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# Fear of the Market or Fear of the Competitor? Ambiguity in a Real Options Game*

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**Abstract**

In this paper we study a two–player investment game with a first mover advantage in continuous time with stochastic payoffs, driven by a geometric Brownian motion. One of the players is assumed to be ambiguous with max–min preferences over a strongly rectangular set of priors. We develop a strategy and equilibrium concept allowing for ambiguity and show that equilibria can be preemptive (a player invests at a point where investment is Pareto dominated by waiting) or sequential (one player invests as if she were the exogenously appointed leader). Following the standard literature, the worst–case prior for the ambiguous player if she is the second mover is obtained by setting the lowest possible trend in the set of priors. However, if the ambiguous player is the first mover, then the worst–case prior can be given by either the lowest or the highest trend in the set of priors. This novel result shows that "worst–case prior" in a setting with geometric Brownian motion and $\kappa$–ambiguity over the drift does not always equate to "lowest trend".

*Keywords:* Real Options, Knightian Uncertainty, Worst–Case Prior, Optimal Stopping, Timing Game

*JEL classification:* C61, C73, D81, L13

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# 1 Introduction

Many, if not most, investment decisions taken by firms are characterized by substantial upfront sunk costs, (partial) irreversibility, and uncertainty over future cash flows (cf. Dixit and Pindyck (1994)). As has been well–recognized since Knight (1921), the uncertainty over future cash flows can seldomly be captured by a unique probability measure. That is to say, there is typically *ambiguity* over the correct probability measure. Extensive experimental evidence has shown that decision makers are typically ambiguity averse (cf. Ellsberg (1961)).

By incorporating an ambiguity aversion axiom into the subjective expected utility framework, Gilboa and Schmeidler (1989) have shown that ambiguity averse decision makers act as if they maximize expected utility over the worst–case prior within a (subjectively chosen) set of priors. In the context of a firm's investment decision it is common to assume that future cash flows develop according to a (continuous–time) stochastic process. In most of the literature it is assumed that cash flows grow at an expected growth rate, augmented with shocks that follow a (continuous–time) random walk. Incorporating ambiguity in such a setting is typically done by assuming that at any time $t$ the expected growth rate is not known, but can take any value in a given set (this is often referred to as *drift ambiguity*). The worst–case in this situation is the lowest possible expected growth rate (cf. Nishimura and Ozaki (2007)). So, in the Gilboa and Schmeidler (1989) framework applied to investment problems, the presence of drift ambiguity leads the firm to act cautiously: by considering the worst possible expected growth rate the firm values future cash flows assuming that nature will act malevolently. One could interpret this as a "fear of the market".

In this paper we extend this kind of analysis by including the effects of competition. In most markets firms are not making investment decisions in isolation; rather decisions are taken in a competitive environment, often oligopolistic in nature. This implies that a firm not only is ambiguous about future cash flows, but also about its competitors' actions. After all, suppose that a firm has just invested in a new technology to obtain a cost advantage, but that its competitor still has the option to invest as well. It is natural to assume that investment by the competitor lowers the first adopter's cash flows. It is similarly innocuous to assume that the competitor will make its investment decision when it expects the future cash flows to be high enough. This implies that, in expectation, the competitor will invest sooner when the expected growth rate of cash flows is higher. This, in turn, means that the worst–case for the first adopter is represented by the earliest possible time, in expectation, that the competitor invest, i.e. the highest possible expected growth rate. One can think of this as a "fear of the competitor".

The problem we address in this paper is two-fold. First, we investigate how these two diametrically opposed "fears" balance: what is *the* worst–case at any given time $t$ when "fear of the market" suggests the lowest possible expected growth rate, but "fear of the competitor" suggests the highest possible expected growth rate? It turns out that we can compute the worst-case prior explicitly: it is either the lowest or the highest expected growth rate. The regions where each of these worst cases dominates the other can, as we show, be computed exactly. Secondly, we investigate the impact of ambiguity on equilibrium investment behavior. In particular, we are interested in (i) constructing an appropriate notion of strategy for timing games with an ambiguous player,[1] and (ii) explore the differences in equilibrium behavior between ambiguous and non–ambiguous players. The latter goal leads us to study an investment game between two firms, one ambiguous and one non–ambiguous.

Our modeling of drift ambiguity follows the seminal contribution of Chen and Epstein (2002), who developed a solid framework for dealing with Gilboa and Schmeidler (1989) max–min preferences in a continuous time multiple prior model of ambiguity. This model has been applied to several problems in economics and finance to gain valuable insights in the consequences of a form of Knightian uncertainty, as opposed to risk, on economic decisions. The main insight of Chen and Epstein (2002) is that in order to find the max–min value of a payoff stream under a particular kind of ambiguity (called *strongly rectangular*) we need to identify the *upper–rim* generator of the set of multiple priors, and value the payoff stream as if this were the true process governing the payoffs. Finding this upper–rim generator is particularly easy if attention is restricted to so-called *$\kappa$-ignorance*, a form of drift ambiguity, where at each point in time the drift is assumed to lie within the *same* compact set.

In this paper, we extend the single–firm Nishimura and Ozaki (2007) model to a timing game between two firms, which both have the option to invest in a project. We assume that one firm is ambiguous about the process governing cash-flows and that the other firm (potentially) has a cost disadvantage.[2] This assumption is made to illustrate the difference an introduction of ambiguity makes compared to a purely risky world in a game theoretic model of investment.

Our main conclusions are as follows. First, contrary to all of the literature on $\kappa$–ignorance in a real options framework, the worst–case prior is not always the lowest possible trend. As in any timing game, an ambiguous player has to consider the payoffs of the leader and follower roles. The payoffs of the latter role

[1]Since the seminal contribution of Fudenberg and Tirole (1985) for deterministic timing games, many attempts to defining equilibria in stochastic timing games have been made such as Thijssen (2010), Thijssen et al. (2012), de Villemeur et al. (2014), Boyarchenko and Levendorski (2014), Azevedo and Paxson (2014), Huisman and Kort (2015).

[2]The assumption that only one firm is ambiguous is not critical. In fact, Section 6 shows that our results can easily be adopted to the case where both firms are ambiguous, possibly to a different degree.

follow along very similar lines as in Nishimura and Ozaki (2007), i.e. the worst–case payoff corresponds to valuing the follower's payoff stream as if the payoffs are driven by the diffusion with the lowest admissible trend under $\kappa$–ignorance. For the leader's payoff, however, the situation is different, because of the interplay between the two opposing forces of "fear of the market" and "fear of the competitor". In Section 3, we use an analysis based on *backward stochastic differential equations* and *g–expectations*, as introduced by Peng (1997), to study which effect dominates. It turns out that for small values of the stochastic process, the worst-case always corresponds to the lowest admissible trend, whereas for higher values the highest admissible trend may represent the worst-case, depending on the underlying parameters. This result also constitutes a contribution to the ambiguity literature, because we provide a very natural setting in which the worst–case prior is non–trivial.

Secondly, in Section 4 we show that equilibria can be of two types. First, there may be preemptive equilibria in which one of the firms invests at a time where it is not optimal for either firm to do so. This type of equilibrium is familiar from the literature ( e.g. Fudenberg and Tirole (1985), Weeds (2002), Pawlina and Kort (2006)) but we use a technique recently developed by Riedel and Steg (2014) to rigorously prove existence of this type of equilibrium. It should be pointed out here that in a preemptive equilibrium it is known a.s. ex ante which firm is going to invest first. This firm will invest at a point in time where its leader value exceeds its follower value, but where its competitor is indifferent between the two roles. A second type of equilibrium that can exist is a sequential equilibrium, in which one firm invests at the same it would if it knew that the other firm could not preempt. Each game always has at least an equilibrium of one of these two types, which can not co–exist. These two types of equilibrium each lead to a clear prediction, a.s., as to which firm invests first. The role of first mover depends crucially on the levels of ambiguity and cost (dis–) advantage, as we show in a numerical analysis.

As mentioned above we obtain our equilibrium results by using techniques developed by Riedel and Steg (2014). It should be pointed out that we cannot simply adopt their strategies to our setting due to the presence of an ambiguous player. In fact, the notion of extended mixed strategy as introduced in Riedel and Steg (2014) presents a conceptual problem here. An extended mixed strategy consists, in essence, of a distribution over stopping times as well as a coordination device that allows players to coordinate in cases where equilibrium considerations require one and only one firm to invest and it is not clear a priori which firm this should be. In our model we need this coordination device as well, but we do not want ambiguity to extend to the uncertainty created by this coordination mechanism, i.e. ambiguity is over payoffs exclusively. This presents problems if we want to define payoffs to the ambiguous firm if it plays a mixture over stopping times. For equilibrium existence, however, such mixtures are not needed, so we choose to restrict attention to what we call *extended pure strategies*, which consist of a stopping time and an element related to the coordination mechanism mentioned above. By making this simplifying assumption, together with strong rectangularity of the set of priors, we can write the worst–case payoff of a pair of extended pure strategies as a sum of worst–cases of leader and follower payoffs.

In Section 5 we provide some comparative statics. In particular, we explore the effect of a change in (i) the degree of ambiguity, (ii) the volatility and (iii) firm 2's cost–disadvantage on equilibrium outcomes. We show numerically that the investment thresholds of the ambiguous firm increase with the degree of ambiguity. Due to the construction of the set of priors via $\kappa$–ignorance, an increase of volatility not only increases the variance of future payoffs, but it also expands the set of priors. It turns out that both firms' investment thresholds rise with the volatility. Due to the effect on the set of priors, however, the thresholds of the ambiguous firm are more affected by a change of the volatility than those of the unambiguous firm. Finally, while Pawlina and Kort (2006) argue that in a purely risky world, the low–cost firm always becomes the leader, we show that this might change if the low–cost firm is sufficiently ambiguous.

## 2 The Model

We follow Pawlina and Kort (2006) in considering two firms that are competing to implement a new technology. Uncertainty in the market is modeled on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$ using a geometric Brownian motion

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t, \tag{1}$$

where $(B_t)_{t\geq 0}$ is a Wiener process. The sunk costs of investment are $I > 0$ for firm 1 and $\eta I$, $\eta > 0$ for firm 2. Typically, we will assume that $\eta > 1$, so that firm 1 has a cost advantage.

The payoff streams are given by processes $(D_{k\ell}X_t)_{t\geq 0}$, where $D_{k\ell}$, $k, \ell = 0, 1$, denotes a scaling factor if the firm's investment status is $k$ ($k = 0$ if the firm has not invested and $k = 1$ if the firm has invested) and the investment status of the competitor is $\ell \in \{0, 1\}$. It is assumed that $D_{10} > D_{11} \geq D_{00} \geq D_{01} \geq 0$, and that there is a first mover advantage, i.e. $D_{10} - D_{00} > D_{11} - D_{01}$.

We assume that, although firm 1 has a cost advantage, it is also ambiguous about the drift $\mu$. Following the recent literature on drift ambiguity in continuous time models, we model priors that the firm considers using a set of density generators. Denoting this set of density generators by $\Theta$, the set of probability measures that constitutes the firm's set of priors is denoted by $\mathscr{P}^{\Theta}$. A process $(\theta_t)_{t\geq 0}$ is a density generator if the process

$\left(M_t^\theta\right)_{t\geq 0}$, where

$$\frac{dM_t^\theta}{M_t^\theta} = -\theta_t dB_t, \quad M_0^\theta = 1, \tag{2}$$

is a $\mathsf{P}$–martingale. Such a process $(\theta_t)_{t\geq 0}$ generates a new measure $\mathsf{P}^\theta$ via the Radon–Nikodym derivative $d\mathsf{P}^\theta/d\mathsf{P} = M_\infty^\theta$.

In order to use density generators as a model for ambiguity the set $\Theta$ needs some more structure. Following Chen and Epstein (2002), the set of density generators, $\Theta$, is chosen as follows. Let $(\Theta_t)_{t\geq 0}$ be a collection of correspondences $\Theta_t : \Omega \twoheadrightarrow \mathbb{R}$, such that

1. There is a compact subset $K \subset \mathbb{R}$, such that $\Theta_t(\omega) \subseteq K$, for all $\omega \in \Omega$ and all $t \in [0, T]$;

2. For all $t \in [0, T]$, $\Theta_t$ is compact-valued and convex-valued;

3. For all $t \in (0, T]$, the mapping $(s, \omega) \mapsto \Theta_s(\omega)$, restricted to $[0, t] \times \Omega$, is $\mathscr{B}[0, t] \times \mathscr{F}_t$-measurable;

4. $0 \in \Theta_t(\omega)$, $dt \otimes d\mathsf{P}$-a.e.

