Solving SODEs with large noise by balanced integration methods

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Introduction

Stochastic ordinary differential equations (SODEs) are used in many applications to model the time-dependent processes, which are exposed to deterministic influences as well as stochastic perturbations. For example, in financial mathematics it is a widely accepted principle to model stock price performance with SODEs, see [30]. Also in biology, physics or chemistry many problems can be formulated in terms of stochastic differential equations (see [2] and [62]). The majority of these SODEs cannot be solved explicitly and numerical methods become increasingly important. Several monographs have been published which set up and analyze numerical methods for stochastic differential equations, see for instance [36], [42], [43], [45].

In this thesis we deal with stochastic differential equations driven by large or rather stiff noise. By stiff noise we understand multiplicative noise terms with large coefficients respectively matrices with large eigenvalues. Strong perturbations frequently appear in stock price performance, in physics, chemical or biological processes. Large noise can certainly contribute to a damping or stabilization of the solutions of SODEs. It will be specified more precisely below.

Let us first consider a homogeneous linear stochastic differential equation in the form

$$
\begin{aligned}
\mathrm{d}X(t) =& AX(t)\,\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\,\mathrm{d}W_r(t), \quad t \in [0, T] \\
X(0) =& X_0,
\end{aligned}
\tag{0.1}
$$

where $A, G_r \in \mathbb{R}^{d \times d}, r = 1, \ldots, m$ and W_r are independent real-valued standard Wiener processes. Details of the analytical setting for SODEs and their solutions will be provided in Chapter 1. If the matrices A and $G_r, r = 1, \ldots, m$ commute, i.e. $AG_r = G_r A$ and $G_r G_l = G_l G_r$ for all r, l, then the explicit solution of (0.1) has the form (see [3], [42])

$$
X(t) = \exp\left((A - \frac{1}{2}\sum_{r=1}^{m} G_r^2)t + \sum_{r=1}^{m} G_r W_r(t)\right) X_0.
\tag{0.2}
$$

For $d = m = 1$ the equation (0.2) is also known as the geometric Brownian motion.

In the literature there are many results on the longtime behavior of solutions to SODEs, see for instance [3], [37], [34], [36], [42], [40]. Quite a few of them focus on the study of the stability of stochastic systems, i.e. the insensitivity of the system state to minor changes in the initial state or system parameters. In particular, they study the asymptotic

stochastic stability in the mean-square sense and the asymptotic stability in the almost sure sense. For definitions we refer to [3], [34], [42]. In the one dimensional case it is shown that the equilibrium position of (0.1), i.e. $X(t) = 0$ with $X_0 = 0$ is asymptotically mean-square stable if and only if $2A + G_1^2 < 0$. Moreover, the equilibrium position is asymptotically stable in the almost sure sense if and only if $A - \frac{1}{2}G_1^2 < 0$. It follows that for sufficiently large G_1 the equilibrium position of (0.1) is asymptotically stable in the almost sure sense but asymptotically unstable in the mean-square sense (see [3], [34], [1]). In the multidimensional case a criterion of the asymptotic mean-square stability of the equilibrium position of (0.1) is derived in [3], [55], [34], [10], [1]. The criterion refers to the stability matrix

$$S = \mathrm{id} \otimes A + A \otimes \mathrm{id} + \sum_{r=1}^{m} G_r \otimes G_r, \tag{0.3}$$

where $\otimes$ denots the Kronecker product (see [10]). The matrix S is derived from the second moment of the solution of (0.1). Properties of the stability matrix are based on the classical theory of deterministic differential equations. If the spectral abscissa of S is negative, then the equilibrium position of (0.1) is asymptotic mean-square stable (see [23] and also [10, Lemma 3.3], [1, Lemma 1]).

The authors from [10] have derived stability matrices for numerical schemes such as the θ-Maruyama method and the θ-Milstein method and explored their asymptotic mean-square stability. We also refer to [25] for results of the exponential stability in the mean-square sense for numerical solutions to stochastic differential equations. In this thesis we are interested in developing methods that treat large noise terms in a more quantitative way on finite time intervals. Nevertheless, for our examples in Section 1.7 we check asymptotic mean-square stability properties of the equilibrium position of (0.1) given in [10, Lemma 3.3] or [1, Lemma 1].

Let us consider the general nonlinear case

$$\begin{aligned} \mathrm{d}X(t) =& f(t, X(t))\, \mathrm{d}t + \sum_{r=1}^{m} g_r(t, X(t))\, \mathrm{d}W_r(t), \quad t \in [0, T] \\ X(0) =& X_0, \end{aligned} \tag{0.4}$$

where $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the drift coefficient function, $g_r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, r = 1, \ldots, m$ are the diffusion coefficient functions and W_r are independent real-valued standard Wiener processes. If we assume that f and $g_r, r = 1. \ldots, m$ are globally Lipschitz continuous and also fulfill a linear growth condition then existence and uniqueness of the solution to (0.4) are guaranteed (see [36], [16], [42]). In the numerical analysis there are many results known for such problems. Since we are interested in the SODEs with stiff noise terms we cite some authors who investigate this problem. For instance, in [44] a balanced implicit method for solving of the SODEs with large noise terms is proposed.

Later, based on in [44] C. Kahl and H. Schurz investigate higher order methods in which the global mean square convergence is shown (see [32]). However, there are several problems with their approach: First, there seems to be no general recipe how to choose their control functions. Second, their error analysis in [32] uses some estimates which need to be corrected (for details see Remark 4.4.8 and Remark 4.5.6).

In our approach we pursue the following idea: If we have m noise terms and the largest of them can be solved exactly, then the remaining terms can be solved numerically without causing large errors. This idea, described in Section 1.4, is based on an orthogonal transform of the Wiener process and a subsequent two-step recursion of the numerical method. Various transformations of stochastic differential equations have been used to analyze their qualitative properties (see [57]). But only few of them seem to be utilized for numerical purposes. For instance, in [1] the authors use the Girsanov transformation of the Wiener process to see the asymptotic mean-square instability of the equilibrium position numerically. In this thesis we use the orthogonal Wiener process transformation to isolate the largest noise term.

Let us consider SODE (0.1) and let $Q \in \mathbb{R}^{m \times m}$ be orthogonal. Further, let us transform the Wiener process as follows:

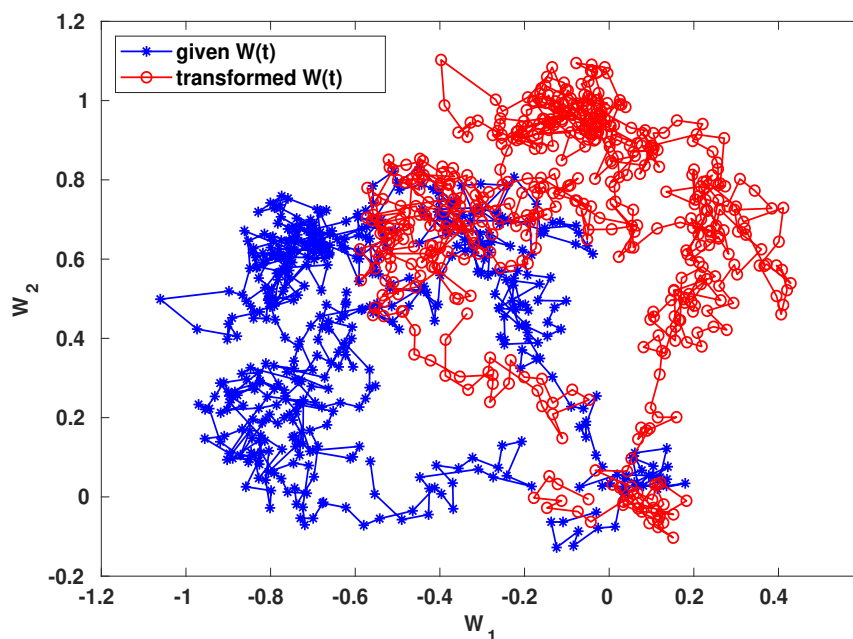$$\tilde{W}(t) = Q^\top W(t), \quad t \in [0, T]. \tag{0.5}$$

Figure 0.1.: Single trajectories of a two-dimensional Wiener process and their orthogonal transformation

Actually, the orthogonal transformation mixes trajectories but it does not change the stochastic nature of the processes (see [18], [57]). Figure 0.1 shows a random walk in $\mathbb{R}^2$ and its transformed version obtained by a rotation. Then the stochastic differential equation (0.1) takes the form

$$\begin{aligned}
\mathrm{d}X(t) =& AX(t)\,\mathrm{d}t + \sum_{k=1}^{m} \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t), \\
X(0) =& X_0,
\end{aligned} \tag{0.6}$$

where $\tilde{G}_k = \sum_{r=1}^{m} Q_{rk} G_r, k = 1, \ldots, m$. Taking the Frobenius norm for the matrices $\tilde{G}_k$ follows

$$\|\tilde{G}_k\|_F^2 = \sum_{r,j=1}^{m} Q_{rk} Q_{jk} \Gamma_{rj}, \quad k = 1, \ldots, m$$

with $\Gamma_{rj} = \mathrm{trace}(G_r^\top G_j)$. By a singular value decomposition of the matrix Γ we are then able to arrange that $\tilde{G}_1$ has the largest Frobenius norm equal to the largest singular value. The access to the largest noise term suggests to split equation (0.6) as follows

$$\begin{aligned}
\mathrm{d}Y(t) =& AY(t)\,\mathrm{d}t + \sum_{k=2}^{m} \tilde{G}_k Y(t)\,\mathrm{d}\tilde{W}_k(t), \quad t \in [0, T] \\
Y(0) =& Y_0,
\end{aligned} \tag{0.7}$$

and

$$\begin{aligned}
\mathrm{d}Z(t) =& \tilde{G}_1 Z(t)\,\mathrm{d}\tilde{W}_1(t), \\
Z(0) =& Z_0.
\end{aligned} \tag{0.8}$$

The splitting idea is also known in the deterministic case (see [21], [9]): A vector field is split into integrable parts and treated separately. The Lie-Trotter and Strang splittings are well-known numerical methods for solving differential equations.

In this thesis we approximate the first SODE (0.7) with the standard Euler-Maruyama or the Milstein schemes. In the second step we solve exactly the SODE (0.8), where the initial value Z_0 is replaced by the result of the first step at a specific time. Our approach is related to the work of Erdoğan and Lord [14]. They treat SODEs and assume that the noise terms commute. Then an exponential integrator is used for explicitly solving the linear part of the SODEs with multiplicative noise terms. Similar problems with uniform estimates for the proof of convergence as in [32] appear and need to be corrected.

Let us denote by $h = (h_1, \ldots, h_N) \in (0, T]^N$ a vector of step sizes with $\sum_{i=1}^{N} h_i = T, N \in \mathbb{N}$. Every vector of step sizes induces a set of time grid points given by

$$\mathcal{T}_h := \{t_n := \sum_{i=1}^{n} h_i \colon n = 0, \ldots, N\}.$$

Then our sample balanced shift noise Euler-type method is given by the two-step recursion

$$\overline{X}_h(t_i) = X_h(t_{i-1}) + AX_h(t_{i-1})h + \sum_{k=2}^{m} \tilde{G}_k X_h(t_{i-1}) \tilde{I}_{(k)}^{t_{i-1},t_i},$$

$$X_h(t_i) = \exp\left(-\frac{1}{2}\tilde{G}_1^2 h + \tilde{G}_1 \tilde{I}_{(1)}^{t_{i-1},t_i}\right) \overline{X}_h(t_i), \quad i = 1, \ldots, N. \tag{0.9}$$

Here we denote the Wiener increments by $\tilde{I}_{(k)}^{t_{i-1},t_i} = \tilde{W}_k(t_i) - \tilde{W}_k(t_{i-1})$ (see [36]). Since we are interested in the convergence of (0.9) in the mean-square sense, we strive to ensure that the second moment of X_h stays bounded. It means that the matrix exponential in (0.9) should not contain very large eigenvalues. To damp the exponential solver we propose the following shift of the deterministic term in (0.8)

$$\mathrm{d}Z(t) = CZ(t)\,\mathrm{d}t + \tilde{G}_1 Z(t)\,\mathrm{d}\tilde{W}_1(t), \quad t \in [0,T],$$

$$Z(0) = Z_0, \tag{0.10}$$

where $C \in \mathbb{R}^{d \times d}$. Moreover, we assume that $C\tilde{G}_1 = \tilde{G}_1 C$. Then the SODE (0.10) has an explicit solution in the form (see [3], [42] or [36])

$$Z(t) = \exp\left((C - \frac{1}{2}\tilde{G}_1^2)t + \tilde{G}_1 \tilde{W}_1(t)\right) Z_0, \quad t \in [0,T]. \tag{0.11}$$

This shift gives us the possibility to choose the matrix C such that the second moment of (0.11) has small values. Therefore, we determine the shift matrix $C = -\frac{1}{2}\tilde{G}_1^2$. Of course, at this shift we get an additional term in (0.7), which can cause stiffness in the drift term. Hence, we assume that the spectrum of the matrix A lies to the left of the imaginary axis of the complex plane. Then our balanced shift noise explicit Euler-type method takes the form

$$\overline{X}_h(t_i) = X_h(t_{i-1}) + A^+ X_h(t_{i-1})h + \sum_{k=2}^{m} \tilde{G}_k X_h(t_{i-1}) \tilde{I}_{(k)}^{t_{i-1},t_i},$$

$$X_h(t_i) = \exp\left(-\tilde{G}_1^2 h + \tilde{G}_1 \tilde{I}_{(1)}^{t_{i-1},t_i}\right) \overline{X}_h(t_i), \quad i = 1, \ldots, N, \tag{0.12}$$

where $A^+ = A + \frac{1}{2}\tilde{G}_1^2$.

In the following we describe the main contents of this thesis. In particular, we discuss several extensions of our basic splitting methods above and we give an overview of the theoretical and numerical in the further chapters. Generally, we consider strongly convergent numerical methods, whose one-step maps satisfy suitable Lipschitz-type conditions, which allow the underlying SODE to have non-globally Lipschitz continuous coefficient functions. For the error analysis of the numerical schemes we use the notion of *B-consistency* and *C-stability* from [5], [6] and study the strong error of convergence

under the *one-sided Lipschitz condition*

$$\langle x_1 - x_2, f(t, x_1) - f(t, x_2)\rangle + \eta \sum_{r=1}^{m} |g_r(t, x_1) - g_r(t, x_2)|^2 \leq L|x_1 - x_2|^2 \qquad (0.13)$$

for all $x_1, x_2 \in \mathbb{R}^d$, $t \in [0, T]$. This condition includes several examples of SODEs with superlinearly growing drift and diffusion coefficient functions, for which the explicit Euler-Maruyama scheme diverges (see [28]). However, in [29] the authors proposed a tamed Euler scheme that has a modified drift term such that it is uniformly bounded. They show that this method converges strongly to the exact solution of the SODE if the drift term is one-sided Lipschitz continuous. An alternative approach is developed in [5], the projected Euler-Maruyama method (PEM), which is based on the standard Euler-Maruyama method and a projection onto a ball in $\mathbb{R}^d$ whose radius is expanding with a negative power of the step size. It is proved that the PEM scheme with the one-sided Lipschitz continuous coefficients is strongly convergent of order $\frac{1}{2}$. Also we refer to [56], [25] for the strong error analysis of the backward Euler scheme and the split-step backward Euler method.

In Chapter 1 we describe our problem setting and general assumptions. As an approach to solving this problem, we present our balanced shift noise Euler-type schemes.

In Chapter 2 we put our focus on the linear case and prove that the balanced shift noise as well as the explicit and the implicit Euler type methods are strongly convergent to the exact solution of the stochastic differential equation. In order to show the advantage of our balanced Euler-type scheme over the standard Euler-Maruyama method we derive sharp error estimates and keep track of the constants that occur. Instead of standard matrix norms we use the so-called logarithmic norm, which gives a more precise bound to the matrix exponential (see [12], [58] or [60]).

In Chapter 3 we treat the nonlinear drift term in the SODE (0.6). We show how to transfer from [5] the cut-off procedure for the explicit case and the splitting for the implicit case. Here we apply known techniques for solving the nonlinear equations under a one-sided Lipschitz condition and prove the strong convergence of the projected and split-step balanced shift noise Euler type schemes.

In Chapter 4 we consider higher order schemes and suggest the projected balanced shift noise Milstein type method. We do not carry out the numerical analysis of this scheme and test it only in our numerical experiments. The numerical experiments suggest that the projected balanced shift noise Milstein-type method converges strongly with order 1 if the diffusion matrices commute. Otherwise the order of the strong convergence of this scheme seems to be only $\frac{1}{2}$. Let us emphasize at this point that our goal is not to achieve a high order of convergence, but rather to treat the problem of the stiff noise term.

Further, returning to the classical balanced Milstein method [32] we prove in Sections 4.4- 4.5 that the balanced Milstein method with one-sided Lipschitz continuous

coefficient functions is strongly convergent of order 1. Our proof repairs some incorrect estimates in [32], which were mentioned above.

In Chapter 5 we present some numerical experiments which support our theoretical results for the discretization of the stochastic Hopf equation and the stochastic Lorenz equation. It is easy to see that the coefficient functions of the Hopf equation satisfy one-sided Lipschitz condition. On the contrary, the drift coefficient functions of the Lorenz equation are not one-sided Lipschitz continuous but satisfy the one-sided linear growth condition (1.7) (see [54], [27]). This point makes this equation interesting. It is shown that the balanced shift noise Euler- and Milstein-type schemes yield good results for the Hopf system but are not very suitable for solving the stochastic Lorenz system.

Our final remark concerns the numerical implementation of the splitting methods proposed and analyzed in this thesis. There are two possibilities to simulate the solution of the transformed equation (0.6). The first one is to simulate a Wiener process for the original equation (0.1) and then transform it via (0.5). This will allow us to obtain numerical solutions that converge strongly to the solutions of the original equation (0.1). However, to simplify our numerical computations we allow to simulate the modified Wiener process $\tilde{W}_r, r = 1, \ldots, m$ directly by a random number generator. In this way we obtain strongly convergent solutions of the modified equation (0.6). When compared with the original equation, these solutions have only the same distribution. Hence they are only suitable for approximating smooth functionals of solutions as it is common in the theory of weak convergence (see [51], [11]).

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1. Balanced integration methods

In this chapter we study known methods and set up a new type of numerical methods for solving stochastic differential equations (SODEs) with the multiplicative noise terms with large coefficients. For deterministic systems with stiff drift terms there is a well-developed theory how to solve such systems efficiently with implicit methods, see for example [22]. However, there seems to be no simple analogue for stiff noisy systems, since a naive implicit treatment of the noise term generally leads to divergence (see [44]).

Therefore, we propose in this chapter a different approach that deals with the problem of the stiff noise terms and we focus on the linear case. The general case is considered in further chapters.

1.1. Problem setting and general assumptions

In this section we introduce the general notations of the stochastic differential equations and their assumptions. In addition, we assume that a one-sided Lipschitz condition is satisfied.

Let $d, m \in \mathbb{N}, T \in (0, \infty)$, and $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ which fulfills the usual conditions (i.e., the filtration is right continuous and each $\mathcal{F}_t$ contains all sets $A \in \mathcal{F}$ with $\mathbf{P}(A) = 0$, see for instance [42],[50]). We consider the solution $X : [0, T] \times \Omega \to \mathbb{R}^d$ to the Itô stochastic differential equation of the form

$$
\begin{aligned}
\mathrm{d}X(t) =& f(t, X(t)) \, \mathrm{d}t + \sum_{r=1}^{m} g_r(t, X(t)) \, \mathrm{d}W_r(t), \\
X(0) =& X_0, \quad t \in [0, T]
\end{aligned}
\tag{1.1}
$$

with the drift and diffusion coefficient functions $f, g_r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and real-valued standard Wiener processes $W_r : [0, T] \times \Omega \to \mathbb{R}, r = 1, \ldots, m$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$. In the following we impose the conditions on the drift and diffusion coefficient functions (see [5], [6]).

Assumption 1.1.1. *The mappings $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g_r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $r = 1, \ldots, m$, are continuous. Furthermore, there exists a positive constant $L \in (0, \infty)$ and a*

parameter value $\eta \in (\frac{1}{2}, \infty)$ such that for all $t \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^d$ it holds

$$\langle x_1 - x_2, f(t, x_1) - f(t, x_2) \rangle + \eta \sum_{r=1}^{m} \left| g_r(t, x_1) - g_r(t, x_2) \right|^2 \leq L |x_1 - x_2|^2. \qquad (1.2)$$

In addition, there exists $q \in [1, \infty)$ such that for all $t, t_1, t_2 \in [0, T]$, $x, x_1, x_2 \in \mathbb{R}^d$ and $r = 1, \ldots, m$ it holds

$$|f(t, x)| \vee |g_r(t, x)| \leq L(1 + |x|^q), \qquad (1.3)$$

$$|f(t_1, x) - f(t_2, x)| \vee |g_r(t_1, x) - g_r(t_2, x)| \leq L(1 + |x|^q)|t_1 - t_2|^{\frac{1}{2}}, \qquad (1.4)$$

$$|f(t, x_1) - f(t, x_2)| \vee |g_r(t, x_1) - g_r(t, x_2)| \leq L(1 + |x_1|^{q-1} + |x_2|^{q-1})|x_1 - x_2|. \qquad (1.5)$$

Here we denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$ and by $\langle \cdot, \cdot \rangle$ the Euclidean innner product. The assumption (1.2) is also called global monotonicity condition. In the case $q = 1$ conditions (1.4) and (1.5) lead to the well-known global Lipschitz condition. The conditions above guarantee the existence and uniqueness of the solution to (1.1) (see [38] or [42]). That means that there exists a unique, **P**-almost surely continuous, and $(\mathcal{F}_t)_{t \in [0, T]}$-adapted stochastic process $X \colon [0, T] \times \Omega \to \mathbb{R}^d$ which satisfies the integral equation

$$X(t) = X_0 + \int_0^t f(s, X(s)) \, \mathrm{d}s + \sum_{r=1}^{m} \int_0^t g_r(s, X(s)) \, \mathrm{d}W_r(s), \quad t \in [0, T]. \qquad (1.6)$$

Moreover, recall from [42, Chap.2.4] if there exist a constant $\alpha_{\mathrm{f}} \geq 0$ and $p \geq 2$ such that

$$\langle x, f(t, x) \rangle + \frac{p-1}{2} \sum_{r=1}^{m} |g_r(t, x)|^2 \leq \alpha_{\mathrm{f}}(1 + |x|^2) \qquad (1.7)$$

for all $x \in \mathbb{R}^d$ and $t \in [0, T]$, then

$$\sup_{t \in [0, T]} \|X(t)\|_{L^p(\Omega; \mathbb{R}^d)} < \infty. \qquad (1.8)$$

The condition (1.7) is also known as *global coercivity condition*.

The following lemma is a generalization of [42, Th.4.1]. We assume that the solution $X(t)$, $t \in [0, T]$ to (1.1) is unique with the initial value $X(0) = X_0$.

Lemma 1.1.2. *Let $X_0 \in L^p(\Omega; \mathbb{R}^d)$ for $p \in [2, \infty)$ and assume that there exist constants $\epsilon \geq 0$ and $\alpha_{\mathrm{f}} \geq 0$ such that for all $x \in \mathbb{R}^d$ and $t \in [0, T]$ it holds*

$$\langle x, f(t, x) \rangle + \frac{p-1}{2} \sum_{r=1}^{m} |g_r(t, x)|^2 \leq \alpha_{\mathrm{f}}(\epsilon^2 + |x|^2). \qquad (1.9)$$

Then

$$\|X(t)\|_{L^p(\Omega; \mathbb{R}^d)} \leq \left(\epsilon + \|X_0\|_{L^p(\Omega; \mathbb{R}^d)} \right) \mathrm{e}^{\alpha_{\mathrm{f}} t}. \qquad (1.10)$$

Proof. For $\epsilon = 1$ we refer to the proof in [42, Th.4.1]. Now, let $\epsilon > 0$. Then by the Itô formula, the Cauchy-Schwarz inequality, the fact that $|x| \leq (\epsilon^2 + |x|^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^d$, and (1.9) we obtain

$$
\begin{aligned}
(\epsilon^2 + |X(t)|^2)^{\frac{p}{2}} =& (\epsilon^2 + |X_0|^2)^{\frac{p}{2}} + p \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} \langle X(s), f(s, X(s)) \rangle \, \mathrm{d}s \\
& + \frac{p(p-2)}{2} \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-4}{2}} |X(s)^\top g_r(t, X(s))|^2 \, \mathrm{d}s \\
& + \frac{p}{2} \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} |g_r(s, X(s))|^2 \, \mathrm{d}s \\
& + p \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} \langle X(s), g_r(s, X(s)) \rangle \, \mathrm{d}W_r(s) \\
\leq& (\epsilon^2 + |X_0|^2)^{\frac{p}{2}} + p \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} \langle X(s), f(s, X(s)) \rangle \, \mathrm{d}s \\
& + \frac{p(p-2)}{2} \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-4}{2}} (\epsilon^2 + |X(s)|^2) |g_r(t, X(s))|^2 \, \mathrm{d}s \\
& + \frac{p}{2} \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} |g_r(s, X(s))|^2 \, \mathrm{d}s \\
& + p \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} \langle X(s), g_r(s, X(s)) \rangle \, \mathrm{d}W_r(s) \\
\leq& (\epsilon^2 + |X_0|^2)^{\frac{p}{2}} \\
& + p \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} \Big(\langle X(s), f(s, X(s)) \rangle + \frac{p-1}{2} \sum_{r=1}^m |g_r(s, X(s))|^2 \Big) \, \mathrm{d}s \\
& + p \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} \langle X(s), g_r(s, X(s)) \rangle \, \mathrm{d}W_r(s) \\
\leq& (\epsilon^2 + |X_0|^2)^{\frac{p}{2}} + p\alpha_{\mathrm{f}} \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p}{2}} \, \mathrm{d}s \\
& + p \sum_{r=1}^m \int_0^t (\epsilon^2 + |X(s)|^2)^{\frac{p-2}{2}} \langle X(s), g_r(s, X(s)) \rangle \, \mathrm{d}W_r(s).
\end{aligned}
$$

Here the last stochastic integral is a local martingale. Therefore, the stochastic Gronwall Lemma A.2.2 applies and we obtain

$$
\mathbb{E}[(\epsilon^2 + |X(t)|^2)^{\frac{p}{2}}] \leq \mathbb{E}[(\epsilon^2 + |X_0|^2)^{\frac{p}{2}}] e^{\alpha_{\mathrm{f}} pt}. \tag{1.11}
$$

Taking the limit as $\epsilon \searrow 0$ shows that (1.11) also holds for $\epsilon = 0$. It remains to take the p-th root and by the Minkovski inequality with respect to the $L^{\frac{p}{2}}(\Omega; \mathbb{R}^d)$-norm we get (1.10). $\qquad\square$

Remark 1.1.3. Let us add to the proof of Theorem 4.1 in [42] that the stochastic integral, obtained by the Itô formula is a local martingale. Therefore, one can apply the stochastic Gronwall Lemma A.2.2, proposed in [53]. In [42] stopping times are used for such an estimate.

Remark 1.1.4. Using ϵ in the proof above avoids studying cases $p = 2$ and $p \geq 4$ as in [39], for instance.

In particular, in the linear case we use the so-called *logarithmic norm*, which is defined below.

Definition 1.1.5. *For the quadratic matrix $B \in \mathbb{R}^{d \times d}$ with the induced matrix norm $|\cdot|$ the logarithmic norm $\mu_2(B)$ is given by*

$$\mu_2(B) = \lambda_{\max}\Big(\frac{B + B^\top}{2}\Big), \tag{1.12}$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $\frac{B+B^\top}{2}$.

We note that the logarithmic norm does not have all properties of the standard norm and can also be negative. For details see Appendix, Section A.4.

Corollary 1.1.6. *In the linear case with $f(x) := Ax$ and $g_r(x) := G_r x, \ r = 1, \ldots, m$ for all $x \in \mathbb{R}^d$ and $A, G_r \in \mathbb{R}^{d \times d}$ it holds*

$$\langle x, Ax \rangle + \frac{p-1}{2} \sum_{r=1}^{m} |G_r x|^2 = x^\top \Big(A + \frac{p-1}{2} \sum_{r=1}^{m} G_r^\top G_r\Big)x \tag{1.13}$$
$$\leq \mu_2(B)|x|^2,$$

where

$$B := A + \frac{p-1}{2} \sum_{r=1}^{m} G_r^\top G_r. \tag{1.14}$$

According to the property of the logarithmic norm (see Appendix, Lemma A.4.2) we can say, that

$$\mu_2(B) \leq |B|.$$

1.2. Transformed Wiener noise

In this section we use an orthogonal transformation of the vector-valued Wiener noise to order the matrix coefficients in the linear noise term. Let us consider a linear homogeneous stochastic differential equation in the form

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\,\mathrm{d}W_r(t), \tag{1.15}$$
$$X(0) = X_0, \quad t \in [0, T],$$

where $A, G_r \in \mathbb{R}^{d \times d}$ and $W_r : [0, T] \times \Omega \to \mathbb{R}, r = 1, \ldots, m$ denote an independent family of real-valued standard Wiener processes defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $(\mathcal{F}_t)_{t \in [0,T]}$.

Let $Q \in \mathbb{R}^{m \times m}$ be an orthogonal matrix and consider for all $t \in [0, T]$ the transformed Wiener process

$$\tilde{W}(t) = Q^\top W(t). \tag{1.16}$$

The random vector $\tilde{W}(t)$ is again a vector of independent real-valued standard Wiener processes, see for example [57, Sec.3.4.4]. Then for all $r = 1, \ldots, m$ we have

$$\mathrm{d}W_r(t) = \sum_{k=1}^{m} Q_{rk} \, \mathrm{d}\tilde{W}_k(t) \tag{1.17}$$

and we rewrite (1.15) in the following form

$$
\begin{aligned}
\mathrm{d}X(t) =& AX(t)\,\mathrm{d}t + \sum_{r=1}^{m} G_r X(t) \sum_{k=1}^{m} Q_{rk} \, \mathrm{d}\tilde{W}_k(t) \\
=& AX(t)\,\mathrm{d}t + \sum_{k=1}^{m} \sum_{r=1}^{m} Q_{rk} G_r X(t) \, \mathrm{d}\tilde{W}_k(t) \\
=& AX(t)\,\mathrm{d}t + \sum_{k=1}^{m} \tilde{G}_k X(t) \, \mathrm{d}\tilde{W}_k(t), \quad t \in [0, T],
\end{aligned}
\tag{1.18}
$$

with the initial value $X(0) = X_0$ and

$$\tilde{G}_k = \sum_{r=1}^{m} Q_{rk} G_r. \tag{1.19}$$

The corresponding integral form is given by

$$X(t) = X_0 + \int_0^t AX(\tau)\,\mathrm{d}\tau + \sum_{k=1}^{m} \int_0^t \tilde{G}_k X(\tau)\,\mathrm{d}\tilde{W}_k(\tau). \tag{1.20}$$

We note that by our derivation the solutions of (1.18) and (1.15) agree pathwise if the Wiener processes are related by (1.17). However, if we consider (1.18) as an SODE with an arbitrarily given Wiener process $\tilde{W}$, then we get a new solution process $\tilde{X}$ which has the same distribution as X, see Chapter 5.

The idea is to determine a matrix Q such that the Frobenius norms of the matrices $\tilde{G}_k$ are ordered:

$$|\tilde{G}_1|_F \geq |\tilde{G}_2|_F \cdots \geq |\tilde{G}_m|_F. \tag{1.21}$$

Calculating the Frobenius norm for all $k = 1, \ldots, m$ yields

$$
\begin{aligned}
|\tilde{G}_k|_F^2 &= \mathrm{trace}\big(\tilde{G}_k^\top \tilde{G}_k\big) \\
&= \mathrm{trace}\Big(\big(\sum_{r=1}^m Q_{rk} G_r\big)^\top \sum_{j=1}^m Q_{jk} G_j\Big) \\
&= \sum_{r,j=1}^m Q_{rk} Q_{jk} \mathrm{trace}\big(G_r^\top G_j\big) \\
&= \sum_{r,j=1}^m Q_{rk} Q_{jk} \Gamma_{rj} \\
&= (Q^\top \Gamma Q)_{kk},
\end{aligned}
\tag{1.22}
$$

where the $m \times m$ matrix $\Gamma = (\Gamma_{rj})_{r,j=1}^m$ is defined by $\Gamma_{rj} := \mathrm{trace}(G_r^\top G_j)$.

Hence the ordering above is achieved by the singular value decomposition (SVD) of the matrix Γ, i.e.,

$$
Q^\top \Gamma Q = D,
$$

where D denotes the diagonal matrix whose entries are the singular values $\gamma_1 \geq \cdots \geq \gamma_m$ of Γ (see for instance, [8]).

The fact that the singular values of Γ are ordered according to size will be used later to suggest numerical approximations to (1.18) with large noise term. It seems that the choice of the Frobenius norm is very useful to get the relation (1.21). However, in our later estimates we use the spectral norm which satisfies $|\tilde{G}_k| \leq |\tilde{G}_k|_F, k = 1, \ldots, m$. Therefore, it is desirable to have the ordering (1.21) with respect to the spectral norm, but we don't have a simple algorithm for this problem.

1.3. Hölder continuity of the solution

In this section we derive a result on the Hölder continuity of the solution of (1.18) with respect to the norm $L^2(\Omega; \mathbb{R}^d)$, which is given by $\|Z\|_{L^2(\Omega; \mathbb{R}^d)} := \big(\mathbb{E}[|Z|^2]\big)^{\frac{1}{2}}$ for all $Z \in L^2(\Omega; \mathbb{R}^d)$.

Let the following block matrices

$$
G = \begin{pmatrix} G_1 \\ \vdots \\ G_m \end{pmatrix} \in \mathbb{R}^{m \times d, d}, \quad \tilde{G} = \begin{pmatrix} \tilde{G}_1 \\ \vdots \\ \tilde{G}_m \end{pmatrix} \in \mathbb{R}^{m \times d, d}.
$$

be given. Then their spectral norm is given by (see [26], [19])

$$
|G| = |G^\top G|^{\frac{1}{2}}
\tag{1.23}
$$

and

$$|\tilde{G}| = |\tilde{G}^\top \tilde{G}|^{\frac{1}{2}}. \tag{1.24}$$

The following lemma simplifies a few estimates.

Lemma 1.3.1. *Let $G_r, r = 1, \ldots, m$ be real-valued $d \times d$-matrices and $\tilde{G}_k, k = 1, \ldots, m$ are given by* (1.19). *Then*

$$|\tilde{G}| = |G|. \tag{1.25}$$

Proof. By (1.19) and (1.24) we obtain

$$
\begin{aligned}
\sum_{k=1}^{m} \tilde{G}_k^\top \tilde{G}_k &= \sum_{k=1}^{m} \Big(\sum_{r=1}^{m} Q_{rk} G_r \Big)^\top \sum_{j=1}^{m} Q_{jk} G_j \\
&= \sum_{r,j=1}^{m} G_r^\top G_j \sum_{k=1}^{m} Q_{kr} Q_{jk} \\
&= \sum_{r,j=1}^{m} G_r^\top G_j \delta_{rj} \\
&= \sum_{k=1}^{m} G_k^\top G_k,
\end{aligned}
$$

where δ_{rj} denotes the Kronecker symbol. Taking the norm and the square-root proves the assertion (1.25).

$\square$

Further, similar to Corollary 1.1.6 with $p = 2$ it holds for all $x \in \mathbb{R}^d$

$$
\begin{aligned}
\langle x, Ax \rangle + \frac{1}{2} \sum_{k=1}^{m} |\tilde{G}_k x|^2 &= x^\top \Big(A + \frac{1}{2} \sum_{k=1}^{m} \tilde{G}_k^\top \tilde{G}_k \Big) x \\
&= x^\top \Big(A + \frac{1}{2} \sum_{k=1}^{m} G_k^\top G_k \Big) x \\
&\leq \mu_2(\tilde{B}) |x|^2,
\end{aligned}
$$

where $\mu_2(\tilde{B})$ is the logarithmic norm of

$$\tilde{B} := A + \frac{1}{2} \sum_{k=1}^{m} G_k^\top G_k. \tag{1.26}$$

In the following, let us denote by

$$\alpha := \mu_2(\tilde{B}) \quad \text{and} \quad \alpha_+ := \max(\alpha, 0). \tag{1.27}$$

Lemma 1.3.2. *The solution* $X \colon [0, T] \times \Omega \to \mathbb{R}^d$ *of* (1.18) *satisfies for* $0 \le s < t \le T$

$$\|X(t) - X(s)\|_{L^2(\Omega;\mathbb{R}^d)} \le C(t - s, t)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t - s|^{\frac{1}{2}}, \tag{1.28}$$

where $C(\delta, \delta_1) = \mathrm{e}^{\alpha + \delta}(|A|\delta_1^{\frac{1}{2}} + |G|)$ *for* $0 \le \delta_1 < \delta \le T$.

Proof. For all $0 \le s \le t \le T$ we have

$$X(t) - X(s) = \int_s^t AX(\tau)\,\mathrm{d}\tau + \sum_{k=1}^m \int_s^t \tilde{G}_k X(\tau)\,\mathrm{d}\tilde{W}_k(\tau).$$

Then by the triangle inequality and the Itô isometry we obtain

$$\|X(t) - X(s)\|_{L^2(\Omega;\mathbb{R}^d)} \le \int_s^t \|AX(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau + \Big\| \sum_{k=1}^m \int_s^t \tilde{G}_k X(\tau)\,\mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$= \int_s^t \|AX(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau + \Big(\sum_{k=1}^m \int_s^t \|\tilde{G}_k X(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}^2\,\mathrm{d}\tau \Big)^{\frac{1}{2}}.$$

By Lemma 1.1.2 we obtain for the first integral

$$\int_s^t \|AX(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau \le |A|\|X_0\|_{L^2(\Omega;\mathbb{R}^d)} \int_s^t \mathrm{e}^{\alpha\tau}\,\mathrm{d}\tau$$
$$\le \mathrm{e}^{\alpha + t}|A|\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t - s|. \tag{1.29}$$

For the second summand we use Lemma 1.3.1 and Lemma 1.1.2

$$\Big(\sum_{k=1}^m \int_s^t \|\tilde{G}_k X(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}^2\,\mathrm{d}\tau \Big)^{\frac{1}{2}} = \Big(\int_s^t \sum_{k=1}^m \|\tilde{G}_k X(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}^2\,\mathrm{d}\tau \Big)^{\frac{1}{2}}$$

$$= \Big(\int_s^t \sum_{k=1}^m \int_\Omega \langle \tilde{G}_k X(\tau), \tilde{G}_k X(\tau) \rangle\,\mathrm{d}\mathbf{P}(\omega)\,\mathrm{d}\tau \Big)^{\frac{1}{2}}$$

$$= \Big(\int_s^t \int_\Omega X(\tau)^T \sum_{k=1}^m \tilde{G}_k^T \tilde{G}_k X(\tau)\,\mathrm{d}\mathbf{P}(\omega)\,\mathrm{d}\tau \Big)^{\frac{1}{2}} \tag{1.30}$$

$$\le \Big| \sum_{k=1}^m G_k^T G_k \Big|^{\frac{1}{2}} \Big(\int_s^t \|X(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}^2\,\mathrm{d}\tau \Big)^{\frac{1}{2}}$$

$$\le |G|\|X_0\|_{L^2(\Omega;\mathbb{R}^d)} \Big(\int_s^t \mathrm{e}^{2\alpha\tau}\,\mathrm{d}\tau \Big)^{\frac{1}{2}}$$

$$\le |G|\mathrm{e}^{\alpha + t}\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t - s|^{\frac{1}{2}}.$$

Altogether, this yields

$$\|X(t) - X(s)\|_{L^2(\Omega;\mathbb{R}^d)} \le \|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha + t}\big(|A||t - s|^{\frac{1}{2}} + |G| \big)|t - s|^{\frac{1}{2}}.$$

$\square$

The Hölder exponent can be increased if we insert the conditional expectation with respect to the σ-field $\mathcal{F}_s$:

Corollary 1.3.3. *The solution $X : [0,T] \times \Omega \to \mathbb{R}^d$ of (1.18) satisfies for $0 \leq s < t \leq T$*

$$\|\mathbb{E}[X(t) - X(s)|\mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \leq C_{\mathrm{cond}}(t)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t - s|,$$

with $C_{\mathrm{cond}}(t) = \mathrm{e}^{\alpha_+ t}|A|$.

Proof. By applying the conditional expectation and the properties of the stochastic integral (see for instance, [42, Th.5.9]) we get

$$\mathbb{E}[X(t) - X(s)|\mathcal{F}_s] = \mathbb{E}\Big[\int_s^t AX(\tau)\,\mathrm{d}\tau + \sum_{k=1}^m \int_s^t \tilde{G}_k X(\tau)\,\mathrm{d}\tilde{W}_k(\tau)|\mathcal{F}_s\Big]$$

$$= \int_s^t \mathbb{E}[AX(\tau)|\mathcal{F}_s]\,\mathrm{d}\tau.$$

Further, we use the fact that

$$\|\mathbb{E}[Z|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}, \tag{1.31}$$

for all $Z \in L^2(\Omega;\mathbb{R}^d)$ and (1.29) and obtain

$$\|\mathbb{E}[X(t) - X(s)|\mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \leq \int_s^t \|\mathbb{E}[AX(\tau)|\mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau$$

$$\leq \int_s^t \|AX(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau$$

$$\leq \mathrm{e}^{\alpha_+ t}|A|\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t - s|.$$

This completes the proof.

$\qquad\square$

We observe that all estimates above involve only terms of the type $\exp(\alpha t), 0 < t \leq T$ where α is a one-sided Lipschitz bound, see (1.9). A dependence of similar type will frequently occur in the following text with the constants getting more and more complicated. Therefore, we use the convention below as a shorthand in the following theorems. Simultaneously, we refer to the precise dependence of constants on the data of the problem.

Convention 1.3.4. *Constants are called of moderate exponential type with respect to the data of the problem if they only contain exponential terms of the form $\exp(\alpha_+ t), t \in [0,T]$, where $\alpha_+ = \alpha$ and α is a one-sided Lipschitz bound.*

1.4. A simple balanced method

In this section we suggest a numerical method which achieves a balancing by a two-step procedure. In the first step we approximate the stochastic differential equation excluding the largest noise of the diffusion term. In the second step we treat the result of the first step as an inhomogeneity and solve a simple stochastic differential equation with the largest noise explicitly. In a sense we use the idea of exponential integrators for SODEs with multiplicative noise, see [14].

Consider the same situation as in Section 1.2. Next, select the first diffusion term with the largest noise and write (1.18) in the form

$$
\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + \tilde{G}_1 X(t)\,\mathrm{d}\tilde{W}_1(t) + \sum_{k=2}^{m} \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t),
$$
$$
X(0) = X_0, \quad t \in [0,T].
$$
(1.32)

For our procedure we do not assume that the matrices A and $\tilde{G}_k$ as well as $\tilde{G}_k, \tilde{G}_j$ for $k \neq j$, $k,j = 1,\dots,m$ commute.

STEP 1: We exclude the largest noise term in (1.32)

$$
\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + \sum_{k=2}^{m} \tilde{G}_k Y(t)\,\mathrm{d}\tilde{W}_k(t), \quad t \in [0,T],
$$
$$
Y(0) = Y_0.
$$
(1.33)

This stochastic differential equation can be approximated, for example, with the well-known Euler-Maruyama scheme or with a higher order method of Milstein type.

