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Doubly Reflected BSDEs and $\mathcal{E}^f$-Dynkin games: beyond the right-continuous case

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Abstract

We formulate a notion of doubly reflected BSDE in the case where the barriers ξ and ζ do not satisfy any regularity assumption. Under a technical assumption (a Mokobodzki-type condition), we show existence and uniqueness of the solution. In the case where ξ is right upper-semicontinuous and ζ is right lower-semicontinuous, the solution is characterized in terms of the value of a corresponding $\mathcal{E}^f$-Dynkin game, i.e. a game problem over stopping times with (non-linear) f-expectation, where f is the driver of the doubly reflected BSDE. In the general case where the barriers do not satisfy any regularity assumptions, the solution of the doubly reflected BSDE is related to the value of "an extension" of the previous non-linear game problem over a larger set of "stopping strategies" than the set of stopping times. This characterization is then used to establish a comparison result and *a priori* estimates with universal constants.

Keywords: Doubly reflected BSDEs; backward stochastic differential equations; Dynkin game; saddle points; f-expectation; nonlinear expectation; game option; stopping time; stopping system.
AMS MSC 2010: Primary 60G40; 93E20; 60H30, Secondary 60G07; 47N10.

1 Introduction

Backward stochastic differential equations (BSDEs) have been introduced in the case of a linear driver in [3], and then generalized to the non-linear case by Pardoux and Peng [33]. The theory of BSDEs provides a useful tool for the study of financial problems such as the pricing of European options among others (cf., e.g., [12] and [13]). When the driver f is non-linear, a BSDE induces a useful family of non-linear operators, first introduced in [13] under the name of *non linear pricing system*, and later called f-*evaluation* (also, f-*expectation*) and denoted by $\mathcal{E}^f$ (cf. [34]). Reflected BSDEs (RBSDEs) are a variant of BSDEs in which the solution is constrained to be greater than or equal to a given process called *obstacle*. RBSDEs have been introduced in [11] in the case of a Brownian filtration and a continuous obstacle, and links with (non-linear) optimal stopping problems with f-expectations have been given in [13]. RBSDEs have been generalized to the case of a not necessarily continuous obstacle and/or a larger filtration than the Brownian one by several authors [21], [5], [27], [15], [28], [37]. In

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all these works, the obstacle has been assumed to be right-continuous. The paper [18] is the first to study RBSDEs beyond the right-continuous case: there, we work under the assumption that the obstacle is only right-uppersemicontinuous. In [19], we address the case where the obstacle does not satisfy any regularity assumption. Existence and uniqueness of the solution in the irregular case is also shown in [30] (in the Brownian framework) by using a different approach. In [18] and [19], links with optimal stopping problems with f-expectations are also provided.

Doubly reflected BSDEs (DRBSDEs) have been introduced by Cvitanic and Karatzas in [6] in the case of continuous barriers and a Brownian filtration. The solutions of such equations are constrained to stay between two adapted processes ξ and ζ, called *barriers*, with $\xi \leq \zeta$ and $\xi_T = \zeta_T$. In the case of non-continuous barriers and/or a larger filtration, DRBSDEs have been studied by several authors, cf. [2], [23], [25], [26], [24], [5], [16], [28], [8]. In all of the above-mentioned works on DRBSDEs, the barriers are assumed to be at least right-continuous.

In the first part of the present paper, we formulate a notion of doubly RBSDEs in the case where the barriers do not satisfy any regularity assumption. We show existence and uniqueness of the solution of these equations. To this purpose, we first consider the case where the driver does not depend on the solution, and is thus given by an adapted process (f_t). We show that in this particular case, the solution of the DRBSDE can be written in terms of the difference of the solutions of a coupled system of two reflected BSDEs. We show that this system (and hence the Doubly Reflected BSDE) admits a solution if and only if the so-called *Mokobodzki's condition* holds (assuming the existence of two strong supermartingales whose difference is between ξ and ζ). We then provide *a priori* estimates for our doubly RBSDEs, by using Gal'chouk-Lenglart's formula (cf. Corollary A.2 in [18]). From these estimates, we derive *the uniqueness of the solution* of the doubly RBSDE associated with driver process (f_t). We then solve the case of a general Lipschitz driver f by using the *a priori* estimates and Banach fixed point theorem.

In the second part of the paper, we focus on links between the solution of the doubly reflected BSDE with irregular barriers from the first part and some related two-stopper-game problems.

Let us first recall the "classical" Dynkin game problem which has been largely studied (cf., e.g., [1] for general results).

Let $\mathcal{T}_0$ denote the set of all stopping times valued in $[0, T]$, where $T > 0$. For each pair $(\tau, \sigma) \in \mathcal{T}_0 \times \mathcal{T}_0$, the terminal time of the game is given by $\tau \wedge \sigma$ and the terminal payoff, or reward, of the game (at time $\tau \wedge \sigma$) is given by

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}. \tag{1.1}$$

The criterion is defined as the (linear) expectation of the pay-off, that is, $E\,[I(\tau, \sigma)]$. It is well-known that, if ξ is right upper-semicontinuous (right u.s.c) and ζ is right lower-semicontinuous (right l.s.c) and satisfy Mokobodzki's condition, this classical Dynkin game has a (common) value, that is, the following equality holds:

$$\inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} E\,[I(\tau, \sigma)] = \sup_{\tau \in \mathcal{T}_0} \inf_{\sigma \in \mathcal{T}_0} E\,[I(\tau, \sigma)]. \tag{1.2}$$

Moreover, under the additional assumptions that ξ and $-\zeta$ are left-uppersemicontinuous along stopping times and $\xi_t < \zeta_t$, $t < T$, there exists a saddle point (cf. [1], [31])[1].

[1] Actually, the strict separability condition on ξ and ζ is not necessary to ensure the existence of a saddle point (cf. Remark 3.8 in [8]). Note also that when ξ and ζ do not satisfy any regularity assumption, there does not necessarily exist a value for the Dynkin game, that is, the equality (1.2) does not necessarily hold.

Furthermore, when the processes ξ and ζ are right-continuous, the (common) value of the classical Dynkin game is equal to the solution at time 0 of the doubly reflected BSDE with driver equal to 0 and barriers (ξ, ζ) (cf. [6],[26],[32]).

In the second part of the present paper, we consider the following generalization of the classical Dynkin game problem: For each pair $(\tau, \sigma) \in \mathcal{T}_0 \times \mathcal{T}_0$, the criterion is defined by $\mathcal{E}^f_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$, where $\mathcal{E}^f_{0,\tau\wedge\sigma}(\cdot)$ denotes the f-expectation at time 0 when the terminal time is $\tau \wedge \sigma$. We refer to this generalized game problem as *$\mathcal{E}^f$-Dynkin game* [2]. This non-linear game problem has been introduced in [8] in the case where ξ and ζ are right-continuous under the name of *generalized Dynkin game*, the term *generalized* referring to the presence of a (non-linear) f-expectation in place of the "classical" linear expectation.

In the second part of the paper, we generalize the results of [8] beyond the right-continuity assumption on ξ and ζ. By using results from the first part of the present paper, combined with some arguments from [8], we show that if ξ is right-u.s.c. and ζ is right-l.s.c. , and if they satisfy Mokobodzki's condition, there exists a (common) value function for the *$\mathcal{E}^f$-Dynkin game*, that is

$$\inf_{\sigma\in\mathcal{T}_0}\sup_{\tau\in\mathcal{T}_0}\mathcal{E}^f_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau\in\mathcal{T}_0}\inf_{\sigma\in\mathcal{T}_0}\mathcal{E}^f_{0,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{1.3}$$

and this common value is equal to the solution at time 0 of the doubly reflected BSDE with driver f and barriers (ξ, ζ) from the first part of the paper. Moreover, under the additional assumption that ξ is left u.s.c. along stopping times and ζ is left l.s.c. along stopping times, we prove that there exists a saddle point for the *$\mathcal{E}^f$-Dynkin game*. Let us note that in the particular case when $f = 0$, our results on existence of a common value and on existence of saddle points correspond to the results from the literature on classical Dynkin games recalled above.

In the final part of the paper, we turn to the interpretation of our Doubly RBSDE in terms of a two-stopper-game in the general case where ξ and ζ do not satisfy any regularity assumption. This is technically a more difficult problem. Indeed, even in the simplest case where $f = 0$, we know from the litterature on classical Dynkin games (cf. e.g. [1]) that the game on stopping times with criterion $E\,[I(\tau,\sigma)]$ does not even (a priori) admit a common value, that is, the equality (1.2) does not necessarily hold; this is true, a fortiori, for the $\mathcal{E}^f$-Dynkin game (with non-linear f). In order to interpret the solution of the doubly reflected BSDE with irregular barriers (ξ, ζ) we formulate "an extension" of the previous $\mathcal{E}^f$-Dynkin game problem over a larger set of "stopping strategies" than the set of stopping times $\mathcal{T}_0$. We show that this extended game has a common value which coincides with the solution of our general DRBSDE with irregular barriers. Using this result, we prove a comparison theorem and *a priori* estimates with universal constants for DRBSDEs with irregular barriers.

The remainder of the paper is organized as follows: In Section 2, we introduce the notation and some definitions. In Section 3, we provide first results on doubly reflected BSDEs associated with a Lipschitz driver and barriers (ξ, ζ) which do not satisfy any regularity assumption; in particular, we show existence and uniqueness of the solution of this equation. Section 4 is dedicated to the interpretation of the solution in terms of a two-stopper game problem, first in the case when ξ is right u.s.c. and ζ is right l.s.c., then in the case where they do not satisfy any regularity assumption. In Section 5, we provide a comparison theorem and *a priori* estimates with universal constants for

[2] Note that this game problem is related to the pricing of game options in imperfect market models (cf. the end of Section 3 for more explanations).

our doubly reflected BSDEs with irregular barriers. The Appendix contains some useful results on reflected BSDEs with an irregular obstacle and also some of the proofs.

2 Preliminaries

Let $T > 0$ be a fixed positive real number. Let ν be a σ-finite positive measure on the measurable space $(E, \mathscr{E}) = (\mathbf{R}^*, \mathcal{B}(\mathbf{R}^*))$. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a one-dimensional Brownian motion W and with an independent Poisson random measure $N(dt, de)$ with compensator $dt \otimes \nu(de)$. We denote by $\tilde{N}(dt, de)$ the compensated process, i.e. $\tilde{N}(dt, de) := N(dt, de) - dt \otimes \nu(de)$. Let $I\!\!F = \{\mathcal{F}_t \colon t \in [0, T]\}$ be the (complete) natural filtration associated with W and N. The space $L^2(\mathcal{F}_T)$ is the space of random variables which are $\mathcal{F}_T$-measurable and square-integrable. For $t \in [0, T]$, we denote by $\mathcal{T}_t$ the set of stopping times τ such that $P(t \leq \tau \leq T) = 1$. More generally, for a given stopping time $\nu \in \mathcal{T}_0$, we denote by $\mathcal{T}_\nu$ the set of stopping times τ such that $P(\nu \leq \tau \leq T) = 1$.

We also use the following notation:

- $\mathcal{P}$ (resp. $\mathcal{O}$) is the predictable (resp. optional) σ-algebra on $\Omega \times [0, T]$.
- L^2_ν is the set of $(\mathscr{E}, \mathcal{B}(\mathbf{R}))$-measurable functions $\ell : E \to \mathbf{R}$ such that $\|\ell\|^2_\nu := \int_E |\ell(e)|^2 \nu(de) < \infty$. For $\ell \in \mathcal{L}^2_\nu$, $\mathcal{k} \in \mathcal{L}^2_\nu$, we define $\langle \ell, \mathcal{k} \rangle_\nu := \int_E \ell(e) \mathcal{k}(e) \nu(de)$.
- $I\!\!H^2$ is the set of $\mathbf{R}$-valued predictable processes ϕ with $\|\phi\|^2_{I\!\!H^2} := E\left[\int_0^T |\phi_t|^2 dt\right] < \infty$.
- $I\!\!H^2_\nu$ is the set of $\mathbf{R}$-valued processes $l : (\omega, t, e) \in (\Omega \times [0, T] \times E) \mapsto l_t(\omega, e)$ which are *predictable*, that is $(\mathcal{P} \otimes \mathscr{E}, \mathcal{B}(\mathbf{R}))$-measurable, and such that $\|l\|^2_{I\!\!H^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty$.

As in [18], we denote by $\mathcal{S}^2$ the vector space of $\mathbf{R}$-valued optional (not necessarily cadlag) processes ϕ such that $|||\phi|||^2_{\mathcal{S}^2} := E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0} |\phi_\tau|^2] < \infty$. By Proposition 2.1 in [18], the mapping $|||\cdot|||_{\mathcal{S}^2}$ is a norm on the space $\mathcal{S}^2$, and $\mathcal{S}^2$ endowed with this norm is a Banach space.
Let $\beta > 0$. For $\phi \in I\!\!H^2$, $\|\phi\|^2_\beta := E[\int_0^T e^{\beta s} \phi_s^2 ds]$. [3] For $l \in I\!\!H^2_\nu$, $\|l\|^2_{\nu,\beta} := E[\int_0^T e^{\beta s} \|l_s\|^2_\nu ds]$. For $\phi \in \mathcal{S}^2$, we define $|||\phi|||^2_\beta := E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0} e^{\beta \tau} \phi_\tau^2]$. We note that $|||\cdot|||_\beta$ is a norm on $\mathcal{S}^2$ equivalent to the norm $|||\cdot|||_{\mathcal{S}^2}$.

Definition 2.1 (Driver, Lipschitz driver). *A function f is said to be a* driver *if*

- $f : \Omega \times [0, T] \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$
  $(\omega, t, y, z, \mathcal{k}) \mapsto f(\omega, t, y, z, \mathcal{k})$ *is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ measurable,*
- $E[\int_0^T f(t, 0, 0, 0)^2 dt] < +\infty$.