The set of density generators is then taken to be,

$$\Theta = \{(\theta_t)_{t\geq 0} \,|\theta_t(\omega) \in \Theta_t(\omega), d\mathsf{P} - \text{a.e., all } t \geq 0\},$$

and the resulting set of measures $\mathscr{P}^\Theta$ is called *strongly-rectangular*. For sets of strongly rectangular priors the following has been obtained by Chen and Epstein (2002):

1. $\mathsf{P} \in \mathscr{P}^\Theta$;

2. All measures in $\mathscr{P}^\Theta$ are uniformly absolutely continuous with respect to $\mathsf{P}$ and are equivalent to $\mathsf{P}$;

3. For every $X \in \mathscr{L}^2(\Omega, \mathscr{F}, \mathsf{P})$, there exists $\mathsf{P}^* \in \mathscr{P}^\Theta$ such that for all $t \geq 0$,

$$\mathsf{E}^{\mathsf{P}^*}[X|\mathscr{F}_t] = \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q[X|\mathscr{F}_t]. \tag{3}$$

Finally, for further reference, define the *upper-rim generator* $(\theta_t^*)_{t\geq 0}$, where

$$\theta_t^* = \arg\max\{\sigma_w(t)\theta_t|\theta_t \in \Theta_t\}. \tag{4}$$

Note that $(\theta_t^*)_{t\geq 0} \in \Theta$.

From Girsanov's theorem it immediately follows that under $\mathsf{P}^\theta \in \mathscr{P}^\Theta$, the process $\left(B_t^\theta\right)_{t\geq 0}$, defined by

$$B_t^\theta = B_t + \int_0^t \theta_s ds,$$

is a $\mathsf{P}^\theta$-Brownian motion and that, under $\mathsf{P}^\theta$, the process $(X_t)_{t\geq 0}$ follows the diffusion

$$\frac{dX_t}{X_t} = \mu^\theta(t)dt + \sigma dB_t^\theta,$$

where

$$\mu^\theta(t) = \mu - \sigma\theta_t.$$

In the remainder we will assume that $\Theta_t = [-\kappa, \kappa]$, for all $t > 0$, for some $\kappa > 0$. Denote $\Delta = [\underline{\mu}, \overline{\mu}] = [\mu - \sigma\kappa, \mu + \sigma\kappa]$. This form of ambiguity is called *$\kappa$–ignorance* (cf. Chen and Epstein (2002)). The advantages of using this definition of ambiguity are that (i) $\Theta$ is strongly rectangular so that the results stated above apply and (ii) the upper–rim generator takes a convenient form, namely $\theta_t^* = \kappa$, for all $t \geq 0$. In addition, it can easily be shown that $\left(B_t^\theta\right)_{t\geq 0}$ is a $\mathsf{P}$-martingale for every $(\theta_t)_{t\geq 0} \in \Theta$.

Note that Cheng and Riedel (2013) show that $\kappa-$ignorance can be applied in an infinite time horizon. In particular, they show that value functions taken under drift ambiguity in the infinite time horizon are nothing but the limits of value functions of finite time horizons $T$ as $T \to \infty$.

In our model, we assume firm 1 to be ambiguity averse in the sense of Gilboa and Schmeidler (1989).

Finally, the discount rate is assumed to be $r > \overline{\mu}$ and to apply to both firms.

# 3 Leader and Follower Value Functions

## 3.1 The Non–Ambiguous Firm

Assume firm 1 becomes the leader at $t$. Then the non–ambiguous firm 2 solves the optimal stopping problem

$$F_2(x_t) = \sup_{\tau_2^F \geq t} \mathsf{E}^{\mathsf{P}} \left[ \int_t^{\tau_2^F} e^{-r(s-t)} D_{01} X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)} D_{11} X_s - e^{-r(\tau_2^F - t)} \eta I \middle| \mathscr{F}_t \right]. \tag{5}$$

Thus, $\tau_2^F$ is the optimal time at which firm 2 invests as a follower.

On the other hand, if the non–ambiguous firm becomes the leader at a certain point in time $t$, its value function is

$$L_2(x_t) = \mathsf{E}^{\mathsf{P}} \left[ \int_t^{\tau_1^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds - \eta I \middle| \mathscr{F}_t \right], \tag{6}$$

where $\tau_1^F$ denotes the optimal time at which the ambiguous firm invests as a follower. From the standard literature on real options games (cf. Pawlina and Kort (2006)) we know that the former value function can be written as

$$F_2(x_t) = \begin{cases} \frac{x_t D_{01}}{r-\mu} + \left( \frac{x_2^F (D_{11} - D_{01})}{r-\mu} - \eta I \right) \left( \frac{x_t}{x_2^F} \right)^{\beta(\mu)}, & \text{if} \quad x_t \leq x_2^F, \\ \frac{x_t D_{11}}{r-\mu} - \eta I & \text{if} \quad x_t > x_2^F, \end{cases} \tag{7}$$

where $\tau_2^F$ is the first hitting time (from below) of an endogenously determined threshold $x_2^F$, i.e

$$\tau_2^F = \inf\{s \geq t | X_s \geq x_2^F\}.$$

The standard procedure of dynamic programming yields that the threshold $x_2^F$ is given by

$$x_2^F = \frac{\beta(\mu)}{\beta(\mu) - 1} \frac{\eta I(r - \mu)}{D_{11} - D_{01}},$$

where $\beta(\mu)$ is the positive root of the fundamental quadratic $1/2\sigma^2\beta(\mu)(\beta(\mu) - 1) + \mu\beta(\mu) - r = 0$, i.e.

$$\beta(\mu) = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1.$$

Similarly, we will show below that firm 1's optimal stopping time in the follower role is the first hitting time (from below) of a threshold $x_1^F < \infty$, i.e.

$$\tau_1^F = \inf\{s \geq t | X_s \geq x_1^F\}.$$

By applying the standard techniques of backward induction and dynamic programming, one can therefore show that the leader value (6) is given by

$$L_2(x_t) = \begin{cases} \frac{x_t D_{10}}{r-\mu} - \eta I + \frac{x_1^F(D_{11}-D_{10})}{r-\mu}\left(\frac{x_t}{x_1^F}\right)^{\beta(\mu)}, & \text{if} \quad x_t \leq x_1^F, \\ \frac{x_t D_{11}}{r-\mu} - I, & \text{if} \quad x_t > x_1^F. \end{cases}$$

Finally, it is possible that both firms invest simultaneously at $t$. One can show that in that case the value function of firm 2 is

$$M_2(x_t) := \mathsf{E}^{\mathsf{P}}\left[\int_t^\infty e^{-r(s-t)} D_{11} X_s ds - \eta I \middle| \mathscr{F}_t\right] = \frac{x_t D_{11}}{r - \mu} - \eta I.$$

### 3.2 The Ambiguous Firm

If ambiguity is introduced, the standard techniques for computing the value functions are not applicable any longer. In our case, where ambiguity is modeled by a strongly rectangular set of density generators, one needs, in contrast to the standard case, to allow for changing priors over time.

The value functions for the ambiguous firm 1 of the follower and leader roles are given by

$$F_1(x_t) := \sup_{\tau_1^F \geq t} \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_1^F} e^{-r(s-t)} D_{01} X_s ds + \int_{\tau_1^F}^\infty e^{-r(s-t)} D_{11} X_s - e^{-r(\tau_1^F - t)} I \middle| \mathscr{F}_t\right] \tag{8}$$

and

$$L_1(x_t) := \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_2^F}^\infty e^{-r(s-t)} D_{11} X_s ds \middle| \mathscr{F}_t\right] - I, \tag{9}$$

respectively.

If the set of priors $\mathscr{P}^\Theta$ is strongly rectangular, it turns out that problem (8) can be reduced to a standard optimal stopping problem and, hence, can be solved by using standard techniques. This reduction is possible due to the following lemma, the proof of which is standard and is, thus, omitted.

**Lemma 1.** *Let $\mathscr{P}^\Theta$ be strongly–rectangular. Then*

$$F_1(x_t) = \sup_{\tau_1^F \geq t} \mathsf{E}^{P^{\theta^*}} \left[ \int_t^{\tau_1^F} e^{-r(s-t)} D_{01} X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds - e^{-r(\tau_1^F - t)} I \middle| \mathscr{F}_t \right], \tag{10}$$

*where $(\theta_t^*)_{t\geq 0}$ is the upper–rim generator* (4).

Hence, for the follower problem of the ambiguous firm, the worst–case is always induced by the worst possible drift $\underline{\mu}$. This observation indeed makes sense, since the actions of the leader have no influence on the decision of the follower once the leader has invested. The problem, therefore, reduces to one of a "monopolistic" decision maker. Nishimura and Ozaki (2007) already showed that for such decisions, the worst–case is always given by the worst possible trend $\underline{\mu}$.

In other words, we find that the follower value of the ambiguous firm can be expressed by

$$F_1(x_t) = \begin{cases} \frac{x_t D_{01}}{r-\underline{\mu}} + \left( \frac{x_1^F (D_{11} - D_{01})}{r - \underline{\mu}} - I \right) \left( \frac{x_t}{x_1^F} \right)^{\beta(\underline{\mu})}, & \text{if } \ x_t \leq x_1^F, \\ \frac{x_t D_{11}}{r-\underline{\mu}} - I & \text{if } \ x_t > x_1^F, \end{cases} \tag{11}$$

where

$$x_1^F = \frac{\beta(\underline{\mu})}{\beta(\underline{\mu}) - 1} \frac{I(r - \underline{\mu})}{D_{11} - D_{01}}.$$

In a similar way, one can argue that for simultaneous investment the value function of firm 1 is induced by the worst–case $\underline{\mu}$ and therefore

$$M_1(x_t) := \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q \left[ \int_t^\infty e^{-r(s-t)} D_{11} X_s ds - I \middle| \mathscr{F}_t \right] = \frac{x_t D_{11}}{r - \underline{\mu}} - I.$$

The next theorem describes the leader value function of the ambiguous firm. Two cases are distinguished there. If the difference $D_{10} - D_{11}$ is sufficiently small, we find that the worst–case is, as before, always induced by $\underline{\mu}$. In case this condition is not satisfied, the worst–case is given by $\underline{\mu}$ for small values $x_t$ up to a certain threshold $x^*$, where it jumps to $\overline{\mu}$. The intuition for this fact can already be derived from equation (9); the lowest trend $\underline{\mu}$ gives the minimal values for the payoff stream $(D_{kl} X_t)$. However, the higher the trend $\mu$ the sooner the stopping time $\tau_2^F$ is expected to be reached. The higher payoff stream $(D_{10} X_t)$ is then sooner replaced by the lower one $(D_{11} X_t)$. If the drop of the payoffs becomes sufficiently small, the former effect always dominates the latter. In this case the worst–case is given by $\underline{\mu}$ for each $x_t$.

**Theorem 1.** *The worst–case for the leader function of the ambiguous firm is always given by the worst possible drift* $\underline{\mu}$ *if and only if the following condition holds*

$$\frac{D_{10}-D_{11}}{D_{10}} \leq \frac{1}{\beta_1(\underline{\mu})}. \tag{12}$$

*In this case, the leader function becomes*

$$L_1(x_t) = \begin{cases} \frac{D_{10}x_t}{r-\underline{\mu}} - \left(\frac{x_t}{x_2^F}\right)^{\beta_1(\underline{\mu})} \frac{D_{11}-D_{10}}{r-\underline{\mu}} x_2^F - I & \text{if } x_t < x_2^F \\ \frac{D_{11}x_t}{r-\underline{\mu}} - I & \text{if } x_t \geq x_2^F. \end{cases} \tag{13}$$

*On the other hand, if* $\frac{D_{10}-D_{11}}{D_{10}} > \frac{1}{\beta_1(\underline{\mu})}$*, then there exists a unique threshold* $x^* \in (0, x_2^F)$ *such that* $\underline{\mu}$ *is the worst–case on* $\{X_t < x^*\}$ *and* $\overline{\mu}$ *is the worst–case on* $\{x^* \leq X_t < x_2^F\}$*. Furthermore, in this case the leader value function is given by*

$$L_1(x_t) = \begin{cases} \frac{D_{10}x_t}{r-\underline{\mu}} - \frac{1}{\beta_1(\underline{\mu})}\frac{D_{10}x^*}{r-\underline{\mu}}\left(\frac{x_t}{x^*}\right)^{\beta_1(\underline{\mu})} - I & \text{if } x_t < x^* \\ \frac{D_{10}x_t}{r-\overline{\mu}} + \frac{(x^*)^{\beta_2(\overline{\mu})}x_t^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}x_t^{\beta_2(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F & \\ + \frac{(x_2^F)^{\beta_1(\overline{\mu})}x_t^{\beta_2(\overline{\mu})} - (x_2^F)^{\beta_2(\overline{\mu})}x_t^{\beta_1(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left[\left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right]x^* - I & \text{if } x^* \leq x_t < x_2^F \\ \frac{D_{11}x_t}{r-\underline{\mu}} - I & \text{if } x_t \geq x_2^F, \end{cases} \tag{14}$$

*where* $\beta_1(\mu) > 1$ *and* $\beta_2(\mu) < 0$ *are the positive and negative roots of the quadratic equation* $1/2\sigma^2\beta(\mu)(\beta(\mu)-1) + \mu\beta(\mu) - r = 0$*, respectively.*

In case the worst–case is not trivially given by the lowest possible trend, the value function contains the terms

$$\frac{(x^*)^{\beta_2(\overline{\mu})}x_t^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}x_t^{\beta_2(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}} \quad \text{and} \quad \frac{(x_2^F)^{\beta_1(\overline{\mu})}x_t^{\beta_2(\overline{\mu})} - (x_2^F)^{\beta_2(\overline{\mu})}x_t^{\beta_1(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}},$$

which admit a clear interpretation: they represent the expected discount factor of the first hitting time of firm 2's follower threshold conditional on it being reached before $x^*$ is reached, and the expected discount factor of the first hitting time of $x^*$ conditional on it being reached before firm 2's follower threshold, respectively.