STEP 2: We consider the Itô equation

$$
\mathrm{d}Z(t) = \tilde{G}_1 Z(t)\,\mathrm{d}\tilde{W}_1(t), \quad t \in [0,T],
$$
$$
Z(0) = Z_0.
$$
(1.34)

Later on, Z_0 will be the result of the first step at a specific time. It is known that the fundamental matrix to (1.34) has the explicit form (see for instance, [3] or [42])

$$
\Phi_0(t,0) = \exp(-\frac{1}{2}\tilde{G}_1^2 t + \tilde{G}_1 \tilde{W}_1(t)),
$$
(1.35)

and the exact solution is given by

$$
Z(t) = \Phi_0(t,0)Z_0, \quad t \in [0,T].
$$
(1.36)

Before we formulate a simple balanced method let us introduce some notation: We define a vector of step sizes $h = (h_1,\dots,h_N) \in (0,T]^N$ with $\sum_{i=1}^{N} h_i = T, N \in \mathbb{N}$ (see Section 2.1). The maximal step size in h is given by

$$
|h| := \max_{i=1,\dots,N} h_i.
$$

Moreover, every vector of step sizes h gives rise to a set of temporal grid points, which given by

$$\mathcal{T}_h := \Big\{ t_n := \sum_{i=1}^{n} h_i : n = 0, \ldots, N \Big\}. \tag{1.37}$$

Further, let $t, s \in [0, T]$ with $s < t$. We use as in [36] the following notation for the stochastic increments:

$$\tilde{I}_{(k)}^{s,t} := \int_{s}^{t} \mathrm{d}\tilde{W}_k(\tau), \quad k = 1, \ldots, m. \tag{1.38}$$

Then *the simple balanced Euler-type method* is given by the following split-step approximation

$$\overline{X}_h(t_i) = X_h(t_{i-1}) + A X_h(t_{i-1}) h_i + \sum_{k=2}^{m} \tilde{G}_k X_h(t_{i-1}) \tilde{I}_{(k)}^{t_{i-1}, t_i},$$
$$X_h(t_i) = \Phi_0(t_i, t_{i-1}) \overline{X}_h(t_i), \quad i = 1, \ldots, N, \tag{1.39}$$

where Z_0 of (1.36) replaced by $\overline{X}_h(t_i)$ and

$$\Phi_0(t_i, t_{i-1}) = \exp\big(-\frac{1}{2} \tilde{G}_1^2 h_i + \tilde{G}_1 \tilde{I}_{(1)}^{t_{i-1}, t_i}\big), \quad i = 1, \ldots, N.$$

This scheme is similar to a method suggested in [14]. The difference is that commutativity of the noise terms is assumed in [14] and then each of them is solved by an exponential integrator. By contrast through our pre-transformation of the Wiener process we have access to the largest noise term, which can be solved exactly. The fact that the matrix $\Phi_0(t_i, t_{i-1})$ contains only one Wiener increment facilitates our estimates. On the other hand we have an additional difficulty since we do not assume the noise terms to commute.

We expect the new scheme to give better numerical approximations than the Euler-Maruyama method in systems with large noise. The diagram below illustrates this for a sample path of a two-dimensional system.

Figure 1.1 shows the simulations of one path generated by a reference solution, the scheme (1.39), and the explicit Euler Maruyama method with step size $h = 2^{-4}$ and parameters

$A = \begin{pmatrix} -0.8 & -1 \\ 0.5 & -1 \end{pmatrix}$, $G_1 = \begin{pmatrix} -3.8 & 0.05 \\ 0.075 & 0.1 \end{pmatrix}$, $G_2 = \begin{pmatrix} -0.3 & -0.05 \\ 0.5 & -2 \end{pmatrix}$, $T = 1$, and the initial value $X_0 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$.

We note that the stability matrix S, defined in (0.3) is given by

$$S = \begin{pmatrix} 0.68 & -0.82 & -0.82 & -0.005 \\ -1.42 & 5.77 & 0.02 & -1.10 \\ -1.42 & 0.02 & 5.77 & -1.10 \\ 0.07 & 0.4 & 0.4 & -2.4 \end{pmatrix}$$

with the eigenvalues $\{0.22, -2.27, 6.12, 5.74\}$. Then the spectral abscissa in this case is equal to 6.12 (see for instance, [20]). Following, the equilibrium position of the linear SODE (1.15) with the given matrices A, G_1 and G_2 is not asymptotically mean-square stable (see [23],[10, Lemma 3.3],[1, Lemma 1]).

The matrices $A, G_1, G_2 \in \mathbb{R}^{2 \times 2}$ do not commute. Therefore, we compute our reference solution by the numerical approximation (1.39) with a step size $\Delta t = 2^{-18}$. In this example we follow the recipe from Section 1.2: We first simulate Wiener increments for the origin SODE (1.15) and then transform them by (1.16). Therefore, we get a strong approximation to (1.15). The transformed matrices are given by

$$\tilde{G}_1 = \begin{pmatrix} 3.81 & -0.05 \\ -0.12 & 0.09 \end{pmatrix} \text{ and } \tilde{G}_2 = \begin{pmatrix} 0.06 & -0.05 \\ 0.49 & -2 \end{pmatrix}.$$

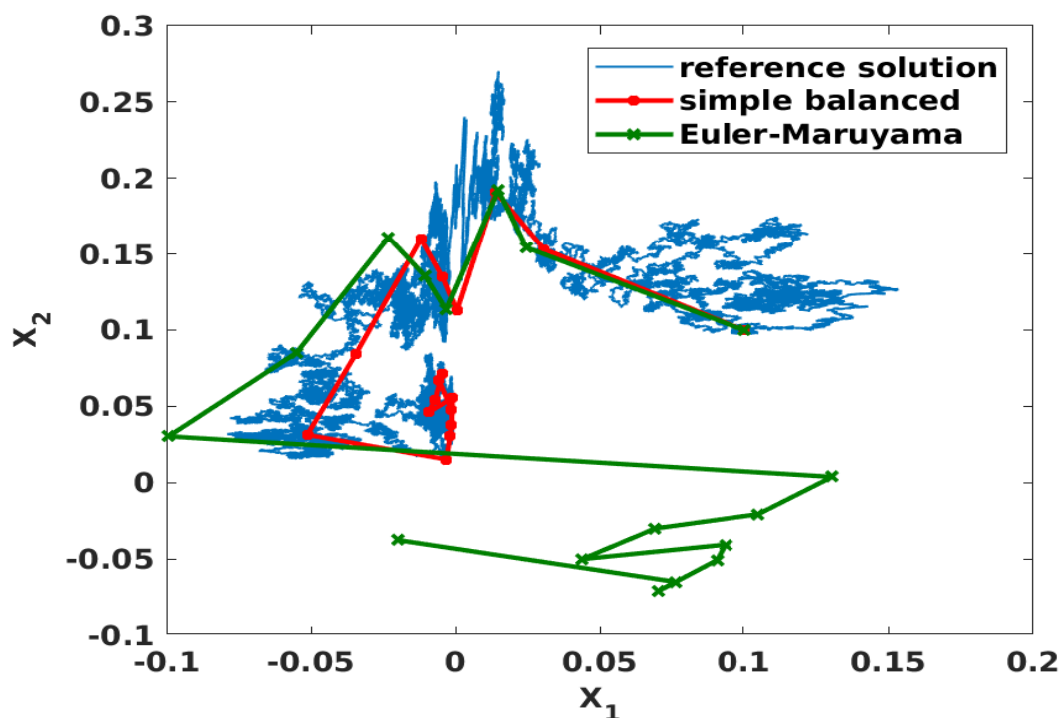Figure 1.1.: Sample trajectories of the simple balanced Euler-type method and Euler-Maruyama scheme with step size $h = 2^{-4}$ and reference solution obtained by $\Delta t = 2^{-18}$. The initial value $X_0 = (0.1, 0.1)^\top$.

Table 1.1 shows an overview of the Frobenius norm and the eigenvalues of $G_1, G_2, \tilde{G}_1$ and $\tilde{G}_2$. One can see that the values in Table 1.1 do not vary significantly from each

other. Therefore, the main effect of (1.39) comes from the exact integrator in the second step.

Table 1.1.: Frobenius norm and eigenvalues

	G_1	G_2	$\tilde{G}_1$	$\tilde{G}_2$
Frobenius norm	3.80	2.08	3.81	2.06
Eigenvalues	-3.80	-0.31	3.81	0.05
	0.10	-1.99	0.09	-1.99

However, the theoretical investigations of this method can not yield better error estimates than other well-known numerical schemes, see Section 1.5 below. It is obvious that for large $\tilde{G}_1$ the second moment of (1.39) increases exponentially.

1.5. Modified solution operator

In this section we modify the solution operator Φ_0 by a shift matrix C. This shift is useful to get good estimates for the solution operator of STEP 2. Motivated by this we keep track of the constants.

To be more precise let $C \in \mathbb{R}^{d \times d}$ and let us consider (1.33) and (1.34) as a two-step form

$$
\begin{aligned}
\mathrm{d}Y(t) =& (A - C)Y(t)\,\mathrm{d}t + \sum_{k=2}^{m} \tilde{G}_k Y(t)\,\mathrm{d}\tilde{W}_k(t), \\
Y(0) =& Y_0,
\end{aligned}
\tag{1.40}
$$

$$
\begin{aligned}
\mathrm{d}Z(t) =& CZ(t)\,\mathrm{d}t + \tilde{G}_1 Z(t)\,\mathrm{d}\tilde{W}_1(t), \quad t \in [0, T], \\
Z(0) =& Z_0.
\end{aligned}
\tag{1.41}
$$

On the continuous level we first solve (1.40) and then (1.41) with initial data $Z_0 = Y(\delta)$, where δ denotes a step size. In order for (1.41) to be explicitly solved the matrices C and $\tilde{G}_1$ should commute, i.e., $C\tilde{G}_1 = \tilde{G}_1 C$. Then the fundamental matrix of (1.41) has the explicit form (see [3], [42])

$$
\Phi_1(t, 0) = \exp\left((C - \frac{1}{2}\tilde{G}_1^2)t + \tilde{G}_1 \tilde{W}_1(t)\right),
\tag{1.42}
$$

for all $t \in [0, T]$ and the exact solution to (1.41) is giving by

$$
Z(t) = \Phi_1(t, 0)Z_0.
\tag{1.43}
$$

For the second step we have the following estimates

Corollary 1.5.1. *Let Φ_1 be given as in (1.42) and let the matrices $C, \tilde{G}_1 \in \mathbb{R}^{d \times d}$ commute. Moreover, let $Z_0 \in L^2(\Omega; \mathbb{R}^d)$. Then for every $t \in [0, T]$ it holds*

$$\|\Phi_1(t, 0)Z_0\|_{L^2} \leq \|Z_0\|_{L^2(\Omega; \mathbb{R}^d)} e^{\mu_2(\hat{B})t}, \tag{1.44}$$

where $\mu_2(\hat{B})$ is the logarithmic norm of the matrix

$$\hat{B} = C + \frac{\tilde{G}_1^\top \tilde{G}_1}{2}. \tag{1.45}$$

In particular, the conditional expectation of (1.43) is given by

$$\mathbb{E}[\Phi_1(t, 0)Z_0 | \mathcal{F}_0] = e^{Ct} Z_0. \tag{1.46}$$

Proof. The assertion (1.44) follows from Lemma 1.1.2. The special case of the condition (1.9) with $p = 2$ and $\epsilon = 0$ yields

$$
\begin{aligned}
\langle x, Cx \rangle + \frac{1}{2}|\tilde{G}_1 x|^2 &= x^\top C x + \frac{1}{2} x^\top \tilde{G}_1^\top \tilde{G}_1 x \\
&= x^\top \big(C + \frac{\tilde{G}_1^\top \tilde{G}_1}{2}\big) x \\
&\leq \mu_2(\hat{B})|x|^2
\end{aligned}
\tag{1.47}
$$

for all $x \in \mathbb{R}^d$. Since Φ_1 is $\mathcal{F}_t$-measurable and $C\tilde{G}_1 = \tilde{G}_1 C$ we obtain

$$
\begin{aligned}
\mathbb{E}\big[\exp\big((C - \frac{1}{2}\tilde{G}_1^2)t + \tilde{G}_1 \tilde{W}_1(t)\big) Z_0 | \mathcal{F}_0\big] \\
= \exp\big((C - \frac{1}{2}\tilde{G}_1^2)t\big) \mathbb{E}\big[\exp\big(\tilde{G}_1 \tilde{W}_1(t)\big)\big] Z_0.
\end{aligned}
\tag{1.48}
$$

Let $V(t) = \exp\big(\tilde{G}_1 \tilde{W}_1(t)\big)$. Then by Itô's formula we have

$$
\begin{aligned}
dV(t) &= \frac{1}{2}\tilde{G}_1^2 V(t)\, dt + \tilde{G}_1 V(t)\, d\tilde{W}_1(t), \\
V(0) &= V_0.
\end{aligned}
$$

The integral form of this equation is given by

$$V(t) = V_0 + \int_0^t \frac{1}{2}\tilde{G}_1^2 V(\tau)\, d\tau + \int_0^t \tilde{G}_1 V(\tau)\, dW_1(\tau).$$

Taking expectation yields

$$\mathbb{E}[V(t)] = \mathbb{E}[V_0] + \frac{1}{2}\tilde{G}_1^2 \int_0^t \mathbb{E}[V(\tau)]\, d\tau.$$

Let denote $\mathbb{E}[V(t)] := \phi(t)$. Taking the t-derivative, we get

$$\phi'(t) = \frac{1}{2}\tilde{G}_1^2 \phi(t),$$

$$\phi(t) = \exp\left(\frac{1}{2}\tilde{G}_1^2 t\right)V_0, \quad t \in [0,T].$$

With $V_0 = \mathrm{id}$ we have

$$\mathbb{E}[\exp\left(\tilde{G}_1 \tilde{W}_1(t)\right)] = \exp\left(\frac{1}{2}\tilde{G}_1^2 t\right). \tag{1.49}$$

Thus, by inserting of (1.49) into (1.48) we get (1.56).

$\square$

In the one-dimensional case one can easily see that, depending on the choice of C, the second moment of (1.43) can either grow exponentially or reduce the exponential growth (see [42, Example 5.5]). Since our problem is multi-dimensional we discuss our choice of the shift matrix C. First of all, this matrix should commute with $\tilde{G}_1$. Second, we strive to avoid large eigenvalues of the matrix $\tilde{B}$ such that the estimate in (1.44) does not grow exponentially. In view of this goal we choose

$$C = -\frac{1}{2}\tilde{G}_1^2. \tag{1.50}$$

Of course, we should allow several types of the matrix $\tilde{G}_1$. In the symmetric case we obtain that $\hat{B} \equiv 0$ and the constant in (1.44) equal to one. If the matrix $\tilde{G}_1$ is skew symmetric, i.e. $\tilde{G}_1^\top = -\tilde{G}_1$ then the eigenvalues of $\tilde{G}_1^2$ are negative and the shift in (1.41) is not necessary, i.e. we should choose $C \equiv 0$.

In fact we shift the stiffness of the diffusion term to the drift term. Therefore, we assume that the spectrum of A lies to the left of the imaginary axis of the complex plane. Moreover, we suggest below an implicit scheme that is well suited for SODEs with the stiff drift term.

We remark that the SODE (1.41) with $C = \frac{1}{2}\tilde{G}_1^2$ represents the Stratonovich integral in the differential form (see [36]). With our choice $C = -\frac{1}{2}\tilde{G}_1^2$ we derive that (1.41) is equivalent to

$$\mathrm{d}Z(t) = -\tilde{G}_1^2 Z(t)\,\mathrm{d}t + \tilde{G}_1 Z(t) \circ \mathrm{d}\tilde{W}_1(t), \quad t \in [0,T]$$
$$Z(0) = Z_0.$$

Let us denote

$$\alpha_1 := \mu_2(-\frac{1}{2}\tilde{G}_1^2) \quad \text{and} \quad \alpha_{1,+} := \max(\alpha_1, 0). \tag{1.51}$$

as the logarithmic norm of the matrix $-\frac{1}{2}\tilde{G}_1^2$. In order to obtain small values in (1.51) the argument $\arg(\lambda_i)$ should be from $[-\frac{\pi}{4};\frac{\pi}{4}] \cup [\frac{3\pi}{4};\frac{5\pi}{4}]$ for all eigenvalues $\lambda_i, i = 1, \ldots$ of the matrix $\tilde{G}_1$. By Lemma A.4.2 we have

$$|\mathrm{e}^{-\frac{1}{2}\tilde{G}_1^2 t}| \le \mathrm{e}^{\mu_2(-\frac{1}{2}\tilde{G}_1^2)t} = \mathrm{e}^{\alpha_1 t}, \quad t \in [0,T]. \tag{1.52}$$

Further, by

$$\alpha_{\mathrm{S}} := \mu_2\Big(\frac{-\tilde{G}_1 + \tilde{G}_1^\top}{2}\tilde{G}_1\Big) \quad\text{and}\quad \alpha_{\mathrm{S},+} := \max(\alpha_{\mathrm{S}}, 0). \tag{1.53}$$

we denote the logarithmic norm of the matrix $\hat{B}$ with $C = -\frac{1}{2}\tilde{G}_1^2$.

Lemma 1.5.2. *Let for every $0 \le s \le t \le T$ and $\tilde{G}_1 \in \mathbb{R}^{d\times d}$ the fundamental matrix Φ_1 be given in the form*

$$\Phi_1(t, s) := \exp(-\tilde{G}_1^2(t - s) + \tilde{G}_1(\tilde{W}_1(t) - \tilde{W}_1(s))). \tag{1.54}$$

Then for all $\mathcal{F}_s$-measurable random variable $Y \in L^2(\Omega; \mathbb{R}^d)$ the estimate holds

$$\|\Phi_1(t, s)Y\|_{L^2(\Omega;\mathbb{R}^d)} \le \|Y\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_{\mathrm{S},+}(t-s)}, \tag{1.55}$$

where α_{S} denotes the logarithmic norm given in (1.53).
In addition, for all $0 \le s \le t \le T$ it holds

$$\|\mathbb{E}[\Phi_1(t, s)Y|\mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \le \|Y\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_1(t-s)}, \tag{1.56}$$

where α_1 is given by (1.51).

Proof. For the proof use Lemma 1.1.2 and Corollary 1.5.1.

$\square$

Remark 1.5.3. We note that the conditional expectation can be estimated by the deterministic part of the equation (1.41). Hence, the estimate (1.56) holds without applying Lemma 1.1.2. Moreover, it holds the following relation of exponents

$$\alpha_1 \le \alpha_{\mathrm{S}} \le \alpha_{\mathrm{S},+}. \tag{1.57}$$

Lemma 1.5.4. *For every $\mathcal{F}_s$-measurable random variable $Y \in L^2(\Omega; \mathbb{R}^d)$ and $0 \le s \le t \le T$ the following estimate holds*

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])\Phi_1(t, s)Y\|_{L^2(\Omega;\mathbb{R}^d)} \le K_{\mathrm{cond}}(t - s)\|Y\|_{L^2(\Omega;\mathbb{R}^d)}|t - s|^{\frac{1}{2}}, \tag{1.58}$$

where $K_{cond}(\delta) = |\tilde{G}_1|\big(\frac{1}{2}|\tilde{G}_1||\delta|^{\frac{1}{2}} + 1\big)\mathrm{e}^{\alpha_{\mathrm{S},+}\delta}$ for $0 \le \delta \le T$.

Proof. From the definition of Φ_1 for all $0 \le s \le t \le T$ we obtain

$$\begin{aligned}
(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])\Phi_1(t, s)Y = &-\frac{1}{2}\int_s^t (\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])\Phi_1(\tau, s)\tilde{G}_1^2 Y \, \mathrm{d}\tau \\
&+ \int_s^t \Phi_1(\tau, s)\tilde{G}_1 Y \, \mathrm{d}\tilde{W}_1(\tau).
\end{aligned}$$

Taking the L^2-norm and using the Itô isometry yields

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])\Phi_1(t,s)Y\|_{L^2(\Omega;\mathbb{R}^d)} \leq \frac{1}{2}\int_s^t \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])\Phi_1(\tau,s)\tilde{G}_1^2 Y\|_{L^2(\Omega;\mathbb{R}^d)}\, \mathrm{d}\tau$$
$$+ \Big\|\int_s^t \Phi_1(\tau,s)\tilde{G}_1 Y\, \mathrm{d}\tilde{W}_1(\tau)\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$= \frac{1}{2}\int_s^t \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])\Phi_1(\tau,s)\tilde{G}_1^2 Y\|_{L^2(\Omega;\mathbb{R}^d)}\, \mathrm{d}\tau$$
$$+ \Big(\int_s^t \|\Phi_1(\tau,s)\tilde{G}_1 Y\|^2_{L^2(\Omega;\mathbb{R}^d)}\, \mathrm{d}\tau\Big)^{\frac{1}{2}}.$$

Since $\mathbb{E}[\cdot|\mathcal{F}_s]$ is an orthogonal projector onto $L^2(\Omega,\mathcal{F}_s,\mathbf{P};\mathbb{R}^d)$, it holds for all $Z \in L^2(\Omega;\mathbb{R}^d)$

$$\|Z\|^2_{L^2(\Omega;\mathbb{R}^d)} = \|\mathbb{E}[Z|\mathcal{F}_s]\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])Z\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Therefore, we obtain

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])Z\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}. \tag{1.59}$$

Using (1.59) and Lemma 1.5.2 yields

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_s])\Phi_1(t,s)Y\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq \frac{1}{2}|\tilde{G}_1^2|\|Y\|_{L^2(\Omega;\mathbb{R}^d)}\int_s^t e^{\alpha_{\mathrm{S},+}(\tau-s)}\, \mathrm{d}\tau$$
$$+ |\tilde{G}_1|\|Y\|_{L^2(\Omega;\mathbb{R}^d)}\Big(\int_s^t e^{2\alpha_{\mathrm{S},+}(\tau-s)}\, \mathrm{d}\tau\Big)^{\frac{1}{2}}$$
$$\leq |\tilde{G}_1|\big(\frac{1}{2}|\tilde{G}_1||t-s|^{\frac{1}{2}} + 1\big)\|Y\|_{L^2(\Omega;\mathbb{R}^d)}e^{\alpha_{\mathrm{S},+}(t-s)}|t-s|^{\frac{1}{2}}.$$

$\square$

The following lemma compares solutions with different initial data.

Lemma 1.5.5. *Let matrix Φ_1 be given as in (1.54) with $\tilde{G}_1 \in \mathbb{R}^{d\times d}$. Then for all $\mathcal{F}_s$-measurable variable $Y \in L^2(\Omega;\mathbb{R}^d)$ and $0 \leq s \leq s_1 < t \leq T$ it holds*

$$\|(\Phi_1(t,s_1) - \Phi_1(t,s))Y\|_{L^2(\Omega;\mathbb{R}^d)} \leq K(t-s,s_1-s)\|Y\|_{L^2(\Omega;\mathbb{R}^d)}|s_1-s|^{\frac{1}{2}}, \tag{1.60}$$

where $K(\delta,\delta_1) = |\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\delta_1^{\frac{1}{2}} + 1)e^{\alpha_{\mathrm{S},+}\delta}$ for $0 \leq \delta_1 < \delta \leq T$.

Proof. For all $0 \leq s \leq s_1 \leq t \leq T$ we get

$$(\Phi_1(t,s_1) - \Phi_1(t,s))Y = \Phi_1(t,s_1)(\mathrm{id} - \Phi_1(s_1,s))Y$$

Let us define

$$\tilde{Y} := (\mathrm{id} - \Phi_1(s_1, s))Y.$$

The random variable $\tilde{Y}$ is $\mathcal{F}_{s_1}$-measurable. Then by Lemma 1.5.2 we obtain

$$\|\Phi_1(t, s_1)\tilde{Y}\|_{L^2(\Omega;\mathbb{R}^d)} \le \|\tilde{Y}\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_{\mathrm{S},+}(t-s_1)}.$$

Further, from the definition of Φ_1 we get for $s \le s_1$

$$\Phi_1(s_1, s)Y = Y - \frac{1}{2}\int_s^{s_1}\Phi_1(\tau, s)\tilde{G}_1^2 Y\,\mathrm{d}\tau + \int_s^{s_1}\Phi_1(\tau, s)\tilde{G}_1 Y\,\mathrm{d}\tilde{W}_1(\tau).$$

Then by the Itô isometry and Lemma 1.5.2 we get

$$\begin{aligned}
\|(\mathrm{id} - \Phi_1(s_1, s))Y\|_{L^2(\Omega;\mathbb{R}^d)} =& \Big\|\frac{1}{2}\int_s^{s_1}\Phi_1(\tau, s)\tilde{G}_1^2 Y\,\mathrm{d}\tau \\
& - \int_s^{s_1}\Phi_1(\tau, s)\tilde{G}_1\,\mathrm{d}\tilde{W}_1(\tau)\Big\|_{L^2(\Omega;\mathbb{R}^d)} \\
\le& \frac{1}{2}\int_s^{s_1}\|\Phi_1(\tau, s)\tilde{G}_1^2 Y\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau \\
& + \Big(\int_s^{s_1}\|\Phi_1(\tau, s)\tilde{G}_1 Y\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau\Big)^{\frac{1}{2}} \\
\le& \|Y\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_{\mathrm{S},+}(s_1-s)}|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1||s_1 - s|^{\frac{1}{2}} + 1)|s_1 - s|^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof.

$\square$

We note that the constants in Lemma 1.5.2, Lemma 1.5.4, and Lemma 1.5.5 are of moderate exponential type in the sense of Convention 1.3.4.

1.6. Reformulation of the linear integral equation

In this section we rewrite the integral equation (1.20) in the form, which is convenient for our later estimates. Let the linear SODE

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + \sum_{k=1}^m \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t), \quad t \in [0, T] \tag{1.61}$$

with the initial data $X(0) = X_0$ be given. Using the shift matrix (1.50) we can write the equation (1.61) in the following form

$$\begin{aligned}
&\mathrm{d}X(t) = (A + \frac{1}{2}\tilde{G}_1^2)X(t)\,\mathrm{d}t - \frac{1}{2}\tilde{G}_1^2 X(t)\,\mathrm{d}t + \tilde{G}_1 X(t)\,\mathrm{d}\tilde{W}_1(t) + \sum_{k=2}^m \tilde{G}_k X(t)\tilde{W}_k(t), \\
&X(0) = X_0, \quad t \in [0, T],
\end{aligned}$$

with $A, \tilde{G}_k \in \mathbb{R}^{d \times d}, k = 1, \ldots, m$. Let denote by

$$A^+ := A + \frac{1}{2}\tilde{G}_1^2. \tag{1.62}$$

Further, let $X(t)$ be a solution of (1.61) and $V(t) = \Phi_1^{-1}(t,0)X(t), t \in [0,T]$. Then by applying Itô's formula to $V(t)$ we obtain

$$
\begin{aligned}
\mathrm{d}V(t) =& \Phi_1^{-1}(t,0)\big(\tilde{G}_1^2\,\mathrm{d}t - \tilde{G}_1\,\mathrm{d}\tilde{W}_1(t) + \frac{1}{2}\tilde{G}_1^2\,\mathrm{d}t\big)X(t) \\
&+ \Phi_1^{-1}(t,0)\big(AX(t)\,\mathrm{d}t + \sum_{k=1}^m \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t)\big) \\
&+ \Phi_1^{-1}(t,0)\big(\tilde{G}_1^2\,\mathrm{d}t - \tilde{G}_1\,\mathrm{d}\tilde{W}_1(t) + \frac{1}{2}\tilde{G}_1^2\,\mathrm{d}t\big)\big(AX(t)\,\mathrm{d}t + \sum_{k=1}^m \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t)\big) \\
=& \Phi_1^{-1}(t,0)\big[(\frac{1}{2}\tilde{G}_1^2 + A)X(t)\,\mathrm{d}t - \tilde{G}_1 X(t)\,\mathrm{d}\tilde{W}_1(t) + \sum_{k=1}^m \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k\big] \\
=& \Phi_1^{-1}(t,0)A^+ X(t)\,\mathrm{d}t + \sum_{k=2}^m \Phi_1^{-1}(t,0)\tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t).
\end{aligned}
\tag{1.63}
$$

For the calculation in the one-dimensional case we refer to [36]. Then the equation (1.63) has the integral form

$$
\begin{aligned}
X(t) =& \Phi_1(t,0)\big(X_0 + \int_0^t \Phi_1^{-1}(\tau,0)A^+ X(\tau)\,\mathrm{d}\tau \\
&+ \sum_{k=2}^m \int_0^t \Phi_1^{-1}(\tau,0)\tilde{G}_k X(\tau)\,\mathrm{d}\tilde{W}_k(\tau)\big).
\end{aligned}
\tag{1.64}
$$

1.7. Balanced shift noise Euler-type methods

In this section we propose three balanced shift noise Euler-type methods. These methods are based on the reformulations from the previous section. We recall from Section 1.4 that $h = (h_1, \ldots, h_N) \in (0,T]^N, N \in \mathbb{N}$ is a vector of step sizes if $\sum_{i=1}^N h_i = T$. Every vector of step sizes h induces a set of temporal grid points $\mathcal{T}_h$, which is given by (1.37) and $|h| := \max_{i \in \{1,\ldots,N\}} h_i$ denotes an upper step size bound. Further, let A^+ be given by (1.62).

Now, we suggest several so called split-step methods, which we denote as *balanced shift noise explicit*, *balanced shift noise implicit*, *balanced shift noise fully implicit* Euler-type

schemes. The first method, in short BSNE, is given by

$$
\begin{aligned}
\overline{X}_h(t_i) =& X_h(t_{i-1}) + A^+ X_h(t_{i-1}) h_i + \sum_{k=2}^m \tilde{G}_k X_h(t_{i-1}) \tilde{I}_{(k)}^{t_{i-1},t_i} \\
X_h(t_i) =& \Phi_1(t_i, t_{i-1}) \overline{X}_h(t_i), \quad 1 \le i \le N
\end{aligned}
\tag{1.65}
$$

with $X_h(0) = X_0$. It is known that for SODEs, which are stiff with respect to the drift term, implicit methods are well-suited. Therefore, the second method uses an implicit first step, i.e. starting with $X_h(0) = X_0$

$$
\begin{aligned}
\overline{X}_h(t_i) =& X_h(t_{i-1}) + A^+ \overline{X}_h(t_i) h_i + \sum_{k=2}^m \tilde{G}_k X_h(t_{i-1}) \tilde{I}_{(k)}^{t_{i-1},t_i} \\
X_h(t_i) =& \Phi_1(t_i, t_{i-1}) \overline{X}_h(t_i), \quad 1 \le i \le N.
\end{aligned}
\tag{1.66}
$$

We call this method *balanced shift noise implicit*, abbreviated as BSNI. Let us note that for numerical analysis of the implicit method BSNI we will need an extra condition: For $\mu_2(A^+) h_i < 1$ the estimate holds

$$
|(\mathrm{id} - A^+ h_i)^{-1}| \le (1 - \mu_2(A^+) h_i)^{-1}, \quad i = 1, \dots, N,
$$

where $\mu_2(A^+)$ is the logarithmic norm of matrix A^+.

Finally, we suggest a fully implicit method Euler type scheme (BSNFI)

$$
\begin{aligned}
\overline{X}_h(t_i) =& X_h(t_{i-1}) + A^+ X_h(t_i) h_i + \sum_{k=2}^m \tilde{G}_k X_h(t_{i-1}) \tilde{I}_{(k)}^{t_{i-1},t_i} \\
X_h(t_i) =& \Phi_1(t_i, t_{i-1}) \overline{X}_h(t_i), \quad 1 \le i \le N,
\end{aligned}
\tag{1.67}
$$

with $X_h(0) = X_0$. Note that (1.67) may be written as

$$
(\mathrm{id} - \Phi_1(t_i, t_{i-1}) A^+ h_i) X_h(t_i) = \Phi_1(t_i, t_{i-1}) \Big(X_h(t_{i-1}) + \sum_{k=2}^m \tilde{G}_k X_h(t_{i-1}) \tilde{I}_{(k)}^{t_{i-1},t_i} \Big).
$$

The theoretical analysis of this method has not been carried out since it is not clear how to guarantee the invertibility of the leading matrix $\mathrm{id} - \Phi_1(t_i, t_{i-1}) A^+ h_i$, and bound its inverse. This is in contrast to the BSNI method above. Nevertheless, this method gives good numerical results and will be used for comparison in the experiments, see Figure 1.3.

In the following picture we compare the behavior of one path of the simple balanced scheme (1.39) and BSNE Euler-type method to reference solution with step size $h = 2^{-4}$ and parameters

$$
A = \begin{pmatrix} -8 & -1 \\ 0.5 & -1 \end{pmatrix}, \ G_1 = \begin{pmatrix} -3.8 & 0.05 \\ 0.075 & 0.1 \end{pmatrix}, \ G_2 = \begin{pmatrix} -0.3 & -0.05 \\ 0.5 & -2 \end{pmatrix}, \ T = 1, \text{ and initial}
$$

value $X_0 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$.

As in the example of Section 1.4 we calculate the stability matrix S (see (0.3))

$$S = \begin{pmatrix} -13.72 & -0.82 & -0.82 & -0.05 \\ -1.42 & -1.43 & 0.02 & -1.10 \\ -1.42 & 0.02 & -1.43 & -1.10 \\ 0.07 & 0.4 & 0.4 & -2.4 \end{pmatrix}.$$

and its eigenvalues: $\{-13.90, -1.81 + 0.71i, -1.81 + 0.71i, -1.45\}$. The spectral abscissa of S is equal to -1.45 (see for instance, [20]). Then the equilibrium position of the linear SODE (1.15) with the given matrices A, G_1 and G_2 is asymptotically mean-square stable. For this we refer to [23],[3],[55],[10],[1].
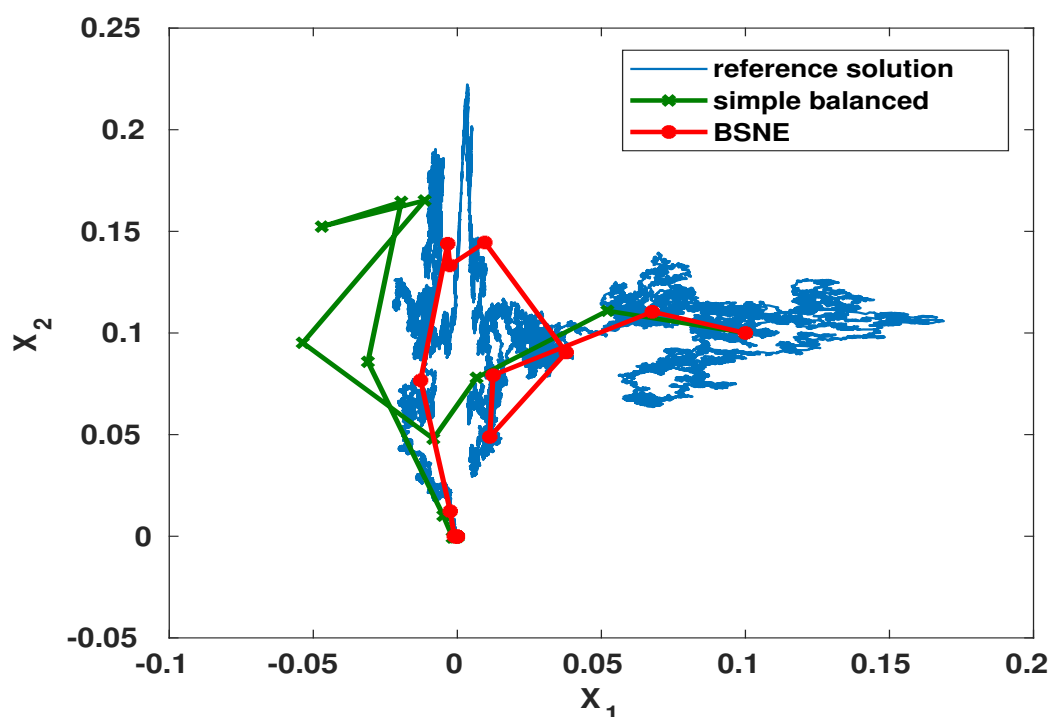

Figure 1.2.: Sample trajectories of the simple balanced method and BSNE Euler-type scheme with step size $h = 2^{-4}$, initial value $X_0 = (0.1, 0.1)^\top$, and $T = 1$.

In this example we take the same parameters as in the example from Section 1.4 with a small difference: We change an entry in the matrix A such that adding the shift matrix (1.50) does not cause very large stiffness in the drift term. Also, we follow here the recipe of the transformed Wiener noise, see Section 1.2. The calculation yields

$$A^+ = \begin{pmatrix} -0.73 & -1.09 \\ 0.26 & -0.99 \end{pmatrix}, \tilde{G}_1 = \begin{pmatrix} 3.81 & -0.05 \\ -0.12 & 0.09 \end{pmatrix}, \tilde{G}_2 = \begin{pmatrix} 0.06 & -0.05 \\ 0.49 & -2 \end{pmatrix}.$$

Table 1.2 shows that the spectrum of $\tilde{G}_1$ lies to the right of the imaginary axis of the complex plane. Thus, due to the shift, we also shift the spectrum of the matrix exponential Φ_1 to the left.

Table 1.2.: Frobenius norm and eigenvalues

	A	A^+		G_1	G_2	$\tilde{G}_1$	$\tilde{G}_2$
Frobenius norm	8.14	1.67		3.80	2.08	3.81	2.06
Eigenvalues	-7.93	$-0.86 + 0.52i$		-3.80	-0.31	3.81	0.05
	-1.07	$-0.86 - 0.52i$		0.10	-1.99	0.09	-1.99

Since matrices $A, G_1, G_2 \in \mathbb{R}^{2 \times 2}$ do not commute we replaced the exact solution of (1.15) by a numerical approximation obtained with a very small step size $\Delta t = 2^{-18}$. Figure 1.2 shows that there can be cases where the error occurs in the approximation (1.39), while the BSNE approximation (1.65) gives a better result. However, the strong error convergence in the mean square sense of the simple balanced method and the BSNE Euler-type scheme shows no difference, see Figure 5.2 and Table 5.2 in Chapter 5.
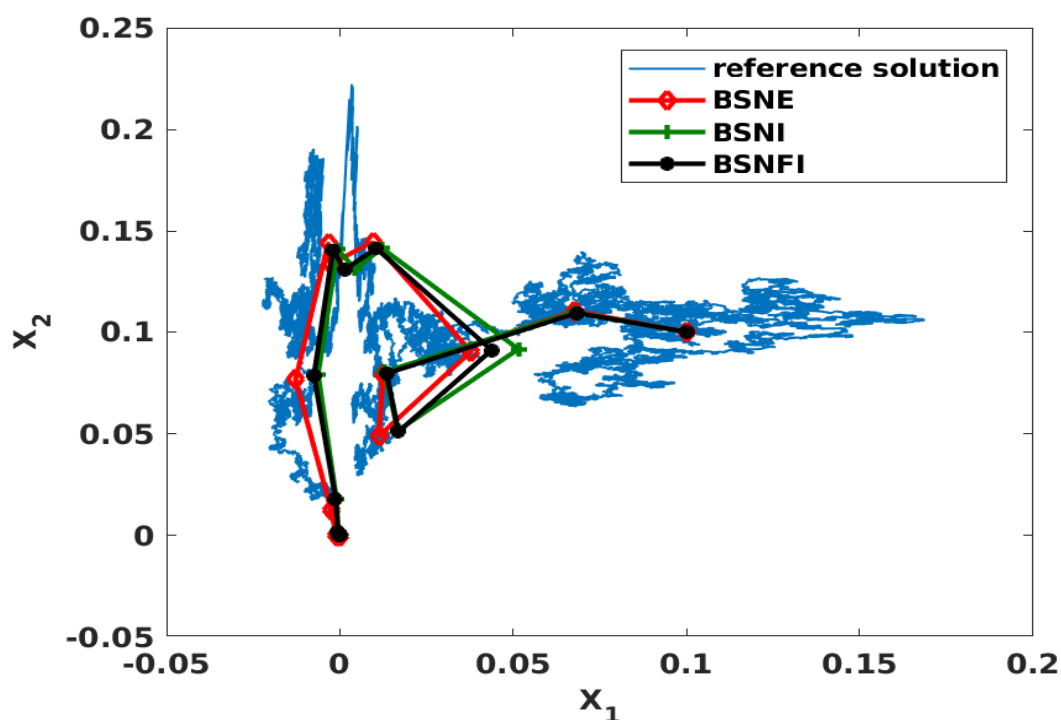

Figure 1.3.: Sample trajectories of the BSNE, BSNI and BSNFI Euler-type schemes with step size $h = 2^{-4}$, initial value $X_0 = (0.1, 0.1)^\top$, and $T = 1$.

The next Figure 1.3 compares a behavior of the three balanced shift noise Euler-type schemes (1.65)-(1.67) to reference solution with step size $h = 2^{-4}$ and the same parameters as above. In addition, $\mu_2(A^+) = -0.43$ and $\mu_2(A^+)h = -0.03$.

The detailed study of orders of convergence and the interplay of large drift and noise terms will be given in Section 2.2 and Section 2.3.

2. Numerical analysis of the balanced shift noise methods

In this chapter we analyze the B-consistency and C-stability of the balanced shift noise methods. These notions are introduced in [5] and [6] and applied to so-called projected Euler-Maruyama and Milstein schemes with nonlinearities satisfying a one-sided Lipschitz estimate. Below we will point out the difference to the schemes considered in this work. We continue to consider the linear stochastic differential equation

$$dX(t) = AX(t)\,\mathrm{d}t + \sum_{k=1}^{m} \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t), \tag{2.1}$$
$$X(0) = X_0, \quad t \in [0, T].$$

2.1. Stochastic B-consistency and C-stability

In this section we recall general definitions and the abstract convergence Theorem 2.1.5, which was proved in [5].

First we introduce some additional notations: Let $\bar{h} \in (0, T]$ be an upper step size bound and define the set $\mathbb{T} := \mathbb{T}(\bar{h}) \subset [0, T) \times (0, \bar{h}]$ as

$$\mathbb{T} := \{(t, \delta) \in [0, T) \times (0, \bar{h}] : t + \delta \leq T\}.$$

We denote by $\mathcal{G}^2(\mathcal{T}_h)$ the space of all adapted and square integrable grid functions

$$\mathcal{G}^2(\mathcal{T}_h) := \{Y : \mathcal{T}_h \times \Omega \to \mathbb{R}^d : Y(t_n) \in L^2(\Omega, \mathcal{F}_{t_n}, \mathbf{P}; \mathbb{R}^d), n = 0, 1, \ldots, N\},$$

for a given vector of step sizes $h \in (0, \bar{h}]^N$. Here $\mathcal{T}_h$ is a set of temporal grid points given by (1.37).

Definition 2.1.1. *Let $\bar{h} \in (0, T]$ be an upper step size bound and $\Psi \colon \mathbb{R}^d \times \mathbb{T} \times \Omega \to \mathbb{R}^d$ be a mapping satisfying the following measurability and integrability condition: For every $(t, \delta) \in \mathbb{T}$ and $Y \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$ it holds*

$$\Psi(Y, t, \delta) \in L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbf{P}; \mathbb{R}^d). \tag{2.2}$$

Then, for every vector of step sizes $h = (h_1, \ldots, h_N) \in (0, \bar{h}]^N$, $N \in \mathbb{N}$, we say that a grid function $X_h \in \mathcal{G}^2(\mathcal{T}_h)$ is generated by the stochastic one-step method $(\Psi, \bar{h}, \xi)$ with

initial condition $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ *if*

$$X_h(t_i) = \Psi(X_h(t_{i-1}), t_{i-1}, h_i), \quad 1 \le i \le N,$$
$$X_h(t_0) = \xi. \tag{2.3}$$

We call Ψ *the one-step map of the method.*

The definition of C-stability already appears in [13] and used in the context of numerical approximation of stiff differential equations. More recent exposition one can find in [22] and [59]. The authors from [5] extend this definition to numerical solutions of stochastic differential equations.

Definition 2.1.2. *A stochastic one-step method* $(\Psi, \bar{h}, \xi)$ *is called stochastically C-stable (with respect to the norm in* $L^2(\Omega; \mathbb{R}^d)$*) if there exist a constant* C_{stab} *and a parameter value* $\eta \in (1, \infty)$ *such that*

$$\|\mathbb{E}[\Psi(Y, t, \delta) - \Psi(Z, t, \delta)|\mathcal{F}_t]\|^2_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \eta \|(\text{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi(Y, t, \delta) - \Psi(Z, t, \delta))\|^2_{L^2(\Omega;\mathbb{R}^d))} \tag{2.4}$$
$$\le (1 + C_{\text{stab}}\delta)\|Y - Z\|^2_{L^2(\Omega;\mathbb{R}^d))}$$

for all $Y, Z \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d))$ *and* $(t, \delta) \in \mathbb{T}$.

The next definition is concerned with the local truncation error.

Definition 2.1.3. *A stochastic one-step method* $(\Psi, \bar{h}, \xi)$ *is called stochastically B-consistent of order* $\gamma > 0$ *to (1.1) if there exist constants* $C_{\text{cons},1}$ *and* $C_{\text{cons},2}$ *such that for every* $(t, \delta) \in \mathbb{T}$ *it holds*

$$\|\mathbb{E}[X(t + \delta) - \Psi(X(t), t, \delta)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \le C_{\text{cons},1}\delta^{\gamma+1} \tag{2.5}$$

$$\|(\text{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(X(t + \delta) - \Psi(X(t), t, \delta))\|_{L^2(\Omega;\mathbb{R}^d)} \le C_{\text{cons},2}\delta^{\gamma+\frac{1}{2}}, \tag{2.6}$$

where $X : [0, T] \times \Omega \to \mathbb{R}^d$.