A driver f is called a Lipschitz driver *if moreover there exists a constant $K \geq 0$ such that $dP \otimes dt$-a.e., for each $(y_1, z_1, \mathcal{k}_1) \in \mathbf{R}^2 \times L^2_\nu$, $(y_2, z_2, \mathcal{k}_2) \in \mathbf{R}^2 \times L^2_\nu$,*

$$|f(\omega, t, y_1, z_1, \mathcal{k}_1) - f(\omega, t, y_2, z_2, \mathcal{k}_2)| \leq K(|y_1 - y_2| + |z_1 - z_2| + \|\mathcal{k}_1 - \mathcal{k}_2\|_\nu).$$

We recall the following definition from [8].

[3] By a slight abuse of notation, we shall also write $\|\phi\|^2_{I\!\!H^2}$ (resp. $\|\phi\|^2_\beta$) for $E\left[\int_0^T |\phi_t|^2 dt\right]$ (resp. $E\left[\int_0^T e^{\beta t} |\phi_t|^2 dt\right]$) in the case of a progressively measurable real-valued process ϕ.

Definition 2.2. *Let $A = (A_t)_{0\leq t\leq T}$ and $A' = (A'_t)_{0\leq t\leq T}$ be two real-valued optional non-decreasing cadlag processes with $A_0 = 0$, $A'_0 = 0$ and $E[A_T] < \infty$ and $E[A'_T] < \infty$. We say that the random measures dA_t and dA'_t are* mutually singular*, and we write $dA_t \perp dA'_t$, if there exists $D \in \mathcal{O}$ such that:*

$$E[\int_0^T \mathbf{1}_{D^c} dA_t] = E[\int_0^T \mathbf{1}_D dA'_t] = 0, \tag{2.1}$$

which can also be written as $\int_0^T \mathbf{1}_{D_t^c} dA_t = \int_0^T \mathbf{1}_{D_t} dA'_t = 0$ a.s., where for each $t \in [0,T]$, D_t is the section at time t *of D, that is, $D_t := \{\omega \in \Omega\,,\, (\omega,t) \in D\}$.*

For real-valued random variables X and X_n, $n \in \mathbb{N}$, the notation "$X_n \uparrow X$" stands for "the sequence (X_n) is nondecreasing and converges to X a.s.".
For a ladlag process ϕ, we denote by ϕ_{t+} and ϕ_{t-} the right-hand and left-hand limit of ϕ at t. We denote by $\Delta_+\phi_t := \phi_{t_+} - \phi_t$ the size of the right jump of ϕ at t, and by $\Delta\phi_t := \phi_t - \phi_{t-}$ the size of the left jump of ϕ at t.

Definition 2.3. *An optional process (ϕ_t) is said to be left upper-semicontinuous (resp. left lower-semicontinuous) along stopping times if for each $\tau \in \mathcal{T}_0$, for each nondecreasing sequence of stopping times (τ_n) such that $\tau_n \uparrow \tau$, a.s., we have $\phi_\tau \geq \limsup_{n\to\infty} \phi_{\tau_n}$ (resp. $\phi_\tau \leq \liminf_{n\to\infty} \phi_{\tau_n}$) a.s.*

Remark 2.1. If the process (ϕ_t) has left limits, (ϕ_t) is left upper-semicontinuous (resp. left lower-semicontinuous) along stopping times if and only if for each predictable stopping time $\tau \in \mathcal{T}_0$, $\phi_{\tau-} \leq \phi_\tau$ (resp. $\phi_{\tau-} \geq \phi_\tau$) a.s.

Definition 2.4 (Strong supermartingale)**.** *An optional process $\phi_. = (\phi_t)$ belonging to $\mathcal{S}^2$ is said to be a* strong supermartingale *if for all $\theta, \theta' \in \mathcal{T}_0$ such that $\theta \geq \theta'$ a.s., $E[\phi_\theta \,|\, \mathcal{F}_{\theta'}] \leq \phi_{\theta'}$ a.s.*

We recall that a strong supermartingale in $\mathcal{S}^2$ is necessarily right upper-semicontinuous (cf., e.g., [7]).

For the easing of the presentation, we define the relation $\geq$ for processes in $\mathcal{S}^2$ as follows: for $\phi, \phi' \in \mathcal{S}^2$, we write $\phi \leq \phi'$, if $\phi_t \leq \phi'_t$ for all $t \in [0,T]$ a.s. Similarly, we define the relations $\leq$ and $=$ on $\mathcal{S}^2$.

3 Doubly Reflected BSDE whose obstacles are irregular

3.1 Definition and first properties

Let $T > 0$ be a fixed terminal time (as before). Let f be a driver. Let $\xi = (\xi_t)_{t\in[0,T]}$ and $\zeta = (\zeta_t)_{t\in[0,T]}$ be two left-limited processes in $\mathcal{S}^2$ such that $\xi_t \leq \zeta_t, 0 \leq t \leq T$, a.s. and $\xi_T = \zeta_T$ a.s. A pair of processes (ξ, ζ) satisfying the previous properties will be called a *pair of admissible barriers*, or a *pair of admissible obstacles*.

Remark 3.2. Let us note that in the following definitions and results we can relax the assumption of existence of left limits for the processes ξ and ζ. All the results still hold true provided we replace the process $(\xi_{t-})_{t\in]0,T]}$ by the process $(\limsup_{s\uparrow t, s<t} \xi_s)_{t\in]0,T]}$ and the process $(\zeta_{t-})_{t\in]0,T]}$ by the process $(\liminf_{s\uparrow t, s<t} \zeta_s)_{t\in]0,T]}$.

Definition 3.1. *A process (Y, Z, k, A, C, A', C') is said to be a solution to the doubly reflected BSDE with parameters (f, ξ, ζ), where f is a driver and (ξ, ζ) is a pair of* *admissible obstacles, if*

$(Y,Z,k,A,C,A',C') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{S}^2)^2 \times (\mathcal{S}^2)^2$ *and a.s. for all* $t \in [0,T]$

$$Y_t = \xi_T + \int_t^T f(s,Y_s,Z_s,k_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E k_s(e)\tilde{N}(ds,de) +$$

$$+ A_T - A_t - (A'_T - A'_t) + C_{T-} - C_{t-} - (C'_{T-} - C'_{t-}), \quad (3.1)$$

$$\xi_t \le Y_t \le \zeta_t, \text{ for all } t \in [0,T] \text{ a.s.}, \quad (3.2)$$

A and A' are nondecreasing right-continuous predictable processes with $A_0 = A'_0 = 0$,

$$\int_0^T \mathbf{1}_{\{Y_{t-} > \xi_{t-}\}} dA_t = 0 \text{ a.s. and } \int_0^T \mathbf{1}_{\{Y_{t-} < \zeta_{t-}\}} dA'_t = 0 \text{ a.s.} \quad (3.3)$$

C and C' are nondecreasing right-continuous adapted purely discontinuous processes with $C_{0-} = C'_{0-} = 0$,

$$(Y_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0 \text{ and } (Y_\tau - \zeta_\tau)(C'_\tau - C'_{\tau-}) = 0 \text{ a.s. for all } \tau \in \mathcal{T}_0, \quad (3.4)$$

$$dA_t \perp dA'_t \text{ and } dC_t \perp dC'_t. \quad (3.5)$$

Here A^c denotes the continuous part of the process A and A^d its discontinuous part. Equations (3.3) and (3.4) are referred to as *minimality conditions* or *Skorokhod conditions.*
Let us note that if (Y,Z,k,A,C,A',C') satisfies the above definition, then the process Y has left and right limits.

Remark 3.3. When A and A' (resp. C and C') are not required to be mutually singular, they can simultaneously increase on $\{\xi_{t-} = \zeta_{t-}\}$ (resp. on $\{\xi_t = \zeta_t\}$). The constraints $dA_t \perp dA'_t$ and $dC_t \perp dC'_t$ will allow us to obtain the uniqueness of the nondecreasing processes A, A', C and C' without the strict separability condition $\xi < \zeta$.
We note also that, due to Eq. (3.1), we have $\Delta C_t - \Delta C'_t = -(Y_{t+} - Y_t) = -\Delta_+ Y_t$. This, together with the condition $dC_t \perp dC'_t$ gives $\Delta C_t = (Y_{t+} - Y_t)^-$ for all t a.s., and $\Delta C'_t = (Y_{t+} - Y_t)^+$ for all t a.s. On the other hand, since in our framework the filtration is quasi-left-continuous, martingales have only totally inaccessible jumps. Hence, for each predictable $\tau \in \mathcal{T}_0$, $\Delta A^d_\tau - \Delta A'^d_\tau = -\Delta Y_\tau$ (cf. Eq. (3.1)). This, together with the condition $dA_t \perp dA'_t$, ensures that for each predictable $\tau \in \mathcal{T}_0$, $\Delta A^d_\tau = (\Delta Y_\tau)^-$ and $\Delta A'^d_\tau = (\Delta Y_\tau)^+$ a.s.
We note also that Y can jump (on the left) at totally inaccessible stopping times; these jumps of Y come from the jumps of the stochastic integral with respect to $\tilde{N}$ in (3.1).

Proposition 3.1. Let f be a driver and (ξ,ζ) be a pair of admissible obstacles.
Let (Y,Z,k,A,C,A',C') be a solution to the doubly reflected BSDE with parameters (f,ξ,ζ).
(i) For each $\tau \in \mathcal{T}_0$, we have

$$Y_\tau = (Y_{\tau+} \vee \xi_\tau) \wedge \zeta_\tau \quad \text{a.s.}$$

(ii) If ξ (resp. ζ) is right continuous, then $C = 0$ (resp. $C' = 0$).
(iii) If ξ (resp. ζ) is left upper-semicontinuous (resp. left lower-semicontinuous) along stopping times, then the process A (resp. A') is continuous.

Proof. Let us show the first assertion. Let $\tau \in \mathcal{T}_0$. By the previous Remark 3.3, we have $\Delta C_\tau = (Y_{\tau+} - Y_\tau)^-$ and $\Delta C'_\tau = (Y_{\tau+} - Y_\tau)^+$ a.s. Since C and C' satisfy the Skorokhod condition (3.4), we have

$$(Y_{\tau+} - Y_\tau)^- = \mathbf{1}_{\{Y_\tau = \xi_\tau\}}(Y_{\tau+} - Y_\tau)^- \text{ and } (Y_{\tau+} - Y_\tau)^+ = \mathbf{1}_{\{Y_\tau = \zeta_\tau\}}(Y_{\tau+} - Y_\tau)^+ \text{ a.s.}$$

Hence, on the set $\{\xi_\tau < Y_\tau < \zeta_\tau\}$, we have $Y_\tau = Y_{\tau^+}$ a.s., which implies that $(Y_{\tau^+} \vee \xi_\tau) \wedge \zeta_\tau = Y_\tau$ a.s. Now, on the set $\{\xi_\tau < Y_\tau = \zeta_\tau\}$, we have $(Y_{\tau^+} - Y_\tau)^- = 0$ a.s., which gives $Y_{\tau^+} \geq Y_\tau = \zeta_\tau \geq \xi_\tau$ a.s., which implies that $(Y_{\tau^+} \vee \xi_\tau) \wedge \zeta_\tau = Y_{\tau^+} \wedge \zeta_\tau = \zeta_\tau = Y_\tau$ a.s. Similarly, on the set $\{\xi_\tau = Y_\tau < \zeta_\tau\}$, we have $(Y_{\tau^+} \vee \xi_\tau) \wedge \zeta_\tau = Y_\tau$ a.s. The first assertion thus holds.

Let us show the second assertion. Suppose that ξ is right-continuous. Let $\tau \in \mathcal{T}_0$. We show $\Delta C_\tau = 0$ a.s. As seen above, we have

$$\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}}(Y_{\tau^+} - Y_\tau)^- = \mathbf{1}_{\{Y_\tau = \xi_\tau\}}(Y_{\tau^+} - \xi_\tau)^- = \mathbf{1}_{\{Y_\tau = \xi_\tau\}}(Y_{\tau^+} - \xi_{\tau^+})^- \text{ a.s.},$$

where the last equality follows from the right-continuity of ξ. Since $Y \geq \xi$, we derive that $\Delta C_\tau = 0$ a.s. This equality being true for all $\tau \in \mathcal{T}_0$, it follows that $C = 0$. Similarly, it can be shown that if ζ is right-continuous, then $C' = 0$. Hence, the second assertion holds.

It remains to show the third assertion. Suppose that ξ is left u.s.c.along stopping times. Let $\tau \in \mathcal{T}_0$ be a predictable stopping time. We show $\Delta A_\tau = 0$ a.s. By the previous Remark 3.3, we have $\Delta A_\tau = (\Delta Y_\tau)^-$ a.s. Since A satisfies the Skorokhod condition (3.3), we have

$$\Delta A_\tau = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(Y_{\tau^-} - Y_\tau)^+ = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(\xi_{\tau^-} - Y_\tau)^+ \leq \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(\xi_\tau - Y_\tau)^+ \text{ a.s.},$$

The (last) inequality in the above computation follows from the inequality $\xi_{\tau^-} \leq \xi_\tau$ a.s., which is due to the assumption of left u.s.c.of ξ (cf. Remark 2.1). Since $\xi \leq Y$, we derive $\Delta A_\tau \leq 0$ a.s., which implies that $\Delta A_\tau = 0$ a.s. This equality being true for every predictable stopping time $\tau \in \mathcal{T}_0$, it follows that A is continuous. Similarly, it can be shown that if ζ is left lower-semicontinuous along stopping times, then A' is continuous, which ends the proof. □

Remark 3.4 (Right-continuous case)**.** It follows from the second assertion in the above proposition that if ξ and ζ are right-continuous, then $C = C' = 0$. In this case, our Definition 3.1 corresponds to the one given in the literature on DRBSDEs (cf. e.g. [8]).