Figure 1 depicts the implications of Theorem (1). In case the drop of the payoff from being the only one who has invested to the situation that both players have invested is sufficiently big, the value $x^*$ distinguishes between the regions where each of the two "fears" dominates.

For the proof of Theorem (1), we need a different approach compared to the standard literature on real option games. We use backward stochastic differential equations and $g$-expectations as introduced by Peng

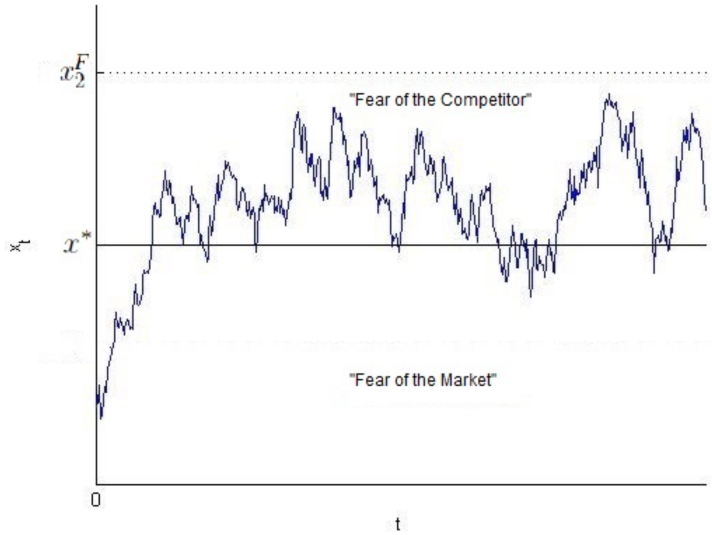


Figure 1: The critical value $x^*$ differentiates between two "regimes".

(1997). The advantage of this approach lies in the fact that we know the value of our problem at the entry point of the follower. This value yields the starting point for a backward stochastic differential equation. The non–linear Feynman–Kac formula reduces the problem to solving a particular non–linear partial differential equation. From this PDE we are eventually able to derive the worst–case prior.

**Proof.**

Denote

$$Y_t := \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_2^F}^\infty e^{-r(s-t)}D_{11}X_s ds \Big| \mathcal{F}_t\right].$$

Applying the time consistency property of a strongly rectangular set of density generators gives

$$\begin{aligned} Y_t &= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_2^F}^\infty e^{-r(s-t)}D_{11}X_s ds \Big| \mathcal{F}_t\right] \\ &= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_2^F}^\infty e^{-r(s-t)}D_{11}X_s ds \Big| \mathcal{F}_{\tau_2^F}\right] \Big| \mathcal{F}_t\right] \\ &= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + e^{-r(\tau_2^F-t)} \inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\left[\int_{\tau_2^F}^\infty e^{-r(s-\tau_2^F)}D_{11}X_s ds \Big| \mathcal{F}_{\tau_2^F}\right] \Big| \mathcal{F}_t\right] \\ &= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + e^{-r(\tau_2^F-t)}\Phi(x_{\tau_2^F}) \Big| \mathcal{F}_t\right], \end{aligned}$$

where

$$\Phi(x_t) := \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[\int_t^\infty e^{-r(s-t)}D_{11}X_s ds \Big| \mathcal{F}_t\Big] = \frac{D_{11}x_t}{r-\underline{\mu}}. \tag{15}$$

Chen and Epstein (2002) show that $Y_t$ solves the BSDE

$$-dY_t = g(Z_t)dt - Z_t dB_t,$$

where, in this case, the *generator*, $g$, is given by

$$g(z) = -\kappa|z| - rY_t + X_t D_{10}.$$

The terminal boundary condition is given by

$$Y_{\tau_2^F} = \Phi(x_2^F),$$

In the terminology of Peng (2013), we say that the leader value is the $g$–expectation of the random variable $e^{-r(\tau_2^F - t)}\Phi(x_2^F)$, i.e.

$$Y_t = \mathsf{E}_g[e^{-r(\tau_2^F - t)}\Phi(x_2^F)|\mathcal{F}_t].$$

Denote the present value of the leader payoff by $L$, i.e.

$$L(x_t) = Y_t.$$

The non–linear Feynman–Kac formula[3] (Peng, 2013, Theorem 3) implies that $L$ solves the non–linear PDE

$$\mathscr{L}_X L(x) + g(\sigma x L'(x)) = 0,$$

where $\mathscr{L}_X$ is the characteristic operator of the SDE (1). Hence, $L$ solves

$$\frac{1}{2}\sigma^2 x^2 L''(x) + \mu x L'(x) - \kappa\sigma x \left|L'(x)\right| - rL(x) + D_{10}x = 0. \tag{16}$$

Expression (16) implies that $\underline{\mu}$ is the worst–case on the set $\{x \leq x_2^F | L'(x) > 0\}$ and $\overline{\mu}$ is the worst–case on $\{x \leq x_2^F | L'(x) < 0\}$.

The unique viscosity solution to the PDE (16) is given by

$$L(\mu, x) = \frac{D_{10}x}{r - \mu} + Ax^{\beta_1(\mu)} + Bx^{\beta_2(\mu)}, \tag{17}$$

where $\mu$ equals either $\underline{\mu}$ or $\overline{\mu}$. The constants $A$ and $B$ are determined by some boundary conditions.

One can easily see that for $x$ close to zero we have $L'(x) > 0$. Now two cases are possible: Either $L'(x) > 0$ for all $x \in [0, x_2^F]$ or we can find (at least) one point $x^*$ at which the worst–case changes from $\underline{\mu}$ to $\overline{\mu}$.

[3]Note that Peng (1991) shows that the non–linear Feynman–Kac formula not only holds for deterministic times but also first exit times like $\tau_2^F$, even if it does not hold a.s. that $\{\tau_2^F < \infty\}$.

Let us first assume that $\underline{\mu}$ is always the worst–case. Since $\beta_2(\underline{\mu}) < 0$, we have $B = 0$. In order to determine the constant $A$, we apply a value matching condition at $x_2^F$ that gives

$$L(\underline{\mu}, x_2^F) = \frac{D_{10}x_2^F}{r-\underline{\mu}} + A_1 {x_2^F}^{\beta_1(\underline{\mu})} = \frac{D_{11}x_2^F}{r-\underline{\mu}}.$$

This implies

$$A_1 = \frac{D_{10} - D_{11}}{r-\underline{\mu}} {x_2^F}^{1-\beta_1(\underline{\mu})},$$

and therefore

$$L(x_t) = \frac{D_{10}x_t}{r-\underline{\mu}} + \left(\frac{x_t}{x_2^F}\right)^{\beta_1(\underline{\mu})} \frac{D_{11} - D_{10}}{r-\underline{\mu}} x_2^F. \tag{18}$$

We get that $\underline{\mu}$ is always the worst–case on $[0, x_2^F]$ if and only if $L'(x) \geq 0$ for all $x \leq x_2^F$. Due to the continuity and concavity of the value function (18), this is equivalent to the condition

$$L'(x_2^F) \geq 0.$$

Therefore,

$$\begin{aligned} L'(x_2^F) = \frac{D_{10}}{r-\underline{\mu}} + \left(\frac{D_{11} - D_{10}}{r-\underline{\mu}}\right) \beta_1(\underline{\mu}) \left(\frac{x_2^F}{x_2^F}\right)^{\beta_1(\underline{\mu})-1} &\geq 0 \\ \iff D_{11} - D_{10} &\geq -\frac{D_{10}}{\beta_1(\underline{\mu})} \\ \iff \frac{D_{10} - D_{11}}{D_{10}} &\leq \frac{1}{\beta_1(\underline{\mu})}. \end{aligned}$$

If this condition is not satisfied, the worst–case changes at some point $x^* < x_2^F$ from $\underline{\mu}$ to $\overline{\mu}$, where $x^*$ is determined by the condition $L'(x^*) = 0$. We denote by $\tilde{L_1}(\underline{\mu}, x)$ the solution to (17) on $[0, x^*]$ and by $\hat{L_1}(\overline{\mu}, x)$ the solution to (17) on $[x^*, x_2^F]$. The unknowns in equation (17) are determined by applying twice a value matching condition and once a smooth pasting condition (see also Cheng and Riedel (2013)). Indeed, it must hold that

1. $\hat{L_1}(\overline{\mu}, x_2^F) = \Phi(x_2^F)$,

2. $\tilde{L_1}(\underline{\mu}, x^*) = \hat{L_1}(\overline{\mu}, x^*)$,

3. $\tilde{L_1'}(\underline{\mu}, x^*) = \hat{L_1'}(\overline{\mu}, x^*)$.

In case $\underline{\mu}$ is not always the worst–case, the unique viscosity solution of (17) is given by

$$L(x_t) = 1_{x_t < x^*} \tilde{L}_1(\underline{\mu}, x_t) + 1_{x_t \geq x^*} \hat{L}_1(\overline{\mu}, x_t),$$

where

$$\tilde{L}_1(\underline{\mu}, x_t) = \frac{D_{10}x_t}{r-\underline{\mu}} - \frac{1}{\beta_1(\underline{\mu})}\frac{D_{10}x^*}{r-\underline{\mu}}\left(\frac{x_t}{x^*}\right)^{\beta_1(\underline{\mu})},$$

and

$$\begin{aligned}\hat{L}_1(\overline{\mu}, x_t) =& \frac{D_{10}x_t}{r-\overline{\mu}} + \frac{(x^*)^{\beta_2(\overline{\mu})}x_t^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}x_t^{\beta_2(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F \\ &+ \frac{(x_2^F)^{\beta_1(\overline{\mu})}x_t^{\beta_2(\overline{\mu})} - (x_2^F)^{\beta_2(\overline{\mu})}x_t^{\beta_1(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x^*.\end{aligned}$$

We can easily verify that $\hat{L}_1$ and $\tilde{L}_1$ satisfy the boundary conditions. Indeed,

$$\begin{aligned}\hat{L}_1(\overline{\mu}, x_2^F) =& \frac{D_{10}x_2^F}{r-\overline{\mu}} + \frac{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F \\ &+ \frac{(x_2^F)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})} - (x_2^F)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x^* \\ =& \frac{D_{10}x_2^F}{r-\overline{\mu}} + \left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F \\ =& \frac{D_{11}x_2^F}{r-\overline{\mu}} \\ =& \Phi(x_2^F).\end{aligned}$$

and

$$\begin{aligned}\hat{L}_1(\overline{\mu}, x^*) =& \frac{D_{10}x^*}{r-\overline{\mu}} + \frac{(x^*)^{\beta_2(\overline{\mu})}(x^*)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x^*)^{\beta_2(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F \\ &+ \frac{(x_2^F)^{\beta_1(\overline{\mu})}(x^*)^{\beta_2(\overline{\mu})} - (x_2^F)^{\beta_2(\overline{\mu})}(x^*)^{\beta_1(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x^* \\ =& \frac{D_{10}x^*}{r-\overline{\mu}} + \left(\left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x^* \\ =& \frac{D_{10}x^*}{r-\underline{\mu}} - \frac{1}{\beta_1(\underline{\mu})}\frac{D_{10}x^*}{r-\underline{\mu}} \\ =& \tilde{L}_1(\underline{\mu}, x^*).\end{aligned}$$

To prove the smooth pasting condition at $x^*$ requires a bit more work. Firstly, we observe that the value $x^*$ is chosen such that it always holds that $\tilde{L}_1'(\underline{\mu}, x^*) = 0$.

The next lemma shows that there exists such a value $x^*$, which is unique and also satisfies $\hat{L}_1(\overline{\mu}, x^*) = 0$.

**Lemma 2.** *If* $\frac{D_{10}-D_{11}}{D_{10}} > \frac{1}{\beta_1(\underline{\mu})}$, *then there exists one and only one value* $x^*$ *that solves* $\hat{L}_1'(\overline{\mu}, x^*) = 0$ *on* $(0, x_2^F]$.