This formulation is given in [5], where the local truncation error is split into deterministic and stochastic part. The conditions (2.5) and (2.6) are already known in the literature and can be found in slightly different form in [43], [45]. Finally, we give the definition of strong convergence.

Definition 2.1.4. *A stochastic one-step method* $(\Psi, \bar{h}, \xi)$ *converges strongly with order* $\gamma > 0$ *to the exact solution of (1.1) if there exists a constant* C *such that for every vector of step sizes* $h \in (0, \bar{h}]^N$ *it holds*

$$\max_{n \in \{0, \dots, N\}} \|X_h(t_n) - X(t_n)\|_{L^2(\Omega;\mathbb{R}^d)} \le C|h|^\gamma.$$

Here X *denotes the exact solution of (1.1) and* $X_h \in \mathcal{G}^2(\mathcal{T}_h)$ *is the grid function generated by* $(\Psi, \bar{h}, \xi)$ *with step sizes* $h \in (0, \bar{h}]^N$.

The following theorem investigates the strong convergence of a stochastic one-step method.

Theorem 2.1.5. *Let the stochastic one-step method* $(\Psi, \overline{h}, \xi)$ *be stochastically C-stable and stochastically B-consistent of order* $\gamma > 0$. *If* $\xi = X_0$, *then there exists a constant* C *depending on* C_{stab}, $C_{\text{cons},1}$, $C_{\text{cons},2}$, T, $\overline{h}$, *and* η *such that for every vector of step sizes* $h \in (0, \overline{h}]^N$ *it holds*

$$\max_{n \in \{0,\dots,N\}} \left\| X(t_n) - X_h(t_n) \right\|_{L^2(\Omega; \mathbb{R}^d)} \leq C|h|^\gamma,$$

where X *denotes the exact solution to* (1.18) *and* X_h *the grid function generated by* $(\Psi, \overline{h}, \xi)$ *with step sizes* h. *In particular,* $(\Psi, \overline{h}, \xi)$ *is strongly convergent of order* γ.

For the proof we refer to [5]. We recall that the constant C is given by (see [5, Th.3.7])

$$C = \left(\mathrm{e}^{\left(1 + C_{\text{stab}}(1 + \overline{h})\right)T} \left(C_{\text{cons},1}^2 (1 + \overline{h}) + C_{\text{cons},2}^2 C_\eta \right) T \right)^{\frac{1}{2}}, \tag{2.7}$$

where $C_\eta = 1 + (\eta - 1)^{-1}$.

2.2. Stochastic B-consistency and C-stability of the BSNE Euler-type method

In this section we derive a strong convergence result for the balanced shift noise explicit (BSNE) Euler-type scheme. Let us first show that this method is a stochastic one-step method in the sense of Definition 2.1.1.

We assume that Assumption 1.1.1 hold. Then for an arbitrary upper size bound $\overline{h} \in (0, 1]$ we define the one-step map $\Psi^{BSNE} \colon \mathbb{R}^d \times \mathbb{T} \times \Omega \to \mathbb{R}^d$ of the balanced shift noise explicit Euler-type method (1.65) by

$$\begin{aligned}
\Psi^{BSNE}(x, t, \delta) :=&\, \Phi_1(t + \delta, t)x + \delta \Phi_1(t + \delta, t)A^+ x \\
&+ \sum_{k=2}^{m} \Phi_1(t + \delta, t)\tilde{G}_k x \tilde{I}_{(k)}^{t, t+\delta},
\end{aligned} \tag{2.8}$$

for all $x \in \mathbb{R}^d$ and $(t, \delta) \in \mathbb{T}$. The matrix Φ_1 is defined by

$$\Phi_1(t + \delta, t) := \exp(-\tilde{G}_1^2 \delta + \tilde{G}_1 \tilde{I}_{(1)}^{t, t+\delta}). \tag{2.9}$$

Recall (1.38) for the definition of the stochastic increments. Now, let $Y \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$ for arbitrary $(t, \delta) \in \mathbb{T}$. Then form Lemma 1.5.2

$$\Phi_1(t + \delta, t)Y \in L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbf{P}; \mathbb{R}^d)$$

and we have that $\Psi^{BSNE}(Y, t, \delta) \colon \Omega \to \mathbb{R}^d$ is an $\mathcal{F}_{t+\delta}/\mathcal{B}(\mathbb{R}^d)$-measurable random variable, which satisfies condition 2.2.

The following estimates will be used for analyzing the consistency error and for establishing stability bounds. We note that the obtained constants are of moderate exponential type in the sense of Convention 1.3.4.

Lemma 2.2.1. *Let X be the exact solution of (2.1) with the initial value $X_0 \in L^2(\Omega; \mathbb{R}^d)$. Then for all $0 \leq t_1 \leq t_2 \leq T$ the estimates hold*

$$\int_{t_1}^{t_2} \|\Phi_1(t_2, \tau) A^+ (X(\tau) - X(t_1))\|_{L^2(\Omega; \mathbb{R}^d)} \, d\tau \leq \mathcal{K}_1(t_2 - t_1, t_2) \|X_0\|_{L^2(\Omega; \mathbb{R}^d)} |t_2 - t_1|^{\frac{3}{2}},$$

(2.10)

$$\int_{t_1}^{t_2} \|(\Phi_1(t_2, \tau) - \Phi_1(t_2, t_1)) A^+ X(t_1)\|_{L^2(\Omega; \mathbb{R}^d)} \, d\tau \leq \mathcal{K}_2(t_2 - t_1, t_1) \|X_0\|_{L^2(\Omega; \mathbb{R}^d)} |t_2 - t_1|^{\frac{3}{2}},$$

(2.11)

where $\mathcal{K}_1, \mathcal{K}_2 : \{(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq T\} \to \mathbb{R}$ are given by

$$\mathcal{K}_1(\delta, \delta_1) = \frac{2}{3} |A^+| (|A| \delta^{\frac{1}{2}} + |G|) \exp(\alpha_{S,+} \delta + \alpha_+ \delta_1),$$

(2.12)

$$\mathcal{K}_2(\delta, \delta_2) = |A^+| |\tilde{G}_1| (\frac{1}{3} |\tilde{G}_1| \delta^{\frac{1}{2}} + \frac{2}{3}) \exp(\alpha_+ \delta_2 + \alpha_{S,+} \delta)$$

(2.13)

for $0 \leq \delta_2 \leq \delta \leq \delta_1 \leq T$. The constants α and α_S are the logarithmic norms defined by (1.27) and (1.53), respectively.

Proof. By Lemma 1.3.2 and Lemma 1.5.2 we obtain

$$\|\Phi_1(t_2, \tau) A^+ (X(\tau) - X(t_1))\|_{L^2(\Omega; \mathbb{R}^d)}$$
$$\leq |A^+| (|A| |t_2 - t_1|^{\frac{1}{2}} + |G|) e^{\alpha_+ t_2} e^{\alpha_{S,+} (t_2 - \tau)} \|X_0\|_{L^2(\Omega; \mathbb{R}^d)} |\tau - t_1|^{\frac{1}{2}}.$$

Thus, by inserting into the integral we obtain

$$\int_{t_1}^{t_2} \|\Phi_1(t_2, \tau) A^+ (X(\tau) - X(t_1))\|_{L^2(\Omega; \mathbb{R}^d)} \, d\tau$$

$$\leq |A^+| (|A| |t_2 - t_1|^{\frac{1}{2}} + |G|) \|X_0\|_{L^2(\Omega; \mathbb{R}^d)} e^{\alpha_+ t_2} \int_{t_1}^{t_2} e^{\alpha_{S,+} (t_2 - \tau)} |\tau - t_1|^{\frac{1}{2}} \, d\tau$$

$$\leq \frac{2}{3} |A^+| \|X_0\|_{L^2(\Omega; \mathbb{R}^d)} (|A| |t_2 - t_1|^{\frac{1}{2}} + |G|) e^{\alpha_{S,+} (t_2 - t_1)} e^{\alpha_+ t_2} |t_2 - t_1|^{\frac{3}{2}}.$$

For the proof of the second estimate we use Lemma 1.1.2 and Lemma 1.5.5. It holds

$$\|(\Phi_1(t_2, \tau) - \Phi_1(t_2, t_1)) A^+ X(t_1)\|_{L^2(\Omega; \mathbb{R}^d)}$$
$$\leq |A^+| \|X_0\|_{L^2(\Omega; \mathbb{R}^2)} e^{\alpha t_1} |\tilde{G}_1| (\frac{1}{2} |\tilde{G}_1| |t_2 - t_1|^{\frac{1}{2}} + 1) e^{\alpha_{S,+} (t_2 - t_1)} |\tau - t_1|^{\frac{1}{2}}.$$

By integrating we obtain

$$|A^+|\|X_0\|_{L^2(\Omega;\mathbb{R}^2)}\mathrm{e}^{\alpha t_1}|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1||t_2-t_1|^{\frac{1}{2}}+1)\mathrm{e}^{\alpha_{\mathrm{S}_+}(t_2-t_1)}\int_{t_1}^{t_2}|\tau-t_1|^{\frac{1}{2}}\,\mathrm{d}\tau$$

$$\leq|A^+||\tilde{G}_1|(\frac{1}{3}|\tilde{G}_1||t_2-t_1|^{\frac{1}{2}}+\frac{2}{3})\|X_0\|_{L^2(\Omega;\mathbb{R}^2)}\exp(\alpha_+t_1+\alpha_{\mathrm{S},+}(t_2-t_1))|t_2-t_1|^{\frac{3}{2}}.$$

Thus, this proves the assertion (2.11).

$\square$

Corollary 2.2.2. *Let X be the exact solution of (2.1) with the initial value $X_0 \in L^2(\Omega;\mathbb{R}^d)$. Then for all $0 \leq t_1 \leq t_2 \leq T$ the estimate holds*

$$\int_{t_1}^{t_2}\|\mathbb{E}[\Phi_1(t_2,\tau)A^+(X(\tau)-X(t_1))|\mathcal{F}_{t_1}]\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau\leq\mathcal{K}_3(t_2-t_1,t_2)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t_2-t_1|^2,$$

where $\mathcal{K}_3 : \{(t_1,t_2) : 0 \leq t_1 \leq t_2 \leq T\} \to \mathbb{R}$ is given by

$$\mathcal{K}_3(\delta,\delta_1) = \frac{1}{2}|A^+||A|\exp\left(\alpha_{1,+}\delta+\alpha_+\delta_1\right) \tag{2.14}$$

for $0 \leq \delta \leq \delta_1 \leq T$. The constants α_1 and α denote the logarithmic norm given by (1.51) and (1.27), respectively.

Proof. By independence of the terms $\Phi_1(t_2,\tau)$ and $X(\tau) - X(t_1)$, Corollary 1.3.3, and (1.56) we get

$$\begin{aligned}\|\mathbb{E}[\Phi_1(t_2,\tau)&A^+(X(\tau)-X(t_1))|\mathcal{F}_{t_1}]\|_{L^2(\Omega;\mathbb{R}^d)}\\&=\|\mathbb{E}[\Phi_1(t_2,\tau)]\mathbb{E}[A^+(X(\tau)-X(t_1))|\mathcal{F}_{t_1}]\|_{L^2(\Omega;\mathbb{R}^d)}\\&\leq|\mathrm{e}^{-\frac{1}{2}\tilde{G}_1^2(t_2-\tau)}|\big\|\mathbb{E}[A^+(X(\tau)-X(t_1))|\mathcal{F}_{t_1}]\big\|_{L^2(\Omega;\mathbb{R}^d)}\\&\leq|A^+|\mathrm{e}^{\alpha_1(t_2-\tau)}\|\mathbb{E}[X(\tau)-X(t_1)|\mathcal{F}_{t_1}]\|_{L^2(\Omega;\mathbb{R}^d)}\\&\leq|A^+||A|\mathrm{e}^{\alpha_1(t_2-\tau)}\mathrm{e}^{\alpha_+\tau}\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|\tau-t_1|.\end{aligned} \tag{2.15}$$

Further, it holds

$$|A^+||A|\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\int_{t_1}^{t_2}\mathrm{e}^{\alpha_1(t_2-\tau)}\mathrm{e}^{\alpha_+\tau}|\tau-t_1|\,\mathrm{d}\tau$$

$$\leq\frac{1}{2}|A^+||A|\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\exp\left(\alpha_{1,+}(t_2-t_1)+\alpha_+t_2\right)|t_2-t_1|^2.$$

$\square$

Let us denote by

$$\tilde{G}^- = \begin{pmatrix}\tilde{G}_2\\\vdots\\\tilde{G}_m\end{pmatrix} \in \mathbb{R}^{m-1\times d,d}.$$

Then the spectral norm of the matrix $\tilde{G}^-$ is given by (see [26], [19])

$$|\tilde{G}^-| = |(\tilde{G}^-)^\top \tilde{G}^-|^{\frac{1}{2}}. \tag{2.16}$$

Lemma 2.2.3. *Let X be the exact solution to (2.1) with initial value $X_0 \in L^2(\Omega; \mathbb{R}^d)$. Then for all $0 \le t_1 \le s \le t_2 \le T$ the estimates hold*

$$\Big\| \sum_{k=2}^m \int_{t_1}^{t_2} \Phi_1(t_2, s) \tilde{G}_k (X(s) - X(t_1)) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^2(\Omega;\mathbb{R}^d)} \tag{2.17}$$

$$\le \mathcal{K}_4(t_2 - t_1, t_2) \|X_0\|_{L^2(\Omega;\mathbb{R}^d)} |t_2 - t_1|,$$

$$\Big\| \sum_{k=2}^m \int_{t_1}^{t_2} (\Phi_1(t_2, s) - \Phi_1(t_2, t_1)) \tilde{G}_k X(t_1) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^2(\Omega;\mathbb{R}^d)} \tag{2.18}$$

$$\le \mathcal{K}_5(t_2 - t_1, t_1) \|X_0\|_{L^2(\Omega;\mathbb{R}^d)} |t_2 - t_1|,$$

where $\mathcal{K}_4, \mathcal{K}_5 : \{(t_1, t_2) : 0 \le t_1 \le t_2 \le T\} \to \mathbb{R}$ are given by

$$\mathcal{K}_4(\delta, \delta_1) = |\tilde{G}^-|(|A|\delta^{\frac{1}{2}} + |G|) \exp\big(\alpha_{S_+}\delta + \alpha_+\delta_1\big), \tag{2.19}$$

$$\mathcal{K}_5(\delta, \delta_2) = |\tilde{G}^-||\tilde{G}_1|(\tfrac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1) \exp\big(\alpha_+\delta_2 + \alpha_{S_+}\delta\big) \tag{2.20}$$

for $0 \le \delta_2 \le \delta \le \delta_1 \le T$.

Remark 2.2.4. We recall that for all $t_1, t_2 \in [0, T]$ the matrix-valued random variable $\Phi_1(t_2, t_1)$ contains only one Wiener increment $\tilde{W}_1(t_2) - \tilde{W}_1(t_1)$ and therefore is independent of the further increments $\tilde{W}_2(t_2) - \tilde{W}_2(t_1), \ldots, \tilde{W}_m(t_2) - \tilde{W}_m(t_1)$. Moreover, the integrals from the left of (2.17) and (2.18) are continuous, square-integrable $(\mathcal{F}_{t_2})_{t_2 \in [0,T]}$-martingales.

Proof of Lemma 2.2.3. First, we use the Itô isometry and obtain

$$\Big\| \sum_{k=2}^m \int_{t_1}^{t_2} \Phi_1(t_2, \tau) \tilde{G}_k (X(\tau) - X(t_1)) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$= \Big(\sum_{k=2}^m \int_{t_1}^{t_2} \| \Phi_1(t_2, \tau) \tilde{G}_k (X(\tau) - X(t_1)) \|^2_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau \Big)^{\frac{1}{2}}$$

and

$$\Big\| \sum_{k=2}^m \int_{t_1}^{t_2} (\Phi_1(t_2, \tau) - \Phi_1(t_2, t_1)) \tilde{G}_k X(t_1)) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$= \Big(\sum_{k=2}^m \int_{t_1}^{t_2} \| (\Phi_1(t_2, \tau) - \Phi_1(t_2, t_1)) \tilde{G}_k X(t_1) \|^2_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau \Big)^{\frac{1}{2}}.$$

By Lemma 1.3.2 and Lemma 1.5.2 it holds for all $0 \leq t_1 \leq s \leq t_2 \leq T$ and $k = 2, \ldots, m$

$$\mathbb{E}[|\Phi_1(t_2, \tau)\tilde{G}_k(X(\tau) - X(t_1))|^2]$$
$$\leq |\tilde{G}_k|^2 e^{2\alpha_{S,+}(t_2 - \tau)}(|A||t_2 - t_1|^{\frac{1}{2}} + |G|)^2 e^{2\alpha + t_2}\mathbb{E}[|X_0|^2]|\tau - t_1|.$$

Therefore, we obtain

$$\Big(\sum_{k=2}^m \int_{t_1}^{t_2} \|\Phi_1(t_2, \tau)\tilde{G}_k(X(\tau) - X(t_1))\|_{L^2(\Omega;\mathbb{R}^d)}^2 \, d\tau\Big)^{\frac{1}{2}}$$
$$\leq \Big(2\sum_{k=2}^m |\tilde{G}_k|^2(|A|^2|t_2 - t_1| + |G|^2)\mathbb{E}[|X_0|^2]e^{2\alpha + t_2}\int_{t_1}^{t_2} e^{2\alpha_{S,+}(t_2 - \tau)}|\tau - t_1| \, d\tau\Big)^{\frac{1}{2}}$$
$$\leq |\tilde{G}^-|(|A||t_2 - t_1|^{\frac{1}{2}} + |G|)e^{\alpha + t_2}e^{\alpha_{S,+}(t_2 - t_1)}\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t_2 - t_1|.$$

For the second expression we use Lemma 1.1.2 and Lemma 1.5.5 and get

$$\mathbb{E}[|(\Phi_1(t_2, \tau) - \Phi_1(t_2, t_1))\tilde{G}_kX(t_1)|^2]$$
$$\leq |\tilde{G}_k|^2|\tilde{G}_1|^2(\frac{1}{2}|\tilde{G}_1||t_2 - t_1|^{\frac{1}{2}} + 1)^2 e^{2\alpha_{S,+}(t_2 - t_1)}e^{2\alpha + t_1}\mathbb{E}[|X_0|^2]|\tau - t_1|.$$

Further, an integration provides

$$\mathbb{E}[|(\Phi_1(t_2, \tau) - \Phi_1(t_2, t_1))\tilde{G}_kX(t_1)|^2]$$
$$\leq \Big(2\sum_{k=2}^m |\tilde{G}_k|^2 e^{2\alpha + t_1}|\tilde{G}_1|^2(\frac{1}{4}|\tilde{G}_1|^2|t_2 - t_1| + 1)e^{2\alpha_S(t_2 - t_1)}\int_{t_1}^{t_2} |\tau - t_1| \, d\tau\Big)^{\frac{1}{2}}$$
$$\leq |\tilde{G}^-||\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1||t_2 - t_1|^{\frac{1}{2}} + 1)e^{\alpha + t_2}e^{\alpha_{S,+}(t_2 - t_1)}\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}|t_2 - t_1|.$$

This completes the proof.

$\square$

The following theorem shows the stochastic B-consistency of the BSNE Euler-type scheme.

Theorem 2.2.5. *Let $\bar{h} \in (0, 1]$ be arbitrary. Then the balanced shift noise explicit Euler-type method $(\Psi^{BSNE}, \bar{h}, X_0)$ for the linear SODE (2.1) is stochastically B-consistent of order $\gamma = \frac{1}{2}$. The constants $C_{\text{cons},1}$ and $C_{\text{cons},2}$ are of moderate exponential type, see (2.22).*

Proof. Let $(t, \delta) \in \mathbb{T}$ be arbitrary. By inserting (1.64) and (2.8) into (2.5) we obtain

$$\|\mathbb{E}[X(t+\delta) - \Psi^{BSNE}(X(t), t, \delta)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$= \Big\|\mathbb{E}\Big[\int_t^{t+\delta} \Phi_1(t + \delta, \tau)A^+X(\tau) \, d\tau + \sum_{k=2}^m \int_t^{t+\delta} \Phi_1(t + \delta, \tau)\tilde{G}_kX(\tau) \, d\tilde{W}_k(\tau)$$

$$- \Phi_1(t+\delta,t)A^+X(t)\delta - \sum_{k=2}^{m} \Phi_1(t+\delta,t)\tilde{G}_k X(t)I_{(k)}^{t,t+\delta}\big|\mathcal{F}_t\big]\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq \int_t^{t+\delta} \|\mathbb{E}[\Phi_1(t+\delta,\tau)A^+(X(\tau)-X(t))|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$

$$+ \int_t^{t+\delta} \|\mathbb{E}[(\Phi_1(t+\delta,\tau)-\Phi_1(t+\delta,t))A^+X(t)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau.$$

Then by Corollary 2.2.2 we get

$$\int_t^{t+\delta} \|\mathbb{E}[\Phi_1(t+\delta,\tau)A^+(X(\tau)-X(t))|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau \leq \mathcal{K}_3(\delta,t+\delta)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta^2,$$

with

$$\mathcal{K}_3(\delta,t+\delta) = \frac{1}{2}|A^+||A|\exp\big(\alpha_{1,+}\delta + \alpha_+(t+\delta)\big).$$

Using the fact that $\|\mathbb{E}[Z|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$ and (2.11) we obtain

$$\int_t^{t+\delta} \|\mathbb{E}[(\Phi_1(t+\delta,\tau)-\Phi_1(t+\delta,t))A^+X(t)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$

$$\leq \int_t^{t+\delta} \|(\Phi_1(t+\delta,\tau)-\Phi_1(t+\delta,t))A^+X(t)\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau \qquad (2.21)$$

$$\leq \mathcal{K}_2(\delta,t)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta^{\frac{3}{2}},$$

where

$$\mathcal{K}_2(\delta,t) = |A^+||\tilde{G}_1|\big(\frac{1}{3}|\tilde{G}_1|\delta^{\frac{1}{2}} + \frac{2}{3}\big)\exp(\alpha_+ t + \alpha_{\mathrm{S},+}\delta).$$

Further, for the second estimate (2.6) we get

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(X(t+\delta) - \Psi^{BSNE}(X(t),t,\delta))\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq \int_t^{t+\delta} \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t+\delta,\tau)A^+(X(\tau)-X(t))\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$

$$+ \int_t^{t+\delta} \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t+\delta,\tau)-\Phi_1(t+\delta,t))A^+X(t)\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$

$$+ \Big\| \sum_{k=2}^{m} \int_t^{t+\delta} \Phi_1(t+\delta,\tau)\tilde{G}_k(X(\tau)-X(t)) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \Big\| \sum_{k=2}^{m} \int_t^{t+\delta} (\Phi_1(t+\delta,\tau)-\Phi_1(t+\delta,t))\tilde{G}_k X(t) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$=: \sum_{i=1}^{4} T_i.$$

Together with $\|(\mathrm{id} - \mathbb{E}[\,\cdot\,|\mathcal{F}_t])Z\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$ and Lemma 2.2.1 we obtain

$$T_1 \leq \int_t^{t+\delta} \|\Phi_1(t+\delta,\tau)A^+(X(\tau) - X(t))\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau$$
$$\leq \mathcal{K}_1(\delta, t+\delta)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta^{\frac{3}{2}}.$$

$\mathcal{K}_1$ is given by

$$\mathcal{K}_1(\delta, t+\delta) = \frac{2}{3}|A^+|(|A|\delta^{\frac{1}{2}} + |G|)\exp\left(\alpha_{\mathrm{S},+}\delta + \alpha_+(t+\delta)\right).$$

The estimate of the second term T_2 is similar to (2.21). Further, by using Lemma 2.2.3 and (2.17) we get

$$T_3 \leq \mathcal{K}_4(\delta, t+\delta)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta,$$

where

$$\mathcal{K}_4(\delta, t+\delta) = |\tilde{G}^-|(|A|\delta^{\frac{1}{2}} + |G|)\exp\left(\alpha_{\mathrm{S},+}\delta + \alpha_+(t+\delta)\right).$$

Finally, by Lemma 2.2.3, inequality (2.18) we have for the last term

$$T_4 \leq \mathcal{K}_5(\delta, t)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta,$$

with

$$\mathcal{K}_5(\delta, t) = |\tilde{G}^-||\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1)\exp\left(\alpha_+ t + \alpha_{\mathrm{S},+}\delta\right).$$

This completes the proof.

$\square$

Remark 2.2.6. The constants $C_{\mathrm{cons},1}$ and $C_{\mathrm{cons},2}$ in (2.5) and (2.6) are given by

$$C_{\mathrm{cons},1} = |A^+|\Big(|\tilde{G}_1|(\frac{1}{3}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + \frac{2}{3})\exp\left(\alpha_+ T + \alpha_{\mathrm{S},+}\bar{h}\right) + \frac{1}{2}|A|\exp\left(\alpha_{1,+}\bar{h} + \alpha_+ T\right)\Big),$$
$$C_{\mathrm{cons},2} = \big(|A^+|\frac{2}{3}(|A|\bar{h}^{\frac{1}{2}} + |G|) + |A^+||\tilde{G}_1|(\frac{1}{3}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + \frac{2}{3})$$
$$+ |\tilde{G}^-|(|A|\bar{h}^{\frac{1}{2}} + |G|) + |\tilde{G}^-||\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1)\big)\exp\left(\alpha_+ T + \alpha_{\mathrm{S},+}\bar{h}\right).$$

$$(2.22)$$

The next step of our numerical analysis is to prove the stochastic C-stability of the BSNE method.

Theorem 2.2.7. *For the linear SODE* (2.1) *with every inital value* $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ *the BSNE Euler-type method is stochastically C-stable. The constant* C_{stab} *in* (2.4) *depends on the data* $\alpha_{\mathrm{S},+}, \eta, |A^+|, |\tilde{G}_1|, |\tilde{G}^-|,$ *and* $\bar{h}$*, see* (2.26).

Proof. Let $(t, \delta) \in \mathbb{T}$ be arbitrary and $Y, Z \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$. Since $\Phi_1(t + \delta, t)$ is $\mathcal{F}_{t+\delta}$-measurable we obtain

$$
\begin{aligned}
&\mathbb{E}[\Psi^{BSNE}(Y, t, \delta) - \Psi^{BSNE}(Z, t, \delta)|\mathcal{F}_t] \\
&\quad = \mathbb{E}[\Phi_1(t + \delta, t)(Y - Z + A^+ Y \delta - A^+ Z \delta)|\mathcal{F}_t],
\end{aligned}
\tag{2.23}
$$

and

$$
\begin{aligned}
&(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{BSNE}(Y, t, \delta) - \Psi^{BSNE}(Z, t, \delta)) \\
&\quad = (\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t + \delta, t)(Y - Z + A^+ Y \delta - A^+ Z \delta) \\
&\qquad + \sum_{k=2}^m \Phi_1(t + \delta, t)(\tilde{G}_k Y - \tilde{G}_k Z)\tilde{I}_{(k)}^{t, t+\delta}.
\end{aligned}
\tag{2.24}
$$

Using (1.56) we obtain

$$
\begin{aligned}
&\|\mathbb{E}[\Psi^{BSNE}(Y, t, \delta) - \Psi^{BSNE}(Z, t, \delta)|\mathcal{F}_t]\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
&\quad = \mathbb{E}[|\mathbb{E}[\Phi_1(t + \delta, t)(Y - Z + A^+ Y \delta - A^+ Z \delta)|\mathcal{F}_t]|^2] \\
&\quad \leq |\mathrm{e}^{-\frac{1}{2}\tilde{G}_1^2 \delta}|^2 \mathbb{E}[|(\mathrm{id} + A^+ \delta)(Y - Z)|^2] \\
&\quad \leq \mathrm{e}^{2\alpha_1 \delta} \mathrm{e}^{2|A^+|\delta} \mathbb{E}[|Y - Z|^2] \\
&\quad \leq \mathrm{e}^{2(\alpha_1 + |A^+|)\delta} \|Y - Z\|_{L^2(\Omega; \mathbb{R}^d)}^2.
\end{aligned}
$$

Here we use the rule: If X is independent of $\mathcal{F}_t$ and Y is $\mathcal{F}_t$-measurable then $\mathbb{E}[XY|\mathcal{F}_t] = \mathbb{E}[X]Y$. Thus, the term (2.23) fulfills the requirement of (2.4). It remains to show that the remaining summand (2.24) allows for a sharper estimate. It holds

$$
\begin{aligned}
&\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{BSNE}(Y, t, \delta) - \Psi^{BSNE}(Z, t, \delta))\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
&\quad = \mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t + \delta, t)(Y - Z + A^+ Y \delta - A^+ Z \delta) \\
&\qquad + \sum_{k=2}^m \Phi_1(t + \delta, t)(\tilde{G}_k Y - \tilde{G}_k Z)\tilde{I}_{(k)}^{t, t+\delta}|^2] \\
&\quad \leq 2\mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t + \delta, t)(Y - Z + A^+ Y \delta - A^+ Z \delta)|^2] \\
&\qquad + 2\mathbb{E}\Big[\Big|\sum_{k=2}^m \Phi_1(t + \delta, t)(\tilde{G}_k Y - \tilde{G}_k Z)\tilde{I}_{(k)}^{t, t+\delta}\Big|^2\Big].
\end{aligned}
\tag{2.25}
$$

Then by Lemma 1.5.4 we obtain

$$
\begin{aligned}
\mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t + \delta, t)(Y - Z + A^+ Y \delta - A^+ Z \delta)|^2] \\
\leq K_{\mathrm{cond}}^2(\delta)\delta \mathbb{E}[|(\mathrm{id} + A^+ \delta)(Y - Z)|^2] \\
\leq K_{\mathrm{cond}}^2(\delta)\delta \mathrm{e}^{2|A^+|\delta}\mathbb{E}[|Y - Z|^2],
\end{aligned}
$$

with $K_{\text{cond}}(\delta) = |\tilde{G}_1|\left(\frac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1\right)e^{\alpha_{\text{S},+}\delta}$. Finally, the Itô isometry and Lemma 1.5.2 yield

$$\mathbb{E}\Big[\Big|\sum_{k=2}^m \Phi_1(t+\delta,t)(\tilde{G}_k Y - \tilde{G}_k Z)\tilde{I}_{(k)}^{t,t+\delta}\Big|^2\Big]$$

$$= \delta \sum_{k=2}^m \mathbb{E}[|\Phi_1(t+\delta,t)(\tilde{G}_k Y - \tilde{G}_k Z)|^2]$$

$$\leq |\tilde{G}^-|^2 \delta e^{2\alpha_{\text{S},+}\delta}\mathbb{E}[|Y - Z|^2].$$

Altogether, this shows

$$\|\mathbb{E}[\Psi^{BSNE}(Y,t,\delta) - \Psi^{BSNE}(Z,t,\delta)|\mathcal{F}_t]\|^2_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \eta\|(\text{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{BSNE}(Y,t,\delta) - \Psi^{BSNE}(Z,t,\delta))\|^2_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq e^{C_{\text{stab}}\delta}\|Y - Z\|^2_{L^2(\Omega;\mathbb{R}^d)},$$

where

$$C_{\text{stab}} = 2\eta\big(|\tilde{G}_1|^2\big(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1\big)^2 + |\tilde{G}^-|^2\big) + 2|A^+| + 2\alpha_{\text{S},+}\big). \tag{2.26}$$

$\square$

Remark 2.2.8. Equation (2.26) shows that the constant C_{stab} is not of moderate type, i.e. in addition to the one-sided Lipschitz constant $\alpha_{\text{S},+}$ the norms $|A^+|, |\tilde{G}_1|$, and $|\tilde{G}^-|$ appear. Hence, the constant (2.7) in the convergence Theorem 2.1.5 is not of moderate exponential type in the sense of Convention 1.3.4. This seems unavoidable in view of the fact that our assumptions allow the solutions to grow exponentially in mean square, see the discussion in Introduction.

Now, the strong convergence of the BSNE Euler-type scheme follows directly from Theorem 2.2.5 and Theorem 2.2.7.

Theorem 2.2.9. *Let $\bar{h} \in (0,1]$. Then the balanced shift noise explicit Euler-type method* $(\Psi^{BSNE}, \bar{h}, X_0)$ *for the linear SODE* (2.1) *is strongly convergent of order* $\gamma = \frac{1}{2}$.

2.3. Stochastic B-consistency and C-stability of the BSNI Euler-type scheme

In this section we analyse the balanced shift noise impicit (BSNI) Euler-type scheme (1.66). With the inverse of $M_h = \text{id} - hA^+$ we rewrite this scheme in the explicit form

$$\overline{X}_h(t_i) := M_h^{-1}\big(X_h(t_{i-1}) + \sum_{k=2}^m \tilde{G}_k X_h(t_{i-1})\tilde{I}_{(k)}^{t_{i-1},t_i}\big)$$

$$X_h(t_i) := \Phi_1(t_i, t_{i-1})\overline{X}_h(t_i), \tag{2.27}$$

$$X_h(0) = X_0, \quad 1 \leq i \leq N,$$

where $\Phi_1(t_i, t_{i-1}) = \exp(-\tilde{G}_1^2 h + \tilde{G}_1 \tilde{I}_{(1)}^{t_{i-1},t_i})$ and $\tilde{I}_{(k)}^{t_{i-1},t_i} = \int_{t_{i-1}}^{t_i} \mathrm{d}\tilde{W}_k(s)$.

The following lemma is a special case of Corollary 4.2 in [5] and Corollary 3.2.2 and plays an important role in the further proofs. The similar result can be found in [58].

Lemma 2.3.1. *Denote $\hat{\alpha} := \mu_2(A^+)$ and assume that $\hat{\alpha}\delta < 1$ for all $\delta \in (0,1]$. Then*

$$|M_\delta^{-1}| \leq (1 - \hat{\alpha}\delta)^{-1}. \tag{2.28}$$

Proof. Consider the matrix $M_\delta^- := A^+\delta - \mathrm{id}$. Using properties 2 and 3 from Lemma A.4.2 we obtain

$$\mu_2(M_\delta^-) = \mu_2(A^+\delta) - \mu_2(\mathrm{id}) = \hat{\alpha}\delta - 1 < 0.$$

Then by Lemma A.4.2, property 7 it holds

$$|(A^+\delta - \mathrm{id})^{-1}| \leq -\frac{1}{\hat{\alpha}\delta - 1},$$

which proves (2.28). $\qquad\square$

Let $(t,\delta) \in \mathbb{T}$ be arbitrary and let $\bar{h} \in (0, \hat{\alpha}^{-1})$ is an upper size bound. Then we define the one-step map $\Psi^{BSNI} \colon \mathbb{R}^d \times \mathbb{T} \times \Omega \to \mathbb{R}^d$ of the balanced shift noise implicit Euler-type method (1.66) by

$$\Psi^{BSNI}(x, t, \delta) = \Phi_1(t + \delta, t) M_\delta^{-1}\left(x + \sum_{k=2}^m \tilde{G}_k x \tilde{I}_{(k)}^{t,t+\delta}\right), \tag{2.29}$$

for all $x \in \mathbb{R}^d$, where $M_\delta = \mathrm{id} - A^+\delta$. By the continuity and boundedness of the mapping $\mathbb{R}^d \ni x \mapsto \Phi_1(t+\delta, t) M_\delta^{-1} x \in \mathbb{R}^d$ we get for all $(t,\delta) \in \mathbb{T}$ and $Y \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$ that

$$\Phi_1(t+\delta, t) M_\delta^{-1} Y \in L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbf{P}; \mathbb{R}^d).$$

Thus, $\Psi^{BSNI}(Y, t, \delta) \colon \Omega \to \mathbb{R}^d$ is an $\mathcal{F}_{t+\delta}/\mathcal{B}(\mathbb{R}^d)$-measurable random variable, which satisfies condition (2.2).

The following lemma is the linear version of a more general nonlinear result from [5, Lemma 4.3].

Lemma 2.3.2. *Let $\bar{h} \in (0, |\hat{\alpha}|^{-1})$ be given and $\delta \in (0, \bar{h}]$. Then for all $x \in \mathbb{R}^d$ the estimates hold*

$$|(\mathrm{id} - M_\delta^{-1})x| \leq C_1 \delta |x|,$$
$$|(\mathrm{id} - M_\delta^{-1} + A^+\delta)x| \leq C_2 \delta^2 |x|,$$

where

$$C_1 = (1 - \hat{\alpha}\bar{h})^{-1}|A^+|, \quad and \quad C_2 = (1 - \hat{\alpha}\bar{h})^{-1}|A^+|^2. \tag{2.30}$$

Proof. Let $z \in \mathbb{R}^d$ be arbitrary. Then by Lemma 2.3.1 we get

$$
\begin{aligned}
|(\mathrm{id} - M_\delta^{-1})x| = |M_\delta^{-1}(M_\delta - \mathrm{id})x| = &|M_\delta^{-1}(\mathrm{id} - A^+\delta - \mathrm{id})x| \\
\leq &(1 - \hat{\alpha}\delta)^{-1}|A^+||\delta||x| \\
\leq &(1 - \hat{\alpha}\bar{h})^{-1}|A^+||\delta||x|.
\end{aligned}
$$

For the second estimate we obtain

$$
\begin{aligned}
|(\mathrm{id} - M_\delta^{-1} + A^+\delta)x| = &|(M_\delta^{-1}(M_\delta - \mathrm{id}) + A^+\delta)x| \\
= &\delta|(\mathrm{id} - M_\delta^{-1})A^+x| \\
\leq &(1 - \hat{\alpha}\bar{h})^{-1}|A^+|^2\delta^2|x|.
\end{aligned}
$$

$\square$

The following theorem provides the stochastic B-consistency of the BSNI Euler-type scheme.

Theorem 2.3.3. *Let $\bar{h} \in (0, |\hat{\alpha}|^{-1})$. Then the balanced shift noise implicit Euler-type method $(\Psi^{BSNI}, \bar{h}, X_0)$ for the linear SODE (2.1) is stochastically B-consistent of order $\gamma = \frac{1}{2}$. The constants $C_{\mathrm{cons},1}$ and $C_{\mathrm{cons},2}$ are of moderate exponential type, see (2.35).*

Proof. First, we note that for arbitrary $(t, \delta) \in \mathbb{T}$ it holds

$$
\begin{aligned}
X(t+\delta) - \Psi^{BSNI}(X(t), t, \delta) = &\int_t^{t+\delta} \Phi_1(t+\delta, \tau)A^+X(\tau) - \Phi_1(t+\delta, t)A^+X(t)\,\mathrm{d}\tau \\
&+ \sum_{k=2}^m \int_t^{t+\delta} (\Phi_1(t+\delta, \tau)\tilde{G}_kX(\tau) - \Phi_1(t+\delta, t)\tilde{G}_kX(t))\,\mathrm{d}\tilde{W}_k(\tau) \\
&+ \Phi_1(t+\delta, t)X(t) + \Phi_1(t+\delta, t)A^+X(t)\delta - \Phi_1(t+\delta, t)M_\delta^{-1}X(t) \\
&+ \sum_{k=2}^m \left(\Phi_1(t+\delta, t)\tilde{G}_kX(t) - \Phi_1(t+\delta, t)M_\delta^{-1}\tilde{G}_kX(t)\right)\tilde{I}_{(k)}^{t,t+\delta}.
\end{aligned}
$$

Then by Definition (2.1.3) we get

$$
\begin{aligned}
&\|\mathbb{E}[X(t+\delta) - \Psi^{BSNI}(X(t), t, \delta)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq \int_t^{t+\delta} \|\mathbb{E}[\Phi_1(t+\delta, \tau)A^+X(\tau) - \Phi_1(t+\delta, t)A^+X(t)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau \\
&\quad + \|\mathbb{E}[\Phi_1(t+\delta, t)X(t) + \Phi_1(t+\delta, t)A^+X(t)\delta - \Phi_1(t+\delta, t)M_\delta^{-1}X(t)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \\
&=: T_1 + T_2.
\end{aligned}
$$

For the first term we have

$$
T_1 \leq \int_t^{t+\delta} \|\mathbb{E}[\Phi_1(t+\delta, \tau)A^+\big(X(\tau) - X(t)\big)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau
$$

$$+ \int_t^{t+\delta} \|\mathbb{E}[(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t))A^+X(t)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau.$$

Then, by Corollary 2.2.2 we get for the first summand

$$\int_t^{t+\delta} \|\mathbb{E}[\Phi_1(t+\delta,\tau)A^+(X(\tau) - X(t))|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau \leq \mathcal{K}_3(\delta,t+\delta)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta^2,$$

with

$$\mathcal{K}_3(\delta,t+\delta) = \frac{1}{2}|A^+||A|\exp(\alpha_{1,+}\delta + \alpha_+(t+\delta)). \tag{2.31}$$

For the second summand we use the fact that $\|\mathbb{E}[Z|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$, for all $Z \in L^2(\Omega;\mathbb{R}^d)$, and (2.11). It follows

$$\int_t^{t+\delta} \|\mathbb{E}[(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t))A^+X(t)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$
$$\leq \int_t^{t+\delta} \|(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t))A^+X(t)\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$
$$\leq \mathcal{K}_2(\delta,t)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta^{\frac{3}{2}},$$

where

$$\mathcal{K}_2(\delta,t) = |A^+||\tilde{G}_1|(\frac{1}{3}|\tilde{G}_1|\delta^{\frac{1}{2}} + \frac{2}{3})\exp(\alpha_+ t + \alpha_{\mathrm{S},+}\delta). \tag{2.32}$$

Further, using Lemma 1.1.2, Lemma 2.3.2 and (1.56) we get

$$T_2 = \|\mathbb{E}[\Phi_1(t+\delta,t)(X(t) + A^+X(t)\delta - M_\delta^{-1}X(t))|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$= \|\exp(-\frac{1}{2}\tilde{G}_1^2\delta)(\mathrm{id} - M_\delta^{-1} + A^+\delta)X(t)\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq |\exp(-\frac{1}{2}\tilde{G}_1^2\delta)|\|(\mathrm{id} - M_\delta^{-1} + A^+\delta)X(t)\|_{L^2(\Omega;\mathbb{R}^d)} \tag{2.33}$$
$$\leq e^{\alpha_1\delta}C_2\delta^2\|X(t)\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq C_2\delta^2\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\exp(\alpha_1\delta + \alpha_+ t),$$

where the constant C_2 is given by (2.30). Next, consider

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(X(t+\delta) - \Psi^{BSNI}(X(t),t,\delta))\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq \int_t^{t+\delta} \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t+\delta,\tau)A^+X(\tau) - \Phi_1(t+\delta,t)A^+X(t)\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$
$$+ \|\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t+\delta,t)(X(t) + A^+X(t)\delta - M_\delta^{-1}X(t))\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \|\sum_{k=2}^m \int_t^{t+\delta}(\Phi_1(t+\delta,\tau)\tilde{G}_kX(\tau) - \Phi_1(t+\delta,t)\tilde{G}_kX(t)) \, \mathrm{d}\tilde{W}_k(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \big\| \sum_{k=2}^{m} \big(\Phi_1(t+\delta,t)\tilde{G}_k X(t) - \Phi_1(t+\delta,t)M^{-1}\tilde{G}_k X(t) \big) \tilde{I}_{(k)}^{t,t+\delta} \big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$=: \sum_{i=1}^{4} S_i.$$

By using of the fact that $\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])Z\|_{L^2(\Omega;\mathbb{R}^d)} \le \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$ and Lemma 2.2.1 we get for S_1

$$S_1 \le \int_t^{t+\delta} \| \Phi_1(t+\delta,\tau)A^+ \big(X(\tau) - X(t) \big) \|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$

$$+ \int_t^{t+\delta} \| \big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t) \big) A^+ X(t) \|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau$$

$$\le \big(\mathcal{K}_1(\delta,t+\delta) + \mathcal{K}_2(\delta,t) \big) \|X_0\|_{L^2(\Omega;\mathbb{R}^d)} \delta^{\frac{3}{2}},$$

where

$$\mathcal{K}_1(\delta,t+\delta) = \frac{1}{2}|A^+|(|A|\delta^{\frac{1}{2}} + |G|) \exp(\alpha_{S,+}\delta + \alpha_+(t+\delta)). \tag{2.34}$$

and $\mathcal{K}_2(\delta,t)$ given in (2.32). For the term S_2 we have

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t+\delta,t)\big(X(t) + A^+ X(t)\delta - M_\delta^{-1} X(t) \big) \|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le \| (\Phi_1(t+\delta,t)\big(X(t) + A^+ X(t)\delta - M_\delta^{-1} X(t) \big) \|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le \mathrm{e}^{\alpha_{S,+}\delta} \| X(t) + A^+ X(t)\delta - M_\delta^{-1} X(t) \|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le \mathrm{e}^{\alpha_{S,+}\delta} C_2 \delta^2 \| X(t) \|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le C_2 \delta^2 \| X_0 \|_{L^2(\Omega;\mathbb{R}^d)} \exp \big(\alpha_{S,+}\delta + \alpha_+ t \big).$$

Here we used Lemma 1.1.2, Lemma 1.5.2, and Lemma 2.3.2. Further, by Lemma 2.2.3 we get

$$S_3 \le \big\| \sum_{k=2}^{m} \int_t^{t+\delta} \Phi_1(t+\delta,\tau)\tilde{G}_k \big(X(\tau) - X(t) \big) \, \mathrm{d}\tilde{W}_k(\tau) \big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \big\| \sum_{k=2}^{m} \int_t^{t+\delta} \big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t) \big) \tilde{G}_k X(t)) \, \mathrm{d}\tilde{W}_k(\tau) \big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le \big(\mathcal{K}_4(\delta,t+\delta) + \mathcal{K}_5(\delta,t) \big) \|X_0\|_{L^2(\Omega;\mathbb{R}^d)} \delta,$$

with

$$\mathcal{K}_4(\delta,t+\delta) = |\tilde{G}^-|(|A|\delta^{\frac{1}{2}} + |G|) \exp \big(\alpha_{S,+}\delta + \alpha_+(t+\delta) \big),$$

and

$$\mathcal{K}_5(\delta,t) = |\tilde{G}^-||\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1) \exp \big(\alpha_+ t + \alpha_{S,+}\delta \big).$$

Finally, by the Itô isometry, Lemma 1.1.2, and Lemma 2.3.2 we obtain

$$
\begin{aligned}
S_4 &= \Big(\sum_{k=2}^{m} \| \Phi_1(t+\delta,t)(\mathrm{id} - M_\delta^{-1}) \tilde{G}_k X(t)\delta \|^2_{L^2(\Omega;\mathbb{R}^d)} \Big)^{\frac{1}{2}} \\
&\leq C_1 |\tilde{G}^-| \exp\big(\alpha_{S,+}\delta + \alpha_+ t \big) \| X_0 \|_{L^2(\Omega;\mathbb{R}^d)} \delta^2,
\end{aligned}
$$

where C_1 is given by (2.30). This completes the proof.