Let $(Y, Z, k, A, C, A', C') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{S}^2)^2 \times (\mathcal{S}^2)^2$ be a solution to the DRBSDE associated with driver f and with a pair of admissible barriers (ξ, ζ). By taking the conditional expectation with respect to $\mathcal{F}_t$ in the equality (3.1), we derive that $Y = H - H'$, where H and H' are the two nonnegative strong supermartingales given by

$$H_t := E[\xi_T^+ + \int_t^T f^+(s, Y_s, Z_s, k_s)ds + A_T - A_t + C_{T-} - C_{t-} \,|\, \mathcal{F}_t];$$

$$H'_t := E[\xi_T^- + \int_t^T f^-(s, Y_s, Z_s, k_s)ds + A'_T - A'_t + C'_{T-} - C'_{t-} \,|\, \mathcal{F}_t].$$

Since $Y = H - H'$ and $\xi \leq Y \leq \zeta$, we get $\xi \leq H - H' \leq \zeta$, which ensures that the following condition holds:

Definition 3.2 (Mokobodzki's condition)**.** *Let $(\xi, \zeta) \in \mathcal{S}^2 \times \mathcal{S}^2$ be a* pair of admissible barriers*. We say that the pair (ξ, ζ) satisfies* Mokobodzki's condition *if there exist two nonnegative strong supermartingales H and H' in $\mathcal{S}^2$ such that:*

$$\xi_t \leq H_t - H'_t \leq \zeta_t \quad 0 \leq t \leq T \quad \text{a.s.} \tag{3.6}$$

Remark 3.5. The above reasoning gives us that Mokobodzki's condition is a necessary condition for the existence of a solution to the DRBSDE.

3.2 The case when f does not depend on the solution

Let us now investigate the question of existence and uniqueness of the solution to the DRBSDE defined above in the case where the driver f does not depend on y, z, and k, that is, $f = (f_t)$, where (f_t) is a process belonging to $I\!H^2$.

3.2.1 Equivalent formulation

We first show that the existence of a solution to the DRBSDE associated with driver process $f = (f_t)$ is equivalent to the existence of a solution to a coupled system of reflected BSDEs.

Let $(Y, Z, k, A, C, A', C') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{S}^2)^2 \times (\mathcal{S}^2)^2$ be a solution to the DRBSDE associated with driver $f(\omega, t)$ and with a pair of admissible barriers (ξ, ζ).
Let $\tilde{Y}_t := Y_t - E[\xi_T + \int_t^T f_s ds \,|\, \mathcal{F}_t]$, for all $t \in [0, T]$. From this definition, together with Eq. (3.1), we get

$$\tilde{Y}_t = X_t^f - X_t^{'f} \text{ for all } t \in [0, T] \text{ a.s.,}$$

where the processes X^f and $X^{'f}$ are defined by

$$X_t^f := E[A_T - A_t + C_{T-} - C_{t-} \,|\, \mathcal{F}_t] \text{ and } X_t^{'f} := E[A'_T - A'_t + C'_{T-} - C'_{t-} \,|\, \mathcal{F}_t], \text{ for all } t \in [0, T]. \tag{3.7}$$

Remark 3.6. Note that X^f and $X^{'f}$ are two nonnegative (right-u.s.c.) strong supermartingales in $\mathcal{S}^2$ such that $X_T^f = X_T^{'f} = 0$ a.s.

By the martingale representation theorem, there exist (π, l) , $(\pi', l') \in I\!H^2 \times I\!H^2_\nu$ such that

$$X_t^f = -\int_t^T \pi_s dW_s - \int_t^T \int_E l_s(e)\tilde{N}(ds, de) + A_T - A_t + C_{T-} - C_{t-}; \tag{3.8}$$

$$X_t^{'f} = -\int_t^T \pi'_s dW_s - \int_t^T \int_E l'_s(e)\tilde{N}(ds, de) + A'_T - A'_t + C'_{T-} - C'_{t-}. \tag{3.9}$$

We introduce the following optional processes:

$$\tilde{\xi}_t^f := \xi_t - E[\xi_T + \int_t^T f_s ds|\mathcal{F}_t], \quad \tilde{\zeta}_t^f := \zeta_t - E[\zeta_T + \int_t^T f_s ds|\mathcal{F}_t], \quad 0 \le t \le T. \tag{3.10}$$

Remark 3.7. Note that $\tilde{\xi}$ and $\tilde{\zeta}$ satisfy $\tilde{\xi}_T^f = \tilde{\zeta}_T^f = 0$ a.s. We also have $\tilde{\xi}^f \in \mathcal{S}^2$ and $\tilde{\zeta}^f \in \mathcal{S}^2$. Indeed, $|\tilde{\xi}_t^f| \le |\xi_t| + E[U|\mathcal{F}_t]$, where $U := |\xi_T| + \int_0^T |f_s|ds$. Now, since $\xi \in \mathcal{S}^2$ and $f \in \mathbb{H}^2$, we have $U \in L^2$. Thus, by Doob's martingale inequalities in L^2, the martingale $(E[U \,|\, \mathcal{F}_t])$ belongs to $\mathcal{S}^2$, which implies that $\tilde{\xi}^f \in \mathcal{S}^2$. Similarly, it can be shown that $\tilde{\zeta}^f \in \mathcal{S}^2$.

From $\xi \le Y \le \zeta$ and the definitions of $\tilde{Y}$, $\tilde{\xi}_t^f$, $\tilde{\zeta}_t^f$, we derive $\tilde{\xi}^f \le \tilde{Y} \le \tilde{\zeta}^f$; since $\tilde{Y}_t = X_t^f - X_t^{'f}$, we have $X_t^f \ge X_t^{'f} + \tilde{\xi}_t^f$ and $X_t^{'f} \ge X_t^f - \tilde{\zeta}_t^f$.

Note that $Y - \xi = \tilde{Y} - \tilde{\xi}^f = X^f - X^{'f} - \tilde{\xi}^f$. The Skorokhod condition (3.4) satisfied by C can thus be written: $\Delta C_\tau(X_\tau^f - X_\tau^{'f} - \tilde{\xi}_\tau^f) = 0$ a.s. We also have $\{Y_{t-} > \xi_{t-}\} = \{X_{t-}^f > X_{t-}^{'f} + \tilde{\xi}_{t-}^f\}$. Hence, the Skorokhod condition (3.3) satisfied by A can be written: $\int_0^T \mathbf{1}_{\{X_{t-}^f > X_{t-}^{'f} + \tilde{\xi}_{t-}^f\}} dA_t = 0$ a.s. It follows that (X^f, π, l, A, C) is the solution of the reflected BSDE associated with driver 0 and obstacle $(X^{'f} + \tilde{\xi}^f)\mathbb{I}_{[0,T)}$ (cf. Prop. 6.3 in the Appendix) [4].

[4] We note that $X^{'f} + \tilde{\xi}^f \in \mathcal{S}^2$ (due to Remarks 3.6 and 3.7). Hence, $(X^{'f} + \tilde{\xi}^f)\mathbb{I}_{[0,T)}$ is an admissible obstacle for RBSDEs.

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By similar arguments we get that (X'^f, π', l', A', C') is the solution of the reflected BSDE associated with driver 0 and obstacle $(X^f - \tilde{\zeta}^f)\mathbb{I}_{[0,T)}$.

We have thus shown that

$$X^f = \mathcal{R}ef[(X'^f + \tilde{\xi}^f)\mathbb{I}_{[0,T)}]; \quad X'^f = \mathcal{R}ef[(X^f - \tilde{\zeta}^f)\mathbb{I}_{[0,T)}], \tag{3.11}$$

where $\mathcal{R}ef$ is the operator induced by the RBSDE with driver 0 (cf. Definition 6.1 in the Appendix). We conclude that the existence of a solution to the DRBSDE with parameters (f, ξ, ζ) (where f is a driver process) implies the existence of a solution to the coupled system of RBSDEs (3.11). We will see in the following proposition that the converse statement also holds true.

Proposition 3.3. *The DRBSDE associated with driver process $f = (f_t) \in I\!H^2$ and with a pair of admissible barriers (ξ, ζ) has a solution if and only if there exist two processes $X_\cdot \in \mathcal{S}^2$ and $X'_\cdot \in \mathcal{S}^2$ satisfying the coupled system of RBSDEs:*

$$X = \mathcal{R}ef[(X' + \tilde{\xi}^f)\mathbb{I}_{[0,T)}]; \quad X' = \mathcal{R}ef[(X - \tilde{\zeta}^f)\mathbb{I}_{[0,T)}]. \tag{3.12}$$

In this case, the optional process Y defined by

$$Y_t := X_t - X'_t + E[\xi_T + \int_t^T f_s ds|\mathcal{F}_t], \; 0 \le t \le T, \text{ a.s.} \tag{3.13}$$

gives the first component of a solution to the DRBSDE.

Proof. The "only if part" of the first assertion has been proved above. Let us prove the "if part" of the first statement, together with the second statement. Let $X_\cdot \in \mathcal{S}^2$ and $X'_\cdot \in \mathcal{S}^2$ be two processes satisfying the coupled system (3.12). Let (π, l, A, C) (resp. (π', l', A', C')) be the vector of the remaining components of the solution to the RBSDE whose first component is X (resp. whose first component is X'). We note that equations (3.8) and (3.9) hold for X and X' (in place of X^f and X'^f). We define the optional process Y as in (3.13).

Since by assumption X and X' belong to $\mathcal{S}^2$, it follows that X and X' are real-valued, which implies that the process Y is well- defined. From (3.13) and the property $X_T = X'_T = 0$ a.s., we get $Y_T = \xi_T$ a.s. From the system (3.12) we get $X_t \ge X'_t + \tilde{\xi}^f_t$ and $X'_t \ge X_t - \tilde{\zeta}^f_t$ for all $t \in [0,T]$ a.s. By using the definitions of $\tilde{\xi}^f$, $\tilde{\zeta}^f$ and Y, we derive that $\xi_t \le Y_t \le \zeta_t$ for all $t \in [0,T]$ a.s.

Moreover, the processes A, C (resp. A', C') satisfy the Skorokhod conditions for RBSDEs. More precisely, for A and C we have: for all $\tau \in \mathcal{T}_0$, $\Delta_+ C_\tau = \mathbf{1}_{\{X_\tau = X'_\tau + \tilde{\xi}^f_\tau\}} \Delta_+ C_\tau$ a.s.; for all predictable $\tau \in \mathcal{T}_0$, $\Delta A_\tau = \mathbf{1}_{\{X_{\tau-} = X'_{\tau-} + \tilde{\xi}^f_{\tau-}\}} \Delta A_\tau$ a.s.; and $\int_0^T \mathbf{1}_{\{X_t > X'_t + \tilde{\xi}^f_t\}} dA^c_t = 0$ a.s. Similar conditions hold for A' and C'.

Now, by using the definitions of $\tilde{\xi}^f$ and Y, we get $\{X_\tau = X'_\tau + \tilde{\xi}^f_\tau\} = \{Y_\tau = \xi_\tau\}$, $\{X_{\tau-} = X'_{\tau-} + \tilde{\xi}^f_{\tau-}\} = \{Y_{\tau-} = \xi_{\tau-}\}$ and $\{X_t > X'_t + \tilde{\xi}^f_t\} = \{Y_t > \xi_t\}$. Combining this with the previous observation gives $\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$ a.s. for all $\tau \in \mathcal{T}_0$ and $\int_0^T \mathbf{1}_{\{Y_{t-} > \xi_{t-}\}} dA_t = 0$ a.s.

By applying the same arguments to A' and C', we get $\Delta C'_\tau = \mathbf{1}_{\{Y_\tau = \zeta_\tau\}} \Delta C'_\tau$ a.s. for all $\tau \in \mathcal{T}_0$ and $\int_0^T \mathbf{1}_{\{Y_{t-} < \zeta_{t-}\}} dA'_t = 0$ a.s.