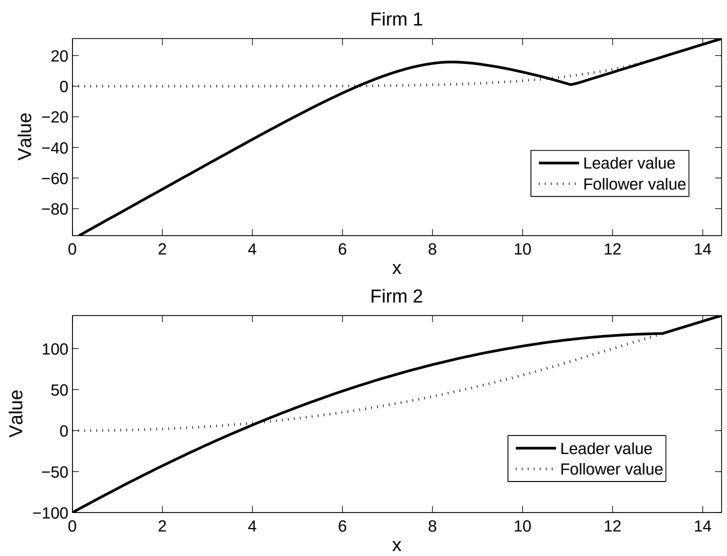


Figure 2: The leader and follower value functions of the ambiguous and non–ambiguous firm.

The proof is reported in the Appendix.

■

**Remark 1.** *The leader value function $L_1$ is always concave on $[0, x_2^F]$ even if the worst–case changes at some point. We prove this fact in the Appendix.*

Figure 2 shows a typical run of the leader and follower value functions of both the ambiguous and the non–ambiguous firm. We observe that the leader value function of firm 1 drops below its follower value function if $x_t$ is close to $x_2^F$. The reason for that is that $x_1^F$ and $x_2^F$ differ (in the illustrated case we have $x_2^F < x_1^F$). That means that the leader and follower value functions hit the shared value function $M$ at different times. This is the case because $x_1^F$ and $x_2^F$ are determined using a different trend. But even if firms use the same prior, in some cases we would observe this pattern, namely if we consider cost–asymmetric firms, i.e. if $\eta > 1$.

### 3.3 Optimal Leader Threshold

Next we want to determine the optimal time to invest as a leader. Suppose firm 2 knows it will not be preempted and searches for the optimal time to invest. It then faces at time $t$ the following optimal stopping problem:

$$L_2^*(x_t) = \sup_{\tau_{L,2}^t \geq t} \mathsf{E}^{\mathsf{P}} \Bigg[ \int_t^{\tau_{L,2}^t} e^{-r(s-t)} D_{00} X_s ds + \int_{\tau_{L,2}^t}^{\tau_1^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds - e^{-r(\tau_{L,2}^t - t)} \eta I \Big| \mathcal{F}_t \Bigg].$$

The solution can be found by applying the standard techniques and is well known from the literature: it is given by

$$\tau_{L,2}^t = \inf\{s \geq t | X_s \geq x_2^L\},$$

where

$$x_2^L = \frac{\beta_1(\mu)}{\beta_1(\mu) - 1} \frac{\eta I (r - \mu)}{D_{10} - D_{00}}.$$

The ambiguous firm solves the following optimal stopping problem

$$L_1^*(x_t) = \sup_{\tau_{L,1}^t \geq t} \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^{Q} \Bigg[ \int_t^{\tau_{L,1}^t} e^{-r(s-t)} D_{00} X_s ds + \int_{\tau_{L,1}^t}^{\tau_2^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds - e^{-r(\tau_{L,1}^t - t)} I \Big| \mathcal{F}_t \Bigg].$$

Again, in order to determine this stopping time for the ambiguous firm, we cannot apply the standard procedure. Nevertheless, the stopping time does not differ from the one of a non–ambiguous firm given a drift $\underline{\mu}$.

**Proposition 1.** *The optimal time to invest as a leader for the ambiguous firm is given by*

$$\tau_{L,1}^t = \inf\{s \geq t | X_s \geq x_1^L\},$$

*where*

$$x_1^L = \frac{\beta_1(\underline{\mu})}{\beta_1(\underline{\mu}) - 1} \frac{I (r - \underline{\mu})}{D_{10} - D_{00}}.$$

For the proof we refer to the Appendix.

# 4 Equilibrium Analysis

The appropriate equilibrium concept for a game with ambiguity as described here is not immediately clear. In this paper, we consider two types of equilibria: *preemptive equilibria* in which firms try to preempt each other at some times where it is sub–optimal to invest, and *sequential equilibria*, where one firm invests at its optimal time.

### 4.1 Strategies and Payoffs

The appropriate notion of subgame perfect equilibrium for our game is developed in Riedel and Steg (2014). Let $\mathscr{T}$ denote the set of stopping times with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. The set $\mathscr{T}$ will act as the set of (pure) strategies. Given the definitions of the leader, follower and shared payoffs above, the timing game is

$$\Gamma = \left\langle (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathsf{P}), \mathscr{P}^{\Theta}, \mathscr{T} \times \mathscr{T}, (L_i, F_i, M_i)_{i=1,2}, (\pi_i)_{i=1,2} \right\rangle,$$

where, for $(\tau_1, \tau_2) \in \mathscr{T} \times \mathscr{T}$,

$$\pi_1(x_0) = \inf_{Q\in\mathscr{P}^{\Theta}} \mathsf{E}^Q[L_1(x_0)1_{\tau_1<\tau_2} + F_1(x_0)1_{\tau_1>\tau_2} + M_1(x_0)1_{\tau_1=\tau_2}], \quad \text{and}$$
$$\pi_2(x_0) = \mathsf{E}^{\mathsf{P}}[L_2(x_0)1_{\tau_1>\tau_2} + F_2(x_0)1_{\tau_1<\tau_2} + M_2(x_0)1_{\tau_1=\tau_2}].$$

The subgame starting at stopping time $\vartheta \in \mathscr{T}$ is the tuple

$$\Gamma^{\vartheta} = \left\langle (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq \vartheta}, \mathsf{P}), \mathscr{P}^{\Theta}, \mathscr{T}_{\vartheta} \times \mathscr{T}_{\vartheta}, (L_i, F_i, M_i)_{i=1,2}, (\pi_i^{\vartheta})_{i=1,2} \right\rangle,$$

where $\mathscr{T}_{\vartheta}$ is the set of stopping times no smaller than $\vartheta$ a.s.,

$$\mathscr{T}_{\vartheta} := \{\tau \in \mathscr{T} | \tau \geq \vartheta, \mathsf{P} - a.s.\},$$

and, for $(\tau_1, \tau_2) \in \mathscr{T}_{\vartheta} \times \mathscr{T}_{\vartheta}$,

$$\pi_1^{\vartheta}(x_{\vartheta}) = \inf_{Q\in\mathscr{P}^{\Theta}} \mathsf{E}^Q[L_1(x_{\vartheta})1_{\tau_1<\tau_2} + F_1(x_{\vartheta})1_{\tau_1>\tau_2} + M_1(x_{\vartheta})1_{\tau_1=\tau_2}|\mathscr{F}_{\vartheta}], \quad \text{and}$$
$$\pi_2^{\vartheta}(x_{\vartheta}) = \mathsf{E}^{\mathsf{P}}[L_2(x_{\vartheta})1_{\tau_1>\tau_2} + F_2(x_{\vartheta})1_{\tau_1<\tau_2} + M_2(x_{\vartheta})1_{\tau_1=\tau_2}|\mathscr{F}_{\vartheta}].$$

As is argued in Riedel and Steg (2014), careful consideration has to be given to the appropriate notion of strategy. They show that the notion of extended mixed strategy is versatile and intuitively appealing. For the subgame $\Gamma^{\vartheta}$ this is a pair of processes $(G^{\vartheta}, \alpha^{\vartheta})$, both taking values in $[0, 1]$, with the following properties.[4]

1. $G^{\vartheta}$ is adapted, has right–continuous and non–decreasing sample paths, with $G^{\vartheta}(s) = 0$ for all $s < \vartheta$, $\mathsf{P} - a.s.$

2. $\alpha^{\vartheta}$ is progressively measurable with right–continuous sample paths whenever its value is in $(0, 1)$, $\mathsf{P} - a.s.$

3. On $\{t \geq \vartheta\}$, it holds that

$$\alpha^{\vartheta}(t) > 0 \Rightarrow G^{\vartheta}(t) = 1, \quad \mathsf{P}\text{-a.s.}$$

[4]Note that the properties below hold for all $Q \in \mathscr{P}^{\Theta}$ if they hold for $\mathsf{P}$, because all measures in $\mathscr{P}^{\Theta}$ are equivalent.

We use the convention that

$$G^{\vartheta}(0-) \equiv 0, \quad G^{\vartheta}(\infty) \equiv 1, \quad \text{and} \quad \alpha^{\vartheta}(\infty) \equiv 1.$$

For our purposes extended mixed strategies are, in fact, more general than necessary. Therefore, we will restrict attention to what we will call *extended pure strategies*. For the subgame $\Gamma^{\vartheta}$ this is a pair of extended mixed strategies $(G_i^{\vartheta}, \alpha_i^{\vartheta})_{i=1,2}$, where $G_i^{\vartheta}$ is restricted to take values in $\{0, 1\}$. In other words, in an extended pure strategy a firm does not mix over stopping times, but potentially mixes over its "investment intensity" $\alpha^{\vartheta}$.

An extended pure strategy for the game $\Gamma$ is then a collection $(G^{\vartheta}, \alpha^{\vartheta})_{\vartheta \in \mathscr{T}}$ of extended pure strategies in subgames $\Gamma^{\vartheta}$, $\vartheta \in \mathscr{T}$ satisfying the time consistency conditions that for all $\vartheta, \nu \in \mathscr{T}$ it holds that

1. $\nu \leq t \in \mathbb{R}_+ \Rightarrow G^{\vartheta}(t) = G^{\vartheta}(\nu-) + (1 - G^{\vartheta}(\nu-))G^{\nu}(t)$, $\mathsf{P}$-a.s. on $\{\vartheta \leq \nu\}$,
2. $\alpha^{\vartheta}(\tau) = \alpha^{\nu}(\tau)$, $\mathsf{P}$-a.s., for all $\tau \in \mathscr{T}$.

The importance of the $\alpha$ component in the definition of extended pure strategy becomes obvious in the definition of payoffs. Essentially $\alpha$ allows both for immediate investment and coordination between firms. It leads to investment probabilities that can be thought of as the limits of conditional stage investment probabilities of discrete–time behavioral strategies with vanishing period length (cf. Fudenberg and Tirole (1985)). In the remainder, let $\hat{\tau}_i^{\vartheta}$ be the first time that $\alpha_i^{\vartheta}$ is strictly positive, and let $\hat{\tau}^{\vartheta}$ be the first time that at least one $\alpha^{\vartheta}$ is non–zero in the subgame $\Gamma^{\vartheta}$, i.e.

$$\hat{\tau}_i^{\vartheta} = \inf\{t \geq \vartheta | \alpha_i^{\vartheta}(t) > 0\}, \quad \text{and} \quad \hat{\tau}^{\vartheta} = \inf\{t \geq \vartheta | \alpha_1^{\vartheta}(t) + \alpha_2^{\vartheta}(t) > 0\},$$

respectively. At time $\hat{\tau}^{\vartheta}$ the extended pure strategies induce a probability measure on the state space

$$\Lambda = \{\ \{\text{Firm 1 becomes the leader}\}, \{\text{Firm 2 becomes the leader}\}, \{\text{Both firms invest simultaneously}\}\ \},$$

for which we will use the shorthand notation

$$\Lambda = \{\ (L, 1), (L, 2), M\ \}.$$

Riedel and Steg (2014) show that the probability measure on $\Lambda$, induced by the pair $(\alpha_1^{\vartheta}, \alpha_1^{\vartheta})$, is given by

$$\lambda_{L,i}^{\vartheta}(\hat{\tau}^{\vartheta}) = \begin{cases} \frac{\alpha_{i,\hat{\tau}^{\vartheta}}^{\vartheta}(1-\alpha_{j,\hat{\tau}^{\vartheta}}^{\vartheta})}{\alpha_{i,\hat{\tau}^{\vartheta}}^{\vartheta}+\alpha_{j,\hat{\tau}^{\vartheta}}^{\vartheta}-\alpha_{i,\hat{\tau}^{\vartheta}}^{\vartheta}\alpha_{j,\hat{\tau}^{\vartheta}}^{\vartheta}} & \text{if } \hat{\tau}_i^{\vartheta} = \hat{\tau}_j^{\vartheta} \text{ and } \alpha_i^{\vartheta}(\hat{\tau}_i^{\vartheta}), \alpha_j^{\vartheta}(\hat{\tau}_i^{\vartheta}) > 0 \\ 1 & \text{if } \hat{\tau}_i^{\vartheta} < \hat{\tau}_j^{\vartheta}, \text{ or } \hat{\tau}_i^{\vartheta} = \hat{\tau}_j^{\vartheta} \text{ and } \alpha_j^{\vartheta}(\hat{\tau}_j^{\vartheta}) = 0 \\ 0 & \text{if } \hat{\tau}_i^{\vartheta} > \hat{\tau}_j^{\vartheta}, \text{ or } \hat{\tau}_i^{\vartheta} = \hat{\tau}_j^{\vartheta} \text{ and } \alpha_j^{\vartheta}(\hat{\tau}_j^{\vartheta}) = 0 \\ \frac{1}{2}\big(\liminf_{t \downarrow \hat{\tau}_i^{\vartheta}} \frac{\alpha_i^{\vartheta}(t)(1-\alpha_j^{\vartheta}(t))}{\alpha_i^{\vartheta}(t)+\alpha_j^{\vartheta}(t)-\alpha_i^{\vartheta}(t)\alpha_j^{\vartheta}(t)} & \text{if } \hat{\tau}_i^{\vartheta} = \hat{\tau}_j^{\vartheta}, \alpha_i^{\vartheta}(\hat{\tau}_i^{\vartheta}) = \alpha_j^{\vartheta}(\hat{\tau}_j^{\vartheta}) = 0, \\ +\limsup_{t \downarrow \hat{\tau}_i^{\vartheta}} \frac{\alpha_i^{\vartheta}(t)(1-\alpha_j^{\vartheta}(t))}{\alpha_i^{\vartheta}(t)+\alpha_j^{\vartheta}(t)-\alpha_i^{\vartheta}(t)\alpha_j^{\vartheta}(t)}\big) & \text{and } \alpha_i^{\vartheta}(\hat{\tau}_i^{\vartheta}+), \alpha_j^{\vartheta}(\hat{\tau}_j^{\vartheta}+) > 0, \end{cases}$$

