proved to be a unique solution of another – more tractable and handy – equation. Using the equation we, moreover, prove that the free-boundary is continuous in our general Lévy process framework. To the best of our knowledge this result appears here for the first time. Finally, we find the explicit form of the optimal boundary even in the non-separable case of a CES (constant elasticity of substitution) operating profit function (see Section 4.2 below), thus leading to a complete characterization of the optimal investment policy of a quite intricate stochastic irreversible investment problem.

The issue of determining the optimal investment boundary of investment problems under Lévy uncertainty has been recently tackled also in [13] and [26]. However the setting therein is simpler than ours since in [13] only separable running profits are considered and in [26] only a one-dimensional model is addressed. Here instead we allow any concave running profit satisfying Assumption 2.1 below and our irreversible investment problem is set up as a two-dimensional degenerate singular stochastic control problem. Moreover, our equation follows from the strong Markov property and the Wiener-Hopf factorization and is not obtained by writing down any integro-differential free-boundary problem (as is done in [26]) nor by imposing any regularity condition of the value function of the associated optimal stopping problem at the boundary itself. In this sense our approach seems to bypass the difficulties related to the validity of the smooth-fit condition in a Lévy setting (see, e.g., [2], [12] and [31]).

The paper is organized as follows. In Section 2.1 we set up the irreversible investment problem, which is then solved in Section 2.2. In Section 2.3 we obtain a characterization of the optimal investment boundary in terms of a base capacity process. Section 3 is devoted to the equations characterizing the optimal investment boundary. Finally, some examples allowing explicit calculations are presented in Section 4.

2 The Optimal Investment Problem

2.1 Setting and Basic Assumptions

As in [22] consider the optimal irreversible investment problem of a firm producing a single good. However, to take into account the fact that empirically the market often exhibits significant skewness and kurtosis we model the uncertain status of the economy (e.g., the demand of the good, or the macroeconomic conditions, or the price of the produced good) by the exponential random process e^X, where $X = \{X_t, t \geq 0\}$ is a real valued Lévy process (other than a compound Poisson process or a subordinator) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For any $x \in \mathbb{R}$, we let $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot|X_0 = x)$ and $\mathbb{E}_x$ the corresponding expectation operator. In the following, we will simply write $\mathbb{P}_0 = \mathbb{P}$ and $\mathbb{E}_0 = \mathbb{E}$.

A Lévy process is a stochastic process with stationary and independent increments, having a.s. càdlàg paths (right-continuous with left limits), and starting from zero at time zero. Each Lévy process is fully characterized by its Lévy triplet (γ, σ, Π), where $\gamma, \sigma \in \mathbb{R}$ and Π is the so called Lévy measure which is concentrated on $\mathbb{R} \setminus \{0\}$ and satisfies

$$\int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty.$$

Moreover, each Lévy process X can be represented as

$$X_t = \gamma t + \sigma B_t + X_t^{(1)} + X_t^{(2)}, \tag{2.1}$$

where B is a standard Brownian motion, $X^{(1)}$ is a zero mean pure jump martingale, $X^{(2)}$ is a compound Poisson process with jumps at least of size one, and all the components in (2.1) are independent. As a consequence of stationary and independent increments, it can also be shown that

$$\mathbb{E}[e^{i\theta X_t}] = e^{-t\Psi(\theta)}, \tag{2.2}$$

for all $t \geq 0$ and $\theta \in \mathbb{R}$, where

$$\Psi(\theta) := -\log \mathbb{E}[e^{i\theta X_1}] = i\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \Big(1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|<1\}}\Big)\Pi(dx)$$

is the Lévy characteristic exponent of X. Well-known Lévy processes are Brownian motion, Poisson process, jump-diffusion processes and the variance-gamma process. We refer to [9] or [25] for a detailed exposition on Lévy processes.

The firm's production capacity is assumed to evolve according to

$$C_t^{y,\nu} := y + \nu_t, \qquad C_0^{y,\nu} := y \geq 0, \tag{2.3}$$

where ν is an (irreversible) investment plan, i.e., a nondecreasing, left-continuous, $(\mathcal{F}_t)$-adapted process such that $\nu_0 = 0$ $\mathbb{P}$-a.s.

The instantaneous profits of the firm are described by the operating profit function $\pi : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, depending on the the status of economy and on the production capacity. The following assumption is taken to be valid throughout the paper:

Assumption 2.1.

1. *The mapping $(z, c) \mapsto \pi(z, c)$ is continuous. Moreover, $c \mapsto \pi(z, c)$ is strictly increasing and strictly concave with continuous and strictly decreasing derivative $\pi_c(z, c) := \frac{\partial}{\partial c}\pi(z, c)$ on $\mathbb{R}_+ \times (0, \infty)$ satisfying*

$$\lim_{c\to 0} \pi_c(z, c) = \infty, \qquad \lim_{c\to\infty} \pi_c(z, c) = \kappa,$$

   for some $0 \leq \kappa < \infty$.

2. *The process $(\omega, t) \mapsto \pi_c\big(e^{x+X_t(\omega)}, y\big)$ is $\mathbb{P}(d\omega) \otimes e^{-rt}dt$ integrable for any $y > 0$.*

Here r is a positive discount factor satisfying

Assumption 2.2. *$r > \kappa$.*

Assumption 2.2 will be needed in the next section to derive the optimal control policy (see Proposition 2.5 below). Moreover, we will see in Remark 2.7 that Assumption 2.2 is necessary to have a nonempty 'no-investment region'.

Remark 2.3. *Assumption 2.1.1 is satisfied with $\kappa = 0$ by the Cobb-Douglas and the logarithmic operating profit functions. On the other hand, in the case of a CES (constant elasticity of substitution) profit function of the form $\pi(z, c) = (\alpha z^\gamma + (1-\alpha)c^\gamma)^{\frac{1}{\gamma}}$, for some $\alpha \in (0, 1)$ and $\gamma \in (0, 1)$ (which reduces to the Cobb-Douglas operating profit, $\pi(z, c) = z^\alpha c^{1-\alpha}$, when $\gamma = 0$) one has $\kappa = (1-\alpha)^{\frac{1}{\gamma}}$. It is also worth noticing that a CES operating profit with $\gamma < 0$ does not fulfill Assumption 2.1.1 because in this case $\lim_{c\to 0} \pi_c(z, c) = (1-\alpha)^{1/\gamma}$.*

For any investment plan ν the expected present value of the future overall net profits is defined as

$$\mathcal{J}_{x,y}(\nu) := \mathbb{E}\bigg\{ \int_0^\infty e^{-rt}\, \pi(e^{x+X_t}, C_t^{y,\nu})\, dt - \int_0^\infty e^{-rt} d\nu_t \bigg\}. \tag{2.4}$$

From now on we will call investment plans admissible if their present value is finite; i.e. if

$$\mathbb{E}\bigg\{ \int_0^\infty e^{-rt} d\nu_t \bigg\} < \infty. \tag{2.5}$$

We will denote by $\mathcal{S}_o$ the set of all admissible investment plans. Due to (2.5) and the positivity of π it holds that $\mathcal{J}_{x,y}(\nu) > -\infty$ for any $\nu \in \mathcal{S}_o$. The firm's manager aims at picking an admissible ν^* such that

$$V(x,y) := \mathcal{J}_{x,y}(\nu^*) = \sup_{\nu \in \mathcal{S}_o} \mathcal{J}_{x,y}(\nu) < \infty, \qquad (x,y) \in \mathbb{R} \times \mathbb{R}_+. \tag{2.6}$$

Since $\pi(z,\cdot)$ is strictly concave, $\mathcal{S}_o$ is convex and $C^{y,\nu}$ is affine in ν, we have that $\mathcal{J}_{x,y}(\cdot)$ is strictly concave on $\mathcal{S}_o$ as well. Consequently, if an optimal solution ν^* to (2.6) does exist, it is unique. We provide the form of the optimal control in the next section.