We now note that the process $(E[\xi_T + \int_t^T f_s ds|\mathcal{F}_t])_{t \in [0,T]}$ (which appears in the definition of Y) corresponds to the first component of the solution to the (non-reflected) BSDE with terminal condition ξ_T and driver f. Hence, there exist $\overline{\pi} \in \mathbb{H}^2$ and $\overline{l} \in \mathbb{H}^2_\nu$ such that

$E[\xi_T + \int_t^T f_s ds|\mathcal{F}_t] = \xi_T + \int_t^T f_s ds - \int_t^T \overline{\pi} dW_s - \int_t^T \int_E \bar{l}_s(e)\tilde{N}(ds,de)$. From this, together with the definition of Y and equations (3.8) and (3.9) for X and $X^{'}$, we obtain

$$Y_t = \xi_T + \int_t^T f_s ds - \int_t^T Z_s dW_s - \int_t^T \int_E k_s(e)\tilde{N}(ds,de) + \alpha_T - \alpha_t + \gamma_{T-} - \gamma_{t-},$$

where $Z := \pi - \pi' + \overline{\pi}$, $k := l - l' + \bar{l}$, $\alpha := A - A^{'}$ and $\gamma := C - C'$.
If $dA_t \perp dA'_t$ and $dC_t \perp dC'_t$, then (Y, Z, k, A, C, A', C') is a solution to the doubly reflected BSDE with parameters (f, ξ, ζ), which gives the desired result.
Otherwise, by the canonical decomposition of RCLL processes with integrable variation (cf. Proposition A.7 in [8]), there exist two nondecreasing right-continuous predictable (resp. optional) processes B and B' (resp. D and D') belonging to $\mathcal{S}^2$ such that $\alpha = B - B'$ (resp. $\gamma = D - D'$) with $dB_t \perp dB'_t$ (resp. $dD_t \perp dD'_t$). Moreover, $dB_t \ll dA_t$, $dB'_t \ll dA'_t$, $dD_t \ll dC_t$ and $dD'_t \ll dC'_t$.
Hence, since $\int_0^T \mathbf{1}_{\{Y_{t-} > \xi_{t-}\}} dA_t = 0$ a.s., we get $\int_0^T \mathbf{1}_{\{Y_{t-} > \xi_{t-}\}} dB_t = 0$ a.s. Similarly, we obtain $\int_0^T \mathbf{1}_{\{Y_{t-} < \zeta_{t-}\}} dB'_t = 0$ a.s. Moreover, since $dD_t \ll dC_t$, the process D is purely discontinuous and $\Delta D_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta D_\tau$ a.s. for all $\tau \in \mathcal{T}_0$. Similarly, D' is purely discontinuous and $\Delta D'_\tau = \mathbf{1}_{\{Y_\tau = \zeta_\tau\}} \Delta D'_\tau$ a.s. for all $\tau \in \mathcal{T}_0$. The nondecreasing RCLL processes D, D' are thus purely discontinuous and satisfy the Skorokhod condition (3.4). The nondecreasing RCLL processes B, B' satisfy the Skorokhod condition (3.3). The process (Y, Z, k, B, D, B', D') is thus a solution to the doubly reflected BSDE with parameters (f, ξ, ζ). □

3.2.2 Existence of a (minimal) solution of the coupled system of RBSDEs

Let $f = (f_t) \in \mathbb{H}^2$ be a driver process (as above). We show the existence of a solution to the system (3.12) under Mokobodzki's condition. To do that, we use Picard's iterations. We set $\mathcal{X}^0 = 0$ and $\mathcal{X}'^0 = 0$, and we define recursively, for each $n \in \mathbb{N}$, the processes:

$$\mathcal{X}^{n+1} := \mathcal{R}ef[(\mathcal{X}'^n + \tilde{\xi}^f)\mathbf{1}_{[0,T)}] \quad ; \quad \mathcal{X}'^{n+1} := \mathcal{R}ef[(\mathcal{X}^n - \tilde{\zeta}^f)\mathbf{1}_{[0,T)}] \tag{3.14}$$

We see, by induction, that the processes $\mathcal{X}^n$ and $\mathcal{X}'^n$ are well-defined; moreover, $\mathcal{X}^n$ and $\mathcal{X}'^n$ are strong supermartingales in $\mathcal{S}^2$. For the sake of simplicity, we have omitted the dependence on f in the notation for $\mathcal{X}^n$ and $\mathcal{X}'^n$.

Proposition 3.2. Assume that the admissible pair (ξ, ζ) satisfies Mokobodzki's condition. The sequences of optional processes $(\mathcal{X}^n_\cdot)_{n\in\mathbb{N}}$ and $(\mathcal{X}'^n_\cdot)_{n\in\mathbb{N}}$ defined above are nondecreasing.
The limit processes

$$\mathcal{X}^f_\cdot := \lim_{n\to+\infty} \mathcal{X}^n_\cdot \text{ and } \mathcal{X}'^f_\cdot := \lim_{n\to+\infty} \mathcal{X}'^n_\cdot \tag{3.15}$$

are nonnegative strong supermartingales in $\mathcal{S}^2$ satisfying the system (3.12) of coupled RBSDEs. Moreover, $\mathcal{X}^f_\cdot, \mathcal{X}'^f_\cdot$ are the *smallest* processes in $\mathcal{S}^2$ satisfying system (3.12). The processes $\mathcal{X}^f, \mathcal{X}'^f$ are also characterized as the *minimal* strong supermartingales in $\mathcal{S}^2$ satisfying the inequalities $\tilde{\xi}^f \le \mathcal{X}^f - \mathcal{X}'^f \le \tilde{\zeta}^f$.

The proof is given in the Appendix.
In the following theorem we summarize some of the properties established so far.

Theorem 3.4. *Let $f = (f_t) \in \mathbb{H}^2$ be a driver process. Let (ξ, ζ) be a pair of admissible barriers. The following assertions are equivalent:*

(i) The pair (ξ, ζ) satisfies Mokobodzki's condition.

(ii) The system (3.12) *of coupled RBSDEs admits a solution.*

(iii) The DRBSDE (3.1) *with driver process f has a solution.*

Proof. The implication $(i) \Rightarrow (ii)$ has been just proved (by using Picard's iterations). The equivalence between (ii) and (iii) has been established in Proposition 3.3. We have noticed that the implication $(iii) \Rightarrow (i)$ holds (in the general case of a Lipschitz driver f) in Remark 3.5. □

3.2.3 Uniqueness of the solution

Let us now investigate the question of uniqueness of the solution to the DRBSDE with driver process $(f_t) \in I\!H^2$. To this purpose, we first state a lemma which will be used in the sequel.

Lemma 3.5 (A priori estimates)**.** *Let $(Y, Z, k, A, C, A', C') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{S}^2)^2 \times (\mathcal{S}^2)^2$ (resp. $(\bar{Y}, \bar{Z}, \bar{k}, \bar{A}, \bar{C}, \bar{A}', \bar{C}') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{S}^2)^2 \times (\mathcal{S}^2)^2$) be a solution to the DRBSDE associated with driver process $f = (f_t) \in I\!H^2$ (resp. $\bar{f} = (\bar{f}_t) \in I\!H^2$) and with a pair of admissible obstacles (ξ, ζ). There exists $c > 0$ such that for all $\varepsilon > 0$, for all $\beta \geq \frac{1}{\varepsilon^2}$ we have*

$$\|k - \bar{k}\|^2_{\nu,\beta} \leq \varepsilon^2 \|f - \bar{f}\|^2_\beta ; \qquad \|Z - \bar{Z}\|^2_\beta \leq \varepsilon^2 \|f - \bar{f}\|^2_\beta ;$$
$$|||Y - \bar{Y}|||^2_\beta \leq 4\varepsilon^2 (1 + 6c^2) \|f - \bar{f}\|^2_\beta. \tag{3.16}$$

The proof, which relies on Gal'chouk-Lenglart's formula (cf. Corollary A.2 in [18]), is given in the Appendix.

We prove below *the uniqueness* of the solution to the DRBSDE associated with the driver process (f_t) and with the admissible pair of barriers (ξ, ζ) satisfying Mokobodzki's condition.

Theorem 3.6. *Let (ξ, ζ) be an admissible pair of barriers satisfying Mokobodzki's condition. Let $f = (f_t) \in I\!H^2$ be a driver process. There exists a unique solution to the DRBSDE* (3.1) *associated with parameters (ξ, ζ, f).*

Proof. Theorem 3.4 yields the existence of a solution. It remains to show the uniqueness. Let (Y, Z, k, A, C, A', C') be a solution of the DRBSDE associated with the driver process (f_t) and the barriers ξ and ζ. By the *a priori estimates* (cf. Lemma 3.5), we derive the uniqueness of (Y, Z, k). By Remark 3.3, we have $\Delta C_t = (Y_{t+} - Y_t)^-$ for all t a.s. and $\Delta C'_t = (Y_{t+} - Y_t)^+$ for all t a.s., which implies the uniqueness of the purely discontinuous processes C and C'. Moreover, since (Y, Z, k, A, C, A', C') satisfies the equation (3.1), it follows that the process $A - A'$ can be expressed in terms of Y, C, C', the integral of the driver process (f_t) with respect to the Lebesgue measure, and the stochastic integrals of Z and k with respect to W and $\tilde{N}$, respectively, which yields the uniqueness of the finite variation process $A - A'$. Now, since $dA_t \perp dA'_t$, the nondecreasing processes A and A' correspond to the (unique) canonical decomposition of this finite variation process, which ends the proof. □

Using the minimality property of $(\mathcal{X}^f, \mathcal{X}'^f)$ (cf. Proposition 3.2), together with the uniqueness property of the solution of the DRBSDE (3.1) with driver process $f = (f_t)$ and Proposition 3.3, we derive that the limit processes $\mathcal{X}^f$ and $\mathcal{X}'^f$ defined by (3.15) can be written in terms of the solution of the DRBSDE.

Proposition 3.7 (Identification of $\mathcal{X}^f$ and $\mathcal{X}'^f$). *Let $\mathcal{X}^f$ and $\mathcal{X}'^f$ be the strong supermartingales defined by* (3.15). *We have a.s.*

$$\mathcal{X}_t^f = E[A_T - A_t + C_{T-} - C_{t-} \mid \mathcal{F}_t] \text{ and } \mathcal{X}_t'^f = E[A'_T - A'_t + C'_{T-} - C'_{t-} \mid \mathcal{F}_t], \text{ for all } t \in [0,T],$$

where A, C, A' and C' are the four last coordinates of the solution of the DRBSDE (3.1) *associated with barriers ξ and ζ, and driver process $f = (f_t)$. Moroever, we have $Y_t = \mathcal{X}_t^f - \mathcal{X}_t'^f + E[\xi_T + \int_t^T f_s ds|\mathcal{F}_t],\ 0 \le t \le T,$ a.s. , where Y is the first coordinate of the solution of the DRBSDE* (3.1).

The proof is given in the Appendix.

3.3 The case of a general Lipschitz driver $f(t, y, z, k)$

We now prove the existence and uniqueness of the solution to the DRBSDE from Definition 3.1 in the case of a general Lipschitz driver.

Theorem 3.8 (Existence and uniqueness of the solution). *Let (ξ, ζ) be a pair of admissible barriers satisfying Mokobodzki's condition and let f be a Lipschitz driver. The DRBSDE with parameters (f, ξ, ζ) from Definition 3.1 admits a unique solution $(Y, Z, k, A, C, A', C') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{S}^2)^2 \times (\mathcal{S}^2)^2$.*

The proof, which relies on the estimates provided in Lemma 3.5 and a fixed point theorem, is given in the Appendix.

4 Doubly reflected BSDEs with irregular barriers and $\mathcal{E}^f$-Dynkin games with irregular rewards

The purpose of this section is to connect our DRBSDE with irregular barriers to a zero-sum game problem between two "stoppers" whose pay-offs are irregular and are assessed by non-linear f-expectations.

In the "classical" case where $f \equiv 0$ (or, more generally, where f is a given process $(f_t) \in \mathbb{H}^2$), this topic has been first studied in [6] in the case of continuous barriers, and in [21] and [22] in the case of right-continuous barriers. The case of right-continuous barriers and a general Lipschitz driver f has been studied in [8].

The following assumption holds in the sequel.

Assumption 4.1. Assume that $dP \otimes dt$-a.s for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$f(t, y, z, k_1) - f(t, y, z, k_2) \ge \langle \gamma_t^{y,z,k_1,k_2}, k_1 - k_2 \rangle_\nu,$$

$$\text{with} \quad \gamma : [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu\,;\ (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable and satisfying the inequalities

$$\gamma_t^{y,z,k_1,k_2}(e) \ge -1 \quad \text{and} \quad \|\gamma_t^{y,z,k_1,k_2}\|_\nu \le C, \tag{4.1}$$

for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$, respectively $dP \otimes dt \otimes d\nu(e)$-a.s. and $dP \otimes dt$-a.s. (where C is a positive constant).

Assumption 4.1 ensures the non decreasing property of $\mathcal{E}^f$ by the comparison theorem for BSDEs with jumps (cf. Theorem 4.2 in [36]).

4.1 The case where ξ is right upper-semicontinuous and ζ is right lower-semicontinuous

In this subsection we focus on the case where ξ is *right upper-semicontinuous* (right u.s.c.) and ζ is *right lower-semicontinuous* (right l.s.c.). We interpret the solution of

our Doubly Reflected BSDE in terms of the value process of a suitably defined zero-sum game problem on *stopping times* with (non-linear) f-expectations.
Let $\xi \in \mathcal{S}^2$ and $\zeta \in \mathcal{S}^2$ be two optional processes (which are not necessarily non negative). We consider a game problem with two players where each of the players' strategy is a stopping time in $\mathcal{T}_0$ and the players payoffs are defined in terms of the given processes ξ and ζ. More precisely, if the first agent chooses $\tau \in \mathcal{T}_0$ as his/her strategy and the second agent chooses $\sigma \in \mathcal{T}_0$, then, at time $\tau \wedge \sigma$ (when the game ends), the pay-off (or reward) is $I(\tau, \sigma)$, where

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}. \tag{4.2}$$

The associated criterion (from time 0 perspective) is defined as the f-evaluation of the pay-off, that is, by $\mathcal{E}^f_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$. The first agent aims at choosing a stopping time $\tau \in \mathcal{T}_0$ which maximizes the criterion. The second agent has the antagonistic objective of choosing a strategy $\sigma \in \mathcal{T}_0$ which minimizes the criterion.

As is usual in stochastic control, we embed the above (game) problem in a dynamic setting, by considering the game from time θ onwards, where θ runs through $\mathcal{T}_0$. From *the perspective of time* θ (where $\theta \in \mathcal{T}_0$ is given), the first agent aims at choosing a strategy $\tau \in \mathcal{T}_\theta$ such that $\mathcal{E}^f_{\theta,\tau\wedge\sigma}[I(\tau,\sigma)]$ be maximal. The second agent has the antagonistic objective of choosing $\sigma \in \mathcal{T}_\theta$ such that $\mathcal{E}^f_{\theta,\tau\wedge\sigma}[I(\tau,\sigma)]$ be minimal.