$\square$

Remark 2.3.4. The constants $C_{\mathrm{cons},1}$ and $C_{\mathrm{cons},2}$ in (2.5) and (2.6) are given by

$$
\begin{aligned}
C_{\mathrm{cons},1} :=& \frac{1}{2} |A^+||A| \exp\big(\alpha_{1,+}\bar{h} + \alpha_+ T \big) + |A^+|^2 (1-\hat{\alpha}\hat{h})^{-1} \exp(\alpha_1 \bar{h} + \alpha_+ T) \\
&+ |A^+||\tilde{G}_1|(\frac{1}{3}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + \frac{2}{3}) \exp\big(\alpha_+ T + \alpha_{S,+}\bar{h} \big), \\
C_{\mathrm{cons},2} :=& \Big(\frac{1}{2}|A^+|(|A|\bar{h}^{\frac{1}{2}} + |G|) + |\tilde{G}^-|(|A|\bar{h}^{\frac{1}{2}} + |G|) + |\tilde{G}^-||\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1) \\
&+ |A^+|(1-\hat{\alpha}\bar{h})^{-1}(|A^+| + |\tilde{G}^-|) \Big) \exp\big(\alpha_{S,+}\bar{h} + \alpha T \big).
\end{aligned}
\tag{2.35}
$$

It remains to show that the BSNI scheme is stochastically C-stable.

Theorem 2.3.5. *Let $\bar{h} \in (0, |\hat{\alpha}|^{-1})$. For the linear SODE* (2.1) *with every initial value $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ the BSNI Euler-type method is stochastically C-stable. The constant C_{stab} in* (2.4) *depend on data $\alpha_{S,+}, \hat{\alpha}, \eta, |\tilde{G}_1|, |\tilde{G}^-|$, and $\bar{h}$, see* (2.39).

Proof. Let $(t,\delta) \in \mathbb{T}$ be arbitrary and $Y, Z \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$. We note that

$$
\mathbb{E}[\Psi^{BSNI}(Y,t,\delta) - \Psi^{BSNI}(Z,t,\delta)|\mathcal{F}_t] = \mathbb{E}[\Phi_1(t+\delta,t)M_\delta^{-1}(Y-Z)|\mathcal{F}_t]
\tag{2.36}
$$

and

$$
\begin{aligned}
(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])&(\Psi^{BSNI}(Y,t,\delta) - \Psi^{BSNI}(Z,t,\delta)) \\
&= (\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t+\delta,t)M_\delta^{-1}(Y-Z) \\
&\quad + \sum_{k=2}^{m} \Phi_1(t+\delta,t)M_\delta^{-1}\tilde{G}_k(Y-Z)\tilde{I}_{(k)}^{t,t+\delta}.
\end{aligned}
\tag{2.37}
$$

Then by (1.56) and Lemma 2.3.1 we obtain for (2.23)

$$
\begin{aligned}
\mathbb{E}[|\mathbb{E}[\Phi_1(t+\delta,t)M_\delta^{-1}(Y-Z)|\mathcal{F}_t]|^2] &\leq |\exp(-\frac{1}{2}\tilde{G}_1^2\delta)|^2 |M_\delta^{-1}|^2 \mathbb{E}[|Y-Z|^2] \\
&\leq e^{2\alpha_1\delta}(1-\hat{\alpha}\delta)^{-2}\mathbb{E}[|Y-Z|^2].
\end{aligned}
$$

As already mentioned in [5], the function $(1-\hat{\alpha}\delta)^{-2}$ is convex and it follows that for all $\delta \in [0,\bar{h}]$ the estimate holds

$$
(1-\hat{\alpha}\delta)^{-2} = \frac{(1-\hat{\alpha}\delta)^2 + 2\hat{\alpha}\delta - \hat{\alpha}^2\delta^2}{(1-\hat{\alpha}\delta)^2} \leq (1 + C_c\delta),
\tag{2.38}
$$

where $C_c = \frac{2\hat{\alpha} - \hat{\alpha}^2 \bar{h}}{(1 - \hat{\alpha}\bar{h})^2}$. Therefore, we get

$$\mathbb{E}[|\mathbb{E}[\Phi_1(t + \delta, t)M_\delta^{-1}(Y - Z)|\mathcal{F}_t]|^2] \leq \exp\left((2\alpha_1 + C_c)\delta\right)\mathbb{E}[|Y - Z|^2].$$

Further, from (2.37) we obtain

$$\mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t + \delta, t)M_\delta^{-1}(Y - Z)$$
$$+ \sum_{k=2}^{m} \Phi_1(t + \delta, t)M_\delta^{-1}\tilde{G}_k(Y - Z)\tilde{I}_{(k)}^{t,t+\delta}|^2]$$
$$\leq 2\mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t + \delta, t)M_\delta^{-1}(Y - Z)|^2]$$
$$+ 2\mathbb{E}[|\sum_{k=2}^{m} \Phi_1(t + \delta, t)M_\delta^{-1}\tilde{G}_k(Y - Z)\tilde{I}_{(k)}^{t,t+\delta}|^2]$$
$$=: T_1 + T_2.$$

By Lemma 1.5.4, Lemma 1.5.2, and Lemma 2.3.1 we get for the first term

$$T_1 \leq 2K_{\mathrm{cond}}^2(\delta)\delta(1 - \hat{\alpha}\delta)^{-2}\mathbb{E}[|Y - Z|^2]$$
$$\leq 2|\tilde{G}_1|^2(\frac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1)^2 \exp\left((2\alpha_{\mathrm{S},+} + C_c)\delta\right)\delta\mathbb{E}[|Y - Z|^2].$$

For the last term we get

$$T_2 = 2\delta \sum_{k=2}^{m} \mathbb{E}|\Phi_1(t + \delta, t)M_\delta^{-1}\tilde{G}_k(Y - Z)|^2]$$
$$\leq 2\delta|\tilde{G}^-|^2 \exp\left((2\alpha_{\mathrm{S},+} + C_c)\delta\right)\mathbb{E}[|Y - Z|^2].$$

Here we used the Itô isometry, Lemma 1.5.2, and (2.38). Altogether, this shows that

$$\|\mathbb{E}[\Psi^{BSNI}(Y, t, \delta) - \Psi^{BSNI}(Z, t, \delta)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}^2$$
$$+ \eta\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{BSNI}(Y, t, \delta) - \Psi^{BSNI}(Z, t, \delta))\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq \mathrm{e}^{C_{\mathrm{stab}}\delta}\|Y - Z\|_{L^2(\Omega;\mathbb{R}^d)}^2.$$

where

$$C_{\mathrm{stab}} = 2\eta\left(|\tilde{G}_1|^2(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1)^2 + |\tilde{G}^-|\right) + 2\alpha_{\mathrm{S},+} + C_c). \tag{2.39}$$

$\square$

The constant C_{stab} is not of moderate type, compare Remark 2.2.8. We conclude this section by stating the strong convergence of the BSNI scheme as obtained from Theorem 2.3.3 and Theorem 2.3.5.

Theorem 2.3.6. *Let $\bar{h} \in (0, |\hat{\alpha}|^{-1})$. Then the balanced shift noise implicit Euler-type method $(\Psi^{BSNI}, \bar{h}, X_0)$ for the linear SODE (2.1) is strongly convergent of order $\gamma = \frac{1}{2}$.*

3. Nonlinearity in the drift term

The balanced shift noise approach from Section 1.7 heavily relies on the linearity of the noise terms. In this chapter we keep this structure but generalize our results to SODEs with a nonlinear drift term. In particular, we follow the approach in [5] and investigate convergence of our new methods under one-sided Lipschitz conditions. It turns out that the BSNE method has to be complemented by a projection or cutoff procedure while the theory for the BSNI method works without such precautions.

3.1. Assumptions and main results

At first we consider a stochastic differential equation in the form

$$
\begin{aligned}
\mathrm{d}X(t) = & f(X(t))\,\mathrm{d}t + \sum_{k=1}^{m} \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t), \\
X(0) = & X_0, \quad t \in [0, T],
\end{aligned}
\tag{3.1}
$$

where $f \colon \mathbb{R}^d \to \mathbb{R}^d$ is the drift coefficient function, $\tilde{G}_k \in \mathbb{R}^{d\times d}, k = 1, \ldots, m$ and $\tilde{W}_k, k = 1, \ldots, m$ are defined as in Section 1.2. Further, we assume that the function f satisfies the conditions below.

Assumption 3.1.1. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be continuous and $\tilde{G}_k \in \mathbb{R}^{d\times d}, k = 1, \ldots, m$. There exists a constant $L > 0$ and a parameter value $\eta \in (\frac{1}{2}, \infty)$ such that for all $x_1, x_2 \in \mathbb{R}^d$ it holds*

$$
\langle x_1 - x_2, f(x_1) - f(x_2)\rangle + \eta \sum_{k=1}^{m} |\tilde{G}_k x_1 - \tilde{G}_k x_2|^2 \le L|x_1 - x_2|^2.
\tag{3.2}
$$

In addition, there exists a constant $q \in (1, \infty)$ such that for all $x, x_1, x_2 \in \mathbb{R}^d$ it holds

$$
\begin{aligned}
|f(x)| \le & L_0(1 + |x|^q), \\
|f(x_1) - f(x_2)| \le & L_1(1 + |x_1|^{q-1} + |x_2|^{q-1})|x_1 - x_2|.
\end{aligned}
\tag{3.3}
\tag{3.4}
$$

We denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$. We recall that Assumption 3.1.1 is sufficient to ensure the existence of a unique solution

to (3.1), i.e., there exists an almost surely continuous and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process $Z:[0,T]\times\Omega\to\mathbb{R}^d$ such that

$$X(t) = X_0 + \int_0^t f(X(\tau))\,\mathrm{d}\tau + \sum_{k=1}^m \int_0^t \tilde{G}_k X(\tau)\,\mathrm{d}\tilde{W}_k(\tau), \tag{3.5}$$

for all $t\in[0,T]$ **P**-almost surely (see for example, [42]). Moreover, we assume that there exist constants $\epsilon\geq 0$, $\alpha_{\mathrm{f}}\geq 0$, and $p\in[2,\infty)$ such that for all $x\in\mathbb{R}^d$ the estimate holds

$$\langle x, f(x)\rangle + \frac{p-1}{2}\sum_{k=1}^m |\tilde{G}_k x|^2 \leq \alpha_{\mathrm{f}}(\epsilon^2 + |x|^2). \tag{3.6}$$

Then the exact solution satisfies (see [42, Th.4.1] and Lemma 1.1.2)

$$\|X(t)\|_{L^p(\Omega;\mathbb{R}^d)} \leq (\epsilon + \|X_0\|_{L^p(\Omega;\mathbb{R}^d)})\mathrm{e}^{\alpha_{\mathrm{f}}t}. \tag{3.7}$$

We note that (3.6) follows from (3.2) with $\epsilon = |f(0)|$, $\eta = \frac{p-1}{2}$ and $\alpha_{\mathrm{f}} = |f(0)|(L + \sqrt{L^2+1})$.

In the following we prove further regularity of the solutions to (3.1) by establishing Hölder regularity with respect to the L^p-norm. The lemma below is analogous to Proposition 5.4 in [5].

Lemma 3.1.2. *Let f satisfies condition 3.3 with $L_0 > 0, q\in(1,\infty)$ and let $\tilde{G}_k \in \mathbb{R}^{d\times d}, k = 1,\ldots,m$. Further, let $p\geq 2$ be given such that the exact solution X satisfies $\sup_{t\in[0,T]}\|X(t)\|_{L^{pq}(\Omega;\mathbb{R}^d)} < \infty$. Then there exists a constant C such that*

$$\|X(t) - X(s)\|_{L^p(\Omega;\mathbb{R}^d)} \leq C\big(1 + \sup_{t\in[0,T]}\|X(t)\|_{L^{pq}(\Omega;\mathbb{R}^d)}^q\big)|t-s|^{\frac{1}{2}} \tag{3.8}$$

for all $0\leq s < t \leq T$.

In addition, if condition (3.6) holds with $\eta = \frac{pq-1}{2}$ for $p\in[2,\infty)$ and $q\in[1,\infty)$ then (3.8) can be estimated by

$$\|X(t) - X(s)\|_{L^p(\Omega;\mathbb{R}^d)} \leq \mathcal{R}(t-s,t)|t-s|^{\frac{1}{2}}, \tag{3.9}$$

where $\mathcal{R}:\{(t,s)\colon 0\leq s\leq t\leq T\}\to\mathbb{R}$ is of moderate size and given by

$$\begin{aligned}
\mathcal{R}(\delta,\delta_1) = &\big(L_0 + L_0(1 + \|X_0\|_{L^{pq}(\Omega;\mathbb{R}^d)}^q)\mathrm{e}^{q\alpha_{\mathrm{f}}\delta_1}\big)\delta^{\frac{1}{2}} \\
&+ C_p|G|\|X_0\|_{L^p(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_{\mathrm{f}}\delta_1}
\end{aligned} \tag{3.10}$$

for $0\leq\delta\leq\delta_1\leq T$. Here C_p denotes the Burkholder-Davis-Gundy constant.

Proof. For the proof of (3.8) we refer to [5]. Further, let $t,s\in[0,T]$ with $s<t$. Then we get

$$\|X(t) - X(s)\|_{L^p(\Omega;\mathbb{R}^d)} \leq \int_s^t \|f(X(\tau))\|_{L^p(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau$$

$$+ \Big\| \sum_{k=1}^{m} \int_s^t \tilde{G}_k X(\tau) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^p(\Omega;\mathbb{R}^d)}.$$

By condition (3.3) and Lemma 1.1.2 we get

$$
\begin{aligned}
\int_s^t \|f(X(\tau))\|_{L^p(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau &\leq L_0 \int_s^t \|1 + |X(\tau)|^q\|_{L^p(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau \\
&\leq L_0 \int_s^t \big(1 + \|X(\tau)\|_{L^{pq}(\Omega;\mathbb{R}^d)}^q\big) \, \mathrm{d}\tau \\
&\leq L_0 \int_s^t \mathrm{d}\tau + L_0(1 + \|X_0\|_{L^{pq}(\Omega;\mathbb{R}^d)}^q) \int_s^t \mathrm{e}^{q\alpha_{\mathrm{f}}\tau} \, \mathrm{d}\tau \\
&\leq \big(L_0 + L_0(1 + \|X_0\|_{L^{pq}(\Omega;\mathbb{R}^d)}^q)\mathrm{e}^{q\alpha_{\mathrm{f}}t}\big)|t - s|.
\end{aligned}
$$

Finally, we use the Burkholder-Davis-Gundy inequality (see [42, Th.7.3]) and Lemma 1.1.2 and obtain

$$
\begin{aligned}
\Big\| \sum_{k=1}^{m} \int_s^t \tilde{G}_k X(\tau) \, \mathrm{d}\tilde{W}_k(\tau) \Big\|_{L^p(\Omega;\mathbb{R}^d)} &\leq C_p \Big(\sum_{k=1}^{m} \int_s^t \|\tilde{G}_k X(\tau)\|_{L^p(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}\tau \Big)^{\frac{1}{2}} \\
&\leq C_p |G| \|X_0\|_{L^p(\Omega;\mathbb{R}^d)} \mathrm{e}^{\alpha_{\mathrm{f}}t} |t - s|^{\frac{1}{2}}.
\end{aligned}
$$

This completes the proof.

$\square$

In the following we assume that function $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfies Assumption 3.1.1 with $L > 0$ and $\eta \in (\frac{1}{2}, \infty)$. In order to formulate the generalization of our balanced shift noise Euler-type methods with nonlinear drift term let us introduce

$$f^+(x) := f(x) + \frac{1}{2}\tilde{G}_1^2 x, \tag{3.11}$$

with $\tilde{G}_1 \in \mathbb{R}^{d \times d}$ given in (1.19). This notation is similar to (1.62).

The following assumption is an extension of Assumption 3.1.1.

Assumption 3.1.3. *Let f^+ be given as in (3.11). There exists a constant $L^+ > 0$ and a parameter value $\eta \in [\frac{1}{2}, \infty)$ such that for all $x_1, x_2 \in \mathbb{R}^d$ it holds*

$$\langle x_1 - x_2, f^+(x_1) - f^+(x_2) \rangle + \eta \sum_{k=2}^{m} |\tilde{G}_k x_1 - \tilde{G}_k x_2|^2 \leq L^+ |x_1 - x_2|^2. \tag{3.12}$$

Moreover, for all $q \in (1, \infty)$ there exist constants $L_0^+, L_1^+ > 0$ such that for all $x, x_1, x_2 \in \mathbb{R}^d$

$$|f^+(x)| \leq L_0^+(1 + |x|^q) \tag{3.13}$$

$$|f^+(x_1) - f^+(x_2)| \leq L_1^+(1 + |x_1|^{q-1} + |x_2|^{q-1})|x_1 - x_2|. \tag{3.14}$$

It is clear that (3.12) implies with $\eta = \frac{1}{2}$

$$\langle x_1 - x_2, f(x_1) - f(x_2) \rangle + \frac{1}{2} \langle x_1 - x_2, \tilde{G}_1^2(x_1 - x_2) \rangle$$
$$+ \frac{1}{2} \sum_{k=2}^{m} \langle \tilde{G}_k(x_1 - x_2), \tilde{G}_k(x_1 - x_2) \rangle$$
$$\leq L^+ |x_1 - x_2|^2.$$

In particular, if $\tilde{G}_1$ is symmetric, then Assumption 3.1.3 hold with $L^+ = L$. Moreover, it holds

$$\langle x_1 - x_2, \tilde{G}_1^2(x_1 - x_2) \rangle \leq \mu_2(\tilde{G}_1^2)|x_1 - x_2|^2 \leq |\tilde{G}_1|^2 |x_1 - x_2|^2.$$

Here we used the first property of the logarithmic norm from Lemma A.4.2.

Now, we define two balanced shift noise methods for the drift nonlinear equations (3.1). The first method is the *projected balanced shift noise explicit* Euler-type method (PBSNE). As already suggested in [5] we use a projection onto a ball in $\mathbb{R}^d$ whose radius is expanding with a suitable negative power of the step size.

Let $h \in (0,1]^N$ be an arbitrary vector of step sizes. The parameter $\beta \in (0, \infty)$ is chosen to be a suitable negative power in dependence of the growth rate q. Then the PBSNE Euler-type method is given by the three-step recursion

$$\begin{aligned}
X_h^\circ(t_i) &:= \min(1, h_i^{-\beta}|X_h(t_{i-1})|^{-1})X_h(t_{i-1}), \\
\overline{X}_h(t_i) &= X_h^\circ(t_i) + f^+(X_h^\circ(t_i))h_i + \sum_{k=2}^{m} \tilde{G}_k X_h^\circ(t_i)\tilde{I}_{(k)}^{t_{i-1},t_i}, \\
X_h(t_i) &= \Phi_1(t_i, t_{i-1})\overline{X}_h(t_i), \quad i = 1, \ldots, N, \\
X_h(0) &= X_0,
\end{aligned} \tag{3.15}$$

where Φ_1 is defined as in the previous chapter by

$$\Phi_1(t_i, t_{i-1}) = \exp(-\tilde{G}_1^2 h_i + \tilde{G}_1 \tilde{I}_{(1)}^{t_{i-1},t_i}).$$

Our aim is to prove the following convergence result:

Theorem 3.1.4. *Let Assumption 3.1.3 hold with growth rate $q \in (1, \infty)$ and let $\bar{h} \in (0,1]$. If $\sup_{\tau \in [0,T]} \|X(\tau)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)} < \infty$, where X denotes the exact solution to (3.1), then the projected balanced shift noise explicit Euler-type method $(\Psi^{PBSNE}, \bar{h}, X_0)$ with $\beta = \frac{1}{2(q-1)}$ is strongly convergent of order $\gamma = \frac{1}{2}$.*

The next method is implicit and given by the recursion

$$
\begin{aligned}
\hat{X}_h(t_i) =& X_h(t_{i-1}) + f^+(\hat{X}_h(t_i))h_i, \\
\overline{X}_h(t_i) =& \hat{X}(t_i) + \sum_{k=2}^{m} \tilde{G}_k \hat{X}(t_i) \tilde{I}_{(k)}^{t_{i-1},t_i}, \\
X_h(t_i) =& \Phi_1(t_i, t_{i-1})\overline{X}_h(t_i), \\
X_h(0) =& X_0, \quad i = 1, \dots, N.
\end{aligned}
\tag{3.16}
$$

We call this method *split-step balanced shift noise implicit* (SSBSNI) Euler-type method. If the one omits the shift term and the stochastic integration step with Φ_1 then one obtains an implicit method studied in [5], [24]. We recall that the recursion in (3.16) evaluates the diffusion term at time t_i in the i-th step. In section 3.5 we will show that the SSBSNI Euler-type method is stochastically C-stable and B-consistent. As a result one obtains the following convergence theorem:

Theorem 3.1.5. *Let Assumption 3.1.3 hold with growth rate $q \in (1, \infty)$ and let $\bar{h} \in (0, \frac{1}{L^+})$. If $\sup_{\tau \in [0,T]} \|X(\tau)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)} < \infty$, where X denotes the exact solution to (3.1), then the split-step balanced shift noise implicit Euler-type method $(\Psi^{SSBSNI}, \bar{h}, X_0)$ is strongly convergent of order $\gamma = \frac{1}{2}$.*

3.2. Solution estimates for nonlinear equations under one-sided Lipschitz conditions

In this section we cite from [5, Chap.4] some known results on the solvability of the nonlinear equations under a one-sided Lipschitz condition.

The first theorem is the so-called Uniform Monotonicity Theorem of nonlinear analysis, which can be found, for instance in [48, Chap. 6.4], [61, Th.C.2].

Theorem 3.2.1. *Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous mapping such that for all $x_1, x_2 \in \mathbb{R}^d$ there exists a constant $c > 0$ with*

$$
\langle g(x_1) - g(x_2), x_1 - x_2 \rangle \geq c|x_1 - x_2|^2.
$$

Then g is a homeomorphism with Lipschitz continuous inverse and for all $y_1, y_2 \in \mathbb{R}^d$ it holds

$$
|g^{-1}(y_1) - g^{-1}(y_2)| \leq \frac{1}{c}|y_1 - y_2|.
$$

For the proof we refer to [5, Th.4.1]. The following corollary is a consequence of Theorem 3.2.1, which was proved in [5].

Corollary 3.2.2. *Let $f^+ : \mathbb{R}^d \to \mathbb{R}^d$ satisfies Assumption 3.1.3 with $L^+, L_0^+, L_1^+ > 0$ and $\eta \in (1, \infty)$. Let $\bar{h} \in (0, \frac{1}{L^+})$ and define the mapping $F_\delta : \mathbb{R}^d \to \mathbb{R}^d$ by $F_\delta(x) = x - f^+(x)\delta$ for all $\delta \in (0, \bar{h}]$. Then the mapping $x \mapsto F_\delta(x)$, $x \in \mathbb{R}^d$ is a homeomorphism.*

In addition, for all $x, x_1, x_2 \in \mathbb{R}^d$ the estimates hold

$$|F_\delta^{-1}(x_1) - F_\delta^{-1}(x_2)| \leq (1 - L^+\delta)^{-1}|x_1 - x_2|, \tag{3.17}$$

$$|F_\delta^{-1}(x)| \leq (1 - L^+\delta)^{-1}(L_0^+\delta + |x|). \tag{3.18}$$

Moreover, for all $x_1, x_2 \in \mathbb{R}^d$ it holds

$$|F_\delta^{-1}(x_1) - F_\delta^{-1}(x_2)|^2 + \eta\delta \sum_{k=2}^m |\tilde{G}_k F_\delta^{-1}(x_1) - \tilde{G}_k F_\delta^{-1}(x_2)|^2$$

$$\leq (1 + C_{\mathrm{H}}\delta)|x_1 - x_2|^2, \tag{3.19}$$

where

$$C_{\mathrm{H}} = \frac{L^+(2 - L^+\bar{h})}{(1 - L^+\bar{h})^2}. \tag{3.20}$$

The poof can be found in [5, Corollary 4.2].

Further, we quote from [5] a useful lemma, which plays an important role in the analysis of the local error of the SSBSNI method.

Lemma 3.2.3. *Let Assumption 3.1.3 hold with $L^+, L_0^+, L_1^+ > 0$ and $\eta \in (1, \infty)$. Let $\bar{h} \in (0, \frac{1}{L^+})$ and for all $\delta \in (0, \bar{h}]$ the mapping $F_\delta : \mathbb{R}^d \to \mathbb{R}^d$ is given by $F_\delta(x) = x - f^+(x)\delta$. Then for all $x \in \mathbb{R}^d$ the estimate holds*

$$|F_\delta^{-1}(x) - x| \leq C_{\mathrm{L}_0}\delta(1 + |x|^q), \tag{3.21}$$

$$|F_\delta^{-1}(x) - x - f^+(x)\delta| \leq C_{\mathrm{L}_1}\delta^2(1 + |x|^{2q-1}), \tag{3.22}$$

where

$$C_{\mathrm{L}_0} = L_0^+(1 + 2^{q-1}(1 - L^+\bar{h})^{-q}), \tag{3.23}$$

$$C_{\mathrm{L}_1} = L_1^+ C_{\mathrm{L}_0}(1 + (1 - L^+\bar{h})^{1-q}(1 + (L_0^+\bar{h})^{q-1}). \tag{3.24}$$

For the proof we refer to [5, Lemma 4.3].

3.3. Reformulation of the nonlinear integral equation

In this section we rewrite the integral equation (3.5) in the same way as in Section 1.6. Let X be a solution to (3.1). Applying Itô's formula to the function $V(t) = \Phi_1(t, 0)^{-1}X(t)$

for all $t \in [0, T]$ provides (compare (1.63))

$$
\begin{aligned}
dV(t) =& \Phi_1^{-1}(t,0)\big(\tilde{G}_1^2\,dt - \tilde{G}_1\,d\tilde{W}_1(t) + \tfrac{1}{2}\tilde{G}_1^2\,dt\big)X(t) \\
& + \Phi_1^{-1}(t,0)\big(f(X(t))\,dt + \sum_{k=1}^{m}\tilde{G}_kX(t)\,d\tilde{W}_k(t)\big) \\
& + \Phi_1^{-1}(t,0)\big(\tilde{G}_1^2\,dt - \tilde{G}_1\,d\tilde{W}_1(t) + \tfrac{1}{2}\tilde{G}_1^2\,dt\big)\big(f(X(t))\,dt + \sum_{k=1}^{m}\tilde{G}_kX(t)\,d\tilde{W}_k(t)\big) \\
=& \Phi_1^{-1}(t,0)\Big(f^+(X(t))\,dt + \sum_{k=2}^{m}\tilde{G}_kX(t)\,d\tilde{W}_k(t)\Big),
\end{aligned}
\tag{3.25}
$$

where f^+ is defined by (3.11). Due to the independence of Φ_1 and the terms: $\tilde{W}_2, \ldots, \tilde{W}_m$ we can rewrite (3.25) in the following integral form

$$
X(t) = \Phi_1(t,0)X_0 + \int_0^t \Phi_1(t,\tau)f^+(X(\tau))\,d\tau + \sum_{k=2}^{m}\int_0^t \Phi_1(t,\tau)\tilde{G}_kX(\tau)\,d\tilde{W}_k(\tau), \tag{3.26}
$$

for all $t \in [0, T]$.

3.4. Stochastic C-stability and B-consistency of the PBSNE method

In Chapter 2 we have shown the stochasic B-consistency and C-stability of the BSNE scheme for linear SODEs. In this section we study the convergence theory of this method with nonlinear deterministic term satisfying Assumption 3.1.3.

Let $\bar{h} \in (0, 1]$ be an arbitrary upper size bound step-size. Then for all $x \in \mathbb{R}^d$ and $(t, \delta) \in \mathbb{T}$ we define the one-step map $\Psi^{PBSNE} \colon \mathbb{R}^d \times \mathbb{T} \times \Omega \to \mathbb{R}^d$ of the PBSNE Euler-type method with the abbreviation $x^\circ := \min(1, \delta^{-\beta}|x|^{-1})x$ by

$$
\begin{aligned}
\Psi^{PBSNE}(x,t,\delta) :=& \Phi_1(t+\delta,t)x^\circ + \delta\Phi_1(t+\delta,t)f^+(x^\circ) \\
& + \sum_{k=2}^{m}\Phi_1(t+\delta,t)\tilde{G}_kx^\circ\tilde{I}_{(k)}^{t,t+\delta},
\end{aligned}
\tag{3.27}
$$

with

$$
\Phi_1(t+\delta,t) = \exp(-\tilde{G}_1^2\delta + \tilde{G}_1\tilde{I}_{(1)}^{t,t+\delta}), \quad (t,\delta) \in \mathbb{T}.
$$

The following proposition shows that (3.27) is a one-step method in the sense of Definition 2.1.1. This result was already proved for the projected Euler-Maruyama (PEM) and projected Milstein (PMil) schemes in [5] and [6].

Proposition 3.4.1. *Let f^+ satisfies Assumption 3.1.3 with $L^+, L_0^+, L_1^+ > 0, q \in (1, \infty)$ and let $\bar{h} \in (0, 1]$. Then for every initial value $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ it holds that $(\Psi^{PBSNE}, \bar{h}, \xi)$ with $\beta \in (0, \infty)$ is a stochastic one-step method.*

Proof. Since, the mapping $x \mapsto \min(1, \delta^{-\beta}|x|^{-1})x, x \in \mathbb{R}^d$ is continuous and bounded, then for all $Y \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$ and arbitrary $(t, \delta) \in \mathbb{T}$ we obtain

$$\min(1, \delta^{-\beta}|Y|^{-1})Y \in L^\infty(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d).$$

By continuity of the function f^+ it holds

$$\Phi_1(t + \delta, t)f^+(\min(1, \delta^{-\beta}|Y|^{-1})Y) \in L^\infty(\Omega, \mathcal{F}_{t+\delta}, \mathbf{P}; \mathbb{R}^d).$$

Therefore, $\Psi^{PBSNE}(Y, t, \delta) \colon \Omega \to \mathbb{R}^d$ is an $\mathcal{F}_{t+\delta}/\mathcal{B}(\mathbb{R}^d)$-measurable random variable, which satisfies condition (2.2).

$\square$

In preparation for the proof of C-stability we quote the result from [5, Lemma 6.2], [6, Lemma 4.2], which states the global Lipschitz continuity of the mapping $\mathbb{R}^d \ni z \mapsto z^\circ \in \mathbb{R}^d$.

Lemma 3.4.2. *For every $\beta \in (0, \infty)$ and $\delta \in (0, 1]$ the mapping $\mathbb{R}^d \ni x \mapsto x^\circ \in \mathbb{R}^d$, defined by $x^\circ := \min(1, \delta^{-\beta}|x|^{-1})x$ is globaly Lipschitz continuous with Lipschitz constant 1. In particular, it holds for all $x_1, x_2 \in \mathbb{R}^d$*

$$|x_1^\circ - x_2^\circ| \leq |x_1 - x_2|. \tag{3.28}$$

The next lemma plays an important role for the stability analysis of the PBSNE Euler-type method. It is similar to Lemma 6.3 in [5].

Lemma 3.4.3. *Let f^+ satisfies Assumption 3.1.3 with $L^+, L_0^+, L_1^+ > 0$, $q \in (1, \infty)$ and $\eta \in (\frac{1}{2}, \infty)$. Consider the mapping $x \mapsto x^\circ$, defined by $x^\circ := \min(1, \delta^{-\beta}|x|^{-1})x$ with $\beta = \frac{1}{2(q-1)}$. Then for all $x_1, x_2 \in \mathbb{R}^d$ the estimate holds*

$$|x_1^\circ - x_2^\circ + \delta(f^+(x_1^\circ) - f^+(x_2^\circ))|^2 + 2\eta\delta \sum_{k=2}^m |\tilde{G}_k(x_1^\circ - x_2^\circ)|^2$$
$$\leq (1 + K\delta)|x_1 - x_2|^2, \tag{3.29}$$

with $\tilde{G}_k \in \mathbb{R}^{d \times d}$ and

$$K = 2L^+ + 9(L_1^+)^2. \tag{3.30}$$

Proof. Expanding the inner product and using condition (3.12) we get

$$
\begin{aligned}
|x_1^\circ - x_2^\circ &+ \delta(f^+(x_1^\circ) - f^+(x_2^\circ))|^2 \\
&= |x_1^\circ - x_2^\circ|^2 + 2\delta\langle x_1^\circ - x_2^\circ, f^+(x_1^\circ) - f^+(x_2^\circ)\rangle \\
&\quad + \delta^2 |f^+(x_1^\circ) - f^+(x_2^\circ)|^2 \\
&\leq (1 + 2L^+\delta)|x_1^\circ - x_2^\circ|^2 - 2\eta\delta\sum_{k=2}^{m} |\tilde{G}_k(x_1^\circ - x_2^\circ)|^2 \\
&\quad + \delta^2 |f^+(x_1^\circ) - f^+(x_2^\circ)|^2.
\end{aligned}
$$

Further, we use the fact that $|x_1^\circ|, |x_2^\circ| \leq \delta^{-\beta}$, condition (3.14), and Lemma 3.4.2 and obtain

$$
\begin{aligned}
|f^+(x_1^\circ) - f^+(x_2^\circ)| &\leq L_1^+(1 + |x_1^\circ|^{q-1} + |x_2^\circ|^{q-1})|x_1^\circ - x_2^\circ| \\
&\leq L_1^+(1 + 2\delta^{-\beta(q-1)})|x_1 - x_2|.
\end{aligned}
$$

Next, we insert $\beta = \frac{1}{2(q-1)}$ and conclude

$$
\begin{aligned}
|x_1^\circ - x_2^\circ &+ \delta(f^+(x_1^\circ) - f^+(x_2^\circ))|^2 + 2\eta\delta\sum_{k=2}^{m} |\tilde{G}_k(x_1^\circ - x_2^\circ)|^2 \\
&\leq (1 + 2L^+\delta)|x_1 - x_2|^2 + \delta^2(L_1^+(1 + 2\delta^{-\frac{1}{2}}))^2|x_1 - x_2|^2 \\
&= (1 + 2L^+\delta + (L_1^+)^2\delta^2 + 4(L_1^+)^2\delta^{\frac{3}{2}} + 4(L_1^+)^2\delta)|x_1 - x_2|^2 \\
&\leq (1 + K\delta)|x_1 - x_2|^2.
\end{aligned}
$$

This completes the proof.

$\square$

Since K contains the constant L_1^+ it is no longer of moderate size.

The following theorem verifies that the projected balanced shift noise Euler-type method is stochastically C-stable.

Theorem 3.4.4. *Let Assumption 3.1.3 hold with $L^+, L_0^+, L_1^+ \in (0, \infty)$, $q \in (1, \infty)$ and $\eta \in (\frac{1}{2}, \infty)$. Then for every initial value $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ the PBSNE Euler-type method with $\beta = \frac{1}{2(q-1)}$ is stochastically C-stable. The constant C_{stab} in (2.4) depend on the data $\alpha_{S,+}, |\tilde{G}_1|, L^+, L_1^+$ and $\bar{h}$, see (3.31).*

Proof. The proof is analogous to the proof of the Theorem 2.2.7 but here we take care of the nonlinear deterministic term. Let $Y, Z \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$ and (t, δ) be arbitrary. We recall the notation $Y^\circ := \min(1, \delta^{-\beta}|Y|^{-1})Y$ and $Z^\circ := \min(1, \delta^{-\alpha}|Z|^{-1})Z$. Then, using the fact that $\mathbb{E}[|\mathbb{E}[Z|\mathcal{F}_t]|^2] \leq \mathbb{E}[|Z|^2]$ and Itô's isometry we get

$$
\mathbb{E}[|\mathbb{E}[\Psi^{PBSNE}(Y, t, \delta) - \Psi^{PBSNE}(Z, t, \delta)|\mathcal{F}_t]|^2]
$$

$$+ \eta \mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{PBSNE}(Y, t, \delta) - \Psi^{PBSNE}(Z, t, \delta))|^2]$$

$$= \mathbb{E}[|\mathbb{E}[\Phi_1(t + \delta, t)(Y^\circ - Z^\circ + f^+(Y^\circ)\delta - f^+(Z^\circ)\delta)|\mathcal{F}_t]|^2]$$

$$+ \eta \mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t + \delta, t)(Y^\circ - Z^\circ + f^+(Y^\circ)\delta - f^+(Z^\circ)\delta)$$

$$+ \sum_{k=2}^{m} \Phi_1(t + \delta, t)\tilde{G}_k(Y^\circ - Z^\circ)\tilde{I}_{(k)}^{t,t+\delta})|^2]$$

$$\leq \mathbb{E}[|\Phi_1(t + \delta, t)(Y^\circ - Z^\circ + f^+(Y^\circ)\delta - f^+(Z^\circ)\delta)|^2]$$

$$+ 2\eta\delta \sum_{k=2}^{m} \mathbb{E}\big[|\Phi_1(t + \delta, t)\tilde{G}_k(Y^\circ - Z^\circ)|^2\big]$$

$$+ 2\eta \mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t + \delta, t)(Y^\circ - Z^\circ + f^+(Y^\circ)\delta - f^+(Z^\circ)\delta)|^2].$$

By Lemma 1.5.2 we get

$$\mathbb{E}[|\Phi_1(t + \delta, t)(Y^\circ - Z^\circ + f^+(Y^\circ)\delta - f^+(Z^\circ)\delta)|^2]$$
$$\leq e^{2\alpha_{S,+}\delta} \mathbb{E}[|Y^\circ - Z^\circ + \delta(f^+(Y^\circ) - f^+(Z^\circ)|^2].$$

The similar estimate applies to the second summand

$$2\eta\delta \sum_{k=2}^{m} \mathbb{E}\big[|\Phi_1(t+\delta, t)\tilde{G}_k(Y^\circ - Z^\circ)|^2\big]$$

$$\leq 2\eta\delta e^{2\alpha_{S,+}\delta} \sum_{k=2}^{m} \mathbb{E}[|\tilde{G}_k(Y^\circ - Z^\circ)|^2].$$

Then by Lemma 3.4.3 we obtain

$$e^{2\alpha_{S,+}\delta}\Big(\mathbb{E}\big[|Y^\circ - Z^\circ + \delta(f^+(Y^\circ) - f^+(Z^\circ))|^2$$

$$+ 2\eta\delta \sum_{k=2}^{m} |\tilde{G}_k(Y^\circ - Z^\circ)|^2\big]\Big)$$

$$\leq e^{2\alpha_{S,+}\delta}(1 + K\delta)\mathbb{E}[|Y - Z|^2]$$

$$\leq e^{(2\alpha_{S,+}+K)\delta}\mathbb{E}[|Y - Z|^2],$$

where K is given by (3.30). For the last summand we obtain by Lemma 1.5.4 and Lemma 3.4.3

$$\mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t + \delta, t)(Y^\circ - Z^\circ + f^+(Y^\circ)\delta - f^+(Z^\circ)\delta)|^2]$$

$$\leq K_{\mathrm{cond}}(\delta)^2 \delta \mathbb{E}[|Y^\circ - Z^\circ + f^+(Y^\circ)\delta - f^+(Z^\circ)\delta|^2]$$

$$\leq K_{\mathrm{cond}}(\delta)^2 \delta(1 + K\delta)\mathbb{E}[|Y - Z|^2]$$

$$\leq K_{\mathrm{cond}}(\delta)^2 \delta e^{K\delta}\mathbb{E}[|Y - Z|^2],$$

where

$$K_{\text{cond}}(\delta) = |\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1)e^{\alpha_{s,+}\delta}.$$

Altogether, this shows that

$$\|\mathbb{E}[\Psi^{PBSNE}(Y,t,\delta) - \Psi^{PBSNE}(Z,t,\delta)|\mathcal{F}_t]\|^2_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \eta\|(\text{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{PBSNE}(Y,t,\delta) - \Psi^{PBSNE}(Z,t,\delta))\|^2_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq e^{C_{\text{stab}}\delta}\|Y - Z\|^2_{L^2(\Omega;\mathbb{R}^d)},$$

where

$$C_{\text{stab}} = 2\eta|\tilde{G}_1|^2(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1)^2 + K + 2\alpha_{S,+}. \tag{3.31}$$

$\square$

Thus, the constant C_{stab} is not of moderate type in the sense of Convention 1.3.4, compare Remark 2.2.8. In the preparation of the proof of consistency we consider the next lemma, which is an analog of Lemma 5.5 in [5].

Lemma 3.4.5. *Let f^+ satisfies Assumption 3.1.3 with $L^+, L_0^+, L_1^+ > 0$ and $q \in (1, \infty)$. Further, let the exact solution X to (3.1) satisfy $\sup_{t \in [0,T]} \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)} < \infty$. Then there exists a constant C such that*

$$\int_s^t \|f^+(X(\tau)) - f^+(X(s))\|_{L^2(\Omega;\mathbb{R}^d)}\, d\tau \leq C\big(1 + \sup_{t \in [0,T]} \|X(t)\|^{2q-1}_{L^{4q-2}(\Omega;\mathbb{R}^d)}\big)|t - s|^{\frac{3}{2}} \tag{3.32}$$

for all $0 \leq s < t \leq T$.

In addition, if condition (3.6) holds with $\eta = \frac{pq-1}{2}$ for $p \in [2, \infty)$ and $q \in [1, \infty)$ then (3.32) can be estimated by

$$\int_s^t \|f^+(X(\tau)) - f^+(X(s))\|_{L^2(\Omega;\mathbb{R}^d)}\, d\tau \leq C_1(t - s, t)|t - s|^{\frac{3}{2}}, \tag{3.33}$$

where $C_1 : \{(t,s) : 0 \leq s \leq t \leq T\} \to \mathbb{R}$ is of moderate size and given by

$$\begin{aligned}
C_1(\delta, \delta_1) = &\frac{2}{3}L_1^+\big(1 + 2(1 + \|X_0\|^{q-1}_{L^{4q-2}(\Omega;\mathbb{R}^d)})e^{(q-1)\alpha_f T}\big)\Big(\big(L_0 \\
&+ L_0(1 + \|X_0\|^q_{L^{4q-2}(\Omega;\mathbb{R}^d)})e^{q\alpha_f \delta_1}\big)\delta^{\frac{1}{2}} \\
&+ C_p|G|\|X_0\|_{L^{\frac{4q-2}{q}}(\Omega;\mathbb{R}^d)}e^{\alpha_f \delta_1}\Big)
\end{aligned} \tag{3.34}$$

for $0 \leq \delta \leq \delta_1 \leq T$. Here C_p denotes the Burkholder-Davis-Gundy constant.