2.2 The Optimal Investment Strategy

In this section we will solve the optimal investment problem (2.6). A very general stochastic irreversible investment problem similar to ours (2.6) has been thoroughly studied in [36], where the shock process is assumed to be a general progressively measurable process, or more recently in [22], in a diffusive setting. It thus follows that some of the following results may be obtained by easily adapting arguments in [22] or [36] (see also [7] and [38]). We will state them for the sake of completeness and to have a self-contained paper, but we will only sketch their proofs referring to the literature for details.

We denote by $\mathcal{T}$ the set of all $\mathcal{F}_t$-stopping times $\tau \in [0,\infty]$ and put $e^{-r\tau(\omega)} = 0$ if $\tau(\omega) = \infty$. Following [22], equation (11) and Theorem 3.2, we have the following characterization of the optimal control ν^*.

Proposition 2.4. *Under Assumption 2.1, a control $\nu^* \in \mathcal{S}_o$ such that $\mathcal{J}_{x,y}(\nu^*) < \infty$ is the unique optimal investment strategy for problem (2.6) if and only if the following first order conditions for optimality*

$$\begin{cases} \mathbb{E}\bigg\{ \displaystyle\int_\tau^\infty e^{-rs} \pi_c(e^{x+X_s}, C_s^{y,\nu^*})\, ds \Big| \mathcal{F}_\tau \bigg\} - e^{-r\tau} \le 0, \quad a.s.\ \forall \tau \in \mathcal{T}, \\[2ex] \mathbb{E}\bigg\{ \displaystyle\int_0^\infty \bigg[\mathbb{E}\bigg\{ \int_t^\infty e^{-rs} \pi_c(e^{x+X_s}, C_s^{y,\nu^*})\, ds \Big| \mathcal{F}_t \bigg\} - e^{-rt} \bigg] d\nu_t^* \bigg\} = 0, \end{cases} \tag{2.7}$$

hold true.

First order conditions (2.7) may be seen as a stochastic, infinite-dimensional generalization of the Kuhn-Tucker conditions from the classical optimization theory. The left-hand side of the inequality in the first condition (2.7) is called the supergradient process (cf. [22], equation (11) and Remark 3.1). It is interpreted as the expected present value of the future overall net marginal profits resulting from an extra unit of investment at time $\tau \in \mathcal{T}$. The intuition behind (2.7) is that when the supergradient is positive at some stopping time, a small extra investment is profitable. On the other hand, investment should not occur when the supergradient is negative, since similarly reducing such an investment would be beneficial. As in [22], Section 3, or [36], Theorem 3.2, the next proposition links the optimal control ν^* to the solution of a suitable Bank-El Karoui's representation problem, see [6], Theorem 1, Theorem 3 and Remark 2.1, related to (2.7).

Proposition 2.5. *Let Assumptions 2.1 and 2.2 hold. Then the equation*

$$\mathbb{E}\Big\{ \int_\tau^\infty e^{-rs} \pi_c\Big(e^{x+X_s}, \sup_{\tau \le u < s} l_u\Big)\, ds \Big| \mathcal{F}_\tau \Big\} = e^{-r\tau}, \qquad \tau \in \mathcal{T}, \tag{2.8}$$

has a unique (up to indistinguishability) strictly positive optional solution with upper right-continuous paths[1]. *Let l^* denote this solution and define*

$$\nu^*_t := (\sup_{0 \le s < t} l^*_s - y) \vee 0, \quad t > 0, \qquad \nu^*_0 := 0. \tag{2.9}$$

If ν^* is admissible and such that $\mathcal{J}_{x,y}(\nu^*) < \infty$ then it is the unique optimal irreversible investment plan for problem (2.6)*.*

Proof. We only sketch roughly the two main steps of the proof and refer to [22] and [36] for details.

Step 1. Here the objective is to prove that (2.8) admits a unique (up to indistinguishability) strictly positive optional solution l^* with upper right-continuous paths. To this end, for κ as in Assumption 2.1, apply the Bank-El Karoui Representation Theorem (cf. [6], Theorem 3 and Remark 2.1) with

$$\hat{T} = +\infty, \qquad \mu(\omega, dt) := e^{-rt} dt \tag{2.10}$$

and

$$f(\omega, t, l) := \begin{cases} \pi_c\left(e^{x+X_t(\omega)}, -\frac{1}{l}\right), & \text{for } l < 0, \\ -l + \kappa\,, & \text{for } l \ge 0, \end{cases} \tag{2.11}$$

to represent the deterministic process $\{e^{-rt},\, t \ge 0\}$, and then use the same arguments as in the proof of Proposition 3.4 in [22].

Step 2. Proceeding as in the proof of Theorem 3.2 in [36], it is easy to see that ν^* of (2.9) satisfies the first order conditions (2.7). Hence by Proposition 2.4 ν^* is optimal if it is admissible and such that $\mathcal{J}_{x,y}(\nu^*) < \infty$. □

Remark 2.6. *Notice that ν^* defined in* (2.9) *is clearly increasing and left-continuous. Moreover, since l^* is optional and hence progressively measurable, then ν^* is progressively measurable by [19], Theorem IV.33, and hence $(\mathcal{F}_t)$-adapted. Therefore to prove that ν^* is admissible it thus remains to show that ν^* satisfies* (2.5)*. Such a condition, as well as the fact that $\mathcal{J}_{x,y}(\nu^*) < \infty$, is usually true if the discount factor r is big enough. In many cases this can be verified once the explicit form of ν^* is known (see Section 4 for examples).*

[1] According to [6], Lemma 4.1 (see also [5], Remark 1.4-(ii)), we call a real valued process ξ upper right-continuous on $[0, T)$ if, for each t, $\xi_t = \limsup_{s \searrow t} \xi_s$ with $\limsup_{s \searrow t} \xi_s := \lim_{\epsilon \downarrow 0} \sup_{s \in [t, (t+\epsilon) \wedge T]} \xi_s$.

2.3 The Base Capacity and the Free-Boundary

It is easy to see that our optimal policy (2.9) coincides with that in Theorem 3.2 of [36] when $\delta = 0$ therein. Following Definition 3.1 in [36], the process l^* is a *base capacity* process, i.e., an index describing the desirable level of capacity at time t. At times t, when the firm's production capacity is strictly above l^*_t, it is optimal to wait as at those times the firm faces excess of capacity. On the other hand, when the capacity level is below l^*_t, the firm should instantaneously invest to reach the level l^*_t. It therefore represents the maximal capacity level for which it is not profitable to delay investment to any future time. Clearly, l^* must be linked to the optimal boundary of an associated optimal timing problem. Such a connection has been recently shown in [22], Theorem 3.9, in a diffusive setting (see also [15] in the context of a one-dimensional irreversible investment problem over a finite time-horizon). In this section it is seen that a similar connection also holds in our Lévy setting.

Similarly as in [22], eq. (25), introduce the optimal stopping problem: find a stopping time τ^* such that for all $(x, y) \in \mathbb{R} \times (0, \infty)$

$$v(x, y) := \mathbb{E}\left\{ \int_0^{\tau^*} e^{-rs}\pi_c(e^{x+X_s}, y)\, ds + e^{-r\tau^*} \right\} = \inf_{\tau \geq 0} \mathbb{E}\left\{ \int_0^{\tau} e^{-rs}\pi_c(e^{x+X_s}, y)\, ds + e^{-r\tau} \right\}. \quad (2.12)$$

Notice that problem (2.12) is the optimal timing problem associated to the irreversible investment problem (2.6) since it may be interpreted as the minimal cost of not investing. Mathematically, problem (2.12) is the one-dimensional, infinite time-horizon, parameter-dependent (as y enters only as a parameter) optimal stopping problem associated to the singular control problem (2.6). In fact, it can be shown (see, e.g., [3], [21] and [24]) that under our assumptions $V_y(x, y) = v(x, y)$ and that $\tau^* := \inf\{t \geq 0 : \nu^*_t > 0\}$, with ν^* optimal for (2.6), is an optimal stopping time for (2.12).