The following notions will be used in the sequel:

Definition 4.1. *Let $\theta \in \mathcal{T}_0$.*

- The upper value $\overline{V}(\theta)$ *and* the lower value $\underline{V}(\theta)$ *of the game at time θ are the random variables defined respectively by*

$$\overline{V}(\theta) := \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_\theta} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \mathcal{E}^f_{\theta,\tau\wedge\sigma}[I(\tau,\sigma)]; \quad \underline{V}(\theta) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_\theta} \mathcal{E}^f_{\theta,\tau\wedge\sigma}[I(\tau,\sigma)]. \tag{4.3}$$

- *We say that* there exists a value *for the game at time θ if $\overline{V}(\theta) = \underline{V}(\theta)$ a.s.*
- *A pair $(\hat\tau, \hat\sigma) \in \mathcal{T}_\theta^2$ is called a* saddle point at time θ *for the game if for all $(\tau,\sigma) \in \mathcal{T}_\theta^2$ we have*

$$\mathcal{E}^f_{\theta,\tau\wedge\hat\sigma}[I(\tau,\hat\sigma)] \leq \mathcal{E}^f_{\theta,\hat\tau\wedge\hat\sigma}[I(\hat\tau,\hat\sigma)] \leq \mathcal{E}^f_{\theta,\hat\tau\wedge\sigma}[I(\hat\tau,\sigma)] \quad \text{a.s.}$$

- *Let $\varepsilon > 0$. A pair $(\hat\tau, \hat\sigma) \in \mathcal{T}_\theta^2$ is called an* ε-saddle point at time θ *for the game if for all $(\tau,\sigma) \in \mathcal{T}_\theta^2$ we have*

$$\mathcal{E}^f_{\theta,\tau\wedge\hat\sigma}[I(\tau,\hat\sigma)] - \varepsilon \leq \mathcal{E}^f_{\theta,\hat\tau\wedge\hat\sigma}[I(\hat\tau,\hat\sigma)] \leq \mathcal{E}^f_{\theta,\hat\tau\wedge\sigma}[I(\hat\tau,\sigma)] + \varepsilon \quad \text{a.s.}$$

The inequality $\underline{V}(\theta) \leq \overline{V}(\theta)$ a.s. is trivially true. As mentioned in the introduction, in the case where the processes ξ and ζ are RCLL, we recover a game problem which appears in [8] under the name of *generalized Dynkin game*. In the case $f = 0$, we have $\mathcal{E}^0_{\theta,\tau\wedge\sigma}[I(\tau,\sigma)] = E[I(\tau,\sigma) \,|\, \mathcal{F}_\theta]$, and, in this case, our game problem corresponds to the classical Dynkin game (cf., e.g., [1]).

We also recall the following definition:

Definition 4.2. *Let $Y \in \mathcal{S}^2$. The process Y is said to be a strong $\mathcal{E}^f$-supermartingale (resp $\mathcal{E}^f$-submartingale), if $\mathcal{E}^f_{\sigma,\tau}[Y_\tau] \leq Y_\sigma$ (resp. $\mathcal{E}^f_{\sigma,\tau}[Y_\tau] \geq Y_\sigma$) a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

Remark 4.8. Recall that Y is right u.s.c.(cf. e.g. Lemma 5.1 in [18]).

Let Y be the first component of the solution to the DRBSDE with parameters (f, ξ, ζ) from Definition 3.1. For each $\theta \in \mathcal{T}_0$ and each $\varepsilon > 0$, we define the stopping times τ_θ^ε and $\sigma_\theta^\varepsilon$ by

$$\tau_\theta^\varepsilon := \inf\{t \geq \theta,\ Y_t \leq \xi_t + \varepsilon\}; \quad \sigma_\theta^\varepsilon := \inf\{t \geq \theta,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{4.4}$$

.

Lemma 4.3. *The process $(Y_t,\, \theta \leq t \leq \tau_\theta^\varepsilon)$ is a strong $\boldsymbol{\mathcal{E}}^f$-submartingale and the process $(Y_t,\, \theta \leq t \leq \sigma_\theta^\varepsilon)$ is a strong $\boldsymbol{\mathcal{E}}^f$-supermartingale.*

The proof is given in the Appendix. We now prove the following result under additional regularity assumptions on ξ and ζ.

Lemma 4.4. *Suppose that ξ is right upper-semicontinuous (resp. ζ is right lower-semicontinuous). We then have*

$$Y_{\tau_\theta^\varepsilon} \leq \xi_{\tau_\theta^\varepsilon} + \varepsilon \quad (\text{resp.} \quad Y_{\sigma_\theta^\varepsilon} \geq \zeta_{\sigma_\theta^\varepsilon} - \varepsilon) \quad \textit{a.s.} \tag{4.5}$$

Proof. Suppose that ξ is right u.s.c. Let us prove that $Y_{\tau_\theta^\varepsilon} \leq \xi_{\tau_\theta^\varepsilon} + \varepsilon$ a.s. By way of contradiction, we suppose $P(Y_{\tau_\theta^\varepsilon} > \xi_{\tau_\theta^\varepsilon} + \varepsilon) > 0$. By the Skorokhod conditions, we have $\Delta C_{\tau_\theta^\varepsilon} = C_{\tau_\theta^\varepsilon} - C_{(\tau_\theta^\varepsilon)-} = 0$ on the set $\{Y_{\tau_\theta^\varepsilon} > \xi_{\tau_\theta^\varepsilon} + \varepsilon\}$. On the other hand, due to Remark 3.3, $\Delta C_{\tau_\theta^\varepsilon} = Y_{\tau_\theta^\varepsilon} - Y_{(\tau_\theta^\varepsilon)+}$. Thus, $Y_{\tau_\theta^\varepsilon} = Y_{(\tau_\theta^\varepsilon)+}$ on the set $\{Y_{\tau_\theta^\varepsilon} > \xi_{\tau_\theta^\varepsilon} + \varepsilon\}$. Hence,

$$Y_{(\tau_\theta^\varepsilon)+} > \xi_{\tau_\theta^\varepsilon} + \varepsilon \text{ on the set } \{Y_{\tau_\theta^\varepsilon} > \xi_{\tau_\theta^\varepsilon} + \varepsilon\}. \tag{4.6}$$

We will obtain a contradiction with this statement. Let us fix $\omega \in \Omega$. By definition of $\tau_\theta^\varepsilon(\omega)$, there exists a non-increasing sequence $(t_n) = (t_n(\omega)) \downarrow \tau_\theta^\varepsilon(\omega)$ such that $Y_{t_n}(\omega) \leq \xi_{t_n}(\omega) + \varepsilon$, for all $n \in I\!N$. Hence, $\limsup_{n\to\infty} Y_{t_n}(\omega) \leq \limsup_{n\to\infty} \xi_{t_n}(\omega) + \varepsilon$. As, by assumption, the process ξ is right-u.s.c., we have $\limsup_{n\to\infty} \xi_{t_n}(\omega) \leq \xi_{\tau_\theta^\varepsilon}(\omega)$. On the other hand, as $(t_n(\omega)) \downarrow \tau_\theta^\varepsilon(\omega)$, we have $\limsup_{n\to\infty} Y_{t_n}(\omega) = Y_{(\tau_\theta^\varepsilon)+}(\omega)$. Thus, $Y_{(\tau_\theta^\varepsilon)+}(\omega) \leq \xi_{\tau_\theta^\varepsilon}(\omega) + \varepsilon$, which is in contradiction with (4.6). We conclude that $Y_{\tau_\theta^\varepsilon} \leq \xi_{\tau_\theta^\varepsilon} + \varepsilon$ a.s. By similar arguments, one can show that if ζ is right-l.s.c., then $Y_{\sigma_\theta^\varepsilon} \geq \zeta_{\sigma_\theta^\varepsilon} - \varepsilon$ a.s. The proof of the lemma is thus complete. □

Using the above lemmas, we show that the game problem defined above has a value, we characterize the value of the game in terms of the (first component of the) solution of the DRBSDE (3.1), and we also show the existence of ε-saddle points.

Theorem 4.5 (Existence and characterization of the value function)**.** *Let f be a Lipschitz driver satisfying Assumption* (4.1)*. Let (ξ, ζ) be an admissible pair of barriers satisfying Mokobodzki's condition, and such that ξ is right u.s.c.and ζ is right l.s.c. Let (Y, Z, k, A, A', C, C') be the solution of the DRBSDE (3.1). There exists a common value function for the $\boldsymbol{\mathcal{E}}^f$-Dynkin game* (4.3)*, and for each stopping time $\theta \in \mathcal{T}_0$, we have*

$$Y_\theta = \overline{V}(\theta) = \underline{V}(\theta) \quad \textit{a.s.} \tag{4.7}$$

Let $\theta \in \mathcal{T}_0$ and let $\varepsilon > 0$. For each $(\tau, \sigma) \in \mathcal{T}_\theta^2$, the stopping times τ_θ^ε and $\sigma_\theta^\varepsilon$, defined by (4.4)*, satisfy the inequalities:*

$$\boldsymbol{\mathcal{E}}^f_{\theta, \tau\wedge\sigma_\theta^\varepsilon}[I(\tau, \sigma_\theta^\varepsilon)] - L\varepsilon \leq Y_\theta \leq \boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon\wedge\sigma}[I(\tau_\theta^\varepsilon, \sigma)] + L\varepsilon \quad \textit{a.s.}, \tag{4.8}$$

where L is a positive constant which only depends on the Lipschitz constant K of f and on the terminal time T. In other terms, the pair $(\tau_\theta^\varepsilon, \sigma_\theta^\varepsilon)$ is an $L\varepsilon$-saddle point at time θ for the $\boldsymbol{\mathcal{E}}^f$-Dynkin game (4.3)*.*

Proof. Let $\theta \in \mathcal{T}_0$ and let $\varepsilon > 0$. Let us show that $(\tau_\theta^\varepsilon, \sigma_\theta^\varepsilon)$ satisfies the inequalities (4.8). By Lemma 4.3, the process $(Y_t,\, \theta \leq t \leq \tau_\theta^\varepsilon)$ is a strong $\boldsymbol{\mathcal{E}}^f$-submartingale. We thus get

$$Y_\theta \leq \boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon\wedge\sigma}[Y_{\tau_\theta^\varepsilon\wedge\sigma}] \quad \text{a.s.} \tag{4.9}$$

Now, by assumption, ξ is right-u.s.c. By Lemma 4.4, we thus get $Y_{\tau_\theta^\varepsilon} \leq \xi_{\tau_\theta^\varepsilon} + \varepsilon$ a.s. Using the inequality $Y \leq \zeta$, we derive

$$Y_{\tau_\theta^\varepsilon\wedge\sigma} \leq (\xi_{\tau_\theta^\varepsilon} + \varepsilon)\mathbf{1}_{\tau_\theta^\varepsilon \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau_\theta^\varepsilon} \leq I(\tau_\theta^\varepsilon, \sigma) + \varepsilon \quad \text{a.s.},$$

where the last inequality follows from the definition of $I(\tau_\theta^\varepsilon, \sigma)$. By using the inequality (4.9) and the nondecreasingness of $\boldsymbol{\mathcal{E}}^f$, we get

$$Y_\theta \;\leq\; \boldsymbol{\mathcal{E}}^f_{\theta,\tau_\theta^\varepsilon\wedge\sigma}[I(\tau_\theta^\varepsilon,\sigma)+\varepsilon] \leq \boldsymbol{\mathcal{E}}^f_{\theta,\tau_\theta^\varepsilon\wedge\sigma}[I(\tau_\theta^\varepsilon,\sigma)] + L\varepsilon \quad \text{a.s.}, \tag{4.10}$$

where the last inequality follows from an estimate on BSDEs (cf. Proposition A.4 in [36]). By Lemma 4.3, the process $(Y_t,\, \theta \leq t \leq \sigma_\theta^\varepsilon)$ is a strong $\boldsymbol{\mathcal{E}}^f$-supermartingale. We thus get

$$Y_\theta \;\geq\; \boldsymbol{\mathcal{E}}^f_{\theta,\tau\wedge\sigma_\theta^\varepsilon}[Y_{\tau\wedge\sigma_\theta^\varepsilon}] \quad \text{a.s.} \tag{4.11}$$

Now, by assumption, ζ is right lower-semicontinuous. Hence, by Lemma 4.4, we have $Y_{\sigma_\theta^\varepsilon} \;\geq\; \zeta_{\sigma_\theta^\varepsilon} - \varepsilon$ a.s. Using similar arguments as above, we derive that $Y_\theta \;\geq\; \boldsymbol{\mathcal{E}}^f_{\theta,\tau\wedge\sigma_\theta^\varepsilon}[I(\tau,\sigma_\theta^\varepsilon)] - L\varepsilon$ a.s, which, together with (4.10), leads to the desired inequalities (4.8).