and

$$\lambda_M^\vartheta(\hat{\tau}^\vartheta) = \begin{cases} 0 & \text{if } \hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta,\ \alpha_i^\vartheta(\hat{\tau}_i^\vartheta) = \alpha_j^\vartheta(\hat{\tau}_i^\vartheta) = 0, \text{ and } \alpha_i^\vartheta(\hat{\tau}_i^\vartheta+), \alpha_j^\vartheta(\hat{\tau}_i^\vartheta+) > 0 \\ \frac{\alpha_{i,\hat{\tau}^\vartheta}^\vartheta \alpha_{j,\hat{\tau}^\vartheta}^\vartheta}{\alpha_{i,\hat{\tau}^\vartheta}^\vartheta + \alpha_{j,\hat{\tau}^\vartheta}^\vartheta - \alpha_{i,\hat{\tau}^\vartheta}^\vartheta \alpha_{j,\hat{\tau}^\vartheta}^\vartheta} & \text{otherwise.} \end{cases}$$

Note the following:

1. If $\hat{\tau}_i^\vartheta < \hat{\tau}_j^\vartheta$ there is no coordination problem: firm $i$ becomes the leader $\lambda$-a.s. at $\hat{\tau}_i^\vartheta$;
2. If $\hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta$, but $\alpha_j^\vartheta(\hat{\tau}_j^\vartheta) = 0$, there is no coordination problem: firm $i$ becomes the leader $\lambda$-a.s. at $\hat{\tau}_i^\vartheta$;
3. In the degenerate case where $\hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta$, $\alpha_i^\vartheta(\hat{\tau}_i^\vartheta) = \alpha_j^\vartheta(\hat{\tau}_j^\vartheta) = 0$, and $\alpha_i^\vartheta(\hat{\tau}_i^\vartheta+), \alpha_j^\vartheta(\hat{\tau}_j^\vartheta+) > 0$, the leader role is assigned at time $\hat{\tau}_i^\vartheta$, effectively on the basis of the flip of a fair coin;
4. Firm 1 is not ambiguous over the measure $\lambda$.

In order to derive the payoffs to firms, let $\tau_{G,i}^\vartheta$ denote the first time that $G_i^\vartheta$ jumps to one, i.e.

$$\tau_{G,i}^\vartheta = \inf\{t \geq \vartheta | G_i^\vartheta(t) > 0\}.$$

The payoff to the ambiguous firm of a pair of extended pure strategies $((G_1, \alpha_1), (G_2, \alpha_2))$ in the subgame $\Gamma^\vartheta$ is given by

$$\begin{aligned} V_1^\vartheta(G_1^\vartheta, \alpha_1^\vartheta, G_2^\vartheta, \alpha_2^\vartheta) := & \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[1_{\tau_{G,1}^\vartheta<\min\{\tau_{G,2}^\vartheta,\hat{\tau}^\vartheta\}}\Big(\int_\vartheta^{\tau_{G,1}^\vartheta} e^{-r(s-\vartheta)}D_{00}X_s ds + \int_{\tau_{G,1}^\vartheta}^{\tau_2^F} e^{-r(s-\vartheta)}D_{10}X_s ds \\ & + \int_{\tau_2^F}^\infty e^{-r(s-\vartheta)}D_{11}X_s ds - e^{-r(\tau_{G,1}^\vartheta-\vartheta)}I\Big)\Big|\mathscr{F}_\vartheta\Big] \\ & + \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[1_{\tau_{G,2}^\vartheta<\min\{\tau_{G,1}^\vartheta,\hat{\tau}^\vartheta\}}\Big(\int_\vartheta^{\tau_{G,2}^\vartheta} e^{-r(s-\vartheta)}D_{00}X_s ds + \int_{\tau_{G,2}^\vartheta}^{\tau_1^F} e^{-r(s-\vartheta)}D_{01}X_s ds \\ & + \int_{\tau_1^F}^\infty e^{-r(s-\vartheta)}D_{11}X_s - e^{-r(\tau_1^F-\vartheta)}I\Big)\Big|\mathscr{F}_\vartheta\Big] \\ & + \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[1_{\tau_{G,1}^\vartheta=\tau_{G,2}^\vartheta<\hat{\tau}^\vartheta}\Big(\int_\vartheta^{\tau_{G,1}^\vartheta} e^{-r(s-\vartheta)}D_{00}X_s ds \\ & + \int_{\tau_{G,1}^\vartheta}^\infty e^{-r(s-\vartheta)}D_{11}X_s ds\Big)\Big|\mathscr{F}_\vartheta\Big] \\ & + \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[1_{\hat{\tau}^\vartheta\leq\min\{\tau_{G,1}^\vartheta,\tau_{G,1}^\vartheta\}}\lambda_{L,1}^\vartheta(\hat{\tau}^\vartheta)\Big(\int_\vartheta^{\hat{\tau}^\vartheta} e^{-r(s-\vartheta)}D_{00}X_s ds \\ & + \int_{\hat{\tau}^\vartheta}^{\tau_2^F} e^{-r(s-\vartheta)}D_{10}X_s ds + \int_{\tau_2^F}^\infty e^{-r(s-\vartheta)}D_{11}X_s ds - e^{-r(\tau_{G,1}^\vartheta-\vartheta)}I\Big)\Big|\mathscr{F}_\vartheta\Big] \end{aligned}$$

$$+\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\Bigg[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{L,2}(\hat\tau^\vartheta)\Bigg(\int_\vartheta^{\hat\tau^\vartheta}e^{-r(s-\vartheta)}D_{00}X_sds$$

$$+\int_{\hat\tau^\vartheta}^{\tau_1^F}e^{-r(s-\vartheta)}D_{01}X_sds+\int_{\tau_1^F}^\infty e^{-r(s-\vartheta)}D_{11}X_s-e^{-r(\tau_1^F-\vartheta)}I\Bigg)\Bigg|\mathscr{F}_\vartheta\Bigg]$$

$$+\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\Bigg[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{M}(\hat\tau^\vartheta)\Bigg(\int_\vartheta^{\hat\tau^\vartheta}e^{-r(s-\vartheta)}D_{00}X_sds$$

$$+\int_{\hat\tau^\vartheta}^\infty e^{-r(s-\vartheta)}D_{11}X_sds\Bigg)\Bigg|\mathscr{F}_\vartheta\Bigg].$$

Hence, the payoff of the ambiguous firm can written as

$$\begin{aligned}
V_1^\vartheta(G_1^\vartheta,\alpha_1^\vartheta,G_2^\vartheta,\alpha_2^\vartheta):=&\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\left[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}L_1(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\left[1_{\tau^\vartheta_{G,2}<\min\{\tau^\vartheta_{G,1},\hat\tau^\vartheta\}}F_1(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\left[1_{\tau^\vartheta_{G,1}=\tau^\vartheta_{G,2}<\hat\tau^\vartheta}M_1(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{L,1}(\hat\tau^\vartheta)L_1(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{L,2}(\hat\tau^\vartheta)F_1(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\inf_{Q\in\mathscr{P}^\Theta}\mathsf{E}^Q\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{M}(\hat\tau^\vartheta)M_1(x_\vartheta)\Big|\mathscr{F}_\vartheta\right].
\end{aligned}$$

In the same way, the payoff for the unambiguous firm can be written as

$$\begin{aligned}
V_2^\vartheta(G_2^\vartheta,\alpha_2^\vartheta,G_1^\vartheta,\alpha_1^\vartheta):=&\mathsf{E}^{\mathsf{P}}\left[1_{\tau^\vartheta_{G,2}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}L_2(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\mathsf{E}^{\mathsf{P}}\left[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}F_2(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\mathsf{E}^{\mathsf{P}}\left[1_{\tau^\vartheta_{G,1}=\tau^\vartheta_{G,2}<\hat\tau^\vartheta}M_2(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\mathsf{E}^{\mathsf{P}}\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{L,2}(\hat\tau^\vartheta)L_2(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\mathsf{E}^{\mathsf{P}}\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{L,1}(\hat\tau^\vartheta)F_2(x_\vartheta)\Big|\mathscr{F}_\vartheta\right]\\
&+\mathsf{E}^{\mathsf{P}}\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,1}\}}\lambda^\vartheta_{M}(\hat\tau^\vartheta)M_2(x_\vartheta)\Big|\mathscr{F}_\vartheta\right].
\end{aligned}$$

### 4.2 Preemptive and Sequential Equilibria

An equilibrium for the subgame $\Gamma^\vartheta$ is a pair of extended pure strategies $\big((\bar{G}_1^\vartheta, \bar{\alpha}_1^\vartheta), (\bar{G}_2^\vartheta, \bar{\alpha}_2^\vartheta)\big)$, such that for each firm $i = 1, 2$ and every extended pure strategy $(G_i^\vartheta, \alpha_i^\vartheta)$ it holds that

$$V_i^\vartheta(\bar{G}_i^\vartheta, \bar{\alpha}_i^\vartheta, \bar{G}_j^\vartheta, \bar{\alpha}_j^\vartheta) \geq V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, \bar{G}_j^\vartheta, \bar{\alpha}_j^\vartheta),$$

for $j \neq i$. A subgame perfect equilibrium is a pair of extended pure strategies $\big((\bar{G}_1, \bar{\alpha}_1), (\bar{G}_2, \bar{\alpha}_2)\big)$, such that for each $\vartheta \in \mathscr{T}$ the pair $\big((\bar{G}_1^\vartheta, \bar{\alpha}_1^\vartheta), (\bar{G}_2^\vartheta, \bar{\alpha}_2^\vartheta)\big)$ is an equilibrium in the subgame $\Gamma^\vartheta$.

There are several types of equilibria of interest in this model. Fix $\vartheta \in \mathscr{T}$. For firm $i$ we denote the optimal time of investment, assuming that the other firm cannot preempt, in the subgame $\Gamma^\vartheta$ by $\tau_{L,i}^\vartheta$, i.e.

$$\tau_{L,i}^\vartheta = \inf\{t \geq \vartheta | X_t \geq x_i^L\}.$$

We also define the *preemption region* as the part of the state space where both firms prefer to be the leader rather than the follower, i.e.

$$\mathcal{P} = \{x \in \mathbb{R}_+ | (L_1(x) - F_1(x)) \wedge (L_2(x) - F_2(x)) > 0\}.$$

The first hitting time of $\mathcal{P}$ in the subgame $\Gamma^\vartheta$ is denoted by $\tau_P^\vartheta$.

We distinguish between two different equilibrium concepts. Lemma (3) establishes existence of a *preemptive equilibrium.*

**Lemma 3.** *(Riedel and Steg (2014)) Suppose $\vartheta \in \mathscr{T}$ satisfies $\vartheta = \tau_P^\vartheta$ $\mathsf{P}-a.s.$ Then $\big((G_1^\vartheta, \alpha_1^\vartheta), (G_2^\vartheta, \alpha_2^\vartheta)\big)$ given by*

$$\alpha_i^\vartheta(t) = \begin{cases} 1 & \text{if } t = \tau_P^t,\ L_t^j = F_t^j, \text{ and } (L_t^i > F_t^i \text{ or } F_t^j = M_t^j) \\ 1_{L_t^1 > F_t^1} 1_{L_t^2 > F_t^2} \frac{L_t^j - F_t^j}{L_t^j - M_t^j} & \text{otherwise,} \end{cases}$$

*for any $t \in [\vartheta, \infty)$ and $G_i^\vartheta = 1_{t \geq \vartheta}$, $i = 1, 2$, $j \in \{1, 2\}$ $i$, are an equilibrium in the subgame at $\vartheta$.*

In this kind of equilibrium both firms try to preempt each other. Investment takes place sooner than it optimally would, i.e. the time one firm would invest without the fear of being preempted. The resulting equilibrium in the latter case is called *sequential equilibrium*. For certain underlying parameters, the preemption time $\tau_P^\vartheta$ is greater than the optimal investment time $\tau_{L,i}^\vartheta$ of some firm $i$. A sequential equilibrium is then given by the next lemma.