Since $v(x, y) \leq 1$, for all $(x, y) \in \mathbb{R} \times (0, \infty)$, the state space splits into

$$\mathcal{S} := \{(x, y) \in \mathbb{R} \times (0, \infty) : v(x, y) = 1\}, \qquad \mathcal{C} := \{(x, y) \in \mathbb{R} \times (0, \infty) : v(x, y) < 1\}.$$

Intuitively $\mathcal{S}$ is the region in which it is optimal to invest immediately (the so-called 'action region' or 'investment region') as therein the marginal value $v = V_y$ equals the marginal cost of the investment. On the other hand, $\mathcal{C}$ is the region in which it is profitable to delay the investment option (the so-called 'inaction region' or 'no-investment region'), as the marginal value $v = V_y$ is strictly less than the marginal cost of investment therein. Since $\pi(z, \cdot)$ is strictly concave, the mapping $y \mapsto v(x, y)$ is decreasing for any $x \in \mathbb{R}$, and therefore

$$b(x) := \sup\{y > 0 : v(x, y) = 1\}, \qquad x \in \mathbb{R}, \quad (2.13)$$

is the boundary between the stopping and continuation regions, i.e. the so called free-boundary. We adopt the convention $b \equiv 0$ if $\{y > 0 : v(x, y) = 1\} = \emptyset$.

Remark 2.7. *Notice that Assumption 2.2 is necessary to have nonempty no-investment region $\mathcal{C}$. Indeed, if $r \leq \kappa$ then*

$$v(x, y) = 1 + \inf_{\tau \geq 0} \mathbb{E}\left\{ \int_0^{\tau} e^{-rs}\Big(\pi_c(e^{x+X_s}, y) - r\Big)\, ds \right\} \geq 1 + \inf_{\tau \geq 0} \mathbb{E}\left\{ \int_0^{\tau} e^{-rs}(\kappa - r)\, ds \right\} = 1,$$

where the inequality above is due to the fact that $\pi_c(z, c) \geq \lim_{c \to \infty} \pi_c(z, c) = \kappa$, (z, c). It thus follows that if $r \leq \kappa$ then $v(x, y) = 1$ for all $(x, y) \in \mathbb{R} \times (0, \infty)$, thus implying $\mathcal{C} = \emptyset$.

As in [22], Assumption 3.6, or [36], Section 5, we now assume that marginal profits are positively affected by improving market conditions.

Assumption 2.8. *The mapping* $z \mapsto \pi_c(z,c)$ *is nondecreasing for any* $c \in (0,\infty)$.

Remark 2.9. *Notice that, if* π *were twice-continuously differentiable, then Assumption 2.8 would be equivalent to requiring* π *to be supermodular (see [39]); that is, for any* $z_1, z_2 \in \mathbb{R}_+$ *and* $c \in (0,\infty)$

$$\pi(z_1 \vee z_2, c) + \pi(z_1 \wedge z_2, c) \geq \pi(z_1, c) + \pi(z_2, c).$$

The Cobb-Douglas and the CES profit functions are well known examples of supermodular profit functions on $(0,\infty) \times (0,\infty)$.

Proposition 2.10. *Let Assumptions 2.1 and 2.8 hold. Then, the value function* v *of optimal stopping problem* (2.12) *is*

- *continuous on* $\mathbb{R} \times (0,\infty)$,
- *such that* $x \mapsto v(x,y)$, $y \in (0,\infty)$, *is nondecreasing.*

Proof. For the continuity, consider a sequence $\{(x_n, y_n) : n \in \mathbb{N}\} \subset \mathbb{R} \times (0,\infty)$ converging to $(x,y) \in \mathbb{R} \times (0,\infty)$. Take $\varepsilon > 0$ and let $\tau^\varepsilon := \tau^\varepsilon(x,y)$ be an ε-optimal stopping time for problem (2.12) with initial values x and y. Then we have

$$v(x,y) - v(x_n,y_n) \geq \mathbb{E}\bigg\{ \int_0^{\tau^\varepsilon} e^{-rt}\big[\pi_c(e^{x+X_t}, y) - \pi_c(e^{x_n+X_t}, y_n)\big]\,dt \bigg\} - \varepsilon. \tag{2.14}$$

Without loss of generality, let $\{x_n : n \in \mathbb{N}\} \subset (x-\epsilon, x+\epsilon)$ for a suitable $\epsilon > 0$ be such that for all $t \geq 0$

$$e^{x-\epsilon+X_t} \leq e^{x_n+X_t} \leq e^{x+\epsilon+X_t}.$$

Taking into account Assumptions 2.1 and 2.8, we can apply the dominated convergence theorem on the right-hand side of (2.14) to get

$$\limsup_{n\to\infty} v(x_n,y_n) \leq v(x,y) + \varepsilon. \tag{2.15}$$

Similarly, taking ε-optimal stopping times $\tau_n^\varepsilon := \tau^\varepsilon(x_n,y_n)$ for problem (2.12) with initial values x_n and y_n one has

$$\begin{aligned} v(x,y) - v(x_n,y_n) &\leq \mathbb{E}\bigg\{ \int_0^{\tau_n^\varepsilon} e^{-rt}\big[\pi_c(e^{x+X_t}, y) - \pi_c(e^{x_n+X_t}, y_n)\big]\,dt \bigg\} + \varepsilon \\ &\leq \mathbb{E}\bigg\{ \int_0^{\infty} e^{-rt}\big|\pi_c(e^{x+X_t}, y) - \pi_c(e^{x_n+X_t}, y_n)\big|\,dt \bigg\} + \varepsilon. \end{aligned} \tag{2.16}$$

Evoking again the dominated convergence theorem yields

$$\liminf_{n\to\infty} v(x_n,y_n) \geq v(x,y) - \varepsilon, \tag{2.17}$$

which together with (2.15) implies the continuity of v.

To verify the second statement, let $\tau^* := \tau^*(x_1, y)$ be an optimal stopping time with initial values x_1 and y. Then for $x_2 < x_1$ we have

$$v(x_1, y) - v(x_2, y) \geq \mathbb{E}\bigg\{ \int_0^{\tau^*} e^{-rt} \big[\pi_c(e^{x_1+X_t}, y) - \pi_c(e^{x_2+X_t}, y)\big] dt \bigg\} \geq 0,$$

since $\pi_c(\cdot, y)$ is supposed to be nondecreasing, see Assumption 2.8, cf. also the proof of Proposition 3.7 in [22]. □

Proposition 2.11. *Under Assumptions 2.1 and 2.8 the free-boundary b defined in* (2.13) *is*

- *nondecreasing,*
- *right-continuous with left limits.*

Proof. The fact that b is nondecreasing can be proved similarly as in [22] (see the proof of Corollary 3.8). From the monotonicity, it clearly follows that b admits right and left limits at any point. To show that b is right-continuous, fix $x \in \mathbb{R}$ and notice that for every $\varepsilon > 0$ we have again by monotonicity of b that $b(x+\varepsilon) \geq b(x)$, which implies $b(x) \leq \lim_{\varepsilon \downarrow 0} b(x+\varepsilon) =: b(x+)$. Consider now the sequence $\{(x+\varepsilon, b(x+\varepsilon)) : \varepsilon > 0\} \subset \mathcal{S}$; one has $\{(x+\varepsilon, b(x+\varepsilon)) : \varepsilon > 0\} \to (x, b(x+))$ when $\varepsilon \downarrow 0$ and $(x, b(x+)) \in \mathcal{S}$, since $\mathcal{S}$ is closed by continuity of v (cf. Proposition 2.10). It then follows that $b(x+) \leq b(x)$ from the definition (2.13) and the proof is complete. □

The next theorem adapts Theorem 3.9 in [22] to our exponential Lévy setting. It connects the base capacity process l^* to the free-boundary b of the optimal stopping problem (2.12) associated with the original control problem.