Proof. For the proof of (3.32) we refer to [5]. From condition (3.14) and Hölder's inequality with exponents $\rho = \frac{2q-1}{q}, \rho' = \frac{2q-1}{q-1}$ we get

$$\|f^+(X(\tau))-f^+(X(s))\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq\|L_1^+(1+|X(\tau)|^{q-1}+|X(s)|^{q-1})|X(\tau)-X(s)|\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq L_1^+\big(1+2\sup_{t\in[0,T]}\|X(t)\|_{L^{2\rho'(q-1)}(\Omega;\mathbb{R}^d)}^{q-1}\big)\|X(\tau)-X(s)\|_{L^{2\rho}(\Omega;\mathbb{R}^d)}$$

We note that $2\rho'(q-1) = 4q-2$. By (3.9) for $p = 2\rho$ we obtain

$$\|X(\tau)-X(s)\|_{L^{2\rho}(\Omega;\mathbb{R}^d)} \leq\Big(\big(L_0+L_0(1+\|X_0\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^q)\big)\mathrm{e}^{q\alpha_{\mathrm{f}}t}\big)|t-s|^{\frac{1}{2}}$$
$$+C_p|G|\|X_0\|_{L^{\frac{4q-2}{q}}(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_{\mathrm{f}}t}\Big)|\tau-s|^{\frac{1}{2}}.$$

Following, by integration we find

$$\int_s^t\|f^+(X(\tau))-f^+(X(s))\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau$$
$$\leq L_1^+\big(1+2(1+\|X_0\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^{q-1})\mathrm{e}^{(q-1)\alpha_{\mathrm{f}}T}\big)\Big(\big(L_0$$
$$+L_0(1+\|X_0\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^q)\mathrm{e}^{q\alpha_{\mathrm{f}}t}\big)|t-s|^{\frac{1}{2}}$$
$$+C_p|G|\|X_0\|_{L^{\frac{4q-2}{q}}(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_{\mathrm{f}}t}\Big)\int_s^t|\tau-s|^{\frac{1}{2}}\,\mathrm{d}\tau$$
$$\leq\frac{2}{3}L_1^+\big(1+2(1+\|X_0\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^{q-1})\mathrm{e}^{(q-1)\alpha_{\mathrm{f}}T}\big)\Big(\big(L_0$$
$$+L_0(1+\|X_0\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^q)\mathrm{e}^{q\alpha_{\mathrm{f}}t}\big)|t-s|^{\frac{1}{2}}$$
$$+C_p|G|\|X_0\|_{L^{\frac{4q-2}{q}}(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_{\mathrm{f}}t}\Big)|t-s|^{\frac{3}{2}}.$$

This completes the proof.

$\square$

Now, we quote Lemma 6.5 from [5], which formalizes a method of proof originating from [24, Th.7.7].

Lemma 3.4.6. *For arbitrary $\beta \in (0,\infty)$ and $\delta \in (0,1]$ consider the mapping $\mathbb{R}^d \ni x \mapsto x^\circ \in \mathbb{R}^d$, which given by $x =: \min(1, \delta^{-\beta}|x|^{-1})x$. Let $L^+ \in (0,\infty)$, $\kappa \in [1,\infty)$ and let $\phi: \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping with*

$$|\phi(x)| \leq L^+(1+|x|^\kappa), \quad x \in \mathbb{R}^d.$$

For some $p \in (2,\infty)$ let $Y \in L^{p\kappa}(\Omega;\mathbb{R}^d)$. Then there exists a constant C_{dif} only depending on L^+ and p such that

$$\|\phi(Y) - \phi(Y^\circ)\|_{L^2(\Omega;\mathbb{R}^d)} \leq C_{\mathrm{dif}}(1+\|Y\|_{L^{p\kappa}(\Omega;\mathbb{R}^d)}^\kappa)^{\frac{p}{2}}\delta^{\frac{1}{2}\beta(p-2)\kappa}.$$

The following theorem investigates stochastic B-consistensy of PBSNE. We recall that the constant C_{dif} is given by

$$C_{\text{dif}} = \Big(\frac{2(2L^+)^p + p - 2}{p}\Big)^{\frac{1}{2}}. \tag{3.35}$$

Theorem 3.4.7. *Let f^+ satisfies Assumption 3.1.3 with $L^+, L_0^+, L_1^+ > 0$ and $q \in (1, \infty)$. Let $\bar{h} \in (0,1]$ be arbitrary. If the exact solution X satisfies $\sup_{t \in [0,T]} \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)} < \infty$, then the PBSNE Euler-type method $(\Psi^{PBSNE}, \bar{h}, X_0)$ is stochastically B-consistent of order $\gamma = \frac{1}{2}$. The constants $C_{\text{cons},1}$ and $C_{\text{cons},2}$ are of moderate exponential type, see (3.37).*

Proof. For arbitrary $(t, \delta) \in \mathbb{T}$ we get

$$
\begin{aligned}
X(t+\delta) - \Psi^{PBSNE}(X(t), t, \delta) = {}& \Phi_1(t+\delta, t)\big(X(t) - X^\circ(t)\big) \\
& + \int_t^{t+\delta} \Phi_1(t+\delta, \tau)\big(f^+(X(\tau)) - f^+(X(t))\big)\,\mathrm{d}\tau \\
& + \int_t^{t+\delta} \big(\Phi_1(t+\delta, \tau) - \Phi_1(t+\delta, t)\big)f^+(X(t))\,\mathrm{d}\tau \\
& + \Phi_1(t+\delta, t)\delta\big(f^+(X(t)) - f^+(X^\circ(t))\big) \\
& + \sum_{k=2}^m \int_t^{t+\delta} \Phi_1(t+\delta, \tau)\big(\tilde{G}_k X(\tau) - \tilde{G}_k X(t)\big)\,\mathrm{d}\tilde{W}_k(\tau) \\
& + \sum_{k=2}^m \int_t^{t+\delta} \big(\Phi_1(t+\delta, \tau) - \Phi_1(t+\delta, t)\big)\tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(\tau) \\
& + \sum_{k=2}^m \Phi_1(t+\delta, t)\big(\tilde{G}_k X(t) - \tilde{G}_k X^\circ(t)\big)\tilde{I}_{(k)}^{t,t+\delta},
\end{aligned}
$$

with $X^\circ(t) = \min(1, \delta^{-\beta}|X(t)|^{-1})X(t)$. By (2.5) and (2.6) we obtain

$$
\begin{aligned}
\big\|\mathbb{E}&[X(t+\delta) - \Psi^{PBSNE}(X(t), t, \delta)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq {}& \big\|\mathbb{E}[\Phi_1(t+\delta, t)(X(t) - X^\circ(t))|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
& + \int_t^{t+\delta} \big\|\mathbb{E}[\Phi_1(t+\delta, \tau)\big(f^+(X(\tau)) - f^+(X(t))\big)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau \\
& + \int_t^{t+\delta} \big\|\mathbb{E}[\big(\Phi_1(t+\delta, \tau) - \Phi_1(t+\delta, t)\big)f^+(X(t))|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau \\
& + \delta\big\|\mathbb{E}[\Phi_1(t+\delta, t)\big(f^+(X(t)) - f^+(X^\circ(t))\big)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
=: {}& \sum_{j=1}^4 S_j,
\end{aligned}
$$

and

$$\|(\mathrm{id}-\mathbb{E}[\cdot|\mathcal{F}_t])(X(t+\delta) - \Psi^{PBSNE}(X(t),t,\delta))\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t+\delta,t)(X(t) - X^\circ(t))\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \int_t^{t+\delta} \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t+\delta,\tau)(f^+(X(\tau)) - f^+(X(t)))\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}\tau$$
$$+ \int_t^{t+\delta} \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t))f^+(X(t)))\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}\tau$$
$$+ \delta\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t+\delta,t)(f^+(X(t)) - f^+(X^\circ(t)))\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \Big\|\sum_{k=2}^m \int_t^{t+\delta} \Phi_1(t+\delta,\tau)(\tilde{G}_k X(\tau) - \tilde{G}_k X(t)) \,\mathrm{d}\tilde{W}_k(\tau)\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \Big\|\sum_{k=2}^m \int_t^{t+\delta} (\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t))\tilde{G}_k X(t) \,\mathrm{d}\tilde{W}_k(\tau)\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \Big\|\sum_{k=2}^m \Phi_1(t+\delta,t)(\tilde{G}_k X(t) - \tilde{G}_k X^\circ(t))\tilde{I}_{(k)}^{t,t+\delta}\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$=: \sum_{j=1}^7 T_j.$$

By applying (1.56) and Lemma 3.4.6 with $\varphi = \mathrm{id}, \kappa = 1, p = 6q - 4$, and $\beta = \frac{1}{2(q-1)}$ we obtain

$$S_1 \leq |\mathrm{e}^{-\frac{1}{2}\tilde{G}_1^2\delta}| \|X(t) - X^\circ(t)\|_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq \mathrm{e}^{\alpha_1\delta} C_{\mathrm{dif}}(1 + \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)}^{3q-2})\delta^{\frac{3}{2}},$$

where C_{dif} is given by (3.35). From Lemma 1.5.2, Lemma 3.4.5, and the fact that $\|\mathbb{E}[Z|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$ we obtain for S_2

$$S_2 \leq \int_t^{t+\delta} \|\Phi_1(t+\delta,\tau)(f^+(X(\tau)) - f^+(X(t)))\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}\tau$$
$$\leq \int_t^{t+\delta} \mathrm{e}^{\alpha_{\mathrm{S},+}(t+\delta-\tau)} \|f^+(X(\tau)) - f^+(X(t))\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}\tau$$
$$\leq \mathrm{e}^{\alpha_{\mathrm{S},+}\delta} C(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^{2q-1})\delta^{\frac{3}{2}}.$$

Further, by Lemma 1.5.5 and condition (3.13) we obtain for the third term

$$
\begin{aligned}
S_3 &\leq \int_t^{t+\delta} \|\big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t)\big) f^+(X(t)))\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau \\
&\leq L_0^+ \|1 + |X(t)|^q\|_{L^2(\Omega;\mathbb{R}^d)} |\tilde{G}_1| (\tfrac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1) \mathrm{e}^{\alpha_{\mathrm{S},+}\delta} \int_t^{t+\delta} |\tau - t|^{\frac{1}{2}} \, \mathrm{d}\tau \qquad (3.36) \\
&\leq L_0^+ (1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{2q}(\Omega;\mathbb{R}^d)}^q) |\tilde{G}_1| (\tfrac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1) \mathrm{e}^{\alpha_{\mathrm{S},+}\delta} \delta^{\frac{3}{2}}.
\end{aligned}
$$

The terms $f^+(X(t))$ and $f^+(X^\circ(t))$ are $\mathcal{F}_t$-measurable and we apply estimate (1.56) and Lemma 3.4.6 with $\varphi = f^+(\cdot)$, $\kappa = q$, $p = \frac{4q-2}{q}$. This yields

$$
\begin{aligned}
S_4 &\leq \delta |\mathrm{e}^{-\frac{1}{2}\tilde{G}_1^2\delta}| \|f^+(X(t)) - f^+(X^\circ(t))\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq \delta \mathrm{e}^{\alpha_1\delta} \|f^+(X(t)) - f^+(X^\circ(t))\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq C_{\mathrm{dif}} \mathrm{e}^{\alpha_1\delta} (1 + \|X(t)\|_{L^{4q-2}}^{2q-1}) \delta^{\frac{3}{2}}.
\end{aligned}
$$

Further, using Lemma 1.5.4 and Lemma 3.4.6 we get

$$
\begin{aligned}
T_1 &\leq K_{\mathrm{cond}}(\delta)\delta^{\frac{1}{2}} \|X(t) - X^\circ(t)\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq |\tilde{G}_1| (\tfrac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1) \mathrm{e}^{\alpha_{\mathrm{S},+}\delta} C_{\mathrm{dif}} (1 + \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)}^{3q-2}) \delta^2.
\end{aligned}
$$

By inequality $\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])Z\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$ we obtain that

$$
T_2 \leq \int_t^{t+\delta} \|\Phi_1(t+\delta,\tau)(f^+(X(\tau)) - f^+(X(t)))\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau
$$

and

$$
T_3 \leq \int_t^{t+\delta} \|\big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t)\big) f^+(X(t)))\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}\tau.
$$

The estimates of T_2 and T_3 are similar to the estimates of the terms S_2 and S_3. Further, by Lemma 1.5.4 and Lemma 3.4.6 with $\varphi = f^+(\cdot)$, $\kappa = q$, $p = \frac{4q-2}{q}$ we get

$$
\begin{aligned}
T_4 &\leq \delta K_{\mathrm{cond}}(\delta)\delta^{\frac{1}{2}} \|f^+(X(t)) - f^+(X^\circ(t))\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq |\tilde{G}_1| (\tfrac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1) \mathrm{e}^{\alpha_{\mathrm{S},+}\delta} C_{\mathrm{dif}} (1 + \|X(t)\|_{L^{4q-2}}^{2q-1})) \delta^2.
\end{aligned}
$$

The terms T_5 and T_6 were already estimated in Section 2.2, Lemma 2.2.3 and obey the following estimates

$$
T_5 \leq \mathcal{K}_4(\delta, t+\delta) \|X_0\|_{L^2(\Omega;\mathbb{R}^d)} \delta,
$$

$$T_6 \leq \mathcal{K}_5(\delta, t)\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\delta,$$

where

$$\mathcal{K}_4(\delta, t + \delta) = |\tilde{G}^-|(|A|\delta^{\frac{1}{2}} + |G|) \exp\left(\alpha_{\mathrm{S},+}\delta + \alpha_+(t + \delta)\right),$$

$$\mathcal{K}_5(\delta, t) = |\tilde{G}^-||\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1) \exp\left(\alpha_+ t + \alpha_{\mathrm{S},+}\delta\right).$$

Finally, using the Itô isometry yields

$$T_7 = \left(\delta \sum_{k=2}^m \|\Phi_1(t + \delta, t)\tilde{G}_k\big(X(t) - X^\circ(t)\big)\|_{L^2(\Omega;\mathbb{R}^d)}^2\right)^{\frac{1}{2}}.$$

As above we use Lemma 1.5.2 and Lemma 3.4.6 with $\varphi = \mathrm{id}$, $\kappa = 1$, $p = 6q - 4$ and obtain

$$T_7 \leq |\tilde{G}^-|\mathrm{e}^{\alpha_{\mathrm{S},+}\delta}C_{\mathrm{dif}}(1 + \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)}^{3q-2})\delta^2.$$

This completes the proof.

$\square$

Remark 3.4.8. The constants $C_{\mathrm{cons},1}$ and $C_{\mathrm{cons},2}$ in (2.5) and (2.6) are given by

$$\begin{aligned}
C_{\mathrm{cons},1} =& \mathrm{e}^{\alpha_1 \bar{h}}C_{\mathrm{dif}}(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)}^{3q-2}) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}}C(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^{2q-1}) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1)L_0^+(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{2q}(\Omega;\mathbb{R}^d)}^q), \\
C_{\mathrm{cons},2} =& \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}C_{\mathrm{dif}}(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)}^{3q-2})\big(|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1) + |\tilde{G}^-|\big) \quad (3.37) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}|\tilde{G}^-|\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\mathrm{e}^{\alpha_+ T}\big(|A|\bar{h}^{\frac{1}{2}} + |G| + |\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1)\big) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}\big(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^{2q-1}\big)\big(C_{\mathrm{dif}}|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1) + C\big) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}L_0^+(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{2q}(\Omega;\mathbb{R}^d)}^q)|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1).
\end{aligned}$$

3.5. Stochastic C-stability and B-consistency of the SSBSNI method

Our next aim is to show that the split-step balanced shift noise implicit Euler-type scheme is convergent with order $\gamma = \frac{1}{2}$. Let us first show that the SSBSNI Euler-type method is a stochastic one-step method in the sense of the Definition 2.1.1.

Consider an arbitrary vector $h \in (0, \bar{h}]^N$ with $\bar{h} \in (0, \frac{1}{L^+})$. By Corollary 3.2.2 there exists a homeomorphism $F_{h_i}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\hat{X}_h(t_i) = F_{h_i}^{-1}(\overline{X}_h(t_{i-1}))$$

is a solution of

$$\overline{X}_h(t_i) = X_h(t_{i-1}) + f^+(\hat{X}_h(t_i))h_i$$

for all $i = 1, \ldots, N$. With this observation the split-step balanced shift noise implicit Euler-type method becomes a one-step method in the sense of Definition 2.1.1 by defining $\Psi^{SSBSNI} : \mathbb{R}^d \times \mathbb{T} \times \Omega \to \mathbb{R}^d$ as follows

$$\Psi^{SSBSNI}(x, t, \delta) = \Phi_1(t + \delta, t)F_\delta^{-1}(x) + \sum_{k=2}^{m} \Phi_1(t + \delta, t)\tilde{G}_k F_\delta^{-1}(x)\tilde{I}_{(k)}^{t,t+\delta}, \qquad (3.38)$$

where $\Phi_1(t + \delta, t) = \exp\left(-\tilde{G}_1^2 \delta + \tilde{G}_1 \tilde{I}_{(1)}^{t,t+\delta} \right)$, $(t, \delta) \in \mathbb{T}$ and $x \in \mathbb{R}^d$.

Proposition 3.5.1. *Let Assumption 3.1.3 hold with $L^+, L_0^+, L_1^+ > 0$ and $q \in (1, \infty)$ and let $\bar{h} \in (0, \frac{1}{L^+})$. For all initial value $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ the tuple $(\Psi^{SSBSNI}, \bar{h}, \xi)$ defines a stochastic one-step method.*

Proof. Let $(t, \delta) \in \mathbb{T}$ and $Y \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$. By Corollary 3.2.2 the mapping $F_\delta^{-1}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the linear growth condition (3.18). It follows

$$F_\delta^{-1}(Y) \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d).$$

Further, by Lemma 1.5.2 we have

$$\Phi_1(t + \delta, t)F_\delta^{-1}(Y) \in L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbf{P}; \mathbb{R}^d).$$

Therefore, $\Psi^{SSBSNI}(Y, t, \delta) : \Omega \to \mathbb{R}^d$ is a well-defined, $\mathcal{F}_{t+\delta}$-measurable random variable in $L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbf{P}; \mathbb{R}^d)$, which satisfies (2.2).

$\square$

The following theorem investigates stochastic C-stability of the split-step balanced shift noise implicit Euler-type method.

Theorem 3.5.2. *Let Assumption 3.1.3 hold with $L^+, L_0^+, L_1^+ > 0$, $q \in (1, \infty)$ and $\eta \in (\frac{1}{2}, \infty)$ and let $\bar{h} \in (0, \frac{1}{L^+})$. Then for all $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ the SSBSNI Euler-type method $(\Psi^{SSBSNI}, \bar{h}, \xi)$ is stochastically C-stable. The constant C_{stab} in (2.4) depends on data $\alpha_{\text{S},+}, \|\tilde{G}_1\|, L^+, C_{\text{H}}, \eta$ and $\bar{h}$, see (3.39).*

Proof. Let fix arbitrary $(t, \delta) \in \mathbb{T}$ and consider $Y, Z \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$. Then we get

$$\mathbb{E}[\Psi^{SSBSNI}(Y, t, \delta) - \Psi^{SSBSNI}(Z, t, \delta) | \mathcal{F}_t] = \mathbb{E}[\Phi_1(t + \delta, t)(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)) | \mathcal{F}_t]$$

and

$$\begin{aligned}
\big(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t]\big)&\big(\Psi^{SSBSNI}(Y, t, \delta) - \Psi^{SSBSNI}(Z, t, \delta)\big) \\
&= \big(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t]\big)\Phi_1(t + \delta, t)(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)) \\
&\quad + \sum_{k=2}^m \Phi_1(t + \delta, t)\tilde{G}_k(F_\delta^{-1}(Y) - F_\delta^{-1}(Z))\tilde{I}_{(k)}^{t,t+\delta}.
\end{aligned}$$

Further, by (2.4), the Itô isometry, and since $\mathbb{E}[|\mathbb{E}[Z|\mathcal{F}_t]|^2] \leq \mathbb{E}[|Z|^2]$ for all $Z \in L^2(\Omega; \mathbb{R}^d)$ we have

$$\begin{aligned}
\mathbb{E}\big[\big|\mathbb{E}[\Phi_1(t + \delta, t)&(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)) | \mathcal{F}_t]\big|^2\big] \\
&+ \eta\mathbb{E}\big[\big|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t + \delta, t)\big(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)\big) \\
&\quad + \sum_{k=2}^m \Phi_1(t + \delta, t)\tilde{G}_k\big(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)\big)\tilde{I}_{(k)}^{t,t+\delta}\big|^2\big] \\
\leq& \mathbb{E}\big[\big|\Phi_1(t + \delta, t)(F_\delta^{-1}(Y) - F_\delta^{-1}(Z))\big|^2\big] \\
&+ 2\eta\mathbb{E}\big[\big|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t + \delta, t)\big(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)\big)\big|^2\big] \\
&+ 2\eta\delta\sum_{k=2}^m \mathbb{E}\big[\big|\Phi_1(t + \delta, t)\tilde{G}_k\big(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)\big)\big|^2\big].
\end{aligned}$$

By Lemma 1.5.2 and Corollary 3.2.2 we obtain

$$\begin{aligned}
\mathbb{E}[|\Phi_1(t + \delta, t)&(F_\delta^{-1}(Y) - F_\delta^{-1}(Z))|^2] \\
&+ 2\eta\delta\sum_{k=2}^m \mathbb{E}\big[\big|\Phi_1(t + \delta, t)\tilde{G}_k\big(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)\big)\big|^2\big] \\
\leq& \mathrm{e}^{2\alpha_{\mathrm{S},+}\delta}\mathbb{E}[|F_\delta^{-1}(Y) - F_\delta^{-1}(Z)|^2] \\
&+ 2\eta\delta\mathrm{e}^{2\alpha_{\mathrm{S},+}\delta}\sum_{k=2}^m \mathbb{E}[|\tilde{G}_k(F_\delta^{-1}(Y) - F_\delta^{-1}(Z))|^2] \\
\leq& \mathrm{e}^{2\alpha_{\mathrm{S},+}\delta}(1 + C_{\mathrm{H}}\delta)\mathbb{E}[|Y - Z|^2] \\
\leq& \mathrm{e}^{(2\alpha_{\mathrm{S},+} + C_{\mathrm{H}})\delta}\mathbb{E}[|Y - Z|^2],
\end{aligned}$$

where C_{H} is given by (3.20). It remains to show that the last term has a sharper estimate. By Lemma 1.5.4 and (3.17) we get

$$\begin{aligned}
\mathbb{E}\big[\big|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])&(\Phi_1(t + \delta, t)\big(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)\big)\big|^2\big] \\
&\leq K_{\mathrm{cond}}^2(\delta)\delta\mathbb{E}[|F_\delta^{-1}(Y) - F_\delta^{-1}(Z)|^2]
\end{aligned}$$

$$\leq |\tilde{G}_1|^2 (\frac{1}{2}|\tilde{G}_1|\delta + 1)^2 \mathrm{e}^{2\alpha_{\mathrm{S},+}\delta}(1 - L^+\delta)^{-2}\delta \mathbb{E}[|Y - Z|^2].$$

Since $(1 - L^+\delta)^{-2}$ is convex, then for all $\delta \in (0, \bar{h}]$ (see Section 2.3)

$$(1 - L^+\delta)^{-2} =\leq 1 + C_{\mathrm{H}}\delta \leq \mathrm{e}^{C_{\mathrm{H}}\delta}.$$

Hence, we obtain

$$\mathbb{E}\big[\big|(\mathrm{id}-\mathbb{E}[\cdot|\mathcal{F}_t])(\Phi_1(t + \delta, t)\big(F_\delta^{-1}(Y) - F_\delta^{-1}(Z)\big)\big|^2\big]$$
$$\leq \delta|\tilde{G}_1|^2(\frac{1}{2}|\tilde{G}_1|\delta + 1)^2 \exp\big((2\alpha_{\mathrm{S},+} + C_{\mathrm{H}})\delta\big)\mathbb{E}[|Y - Z|^2].$$

Altogether, this shows that

$$\|\mathbb{E}[\Psi^{SSBSNI}(Y, t, \delta) - \Psi^{SSBSNI}(Z, t, \delta)|\mathcal{F}_t]\|^2_{L^2(\Omega;\mathbb{R}^d)}$$
$$+ \eta\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{SSBSNI}(Y, t, \delta) - \Psi^{SSBSNI}(Z, t, \delta))\|^2_{L^2(\Omega;\mathbb{R}^d)}$$
$$\leq \mathrm{e}^{C_{\mathrm{stab}}\delta}\|Y - Z\|^2_{L^2(\Omega;\mathbb{R}^d)},$$

where

$$C_{\mathrm{stab}} = 2\eta|\tilde{G}_1|^2(\frac{1}{2}|\tilde{G}_1|\bar{h} + 1)^2 + 2\alpha_{\mathrm{S},+} + C_{\mathrm{H}} \tag{3.39}$$

with C_{H} given by (3.20).

$\square$

The constant C_{stab} is not of moderate type, see Remark 2.2.8. In the following theorem we prove B-consistency of the SSBSNI Euler-type scheme.

Theorem 3.5.3. *Let f^+ satisfies Assumption 3.1.3 with $L^+, L_0^+, L_1^+ > 0$ and $q \in (1, \infty)$ and let $\bar{h} \in (0, \frac{1}{L^+})$. If the exact solution X satisfies $\sup_{t\in[0,T]} \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)} < \infty$, then the SSBSNI Euler-type method $(\Psi^{SSBSNI}, \bar{h}, X_0)$ is stochastically B-consistent of order $\gamma = \frac{1}{2}$. The constants $C_{\mathrm{cons},1}$ and $C_{\mathrm{cons},2}$ are of moderate exponential type, see (3.40).*

Proof. By inserting (3.26) and (3.38) for arbitrary $(t, \delta) \in \mathbb{T}$ we obtain

$$X(t + \delta)-\Psi^{SSBSNI}(X(t), t, \delta) = \int_t^{t+\delta} \Phi_1(t + \delta, \tau)\big(f^+(X(\tau)) - f^+(X(t))\big)\,\mathrm{d}\tau$$
$$+ \int_t^{t+\delta} \big(\Phi_1(t + \delta, \tau) - \Phi_1(t + \delta, t)\big)f^+(X(t))\,\mathrm{d}\tau$$
$$+ \Phi_1(t + \delta, t)\big(X(t) + f^+(X(t))\delta - F_\delta^{-1}(X(t))\big)$$
$$+ \sum_{k=2}^m \int_t^{t+\delta} \Phi_1(t + \delta, \tau)\tilde{G}_k\big(X(\tau) - X(t)\big)\,\mathrm{d}\tilde{W}_k(\tau)$$

$$+ \sum_{k=2}^{m} \int_t^{t+\delta} \big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t)\big)\tilde{G}_k X(t)\, \mathrm{d}\tilde{W}_k(\tau)$$

$$+ \sum_{k=2}^{m} \Phi_1(t+\delta,t)\tilde{G}_k\big(X(t) - F_\delta^{-1}(X(t))\big)\tilde{I}_{(k)}^{t,t+\delta}.$$

For the proof of (2.5) we have to estimate three summands:

$$\big\|\mathbb{E}[|X(t+\delta) - \Psi^{SSBSNI}(X(t),t,\delta)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq \int_t^{t+\delta} \big\|\mathbb{E}[\Phi_1(t+\delta,\tau)\big(f^+(X(\tau) - f^+(X(t)))\big)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)}\, \mathrm{d}\tau$$

$$+ \int_t^{t+\delta} \big\|\mathbb{E}[\big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t)\big)f^+(X(t))|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)}\, \mathrm{d}\tau$$

$$+ \big\|\mathbb{E}[\Phi_1(t+\delta,t)\big(X(t) + f^+(X(t))\delta - F_\delta^{-1}(X(t))\big)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$=: \sum_{j=1}^{3} S_j.$$

Since the estimate of S_1 and S_2 has been done in the section 3.4, Theorem 3.4.7 it remains to show that the estimate of the summand S_3 satisfies (2.5). By (1.56) and Lemma 3.2.3 we obtain

$$S_3 \leq \mathrm{e}^{\alpha_1\delta}C_{\mathrm{L}_1}\delta^2\|1 + |X(t)|^{2q-1}\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}$$

$$\leq \mathrm{e}^{\alpha_1\delta}C_{\mathrm{L}_1}\delta^2\big(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^{2q-1}\big),$$

where C_{L_1} is given by (3.24). Further, the proof of (2.6) is obtained as follows

$$\big\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\big(X(t+\delta) - \Psi^{SSBSNI}(X(t),t,\delta)\big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq \int_t^{t+\delta} \big\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t+\delta,\tau)\big(f^+(X(\tau)) - f^+(X(t))\big)\big\|_{L^2(\Omega;\mathbb{R}^d)}\, \mathrm{d}\tau$$

$$+ \int_t^{t+\delta} \big\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t)\big)f^+(X(t))\big\|_{L^2(\Omega;\mathbb{R}^d)}\, \mathrm{d}\tau$$

$$+ \big\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Phi_1(t+\delta,t)\big(X(t) + f^+(X(t))\delta - F_\delta^{-1}(X(t))\big)\big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \Big\|\sum_{k=2}^{m} \int_t^{t+\delta} \Phi_1(t+\delta,\tau)\tilde{G}_k\big(X(\tau) - X(t)\big)\, \mathrm{d}\tilde{W}_k(\tau)\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \Big\|\sum_{k=2}^{m} \int_t^{t+\delta} \big(\Phi_1(t+\delta,\tau) - \Phi_1(t+\delta,t)\big)\tilde{G}_k X(t)\, \mathrm{d}\tilde{W}_k(\tau)\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \Big\|\sum_{k=2}^{m} \Phi_1(t+\delta,t)\tilde{G}_k\big(X(t) - F_\delta^{-1}(X(t))\big)\tilde{I}_{(k)}^{t,t+\delta}\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$=: \sum_{j=1}^{6} T_j.$$

The estimates of T_1, T_2, T_4 and T_5 have already been shown in Theorem 3.4.7. Therefore, we consider only T_3 and T_6. Using Lemma 1.5.4 and Lemma 3.2.3 yields

$$
\begin{aligned}
T_3 &\leq K_{\mathrm{cond}}(\delta)\delta^{\frac{1}{2}}C_{\mathrm{L}_1}\delta^2\|1 + |X(t)|^{2q-1}\|_{L^{4q-2}(\Omega;\mathbb{R}^d)} \\
&\leq |\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\delta^{\frac{1}{2}} + 1)\mathrm{e}^{\alpha_{\mathrm{S},+}\delta}C_{\mathrm{L}_1}\big(1 + \sup_{t\in[0,T]}\|X(t)\|^{2q-1}_{L^{4q-2}(\Omega;\mathbb{R}^d)}\big)\delta^{\frac{5}{2}}.
\end{aligned}
$$

Finally, by the Itô isometry, Lemma 1.5.2, and Lemma 3.2.3 we obtain

$$
\begin{aligned}
T_6 &= \Big(\delta\sum_{k=2}^{m}\|\Phi_1(t+\delta,t)\tilde{G}_k\big(X(t) - F_\delta^{-1}(X(t))\big)\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big)^{\frac{1}{2}} \\
&\leq |\tilde{G}^-|\mathrm{e}^{\alpha_{\mathrm{S},+}\delta}C_{\mathrm{L}_0}\|1 + |X(t)|^q\|_{L^{2q}}\delta \\
&\leq |\tilde{G}^-|\mathrm{e}^{\alpha_{\mathrm{S},+}\delta}C_{\mathrm{L}_0}(1 + \sup_{t\in[0,T]}\|X(t)\|^q_{L^{2q}})\delta,
\end{aligned}
$$

where C_{L_0} is given by (3.23). Since $2q \leq 4q - 2$ for $q \geq 1$, this completes the proof. $\qquad\square$

Remark 3.5.4. The constants $C_{\mathrm{cons},1}$ and $C_{\mathrm{cons},2}$ in (2.5) and (2.6) are given by

$$
\begin{aligned}
C_{\mathrm{cons},1} =&(C\mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}} + C_{\mathrm{L}_1}\mathrm{e}^{\alpha_1\bar{h}})(1 + \sup_{t\in[0,T]}\|X(t)\|^{2q-1}_{L^{4q-2}(\Omega;\mathbb{R}^d)}) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}L_0^+(1 + \sup_{t\in[0,T]}\|X(t)\|^q_{L^{2q}(\Omega;\mathbb{R}^d)})|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1), \\
C_{\mathrm{cons},2} =&\mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}\big(1 + \sup_{t\in[0,T]}\|X(t)\|^{2q-1}_{L^{4q-2}(\Omega;\mathbb{R}^d)}\big)\big(C_{\mathrm{L}_1}|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1) + C\big) \qquad (3.40) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}(1 + \sup_{t\in[0,T]}\|X(t)\|^q_{L^{2q}(\Omega;\mathbb{R}^d)})(L_0^+|\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1) + C_{\mathrm{L}_0}|\tilde{G}^-|) \\
&+ \mathrm{e}^{\alpha_{\mathrm{S},+}\bar{h}}\mathrm{e}^{\alpha_+T}\|X_0\|_{L^2(\Omega;\mathbb{R}^d)}\big(|A|\bar{h}^{\frac{1}{2}} + |\tilde{G}_1|(\frac{1}{2}|\tilde{G}_1|\bar{h}^{\frac{1}{2}} + 1)\big),
\end{aligned}
$$

where C_{L_0} and C_{L_1} are given by (3.23) and (3.24), respectively.

4. Balanced higher order methods

So far, the focus of the thesis has been to isolate the largest noise term and to analyze of the numerical methods, in which the largest noise is integrated by an extra step. Of course, the question arises whether this approach can be used to implement the methods of higher order, $\gamma > \frac{1}{2}$. At the beginning, we shall introduce a Milstein version of our PBSNE method, in which the intermediate step is realized by a method of higher order. However, this does not lead to a higher order method in the general case, since the split into steps avoids evaluating double stochastic integrals. But these are essential for a higher order method (see [46]).

For the reasons stated above, we analyze in this chapter the classical balanced method proposed in [44] and [32]. For this method a higher order is achievable, if it is assumed that double stochastic integrals can be evaluated accurately. However, there is no systematic rule how to determine weight functions within a damping matrix.

In Chapter 5 we will present some numerical experiments that show in the case of commutative noise (see [36]) both approaches are suitable to approximate stiff stochastic differential equations.

4.1. Projected balanced shift noise Milstein-type method

In this section we extend our previous methods from Section 3.1 by using a higher order approximation for the intermediate step. Consider the same situation as in Chapter 3 with the stochastic differential equation

$$
\begin{aligned}
\mathrm{d}X(t) =& f(t, X(t))\,\mathrm{d}t + \sum_{k=1}^{m} \tilde{G}_k X(t)\,\mathrm{d}\tilde{W}_k(t), \\
X(0) =& X_0, \quad t \in [0, T],
\end{aligned}
\tag{4.1}
$$

where $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{G}_k \in \mathbb{R}^{d \times d}, k = 1, \ldots, m$ are given by (1.19). In the following we denote double Itô integrals as in [36]

$$
I_{(r_1, r_2)}^{s,t} = \int_s^t \int_s^z \mathrm{d}W_{r_1}(\tau)\,\mathrm{d}W_{r_2}(z),
$$

for all $0 \leq s < t \leq T$.

Furthermore, let $h \in (0, \bar{h}]^N$ be an arbitrary vector with $\bar{h} \in (0, 1]$ and let $\beta \in (0, \infty)$. Then the projected balanced shift noise Milstein-type scheme (PBSNM) is given by the three-step recursion

$$
\begin{aligned}
X_h^\circ(t_i) :=& \min(1, h_i^{-\beta} |X_h(t_{i-1})|^{-1}) X_h(t_{i-1}), \\
\overline{X}_h(t_i) =& X_h^\circ(t_i) + f^+(X_h^\circ(t_i)) h_i + \sum_{k=2}^{m} \tilde{G}_k X_h^\circ(t_i) \tilde{I}_{(k)}^{t_{i-1}, t_i} + \sum_{k_1, k_2 = 2}^{m} \tilde{G}_{k_1} \tilde{G}_{k_2} X_h^\circ(t_i) \tilde{I}_{(k_1, k_2)}^{t_{i-1}, t_i}, \\
X_h(t_i) =& \Phi_1(t_i, t_{i-1}) \overline{X}_h(t_i), \\
X_h(0) =& X_0,
\end{aligned}
\tag{4.2}
$$

for all $i = 1, \dots, N$.

The numerical analysis of this scheme has not yet been carried out in detail, but we will use it for comparison in our numerical simulations, see Chapter 5.

4.2. The classical balanced Milstein method

Our interest is to verify that the classical balanced Milstein method (BMM), proposed in [32], fits into our convergence concept from Section 2.1 provided the nonlinearity satisfies a one-sided Lipschitz condition as in (1.2).

Let us consider the stochastic ordinary differential equation

$$
\begin{aligned}
\mathrm{d}X(t) =& f(t, X(t)) \, \mathrm{d}t + \sum_{r=1}^{m} g_r(t, X(t)) \, \mathrm{d}W_r(t), \\
X(0) =& X_0, \quad t \in [0, T],
\end{aligned}
\tag{4.3}
$$

where $f, g_r \colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, r = 1, \dots, m$ are drift and diffusion coefficient functions and $W_r, r = 1, \dots, m$ are independent standard real Wiener processes.

For an arbitrary vector $h \in (0, \bar{h}]^N$ with $\bar{h} \in (0, 1]$ the approximation to (4.3) is given by

$$
\begin{aligned}
X_h(t_i) =& X_h(t_{i-1}) + f(t_{i-1}, X_h(t_{i-1})) h_i \\
&+ \sum_{r=1}^{m} g^r(t_{i-1}, X_h(t_{i-1})) I_{(r)}^{t_{i-1}, t_i} + \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(t_{i-1}, X_h(t_{i-1})) I_{(r_1, r_2)}^{t_{i-1}, t_i} \\
&+ \left(d^0(t_{i-1}, X_h(t_{i-1})) h_i + \sum_{l=1}^{m} d^l(t_{i-1}, X_h(t_{i-1})) I_{(l,l)}^{t_{i-1}, t_i} \right) \left(X_h(t_{i-1}) - X_h(t_i) \right),
\end{aligned}
\tag{4.4}
$$

for all $i = 1, \dots, N$. This is called the balanced MIlstein method (see [32]). In (4.4) we use the same notation as in [6] for some coefficients of stochastic Taylor expansions:

$$
g^{r_1, r_2}(t, x) := \frac{\partial g^{r_1}}{\partial x}(t, x) g^{r_2}(t, x),
\tag{4.5}
$$

for all $x \in \mathbb{R}^d$ and $t \in [0, T]$. Furthermore, $d^l, l = 0, \ldots, m$ denote $d \times d$-matrix valued weight functions. This scheme can be rewritten in one-step form as follows

$$
\begin{aligned}
X_h(t_i) =& X_h(t_{i-1}) + M_{X_h, t_{i-1}, t_i}^{-1} \Big(f(t_{i-1}, X_h(t_{i-1})) h_i \\
& + \sum_{r=1}^{m} g^r(t_{i-1}, X_h(t_{i-1})) I_{(r)}^{t_{i-1}, t_i} + \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(t_{i-1}, X_h(t_{i-1})) I_{(r_1, r_2)}^{t_{i-1}, t_i} \Big),
\end{aligned} \tag{4.6}
$$

with the damping matrix

$$
M(X_h, t_{i-1}, t_i) = \mathrm{id} + d^0(t_{i-1}, X_h(t_{i-1})) h_i + \sum_{l=1}^{m} d^l(t_{i-1}, X_h(t_{i-1})) I_{(l,l)}^{t_{i-1}, t_i}. \tag{4.7}
$$

This method was first studied by C. Kahl and H.Schurz in [32]. They have shown under global Lipschitz condition the global mean square convergence of the balanced Milstein method. We note, however that some of estimates there are not correct as written, see Remark 4.4.8 and Remark 4.5.6 below.

In fact, strong convergence of order 1 is true under global Lipschitz condition and even more can be shown: in [31] it is proved that under these conditions the BMM scheme is bistable and consistent of order 1 which implies strong convergence. Here we are interested in the cases where the global Lipschitz condition is not fulfilled. The one-sided Lipschitz condition weakens our assumptions on the function f by using a projection onto a ball in $\mathbb{R}^d$ whose radius is expanding with a negative power of the step size (see in Section 3.1). Moreover, we assume that the matrices $d^l \in \mathbb{R}^{d \times d}, l = 0, \ldots, m$ are constant. It is possible to treat matrices which depend on $X_h(t_i)$ in a Lipschitz continuous way, but this involves tricky calculations which we try to avoid here.

Now, we consider a modified form of the classical balanced Milstein method: Let $h \in (0, \bar{h}]^N$ be an arbitrary vector of step sizes and $\bar{h} \in (0, 1]$. Then for a given parameter $\beta \in (0, \infty)$ the projected balanced Milstein method (PBMM) is defined by

$$
\begin{aligned}
\overline{X}_h(t_i) =& \min(1, h_i^{-\beta} |X_h(t_{i-1})|^{-1}) X_h(t_{i-1}), \\
X_h(t_i) =& \overline{X}_h(t_i) + M_{t_{i-1}, t_i}^{-1} \Big(f(t_{i-1}, \overline{X}_h(t_i)) h_i + \sum_{r=1}^{m} g_r(t_{i-1}, \overline{X}_h(t_i)) I_{(r)}^{t_{i-1}, t_i} \\
& + \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(t_{i-1}, \overline{X}_h(t_i)) I_{(r_1, r_2)}^{t_{i-1}, t_i} \Big),
\end{aligned} \tag{4.8}
$$

where

$$
M(t_{i-1}, t_i) = \mathrm{id} + d^0 h_i + \sum_{l=1}^{m} d^l I_{(l,l)}^{t_{i-1}, t_i}.
$$

The convergence of (4.8) and numerical results will be studied in the sections below.

4.3. Preliminaries

In this section we describe our assumptions on the stochastic differential equation

$$
\begin{aligned}
\mathrm{d}X(t) =& f(t, X(t))\,\mathrm{d}t + \sum_{r=1}^{m} g_r(t, X(t))\,\mathrm{d}W_r(t), \\
X(0) =& X_0, \quad t \in [0, T],
\end{aligned}
\tag{4.9}
$$

where $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the drift diffusion coefficient function, $g_r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, r = 1, \ldots, m$ are the diffusion coefficient functions. By $W_r \colon [0, T] \times \Omega \to \mathbb{R}$ we denote an independent family of real-valued standard Wiener processes on the probability space $(\Omega, \mathcal{F}, P)$. Since we consider methods of higher order we have to extend the assumptions from Section 1.1 by assumptions on the derivatives of the coefficient functions.

In the following we denote by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ the Euclidean inner product and the Euclidean norm on $\mathbb{R}^d$, respectively. For sufficiently smooth function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ we denote by $\frac{\partial f}{\partial x}(t, x) \in \mathbb{R}^{d \times d}$ the Jacobian matrix of the mapping $x \mapsto f(t, x) \in \mathbb{R}^d$ for all $x \in \mathbb{R}^d$ and $t \in [0, T]$.