**Lemma 4.** *Suppose $\vartheta = \tau_{L,i}^\vartheta < \tau_P^\vartheta$, $\mathsf{P}$-a.s. for one $i \in \{1, 2\}$. Then $\big((G_1^\vartheta, \alpha_1^\vartheta), (G_2^\vartheta, \alpha_2^\vartheta)\big)$ given by*

$$\alpha_i^\vartheta(\vartheta) = 1, G_i^\vartheta(t) = 0 \text{ for all } t < \vartheta,\ \ G_j^\vartheta(t) = 0 \text{ for all } t \leq \vartheta$$

*is an equilibrium in the subgame at $\vartheta$.*

**Proof.** The stopping time $\tau_{L,i}^{\vartheta}$ is determined in Proposition (1) as the stopping time that maximizes the leader payoff. Hence, without the threat of being preempted by its opponent, i.e. $\tau_{L,i}^{\vartheta} < \tau_P^{\vartheta}$ $\mathsf{P}$-a.s., it is not optimal to deviate for firm $i$. Firm $j$ does not want to stop before $\tau_{L,i}^{\vartheta}$ as its payoff of becoming the leader is strictly smaller than becoming the follower up to $\tau_P^{\vartheta}$. ■

Now, we are finally able to formulate a subgame perfect equilibrium for our game.

**Theorem 2.** *There exists a subgame perfect equilibrium $((G_1, \alpha_1), (G_2, \alpha_2))$, where for each $\vartheta \in \mathscr{T}$, $\alpha_i^{\vartheta}$ and $G_1^{\vartheta}$ given by*

*(i) Lemma (3) if either $\vartheta \geq \tau_P^{\vartheta}$ $\mathsf{P}$-a.s. or $\tau_P^{\vartheta} \leq \tau_{L,i}^{\vartheta}$ $\mathsf{P}$-a.s.*

*(ii) Lemma (4) otherwise (i.e. $\vartheta < \tau_P^{\vartheta}$ $\mathsf{P}$-a.s. and $\tau_P^{\vartheta} > \tau_{L,i}^{\vartheta}$ $\mathsf{P}$-a.s.).*

**Proof.** Optimality for case (ii) follows along the same lines as in the proof of Lemma (4).

If $\vartheta \geq \tau_P^{\vartheta}$ $\mathsf{P} - a.s.$, then optimality for case (i) follows directly from Lemma (3). What remains to prove is that, in case $\vartheta < \tau_P^{\vartheta}$ $\mathsf{P} - a.s.$, neither of the firms wants to invest sooner than $\tau_P^{\vartheta}$.

We start with firm 2. Suppose that firm 1 plays the preemption equilibrium strategy. Then if firm 2 plays the preemption strategy, its payoff is $V_2(x) = \mathsf{E}_x[e^{-r\tau_P} L_2(x_P)]$, for any $x < x_P$. This is the case, because, either the other firm is indifferent between the leader and follower role at $x_P$, in which case firm 2 becomes the leader, or firm 2 is indifferent in which case $F_2(x_P) = L_2(x_P)$.

Note that we have $V_2(x) = \mathsf{E}_x[e^{-r\tau_P} L_2(x_P)] = \left(\frac{x}{x_P}\right)^{\beta_1(\mu)} L_2(x_P)$ (cf. Dixit and Pindyck (1994, Chapter 9, Appendix A)). $V_2$ is a strictly increasing function, with $V_2(x_P) = L_2(x_P)$ and $V_2(0) = 0 > L_2(0)$, so that $V_2(x) > L_2(x)$ for any $x < x_P$.

The only deviations $\hat{\tau}$ that could potentially give a higher payoff have $\hat{\tau} < \tau_P$, $\mathsf{P}$-a.s. Consider the first hitting time $\hat{\tau}$ of some $\hat{x} < x_P$. Let $\hat{V}_2$ denote the payoff to firm 2 of this strategy (while the other firm plays its preemption strategy). For $\hat{x} \leq x < x_P$, it holds that $\hat{V}_2(x) = L_2(x) < V_2(x)$.

For $x < \hat{x}$, note that $\hat{V}_2(x) = \left(\frac{x}{\hat{x}}\right)^{\beta_1(\mu)} L_2(\hat{x}) = \frac{L_2(\hat{x})}{\hat{x}^{\beta_1(\mu)}} x^{\beta_1(\mu)}$. Consider the mapping $x \mapsto \frac{L_2(x)}{x^{\beta_1(\mu)}}$. This function attains its maximum at $x_2^L > x_P$. Therefore, its derivative is positive on $(0, x_P)$, implying that $V_2(x) > \hat{V}_2(x)$. Any stopping time $\tau$ can be written as a mixture of first hitting times. So, no stopping time $\hat{\tau}$ with $\hat{\tau} < \tau_P$, $\mathsf{P}$-a.s. yields a higher payoff than $\tau_P$.

For firm 1, the argument is similar after realizing that $V_1(x) = \frac{L_1(x_P)}{x_P^{\beta_1(\underline{\mu})}} x^{\beta_1(\underline{\mu})}$ and $\hat{V}_1(x) = \frac{L_1(\hat{x})}{\hat{x}^{\beta_1(\underline{\mu})}} x^{\beta_1(\underline{\mu})}$. This holds because $x_P < x_1^L < x^*$, so that $\underline{\mu}$ is the trend under the worst–case measure for every $x \in (0, x_P]$.

■

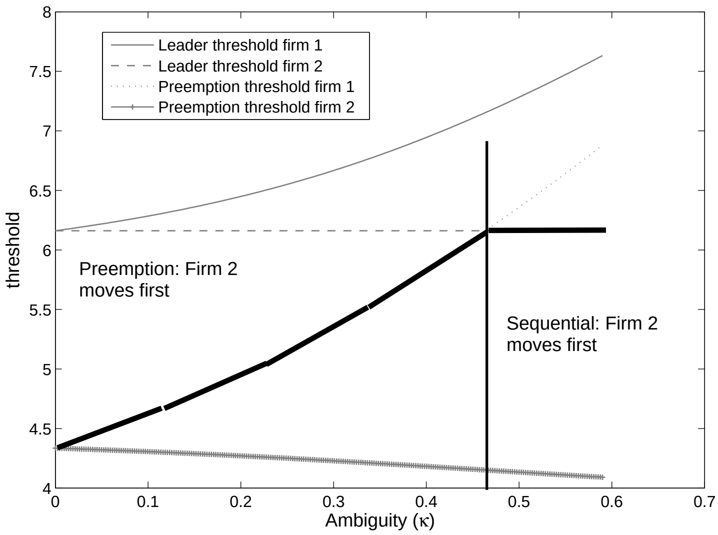


Figure 3: Thresholds for varying $\kappa$, with $D_{10} = 1.8$, $D_{11} = 1$, $D_{01} = 0$, $D_{00} = 0$, $r = 0.1$, $\sigma = 0.1$, $\mu = 0.04$, $I = 100$ and $\eta = 1$. The black line indicates the threshold for first investment in equilibrium.

# 5 Comparative Statics

In this section, we analyze the sensitivity of equilibria with respect to a change of the degree of ambiguity, $\kappa$; the volatility, $\sigma$; and the cost difference, $\eta$, respectively.

## 5.1 Comparative Statics With Respect to $\kappa$

Nishimura and Ozaki (2007) argued that in a monopolistic model where the firm faces $\kappa$–ignorance, an increase in $\kappa$ postpones investment and decreases the profit.

In our duopoly framework, we observe that both the leader and the follower value function of the ambiguous firm decrease with an increase of $\kappa$.

For equilibrium outcomes it is important to investigate how investment times (or thresholds) vary with a change of $\kappa$. We find that the follower investment threshold of the ambiguous firm rises if $\kappa$ increases, as in Nishimura and Ozaki (2007). Hence, the non–ambiguous firm's payoff increases as it enjoys the benefits of being the only one who has invested for a longer time. Further, we easily see that $x_1^L$ increases with $\kappa$.

To see what happens to the preemption time $\tau_P^1 := \inf\{t \geq 0 | L_1(x_t) \geq F_1(x_t)\}$, we need to consider $L_1 - F_1$. Both functions $L_1$ and $F_1$ decline by a decrease of $\kappa$. However, due to the complexity of the ambiguous firm's leader value function, it is not possible to come up with an analytic result about which function decreases more. For this reason, we consider some numerical examples which suggest that the

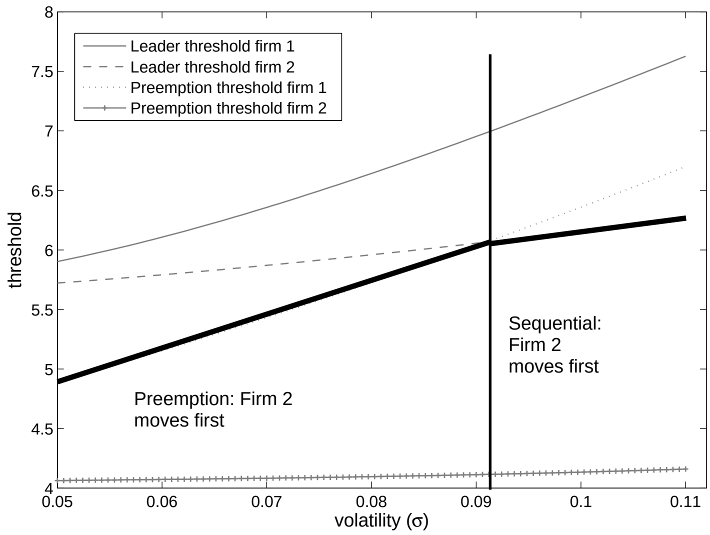


Figure 4: Thresholds for varying $\sigma$, with $D_{10} = 1.8$, $D_{11} = 1$, $D_{01} = 0$, $D_{00} = 0$, $r = 0.1$, $\kappa = 0.5$, $\mu = 0.04$. $I = 100$ and $\eta = 1$. The black line indicates the threshold for first investment in equilibrium.

leader function is more affected by a change of $\kappa$ than the follower function.

Figure 3 depicts the change of the leader thresholds as well as the preemption thresholds of both firms with respect to $\kappa$. Starting with completely symmetric firms ($\eta = 1$ and $\kappa = 0$), Figure 3 shows that both the preemption threshold and the leader threshold of firm 1 increase with $\kappa$. This indicates that $L_1$ decreases more in $\kappa$ than $F_1$. This observation makes sense; if it were the other way around, firm 1 could benefit from an increase of $\kappa$. Indeed, if firm 1's preemption threshold would decrease more than firm 2's, firm 1 might benefit by receiving the leader role for ever bigger $\kappa$.

Note that there is a qualitative change of equilibrium around $\kappa \approx .48$. For smaller values of $\kappa$ there is a preemption equilibrium, where firm 2 moves first at the preemption threshold of firm 1. For larger values of $\kappa$, firm 1's preemption threshold is so high that firm 2 can invest at its leader threshold, i.e. there is a sequential equilibrium.

## 5.2 Comparative Statics With Respect to $\sigma$

Comparative statics with respect to the volatility $\sigma$ are even more complex, because a change in $\sigma$ affects not only the volatility but also the interval of possible trends, since $[\underline{\mu}, \overline{\mu}] = [\mu - \sigma\kappa, \mu + \sigma\kappa]$. Note that a change in $\sigma$ and a change in $\kappa$ of the same magnitude have exactly the same impact on the interval of possible trends.

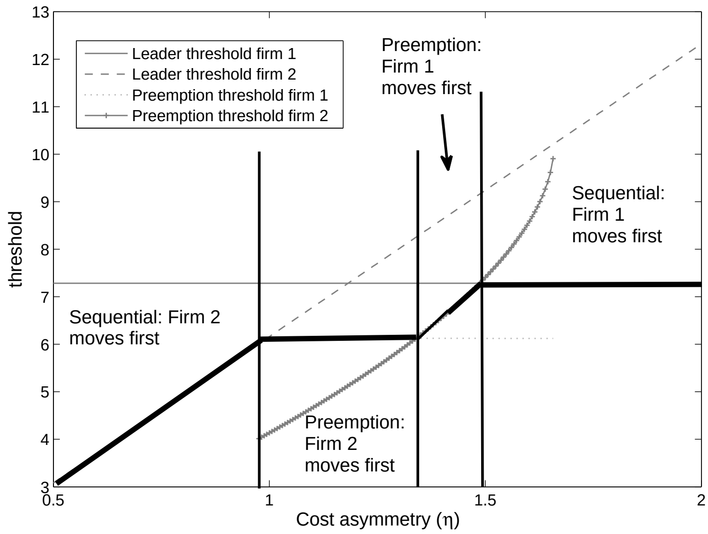


Figure 5: Thresholds for varying $\eta$ with $D_{10} = 1.8$, $D_{11} = 1$, $D_{01} = 0$, $D_{00} = 0$, $r = 0.1$, $\sigma = 0.1$, $\mu = 0.04$, $I = 100$ and $\kappa = 0.5$. The black line indicates the threshold for first investment in equilibrium.

From the standard literature on real options it is well known that an increase of $\sigma$ increases the investment threshold of a monopolistic firm in a purely risky environment (cf. Nishimura and Ozaki (2007)).