Theorem 2.12. *Let l^* be the unique optional solution of (2.8) and b the free-boundary defined in (2.13). Under Assumptions 2.1, 2.2 and 2.8 one has*

$$l_t^* = b(x + X_t), \qquad t \geq 0. \tag{2.18}$$

Proof. First of all, since b is Borel-measurable (being monotone) and X is optional it follows that the process on the right-hand side of (2.18) is optional. Moreover, $t \mapsto b(x + X_t)$ is upper right-continuous since b is upper-semicontinuous (being nondecreasing and right-continuous by Proposition 2.11) and $t \mapsto X_t$ is right-continuous.

To prove (2.18), the arguments in [22] (see the proof of Proposition 3.4), are easily adapted to the present case. Hence, we have $l_t^* = -\frac{1}{\xi_t^*}$, where the process ξ^* admits the following representation (cf. also [6], formula (23) on page 1049)

$$\xi_t^* = \sup\left\{ l < 0 : \operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}\bigg\{ \int_t^\tau e^{-r(s-t)} \pi_c\Big(e^{x+X_s}, -\frac{1}{l}\Big)\, ds + e^{-r(\tau-t)} \Big| \mathcal{F}_t \bigg\} = 1 \right\}. \tag{2.19}$$

To take care of the conditional expectation in (2.19) it is convenient to proceed as in the proof of Theorem 3.9 in [22], and work on the canonical probability space $(\overline{\Omega}, \overline{\mathbb{P}})$. However, to take into account our Lévy setting we let $\overline{\Omega} := \mathcal{D}_0([0, \infty))$ be the Skorohod space of all càdlàg functions $\overline{\omega}$ on $[0, \infty)$ such that $\omega_0 = 0$, endowed with Skorohod's topology and let $\mathcal{F}$ denote its Borel σ-field. Moreover, $\overline{\mathbb{P}}$ is the probability measure on $\overline{\Omega}$ under which the coordinate process $X_u(\overline{\omega}) = \overline{\omega}_u$, $u \geq 0$, is a Lévy process and the shift operator $\theta_u : \overline{\Omega} \mapsto \overline{\Omega}$ is defined by $\theta_u(\overline{\omega})(s) = \overline{\omega}_{u+s}$, for

$\overline{\omega} \in \overline{\Omega}$ and $u, s \geq 0$. Finally, we denote by $(\mathcal{F}_u)_{u\geq 0}$ the filtration, where $\mathcal{F}_u$ is generated by $s \mapsto \overline{\omega}_s$, $s \leq u$, and augmented by the $\overline{\mathbb{P}}$-null sets. By Theorem 103, p. 151 in [19] – based on Galmarino's test – any stopping time $\tau \in \mathcal{T}$, $\tau \geq t$, can be written as $\tau(\overline{\omega}) = t + \tau'(\overline{\omega}, \theta_t(\overline{\omega}))$, with $\tau' : \overline{\Omega} \times \overline{\Omega} \mapsto [0, \infty]$, $\mathcal{F}_t \otimes \mathcal{F}_\infty$-measurable and such that $\tau'(\overline{\omega}, \cdot)$ is a stopping time for each $\overline{\omega} \in \overline{\Omega}$. In this way, defining the $\mathcal{F}_t \otimes \mathcal{F}_\infty$-measurable positive random variable

$$Z(\omega, \omega^{'}) := \int_0^{\tau'(\omega,\omega^{'})} e^{-ru}\pi_c\Big(e^{x+\omega^{'}_u}, -\frac{1}{l}\Big)\, du + e^{-r\tau'(\omega,\omega^{'})}, \tag{2.20}$$

and setting $Z^t(\overline{\omega}) := Z(\overline{\omega}, \theta_t(\overline{\omega}))$, after a simple change of variable the term inside the conditional expectation in (2.19) equals $Z^t(\overline{\omega})$. More precisely,

$$\begin{aligned}\int_t^{\tau(\overline{\omega})} & e^{-r(s-t)}\pi_c\Big(e^{x+X_s(\overline{\omega})}, -\frac{1}{l}\Big)\, ds + e^{-r(\tau(\overline{\omega})-t)} \\ &= \int_0^{\tau'(\overline{\omega},\theta_t(\overline{\omega}))} \mathrm{e}^{-ru}\pi_c\Big(e^{x+\theta_t(\overline{\omega})(u)}, -\frac{1}{l}\Big)\, du + \mathrm{e}^{-r\tau'(\overline{\omega},\theta_t(\overline{\omega}))} \\ &= Z^t(\overline{\omega}).\end{aligned}$$

An application of the strong Markov property (see, e.g., Exercise 3.19 at p. 111 of [35]) thus implies

$$\mathbb{E}\{Z^t \,|\, \mathcal{F}_t\}(\overline{\omega}) = \mathbb{E}_{X_t(\overline{\omega})}\{Z(\overline{\omega}, \cdot)\},$$

for all $\overline{\omega} \in \overline{\Omega}$. Recalling the definition of v in (2.12) and using (2.20) it holds for all $\overline{\omega} \in \overline{\Omega}$

$$\xi^*_t(\overline{\omega}) = \sup\{l < 0 \;:\; v(x + X_t(\overline{\omega}), -\frac{1}{l}) = 1\}.$$

Finally, employing arguments as those in [22] (see the proof of Theorem 3.9) we may write for each $\overline{\omega} \in \overline{\Omega}$ and $t \geq 0$

$$l^*_t(\overline{\omega}) = -\frac{1}{\xi^*_t(\overline{\omega})} = \sup\{y > 0 \;:\; v(x + X_t(\overline{\omega}), y) = 1\} = b(x + X_t(\overline{\omega})),$$

where the last equality above follows from (2.13). This completes the proof.

□

At this point it is clear that if ν^* of (2.9) is admissible and such that $\mathcal{J}_{x,y}(\nu^*) < \infty$, hence optimal, then the free-boundary b of the optimal stopping problem (2.12) is indeed the optimal investment boundary of problem (2.6). Moreover, then also the continuation region $\mathcal{C}$ and the stopping region $\mathcal{S}$ are the inaction and the action region, respectively. In fact, due to Theorem 2.12 the optimal control ν^* of (2.9) can be expressed as

$$\nu^*_t = \sup_{0\leq s<t} (b(x + X_s) - y) \vee 0, \quad t > 0, \qquad \nu^*_0 = 0; \tag{2.21}$$

i.e., it is the least effort needed at time t to reflect the production capacity at the (random) time-dependent boundary $l^*_t = b(x + X_t)$, $t \geq 0$.

Remark 2.13. *Combining Theorem 2.12 and Proposition 2.11 we recover [36], Theorem 5.1, in which, via an argument different than ours, it is shown that the base capacity is monotonically increasing in the underlying shock process; namely, if l^* is the base capacity associated with a Lévy process X and $\tilde{l}^*$ is the base capacity associated with another Lévy process $\widetilde{X}$ such that $\widetilde{X}_t \leq X_t$ for all $t \geq 0$ a.s., then $\tilde{l}^*_t \leq l^*_t$ for all $t \geq 0$ a.s.*

3 The Equation for the Optimal Investment Boundary

In this section we present our main result. Using Propositions 2.5 and 2.12 and the Wiener-Hopf factorization we firstly derive an integral equation for the optimal investment boundary b (see Theorem 3.1 below). It is shown that if the Lévy process hits every point in $\mathbb{R}$ with positive probability then this equation has a unique solution. In Theorem 3.3 another simpler equation is presented which anyway characterizes the optimal investment boundary. Using such equation we, moreover, show that the boundary is continuous. To the best of our knowledge a proof of the continuity of the free-boundary in infinite time-horizon, one-dimensional parameter-dependent optimal stopping problems of type (2.12) for exponential Lévy processes appears here for the first time.

To simplify exposition, in the rest of this section we will assume that ν^* of (2.9) is admissible and such that $\mathcal{J}_{x,y}(\nu^*) < \infty$, and hence optimal. This way the free-boundary b of the optimal stopping problem (2.12) is indeed the optimal investment boundary of problem (2.6).