Assumption 4.3.1. *The functions $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g_r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, r = 1, \ldots, m$ are continuously differentiable. Furthermore, there exists a positive constant L and a parameter value $\eta \in (\frac{1}{2}, \infty)$ such that for all $t \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^d$ it holds*

$$
\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle + \eta \sum_{r=1}^{m} \big| g_r(t, x_1) - g_r(t, x_2) \big|^2 \le L|x_1 - x_2|^2.
\tag{4.10}
$$

In addition, there exists $q \in [2, \infty)$ such that for all $t \in [0, T]$ and $x, x_1, x_2 \in \mathbb{R}^d$ it holds

$$
\left| \frac{\partial f}{\partial t}(t, x) \right| \le L(1 + |x|)^q,
\tag{4.11}
$$

$$
\left| \frac{\partial g_r}{\partial t}(t, x) \right| \le L(1 + |x|)^{\frac{q+1}{2}}, \quad r = 1, \ldots, m,
\tag{4.12}
$$

$$
\left| \frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2) \right| \le L(1 + |x_1| + |x_2|)^{q-2}|x_1 - x_2|
\tag{4.13}
$$

$$
\left| \frac{\partial g_r}{\partial x}(t, x_1) - \frac{\partial g_r}{\partial x}(t, x_2) \right| \le L(1 + |x_1| + |x_2|)^{\frac{q-3}{2}}|x_1 - x_2|, \quad r = 1, \ldots, m.
\tag{4.14}
$$

Moreover, there exists a constant $q \in [1, \infty)$ such that for all $t, t_1, t_2 \in [0, T]$ and $x, x_1, x_2 \in \mathbb{R}^d$ it holds

$$
|f(t, x)| \le L(1 + |x|)^q,
\tag{4.15}
$$

$$
\left| \frac{\partial f}{\partial x}(t, x) \right| \le L(1 + |x|)^{q-1},
\tag{4.16}
$$

$$
|f(t_1, x) - f(t_2, x)| \le L(1 + |x|)^q|t_1 - t_2|,
\tag{4.17}
$$

$$|f(t,x_1) - f(t,x_2)| \leq L(1 + |x_1| + |x_2|)^{q-1}|x_1 - x_2|. \tag{4.18}$$

and for all $r = 1, \ldots, m$

$$|g_r(t,x)| \leq L(1 + |x|)^{\frac{q+1}{2}}, \tag{4.19}$$

$$\left|\frac{\partial g_r}{\partial x}(t,x)\right| \leq L(1 + |x|)^{\frac{q-1}{2}}, \tag{4.20}$$

$$|g_r(t_1,x) - g_r(t_2,x)| \leq L(1 + |x|)^{\frac{q+1}{2}}|t_1 - t_2|, \tag{4.21}$$

$$|g_r(t,x_1) - g_r(t,x_2)| \leq L(1 + |x_1| + |x_2|)^{\frac{q-1}{2}}|x_1 - x_2|. \tag{4.22}$$

We recall that condition (4.10) is also called a global monotonicity condition. In addition, we use the weights $(1 + |x|)^q$ instead of $1 + |x|^q$, compare conditions (1.3)-(1.5) from Section 1.1 with (4.15), (4.17), (4.18), (4.19), (4.21) and (4.22). For $q \geq 0$ this makes no difference, but if $2 \leq q < 3$ then the Lipschitz constants in condition (4.14) are required to decrease .

Now, we extend Assumption 4.3.1: The mappings g^{r_1,r_2} satisfy the polynomial growth condition

$$|g^{r_1,r_2}(t,x)| \leq L(1 + |x|)^q, \quad r_1, r_2 = 1, \ldots, m, \tag{4.23}$$

for all $x \in \mathbb{R}^d, t \in [0,T]$, and the local Lipschitz condition

$$|g^{r_1,r_2}(t,x_1) - g^{r_1,r_2}(t,x_2)| \leq L(1 + |x_1| + |x_2|)^{q-1}|x_1 - x_2|, \quad r_1, r_2 = 1, \ldots, m, \tag{4.24}$$

for all $x_1, x_2 \in \mathbb{R}^d, t \in [0,T]$.

The Assumption 4.3.1 is also sufficient to ensure the existence of a unique solution to (4.9), see for instance [38], [42] or [50, Chap.3]. We recall that an almost surely continuous and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process $X \colon [0,T] \times \Omega \to \mathbb{R}^d$ is a solution of (4.3) if it satisfies the integral equation

$$X(t) = X_0 + \int_0^t f(s, X(s))\,\mathrm{d}s + \sum_{r=1}^m \int_0^t g_r(s, X(s))\,\mathrm{d}W_r(s), \quad t \in [0,T]. \tag{4.25}$$

In addition, the exact solution has finite p-th moments, that is

$$\sup_{t\in[0,T]} \|X(t)\|_{L^p(\Omega;\mathbb{R}^d)} < \infty, \tag{4.26}$$

if there exist a constants $C > 0$ and $p \in [2,\infty)$ such that

$$\langle f(t,x), x \rangle + \frac{p-1}{2} \sum_{r=1}^m |g_r(t,x)|^2 \leq \alpha_{\mathrm{f}}(1 + |x|^2) \tag{4.27}$$

for all $x \in \mathbb{R}^d, t \in [0,T]$. For the proof see [42, Th.4.1] and Lemma 1.1.2.

4.4. C-stability of the projected balanced Milstein method

In this section we prove stochastic C-stability of the projected balanced Milstein scheme. Throughout we assume that Assumption 4.3.1 is satisfied with grow rate $q \in [2, \infty)$.

At first, define the matrix

$$M_{s,t} = \mathrm{id} + d^0 I_{(0)}^{s,t} + \sum_{l=1}^{m} d^l I_{(l,l)}^{s,t}, \quad 0 \leq s \leq t \leq T, \tag{4.28}$$

where $d^l \in \mathbb{R}^{d \times d}$, $l = 0, \ldots, m$.

The following condition states the uniform boundedness of the inverse of $M_{s,t}$.

Assumption 4.4.1. *For the chosen matrices $d^l \in \mathbb{R}^{d \times d}$, $l = 0, \ldots, m$ there exists an inverse $M_{s,t}^{-1}, 0 \leq s \leq t \leq T$ and a constant $K_M > 0$, such that*

$$|M_{s,t}^{-1}| \leq K_{\mathrm{M}}. \tag{4.29}$$

The next lemma is quoted from [32] and provides sufficient conditions on the matrices $d^l, l = 0, \ldots, m$, which guarantee Assumption 4.4.1.

Lemma 4.4.2. *The Assumption 4.4.1 with the constant K_{M} are satisfied by weight matrices d^l, $l = 0, \ldots, m$ if*

$$d^0 - \frac{1}{2} \sum_{l=1}^{m} d^l \quad \textit{is positive semi-definite} \tag{4.30}$$

$$\textit{and} \quad d^l \quad \textit{is positive semi-definite for all} \quad l = 1, \ldots, m. \tag{4.31}$$

For details we refer to [32].

Proof. The proof of (4.29) follows from the fact that for all $0 \leq s \leq t \leq T$ the double integral of the matrix M can be written as

$$I_{(l,l)}^{s,t} = \frac{1}{2}((I_{(l)}^{s,t})^2 - h), \quad l = 1, \ldots, m,$$

where $h = t - s$. By inserting in (4.28) we get

$$M_{s,t} = \mathrm{id} + (d^0 - \frac{1}{2} \sum_{l=1}^{m} d^l)h + \frac{1}{2} \sum_{l=1}^{m} d^l (I_{(l)}^{s,t})^2.$$

This representation of the matrix $M_{s,t}$ with (4.30) and (4.31) guarantees the existence of the inverse matrix $M_{s,t}^{-1}$ and the estimate $|M_{s,t}^{-1}| \leq K_{\mathrm{M}} \leq 1$.

$\square$

The following lemma provides estimates of products of multiple integrals needed later on.

Lemma 4.4.3. *Let Assumptions 4.4.1 hold and let $Y \in L^2(\Omega, \mathcal{F}_s; \mathbb{R}^d)$. Then there exist constants $C_{(\mathrm{r})}$, $C_{(\mathrm{r}_1, \mathrm{r}_2)}$, $C_{(\mathrm{r},\mathrm{l},\mathrm{l})}$, and $C_{(\mathrm{r}_1, \mathrm{r}_2, \mathrm{l}, \mathrm{l})}$ such that for all $0 \le s < t \le T$ and $r, r_1, r_2, l = 1, \ldots, m$ the estimates hold*

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(r)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \le C_{(\mathrm{r})} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^{\frac{3}{2}}, \tag{4.32}$$

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(r_1,r_2)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \le C_{(\mathrm{r}_1,\mathrm{r}_2)} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^2, \tag{4.33}$$

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(l,l)}^{s,t} I_{(r)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \le C_{(\mathrm{r},\mathrm{l},\mathrm{l})} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^{\frac{5}{2}}, \tag{4.34}$$

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(l,l)}^{s,t} I_{(r_1,r_2)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \le C_{(\mathrm{r}_1,\mathrm{r}_2,\mathrm{l},\mathrm{l})} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^3. \tag{4.35}$$

Proof. For all $0 \le s \le t \le T$ and $r = 1, \ldots, m$ we have

$$\mathbb{E}\big[M_{s,t}^{-1} I_{(r)}^{s,t} Y | \mathcal{F}_s\big] = \mathbb{E}\big[M_{s,t}^{-1} I_{(r)}^{s,t} Y - I_{(r)}^{s,t} Y | \mathcal{F}_s\big] = \mathbb{E}[(M_{s,t}^{-1} - \mathrm{id}) I_{(r)}^{s,t} Y | \mathcal{F}_s]. \tag{4.36}$$

Using the inequality $\|\mathbb{E}[Z|\mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \le \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$, Assumptions 4.4.1, Lemma A.3.1 and the fact that $\mathbb{E}[|I_{(r)}^{s,t}|^2] = t - s, r = 1, \ldots, m$ for $0 \le s < t \le T$ we get

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(r)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;R^d)} \le \|(M_{s,t}^{-1} - \mathrm{id}_{\mathbb{R}^d}) I_{(r)}^{s,t} Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$= \|M_{s,t}^{-1} \Big(d^0 I_{(0)}^{s,t} + \sum_{l=1}^m d^l I_{(l,l)}^{s,t}\Big) I_{(r)}^{s,t} Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le K_{\mathrm{M}} |d^0| \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^{\frac{3}{2}}$$

$$+ m K_{\mathrm{M}} \max_{l=1,\ldots,m} |d^l| \sum_{l=1}^m (\mathbb{E}[|I_{(l,l)}^{s,t} I_{(r)}^{s,t}|^2])^{\frac{1}{2}} \|Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le C_{(\mathrm{r})} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^{\frac{3}{2}}.$$

In the same way and the fact that $\mathbb{E}[|I_{(r_1,r_2)}^{s,t}|^2] = \frac{1}{2}(t-s)^2, r_1, r_2 = 1, \ldots, m$ we get the second estimate

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(r_1,r_2)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;R^d)} \le \|(M_{s,t}^{-1} - \mathrm{id}_{\mathbb{R}^d}) I_{(r_1,r_2)}^{s,t} Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le \frac{1}{\sqrt{2}} K_{\mathrm{M}} |d^0| (t-s)^2 \|Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ m K_{\mathrm{M}} \max_{l=1,\ldots,m} |d^l| \sum_{l=1}^m (\mathbb{E}[|I_{(l,l)}^{s,t} I_{(r_1,r_2)}^{s,t}|^2])^{\frac{1}{2}} \|Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\le C_{(\mathrm{r}_1,\mathrm{r}_2)} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^2.$$

A similar estimate holds also for the proof of (4.34).

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(l,l)}^{s,t} I_{(r)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)} \le \|(M_{s,t}^{-1} - \mathrm{id}_{\mathbb{R}^d}) I_{(l,l)}^{s,t} I_{(r)}^{s,t} Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$= \|M_{s,t}^{-1}\Big(d^0 I_{(0)}^{s,t} + \sum_{l=1}^m d^l I_{(l,l)}^{s,t}\Big) I_{(l,l)}^{s,t} I_{(r)}^{s,t} Y\|_{L^2(\Omega;R^d)}$$

$$\leq K_{\mathrm{M}} |d^0|(t-s)\big(\mathbb{E}\big[|I_{(l,l)}^{s,t} I_{(r)}^{s,t}|^2\big]\big)^{\frac{1}{2}} \|Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ mK_{\mathrm{M}} \max_{l=1,\dots,m} |d^l| \sum_{l=1}^m \big(\mathbb{E}\big[|\big(I_{(l,l)}^{s,t}\big)^2|^4\big]\big)^{\frac{1}{2}}\big(\mathbb{E}\big[|I_{(r)}^{s,t}|^4\big]\big)^{\frac{1}{2}} \|Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq C_{(\mathrm{r,l,l})} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^{\frac{5}{2}}.$$

For the last estimate we get

$$\|\mathbb{E}[M_{s,t}^{-1} I_{(l,l)}^{s,t} I_{(r_1,r_2)}^{s,t} Y | \mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq \|M_{s,t}^{-1}\Big(d^0 I_{(0)}^{s,t} + \sum_{l=1}^m d^l I_{(l,l)}^{s,t}\Big) I_{(l,l)}^{s,t} I_{(r_1,r_2)}^{s,t} Y\|_{L^2(\Omega;R^d)}$$

$$\leq K_{\mathrm{M}} |d^0|(t-s)\big(\mathbb{E}\big[|I_{(l,l)}^{s,t} I_{(r_1,r_2)}^{s,t}|^2\big]\big)^{\frac{1}{2}} \|Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ mK_{\mathrm{M}} \max_{l=1,\dots,m} |d^l| \sum_{l=1}^m \big(\mathbb{E}\big[|\big(I_{(l,l)}^{s,t}\big)^2|^4\big]\big)^{\frac{1}{2}}\big(\mathbb{E}\big[|I_{(r_1,r_2)}^{s,t}|^4\big]\big)^{\frac{1}{2}} \|Y\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq C_{(\mathrm{r_1,r_2,l,l})} \|Y\|_{L^2(\Omega;\mathbb{R}^d)} (t-s)^3.$$

This completes the proof.

$\square$

For the definition of the one-step map of the PBMM scheme we introduce, as in Section 3.1, the following notation: Let $\bar{h} \in (0,1]^N$ be an arbitrary vector of step sizes. The parameter $\beta \in (0,\infty)$ is chosen to be suitable negative power in dependence of the growth rate q. Further, for all $\delta \in (0,\bar{h}]$ let us denote the projection of $x \in \mathbb{R}^d$ onto the ball of radius $\delta^{-\beta}$ by

$$x^\circ := \min(1, \delta^{-\beta} |x|^{-1})x. \tag{4.37}$$

Then the one-step map $\Psi^{PBMM} \colon \mathbb{R}^d \times \mathbb{T} \times \Omega \to \mathbb{R}^d$ of the projected balanced Milstein method is given by

$$\Psi^{PBMM}(x,t,\delta) := x^\circ + M_{t,t+\delta}^{-1}\Big[f(t,x^\circ)\delta + \sum_{r=1}^m g_r(t,x^\circ) I_{(r)}^{t,t+\delta} + \sum_{r_1,r_2=1}^m g^{r_1,r_2}(t,x^\circ) I_{(r_1,r_2)}^{t,t+\delta}\Big],$$

$$\tag{4.38}$$

for all $x \in \mathbb{R}^d$ and $(t,\delta) \in \mathbb{T}$.

The proposition below is an analog to the Proposition 4.1 in [6] und shows that the PBMM is a stochastic one-step method in the sense of the Definition 2.1.1.

Proposition 4.4.4. *Let the functions f and $g_r, r = 1, \ldots, m$ satisfy Assumption 4.3.1 with constants $L \in (0, \infty)$, $q \in [2, \infty)$ and let Assumption 4.4.1 hold. Further, let $\bar{h} \in (0, 1]$. Then for every $\beta \in (0, \infty)$ and inital value $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ it holds that $(\Psi^{PBMM}, \bar{h}, \xi)$ is a stochasic one-step method.*

In addition, there exists a constant C_0 only depending on L and m such that

$$\|\mathbb{E}[\Psi^{PBMM}(0, t, \delta)|\mathcal{F}_t]\|_{L^2(\Omega; \mathbb{R}^d)} \leq C_0 \delta, \tag{4.39}$$

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Psi^{PBMM}(0, t, \delta)\|_{L^2(\Omega; \mathbb{R}^d)} \leq C_0 \delta^{\frac{1}{2}} \tag{4.40}$$

for all $(t, \delta) \in \mathbb{T}$.

Proof. For the proof we fix arbitrary $(t, \delta) \in \mathbb{T}$ and let $Y \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$. Since the mapping $x \mapsto \min(1, \delta^{-\beta}|x|^{-1})x$, for every $x \in \mathbb{R}^d$ is continuous and bounded it holds

$$Y^\circ \in L^\infty(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d).$$

Further, by the smoothness of f and $g_r, r = 1, \ldots, m$, conditions (4.15), (4.19), (4.23) and Assumption 4.4.1 it follows that for all $r, r_1, r_2 = 1, \ldots, m$

$$M_{t,t+\delta}^{-1} f(t, Y^\circ), M_{t,t+\delta}^{-1} g_r(t, Y^\circ), M_{t,t+\delta}^{-1} g^{r_1, r_2}(t, Y^\circ) \in L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbf{P}; \mathbb{R}^d).$$

Thus, $\Psi^{PBMM}(Y, t, \delta)$ is an $\mathcal{F}_{t+\delta}$-measurable random variable, which satisfies condition (2.2). To show (4.39) we use Assumption 4.4.1, conditions (4.15), (4.19), (4.23), and Lemma 4.4.3 and obtain

$$\|\mathbb{E}[\Psi^{PBMM}(0, t, \delta)\mathcal{F}_t]\|_{L^2(\Omega; \mathbb{R}^d)} \leq \|\mathbb{E}[M_{t,t+\delta}^{-1} f(t, 0)\delta|\mathcal{F}_t]\|_{L^2(\Omega; \mathbb{R}^d)}$$

$$+ \Big\|\sum_{r=1}^m M_{t,t+\delta}^{-1} g_r(0, t) I_{(r)}^{t,t+\delta}|\mathcal{F}_t\Big\|_{L^2(\Omega; \mathbb{R}^d)} + \Big\|\sum_{r_1, r_2=1}^m M_{t,t+\delta}^{-1} g^{r_1, r_2}(t, 0) I_{(r_1, r_2)}^{t,t+\delta}|\mathcal{F}_t\Big\|_{L^2(\Omega; \mathbb{R}^d)}$$

$$\leq L K_\mathrm{M} \delta + L C_{(\mathrm{r})} \delta^{\frac{3}{2}} + L C_{(\mathrm{r_1, r_2})} \delta^2.$$

Since $\delta \leq \bar{h} \leq 1$ this verifies (4.39). It remains to show (4.40). Using the fact that $\|(\mathrm{id} - \mathbb{E}[\cdot|F_t])Z\|_{L^2(\Omega; \mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega; \mathbb{R}^d)}$ for all $Z \in L^2(\Omega; \mathbb{R}^d)$ yields

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\Psi^{PBMM}(0, t, \delta)\|_{L^2(\Omega; \mathbb{R}^d)} \leq \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])M_{t,t+\delta}^{-1} f(t, 0)\delta\|_{L^2(\Omega; \mathbb{R}^d)}$$

$$+ \Big\|\sum_{r=1}^m (\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])M_{t,t+\delta}^{-1} g_r(t, 0) I_{(r)}^{t,t+\delta}\Big\|_{L^2()\Omega; \mathbb{R}^d}$$

$$+ \Big\|\sum_{r_1, r_2=1}^m (\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])M_{t,t+\delta}^{-1} g^{r_1, r_2}(t, 0) I_{(r_r, r_2)}^{t,t+\delta}\Big\|_{L^2()\Omega; \mathbb{R}^d}$$

$$\leq \|M_{t,t+\delta}^{-1} f(t, 0)\delta\|_{L^2(\Omega; \mathbb{R}^d)} + \Big\|\sum_{r=1}^m M_{t,t+\delta}^{-1} g_r(t, 0) I_{(r)}^{t,t+\delta}\Big\|_{L^2()\Omega; \mathbb{R}^d}$$

$$+ \| \sum_{r_1,r_2=1}^{m} M_{t,t+\delta}^{-1} g^{r_1,r_2}(t,0) I_{(r_r,r_2)}^{t,t+\delta} \|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq 2LK_{\mathrm{M}}\delta + LK_{\mathrm{M}}\delta^{\frac{1}{2}}.$$

Here we used Assumption 4.4.1, conditions (4.15), (4.19), and (4.23).

$\square$

As a preparation for the proof of C-stability we consider the following estimates:

Lemma 4.4.5. *Let* f *and* $g_r, r = 1, \ldots, m$ *satisfy Assumption 4.3.1 with* $L > 0, \eta \in (\frac{1}{2}, \infty)$ *and* $q \in [2, \infty)$ *and let Assumption 4.4.1 hold. Consider the mapping* $\mathbb{R}^d \ni x \mapsto x^\circ \in \mathbb{R}^d$ *defined in* (4.37) *with parameter* $\beta = \frac{1}{2(q-1)}$ *and* $\delta \in (0,1]$. *Then there exist constants* $K_1, K_2,$ *and* K_3 *such that for all* $x_1, x_2 \in \mathbb{R}^d$ *and* $(t,\delta) \in \mathbb{T}$ *the following estimates hold*

$$\left| \mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\delta(f(t,x_1^\circ) - f(t,x_2^\circ)) \big] \right| \leq K_1 \delta^{\frac{3}{2}} |x_1 - x_2|, \quad (4.41)$$

$$\left| \mathbb{E}\big[\sum_{r=1}^{m} (M_{t,t+\delta}^{-1} - \mathrm{id})(g_r(t,x_1^\circ) - g_r(t,x_2^\circ)) I_{(r)}^{t,t+\delta} \big] \right| \leq K_2 \delta^{\frac{5}{4}} |x_1 - x_2|, \quad (4.42)$$

$$\left| \mathbb{E}\big[\sum_{r_1,r_2=1}^{m} (M_{t,t+\delta}^{-1} - \mathrm{id})(g^{r_1,r_2}(t,x_1^\circ) - g^{r_1,r_2}(t,x_2^\circ)) I_{(r_1,r_2)}^{t,t+\delta} \big] \right| \leq K_3 \delta^{\frac{3}{2}} |x_1 - x_2|. \quad (4.43)$$

Proof. Using Hölder's inequality, the fact that $\mathbb{E}[|I_{(l,l)}^{t,t+\delta}|^2] = \frac{1}{2}\delta^2$, and condition (4.18) yields

$$\left| \mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\delta(f(t,x_1^\circ) - f(t,x_2^\circ)) \big] \right|$$

$$\leq \delta \mathbb{E}[|M_{t,t+\delta}^{-1}(d^0\delta + \sum_{l=1}^{m} I_{(l,l)}^{t,t+\delta} d^l)(f(t,x_1^\circ) - f(t,x_2^\circ))|]$$

$$\leq K_{\mathrm{M}}|d^0|\delta^2 |f(t,x_1^\circ) - f(t,x_2^\circ)|$$

$$+ \delta \sum_{l=1}^{m} (\mathbb{E}[|M_{t,t+\delta}^{-1}|^2])^{\frac{1}{2}} (\mathbb{E}[|I_{(l,l)}^{t,t+\delta}|^2])^{\frac{1}{2}} |d^l(f(t,x_1^\circ) - f(t,x_2^\circ))|$$

$$\leq K_{\mathrm{M}}\delta^2(|d^0| + \frac{1}{\sqrt{2}}m \max_{l=1,\ldots,m}|d^l|)L(1 + |x_1^\circ| + |x_2^\circ|)^{q-1}|x_1^\circ - x_2^\circ|.$$

Since $|x_1^\circ|, |x_2^\circ| \leq \delta^{-\beta}$ with $\delta \in (0,1]$ and $\beta(q-1) = \frac{1}{2}$ we obtain that $\delta^{\frac{1}{2}}(1 + 2\delta^{-\beta})^{q-1} \leq 3^{q-1}$. Then by Lemma 3.4.2 it holds

$$\left| \mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\delta(f(t,x_1^\circ) - f(t,x_2^\circ)) \big] \right|$$

$$\leq K_{\mathrm{M}}\delta^2(|d^0| + \frac{1}{\sqrt{2}}m \max_{l=1,\ldots,m}|d^l|)L(1 + 2\delta^{-\beta})^{q-1}|x_1 - x_2|$$

$$\leq 3^{q-1}K_{\mathrm{M}}L(|d^0| + \frac{1}{\sqrt{2}}m \max_{l=1,\dots,m}|d^l|)\delta^{\frac{3}{2}}|x_1 - x_2|.$$

To show (4.42) we use the Hölder inequality, the fact that $\mathbb{E}[|I_{(r)}^{t,t+\delta}|^2] = \delta$, and Lemma A.3.1:

$$\left|\mathbb{E}\left[(M_{t,t+\delta}^{-1}-\mathrm{id})\sum_{r=1}^{m}(g_r(t,x_1^\circ) - g_r(t,x_2^\circ))I_{(r)}^{t,t+\delta}\right]\right|$$

$$\leq \mathbb{E}\left[\left|M_{t,t+\delta}^{-1}(d^0\delta + \sum_{l=1}^{m}I_{(l,l)}^{t,t+\delta}d^l)\sum_{r=1}^{m}(g_r(t,x_1^\circ) - g_r(t,x_2^\circ))I_{(r)}^{t,t+\delta}\right|\right]$$

$$\leq \delta \sum_{r=1}^{m}(\mathbb{E}[|M_{t,t+\delta}^{-1}|^2])^{\frac{1}{2}}(\mathbb{E}[|I_{(r)}^{t,t+\delta}|^2])^{\frac{1}{2}}|d^0(g_r(t,x_1^\circ) - g_r(t,x_2^\circ))|$$

$$+ \sum_{l=1}^{m}\sum_{r=1}^{m}(\mathbb{E}[|M_{t,t+\delta}^{-1}|^2])^{\frac{1}{2}}(\mathbb{E}[|I_{(l,l)}^{t,t+\delta}I_{(r)}^{t,t+\delta}|^2])^{\frac{1}{2}}|d^l(g_r(t,x_1^\circ) - g_r(t,x_2^\circ))|$$

$$\leq \delta^{\frac{3}{2}}mK_{\mathrm{M}}L\left(|d^0| + mK_{\mathrm{mult}}\max_{l=1,\dots,m}|d^l|\right)\left(1 + |x_1^\circ| + |x_2^\circ|\right)^{\frac{q-1}{2}}|x_1^\circ - x_2^\circ|$$

$$\leq 3^{\frac{q-1}{2}}mK_{\mathrm{M}}L\left(|d^0| + mK_{\mathrm{mult}}\max_{l=1,\dots,m}|d^l|\right)\delta^{\frac{5}{4}}|x_1 - x_2|,$$

where K_{mult} is a constant, obtained by multiplying of the Wiener increments (see Lemma A.3.1). Here we used Lemma 3.4.2 and the fact that for $\frac{\beta(q-1)}{2} = \frac{1}{4}$ and $\delta \in (0,1)]$ it follows that $\delta^{\frac{1}{4}}(1+2\delta^{-\beta})^{\frac{q-1}{2}} \leq 3^{\frac{q-1}{2}}$. It remains to show (4.43). By the Hölder inequality, condition (4.24), Lemma 3.4.2, and Lemma A.3.1 it holds

$$\left|\mathbb{E}\left[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t,x_1^\circ) - g^{r_1,r_2}(t,x_2^\circ))I_{(r_1,r_2)}^{t,t+\delta}\right]\right|$$

$$\leq \mathbb{E}\left[\left|M_{t,t+\delta}^{-1}(d^0\delta + \sum_{l=1}^{m}I_{(l,l)}^{t,t+\delta}d^l)\sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t,x_1^\circ) - g^{r_1,r_2}(t,x_2^\circ))I_{(r_1,r_2)}^{t,t+\delta}\right|\right]$$

$$\leq \delta \sum_{r_1,r_2=1}^{m}(\mathbb{E}[|M_{t,t+\delta}^{-1}|^2])^{\frac{1}{2}}(\mathbb{E}[|I_{(r_1,r_2)}^{t,t+\delta}|^2])^{\frac{1}{2}}|d^0(g^{r_1,r_2}(t,x_1^\circ) - g^{r_1,r_2}(t,x_2^\circ))|$$

$$+ \sum_{l=1}^{m}\sum_{r_1,r_2=1}^{m}(\mathbb{E}[|M_{t,t+\delta}^{-1}|^2])|^{\frac{1}{2}}(\mathbb{E}[|I_{(l,l)}^{t,t+\delta}I_{(r_1,r_2)}^{t,t+\delta}|^2])^{\frac{1}{2}}|d^l(g^{r_1,r_2}(t,x_1^\circ) - g^{r_1,r_2}(t,x_2^\circ))|$$

$$\leq \delta^2 m^2 K_{\mathrm{M}}L\left(\frac{1}{\sqrt{2}}|d^0| + mK_{\mathrm{mult}}\max_{l=1,\dots,m}|d^l|\right)\left(1 + |x_1^\circ| + |x_2^\circ|\right)^{q-1}|x_1^\circ - x_2^\circ|.$$

From $\delta^{\frac{1}{2}}(1 + 2\delta^{-\beta})^{q-1} \leq 3^{q-1}$ we obtain

$$\left|\mathbb{E}\left[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t,x_1^\circ) - g^{r_1,r_2}(t,x_2^\circ))I_{(r_1,r_2)}^{t,t+\delta}\right]\right|$$

$$\leq 3^{q-1}m^2 K_{\mathrm{M}}L\left(\frac{1}{\sqrt{2}}|d^0| + mK_{\mathrm{mult}}\max_{l=1,\dots,m}|d^l|\right)\delta^{\frac{3}{2}}|x_1 - x_2|.$$

This completes the proof.

$\square$

Corollary 4.4.6. *Consider the same situation as in Lemma 4.4.5. Then there exists constants K_4, K_5, and K_6 such that for all $(t, \delta) \in \mathbb{T}$ the following estimates hold*

$$\mathbb{E}\big[\big|\big(M_{t,t+\delta}^{-1} - \mathrm{id}\big)\delta(f(t, x_1^\circ) - f(t, x_2^\circ))\big|^2\big] \leq K_4\delta^3|x_1 - x_2|^2], \quad (4.44)$$

$$\mathbb{E}\big[\big|\sum_{r=1}^{m}\big(M_{t,t+\delta}^{-1} - \mathrm{id}\big)(g_r(t, x_1^\circ) - g_r(t, x_2^\circ))I_{(r)}^{t,t+\delta}\big|^2\big] \leq K_5\delta^{\frac{5}{2}}|x_1 - x_2|^2, \quad (4.45)$$

$$\mathbb{E}\big[\big|\sum_{(r_1,r_2)}\big(M_{t,t+\delta}^{-1} - \mathrm{id}\big)(g^{r_1,r_2}(t, x_1^\circ) - g^{r_1,r_2}(t, x_2))I_{(r_1,r_2)}^{t,t+\delta}\big|^2\big] \leq K_6\delta^3|x_1 - x_2|^2. \quad (4.46)$$

Proof. The proof is similar to the proof of Lemma 4.4.5 if we square the estimates there.

$\square$

In the next theorem we show that the PBMM scheme is stochastically C-stable.

Theorem 4.4.7. *Let the functions f and $g_r, r = 1, \dots, m$ satisfy Assumption 4.3.1 with $L > 0, q \in [2, \infty)$ and $\eta \in (\frac{1}{2}, \infty)$ and let Assumption 4.4.1 hold. Further, let $\bar{h} \in (0, 1]$. Then for every $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathbb{R}^d)$ the projected balanced Milstein method $(\Psi^{PBMM}, \bar{h}, \xi)$ with $\beta = \frac{1}{2(q-1)}$ is stochastically C-stable.*

Proof. First consider for arbitrary $(t, \delta) \in \mathbb{T}$ and $Y, Z \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$

$$\mathbb{E}[\Psi^{PBMM}(Y, t, \delta) - \Psi^{PBMM}(Z, t, \delta)|\mathcal{F}_t] = Y^\circ - Z^\circ + \delta(f(t, Y^\circ) - f(t, Z^\circ))$$

$$+ \mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\big(\delta(f(t, Y^\circ) - f(t, Z^\circ)) + \sum_{r=1}^{m}(g_r(t, Y^\circ) - g_r(t, Z^\circ))I_{(r)}^{t,t+\delta}$$

$$+ \sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}\big)|\mathcal{F}_t]$$

$$(4.47)$$

and

$$(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{PBMM}(Y, t, \delta) - \Psi^{PBMM}(Z, t, \delta)) = \sum_{r=1}^{m}(g_r(t, Y^\circ) - g_r(t, Z^\circ))I_{(r)}^{t,t+\delta}$$

$$+ \sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}$$

$$+ (\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(M_{t,t+\delta}^{-1} - \mathrm{id})\big(\delta(f(t, Y^\circ) - f(t, Z^\circ))$$

$$+ \sum_{r=1}^{m}(g_r(t, Y^\circ) - g_r(t, Z^\circ))I_{(r)}^{t,t+\delta} + \sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}\big).$$

$$(4.48)$$

Hence it follows

$$
\begin{aligned}
\mathbb{E}\big[&\big|\mathbb{E}[\Psi^{PBMM}(Y,t,\delta) - \Psi^{PBMM}(Z,t,\delta)|\mathcal{F}_t]\big|^2\big] \\
&= \mathbb{E}\big[|Y^\circ - Z^\circ|^2 + 2\delta\langle Y^\circ - Z^\circ, f(t,Y^\circ) - f(t,Z^\circ)\rangle \\
&\quad + 2\langle Y^\circ - Z^\circ, \mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})|\mathcal{F}_t\big](\delta(f(t,Y^\circ) - f(t,Z^\circ)))\rangle \\
&\quad + 2\langle Y^\circ - Z^\circ, \mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r=1}^m (g_r(t,Y^\circ) - g_r(t,Z^\circ))I_{(r)}^{t,t+\delta}|\mathcal{F}_t]\rangle \\
&\quad + 2\langle Y^\circ - Z^\circ, \mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r_1,r_2=1}^m (g^{r_1,r_2}(t,Y^\circ) - g^{r_1,r_2}(t,Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}|\mathcal{F}_t]\rangle \\
&\quad + \big|\delta(f(t,Y^\circ) - f(t,Z^\circ)) + \mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\big(\delta(f(t,Y^\circ) - f(t,Z^\circ)) \\
&\quad + \sum_{r=1}^m (g_r(t,Y^\circ) - g_r(t,Z^\circ))I_{(r)}^{t,t+\delta} \\
&\quad + \sum_{r_1,r_2=1}^m (g^{r_1,r_2}(t,Y^\circ) - g^{r_1,r_2}(t,Z^\circ))I_{(r_1,r_2)}^{t,t+\delta})|\mathcal{F}_t\big]\big|^2\big].
\end{aligned}
$$

By condition (4.10) and Lemma 3.4.2 we get for the second term that

$$
\begin{aligned}
2\delta\mathbb{E}&\big[\langle Y^\circ - Z^\circ, f(t,Y^\circ) - f(t,Z^\circ)\rangle\big] \\
&\leq 2\delta\mathbb{E}\big[L|Y^\circ - Z^\circ|^2 - \eta\sum_{r=1}^m |g_r(t,Y^\circ) - g_r(t,Z^\circ)|^2\big] \\
&\leq 2\delta\mathbb{E}\big[L|Y - Z|^2 - \eta\sum_{r=1}^m |g_r(t,Y^\circ) - g_r(t,Z^\circ)|^2\big].
\end{aligned}
$$

By an application of the Cauchy-Schwarz inequality we get for the three next terms

$$
\begin{aligned}
2\langle Y^\circ - Z^\circ, &\mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})(\delta(f(t,Y^\circ) - f(t,Z^\circ))|\mathcal{F}_t]\rangle \\
&+ 2\langle Y^\circ - Z^\circ, \mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r=1}^m (g_r(t,Y^\circ) - g_r(t,Z^\circ))I_{(r)}^{t,t+\delta}|\mathcal{F}_t]\rangle \\
&+ 2\langle Y^\circ - Z^\circ, \mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r_1,r_2=1}^m (g^{r_1,r_2}(t,Y^\circ) - g^{r_1,r_2}(t,Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}|\mathcal{F}_t]\rangle \\
&\leq 2\big|Y^\circ - Z^\circ\big|\big|\mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\delta(f(t,Y^\circ) - f(t,Z^\circ))]\big| \\
&\quad + 2\big|Y^\circ - Z^\circ\big|\big|\mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r=1}^m (g_r(t,Y^\circ) - g_r(t,Z^\circ))I_{(r)}^{t,t+\delta}]\big| \\
&\quad + 2\big|Y^\circ - Z^\circ\big|\big|\mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r_1,r_2=1}^m (g^{r_1,r_2}(t,Y^\circ) - g^{r_1,r_2}(t,Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}]\big|.
\end{aligned}
$$

Hence, using Lemma 3.4.2 and Lemma 4.4.5, one can derive that

$$2\mathbb{E}\Big[\big|Y^\circ - Z^\circ\big|\big|\mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\delta(f(t,Y^\circ) - f(t,Z^\circ))\big]\big|$$

$$+ \big|Y^\circ - Z^\circ\big|\big|\mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r=1}^{m}(g_r(t,Y^\circ) - g_r(t,Z^\circ))I_{(r)}^{t,t+\delta}\big]\big|$$

$$+ \big|Y^\circ - Z^\circ\big|\big|\mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t,Y^\circ) - g^{r_1,r_2}(t,Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}\big]\big|\Big]$$

$$\leq 2(K_1\delta^{\frac{3}{2}} + K_2\delta^{\frac{5}{4}} + K_3\delta^{\frac{3}{2}})\mathbb{E}[|Y - Z|^2]$$

Further, we consider the next summand

$$\mathbb{E}\Big[\big|\delta(f(t,Y^\circ) - f(t,Z^\circ)) + \mathbb{E}\big[(M_{t,t+\delta}^{-1} - \mathrm{id})\big(\delta(f(t,Y^\circ) - f(t,Z^\circ))$$

$$+ \sum_{r=1}^{m}(g_r(t,Y^\circ) - g_r(t,Z^\circ))I_{(r)}^{t,t+\delta} + \sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t,Y^\circ) - g^{r_1,r_2}(t,Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}\big)\big|\mathcal{F}_t\big]\big|^2\Big]$$

$$\leq 4\delta^2\mathbb{E}\big[\big|f(t,Y^\circ) - f(t,Z^\circ)\big|^2\big] + 4\mathbb{E}\big[\big|\mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\delta(f(t,Y^\circ) - f(t,Z^\circ))|\mathcal{F}_t]\big|^2\big]$$

$$+ 4\mathbb{E}\big[\big|\mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r=1}^{m}(g_r(t,Y^\circ) - g_r(t,Z^\circ))I_{(r)}^{t,t+\delta}|\mathcal{F}_t]\big|^2\big]$$

$$+ 4\mathbb{E}\big[\big|\mathbb{E}[(M_{t,t+\delta}^{-1} - \mathrm{id})\sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t,Y^\circ) - g^{r_1,r_2}(t,Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}|\mathcal{F}_t\big|^2\big]$$

$$:= \sum_{j=1}^{4} T_j.$$

By condition (4.18), Lemma 3.4.2 and the fact that $|Y^\circ|, |Z^\circ| \leq \delta^{-\beta}$ with $\beta = \frac{1}{2(q-1)}$ for all $\delta \in (0,1]$ we get for the first summand

$$T_1 \leq 4\delta^2 L^2\mathbb{E}[(1 + |Y^\circ| + |Z^\circ|)^{2(q-1)}|Y^\circ - Z^\circ|^2]$$

$$\leq 4\delta^2 L^2(1 + 2\delta^{-\beta})^{2(q-1)}\mathbb{E}[|Y^\circ - Z^\circ|^2]$$

$$\leq 3^{2(q-1)}4L^2\delta\mathbb{E}[|Y - Z|^2].$$

Using the fact that $\mathbb{E}[|\mathbb{E}[Z|\mathcal{F}_t]|^2] \leq \mathbb{E}[|Z|^2]$ for all $Z \in L^2(\Omega; \mathbb{R}^d)$ and Corollary 4.4.6 we get that

$$T_2 \leq 4K_4\delta^3\mathbb{E}[|Y - Z|^2], \quad T_3 \leq 4K_5\delta^{\frac{5}{2}}\mathbb{E}[|Y - Z|^2], \quad \text{and} \quad T_4 \leq 4K_6\delta^3\mathbb{E}[|Y - Z|^2].$$

So, we showed that

$$\mathbb{E}\big[\big|\mathbb{E}[\Psi^{PBMM}(Y,t,\delta) - \Psi^{PBMM}(Z,t,\delta)|\mathcal{F}_t]\big|^2\big] \leq (1 + C\delta)\mathbb{E}[|Y - Z|^2]$$

$$- 2\eta\delta\sum_{r=1}^{m}\mathbb{E}[|g_r(t,Y^\circ) - g_r(t,Z^\circ)|^2], \tag{4.49}$$

where C depends on $K_1, K_2, K_3, K_4, K_5, K_6, L, q$ and $\bar{h}$. Further, let us consider

$$\mathbb{E}\big[\big|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{PBMM}(Y, t, \delta) - \Psi^{PBMM}(Z, t, \delta))\big|^2\big]$$

$$\leq 2\mathbb{E}\big[\big|\sum_{r=1}^{m}(g_r(t, Y^\circ) - g_r(t, Z^\circ))I_{(r)}^{t,t+\delta} + \sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}\big|^2\big]$$

$$+ 2\mathbb{E}\big[\big|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(M_{t,t+\delta}^{-1} - \mathrm{id})\big(\delta(f(t, Y^\circ) - f(t, Z^\circ))$$

$$+ \sum_{r=1}^{m}(g_r(t, Y^\circ) - g_r(t, Z^\circ))I_{(r)}^{t,t+\delta} + \sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}\big)\big|^2\big]$$

$$=: S_1 + S_2.$$

Since the stochastic increments are pairwise uncorrelated, i.e. $\mathrm{cov}\big(I_{(r)}^{t,t+\delta}, I_{(r_1,r_2)}^{t,t+\delta}\big) = 0$ and independent of Y° and Z° we get for the first summand (see for example, [42, Chap.1], [49])

$$S_1 = 2\mathbb{E}\big[\big|\sum_{r=1}^{m}(g_r(t, Y^\circ) - g_r(t, Z^\circ))I_{(r)}^{t,t+\delta}\big|^2\big]$$

$$+ 2\mathbb{E}\big[\big|\sum_{r_1,r_2=1}^{m}(g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ))I_{(r_1,r_2)}^{t,t+\delta}\big)\big|^2\big]$$

$$= 2\delta \sum_{r=1}^{m}\mathbb{E}[|g_r(t, Y^\circ) - g_r(t, Z^\circ)|^2]$$

$$+ 2\delta^2 \sum_{r_1,r_2=1}^{m}\mathbb{E}[|g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ)|^2].$$

Here we used the variance relation $V(x + y) = V(x) + V(y)$ for uncorrelated variables $x, y \in L^2(\Omega; \mathbb{R}^d)$ and the Itô isometry. Thus, we arrive at

$$\mathbb{E}\big[\big|\mathbb{E}[\Psi^{PBMM}(Y, t, \delta) - \Psi^{PBMM}(Z, t, \delta)|\mathcal{F}_t]\big|^2\big]$$

$$+ \eta\mathbb{E}\big[\big|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\Psi^{PBMM}(Y, t, \delta) - \Psi^{PBMM}(Z, t, \delta))\big|^2\big]$$

$$\leq (1 + C\delta)\mathbb{E}[|Y - Z|^2] + 2\eta\delta^2 \sum_{r_1,r_2=1}^{m}\mathbb{E}[|g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ)|^2] + \eta S_2.$$

Using the fact that $|Y^\circ|, |Z^\circ| \leq \delta^{-\beta}$ with $2\beta(q-1) = 1$ and $\delta \in (0, 1]$ we get

$$2\eta\delta^2 \sum_{r_1,r_2=1}^{m}\mathbb{E}[|g^{r_1,r_2}(t, Y^\circ) - g^{r_1,r_2}(t, Z^\circ)|^2] \leq 3^{2(q-1)}2\eta m^2 L^2\delta\mathbb{E}[|Y - Z|^2].$$

Here we applied condition (4.24) and Corollary 3.4.2. Further, by $\mathbb{E}[|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])Z|^2] \leq \mathbb{E}[|Z|^2]$ for all $Z \in L^2(\Omega; \mathbb{R}^d)$ and Lemma 4.4.6 we obtain

$$S_2 \leq 2\mathbb{E}\big[\big|(M_{t,t+\delta}^{-1} - \mathrm{id})\big(\delta(f(t, Y^\circ) - f(t, Z^\circ))$$

$$+ \sum_{r=1}^{m} (g_r(t, Y^{\circ}) - g_r(t, Z^{\circ})) I_{(r)}^{t,t+\delta} + \sum_{r_1, r_2=1}^{m} (g^{r_1, r_2}(t, Y^{\circ}) - g^{r_1, r_2}(t, Z^{\circ})) I_{(r_1, r_2)}^{t,t+\delta} \Big) \Big|^2 \Big]$$

$$\leq 6 \mathbb{E} \Big[\big| (M_{t,t+\delta}^{-1} - \mathrm{id}) \delta (f(t, Y^{\circ}) - f(t, Z^{\circ})) \big|^2 \Big]$$

$$+ 6 \mathbb{E} \Big[\big| (M_{t,t+\delta}^{-1} - \mathrm{id}) \sum_{r=1}^{m} (g_r(t, Y^{\circ}) - g_r(t, Z^{\circ})) I_{(r)}^{t,t+\delta} \big|^2 \Big]$$

$$+ 6 \mathbb{E} \Big[\big| (M_{t,t+\delta}^{-1} - \mathrm{id}) \sum_{r_1, r_2=1}^{m} (g^{r_1, r_2}(t, Y^{\circ}) - g^{r_1, r_2}(t, Z^{\circ})) I_{(r_1, r_2)}^{t,t+\delta} \big|^2 \Big]$$

$$\leq 6 \big(K_4 \delta^3 + K_5 \delta^{\frac{5}{2}} + K_6 \delta^3 \big) \mathbb{E}[|Y - Z|^2].$$

This completes the proof.