Figure 4 shows what happens to the investment thresholds in our framework. All thresholds for both firms increase with the volatility. Due to the effect on the interval of possible trends, however, firm 1's thresholds rise much stronger.

There is a qualitative change of equilibrium around $\sigma \approx .091$. For smaller values of $\sigma$ there is a preemption equilibrium, where firm 2 moves first at the preemption threshold of firm 1. For larger values of $\sigma$, firm 1's preemption threshold is so high that firm 2 can invest at its leader threshold, i.e. there is a sequential equilibrium.

### 5.3 Comparative Statics With Respect to $\eta$

In a purely risky framework, the firm that has the lower investment cost always becomes the leader (cf. Pawlina and Kort (2006)). This result, however, might change if ambiguity is introduced. Figure 5 shows that even if the the non–ambiguous firm has a higher cost of investment, it might become the leader anyway. Ambiguity, therefore, might outbalance the cost advantage of firm 1.

From Figure 5 we can observe that the preemption threshold as well as the leader threshold of firm 2 increase with $\eta$. To the far right, there does not even exist a preemption threshold anymore, as the cost disadvantage is so big that firm 2's leader function always lies below its follower function on $[0, x_1^F]$. Firm 1's leader threshold is unaffected by a change of $\eta$. Its preemption threshold, however, is (slightly) decreasing. The reason for this fact might not be obvious in case condition (12) is not satisfied. First note that firm 1's follower function is not affected by a change of $\eta$. Further note that the preemption point can only lie in the region where $L_1$ is increasing. This means that, if the worst–case changes at some point, then the preemption point is smaller than $x^*$. Thus, the function needed to be considered is

$$\frac{D_{10}x_t}{r-\underline{\mu}} - \frac{1}{\beta_1(\underline{\mu})}\frac{D_{10}x^*}{r-\underline{\mu}}\left(\frac{x_t}{x^*}\right)^{\beta_1(\underline{\mu})} - I.$$

This function is also not directly affected by a change of $\eta$. Yet, due to the fact that $x_2^F$ increases with $\eta$, $L_1$ increases in the region $[x^*, x_2^F]$. Since the smooth pasting condition has to be fulfilled, this implies that $x^*$ moves to the left. This, however, means that $L_1$ is also increasing in the region before $x^*$ is reached. This implies that the preemption threshold of firm 1 is decreasing.

There are several points where the qualitative nature of equilibrium changes. For small values of $\eta$, firm 2 is the first firm to invest and it does so at its leader threshold; this represents a sequential equilibrium. In this region, no preemption thresholds exist, because firm 2's advantage is so great that firm 1 would never wish to preempt. For values of $\eta$ approximately in the interval $[.95, 1.35]$, firm 2 invests first at firm 1's preemption threshold in a preemptive equilibrium. For even larger values of $\eta$, approximately in the interval $[1.35, 1.5]$, the cost disadvantage becomes large enough relative to firm 1's ambiguity that the role of first mover switches: firm 1 invests first at firm 2's preemption threshold in a preemptive equilibrium. Finally, for $\eta > 1.5$, the cost disadvantage is so large that firm 2's preemption threshold lies above firm 1's leader threshold, so that firm 1 invests first at its leader threshold in a sequential equilibrium.

# 6 The Case where Both Firms are Ambiguous

We want to emphasize that our analysis is independent of the assumption that only one of the firms is ambiguous. Throughout the paper, this assumption is made in order to elaborate the difference that an introduction of ambiguity makes in contrast to a purely risky world.

We may very well allow for both firms to be ambiguous about the trend of the underlying dynamics. We even do not need to require that the firms have the same degree of ambiguity (same $\kappa$).

In fact, for the analysis of the worst–case prior, it is only required that the degree of ambiguity and the cost of investment of each player are common knowledge (such that each firm is able to compute the follower threshold of its competitor). The determination of the follower and leader value functions of a second

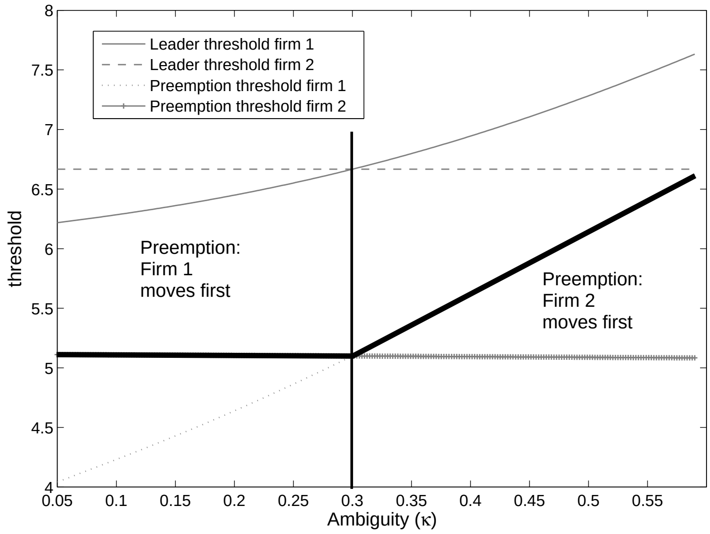


Figure 6: Thresholds for varying $\kappa_1$, with $D_{10} = 1.8$, $D_{11} = 1$, $D_{01} = 0$, $D_{00} = 0$, $r = 0.1$, $\sigma = 0.1$, $\mu = 0.04$, $\eta = 1$, $I = 100$ and $\kappa_2 = 0.3$. The black line indicates the threshold for first investment in equilibrium.

ambiguous firm is completely analogous to the analysis in Section 3.2. Recall that ambiguity is assumed not to be about strategies but about payoffs exclusively. This implies, knowing the new value functions, the equilibrium analysis follows along the same lines as presented in Section 4.

In Figure 6, we draw firms' thresholds for the case that both players are $\kappa$–ignorant, possibly to a different degree. The firms are assumed to be symmetric in terms of the investment costs. The degree of ambiguity for firm 2 is $\kappa_2 = 0.3$. We vary the degree of ambiguity for the first firm and see that both the preemption threshold and the leader threshold of firm 1 are strictly increasing, whereas the preemption threshold as well as the leader threshold of firm 2 are slightly decreasing.

We now only get preemptive equilibria: firm 1 preempts firm 2 for small values of $\kappa_1$, whereas firm 2 preempts firm 1 for larger values of $\kappa_1$. Note that the domain of $\kappa_1$ is bounded by the condition that $r > \overline{\mu}$, i.e. that $\kappa_1 < (r - \mu)/\sigma$. This means that Figure 6 can not be extended beyond $\kappa_1 \approx .6$. So, while one might expect that for $\kappa_1 > .6$ firm 2 invests first in a sequential equilibrium, this can not be verified.

# Appendix

## A Proof of Lemma (2)

In this section, we show that if the worst–case for the leader value is not always given by the worst possible trend, there exists a unique value $x^*$ at which the worst–case changes from $\underline{\mu}$ to $\overline{\mu}$.

**Proof.** The critical value $x^*$ is found by applying the smooth pasting condition $\hat{L}_1(\overline{\mu}, x^*) = 0$. The first derivative of $\hat{L}_1$ is given by

$$\begin{aligned}\hat{L}_1'(\overline{\mu}, x) =& \frac{D_{10}}{r-\overline{\mu}} + \frac{\beta_1(\overline{\mu})(x^*)^{\beta_2(\overline{\mu})}x^{\beta_1(\overline{\mu})-1} - \beta_2(\overline{\mu})(x^*)^{\beta_1(\overline{\mu})}x^{\beta_2(\overline{\mu})-1}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F \\ &+ \frac{\beta_2(\overline{\mu})(x_2^F)^{\beta_1(\overline{\mu})}x^{\beta_2(\overline{\mu})-1} - \beta_1(\overline{\mu})(x_2^F)^{\beta_2(\overline{\mu})}x^{\beta_1(\overline{\mu})-1}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left[\left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right]x^*.\end{aligned}$$

In order to prove the existence of $x^*$, we will show that if $x^* \uparrow x_2^F$, $\hat{L}_1'(\overline{\mu}, x^*)$ becomes negative, and if $x^* \downarrow 0$, $\hat{L}_1'(\overline{\mu}, x^*)$ becomes positive.

We have

$$\begin{aligned}\hat{L}_1'(\overline{\mu}, x^*) =& \frac{D_{10}}{r-\overline{\mu}} + \frac{(\beta_1(\overline{\mu}) - \beta_2(\overline{\mu}))(x^*)^{\beta_1(\overline{\mu})+\beta_2(\overline{\mu})-1}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F \\ &+ \frac{\beta_2(\overline{\mu})(x_2^F)^{\beta_1(\overline{\mu})}(x^*)^{\beta_2(\overline{\mu})} - \beta_1(\overline{\mu})(x_2^F)^{\beta_2(\overline{\mu})}(x^*)^{\beta_1(\overline{\mu})}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}\left[\left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right].\end{aligned}$$

Clearly, $\lim_{x^*\downarrow x_2^F} \hat{L}_1'(\overline{\mu}, x^*)$ has the same sign as the following expression.

$$\begin{aligned}&\frac{D_{10}}{r-\overline{\mu}}\left((x_2^F)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x_2^F)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}\right) \\ &+ (\beta_1(\overline{\mu}) - \beta_2(\overline{\mu}))\,(x_2^F)^{\beta_1(\overline{\mu})+\beta_2(\overline{\mu})}\left[\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}} - \left(1-\frac{1}{\beta_1(\underline{\mu})}\right)\frac{D_{10}}{r-\underline{\mu}} + \frac{D_{10}}{r-\overline{\mu}}\right].\end{aligned} \tag{A.1}$$

Using the fact that $\frac{1}{\beta_1(\underline{\mu})} < \frac{D_{10}-D11}{D_{10}}$ yields that (A.1) is smaller than

$$(\beta_1(\overline{\mu}) - \beta_2(\overline{\mu}))\,(x_2^F)^{\beta_1(\overline{\mu})+\beta_2(\overline{\mu})}\frac{1}{r-\underline{\mu}}\,(D_{11} - D_{10} + D_{10} - D_{11}) = 0. \tag{A.2}$$

Considering the case $x^* \downarrow 0$, one can easily see that $\lim_{x^*\downarrow 0} \hat{L}_1'(\overline{\mu}, x^*)$ has the same sign as

$$\begin{aligned}\lim_{x^*\downarrow 0}\Bigg\{&\frac{D_{10}}{r-\overline{\mu}}\left((x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}\right) \\ &+ (\beta_1(\overline{\mu}) - \beta_2(\overline{\mu}))\,(x^*)^{\beta_1(\overline{\mu})+\beta_2(\overline{\mu})-1}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F\end{aligned}$$

$$
+ \beta_2(\overline{\mu})(x_2^F)^{\beta_1(\overline{\mu})}(x^*)^{\beta_2(\overline{\mu})} - \beta_1(\overline{\mu})(x_2^F)^{\beta_2(\overline{\mu})}(x^*)^{\beta_1(\overline{\mu})} \left( \left(1 - \frac{1}{\beta_1(\underline{\mu})}\right) \frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}} \right) \Bigg\}
$$

$$
= \lim_{x^* \downarrow 0} \Bigg\{ (x^*)^{\beta_2(\overline{\mu})} \Bigg( \frac{D_{10}}{r-\overline{\mu}} \left( (x_2^F)^{\beta_2(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})-\beta_2(\overline{\mu})} \right)
$$

$$
+ (\beta_1(\overline{\mu}) - \beta_2(\overline{\mu}))(x^*)^{\beta_1(\overline{\mu})-1} \left( \frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}} \right) x_2^F
$$

$$
+ \left( \beta_2(\overline{\mu})(x_2^F)^{\beta_1(\overline{\mu})} - \beta_1(\overline{\mu})(x^*)^{\beta_1(\overline{\mu})-\beta_2(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})} \right) \left( \left(1 - \frac{1}{\beta_1(\underline{\mu})}\right) \frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}} \right) \Bigg) \Bigg\}
$$

$$
= \lim_{x^* \downarrow 0} \underbrace{(x^*)^{\beta_2(\overline{\mu})}}_{\to +\infty} \Bigg\{ \underbrace{\frac{D_{10}}{r-\overline{\mu}}}_{>0} \Bigg( (x_2^F)^{\beta_2(\overline{\mu})} - \underbrace{(x^*)^{\beta_1(\overline{\mu})-\beta_2(\overline{\mu})}}_{\to 0} \Bigg)
$$

$$
+ \underbrace{(\beta_1(\overline{\mu}) - \beta_2(\overline{\mu}))(x^*)^{\beta_1(\overline{\mu})-1}}_{\to 0} \left( \frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}} \right) x_2^F
$$

$$
+ \Bigg( \underbrace{\beta_2(\overline{\mu})(x_2^F)^{\beta_1(\overline{\mu})}}_{<0} - \underbrace{\beta_1(\overline{\mu})(x^*)^{\beta_1(\overline{\mu})-\beta_2(\overline{\mu})}}_{\to 0} \Bigg) \underbrace{\left( \left(1 - \frac{1}{\beta_1(\underline{\mu})}\right) \frac{D_{10}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}} \right)}_{<0} \Bigg\}.
$$

Therefore, we get $\hat{L}_1'(\overline{\mu}, x^*) > 0$ for $x^*$ close to 0. Due to continuity of $L_2'$ on $[0, x_2^F]$, we can find in that region a solution to $\hat{L}_1'(\overline{\mu}, x^*) = 0$.