Theorem 3.1. *Let Assumptions 2.1, 2.2 and 2.8 hold. Let $M_t := \sup_{0\le u\le t} X_u$, $I_t := \inf_{0\le u\le t} X_u$. Moreover, let T_r denote an exponentially distributed random time with parameter r independent of X. Then, the optimal investment boundary b between the inaction and the action region is a positive nondecreasing right-continuous with left limits solution to the integral equation*

$$\int_0^\infty \mathbb{E}\Big\{\pi_c\Big(e^{y+z+I_{T_r}}, f(y+z)\Big)\Big\}\mathbb{P}(M_{T_r}\in dz) = r. \tag{3.1}$$

Moreover, if the Lévy process X hits every point of $\mathbb{R}$ with positive probability, then the solution of (3.1) *is unique.*

Proof. Since l^* solves (2.8) and $l^*_t = b(x+X_t)$ (cf. Theorem 2.12), then b satisfies

$$\begin{aligned} r &= \mathbb{E}\Big\{\int_\tau^\infty re^{-r(s-\tau)}\pi_c\Big(e^{x+X_s}, \sup_{\tau\le u<s} b(x+X_u)\Big)ds\Big|\mathcal{F}_\tau\Big\} \\ &= \mathbb{E}\Big\{\int_0^\infty re^{-rt}\pi_c\Big(e^{x+X_\tau+(X_{t+\tau}-X_\tau)}, b\big(\sup_{0\le u<t}(x+X_\tau+X_{u+\tau}-X_\tau)\big)\Big)dt\Big|\mathcal{F}_\tau\Big\}, \end{aligned} \tag{3.2}$$

for any $\tau\in\mathcal{T}$, where in the second equality we have used the fact that b is nondecreasing. Using the independence of increments and the strong Markov property of X it is seen that (3.2) is equivalent with

$$\mathbb{E}_y\Big\{\int_0^\infty re^{-rt}\pi_c\Big(e^{X_t}, b(M_t)\Big)dt\Big\} = r, \qquad \forall y\in\mathbb{R},$$

and, furthermore, with

$$\mathbb{E}_y\Big\{\pi_c\Big(e^{X_{T_r}}, b(M_{T_r})\Big)\Big\} = r, \qquad \forall y\in\mathbb{R}. \tag{3.3}$$

But now, by the Wiener-Hopf factorization (cf. [25], Chapter 6) we know that $X_{T_r}-M_{T_r}$ is independent of M_{T_r} and $X_{T_r}-M_{T_r}$ has the same law as $\hat{I}_{T_r}$ with $\hat{I}$ an independent copy of I, and then we can write from (3.3)

$$\begin{aligned} r = \mathbb{E}_y\Big\{\pi_c\Big(e^{X_{T_r}}, b(M_{T_r})\Big)\Big\} &= \mathbb{E}\Big\{\mathbb{E}\Big\{\pi_c\Big(e^{y+X_{T_r}}, b(y+M_{T_r})\Big)\Big|M_{T_r}\Big\}\Big\} \\ &= \mathbb{E}\Big\{\mathbb{E}\Big\{\pi_c\Big(e^{y+M_{T_r}+(X_{T_r}-M_{T_r})}, b(y+M_{T_r})\Big)\Big|M_{T_r}\Big\}\Big\} \\ &= \int_0^\infty \mathbb{E}\Big\{\pi_c\Big(e^{y+z+I_{T_r}}, b(y+z)\Big)\Big\}\mathbb{P}(M_{T_r}\in dz), \end{aligned}$$

where spatial homogeneity of X has been used for the second equality above.

Finally, if $\tau^{\{x_o\}} := \inf\{t \geq 0 : X_t = x_o\} < \infty$ with positive probability for all $x_o \in \mathbb{R}$, then the uniqueness of a positive, nondecreasing right-continuous with left limits b satisfying (3.1) can be proved arguing by contradiction as in [22] (see the proof of Theorem 3.11).

□

Remark 3.2. *Sufficient conditions ensuring that the Lévy process X hits every point of $\mathbb{R}$ with positive probability can be found, e.g., in Theorem 7.12 of [25]. Examples of Lévy processes with such a property are any Lévy process with Gaussian component (including the case of jump-diffusion processes that play an important role in Financial Economics) or symmetric α-stable Lévy processes with $\alpha \in (1,2)$ (see Section 7.5 and Exercise 7.6 in [25]). Concerning spectrally one-sided Lévy processes one has that the set $C := \{x \in \mathbb{R} : \mathbb{P}(\tau^{\{x\}} < +\infty) > 0\}$ as defined in Theorem 7.12 of [25] is not empty (since spectrally one-sided Lévy processes creep) and therefore condition (7.21) therein is satisfied and (i)-(iii) apply accordingly.*

The Wiener-Hopf factorization in the proof of Theorem 3.1 replaces in our Lévy setting the use of the joint law of the position of a regular, one-dimensional diffusion and its running supremum evaluated at an independent exponential time (cf. [17] p. 185 and [11] p. 26) exploited in [22], Theorem 3.11, in the diffusive setting. It is also worth noticing that the Wiener-Hopf factorization has been recognized as a useful tool for solving one-dimensional, infinite time-horizon, optimal stopping problems for Lévy processes as shown in [12], [16], [18], [28], [29], [37], among others.

The next theorem represents our main result. It shows that in order to have a solution to (3.1) it suffices to find a solution of a simpler equation.

Theorem 3.3. *Under Assumptions 2.1, 2.2 and 2.8, there exists a unique positive function $\hat{b}$ satisfying the equation*

$$\mathbb{E}\Big\{\pi_c\Big(e^{u+I_{T_r}}, f(u)\Big)\Big\} = r, \qquad u \in \mathbb{R}. \tag{3.4}$$

The function $\hat{b}$ is nondecreasing and continuous. Moreover, if the Lévy process X hits every point of $\mathbb{R}$ with positive probability, then $\hat{b}$ is the optimal investment boundary between the inaction and the action region.

Proof. We start with showing that (3.4) admits at most one positive solution $\hat{b}$ such that it is nondecreasing and continuous. Define the function

$$\Phi(u,y) := \mathbb{E}\Big\{\pi_c\Big(e^{u+I_{T_r}}, y\Big)\Big\} - r, \quad (u,y) \in \mathbb{R} \times (0,\infty). \tag{3.5}$$

It is not hard to see that $\Phi(u,\cdot)$ is (strictly) decreasing for any $u \in \mathbb{R}$ due to Assumption 2.1. Moreover, thanks again to Assumption 2.1, we can apply the monotone convergence theorem to show that $\Phi(u,\cdot)$ is continuous on $(0,\infty)$, $\lim_{y\downarrow 0}\Phi(u,y) = \infty$ and $\lim_{y\uparrow\infty}\Phi(u,y) = \kappa - r < 0$ for any $u \in \mathbb{R}$, where the last limit is strictly negative by Assumption 2.2. Hence there exists a unique positive $\hat{b}(u)$, $u \in \mathbb{R}$, solving (3.4).

To prove that $\hat{b}$ is nondecreasing, fix $\varepsilon > 0$ and notice that (3.5) and the fact that $\hat{b}$ solves (3.4) imply

$$\begin{aligned} 0 = \Phi(u+\varepsilon, \hat{b}(u+\varepsilon)) - \Phi(u,\hat{b}(u)) &= \Phi(u+\varepsilon,\hat{b}(u+\varepsilon)) - \Phi(u,\hat{b}(u+\varepsilon)) \\ &+ \Phi(u,\hat{b}(u+\varepsilon)) - \Phi(u,\hat{b}(u)) \geq \Phi(u,\hat{b}(u+\varepsilon)) - \Phi(u,\hat{b}(u)), \end{aligned}$$

where the inequality above follows since, by Assumption 2.8, $\Phi(\cdot, y)$ is nondecreasing for any $y \in (0,\infty)$. Hence, one has $\Phi(u, \hat{b}(u+\varepsilon)) - \Phi(u, \hat{b}(u)) \leq 0$ and therefore $\hat{b}(u+\varepsilon) \geq \hat{b}(u)$, $u \in \mathbb{R}$, because $y \mapsto \Phi(u, y)$ is nonincreasing.