$\square$

Remark 4.4.8. The proof of stability of the classical BMM in [32, Th. 3.1] uses the equality

$$\mathbb{E}[|\Psi^{BMM}(Y, t, \delta)|^2] = \mathbb{E}[|Y|^2] + \mathbb{E}[2\langle Y, M_{t,t+\delta}^{-1} f(t, Y) \rangle] \delta + T_1 + T_2$$

$$+ \mathbb{E} \Big[\big| M_{t,t+\delta}^{-1} \big(f(t, Y) \delta + \sum_{r=1}^{m} g_r(t, Y) I_{(r)}^{t,t+\delta} + \sum_{r_1, r_2=1}^{m} g^{r_1, r_2}(t, Y) I_{(r_1, r_2)}^{t,t+\delta} \big) \big|^2 \Big],$$

where

$$T_1 = \mathbb{E}[2\langle Y, M_{t,t+\delta}^{-1} \sum_{r=1}^{m} g_r(t, Y) I_{(r)}^{t,t+\delta} \rangle]$$

$$\text{and} \quad T_2 = \mathbb{E}[2\langle Y, M_{t,t+\delta}^{-1} \sum_{r_1, r_2=1}^{m} g_r(t, Y) I_{(r_1, r_2)}^{t,t+\delta} \rangle].$$

It is claimed in [32] that T_1 and T_2 vanish due to " well-known martingale properties ". We could not decide whether this is true in general. The problem is that the elements of $M_{t,t+\delta}^{-1}$ and $I_{(r)}^{t,t+\delta}, I_{(r_1, r_2)}^{t,t+\delta}, r, r_1, r_2 = 1, \ldots, m$ are correlated. In the previous paper [44] on balanced Euler-type methods the authors give a symmetry argument why T_1 wanishes in the scalar case. In the proof of Theorem 4.4.7 we cure this problem by further estimating the terms T_1 and T_2.

4.5. B-consistency of the projected balanced Milstein method

In this section we show that the PBMM scheme is stochatically B-consistent of order $\gamma = 1$. We prove the following theorem:

Theorem 4.5.1. *Let the functions f and $g_r, r = 1, \ldots, m$ satisfy Assumption 4.3.1 with $L > 0$ and $q \in [2, \infty)$ and let Assumption 4.4.1 hold. Further, let $\bar{h} \in (0, 1]$. If the exact solution satisfies $\sup_{\tau \in [0,T]} \|X(\tau)\|_{L^{8q-6}} < \infty$, then the projected balanced Milstein method $(\Psi^{PBMM}, \bar{h}, X_0)$ with $\beta = \frac{1}{2(q-1)}$ is stochastically B-consistent of order $\gamma = 1$.*

Before we prove this result we quote the following lemmas from [5] and [6].

Lemma 4.5.2. *Let Assumption 4.3.1 be satisfied with $L > 0$ and $q \in [2, \infty)$. Further, let the exact solution X to the SODE (4.9)) satisfies $\sup_{t \in [0,T]} \|X(t)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)} < \infty$. Then, there exists a constant C such that for all $t_1, t_2, s \in [0, T]$ with $0 \le t_1 \le s \le t_2 \le T$ it holds*

$$\int_{t_1}^{t_2} \big\| f(\tau, X(\tau)) - f(s, X(t_1)) \big\|_{L^2(\Omega; \mathbb{R}^d)} \, \mathrm{d}\tau$$
$$C\big(1 + \sup_{t \in [0,T]} \big\| X(t) \big\|_{L^{4q-2}(\Omega; \mathbb{R}^d)}^{2q-1}\big)\big|t_1 - t_2\big|^{\frac{3}{2}}.$$

The proof can be found in [5, Lemma 5.5]. If we insert the conditional expectation with respect to the σ-field $\mathcal{F}_{t_1}$ then the Hölder exponent increases:

Lemma 4.5.3. *Let Assumption 4.3.1 be satisfied with $L > 0$ and $q \in [2, \infty)$. Further, let the exact solution X to the SODE (4.9)) satisfies $\sup_{t \in [0,T]} \|X(t)\|_{L^{6q-4}(\Omega; \mathbb{R}^d)} < \infty$. Then, there exists a constant C such that for all $t_1, t_2, s \in [0, T]$ with $0 \le t_1 \le s \le t_2 \le T$ it holds*

$$\int_{t_1}^{t_2} \big\| \mathbb{E}\big[f(\tau, X(\tau)) - f(s, X(t_1)) \big| \mathcal{F}_{t_1} \big] \big\|_{L^2(\Omega; \mathbb{R}^d)} \, \mathrm{d}\tau$$
$$C\big(1 + \sup_{t \in [0,T]} \big\| X(t) \big\|_{L^{6q-4}(\Omega; \mathbb{R}^d)}^{3q-2}\big)\big|t_1 - t_2\big|^2.$$

For the proof we refer to [6, Lemma 5.6].

Lemma 4.5.4. *Let Assumption 4.3.1 be satisfiedwith $L > 0$ and $q \in [2, \infty)$. Further, let the exact solution X to the SODE (4.9) satisfies $\sup_{t \in [0,T]} \|X(t)\|_{L^{6q-4}(\Omega; \mathbb{R}^d)} < \infty$. Then, there exists a constant C such that for all $r = 1, \ldots, m$ and $t_1, t_2, s \in [0, T]$ with $0 \le t_1 \le s \le t_2 \le T$ it holds*

$$\Big\| \int_{t_1}^{t_2} g_r(\tau, X(\tau)) - g_r(s, X(t_1)) \, \mathrm{d}W^r(\tau) - \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(s, X(t_1)) I_{(r_1, r_2)}^{t_1, t_2} \Big\|_{L^2(\Omega; \mathbb{R}^d)}$$
$$\le C\big(1 + \sup_{t \in [0,T]} \big\| X(t) \big\|_{L^{6q-4}(\Omega; \mathbb{R}^d)}^{3q-2}\big)\big|t_1 - t_2\big|^{\frac{3}{2}}.$$

The proof can be found in [6, Lemma 5.7]

Lemma 4.5.5. *Let f and g_r, $r = 1, \ldots, m$ satisfy Assumption 4.3.1 with $L > 0$ and $q \in [2, \infty)$ and let Assumption 4.4.1 hold. Further, let the exact solution X to the SODE (4.9) satisfies $\sup_{t \in [0,T]} \big\| X(t) \big\|_{L^{2q}(\Omega;\mathbb{R}^d)} < \infty$. Then, there exists a constant C such that for all $s, t \in [0, T]$ with $0 \leq s \leq t \leq T$ the following estimate holds*

$$
\begin{aligned}
\Big\| \mathbb{E} \big[(\mathrm{id} - M_{s,t}^{-1}) \big(f(s, X(s)) I_{(0)}^{s,t} + \sum_{r=1}^{m} g_r(s, X(s)) I_{(r)}^{s,t} \\
+ \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(s, X(s)) I_{(r_1, r_2)}^{s,t} \big| \mathcal{F}_s \big] \Big\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq C \big(1 + \sup_{t \in [0,T]} \| X(t) \|_{L^{2q}(\Omega;\mathbb{R}^d)}^q \big) |t - s|^2.
\end{aligned}
\tag{4.50}
$$

Remark 4.5.6. In the proof of the mean consistency of the classical BMM in [32, Th.2.4] it is claimed that

$$
\begin{aligned}
\big| \mathbb{E}[(\mathrm{id} - M_{s,t}^{-1})(f(s, X(s)) I_{(0)}^{s,t} + \sum_{r=1}^{m} g_r(s, X(s)) I_{(r)}^{s,t} \\
+ \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(s, X(s)) I_{(r_1, r_2)}^{s,t} | \mathcal{F}_s] \big| \\
\leq \big| M_{s,t}^{-1} \big| \big| \mathbb{E} \big[(M_{s,t} - \mathrm{id}) \big(f(s, X(s)) I_{(0)}^{s,t} + \sum_{r=1}^{m} g_r(s, X(s)) I_{(r)}^{s,t} \\
+ \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(s, X(s)) I_{(r_1, r_2)}^{s,t} \big) \big] \big|.
\end{aligned}
$$

The authors justify this by using the " discrete Hölder inequality ". However, $M_{s,t}^{-1}$ is only $\mathcal{F}_t$-measurable and not $\mathcal{F}_s$-measurable. Hence, can not be taken out from the conditional expectation. On the other hand, if one wants to estimate the term, then the product $|M_{s,t}^{-1}| \mathbb{E}[|\cdots|]$ appears. The following proof contains a suitable correction, which uses arguments similar to those in Remark 4.4.8.

Proof of Lemma 4.5.5. For arbitrary $r = 1, \ldots, m$ we obtain

$$
\begin{aligned}
\Big\| \mathbb{E} \big[(\mathrm{id} - M_{s,t}^{-1}) \big(f(s, X(s)) I_{(0)}^{s,t} + \sum_{r=1}^{m} g_r(s, X(s)) I_{(r)}^{s,t} \\
+ \sum_{r_1, r_2 = 1}^{m} g^{r_1, r_2}(s, X(s)) I_{(r_1, r_2)}^{s,t} | \mathcal{F}_s \big] \Big\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq \| \mathbb{E}[M_{s,t}^{-1} \big(d^0 I_{(0)}^{s,t} + \sum_{l=1}^{m} d^l I_{(l,l)}^{s,t} \big) f(s, X(s)) I_{(0)}^{s,t} | \mathcal{F}_s] \|_{L^2(\Omega;\mathbb{R}^d)} \\
+ \| \mathbb{E}[M_{s,t}^{-1} \big(d^0 I_{(0)}^{s,t} + \sum_{l=1}^{m} d^l I_{(l,l)}^{s,t} \big) \sum_{r=1}^{m} g_r(s, X(s)) I_{(r)}^{s,t} | \mathcal{F}_s] \|_{L^2(\Omega;\mathbb{R}^d)}
\end{aligned}
$$

$$+ \|\mathbb{E}[M_{s,t}^{-1}\big(d^0 I_{(0)}^{s,t} + \sum_{l=1}^{m} d^l I_{(l,l)}^{s,t}\big) \sum_{r_1,r_2=1}^{m} g^{r_1,r_2}(s, X(s)) I_{(r_1,r_2)}^{s,t} | \mathcal{F}_s]\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$=: \sum_{j=1}^{3} T_j.$$

By condition (4.15), Assumption 4.4.1, and and the fact that $\mathbb{E}[|I_{(l,l)}^{s,t}|^2] = \frac{1}{2}(t-s)^2$ for $0 \leq s < t \leq T$ it holds

$$\begin{aligned}
T_1 \leq & \|M_{s,t}^{-1} d^0 I_{(0)}^{s,t} f(s, X(s)) I_{(0)}^{s,t}\|_{L^2(\Omega;\mathbb{R}^d)} \\
& + \sum_{l=1}^{m} \|M_{s,t}^{-1} I_{(l,l)}^{s,t} d^l f(s, X(s)) I_{(0)}^{s,t}\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq & K_{\mathrm{M}}|t-s|^2(|d^0| + \frac{1}{\sqrt{2}} m \max_{l=1,\dots,m} |d^l|) L\|(1 + |X(s)|)^q\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq & K_{\mathrm{M}}L|t-s|^2(|d^0| + \frac{1}{\sqrt{2}} m \max_{l=1,\dots,m} |d^l|)\big(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{2q}(\Omega;\mathbb{R}^d)}^q\big).
\end{aligned}$$

For the second term we use (4.32), (4.34), and condition (4.19) and get

$$\begin{aligned}
T_2 \leq & \sum_{r=1}^{m} \big\|\mathbb{E}[M_{s,t}^{-1} I_{(r)}^{s,t} d^0 I_{(0)}^{s,t} g_r(s, X(s))|\mathcal{F}_s]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
& + \sum_{l=1}^{m}\sum_{r=1}^{m} \big\|\mathbb{E}[M_{s,t}^{-1} I_{(l,l)}^{s,t} I_{(r)}^{s,t} d^l g_r(s, X(s))|\mathcal{F}_s]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq & |d^0|m C_{(\mathrm{r})}|t-s|^{\frac{5}{2}} L\|(1 + |X(s)|)^{\frac{q+1}{2}}\|_{L^2(\Omega;\mathbb{R}^d)} \\
& + m^2 \max_{l=1,\dots,m} |d^l| C_{(\mathrm{r,l,l})}|t-s|^{\frac{5}{2}} L\|(1 + |X(s)|)^{\frac{q+1}{2}}\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq & Lm|t-s|^{\frac{5}{2}}(|d^0|C_{(\mathrm{r})} + m C_{(\mathrm{r,l,l})} \max_{l=1,\dots,m} |d^l|)\big(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{q+1}(\Omega;\mathbb{R}^d)}^{\frac{q+1}{2}}\big).
\end{aligned}$$

Since $q + 1 \leq 2q$ for $q \geq 1$, it follows that T_2 satisfies the estimate (4.50). Finally, by Assumption (4.4.1), estimations (4.33), (4.35), and condition (4.23) we get for the last term

$$\begin{aligned}
T_3 \leq & \sum_{r_1,r_2=1}^{m} \big\|\mathbb{E}[M_{s,t}^{-1} I_{(r_1,r_2)}^{s,t} d^0 I_{(0)}^{s,t} g^{r_1,r_2}(s, X(s))|\mathcal{F}_s]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
& + \sum_{l=1}^{m}\sum_{r_1,r_2=1}^{m} \big\|\mathbb{E}[M_{s,t}^{-1} I_{(l,l)}^{s,t} I_{(r_1,r_2)}^{s,t} d^l g^{r_1,r_2}(s, X(s))|\mathcal{F}_s]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq & m^2 C_{(\mathrm{r_1,r_2})}|t-s|^3|d^0|L\|(1 + |X(s)|)^q\|_{L^2(\Omega;\mathbb{R}^d)} \\
& + m^3 C_{(\mathrm{r_1,r_2,l,l})}|t-s|^3 \max_{l=1,\dots,m} |d^l||L\|(1 + |X(s)|)^q\|_{L^2(\Omega;\mathbb{R}^d)}
\end{aligned}$$

$$\leq L m^2 |t - s|^3 \big(|d^0| C_{(\mathrm{r_1,r_2})} + m \max_{l=1,\ldots,m} |d^l| C_{(\mathrm{r_1,r_2,l,l})} \big) \big(1 + \sup_{t \in [0,T]} \|X(t)\|^q_{L^{2q}(\Omega;\mathbb{R}^d)} \big).$$

This proves the assertion (4.50).

<div style="text-align:right">□</div>

Proof of Theorem 4.5.1. By using of (4.25) and (4.38) we obtain

$$X(t+\delta) - \Psi^{PBMM}(X(t),t,\delta) = X(t) - X^\circ(t) + \int_t^{t+\delta} f(\tau, X(\tau)) - f(t, X(t)) \, d\tau$$

$$+ \sum_{r=1}^m \int_t^{t+\delta} g_r(\tau, X(\tau)) - g_r(t, X(t)) \, dW_r(\tau) - \sum_{r_1,r_2=1}^m g^{r_1,r_2}(t, X(t)) I^{t,t+\delta}_{(r_1,r_2)}$$

$$+ M^{-1}_{t,t+\delta} \delta \big(f(t, X(t)) - f(t, X^\circ) \big) + M^{-1}_{t,t+\delta} \sum_{r=1}^m \big(g_r(t, X(t)) - g_r(t, X^\circ(t)) \big) I^{t,t+\delta}_{(r)}$$

$$+ M^{-1}_{t,t+\delta} \sum_{r_1,r_2=1}^m \big(g^{r_1,r_2}(t, X(t)) - g^{r_1,r_2}(t, X^\circ(t)) \big) I^{t,t+\delta}_{(r_1,r_2)}$$

$$+ \big(\mathrm{id} - M^{-1}_{t,t+\delta} \big) \Big(\delta f(t, X(t)) + \sum_{r=1}^m g_r(t, X(t)) I^{t,t+\delta}_{(r)} + \sum_{r_1,r_2=1}^m g^{r_1,r_2}(t, X(t)) I^{t,t+\delta}_{(r_1,r_2)} \Big).$$

Further, it holds

$$\|\mathbb{E}[X(t+\delta) - \Psi^{PBMM}(X(t),t,\delta)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq \|X(t) - X^\circ(t)\|_{L^2(\Omega;\mathbb{R}^d)} + \int_t^{t+\delta} \|\mathbb{E}[f(\tau, X(\tau)) - f(t, X(t))|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)} \, d\tau$$

$$+ \|\mathbb{E}[M^{-1}_{t,t+\delta} \delta \big(f(t, X(t)) - f(t, X^\circ) \big)|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \|\mathbb{E}[M^{-1}_{t,t+\delta} \sum_{r=1}^m \big(g_r(t, X(t)) - g_r(t, X^\circ(t)) \big) I^{t,t+\delta}_{(r)}|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \|\mathbb{E}[M^{-1}_{t,t+\delta} \sum_{r_1,r_2=1}^m \big(g^{r_1,r_2}(t, X(t)) - g^{r_1,r_2}(t, X^\circ(t)) \big) I^{t,t+\delta}_{(r_1,r_2)}|\mathcal{F}_t]\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+ \Big\| \mathbb{E}\big[\big(\mathrm{id} - M^{-1}_{t,t+\delta} \big) \Big(\delta f(t, X(t)) + \sum_{r=1}^m g_r(t, X(t)) I^{t,t+\delta}_{(r)}$$

$$+ \sum_{r_1,r_2=1}^m g^{r_1,r_2}(t, X(t)) I^{t,t+\delta}_{(r_1,r_2)} \Big)|\mathcal{F}_t \big] \Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$=: \sum_{j=1}^6 S_j.$$

For the first term it holds by Lemma 3.4.6 with $\varphi = \mathrm{id}, \kappa = 1, p = 8q - 6$ and $\beta = \frac{1}{2(q-1)}$

$$S_1 \leq C_{\mathrm{dif}} (1 + \|X(t)\|^{4q-3}_{L^{8q-6}(\Omega;\mathbb{R}^d)}) \delta^2,$$

since $\frac{1}{2}\beta(p-2)\kappa = 2$. For the second term we have by Lemma 4.5.3

$$S_2 \leq C\big(1 + \sup_{t\in[0,T]} \|X(t)\|^q_{L^{2q}(\Omega;\mathbb{R}^d)}\big)\delta^2.$$

An application of Lemma 3.4.6 with $\varphi = f(t,\cdot)$, $\kappa = q$ and $p = \frac{6q-4}{q}$ and since $\frac{1}{2}\beta(p-2)\kappa = 1$ we obtain

$$
\begin{aligned}
S_3 \leq & \|M^{-1}_{t,t+\delta}\delta\big(f(t,X(t)) - f(t,X^\circ(t))\big)\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq & K_{\mathrm{M}}\delta\|f(t,X(t)) - f(t,X^\circ(t))\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq & K_{\mathrm{M}}C_{\mathrm{dif}}\big(1 + \|X(t)\|^{3q-2}_{L^{6q-4}(\Omega;\mathbb{R}^d)}\big)\delta^2.
\end{aligned}
\tag{4.51}
$$

Further, using estimate (4.32) we obtain for S_4

$$
\begin{aligned}
S_4 &= \sum_{r=1}^{m} \big\|\mathbb{E}[M^{-1}_{t,t+\delta}I^{t,t+\delta}_{(r)}\big(g_r(t,X(t)) - g_r(t,X^\circ(t))\big)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq C_{(\mathrm{r})}\delta^{\frac{3}{2}} \sum_{r=1}^{m} \|g_r(t,X(t)) - g_r(t,X^\circ(t))\|_{L^2(\Omega;\mathbb{R}^d)}.
\end{aligned}
$$

By applying Lemma 3.4.6 with $\varphi = g_r(t,\cdot)$, $\kappa = \frac{q+1}{2}$ and $p = \frac{10q-6}{q+1}$ we get

$$
\begin{aligned}
S_4 &\leq C_{(\mathrm{r})}m\delta^{\frac{3}{2}}C_{\mathrm{dif}}\big(1 + \|X(t)\|_{L^{5q-3}}\big)^{\frac{5q-3}{2}}\delta \\
&\leq C\big(1 + \sup_{t\in[0,T]} \|X(t)\|^{\frac{5q-3}{2}}_{L^{5q-3}(\Omega;\mathbb{R}^d)}\big)\delta^{\frac{5}{2}},
\end{aligned}
\tag{4.52}
$$

and we note that $5q - 3 \leq 8q - 6$ for $q \geq 1$. Similarly, by estimate (4.33) we obtain

$$
\begin{aligned}
S_5 &= \sum_{r_1,r_2=1}^{m} \big\|\mathbb{E}[M^{-1}_{t,t+\delta}I^{t,t+\delta}_{(r_1,r_2)}\big(g^{r_1,r_2}(t,X(t)) - g^{r_1,r_2}(t,X^\circ(t))\big)|\mathcal{F}_t]\big\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq C_{(\mathrm{r_1,r_2})}\delta^2 \sum_{r_1,r_2=1}^{m} \|g^{r_1,r_2}(t,X(t)) - g^{r_1,r_2}(t,X^\circ(t))\|_{L^2(\Omega;\mathbb{R}^d)}.
\end{aligned}
$$

Then, by Lemma 3.4.6 with $\varphi = g^{r_1,r_2}(t,\cdot)$, $\kappa = q$ and $p = \frac{4q-2}{q}$ it holds

$$
\begin{aligned}
S_5 &\leq C_{(\mathrm{r_1,r_2})}m^2\delta^2 C_{\mathrm{dif}}\big(1 + \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}\big)^{2q-1}\delta^{\frac{1}{2}} \\
&\leq C_{(\mathrm{r_1,r_2})}C_{\mathrm{dif}}m^2\big(1 + \sup_{t\in[0,T]} \|X(t)\|^{2q-1}_{L^{4q-2}(\Omega;\mathbb{R}^d)}\big)\delta^{\frac{5}{2}}.
\end{aligned}
$$

By applying Lemma 4.5.5 we get for the last summand

$$S_6 \leq C\big(1 + \sup_{t\in[0,T]} \|X(t)\|^q_{L^{2q}(\Omega;\mathbb{R}^d)}\big)\delta^2.$$

This proves (2.5). It remains to show (2.6). It holds

$$\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(X(t+\delta) - \Psi^{PBMM}(X(t),t,\delta)\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\int_t^{t+\delta} f(\tau, X(\tau))\,\mathrm{d}\tau\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+\sum_{r=1}^m\|\int_t^{t+\delta} g_r(\tau, X(\tau)) - g_r(t, X(t))\,\mathrm{d}W_r(\tau) - \sum_{r_2}^m g^{r,r_2}(t, X(t))I_{(r_2,r)}^{t,t+\delta}\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])M_{t,t+\delta}^{-1}\delta\big(f(t, X(t)) - f(t, X^\circ(t))\big)\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+\sum_{r=1}^m\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])M_{t,t+\delta}^{-1}\big(g_r(t, X(t)) - g_r(t, X^\circ(t))\big)I_{(r)}^{t,t+\delta}\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+\sum_{r_1,r_2=1}^m\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])M_{t,t+\delta}^{-1}\big(g^{r_1,r_2}(t, X(t)) - g^{r_1,r_2}(t, X^\circ(t))\big)I_{(r_1,r_2)}^{t,t+\delta}\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$+\left\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])(\mathrm{id} - M_{t,t+\delta}^{-1})\Big(\delta f(t, X(t)) + \sum_{r=1}^m g_r(t, X(t))I_{(r)}^{t,t+\delta}\right.$$

$$\left.+\sum_{r_1,r_2=1}^m g^{r_1,r_2}(t, X(t))I_{(r_1,r_2)}^{t,t+\delta}\Big)\right\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$=:\sum_{j=1}^6 Q_j.$$

For the first summand we have

$$Q_1 = \|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])\int_t^{t+\delta} f(\tau, X(\tau)) - f(t, X(t))\,\mathrm{d}\tau\|_{L^2(\Omega;\mathbb{R}^d)},$$

since $\mathbb{E}[f(t, X(t))|\mathcal{F}_t] = f(t, X(t))$. Using the fact that $\|(\mathrm{id} - \mathbb{E}[\cdot|\mathcal{F}_t])Z\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$ and Lemma 4.5.2 we obtain

$$Q_1 \leq \int_t^{t+\delta} \|f(t, X(\tau)) - f(t, X(t))\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}\tau$$

$$\leq C\big(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{4q-2}(\Omega;R^d)}^{2q-1}\big)\delta^{\frac{3}{2}}.$$

For the second term we apply Lemma 4.5.3 and get

$$Q_2 \leq C\big(1 + \sup_{t\in[0,T]} \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)}^{3q-2}\big)\delta^{\frac{3}{2}}.$$

In the same way and with (4.51) we have for the next term

$$Q_3 \leq K_M C_{\mathrm{dif}}\big(1 + \|X(t)\|_{L^{6q-4}(\Omega;\mathbb{R}^d)}^{3q-2}\big)\delta^2.$$

Further, by Lemma 3.4.6 with $\varphi = g_r(t, \cdot), \kappa = \frac{q+1}{2}$ and $p = \frac{10q-6}{q+1}$ we get

$$Q_4 \leq \sum_{r=1}^{m} \|M_{t,t+\delta}^{-1}\big(g_r(t, X(t)) - g_r(t, X^\circ(t))\big)I_{(r)}^{t,t+\delta}\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq K_{\mathrm{M}}C_{\mathrm{dif}}\big(1 + \sup_{t \in [0,T]} \|X(t)\|_{L^{5q-3}(\Omega;\mathbb{R}^d)}^{\frac{5q-3}{2}}\big)\delta^{\frac{3}{2}},$$

since $\mathbb{E}[|I_{(r)}^{t,t+\delta}|^2] = \delta$ and $\frac{1}{2}\beta(p-2)\kappa = 1$. A similar estimate holds also for Q_5 with $\varphi = g^{r_1,r_2}(t, \cdot), \kappa = q$ and $p = \frac{4q-2}{q}$

$$Q_5 \leq \sum_{r_1,r_2=1}^{m} \|M_{t,t+\delta}^{-1}\big(g^{r_1,r_2}(t, X(t)) - g^{r_1,r_2}(t, X^\circ(t))\big)I_{(r_1,r_2)}^{t,t+\delta}\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq \frac{1}{\sqrt{2}}K_{\mathrm{M}}C_{\mathrm{dif}}\big(1 + \sup_{t \in [0,T]} \|X(t)\|_{L^{4q-2}(\Omega;\mathbb{R}^d)}^{2q-1}\big)\delta^{\frac{3}{2}}.$$

In this case $\frac{1}{2}\beta(p-2)\kappa = \frac{1}{2}$ and $\mathbb{E}[|I_{(r_1,r_2)}^{t,t+\delta}|^2] = \frac{1}{2}\delta^2$. Finally, using $\|(\mathrm{id}-\mathbb{E}[\cdot|\mathcal{F}_t])Z\|_{L^2(\Omega;\mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $Z \in L^2(\Omega;\mathbb{R}^d)$ and Lemma 4.5.5 yield

$$Q_6 \leq \Big\|(\mathrm{id} - M_{t,t+\delta}^{-1})\Big(\delta f(t, X(t)) + \sum_{r=1}^{m} g_r(t, X(t))I_{(r)}^{t,t+\delta}$$

$$+ \sum_{r_1,r_2=1}^{m} g^{r_1,r_2}(t, X(t))I_{(r_1,r_2)}^{t,t+\delta}\Big)\Big\|_{L^2(\Omega;\mathbb{R}^d)}$$

$$\leq C\big(1 + \sup_{t \in [0,T]} \|X(t)\|_{L^{2q}(\Omega;\mathbb{R}^d)}^q\big)\delta^2.$$

This completes the proof.

$\square$

Remark 4.5.7. If the weight coefficients $d^l \in \mathbb{R}^{d \times d}, l = 0, \ldots, m$ are non-constant then the proof of stochastic C-stability and B-consistency needs more delicate estimates. And it can happen that one needs even higher moments of the solutions than before.

5. Numerical results

In this chapter we present several numerical examples. On one hand they show strengths and weaknesses of the proposed method, and on the other hand they are designed to illustrate the strong convergence results from previous sections.

We note that we have two ways to compute the transformed schemes. The first way is to follow exactly the Wiener transformation described in Section 1.2. In this case we get strong approximations of the (1.15). The second way is to determine the Wiener processes for the transformed equation (1.18) directly by a random number generator and avoid the transformation. In this case we obtain only weak approximation of (1.15). Anyway, in both cases we have the same distribution. In order to simplify the computations we use for our numerical tests the second method.

First, we turn to the example in the linear case, in which we compared the simple balanced method (1.39) and the Euler-Maruyama scheme, see Section 1.4. Table 5.1 and Figure 5.1 show the estimated strong error of convergence for seven different equidistant step sizes $h = 2^{k-11}$, $k = 1, \ldots, 7$. For simplicity we only estimate the error at the end time $T = 1$, that is

$$\text{error} = \left(\mathbb{E}[|X_h(T) - X(T)|^2] \right)^{\frac{1}{2}}, \tag{5.1}$$

where $X_h(T)$ denotes the numerical approximation of the reference solution $X(T)$. The expected value is estimated by a Monte Carlo simulation based on 10^6 sample paths.

Table 5.1.: Estimated errors and EOCs for the approximations of the linear SODE (1.18).

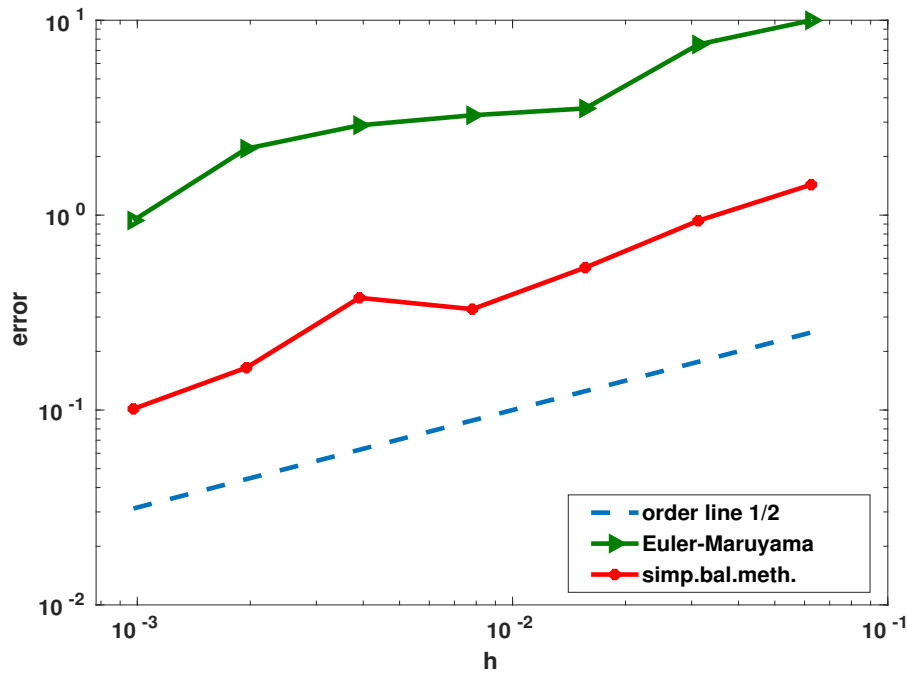| h	Simp.bal.meth.		Euler-Maruyama	
	error	EOC	error	EOC
2^{-4}	1.43300		9.98300	
2^{-5}	0.93560	0.62	7.50900	0.41
2^{-6}	0.53790	0.80	3.52700	1.09
2^{-7}	0.32920	0.70	3.25500	0.12
2^{-8}	0.37660	-0.19	2.89000	0.17
2^{-9}	0.16490	1.19	2.19200	0.40
2^{-10}	0.10130	0.70	0.93910	1.22

Figure 5.1.: Strong convergence errors for the approximation of the linear SODE (1.18).

As before the parameter values are

$$A = \begin{pmatrix} -0.8 & -1 \\ 0.5 & -1 \end{pmatrix}, \ G_1 = \begin{pmatrix} -3.8 & 0.05 \\ 0.075 & 0.1 \end{pmatrix}, \ G_2 = \begin{pmatrix} -0.3 & -0.05 \\ 0.5 & -2 \end{pmatrix}, \ T = 1,$$ and the initial

value $X_0 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$. Table 5.1 contains the estimates of the errors and the corresponding experimental order of convergence:

$$\text{EOC} = \frac{\log(\text{error}(h_i)) - \log(\text{error}(h_{i-1}))}{\log(h_i) - \log(h_{i-1})}, \quad i = 2, \ldots, k.$$

This example shows that the Euler-Maruyama scheme gives very large errors comparing with the simple balanced method. This indicates that the latter method is more suitable to capture the solution behavior of SODEs with large noise.

In our next numerical example we compare the simple balanced method (1.39) and the balanced shift noise Euler-type scheme (1.65), see section 1.7. Here we use the same parameters as above except the matrix A, which is given by

$$A = \begin{pmatrix} -8 & -1 \\ 0.5 & -1 \end{pmatrix}.$$

Table 5.2.: Estimated errors and EOCs for the approximations of the linear SODE (1.18).

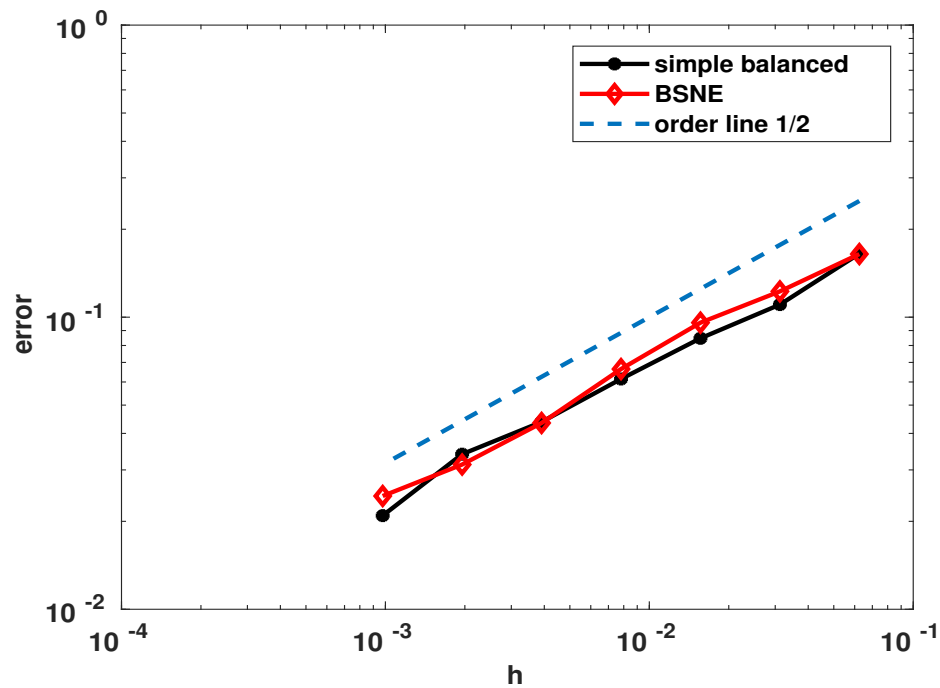| h	Simp.bal.meth.		BSNE	
	error	EOC	error	EOC
2^{-4}	0.16499		0.16451	
2^{-5}	0.11045	0.58	0.12256	0.42
2^{-6}	0.08467	0.38	0.09576	0.36
2^{-7}	0.06143	0.46	0.06643	0.53
2^{-8}	0.04379	0.49	0.04343	0.61
2^{-9}	0.03396	0.37	0.03130	0.47
2^{-10}	0.02092	0.70	0.02442	0.36

Figure 5.2.: Strong convergence errors for the approximation of the linear SODE (1.18).

In contrast to the path-wise convergence (see Figure 1.2), one can see that the convergence in the mean square sense (5.1) delivers almost no difference in the error estimate for both numerical approximations. In Figure 5.2 and Table 5.2 one clearly observes

strong order $\gamma = \frac{1}{2}$ for both methods.

5.1. Stochastic Hopf equation

Consider the following two-dimensional system of stochastic differential equations

$$
\begin{aligned}
\mathrm{d}x(t) = {} & \left[\begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} \mu - x_1^2 - x_2^2 \\ \theta \end{pmatrix} - \frac{1}{2} G_2^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \mathrm{d}t \\
& + G_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mathrm{d}W_1(t) + G_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mathrm{d}W_2(t),
\end{aligned}
\tag{5.2}
$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $\mu, \theta \in \mathbb{R}$ and $G_1, G_2 \in \mathbb{R}^{2\times 2}$. As in [6, Section 8] we call this the stochastic Hopf system since variation of the parameter μ drives the system through a stochastic Hopf bifurcation (see [3, Ch.9.4.2]). In [6, Section 8] the matrices G_1 and G_2 were assumed to commute while here we consider the general noncommuting case. Since f is cubic with uniform upper Lipschitz bound and the diffusion coefficients are globally Lipschitz continuous, the Assumption 1.1.1 as well as Assumption 4.3.1 are fulfilled with the growth rate $q = 3$.

In our tests the SODE (5.2) is discretized by the PBSNE Euler-type scheme, the SSBSNI Euler-type method, the projected Euler-Maruyama (PEM) scheme, proposed in [5], the PBSNM method, and the PBMM scheme.

5.1.1. PBSNE and SSBSNI Euler-type schemes

In this section we test the projected balanced shift noise Euler-type method (3.15) and the split-step balanced shift noise Euler-type scheme (3.16). In addition, we compare with the projected Euler-Maruyama method (see [5])

$$
\overline{X}_h(t_i) = \min(1, h_i^{-\beta} |X_h(t_{i-1})|^{-1}) X_h(t_{i-1}),
$$

$$
X_h(t_i) = \overline{X}_h(t_i) + h_i f(\overline{X}(t_i)) + \sum_{r=1}^{m} G_r \overline{X}_h(t_i) I_{(r)}^{t_{i-1}, t_i},
$$

for all $i = 1, \ldots, N, h \in (0, 1]^N, \beta = \frac{1}{2(q-1)}$ and $X_h(0) := X_0$.

Figure 5.3 shows the simulation of a single path of the reference solution and the PBSNE Euler-type method with step-size $h = 2^{-4}$ and parameters

$G_1 = \begin{pmatrix} -0.5 & 1 \\ 0.5 & 1 \end{pmatrix}$, $G_2 = \begin{pmatrix} 2 & 0.5 \\ -4.5 & 2 \end{pmatrix}$, $\beta = \frac{1}{4}, \mu = -0.5, \theta = 1, T = 2^4$, and the initial value $X_0 = \begin{pmatrix} 1.35 \\ 1.35 \end{pmatrix}$.
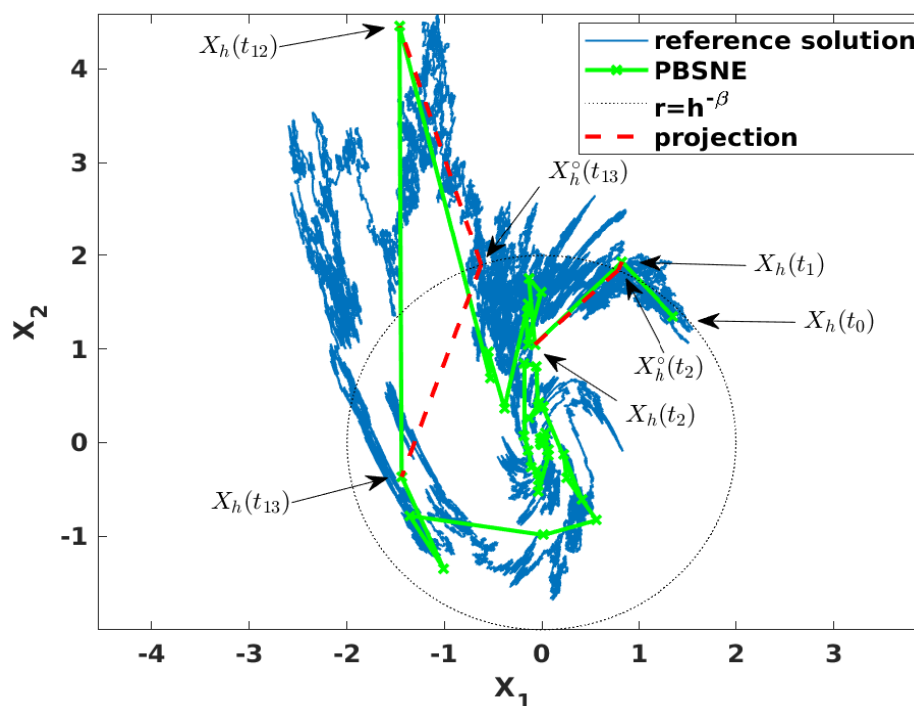
Figure 5.3.: Sample trajectory of the PBSNE Euler-type method with step size $h = 2^{-4}$, initial value $X_0 = (1.35, 1.35)^\top$, and $T = 2^4$.

Since there is no explicit expression for the solution of (5.2) we replace the exact solution by a reference approximation (3.15) with very small step-size $\Delta t = 2^{-18}$. The transformed matrices are given by

$$\tilde{G}_1 = \begin{pmatrix} 2.01 & 0.47 \\ -4.51 & 1.97 \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} -0.44 & 1.01 \\ -0.37 & 1.06 \end{pmatrix}.$$

Table 5.3 provides an overview of the Frobenius norm and eigenvalues of the diffusion matrices $G_1, G_2, \tilde{G}_1$, and $\tilde{G}_2$.

Table 5.3.: Frobenius norm and eigenvalues

	G_1	G_2	$\tilde{G}_1$	$\tilde{G}_2$
Frobenius norm	1.58	5.33	5.34	1.57
Eigenvalues	-0.78	$2 + 1.5i$	$1.99 + 1.45i$	-0.66
	1.28	$2 - 1.5i$	$1.99 - 1.45i$	1.27

As already mentioned in [5], [6] we are interested in trajectories of the PBSNE method which do not coincide with trajectories generated by the balanced shift noise explicit

Euler-type scheme. This event occurs when the scheme leaves the sphere of radius $h^{-\beta}$, i.e.

$$\#\text{-Proj.} = \{i = 1, \ldots, N \colon |X_h^{PBSNE}(t_i)| > h^{-\beta}\}. \tag{5.3}$$

In the Figure 5.3 one can see that the trajectory of the PBSNE scheme crosses the circle of radius $h^{-\beta} = 2$ twice: in the first and in the twelfth steps. The intermediate values $X_h^\circ(t_2)$ and $X_h^\circ(t_{13})$ are connected by dashed lines.

Figure 5.4 and Table 5.4 show the results of the strong error convergence for the PBSNE method, the SSBSNI scheme, and PEM method. All three methods converge with the strong order $\gamma = \frac{1}{2}$. The parameters and initial value are as in Figure 5.3. Nonlinear equations in the SSBSNI scheme are solved by the Newton method with three iteration steps. The number of samples for which the trajectories of the PBSNE method, SSBSNI scheme, and PEM method leave the sphere of radius $h^{-\beta}$ is given in the fourth and ninth column of Table 5.4.
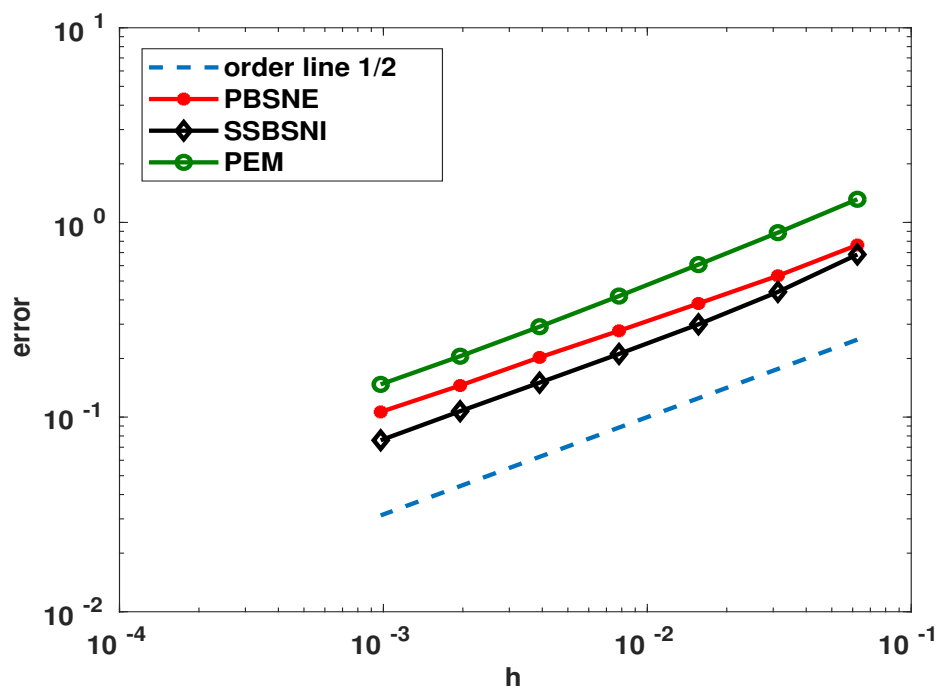

Figure 5.4.: Strong convergence errors for the approximation of (5.2).