The uniqueness of $x^*$ is automatically given by the uniqueness of the solution to PDE (16).

■

# B Concavity of $L_1$

In this section we prove that the leader function of the ambiguous firm is concave on $[0, x_2^F]$ . In case the worst–case prior is always induced by the lowest possible trend, this statement is trivial. The next proof shows that concavity is not lost even if the worst–case changes at some point.

**Proof.** Suppose condition (12) is not satisfied (i.e. $\underline{\mu}$ is not always the worst–case). The concavity of $L_1(x)$ for $x < x^*$ is trivial. We therefore consider the second derivative of $L_1(x)$ in the interval $[x^*, x_2^F)$.

$$
\hat{L}_1''(\overline{\mu}, x) = \frac{\beta_1(\overline{\mu})(\beta_1(\overline{\mu})-1)(x^*)^{\beta_2(\overline{\mu})}x^{\beta_1(\overline{\mu})-2} - \beta_2(\overline{\mu})(\beta_2(\overline{\mu})-1)(x^*)^{\beta_1(\overline{\mu})}x^{\beta_2(\overline{\mu})-2}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}
$$

$$
\times \left( \frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}} \right) x_2^F
$$

$$
+ \frac{\beta_2(\overline{\mu})(\beta_2(\overline{\mu})-1)(x_2^F)^{\beta_1(\overline{\mu})}x^{\beta_2(\overline{\mu})-2} - \beta_1(\overline{\mu})(\beta_1(\overline{\mu})-1)(x_2^F)^{\beta_2(\overline{\mu})}x^{\beta_1(\overline{\mu})-2}}{(x^*)^{\beta_2(\overline{\mu})}(x_2^F)^{\beta_1(\overline{\mu})} - (x^*)^{\beta_1(\overline{\mu})}(x_2^F)^{\beta_2(\overline{\mu})}}
$$

$$\times \left[ \left(1 - \frac{1}{\beta_1(\underline{\mu})}\right) \frac{D_{10}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right] x^*.$$

Now, we have

$$\begin{aligned}
&\beta_1(\overline{\mu})(\beta_1(\overline{\mu}) - 1)x^{\beta_1(\overline{\mu})-2}\Bigg[ \left( \frac{D_{11}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right) x_2^F (x^*)^{\beta_2(\overline{\mu})} \\
&\qquad\qquad - \left( \left(1 - \frac{1}{\beta_1(\underline{\mu})}\right) \frac{D_{10}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right) x^* (x_2^F)^{\beta_2(\overline{\mu})} \Bigg] \\
&< \beta_1(\overline{\mu})(\beta_1(\overline{\mu}) - 1)x^{\beta_1(\overline{\mu})-2} x^* (x_2^F)^{\beta_2(\overline{\mu})} \left[ \left( \frac{D_{11}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right) - \left( \left(1 - \frac{1}{\beta_1(\underline{\mu})}\right) \frac{D_{10}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right) \right] \\
&= \beta_1(\overline{\mu})(\beta_1(\overline{\mu}) - 1)x^{\beta_1(\overline{\mu})-2} x^* (x_2^F)^{\beta_2(\overline{\mu})} \left[ \frac{D_{11}}{r - \underline{\mu}} - \frac{D_{10}}{r - \underline{\mu}} + \frac{1}{\beta_1(\underline{\mu})} \frac{D_{10}}{r - \underline{\mu}} \right] \\
&< \beta_1(\overline{\mu})(\beta_1(\overline{\mu}) - 1)x^{\beta_1(\overline{\mu})-2} x^* (x_2^F)^{\beta_2(\overline{\mu})} \frac{1}{r - \underline{\mu}} \left[ D_{11} - D_{10} + D_{10} - D_{11} \right] \\
&= 0,
\end{aligned}$$

where we used the fact that $x^*(x_2^F)^{\beta_2(\overline{\mu})} < (x^*)^{\beta_2(\overline{\mu})}(x_2^F)$ (because $x^* < x_2^F$ and $\beta_2(\overline{\mu}) < 0$) and $\frac{D_{10} - D_{11}}{D_{10}} > \frac{1}{\beta_1(\underline{\mu})}$.

In a similar, way we can show that

$$\begin{aligned}
&\beta_2(\overline{\mu})(\beta_2(\overline{\mu}) - 1)x^{\beta_2(\overline{\mu})-2}\Bigg[ - \left( \frac{D_{11}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right) x_2^F (x^*)^{\beta_1(\overline{\mu})} \\
&\qquad\qquad + \left( \left(1 - \frac{1}{\beta_1(\underline{\mu})}\right) \frac{D_{10}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right) x^* (x_2^F)^{\beta_1(\overline{\mu})} \Bigg] \\
&< 0,
\end{aligned}$$

which proves the concavity of $L_1$.

■

# C Proof of Proposition (1)

The proof follows along similar lines to the proof of Theorem (1). We use the same procedure, but now we consider the value function in the continuation region, i.e. before any investment has taken place. Applying the BSDE approach with different value matching and smooth pasting conditions eventually yields the desired stopping time.

**Proof.**

Denote

$$Y_t = \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[\int_t^{\tau^t_{L,1}} e^{-r(s-t)}D_{00}X_s ds + \int_{\tau^t_{L,1}}^{\tau^F_2} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau^F_2}^{\infty} e^{-r(s-t)}D_{11}X_s ds\Big|\mathcal{F}_t\Big].$$

Using the time consistency property of a strongly rectangular set of density generators yields

$$\begin{aligned}
Y_t &= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[\int_t^{\tau^t_{L,1}} e^{-r(s-t)}D_{00}X_s ds + \int_{\tau^t_{L,1}}^{\tau^F_2} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau^F_2}^{\infty} e^{-r(s-t)}D_{11}X_s ds\Big|\mathcal{F}_t\Big] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\left[\int_t^{\tau^t_{L,1}} e^{-r(s-t)}D_{00}X_s ds + \int_{\tau^t_{L,1}}^{\tau^F_2} e^{-r(s-t)}D_{10}X_s ds\right.\right. \\
&\qquad\qquad \left.\left. + \int_{\tau^F_2}^{\infty} e^{-r(s-t)}D_{11}X_s ds\Big|\mathcal{F}_{\tau^t_{L,1}}\right]\Big|\mathcal{F}_t\right] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau^t_{L,1}} e^{-r(s-t)}D_{00}X_s ds + e^{-r(\tau^t_{L,1}-t)} \inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\left[\int_{\tau^t_{L,1}}^{\tau^F_2} e^{-r(s-\tau^t_{L,1})}D_{10}X_s ds\right.\right. \\
&\qquad\qquad \left.\left. + \int_{\tau^F_2}^{\infty} e^{-r(s-\tau^t_{L,1})}D_{11}X_s ds\Big|\mathcal{F}_{\tau^t_{L,1}}\right]\Big|\mathcal{F}_t\right] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau^t_{L,1}} e^{-r(s-t)}D_{00}X_s ds + L_1(x_{\tau^t_{L,1}})\Big|\mathcal{F}_t\right].
\end{aligned}$$

Following Chen and Epstein (2002), $Y_t$ solves the BSDE

$$-dY_t = g(Z_t)dt - Z_t dB_t,$$

for the generator

$$g(z) = -\kappa|z| - rY_t + X_t D_{00}.$$

The boundary condition is given by

$$Y_{\tau^t_{L,1}} = L(x^L_1),$$

where $L(x^L_1)$ is given by Theorem (1) and $x^L_1 = x_{\tau^t_{L,1}}$.

Denote the present value of the leader payoff by $\Lambda$, i.e.

$$\Lambda(x_t) = Y_t.$$

The non–linear Feynman–Kac formula implies that $\Lambda$ solves the non–linear PDE

$$\mathscr{L}_X\Lambda(x) + g(\sigma x\Lambda'(x)) = 0.$$

Hence, $\Lambda$ solves

$$\frac{1}{2}\sigma^2 x^2 \Lambda''(x) + \mu x \Lambda'(x) - \kappa\sigma x \left|\Lambda'(x)\right| - r\Lambda(x) + D_{00}x = 0. \tag{C.1}$$

In the continuation region the leader function has to be increasing, hence $\Lambda' > 0$. This implies that $\underline{\mu}$ is the worst–case in the continuation region.

Therefore, equation (C.1) becomes

$$\frac{1}{2}\sigma^2 x^2 \Lambda''(x) + (\mu - \kappa\sigma)x\Lambda'(x) - r\Lambda(x) + D_{00}x = \frac{1}{2}\sigma^2 x^2 \Lambda''(x) + \underline{\mu} x\Lambda'(x) - r\Lambda(x) + D_{00}x = 0.$$

The general increasing solution to this PDE is

$$\Lambda(x) = \frac{D_{00}x}{r - \underline{\mu}} + A_2 x^{\beta_1(\underline{\mu})}.$$

We have to distinguish two cases here. Either the condition given in Theorem (1) holds which means that the boundary condition takes the form (13) or the boundary condition becomes (14).

We will show that for both cases, the optimal threshold to invest becomes

$$x_1^L = \frac{\beta_1(\underline{\mu})}{\beta_1(\underline{\mu}) - 1} \frac{I(r - \underline{\mu})}{D_{10} - D_{00}}. \tag{C.2}$$

If condition (12) is satisfied, the boundary condition is given by

$$L_1(x_1^L) = \frac{D_{10}x_1^L}{r - \underline{\mu}} + \left(\frac{x_1^L}{x_2^F}\right)^{\beta_1(\underline{\mu})} \frac{D_{11} - D_{10}}{r - \underline{\mu}} x_2^F - I.$$

Otherwise, the boundary condition is given by

$$L_1(x_1^L) = \frac{D_{10}x_1^L}{r - \underline{\mu}} - \frac{1}{\beta_1(\underline{\mu})} \frac{D_{10}x^*}{r - \underline{\mu}} \left(\frac{x_1^L}{x^*}\right)^{\beta_1(\underline{\mu})} - I.$$

In addition to the value matching condition, we apply a smooth pasting condition. Here, smooth pasting implies that the derivatives of the value function $\Lambda$ and $L$ coincide at $x_{\tau_{L,1}^t}$, i.e.

$$\Lambda'(x_{\tau_{L,1}^t}) = L_1'(x_{\tau_{L,1}^t}). \tag{C.3}$$

This condition ensures differentiability at the investment threshold.

Applying condition (C.3) gives

$$\frac{D_{00}}{r - \underline{\mu}} + \beta_1(\underline{\mu}) A_2 {x_1^L}^{\beta_1(\underline{\mu})-1} = \frac{D_{10}}{r - \underline{\mu}} + \beta_1(\underline{\mu}) A_1 {x_1^L}^{\beta_1(\underline{\mu})-1},$$

where

$$A_1 = \left(\frac{1}{x_2^F}\right)^{\beta_1(\underline{\mu})-1} \frac{D_{11} - D_{10}}{r - \underline{\mu}}$$

in the first case and

$$A_1 = -\frac{1}{\beta_1(\underline{\mu})}\frac{D_{10}x^*}{r-\underline{\mu}}\left(\frac{1}{x^*}\right)^{\beta_1(\underline{\mu})}$$

in the second.

Hence,

$$A_2 = \frac{D_{10}-D_{00}}{r-\underline{\mu}}\frac{1}{\beta_1(\underline{\mu})}\frac{1}{x_1^{L\,\beta_1(\underline{\mu})-1}} + A_1.$$

Applying the value matching condition finally yields

$$\frac{D_{00}x_1^L}{r-\underline{\mu}} + \left(\frac{D_{10}-D_{00}}{r-\underline{\mu}}\frac{1}{\beta_1(\underline{\mu})}\frac{1}{x_1^{L\,\beta_1(\underline{\mu})-1}} + A_1\right)x_1^{L\,\beta_1(\underline{\mu})} = \frac{D_{10}x_1^L}{r-\underline{\mu}} + A_1 x_1^{L\,\beta_1(\underline{\mu})} - I$$

$$\Longleftrightarrow \frac{D_{10}-D_{00}}{r-\underline{\mu}}x_1^L - \frac{D_{10}-D_{00}}{r-\underline{\mu}}\frac{1}{\beta_1(\underline{\mu})}x_1^L = I$$

$$\Longleftrightarrow \frac{\beta_1(\underline{\mu})-1}{\beta_1(\underline{\mu})}\frac{D_{10}-D_{00}}{r-\underline{\mu}}x_1^L = I,$$

and therefore, for both cases, it holds that

$$x_1^L = \frac{\beta_1(\underline{\mu})}{\beta_1(\underline{\mu})-1}\frac{I(r-\underline{\mu})}{D_{10}-D_{00}}.$$

■

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