Now, since inequality (4.10) holds for all $\sigma \in \mathcal{T}_\theta$, it follows that

$$Y_\theta \leq \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_\theta} \boldsymbol{\mathcal{E}}^f_{\theta,\tau_\theta^\varepsilon\wedge\sigma}[I(\tau_\theta^\varepsilon,\sigma)] + L\varepsilon \leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_\theta} \boldsymbol{\mathcal{E}}^f_{\theta,\tau\wedge\sigma}[I(\tau,\sigma)] + L\varepsilon \;\; \text{a.s.}$$

From this, together with the definition of $\underline{V}(\theta)$ (cf. (4.3)), we obtain $Y_\theta \;\leq\; \underline{V}(\theta) + L\varepsilon$ a.s. Similarly, we show that $\overline{V}(\theta) - L\varepsilon \;\leq\; Y_\theta$ a.s. for all $\varepsilon > 0$. We thus get $\overline{V}(\theta) \;\leq\; Y_\theta \;\leq\; \underline{V}(\theta)$ a.s. This, together with the inequality $\underline{V}(\theta) \leq \overline{V}(\theta)$ a.s., yields $\underline{V}(\theta) = Y_\theta = \overline{V}(\theta)$ a.s. □

We will now show the existence of saddle points under an additional regularity assumption on the barriers. Let (Y, Z, k, A, A', C, C') be the solution of the DRBSDE (3.1). For each $\theta \in \mathcal{T}_0$, we introduce the following stopping times:

$$\tau_\theta^* := \inf\{t \geq S,\ Y_t = \xi_t\}; \quad \sigma_\theta^* := \inf\{t \geq S,\ Y_t = \zeta_t\}, \tag{4.12}$$

and

$$\overline{\tau}_\theta := \inf\{t \geq S,\ A_t > A_\theta \text{ or } C_{t-} > C_{\theta-}\}; \quad \overline{\sigma}_\theta := \inf\{t \geq S,\ A'_t > A'_\theta \text{ or } C'_{t-} > C'_{\theta-}\}. \tag{4.13}$$

Theorem 4.6 (Existence of saddle points)**.** *Let the assumptions of the previous theorem hold. We assume moreover that ξ is left u.s.c.and ζ is left l.s.c.along stopping times. Then, for each $\theta \in \mathcal{T}_0$, the pairs of stopping times $(\tau_\theta^*, \sigma_\theta^*)$ and $(\overline{\tau}_\theta, \overline{\sigma}_\theta)$, defined by* (4.12) *and* (4.13)*, are saddle points at time θ for the $\boldsymbol{\mathcal{E}}^f$-Dynkin game.*

Proof. The proof of the theorem is given in the Appendix. □

Classical Dynkin game with irregular rewards In this paragraph, we consider the particular case where $f \equiv 0$, that is, the case where the f-expectation reduces to the classical linear expectation. Let (ξ, ζ) be an admissible pair of barriers satisfying Mokobodzki's condition and such that ξ is right u.s.c.and ζ are right l.s.c. (as in Theorem 4.5). Let $\theta \in \mathcal{T}_0$. For $\tau \in \mathcal{T}_\theta$ and $\sigma \in \mathcal{T}_\theta$, it holds $\boldsymbol{\mathcal{E}}^0_{\theta,\tau\wedge\sigma}[I(\tau,\sigma)] = E\,[I(\tau,\sigma)\,|\,\mathcal{F}_\theta]$. The *upper* and *lower values* at time θ are then given by

$$\overline{V}(\theta) := \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_\theta} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} E\,[I(\tau,\sigma)\,|\,\mathcal{F}_\theta]; \quad \underline{V}(\theta) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_\theta} E\,[I(\tau,\sigma)\,|\,\mathcal{F}_\theta], \tag{4.14}$$

We thus recover the classical Dynkin game on stopping times (with linear expectations) recalled in the introduction (cf., e.g., [4] and [1]). In [1], it has been shown that this classical Dynkin game has a value. From our Theorem 4.5, we derive an infinitesimal characterization of the value of this game. From Theorem 4.6, we derive the existence of saddle points under the additional regularity assumption of the reward processes.

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Corollary 4.1. There exists a process $Y \in \mathcal{S}^2$ which aggregates the common value function, i.e., Y is such that for all $\theta \in \mathcal{T}_0$, $Y_\theta = \overline{V}(\theta) = \underline{V}(\theta)$ a.s. Moreover, the process Y is equal to the first component of the solution (Y, Z, k, A, A', C, C') of the DRBSDE (3.1) associated with driver $f = 0$ and with barriers ξ and ζ.

If, moreover, ξ left u.s.c.and ζ are left l.s.c.along stopping times, then for each $\theta \in \mathcal{T}_0$, the pairs of stopping times $(\tau^*_\theta, \sigma^*_\theta)$ and $(\overline{\tau}_\theta, \overline{\sigma}_\theta)$, defined by (4.12) and (4.13), are saddle points at time θ for the Dynkin game (4.14).

Game options In this paragraph, we briefly discuss how the results of this section can be applied to the problem of pricing of game options in some market models with imperfections.
We recall that a game option is a financial instrument which gives the buyer the right *to exercise* at any stopping time $\tau \in \mathcal{T}$ and the seller the right *to cancel* at any stopping time $\sigma \in \mathcal{T}$. If the buyer exercises at time τ before the seller cancels, then the seller pays to the buyer the amount ξ_τ; if the seller cancels at time σ before the buyer exercises, the seller pays to the buyer the amount ζ_σ at the cancellation time σ. The difference $\zeta - \xi \geq 0$ corresponds to a penalty which the seller pays to the buyer in the case of an early cancellation of the contract. Thus, if the seller chooses a cancellation time σ and the buyer chooses an exercise time τ, the former pays to the latter the payoff $I(\tau, \sigma)$ (defined in (1.1)) at time $\tau \wedge \sigma$. In the seminal paper [14], Kifer relates the problem of pricing of game options in a frictionless complete market model to the theory of "classical" Dynkin games (with " classical" linear expectations). Since Kifer's work [14], it is well-known that if the market model is complete and if the processes ξ and ζ are right-continuous and satisfy Mokobodzki's condition, then the price of the game option (up to a discount factor) is equal to the common value of the classical Dynkin game from equation (1.2), where the expectation is taken under the unique martingale measure of the model. Let us also recall that, in a complete market model, the expectation under the unique martingale measure corresponds (up to discounting) to the pricing functional for European-type options.
In market models *with imperfections* however, pricing rules for European-type options are in general no longer linear (cf, e.g. the notion of *non linear pricing system* introduced in [13] or the notion of *pricing rule* introduced in [29]). In a large class of market models with imperfections, European options can be priced via an f-expectation/evaluation, where f is a *nonlinear* driver in which the imperfections are encoded (cf. [13] where also several concrete examples of imperfections are provided). In such a framework, the problem of pricing of game options has been considered in [9]: when ξ and ζ are right-continuous and satisfy Mokobodzki's condition, the common value of the *$\mathcal{E}^f$-Dynkin game* from equation (1.3) is shown to be equal to the "seller's price" of the game option (cf. Theorem 3.12 in [9]).
Using Theorem 4.5 and Proposition 3.1 of the present paper, we can show that the result of [9] can be generalized to the case where the assumption of right-continuity is replaced by the weaker assumption of right upper-semicontinuity of ξ and right lower-semicontinuity of ζ.

4.2 The general irregular case

In this subsection (ξ, ζ) is an admissible pair of barriers satisfying Mokobodzki's condition. Contrary to the previous subsection, here we do not make any regularity assumptions on the pair (ξ, ζ). In this general case, we will interpret the DRBSDE with a pair of obstacles (ξ, ζ) in terms of the value of "an extension" of the zero-sum game of the previous subsection over a larger set of "stopping strategies" than the set of stopping times $\mathcal{T}_0$. To this purpose we introduce the following notion of *stopping system*.

Definition 4.1. Let $\tau \in \mathcal{T}_0$ be a stopping time (in the usual sense). Let H be a set in $\mathcal{F}_\tau$. Let H^c denote its complement in Ω. The pair $\rho = (\tau, H)$ is called a *stopping system* if $H^c \cap \{\tau = T\} = \varnothing$.

By taking $H = \Omega$ in the above definition, we see that the notion of a stopping system generalizes that of a stopping time (in the usual sense).

Remark 4.9. A stopping system is an example of divided stopping time (from the French "temps d'arrêt divisé") in the sense of [10] or [1].

We denote by $\mathcal{S}_0$ the set of all stopping systems; for a stopping time $\theta \in \mathcal{T}_0$, we denote by $\mathcal{S}_\theta$ the set of stopping systems $\rho = (\tau, H)$ such that such that $\theta \leq \tau$.

For an optional *right-limited* process ϕ and a stopping system $\rho = (\tau, H)$, we define ϕ_ρ by

$$\phi_\rho := \phi_\tau \mathbf{1}_H + \phi_{\tau^+} \mathbf{1}_{H^c}.$$

In the particular case where $\rho = (\tau, \Omega)$, we have $\phi_\rho = \phi_\tau$, so the notation is consistent.

For an optional (not necessarily right-limited) process ϕ and for a stopping system $\rho = (\tau, H)$, we set

$$\overset{u}{\phi_\rho} := \phi_\tau \mathbf{1}_H + \bar{\phi}_\tau \mathbf{1}_{H^c} \text{ and } \overset{l}{\phi_\rho} := \phi_\tau \mathbf{1}_H + \underline{\phi}_\tau \mathbf{1}_{H^c},$$

where $(\bar{\phi}_t)$ (resp. $(\underline{\phi}_t)$) denotes the right upper- (resp. right lower-) semicontinuous envelope of the process ϕ, defined by $\bar{\phi}_t := \limsup_{s \downarrow t, s > t} \xi_s$ (resp. $\underline{\phi}_t := \liminf_{s \downarrow t, s > t} \xi_s$), for all $t \in [0, T[$ (cf., e.g., [10, page 133]). The process $\bar{\phi}$ (resp. $(\underline{\phi}_t)$) is progressive and right upper- (resp. right lower-) semicontinuous.
Note that when ϕ is right-limited, we have $\overset{u}{\phi_\rho} = \overset{l}{\phi_\rho} = \phi_\rho$.
Moreover, in the particular case where $\rho = (\tau, \Omega)$, we have $\overset{u}{\phi_\rho} = \overset{l}{\phi_\rho} = \phi_\tau$, so the notation is consistent.
With the help of the above definitions and notation we formulate an extension of the zero-sum game problem from Subsection 4.1 where the set of "stopping strategies" of the agents is the set of stopping systems. More precisely, for two stopping systems $\rho = (\tau, H) \in \mathcal{S}_0$ and $\delta = (\sigma, G) \in \mathcal{S}_0$, we define the pay-off $I(\rho, \delta)$ by

$$I(\rho, \delta) := \overset{u}{\xi_\rho} \mathbf{1}_{\tau \leq \sigma} + \overset{l}{\zeta_\delta} \mathbf{1}_{\sigma < \tau}. \tag{4.15}$$

We note that, by definition, $I(\rho, \delta)$ is an $\mathcal{F}_{\tau \wedge \sigma}$-measurable random variable. As in the previous subsection, the pay-off is assessed by an f-expectation, where f is a Lipschitz driver. Let $\theta \in \mathcal{T}_0$ be a stopping time. The upper and lower value of the game at time θ are defined by:

$$\overline{\boldsymbol{V}}(\theta) := \operatorname*{ess\,inf}_{\delta = (\sigma, G) \in \mathcal{S}_\theta} \operatorname*{ess\,sup}_{\rho = (\tau, H) \in \mathcal{S}_\theta} \boldsymbol{\mathcal{E}}^f_{\theta, \tau \wedge \sigma}[I(\rho, \delta)]; \quad \underline{\boldsymbol{V}}(\theta) := \operatorname*{ess\,sup}_{\rho = (\tau, H) \in \mathcal{S}_\theta} \operatorname*{ess\,inf}_{\delta = (\sigma, G) \in \mathcal{S}_\theta} \boldsymbol{\mathcal{E}}^f_{\theta, \tau \wedge \sigma}[I(\rho, \delta)]. \tag{4.16}$$

The other definitions from Definition 4.1 are generalized to the above framework in a similar manner, by replacing the set of stopping times $\mathcal{T}_\theta$ by the set of stopping systems $\mathcal{S}_\theta$. We will refer to this game problem as "extended" $\boldsymbol{\mathcal{E}}^f$-Dynkin game (over the set of stopping systems). We will show that, for any $\theta \in \mathcal{T}_0$, the "extended" $\boldsymbol{\mathcal{E}}^f$-Dynkin game defined above has a value $\boldsymbol{V}(\theta)$, that is, we have $\boldsymbol{V}(\theta) = \overline{\boldsymbol{V}}(\theta) = \underline{\boldsymbol{V}}(\theta)$ a.s., and that this (common) value coincides with the first component of the solution (at time θ) to the DRBSDE with driver f and obstacles (ξ, ζ); we also show the existence of *ε-optimal* stopping systems.

Let (Y, Z, k, A, A', C, C') be the solution of the DRBSDE (3.1). Let us give some definitions. For each $\theta \in \mathcal{T}_0$ and each $\varepsilon > 0$, we define the sets

$$A^\varepsilon := \{(\omega, t) \in \Omega \times [0, T] : Y_t \leq \xi_t + \varepsilon\} \quad B^\varepsilon := \{(\omega, t) \in \Omega \times [0, T] : Y_t \geq \zeta_t - \varepsilon\}.$$

We recall that the stopping times τ_θ^ε and $\sigma_\theta^\varepsilon$ have been defined as the *dÃl'buts* after θ of the sets A^ε and B^ε (cf. Eq. (4.4)). We now set

$$H^\varepsilon := \{\omega \in \Omega : (\omega, \tau_\theta^\varepsilon(\omega)) \in A^\varepsilon\} \quad G^\varepsilon := \{\omega \in \Omega : (\omega, \sigma_\theta^\varepsilon(\omega)) \in B^\varepsilon\}$$

and we define the stopping systems

$$\rho_\theta^\varepsilon := (\tau_\theta^\varepsilon, H^\varepsilon) \text{ and } \delta_\theta^\varepsilon := (\sigma_\theta^\varepsilon, G^\varepsilon). \tag{4.17}$$

The following lemma uses an additional piece of notation.
For an optional *right-limited* process ϕ, and for two stopping systems $\rho = (\tau, H) \in \mathcal{S}_0$ and $\delta = (\sigma, G) \in \mathcal{S}_0$, we set

$$\phi_{\rho \wedge \delta} := \phi_\rho \mathbf{1}_{\tau \leq \sigma} + \phi_\delta \mathbf{1}_{\sigma < \tau}.$$

Remark 4.10. For general stopping systems, the above notation is not symmetric (i.e. the equality $\phi_{\rho \wedge \delta} = \phi_{\delta \wedge \rho}$ is not necessarily true). In the particular case where $\rho = (\tau, \Omega)$ and $\delta = (\sigma, \Omega)$ (i.e. the particular case of stopping times), we have $\phi_{\rho \wedge \delta} = \phi_{\tau \wedge \sigma}$, where $\tau \wedge \sigma$ is the usual notation for the minimum of the two stopping times τ and σ, and we have the equality $\phi_{\rho \wedge \delta} = \phi_{\tau \wedge \sigma} = \phi_{\sigma \wedge \tau} = \phi_{\delta \wedge \rho}$.