We next show the continuity of $\hat{b}$. We consider first its left-continuity. Fix $u \in \mathbb{R}$, take a sequence $\{u_n : n \in \mathbb{N}\} \subset \mathbb{R}$ such that $u_n \uparrow u$ as $n \uparrow \infty$, and define $\hat{b}(u-) := \lim_{n\uparrow\infty} \hat{b}(u_n)$. Without loss of generality, we may think that $u \geq u_n \geq u - \epsilon$, for a suitable $\epsilon > 0$, so that $\hat{b}(u_n) \geq \hat{b}(u-\epsilon)$ by the monotonicity of $\hat{b}$ and the fact that $e^{u_n + I_{T_r}} \leq e^u$ since $I_{T_r} \leq 0$. It thus follows from the concavity of $c \mapsto \pi(z,c)$ and the monotonicity of $z \mapsto \pi_c(z,c)$ (cf. Assumptions 2.1 and 2.8, respectively) that $\pi_c(e^{u_n+I_{T_r}}, \hat{b}(u_n)) \leq \pi_c(e^u, \hat{b}(u-\epsilon))$. Note that for $\{u_n : n \in \mathbb{N}\}$ as above we have

$$r = \mathbb{E}\Big\{\pi_c\Big(e^{u_n+I_{T_r}}, \hat{b}(u_n)\Big)\Big\}, \qquad n \in \mathbb{N}.$$

Letting $n \uparrow \infty$ yields

$$r = \lim_{n\uparrow\infty} \mathbb{E}\Big\{\pi_c\Big(e^{u_n+I_{T_r}}, \hat{b}(u_n)\Big)\Big\} = \mathbb{E}\Big\{\pi_c\Big(e^{u+I_{T_r}}, \hat{b}(u-)\Big)\Big\}, \tag{3.6}$$

where the dominated convergence theorem and joint continuity of π (cf. Assumption 2.1) are used. We then conclude that $\hat{b}(u-) = \hat{b}(u)$ by the uniqueness of the solution of (3.4). On the other hand, taking a sequence $\{u_n : n \in \mathbb{N}\} \subset \mathbb{R}$ such that $u_n \downarrow u$ as $n \uparrow \infty$ and following similar arguments as above, one can prove also right-continuity of $\hat{b}$. Consequently, $\hat{b}$ is continuous.

Clearly, the positive, nondecreasing continuous $\hat{b}$ solving (3.4) also solves (3.1), and then the positive, upper right-continuous optional process $\hat{l}_t := \hat{b}(x + X_t)$ solves (2.8). Hence, by Proposition 2.5 and Theorem 2.12 we have $\hat{l}_t := \hat{b}(x + X_t) = b(x + X_t) = l^*_t$, up to the indistinguishability. The proof is completed by arguing similarly as in the proof of Theorem 3.1 that $\hat{b} = b$.

□

As a remarkable by-product of Theorem 3.3 we have the following result.

Corollary 3.4. *If the Lévy process X hits every point of $\mathbb{R}$ with positive probability, then the optimal investment boundary b between the inaction and the action region is continuous.*

Remark 3.5. *It is worth noting that in the separable case $\pi(z,c) = zG(c)$, with G continuously differentiable, increasing, strictly concave and satisfying Inada conditions, equation* (3.4) *easily translates into equation (15) of [13] (see also [36], Example 3.3), where H therein is the (generalized) inverse of b. However, our result is much more general than that of [13]. Differently to [13] we have indeed not assumed that investment strategies stay bounded above and, moreover, our equation* (3.4) *holds for every operating profit satisfying Assumption 2.1, hence not necessarily separable. According to the discussion in [13], Section III, equation* (3.4) *may be seen as a correction to the Net Present Value rule, or Marshallian law, taking into account the irreversibility of investment strategies.*

Remark 3.6. *Combining Theorem 2.12 and Corollary 3.4, we find that if X hits every point of $\mathbb{R}$ with positive probability, then the base capacity process l^* of problem* (2.6)*, which in general is only known to be upper right-continuous by [6], has indeed the same path regularity as X, namely it has at least càdlàg paths.*

To some extent, our equation (3.4) may be interpreted as a substitute to the free-boundary value problem which one usually writes down to characterize the solution to an optimal stopping problem (see [32] for a review). In the case of optimal stopping problems with Lévy uncertainty it is still possible to derive a free-boundary problem (see, e.g., [12]), even if one has to pay attention to the sense in which the associated integro-differential operator is understood, and to which are the suitable regularity properties of the value function to be imposed at the boundary. It has been in fact noticed (see, e.g., [2], [12] and [31]) that the smooth-fit property of the value function of an optimal stopping problem (i.e. its C^1-property at the optimal boundary) may fail in a Lévy setting. Our equation (3.4), instead, is not derived from any free-boundary problem but it follows immediately from (3.1), thanks to the backward equation (2.8) for $l^* = b(x+X)$, the Wiener-Hopf factorization and the strong Markov property of X. It thus represents a very useful tool to determine the optimal investment boundary for the whole class of irreversible investment problems of type (2.6), under the assumption that the Lévy process hits every point of $\mathbb{R}$ with positive probability. In the next section we will show how to analytically solve equation (3.4) even in the non trivial case with a non-separable profit function.

4 Explicit Results

In this section we derive the explicit form of the optimal investment boundary of the irreversible investment problem (2.6) for the Cobb-Douglas and the CES (constant elasticity of substitution) operating profit functions, that is, for $\pi(z,c) = z^\alpha c^\beta$ with $\alpha, \beta \in (0,1)$, and $\pi(z,c) = (\alpha z^\gamma + (1-\alpha)c^\gamma)^{\frac{1}{\gamma}}$, with $\alpha, \gamma \in (0,1)$, respectively. Moreover, we will assume throughout this section that the Lévy process X hits any point of $\mathbb{R}$ with positive probability, so to have the optimal investment boundary as the unique solution of equation (3.4) (cf. Theorem 3.3).

Recall that T_r is an exponentially distributed random time with parameter r independent of X, $M_t := \sup_{0\le u\le t} X_u$ and $I_t := \inf_{0\le u\le t} X_u$. The notation $\widehat{\Psi}$ is used for the logarithm of the Laplace transform of X_1 (when well defined), i.e.,

$$\widehat{\Psi}(\lambda) := \log \mathbb{E}\{e^{\lambda X_1}\}.$$

4.1 Cobb-Douglas Operating Profit

Assume that the operating profit function is of the Cobb-Douglas type; that is, $\pi(z,c) = z^\alpha c^\beta$ for $\alpha, \beta \in (0,1)$.

Proposition 4.1. *Assume that $\widehat{\Psi}(\frac{\alpha}{1-\beta}) \vee \widehat{\Psi}(\alpha+\beta)$ is well defined and that (cf. also Assumption 2.2)*

$$r > 0 \vee \widehat{\Psi}(\frac{\alpha}{1-\beta}) \vee \widehat{\Psi}(\alpha+\beta). \tag{4.1}$$

Then for a Cobb-Douglas operating profit the optimal investment boundary is

$$b(x) = (\vartheta e^x)^{\frac{\alpha}{1-\beta}}, \qquad x \in \mathbb{R}, \tag{4.2}$$

with ϑ given by

$$\vartheta := \left(\frac{\beta \mathbb{E}\{e^{\alpha I_{T_r}}\}}{r}\right)^{\frac{1}{\alpha}}. \tag{4.3}$$

Proof. In this case equation (3.4) has the form

$$r = \beta e^{\alpha x} \mathbb{E}\{e^{\alpha I_{T_r}}\} b^{\beta-1}(x). \tag{4.4}$$

Taking $b(x) = (\vartheta e^x)^{\frac{\alpha}{1-\beta}}$ it is easy to see that (4.4) above is solved for ϑ as in (4.3).