The error estimates, given by (5.1) at the final time $T = 1$ with seven different equidistant step sizes $h = 2^{k-11}$, $k = 1, \ldots, 7$ are based on Monte Carlo simulations with 10^6 samples. In Figure 5.4 it can be seen that the PEM scheme gives larger errors than the

PBSNE method and the SSBSNI scheme. Actually, the explicit solution of the SODE with the largest noise term plays an important role in the damping of large amplitudes and leads to better results in the approximations. In the PEM method all noise terms are approximated with the Euler-Maruyama scheme.

Table 5.4.: Estimated errors and EOCs for the approximations of (5.2)

	PBSNE			SSBSNI		PEM		
h	error	EOC	#-proj.	error	EOC	error	EOC	#-proj.
2^{-4}	0.76556		73766	0.68454		1.31484		268618
2^{-5}	0.53193	0.53	28017	0.43965	0.64	0.88604	0.57	61672
2^{-6}	0.38325	0.47	10085	0.29964	0.55	0.60729	0.54	14702
2^{-7}	0.27743	0.47	3005	0.21107	0.51	0.41853	0.54	3529
2^{-8}	0.20258	0.45	797	0.15026	0.49	0.29125	0.52	859
2^{-9}	0.14526	0.48	200	0.10721	0.49	0.20543	0.50	189
2^{-10}	0.10627	0.45	49	0.07600	0.50	0.14724	0.48	48

5.1.2. PBSNM and PBMM schemes

In the next step we compare the projected balanced shift noise Milstein-type scheme (4.2) and the projected balanced Milstein method (4.8).

Since $m = 2$ both steps of the split-step method involve at most one double stochastic integral and no mixed integrals. As in [36] we evaluate the double integral by

$$\frac{1}{2}\big((\tilde{I}_{(2)}^{t_{i-1},t_i})^2 - h_i\big), \quad i = 1, \ldots, N.$$

As we already noted in Chapter 4 the splitting into steps avoids in this example evaluating the iterated Itô integrals $\tilde{I}_{(1,2)}^{t_{i-1},t_i}$ and $\tilde{I}_{(2,1)}^{t_{i-1},t_i}$. In [46] it was proved that without evaluating iterated stochastic integrals the higher order of convergence is not possible. Table 5.5 and Figure 5.5 show that the strong order of convergence for the balanced shift noise Milstein-type is only one half.

Here we use the same parameters as in Section 5.1.1 for $T = 1$ and the initial value $X_0 = \begin{pmatrix} 1.35 \\ 1.35 \end{pmatrix}$. The estimates of the errors are based on the Monte Carlo simulation with the same number 10^6 of samples.

Table 5.5.: Estimated errors and EOCs for the approximations of (4.1)

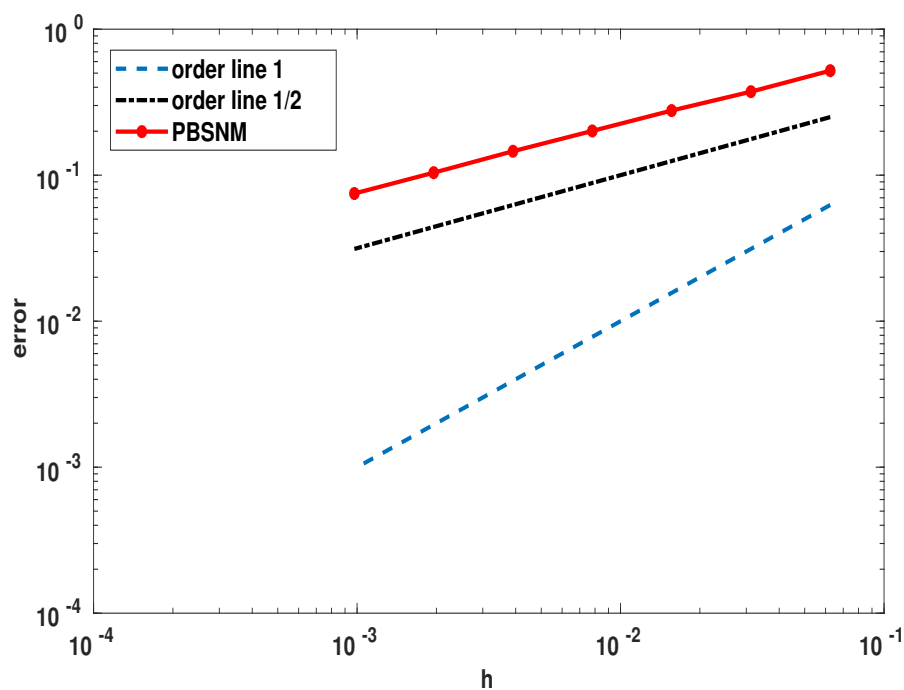|	PBSNM		
h	error	EOC	#-proj.
2^{-4}	0.51966		35791
2^{-5}	0.37318	0.48	17383
2^{-6}	0.27702	0.43	7609
2^{-7}	0.20089	0.46	2722
2^{-8}	0.14588	0.46	718
2^{-9}	0.10400	0.49	183
2^{-10}	0.07490	0.47	41

Figure 5.5.: Strong convergence errors for the approximation of (4.1)

However, if the noise terms commute, then the strong order of convergence is one. This can be seen in Table 5.6 and Figure 5.6. Here we present the comparison of the PBSNM method and the PBMM scheme. For this experiment we use the following parameter values:

$G_1 = \begin{pmatrix} -0.5 & 0 \\ 0 & 1 \end{pmatrix}$, $G_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\beta = \frac{1}{4}, \mu = -0.5, \theta = 1, T = 1$, and the initial value

$X_0 = \begin{pmatrix} 1.35 \\ 1.35 \end{pmatrix}$. The transformed matrices are given by

$\tilde{G}_1 = \begin{pmatrix} 1.91 & 0 \\ 0 & 2.12 \end{pmatrix}$ and $\tilde{G}_2 = \begin{pmatrix} -0.78 & 0 \\ 0 & 0.70 \end{pmatrix}$. It is obvious that the PBSNM method has smaller errors than the PBMM scheme.

Table 5.6.: Estimated errors and EOCs for the approximations of (4.1)

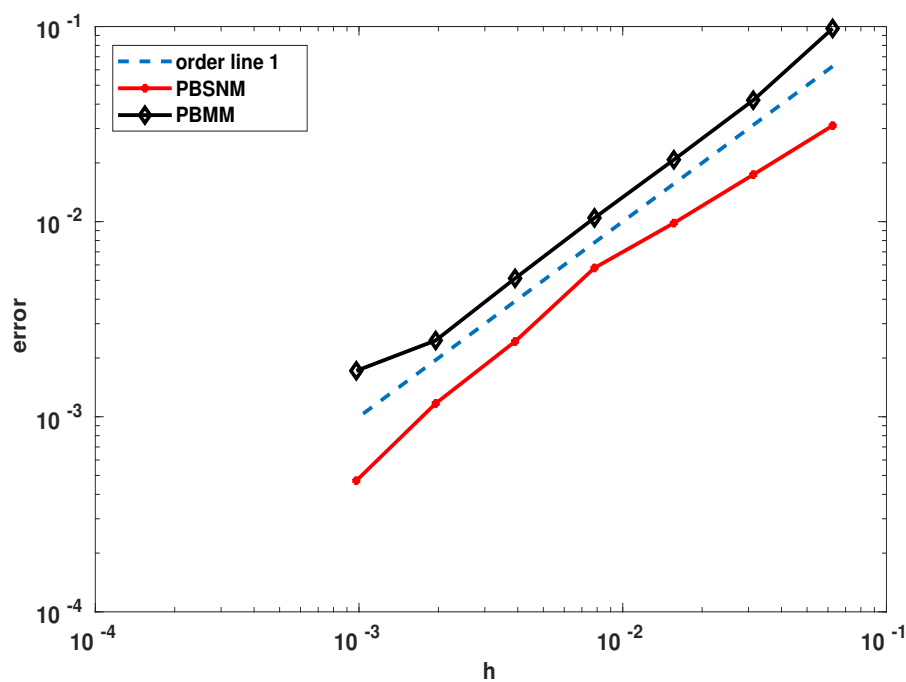| h	PBSNM			PBMM		
	error	EOC	#-proj.	error	EOC	#-proj.
2^{-4}	0.03104		1072	0.09771		830
2^{-5}	0.01741	0.83	415	0.04192	1.22	420
2^{-6}	0.00983	0.82	123	0.02076	1.01	161
2^{-7}	0.00579	0.76	51	0.01045	0.99	49
2^{-8}	0.00243	1.25	8	0.00512	1.03	13
2^{-9}	0.00117	1.06	2	0.00246	1.06	0
2^{-10}	0.00047	1.32	0	0.00172	0.51	1

Figure 5.6.: Strong convergence errors for the approximation of (4.1)

For the implementation of the PBMM method we use the following weight matrices

$$d_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; d_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } d_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

As in the case of the Euler-type methods we can see that the PBSNM method has smaller errors than the PBMM scheme.

Our next task is to implement the projected balanced Milstein method (4.8) if the noise terms do not commute. Except the integrals $I_{(1,1)}^{t_{i-1},t_i}$ and $I_{(2,2)}^{t_{i-1},t_i}$ we should also generate the integrals $I_{(1,2)}^{t_{i-1},t_i}$ and $I_{(2,1)}^{t_{i-1},t_i}$. There are several publications for evaluating iterated stochastic integrals, see for example [35], [63], [52], [17]. In our numerical tests we don't focus on excellent calculations of double Itô integrals, but we only check whether the theoretical statements of the PBMM scheme also apply in practice. Therefore, we generate the iterated integrals as well as the reference solution of (5.2) by the Euler-Maruyama method with a very small step size $\Delta t = 2^{-18}$. For computing of $I_{(1,2)}^{0,t}$ we have to implement the following system for $t \in [0, T]$

$$\begin{aligned} dX_1(t) &= dW_1(t), \quad X_1(0) = 0, \\ dX_2(t) &= X_1(t) \, dW_2(t), \quad X_2(0) = 0. \end{aligned}$$

Using the relation (see [36])

$$I_{(1,2)} + I_{(2,1)} = I_{(1)} I_{(2)}$$

we determine the second iterated Itô integral $I_{(2,1)}$. It is certainly a very expensive procedure, but we accept that to get the expected order of convergence 1 of the PBMM scheme.

Table 5.7.: Estimated errors and EOCs for the approximations of (4.1)

	PBMM		
h	error	EOC	#-proj.
2^{-4}	0.11289		830
2^{-5}	0.04777	1.24	319
2^{-6}	0.02500	0.93	120
2^{-7}	0.01348	0.89	42
2^{-8}	0.00749	0.85	8
2^{-9}	0.00429	0.80	2
2^{-10}	0.00316	0.44	1

Table 5.7 and Figure 5.7 show the results of the strong convergence for the projected balanced Milstein method with the parameter values

$G_1 = \begin{pmatrix} 0 & 1 \\ 0.5 & 1 \end{pmatrix}$, $G_2 = \begin{pmatrix} 2 & 0.5 \\ 0 & 2 \end{pmatrix}$, $\beta = \frac{1}{4}, \mu = -0.5, \theta = 1, T = 1$, and the initial value

$X_0 = \begin{pmatrix} 1.35 \\ 1.35 \end{pmatrix}$. The matrices d_0, d_1 and d_2 are given as above.
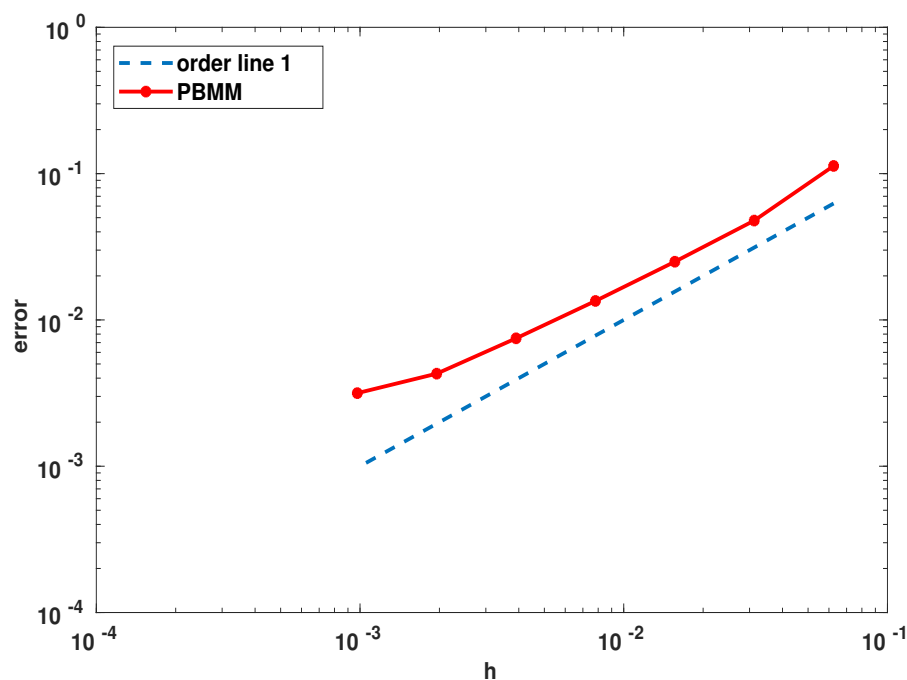

Figure 5.7.: Strong convergence errors for the approximation of (4.1)

Our estimate of the errors are based on a Monte Carlo simulation with 10^6 sample paths. The numerical results confirm the theoretical order of convergence, though with some loss towards smaller step sizes for the PBMM scheme.

5.2. Some experiments for the stochastic Lorenz system

The next example is a three-dimensional system, goes back to Lorenz [41], which is showing chaotic features already in the deterministic case. The stochastic version of the Lorenz system was already studied in [54]. Let us consider for a triple $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ and $G_1, G_2, G_3 \in \mathbb{R}^{3 \times 3}$ the following system

$$dx(t) = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ \lambda_2 & -1 & -x_1 \\ x_2 & 0 & -\lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} dt + G_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} dW_1(t)$$
$$+ G_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} dW_2(t) + G_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} dW_3(t)$$

$$(5.4)$$

for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. It is known that apart from the global Lipschitz continuous diffusion coefficient in (5.4), the drift coefficient is not globally one-sided Lipschitz continuous. But it satisfies the global coercivity condition (1.7), see [54], [27]. Therefore, it is an interesting example where only part of our assumptions are satisfied.

There is no explicit solution available, hence we replace the exact solution in (5.4) by the numerical approximation (3.15) with a very fine step size $\Delta t = 2^{-18}$. The implicit scheme is again implemented by solving the nonlinear equation by the Newton method with three iteration steps. Figure 5.10 shows the simulation of the single paths of the reference solution, the PBSNE and SSBSNI Euler-type schemes with parameters

$$G_1 = \begin{pmatrix} 2^{-3} & 2^{-5} & 0 \\ 2^{-10} & 2^{-7} & 0 \\ 0 & 0 & 0 \end{pmatrix}, G_2 = \begin{pmatrix} 0 & 2^{-5} & 2^{-4} \\ 0 & 2^{-7} & 2^{-10} \\ 0 & 0 & 0 \end{pmatrix}, G_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^{-4} & 2^{-7} \\ 0 & 2^{-4} & 2^{-6} \end{pmatrix},$$

$X_0 = \begin{pmatrix} 2^{-7} \\ 2^{-7} \\ 2^{-7} \end{pmatrix}$, and $T = 2^6$. The transformed matrices are given by

$$\tilde{G}_1 = \begin{pmatrix} -0.124 & -0.034 & -0.006 \\ -0.001 & -0.012 & 0 \\ 0 & -0.004 & -0.001 \end{pmatrix}, \tilde{G}_2 = \begin{pmatrix} 0.009 & -0.002 & -0.008 \\ 0 & -0.062 & -0.008 \\ 0 & -0.061 & -0.016 \end{pmatrix},$$

and

$$\tilde{G}_3 = \begin{pmatrix} 0.010 & -0.028 & -0.061 \\ 0 & 0.001 & 0 \\ 0 & 0.009 & 0.002 \end{pmatrix}.$$

An overview of the Frobenius norm and eigenvalues is shown in Table 5.8.

Table 5.8.: Frobenius norm and eigenvalues

	G_1	G_2	G_3	$\tilde{G}_1$	$\tilde{G}_2$	$\tilde{G}_3$
Frobenius norm	0.13	0.07	0.09	0.13	0.09	0.07
Eigenvalues	0.13	0	0.01	−0.13	0.01	0.01
	0.01	0.01	0.07	−0.01	−0.07	0
	0	0	0	0	−0.01	0.01

The stochastic Lorenz system is analyzed theoretically in [54], [29]. Practically, un-bounded trajectories arise when the noise terms are not very small. In the Figure 5.10 (b) one can see that the SSBSNI scheme produces for $h = 2^{-6}$ large errors in contrast to the PBSNE method.

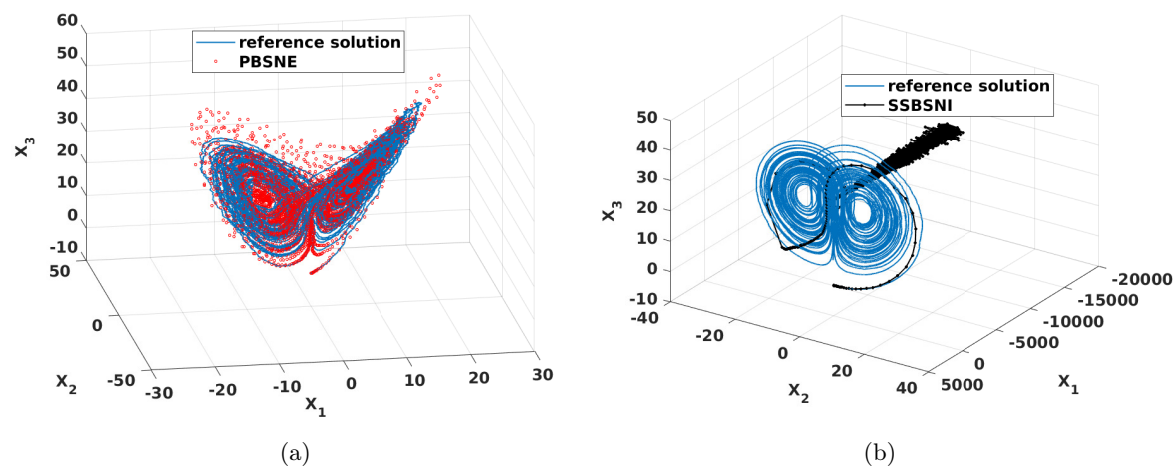(a)   (b)

Figure 5.8.: Sample trajectories of reference solution, PBSNE and SSBSNI Euler-type methods with step size $h = 2^{-6}$, initial value $X_0 = (2^{-7}, 2^{-7}, 2^{-7})^\top$, and $T = 2^6$.

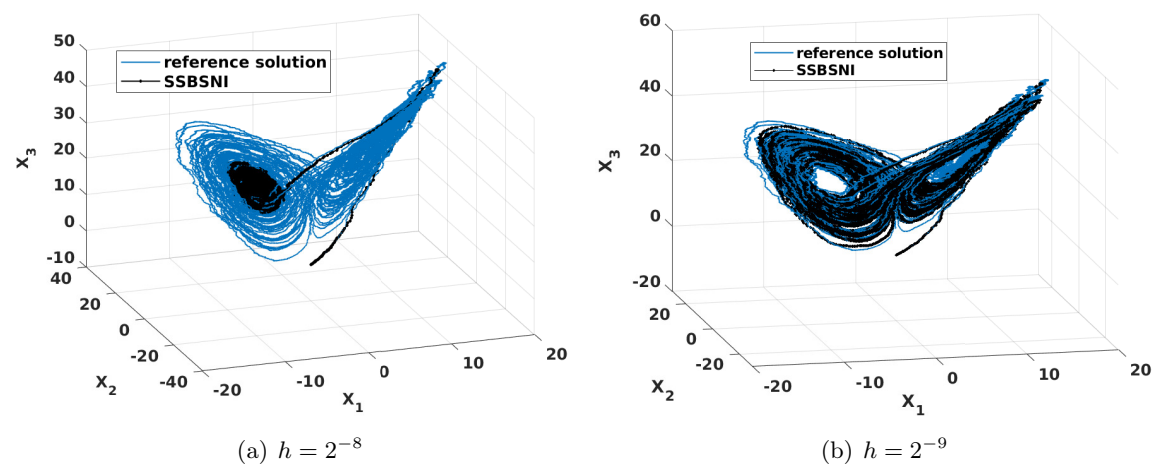(a) $h = 2^{-8}$ (b) $h = 2^{-9}$

Figure 5.9.: Sample trajectories of reference solution and SSBSNI Euler-type method with several step sizes, initial value $X_0 = (2^{-7}, 2^{-7}, 2^{-7})^\top$, and $T = 2^6$.

Figure 5.9 shows that the split-step scheme creates only for a step size as small as

$h = 2^{-9}$. This show that the SSBSNI method is inefficient for the stochastic Lorenz system.

The simulation of a single path of the PBSNE method with a step size $h = 2^{-6}$ for $T = 1$ shows that already on a small time interval the PBSNE scheme deviates significantly from the reference solution. It is known from the deterministic case that the Lorenz system is sensitive to the initial conditions. In the stochastic case this sensitivity increases even more.
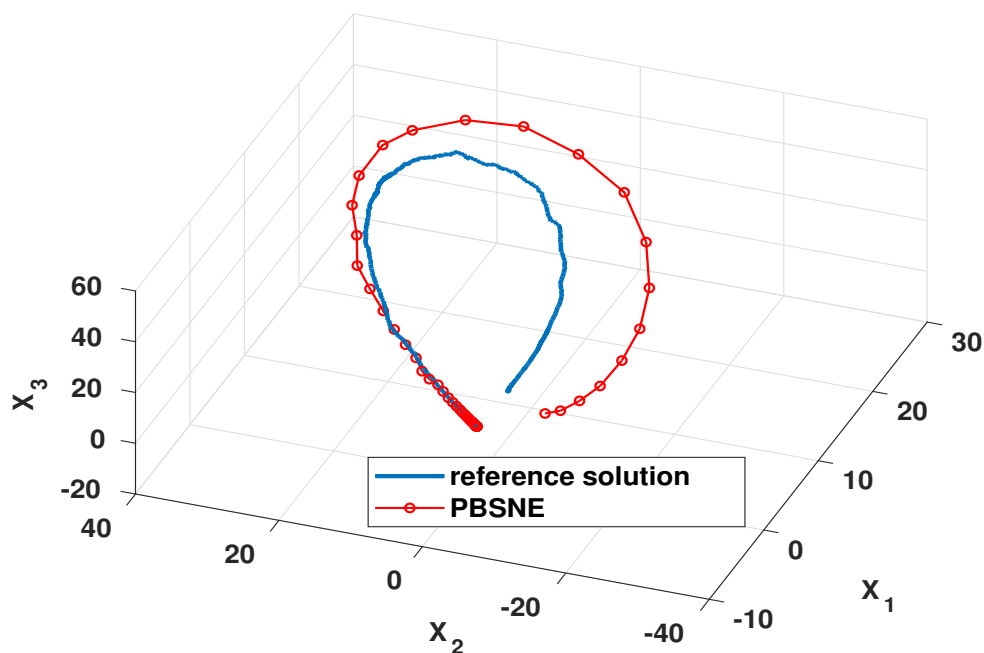

Figure 5.10.: Sample trajectory of PBSNE Euler-type method with step size $h = 2^{-6}$, initial value $X_0 = (2^{-7}, 2^{-7}, 2^{-7})^\top$, and $T = 1$.

Table 5.9.: Estimated errors and EOCs for the approximations of (5.4)

	PBSNE			SSBSNI	
h	error	EOC	#-proj.	error	EOC
2^{-10}	9.89413		0	10.17977	
2^{-11}	6.13433	0.69	0	5.97699	0.77
2^{-12}	3.24605	0.92	0	3.04421	0.97
2^{-13}	1.62101	1.00	0	1.75943	0.79
2^{-14}	0.78605	1.04	0	1.37597	0.35

Table 5.9 and Figure 5.11 show the strong convergence errors of the PBSNE scheme and the SSBSNI method with five different steps $h = 2^{k-15}, k = 1, \ldots, 5$. The strong error is measured at the endpoint $T = 1$ by (5.1) with a Monte Carlo simulation using 10^6 samples.
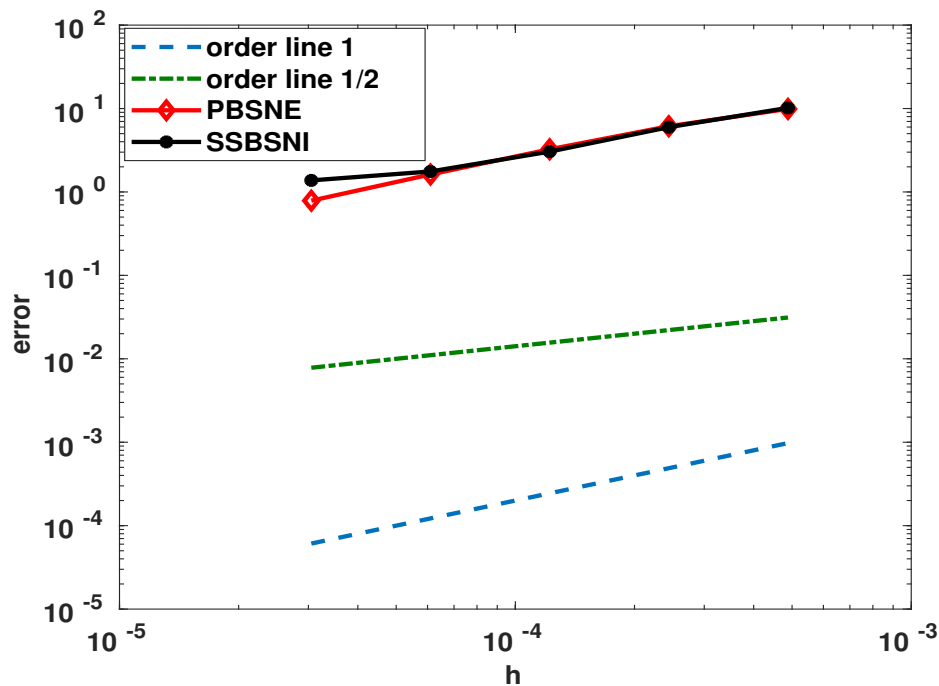

Figure 5.11.: Strong convergence errors for the approximation of (5.2).

Perhaps the perturbation in stochastic terms is neglected here and we can see only the strong convergence of the deterministic system, and for this system both methods produce a large error for every step size. According to our experiments it seems that the balanced shift noise methods (3.15) and (3.16) are not well suited to the approximation of the stochastic Lorenz system given by (5.4).

Conclusions

In this thesis, we have dealt with numerical solutions for stochastic differential equations with large noise. We have analyzed our proposal for the solution of these problems analytically as well as numerically.

As shown in Section 1.2 an orthogonal transformation of the Wiener process gives us the opportunity to isolate the largest noise term. Our approach is then to split the numerical integration into two steps:

1. Solving the SODE without the largest noise term by the Euler-Maruyama scheme or the Milstein method,

2. Solving explicitly the SODE with the largest noise term.

The theoretical and numerical results lead to several open questions:

- How do the suggested methods work if the diffusion coefficient functions are not autonomous or even nonlinear?

- Is there an optimal choice for the shift matrix C (see (1.40) and (1.41))?

As in [5] the convergence theory of numerical methods was based on the study of stochastic C-stability and stochastic B-consistency under the one-sided Lipschitz condition. By keeping track of the constants we have tried to derive sharper estimates. It is shown that the constants $C_{\text{cons},1}$ and $C_{\text{cons},2}$ in (2.5) and (2.6) are of moderate type in the sense of Convention 1.3.4. On the other hand, the C-stability constant C_{stab} is not of moderate type, because our assumptions allow the solutions to grow exponentially in mean square. This leads to the further questions:

- Is it possible to improve this result under stronger assumptions?

- Can the exponents of the exponential terms be replaced by the logarithmic norm of the stability matrix S (see (0.3))?

The experiments in Chapter 5 have shown that in general our method of Milstein-type converges strongly with order $\frac{1}{2}$ and therefore does not belong to the methods of higher order. On the contrary, the projected balanced Milstein method is strongly convergent of order 1. Therefore, there is interest in the investigating the weak convergence of these methods.

A. Appendix

A.1. Itô formula

In this section we recall some known results from stochastic analysis.

Definition A.1.1. *Let $(W(t))_{t \geq 0}$ be an m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. A d-dimensional Itô process is an $\mathbb{R}^d$-valued continuous adapted process $X(t)$ on $t \geq 0$ of the form*

$$X(t) = X_0 + \int_0^t f(\tau) \, d\tau + \int_0^t g(\tau) \, dW(\tau),$$

where $f = (f_1, \ldots, f_d)^T \in L^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g = (g_{ij})_{d \times m} \in L^2(\mathbb{R}_+; \mathbb{R}^{d \times m})$. The integral is formally written with stochastic differentials as follows

$$dX(t) = f(t) \, dt + g(t) \, dW(t). \tag{A.1}$$

Theorem A.1.2 (The multi-dimensional Itô formula)**.** *Let $X(t)$ be a d-dimensional Itô process on $t \geq 0$ with the stochastic differential (A.1). Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Then $V(X(t), t)$ is again an Itô process with the stochastic differential given by*

$$
\begin{aligned}
dV(X(t), t) = & [V_t(X(t), t) + V_x(X(t), t)f(t) \\
& + \frac{1}{2}\mathrm{trace}(g^T(t)V_{xx}(X(t), t)g(t))] \, dt + V_x(X(t), t)g(t) \, dW(t).
\end{aligned}
\tag{A.2}
$$

The proof of the Theorem A.1.2 can be found, for example, in [15], [36] or [42].

Remark A.1.3 (Version for Wiener processes)**.** Let $(W(t))_{t \geq 0}$ be a standard Wiener process and $h : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function. Then it holds

$$h(W(t)) = h(W_0) + \int_0^t h'(W(\tau)) \, dW + \frac{1}{2} \int_0^t h''(W(\tau)) \, d\tau.$$

Also, for the process $Y(t) = h(W(t)), t \geq 0$ this formula can be written in the differential notation

$$dY(t) = h'(W(t)) \, dW(t) + \frac{1}{2}h''(W(t)) \, dt.$$

A.2. Stochastic Gronwall lemma

In this section we prove an stochastic version of the Gronwall lemma in continuous time, proposed in [53]. The proof of this lemma is taken from the book [7] that is currently in preparation.

Definition A.2.1. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space. A right continuous adapted process* $M = \{M(t)\}_{t \in [0,T]}$ *is called a local martingale if there exists a non-decreasing sequence* $\{\tau_n\}_{n \in \mathbb{N}}$ *of stopping times with* $\mathbf{P}(\tau_n \to \infty$ *as* $n \to \infty) = 1$ *such that every* $\{M_{\tau_n \wedge t} - M_0\}_{t \in [0,T]}$ *is a martingale.*

This definition can be found in [42]. In the following, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a filtered probability space and $T \in (0, \infty)$.

Lemma A.2.2. *Let* $Z, H : [0, T] \times \Omega \to \mathbb{R}$ *be nonnegative, adapted processes with continuous paths and assume that* $\psi : [0, T] \to [0, \infty)$ *is integrable and nonnegative. Let* $M : [0, T] \times \Omega \to \mathbb{R}$ *be a continuous local martingale with* $M(0) = 0$. *Suppose that for every* $t \in [0, T]$ *it holds*

$$Z(t) \leq H(t) + \int_0^t \psi(s) Z(s) \, \mathrm{d}s + M(t). \tag{A.3}$$

Then

$$\mathbb{E}[Z(t)] \leq \exp\left(\int_0^t \psi(s) \, \mathrm{d}s \right) \mathbb{E}[\sup_{s \in [0,t]} H(s)]. \tag{A.4}$$

Proof. Let for $t \in [0, T]$ define

$$\Phi(t) := \int_0^t \psi(s) \, \mathrm{d}s.$$

We note that Φ is a deterministic process. Further, let us assume that the process H is non-decreasing and, hence, of bounded variation. Otherwise we replace $H(t)$ by $\sup_{s \in [0,t]} H(s)$. Next, let define two auxiliary processes $\overline{Z}, \underline{\psi} : [0, T] \times \Omega \to [0, \infty)$ by

$$\overline{Z}(t) := H(t) + \int_0^t \psi(s) Z(s) \, \mathrm{d}s + M(t),$$

$$\underline{\psi}(t) := \mathbb{I}_{Z(t) \neq 0}(t) \psi(t) \frac{Z(t)}{\overline{Z}(t)},$$

for all $t \in [0, T]$. By (A.3) we have that $0 \leq Z(t) \leq \overline{Z}(t)$ **P**-almost surely. Following, $\underline{\psi}$ is well-defined and $\underline{\psi}(t) \leq \psi(t)$. Consequently, we define by

$$\underline{\Phi}(t) := \int_0^t \underline{\psi}(s) \, \mathrm{d}s$$

for every $t \in [0, T]$. We have that $\underline{\Phi}(t) \leq \Phi(t)$ almost surely. By setting $\overline{Z}$ and $\underline{\psi}$ in (A.4) we obtain

$$\overline{Z}(t) = H(t) + \int_0^t \underline{\psi}(s)\overline{Z}(s)\,\mathrm{d}s + M(t), \quad \mathbf{P}\text{-almost surely.}$$

It is obvious that $\overline{Z}$ is an almost surely continuous semimartingale. By the Itô formula we get for the semimartingales that (see [33, Th.17.18])

$$\exp(-\underline{\Phi}(t))\overline{Z}(t) = H(0) + \int_0^t \exp(-\underline{\Phi}(s))\,\mathrm{d}H(s) + \int_0^t \exp(-\underline{\Phi}(s))\,\mathrm{d}M(s).$$

Obeserve that $\underline{\Phi}(t) \geq 0$, then it follows that $\exp(-\underline{\Phi}(t)) \leq 1$. Further, let $(\tau_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of stopping times with $\mathbf{P}(\tau_n \to \infty \text{ as } n \to \infty) = 1$, such that the stopped process $(M(t \wedge \tau_n))_{t \in [0,T]}$ is a martingale for all $n \in \mathbb{N}$. Then for all $t \in [0, T]$ and $n \in \mathbb{N}$ we obtain

$$\mathbb{E}[\exp(-\underline{\Phi}(t \wedge \tau \wedge \tau_n))\overline{Z}(t \wedge \tau \wedge \tau_n)] \leq \mathbb{E}[H(t \wedge \tau \wedge \tau_n)] \leq \mathbb{E}[H(t)],$$

since $\exp(\underline{\Phi}(t)) \leq 1$ and H is non-decreasing. Further, by Fatou's lemma we get

$$\begin{aligned}
\mathbb{E}[\exp(-\underline{\Phi}(t \wedge \tau))\overline{Z}(t \wedge \tau)] =& \mathbb{E}[\liminf_{n \to \infty} \exp(-\underline{\Phi}(t \wedge \tau \wedge \tau_n))\overline{Z}(t \wedge \tau \wedge \tau_n)] \\
\leq& \lim_{n \to \infty} \mathbb{E}[\exp(-\underline{\Phi}(t \wedge \tau \wedge \tau_n))\overline{Z}(t \wedge \tau \wedge \tau_n)] \\
\leq& \mathbb{E}[H(t)].
\end{aligned}$$

Since $\underline{\Phi}(t) \leq \Phi(t)$ almost surely and $\overline{Z}$ is non-negative it holds with $\tau = t$

$$\begin{aligned}
\mathbb{E}[\overline{Z}(t)] =& \mathbb{E}[\exp(\underline{\Phi}(t))\exp(-\underline{\Phi}(t))\overline{Z}(t)] \\
\leq& \exp(\Phi(t))(\mathbb{E}[\exp(-\underline{\Phi}(t))\underline{Z}(t)] \\
\leq& \exp(\Phi(t))\mathbb{E}[H(t)]).
\end{aligned}$$

Then, by $Z(t) \leq \overline{Z}(t)$ follows (A.4).

$\square$

A.3. Higher order estimates for Itô multi-integrals

The following lemma is a general result for the products of multiple Itô integrals. Here we use the same notation for the multi-indices as in [36]. For example, the hierarchical set of the classical Milstein method is given by

$$\mathcal{A}_1 = \{\alpha \in \mathcal{A} \colon 1 \leq l(\alpha) + n(\alpha) \leq 2\} = \{(0), (1), \ldots, (m)\} \cup \{(i,j) | i, j = 1, \ldots, m\},$$

where $\mathcal{A}$ is the set of multi-indices, $l = l(\alpha)$ is the length of α and $n(\alpha)$ is the number of components in α with $\alpha_i = 0, i = 1, \ldots, l$.

Lemma A.3.1. *Assume that $0 \leq t - s \leq 1$. Then there exists a real constant $K_{\mathrm{mult}} = K_{\mathrm{mult}}(\alpha, \alpha', p)$ for all $\alpha = (\alpha_1, ..., \alpha_\ell) \in \mathcal{A}$, $\alpha' = (\alpha'_1, ..., \alpha'_\ell) \in \mathcal{A}$ and $p = 1, 2, ...,$ such that*

$$\mathbb{E}\big[(I_\alpha^{s,t} I_{\alpha'}^{s,t})^{2p}\big] \leq K_{\mathrm{mult}}(t-s)^{p(\ell(\alpha)+n(\alpha)+\ell(\alpha')+n(\alpha'))}. \tag{A.5}$$

Proof. It is convenient, to introduce the rescaled and shifted Brownian motion for $r \in [0, 1]$

$$W_{[s,t]}^{\alpha_i}(r) = (t-s)^{-(1+\delta_{\alpha_i,0})/2}\big(W^{\alpha_i}(s + r(t-s)) - W^{\alpha_i}(s)\big), \; t > s, \; i = 0, ..., m, \tag{A.6}$$

where $\delta_{\alpha_i,0}$ is the Kronecker symbol. In the case $\alpha_i = 0$ the increment is $dW_{[s,t]}^0(r) = dW^0(r) = dr$. Let $n(\alpha) = \sum_{i=1}^{\ell} \delta_{\alpha_i,0}$. By shifting and the time change formula for Itô-integrals (see Th.8.5.7 in [47]) we have

$$I_\alpha^{s,t} = (t-s)^{(\ell(\alpha)+n(\alpha))/2} I_{[\alpha]}^{s,t}, \tag{A.7}$$

where

$$I_{[\alpha]}^{s,t} = \int_0^1 s_1^{(1+\delta_{\alpha_1,0})/2} \int_0^1 s_2^{(1+\delta_{\alpha_2,0})/2} \cdots \int_0^1 dW_{[s,s+s_{\ell-1}(t-s)]}^{\alpha_1}(s_\ell) \cdots dW_{[s,t]}^{\alpha_\ell}(s_1). \tag{A.8}$$

Then for all $\alpha, \alpha' \in \mathcal{A}$ it holds

$$\mathbb{E}\big[(I_\alpha^{s,t} I_{\alpha'}^{s,t})^{2p}\big] = (t-s)^{p(\ell(\alpha)+n(\alpha)+\ell(\alpha')+n(\alpha'))}\mathbb{E}\big[(I_{[\alpha]}^{s,t} I_{[\alpha']}^{s,t})^{2p}\big]. \tag{A.9}$$

For the estimate of the expectation of the product of iterated stochastic integrals we apply the Cauchy-Schwarz inequality

$$\mathbb{E}\big[(I_{[\alpha]}^{s,t} I_{[\alpha']}^{s,t})^{2p}\big] \leq \big(\mathbb{E}\big[(I_{[\alpha]}^{s,t})^{4p}\big]\big)^{\frac{1}{2}} \big(\mathbb{E}\big[(I_{[\alpha']}^{s,t})^{4p}\big]\big)^{\frac{1}{2}}.$$

Now, by Burkholder-Davis-Gundy inequality (see for example in [4]) we obtain for all $\alpha \in \mathcal{A}$, $p = 1, 2, ...,$

$$\mathbb{E}\big[(I_{[\alpha]}^{s,t})^{4p}\big] \leq K_{BDG}\mathbb{E}\big[\big(\int_0^1 (s_1^{(1+\delta_{\alpha_1,0})/2} \cdots \int_0^1 \cdots dW_{[s,s+s_1(t-s)]}^{\alpha_{\ell-1}}(s_2))^2 ds_1\big)^{2p}\big]$$

$$\leq K_{BDG}\int_0^1 s_1^{2p(1+\delta_{\alpha_1,0})}\mathbb{E}\big[\big(\int_0^1 s_2^{(1+\delta_{\alpha_2,0})/2} \cdots \int_0^1 \cdots dW_{[s,s+s_1(t-s)]}^{\alpha_{\ell-1}}(s_2)\big)^{4p}\big]ds_1$$

$$\leq \cdots$$

$$\leq K_{BDG}^{\ell-1}\frac{(4p-1)!!}{(2p)^{\ell-1}\prod_{j=1}^{\ell-1}(1+\delta_{\alpha_j,0})},$$

where K_{BDG} is the Burkholder-Davis-Gundy constant. Therefore also for all $\alpha' \in \mathcal{A}$ we obtain

$$\mathbb{E}\big[\big(I_{[\alpha']}^{s,t}\big)^{4p}\big] \leq K_{BDG}^{\ell-1}\frac{(4p-1)!!}{(2p)^{\ell-1}\prod_{j=1}^{\ell-1}(1+\delta_{\alpha'_j,0})}, \quad p = 1, 2, ...,$$

which proves the assertion (A.5). $\qquad\square$

A.4. Logarithmic norm and its properties

The logarithmic norm appears in various applications: in differential equations, numerical analysis, or in matrix theory. The classical definition was independently introduced in 1958 by Dahlquist and Lozinskii, see more, for example, in [58].

Definition A.4.1. *For the quadratic matrix M with an induced matrix norm $|\cdot|$ the associated logarithmic norm $\mu(M)$ is defined by*

$$\mu(M) = \lim_{h \to 0^+} \frac{|\mathrm{id} + Mh| - 1}{h}, \tag{A.10}$$

where id *is identity matrix, and* $h > 0$.

It is known that the limit in (A.10) exists and convergence to the limit is monotonic, see in [12]. As an alternative, the logarithmic norm can be defined as follows

$$\mu(M) = \sup_{z \neq 0} \frac{Re\langle z, Mz \rangle}{\langle z, z \rangle}$$

for all $z \in \mathbb{C}^d$. The following lemma summarizes well-known results of properties of the logarithmic norm. These may be found, for example in [58], [60].

Lemma A.4.2. *Let M and P be quadratic matrices and $\alpha(M)$ is the maximal real part of the eigenvalues of M. Denote by λ a real number and by z a complex number. Then the following properties hold*

1. $\mu(M) \leq |M|$,
2. $\mu(\lambda M) = |\lambda| \mu(\mathrm{sgn}(\lambda) M)$,
3. $\mu(M + P) \leq \mu(M) + \mu(P)$,
4. $\alpha(M) \leq \mu(M)$,
5. $|\mathrm{e}^{Mt}| \leq \mathrm{e}^{\mu(M)t}$,
6. $\mu(M + z\,\mathrm{id}) = \mu(M) + Re z$,
7. $If \ \mu(M) < 0, \quad then \quad |M^{-1}| \leq -\dfrac{1}{\mu(M)}.$

For the most common norms $|\cdot|_p, p = 1, 2, \infty$ the logarithmic norm may be expressed as follows:

- $\mu_\infty(M) = \sup_i (Re(m_{ii}) + \sum\limits_{j,j \neq i} |m_{ij}|)$,

- $\mu_1(M) = \sup_j (Re(m_{jj}) + \sum\limits_{i,i \neq j} |m_{ij}|)$,

- $\mu_2(M) = \lambda_{\max}\left(\frac{M + M^H}{2}\right)$.

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