The following lemma is to be compared with Lemmas 4.3 and 4.4. In the general irregular case where ξ is not necessarily right upper-semicontinuous and ζ is not necessarily left lower-semicontinuous, the inequalities (4.5) from Lemma 4.4 do not necessarily hold true. In this case, working with the "regularized" processes ξ^u and ζ^l and with stopping systems (instead of stopping times) allows us to have inequalities which are analogous to those of Lemma 4.4, as well as some properties which are, in a certain sense (cf. Remark 4.11 below), analogous to those of Lemma 4.3.

Lemma 4.7. *Let (ξ, ζ) be an admissible pair of barriers satisfying Mokobodzki's condition. Let (Y, Z, k, A, A', C, C') be the solution of the DRBSDE (3.1). The following assertions hold:*

1. *We have*
$$Y_{\rho_\theta^\varepsilon} \leq \overset{u}{\xi}_{\rho_\theta^\varepsilon} + \varepsilon \quad \text{and} \quad Y_{\delta_\theta^\varepsilon} \geq \overset{l}{\zeta}_{\delta_\theta^\varepsilon} - \varepsilon \quad a.s. \tag{4.18}$$
2. *For all stopping systems $\rho = (\tau, H)$ and $\delta = (\sigma, G)$, we have*
$$\boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[Y_{\rho_\theta^\varepsilon \wedge \delta}] \geq Y_\theta \quad \textit{and} \quad \boldsymbol{\mathcal{E}}^f_{\theta, \tau \wedge \sigma_\theta^\varepsilon}[Y_{\rho \wedge \delta_\theta^\varepsilon}] \leq Y_\theta \quad a.s. \tag{4.19}$$

Remark 4.11. Note that the inequalities (4.19) are the analogue, for the stopping systems ρ_θ^ε, δ, $\delta_\theta^\varepsilon$ and ρ, of the inequalities (4.9) and (4.11) satisfied by the stopping times τ_θ^ε, σ, $\sigma_\theta^\varepsilon$ and τ.

Proof. Let us prove the first point. On the set H^ε, we have $Y_{\rho_\theta^\varepsilon} = Y_{\tau_\theta^\varepsilon} \leq \xi_{\tau_\theta^\varepsilon} + \varepsilon = \overset{u}{\xi}_{\rho_\theta^\varepsilon} + \varepsilon$, where we have used the definitions of ρ_θ^ε, $Y_{\rho_\theta^\varepsilon}$, $\overset{u}{\xi}_{\rho_\theta^\varepsilon}$ and H^ε. On the complement $H^{\varepsilon, c}$, we have:

$$Y_{\rho_\theta^\varepsilon} = Y_{\tau_\theta^\varepsilon +} \text{ and } \overset{u}{\xi}_{\rho_\theta^\varepsilon} = \bar{\xi}_{\tau_\theta^\varepsilon}. \tag{4.20}$$

.

On the other hand, by definitions of τ_θ^ε and of $H^{\varepsilon,c}$, for a.e. $\omega \in \Omega$, there exists a decreasing sequence $(t_n) := (t_n(\omega))$ such that $t_n(\omega) \downarrow\downarrow \tau_\theta^\varepsilon(\omega)$ and $Y_{t_n} \leq \xi_{t_n} + \varepsilon$, for all $n \in \mathbb{N}$. Hence, $\limsup_{n\to\infty} Y_{t_n}(\omega) \leq \limsup_{n\to\infty} \xi_{t_n}(\omega) + \varepsilon$. Now, be definition of $\bar{\xi}$, we have $\limsup_{n\to\infty} \xi_{t_n}(\omega) \leq \bar{\xi}_{\tau_\theta^\varepsilon}(\omega)$. On the other hand, we have $\limsup_{n\to\infty} Y_{t_n}(\omega) = Y_{\tau_\theta^\varepsilon+}(\omega)$. Hence, $Y_{\tau_\theta^\varepsilon+}(\omega) \leq \bar{\xi}_{\tau_\theta^\varepsilon}(\omega) + \varepsilon$. This inequality, together with (4.20) gives that $Y_{\rho_\theta^\varepsilon} \leq \overset{u}{\xi}_{\rho_\theta^\varepsilon} + \varepsilon$ a.s. on $H^{\varepsilon,c}$. We thus derive the desired result, namely $Y_{\rho_\theta^\varepsilon} \leq \overset{u}{\xi}_{\rho_\theta^\varepsilon} + \varepsilon$ a.s. on Ω.

Let us prove the second inequality. On the set G^ε, we have $Y_{\delta_\theta^\varepsilon} = Y_{\sigma_\theta^\varepsilon} \geq \zeta_{\sigma_\theta^\varepsilon} - \varepsilon = \overset{l}{\zeta}_{\delta_\theta^\varepsilon} - \varepsilon$, where we have used the definitions of $\delta_\theta^\varepsilon$, $Y_{\delta_\theta^\varepsilon}$, $\overset{l}{\zeta}_{\delta_\theta^\varepsilon}$ and G^ε. On the complement $G^{\varepsilon,c}$, we have

$$Y_{\delta_\theta^\varepsilon} = Y_{\sigma_\theta^\varepsilon+} \text{ and } \overset{l}{\zeta}_{\delta_\theta^\varepsilon} = \underline{\zeta}_{\sigma_\theta^\varepsilon}. \tag{4.21}$$

Now, for a.e. $\omega \in \Omega$, there exists a decreasing sequence $(t_n) := (t_n(\omega))$ such that $t_n(\omega) \downarrow\downarrow \sigma_\theta^\varepsilon(\omega)$ and $Y_{t_n}(\omega) \geq \zeta_{t_n}(\omega) - \varepsilon$, for all $n \in \mathbb{N}$. Hence,

$$\liminf_{n\to\infty} Y_{t_n}(\omega) \geq \liminf_{n\to\infty} \zeta_{t_n}(\omega) - \varepsilon. \tag{4.22}$$

Now, $\liminf_{n\to\infty} Y_{t_n}(\omega) = Y_{\sigma_\theta^\varepsilon+}(\omega)$. Moreover, by definition of $\underline{\zeta}$, we have $\liminf_{n\to\infty} \zeta_{t_n}(\omega) \geq \underline{\zeta}_{\sigma_\theta^\varepsilon}(\omega)$. Hence, by (4.22), we get $Y_{\sigma_\theta^\varepsilon+}(\omega) \geq \underline{\zeta}_{\sigma_\theta^\varepsilon}(\omega) - \varepsilon$. Using (4.21), we derive that on $G^{\varepsilon,c}$, $Y_{\delta_\theta^\varepsilon} \geq \overset{l}{\zeta}_{\delta_\theta^\varepsilon} - \varepsilon$ a.s. We have thus shown that $Y_{\delta_\theta^\varepsilon} \geq \overset{l}{\zeta}_{\delta_\theta^\varepsilon} - \varepsilon$ a.s. on Ω.

Let us prove now the first inequality of (4.19). We have

$$Y_{\rho_\theta^\varepsilon \wedge \delta} = Y_{\rho_\theta^\varepsilon} \mathbf{1}_{\tau_\theta^\varepsilon \leq \sigma} + Y_\delta \mathbf{1}_{\sigma < \tau_\theta^\varepsilon}.$$

For *the first term* of the second member of the equality, we have $Y_{\rho_\theta^\varepsilon} = Y_{\tau_\theta^\varepsilon} \mathbf{1}_{H^\varepsilon} + Y_{\tau_\theta^\varepsilon+} \mathbf{1}_{H^{\varepsilon,c}}$. Now, on $H^{\varepsilon,c}$, we have $Y_{\tau_\theta^\varepsilon} > \xi_{\tau_\theta^\varepsilon} + \varepsilon$. The Skorokhod condition thus gives $\Delta C_{\tau_\theta^\varepsilon} = 0$. This, together with Remark 3.3, gives $(Y_{\tau_\theta^\varepsilon+} - Y_{\tau_\theta^\varepsilon})^- = 0$. Hence, $Y_{\tau_\theta^\varepsilon+} \geq Y_{\tau_\theta^\varepsilon}$ on $H^{\varepsilon,c}$. Hence, $Y_{\rho_\theta^\varepsilon} \geq Y_{\tau_\theta^\varepsilon}$. For *the second term*, we have $Y_\delta \mathbf{1}_{\sigma < \tau_\theta^\varepsilon} = (Y_\sigma \mathbf{1}_H + Y_{\sigma+} \mathbf{1}_{H^c}) \mathbf{1}_{\sigma < \tau_\theta^\varepsilon}$. By using the fact that Y is a strong $\boldsymbol{\mathcal{E}}^f$-submartingale on $[\theta, \tau_\theta^\varepsilon]$ (cf. Lemma 4.3), we have $Y_{\sigma+} \geq Y_\sigma$ on $\{\sigma < \tau_\theta^\varepsilon\}$. Hence, $(Y_\sigma \mathbf{1}_H + Y_{\sigma+} \mathbf{1}_{H^c}) \mathbf{1}_{\sigma < \tau_\theta^\varepsilon} \geq Y_\sigma \mathbf{1}_{\sigma < \tau_\theta^\varepsilon}$. By combining the two terms, we get $Y_{\rho_\theta^\varepsilon \wedge \delta} \geq Y_{\tau_\theta^\varepsilon} \mathbf{1}_{\tau_\theta^\varepsilon \leq \sigma} + Y_\sigma \mathbf{1}_{\sigma < \tau_\theta^\varepsilon} = Y_{\tau_\theta^\varepsilon \wedge \sigma}$. Using this and the nondecreasingness of $\boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[\cdot]$, we obtain $\boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[Y_{\rho_\theta^\varepsilon \wedge \delta}] \geq \boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[Y_{\tau_\theta^\varepsilon \wedge \sigma}]$. As Y is a strong $\boldsymbol{\mathcal{E}}^f$-submartingale on $[\theta, \tau_\theta^\varepsilon]$ (cf. Lemma 4.3), we get $\boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[Y_{\tau_\theta^\varepsilon \wedge \sigma}] \geq Y_\theta$, from which we conclude that $\boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[Y_{\rho_\theta^\varepsilon \wedge \delta}] \geq Y_\theta$. The proof of the second inequality of (4.19) is similar. $\square$

With the help of the previous lemma, we establish the following inequalities which are to be compared with the inequalities (4.8) from Theorem 4.5.

Lemma 4.8. *The following inequalities hold:*

$$\boldsymbol{\mathcal{E}}^f_{\theta, \tau \wedge \sigma_\theta^\varepsilon}[I(\rho, \delta_\theta^\varepsilon)] - L\varepsilon \ \leq \ Y_\theta \ \leq \ \boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[I(\rho_\theta^\varepsilon, \delta)] + L\varepsilon \quad \textit{a.s.}, \tag{4.23}$$

Proof. Let $\theta \in \mathcal{T}_0$ and let $\varepsilon > 0$. We first show the right-hand inequality. By Lemma 4.7,

$$Y_\theta \ \leq \ \boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^\varepsilon \wedge \sigma}[Y_{\rho_\theta^\varepsilon \wedge \delta}] \quad \text{a.s.} \tag{4.24}$$

By definition of $Y_{\rho_\theta^\varepsilon \wedge \delta}$, we have

$$Y_{\rho_\theta^\varepsilon \wedge \delta} \ = \ Y_{\rho_\theta^\varepsilon} \mathbf{1}_{\tau_\theta^\varepsilon \leq \sigma} + Y_\delta \mathbf{1}_{\sigma < \tau_\theta^\varepsilon} \quad \text{a.s.}$$

Now, $Y_{\rho_\theta^\varepsilon} \ \leq \ \overset{u}{\xi}_{\rho_\theta^\varepsilon} + \varepsilon$ (cf. Lemma 4.7). Moreover, since $Y \leq \zeta$ and since Y is right-limited, we have $Y_\delta = \overset{l}{Y}_\delta \leq \overset{l}{\zeta}_\delta$. We thus get

$$Y_{\rho_\theta^\varepsilon \wedge \delta} \ \leq \ (\overset{u}{\xi}_{\rho_\theta^\varepsilon} + \varepsilon) \mathbf{1}_{\tau_\theta^\varepsilon \leq \sigma} + \overset{l}{\zeta}_\delta \mathbf{1}_{\sigma < \tau_\theta^\varepsilon} \ \leq \ I(\rho_\theta^\varepsilon, \delta) + \varepsilon \quad \text{a.s.}$$

.

where the last inequality follows from the definition of $I(\rho_\theta^\varepsilon, \delta)$. By using the inequality (4.24) and the nondecreasingness of $\boldsymbol{\mathcal{E}}^f$, we derive

$$Y_\theta \leq \boldsymbol{\mathcal{E}}^f_{\theta,\tau_\theta^\varepsilon\wedge\sigma}[I(\rho_\theta^\varepsilon,\delta)+\varepsilon] \leq \boldsymbol{\mathcal{E}}^f_{\theta,\tau_\theta^\varepsilon\wedge\sigma}[I(\rho_\theta^\varepsilon,\delta)] + L\varepsilon \quad \text{a.s.}, \tag{4.25}$$

where the last inequality follows from an estimate on BSDEs (cf. Proposition A.4 in [36]). Using similar arguments, it can be shown that $Y_\theta \geq \boldsymbol{\mathcal{E}}^f_{\theta,\tau\wedge\sigma_\theta^\varepsilon}[I(\rho,\delta_\theta^\varepsilon)] - L\varepsilon$ a.s, which, together with (4.25), leads to the desired inequalities (4.23). □

In the following theorem we show that the "extended" $\boldsymbol{\mathcal{E}}^f$-Dynkin game has a value which coincides with the first component of the DRBSDE with irregular barriers.