The arguments employed in [36] (see the proof of Theorem 7.2) are easily adapted to our case with $\alpha \neq 1-\beta$ to show that if (4.1) holds then $\nu^*_t := \sup_{0 \leq s < t}(b(x+X_s) - y) \vee 0$, $t > 0$, and $\nu^*_0 := 0$, (cf. (2.21)) is admissible and $\mathcal{J}_{x,y}(\nu^*) < \infty$. Therefore, ν^* is optimal for problem (2.6) and, hence, b as given in (4.2) is the optimal investment boundary. □

Remark 4.2. *The result of Proposition 4.1 is in line with the findings of Proposition 7.1 and Theorem 7.2 in [36], in which the base capacity process l^* has been explicitly determined in the case of Lévy processes and Cobb-Douglas profits.*

4.2 CES Operating Profit

We turn now to the case with a non-separable operating profit of the CES (constant elasticity of substitution) type, that is, $\pi(z,c) = (\alpha z^\gamma + (1-\alpha)c^\gamma)^{\frac{1}{\gamma}}$ for some $\alpha \in (0,1)$. Moreover, to meet Assumption 2.1.1 let $\gamma \in (0,1)$ to have $\lim_{c\to 0} \pi_c(z,c) = 0$ and $\kappa := \lim_{c\to\infty} \pi_c(z,c) = (1-\alpha)^{\frac{1}{\gamma}}$. To the best of our knowledge, the explicit form of the optimal investment boundary of problem (2.6) for a non-separable profit of CES type and exponential Lévy processes appears here for the first time.

Proposition 4.3. *Assume that $\widehat{\Psi}(1)$ is well defined and that (cf. also Assumption 2.2)*

$$r > (1-\alpha)^{\frac{1}{\gamma}} \vee \widehat{\Psi}(1). \tag{4.5}$$

Then for a CES operating profit the optimal investment boundary is given by

$$b(x) = Ke^x, \qquad x \in \mathbb{R}, \tag{4.6}$$

where the constant K (depending on γ, α and r) is the unique positive solution to

$$\mathbb{E}\Big\{\Big(1 + \Big(\frac{\alpha}{1-\alpha}\Big)e^{\gamma I_{T_r}} K^{-\gamma}\Big)^{\frac{1-\gamma}{\gamma}}\Big\} = \frac{r}{(1-\alpha)^{\frac{1}{\gamma}}}. \tag{4.7}$$

Proof. In this case equation (3.4) becomes

$$\frac{r}{(1-\alpha)^{\frac{1}{\gamma}}} = \mathbb{E}\Big\{\Big(1 + \Big(\frac{\alpha}{1-\alpha}\Big)e^{\gamma(x+I_{T_r})} b^{-\gamma}(x)\Big)^{\frac{1-\gamma}{\gamma}}\Big\}. \tag{4.8}$$

Comparing (4.8) and (4.7) it is seen that $b(x) = Ke^x$ is a natural candidate for the optimal boundary. To validate our candidate we firstly have to show that (4.7) actually admits at most one positive solution. Define the function $F : (0,\infty) \mapsto \mathbb{R}$ as

$$F(u) := \mathbb{E}\Big\{\Big(1 + \Big(\frac{\alpha}{1-\alpha}\Big)e^{\gamma I_{T_r}} u^{-\gamma}\Big)^{\frac{1-\gamma}{\gamma}}\Big\} - \frac{r}{(1-\alpha)^{\frac{1}{\gamma}}}.$$

It is clear that $u \mapsto F(u)$ is strictly decreasing, continuous and because $0 < \gamma < 1$ it holds

$$\lim_{u \downarrow 0} F(u) \geq \lim_{u \downarrow 0} u^{\gamma-1} \mathbb{E}\Big\{ \Big(\Big(\frac{\alpha}{1-\alpha} \Big) e^{\gamma I_{T_r}} \Big)^{\frac{1-\gamma}{\gamma}} \Big\} - \frac{r}{(1-\alpha)^{\frac{1}{\gamma}}} = \infty.$$

Moreover, since $0 < \gamma < 1$ one also has $0 \leq e^{\gamma I_{T_r}} \leq 1$ and then

$$\lim_{u \uparrow \infty} F(u) \leq \lim_{u \uparrow \infty} \Big(1 + \Big(\frac{\alpha}{1-\alpha} \Big) u^{-\gamma} \Big)^{\frac{1-\gamma}{\gamma}} - \frac{r}{(1-\alpha)^{\frac{1}{\gamma}}} = 1 - \frac{r}{(1-\alpha)^{\frac{1}{\gamma}}} < 0,$$

where the last inequality is due to the fact that $r > (1-\alpha)^{\frac{1}{\gamma}}$ by the assumption. It thus follows that $F(u) = 0$ admits at most one positive solution.

To complete the proof, we have to show that $\nu^*_t := \sup_{0 \leq s < t}(b(x+X_s)-y) \vee 0$, $t > 0$, $\nu^*_0 := 0$, (cf. (2.21)) is admissible and such that $\mathcal{J}_{x,y}(\nu^*) < \infty$; hence, optimal for problem (2.6). Clearly, ν^* is $(\mathcal{F}_t)$-adapted, left-continuous and nondecreasing. By the Wiener-Hopf factorization

$$\mathbb{E}\big\{ e^{M_{T_r}} \big\} \mathbb{E}\big\{ e^{I_{T_r}} \big\} = \frac{r}{r - \widehat{\Psi}(1)}, \tag{4.9}$$

Then recalling (4.5) and using (4.9) we obtain

$$\begin{aligned} \mathbb{E}\Big\{ \int_0^\infty r e^{-rt} \nu^*_t dt \Big\} &\leq K \mathbb{E}\Big\{ \int_0^\infty r e^{-rt} \sup_{0 \leq s < t} e^{x+X_s} dt \Big\} = K \mathbb{E}\Big\{ \int_0^\infty r e^{-rt} e^{x+M_t} dt \Big\} \\ &= K e^x \mathbb{E}\big\{ e^{M_{T_r}} \big\} < \infty. \end{aligned}$$

Consequently, integrating by parts yields

$$\mathbb{E}\Big\{ \int_0^\infty e^{-rt} d\nu^*_t \Big\} < \infty, \tag{4.10}$$

i.e., ν^* is admissible. Next consider

$$\begin{aligned} &\mathbb{E}\Big\{ \int_0^\infty e^{-rt} \pi(e^{x+X_t}, y + \nu^*_t) dt \Big\} \\ &\qquad \leq \frac{1}{r} \mathbb{E}\Big\{ \int_0^\infty r e^{-rt} \Big(e^{\gamma(x+X_t)} + \big(y + \sup_{0 \leq s < t} K e^{x+X_s} \big)^\gamma \Big)^{\frac{1}{\gamma}} dt \Big\} \\ &\qquad \leq \frac{2^{\frac{1-\gamma}{\gamma}}}{r} \mathbb{E}\Big\{ \int_0^\infty r e^{-rt} \Big(e^{x+X_t} + y + \sup_{0 \leq s < t} K e^{x+X_s} \Big) dt \Big\} \\ &\qquad = \frac{2^{\frac{1-\gamma}{\gamma}}}{r} \Big(y + e^x \mathbb{E}\big\{ e^{X_{T_r}} \big\} + K e^x \mathbb{E}\big\{ e^{M_{T_r}} \big\} \Big) < \infty, \end{aligned} \tag{4.11}$$

where we have used again (4.5) and (4.9). Combining (4.10) and (4.11) shows that $\mathcal{J}_{x,y}(\nu^*) < \infty$ (cf. (2.4)).

It thus follows that ν^*_t is optimal for problem (2.6) and b in (4.6) is the optimal investment boundary. □

Remark 4.4. *Clearly, the case $\gamma = \frac{1}{n}$, $n \geq 2$, discussed in [22] in a diffusive setting, is a particular case of the one studied in Proposition 4.3 and it follows that in such a case $b(x) = Ke^x$ for some positive constant $K := K(n, \alpha, r)$ solving equation (4.7). An application of the binomial expansion (see also [22], Section 4.2) reduces equation (4.7) for the constant K to the following polynomial equation of order $n-1$*

$$\sum_{j=1}^{n-1} \binom{n-1}{j} A_{j,n} \Big(\frac{\alpha}{1-\alpha}\Big)^j K^{-\frac{j}{n}} - \left[\frac{r}{(1-\alpha)^{\frac{1}{\gamma}}} - 1\right] = 0$$

with $A_{j,n} := \mathbb{E}\{e^{\frac{j}{n} I_{T_r}}\}$. Such a polynomial equation admits a unique positive solution thanks to Descartes' rule of signs since $r > (1-\alpha)^{\frac{1}{\gamma}}$.

Acknowledgments. Giorgio Ferrari thanks Frank Riedel for useful discussions. Paavo Salminen thanks the Center for Mathematical Economics, Bielefeld University for the hospitality and the support during the stay in Bielefeld and Andreas Kyprianou for a stimulating exchange via e-mail.

References

[1] ABEL, A.B., EBERLY, J.C. (1996). Optimal Investment with Costly Reversibility. *Rev. Econ. Stud.* **63** 581–593.

[2] ALILI, L., KYPRIANOU, A.E. (2005). Some Remarks on First Passage of Lévy Processes, the American Put and Pasting Principles. *Ann. Appl. Probab.* **15(3)** 2062–2080.

[3] BALDURSSON, F.M., KARATZAS, I. (1997). Irreversible Investment and Industry Equilibrium. *Finance Stoch.* **1** 69–89.

[4] BANK, P., RIEDEL, F. (2001). Optimal Consumption Choice with Intertemporal Substitution. *Ann. Appl. Probab.* **11** 750–788.

[5] BANK, P., FÖLLMER, H. (2002). American Options, Multi-armed Bandits, and Optimal Consumption Plans: a unifying view. In *Paris-Princeton Lectures on Mathematical Finance*. Lecture Notes in Math. **1814** 1–42. Springer-Verlag. Berlin.

[6] BANK, P., EL KAROUI, N. (2004). A Stochastic Representation Theorem with Applications to Optimization and Obstacle Problems. *Ann. Probab.* **32** 1030–1067.

[7] BANK, P. (2005). Optimal Control under a Dynamic Fuel Constraint. *SIAM J. Control Optim.* **44** 1529–1541.

[8] BENTOLILA, S., BERTOLA, G. (1990). Firing Costs and Labour Demand: How Bad is Eurosclerosis? *Rev. Econom. Stud.* **57(3)** 381–402.

[9] BERTOIN, J. (1996). *Lévy Processes*. Cambridge University Press.

[10] BERTOLA, G. (1998). Irreversible Investment. *Res. Econ.* **52** 3–37.

[11] BORODIN, A.N., SALMINEN, P. (2002). *Handbook of Brownian Motion - Facts and Formulae, 2nd. ed.* Birkhäuser Verlag. Basel.

[12] BOYARCHENKO, S., LEVENDORSKII, S. Z. (2002). Perpetual American Options under Lévy Processes. *SIAM J. Control Optim.* **40** 1663-1696.

[13] BOYARCHENKO, S. (2004). Irreversible Decisions and Record-Setting News Principles. *Am. Econ. Rev.* **94(3)** 557–568.

[14] CHIAROLLA, M.B., HAUSSMANN, U.G. (2009). On a Stochastic Irreversible Investment Problem. *SIAM J. Control Optim.* **48** 438–462.

[15] CHIAROLLA, M.B., FERRARI, G. (2014). Identifying the Free-Boundary of a Stochastic, Irreversible Investment Problem via the Bank-El Karoui Representation Theorem. *SIAM J. Control Optim.* **52(2)** 1048–1070.

[16] CHRISTENSEN, S., SALMINEN, P. and TA, Q.B. (2013). Optimal Stopping of Strong Markov Processes. *Stochastic Process. Appl.* **123(3)** 1138–1159.

[17] CSÁKI, E., FÖLDES, A. and SALMINEN, P. (1987). On the Joint Distribution of the Maximum and its Location for a Linear Diffusion. *Ann. I. H. Poincaré*, Section B **23(2)** 179–194.

[18] DELIGIANNIDIS, G., LE H. and UTEV, S. (2009). Optimal Stopping for Processes with Independent Increments, and Applications. *J. Appl. Probab.* **46(4)** 1130–1145.

[19] DELLACHERIE, C., MEYER, P. (1978). *Probabilities and Potential. Chapters I–IV.* North-Holland Mathematics Studies **29**.

[20] DIXIT, A.K., PINDYCK, R.S. (1994). *Investment under Uncertainty.* Princeton University Press. Princeton.

[21] EL KAROUI, N., KARATZAS, I. (1991). A New Approach to the Skorohod Problem and its Applications. *Stoch. Stoch. Rep.* **34** 57–82.

[22] FERRARI, G. (2013). On an Integral Equation for the Free-Boundary of Stochastic, Irreversible Investment Problems. **arXiv**:1211.0412v1, forthcoming on Ann. Appl. Probab.

[23] KARATZAS, I. (1981). The Monotone Follower Problem in Stochastic Decision Theory. *Appl. Math. Optim.* **7** 175–189.

[24] KARATZAS, I., SHREVE, S.E. (1984). Connections between Optimal Stopping and Singular Stochastic Control I. Monotone Follower Problems. *SIAM J. Control Optim.* **22** 856–877.

[25] KYPRIANOU, A.E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer.

[26] LIANG, J., YANG, M. and JIANG, L. (2013). A Closed-Form Solution for the Exercise Strategy in Real Options Model with a Jump-Diffusion Process. *SIAM J. Appl. Math.* **73(1)** 549–571

[27] MCDONALD, R., SIEGEL, D. (1986). The Value of Waiting to Invest. *Q. J. Econ.* **101** 707–727.

[28] MORDECKI, E. (2002). Optimal Stopping and Perpetual Options for Lévy Processes. *Finance Stoch.* **6(4)** 473–493.

[29] MORDECKI, E., SALMINEN, P. (2007). Optimal Stopping of Hunt and Lévy Processes. *Stochastics* **79(3–4)** 233–251.

[30] ØKSENDAL, A. (2000). Irreversible Investment Problems. *Finance Stoch.* **4** 223–250.

[31] PESKIR, G., SHIRYAEV, A. (2000). Sequential Testing Problems for Poisson Processes. *Ann. Stat.* **28** 837–859.

[32] PESKIR, G., SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics ETH, Birkhauser.

[33] PHAM, H. (2006). Explicit Solution to an Irreversible Investment Model with a Stochastic Production Capacity. In: *From Stochastic Analysis to Mathematical Finance, Festschrift for Albert Shiryaev* (Y. Kabanov and R. Liptser eds.). Springer.

[34] PINDYCK, R.S. (1988). Irreversible Investment, Capacity Choice, and the Value of the Firm. *Am. Econ. Rev.* **78** 969–985.

[35] REVUZ, D., YOR, M. (1999). *Continuous Martingales and Brownian Motion*. Springer-Verlag. Berlin.

[36] RIEDEL, F., SU, X. (2011). On Irreversible Investment. *Finance Stoch.* **15(4)** 607–633.

[37] SALMINEN, P. (2011). Optimal Stopping, Appel Polynomials and Wiener-Hopf Factorization. *Stochastics* **83(4–6)** 611–622.

[38] STEG, J.H. (2012). Irreversible Investment in Oligopoly. *Finance Stoch.* **16(2)** 207–224.

[39] TOPKIS, D.M. (1978). Minimizing a Submodular Function on a Lattice. *Oper. Res.* **26** 305–321.