Theorem 4.9 (Existence of a value and characterization)**.** *Let f be a Lipschitz driver satisfying Assumption* (4.1)*. Let (ξ,ζ) be an admissible pair of barriers satisfying Mokobodzki's condition. Let (Y,Z,k,A,A',C,C') be the solution of the DRBSDE (3.1). There exists a common value for the "extended" $\boldsymbol{\mathcal{E}}^f$-Dynkin game, and for each stopping time $\theta \in \mathcal{T}_0$, we have*

$$\underline{\boldsymbol{V}}(\theta) = Y_\theta = \overline{\boldsymbol{V}}(\theta) \text{ a.s.}$$

Proof. The proof relies on Lemma 4.8. Since the right-hand inequality in (4.23) from Lemma 4.8 holds for all $\delta = (\sigma, G) \in \mathcal{S}_\theta$, we have

$$Y_\theta \leq \operatorname*{ess\,inf}_{\delta=(\sigma,G)\in\mathcal{S}_\theta} \boldsymbol{\mathcal{E}}^f_{\theta,\tau_\theta^\varepsilon\wedge\sigma}[I(\rho_\theta^\varepsilon,\delta)] + L\varepsilon \leq \operatorname*{ess\,sup}_{\rho=(\tau,H)\in\mathcal{S}_\theta} \operatorname*{ess\,inf}_{\delta=(\sigma,G)\in\mathcal{S}_\theta} \boldsymbol{\mathcal{E}}^f_{\theta,\tau\wedge\sigma}[I(\rho,\delta)] + L\varepsilon \text{ a.s.}$$

From this, together with the definition of $\underline{\boldsymbol{V}}(\theta)$ (cf. (4.3)), we obtain $Y_\theta \leq \underline{\boldsymbol{V}}(\theta) + L\varepsilon$ a.s. Similarly, we show that $\overline{\boldsymbol{V}}(\theta) - L\varepsilon \leq Y_\theta$ a.s. for all $\varepsilon > 0$. We thus get $\overline{\boldsymbol{V}}(\theta) \leq Y_\theta \leq \underline{\boldsymbol{V}}(\theta)$ a.s. This, together with the inequality $\underline{\boldsymbol{V}}(\theta) \leq \overline{\boldsymbol{V}}(\theta)$ a.s., yields $\underline{\boldsymbol{V}}(\theta) = Y_\theta = \overline{\boldsymbol{V}}(\theta)$ a.s. The proof is thus complete. □

5 Two useful corollaries

Using the characterization of the solution of the nonlinear DRBSDE as the value function of the "extended" $\boldsymbol{\mathcal{E}}^f$-Dynkin game (over the set of stopping systems) from Theorem 4.9, we now establish a comparison theorem and *a priori* estimates with universal constants (i.e. depending only on the terminal time T and the common Lipschitz constant K) for DRBSDEs with completely irregular barriers.

Corollary 5.2 (Comparison theorem for DRBSDEs.)**.** Let (ξ^1,ζ^1) and (ξ^2,ζ^2) be two admissible pairs of barriers satisfying Mokobodzki's condition. Let f^1, f^2 be Lipschitz drivers satisfying Assumption 4.1. For $i = 1,2$, let $(Y^i, Z^i, k^i, A^i, A'^i, C^i, C'^i)$ be the solution of the DRBSDE associated with driver f^i and barriers ξ^i, ζ^i.
Assume that $\xi^2 \leq \xi^1$ and $\zeta^2 \leq \zeta^1$ and $f^2(t, Y_t^2, Z_t^2, k_t^2) \leq f^1(t, Y_t^2, Z_t^2, k_t^2)\ dP \otimes dt$-a.s.
Then, we have $Y^2 \leq Y^1$.

Proof. **Step 1**: Let us first assume that $\xi^2 \leq \xi^1$, $\zeta^2 \leq \zeta^1$, and that $f^2(t,y,z,\kappa) \leq f^1(t,y,z,\kappa)$ for all $(y,z,\kappa) \in \mathbf{R}^2 \times L^2_\nu$, $dP \otimes dt$-a.s. Let $\theta \in \mathcal{S}_0$. For $i = 1,2$ and for all stopping systems $\rho = (\tau, H) \in \mathcal{S}_\theta, \delta = (\sigma, G) \in \mathcal{S}_\theta$, let $\boldsymbol{\mathcal{E}}^i_{\cdot,\tau\wedge\sigma}[I^i(\rho,\delta)]$ be the first coordinate of the solution of the BSDE associated with driver f^i, terminal time $\tau\wedge\sigma$ and terminal condition $I^i(\rho,\delta) = (\xi^i)^u_\rho \mathbf{1}_{\tau\leq\sigma} + (\zeta^i)^l_\delta \mathbf{1}_{\sigma<\tau}$. Since $\xi^2 \leq \xi^1$ and $\zeta^2 \leq \zeta^1$, we have $I^2(\rho,\delta) \leq I^1(\rho,\delta)$ a.s. Since, moreover $f^2 \leq f^1$, the comparison theorem for BSDEs gives: for all stopping systems $\rho = (\tau,H) \in \mathcal{S}_\theta, \delta = (\sigma,G) \in \mathcal{S}_\theta$, $\boldsymbol{\mathcal{E}}^2_{\theta,\tau\wedge\sigma}[I^2(\rho,\delta)] \leq \boldsymbol{\mathcal{E}}^1_{\theta,\tau\wedge\sigma}[I^1(\rho,\delta)]$ a.s. Taking the essential supremum over ρ in $\mathcal{S}_\theta$ and the essential infimum over δ in $\mathcal{S}_\theta$

.

in this inequality, and using the characterization of the solution of the DRBSDE with obstacles (ξ,ζ) as the value function of the "extended" $\boldsymbol{\mathcal{E}}^f$-Dynkin game (cf. Theorem 4.9), we obtain:

$$Y_\theta^2 = \operatorname*{ess\,inf}_{\delta=(\sigma,G)\in\mathcal{S}_\theta} \operatorname*{ess\,sup}_{\rho=(\tau,H)\in\mathcal{S}_\theta} \boldsymbol{\mathcal{E}}^2_{\theta,\tau\wedge\sigma}[I^2(\rho,\delta)] \leq \operatorname*{ess\,inf}_{\delta=(\sigma,G)\in\mathcal{S}_\theta} \operatorname*{ess\,sup}_{\rho=(\tau,H)\in\mathcal{S}_\theta} \boldsymbol{\mathcal{E}}^1_{\theta,\tau\wedge\sigma}[I^1(\rho,\delta)] = Y_\theta^1 \quad \text{a.s.}$$

Since this inequality holds for each $\theta\in\mathcal{T}_0$, we get $Y^2\leq Y^1$.

Step 2: We now place ourselves under the assumptions of the theorem (which are weaker than those made in Step 1). Let $\tilde f$ be the process defined by $\tilde f_t := f^2(t,Y_t^2,Z_t^2,k_t^2) - f^1(t,Y_t^2,Z_t^2,k_t^2)$, which, by assumption, is non positive. Note that (Y^2,Z^2,k^2) is the solution of the DRBSDE associated with barriers ξ^2,ζ^2 and driver $f^1(t,y,z,\kappa)+\tilde f_t$. We have $f^1(t,y,z,\kappa)+\tilde f_t \leq f^1(t,y,z,\kappa)$ for all (y,z,κ). By Step 1 applied to the driver f^1 and the driver $f^1(t,y,z,\kappa)+\tilde f_t$ (instead of f^2), we get $Y^2\leq Y^1$. $\square$

Using Theorem 4.9 and Lemma 4.8, we prove the following estimates for the spread of the solutions of two DRBSDEs with completely irregular barriers.

Corollary 5.3 (A priori estimates for DRBSDEs)**.** Let (ξ^1,ζ^1) and (ξ^2,ζ^2) be two admissible pairs of barriers satisfying Mokobodzki's condition. Let f^1,f^2 be Lipschitz drivers satisfying Assumption 4.1 with common Lipschitz constant $C>0$. For $i=1,2$, let Y^i be the solution of the DRBSDE associated with driver f^i and barriers ξ^i, ζ^i.
Let $\tilde Y := Y^1-Y^2$, $\tilde\xi := \xi^1-\xi^2$, $\tilde\zeta := \zeta^1-\zeta^2$. Let $\eta,\beta>0$ with $\beta\geq \dfrac{3}{\eta}+2C$ and $\eta\leq\dfrac{1}{C^2}$.
Setting $\delta f_s := f^2(t,Y_s^2,Z_s^2,k_s^2) - f^1(t,Y_s^2,Z_s^2,k_s^2)$, $0\leq s\leq T$, for each $\theta\in\mathcal{T}_0$, we have

$$(\tilde Y_\theta)^2 \leq e^{\beta(T-\theta)}E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta}\tilde\xi_\tau^{\,2} + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta}\tilde\zeta_\tau^{\,2}|\mathcal{F}_\theta] + \eta E[\int_\theta^T e^{\beta(s-\theta)}(\delta f_s)^2 ds|\mathcal{F}_\theta] \quad \text{a.s.} \tag{5.1}$$

Proof. The proof is divided into two steps.

Step 1: For $i=1,2$ and for all *stopping systems* $\rho=(\tau,H),\delta=(\sigma,G)\in\mathcal{S}_\theta$, let $(X^{i,\rho,\delta}$, $\pi^{i,\rho,\delta},l^{i,\rho,\delta})$ be the solution of the BSDE associated with driver f^i, terminal time $\tau\wedge\sigma$ and terminal condition $I^i(\rho,\delta)$, where $I^i(\rho,\delta) = (\xi^i)^u_\rho\mathbf{1}_{\tau\leq\sigma} + (\zeta^i)^l_\delta\mathbf{1}_{\sigma<\tau}$. Set $\tilde X^{\rho,\delta} := X^{1,\rho,\delta}-X^{2,\rho,\delta}$ and $\tilde I(\rho,\delta) := I^1(\rho,\delta)-I^2(\rho,\delta) = ((\xi^1)^u_\rho-(\xi^2)^u_\rho)\mathbf{1}_{\tau\leq\sigma} + ((\zeta^1)^l_\delta-(\zeta^2)^l_\delta)\mathbf{1}_{\sigma<\tau}$.
By an estimate on BSDEs (see Proposition $A.4$ in [37]), for each $\theta\in\mathcal{T}_0$, we have a.s.:

$$(\tilde X_\theta^{\tau,\delta})^2 \leq e^{\beta(T-\theta)}E[\tilde I(\rho,\delta)^2 \mid \mathcal{F}_\theta] + \eta E[\int_\theta^T e^{\beta(s-\theta)}[(f^1-f^2)(s,X_s^{2,\rho,\delta},\pi_s^{2,\rho,\delta},l_s^{2,\rho,\delta})]^2 ds \mid \mathcal{F}_\theta].$$

From this, together with the definitions of $(\xi^i)^u_\rho$ and $(\zeta^i)^l_\delta$, we derive

$$(\tilde X_\theta^{\rho,\delta})^2 \leq e^{\beta(T-\theta)}E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta}\tilde\xi_\tau^{\,2} + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta}\tilde\zeta_\tau^{\,2}|\mathcal{F}_\theta] + \eta E[\int_\theta^T e^{\beta(s-\theta)}(\tilde f_s)^2 ds|\mathcal{F}_\theta] \quad \text{a.s.} \tag{5.2}$$

where $\tilde f_s := \sup_{y,z,\kappa}|f^1(s,y,z,\kappa)-f^2(s,y,z,\kappa)|$.
For each $\varepsilon>0$, let $\rho_\theta^{1,\varepsilon}$ (resp. $\delta_\theta^{2,\varepsilon}$) be the *stopping system* ρ_θ^ε (resp. $\delta_\theta^\varepsilon$) associated with (Y^1,ξ^1) (resp. (Y^2,ζ^2)) defined by (4.17). By using inequality (4.23) in Lemma 4.8, we obtain that for all $\varepsilon>0$ and for all *stopping systems* ρ, $\delta\in\mathcal{S}_\theta$,

$$Y_\theta^1 - Y_\theta^2 \leq X_\theta^{1,\rho_\theta^{1,\varepsilon},\delta} - X_\theta^{2,\rho,\delta_\theta^{2,\varepsilon}} + 2L\varepsilon \quad \text{a.s.}$$

Applying this inequality to the *stopping systems* $\rho=\rho_\theta^{1,\varepsilon}$ and $\delta=\delta_\theta^{2,\varepsilon}$, we get

$$Y_\theta^1 - Y_\theta^2 \leq X_\theta^{1,\rho_\theta^{1,\varepsilon},\delta_\theta^{2,\varepsilon}} - X_\theta^{2,\rho_\theta^{1,\varepsilon},\delta_\theta^{2,\varepsilon}} + 2L\varepsilon \leq |X_\theta^{1,\rho_\theta^{1,\varepsilon},\delta_\theta^{2,\varepsilon}} - X_\theta^{2,\rho_\theta^{1,\varepsilon},\delta_\theta^{2,\varepsilon}}| + 2L\varepsilon \quad \text{a.s.}$$

This inequality together with (5.2) gives

$$Y^1_\theta - Y^2_\theta \leq \sqrt{e^{\beta(T-\theta)} E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \tilde{\xi_\tau}^2 + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \tilde{\zeta_\tau}^2 |\mathcal{F}_\theta] + \eta E[\int_\theta^T e^{\beta(s-\theta)} (\tilde{f}_s)^2 ds|\mathcal{F}_\theta]} + 2L\varepsilon \quad \text{a.s.}$$

By symmetry, the last inequality is also verified by $Y^2_\theta - Y^1_\theta$. Since this holds for all $\varepsilon > 0$, we derive that

$$(\tilde{Y_\theta})^2 \leq e^{\beta(T-\theta)} E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \tilde{\xi_\tau}^2 + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \tilde{\zeta_\tau}^2 |\mathcal{F}_\theta] + \eta E[\int_\theta^T e^{\beta(s-\theta)} (\tilde{f}_s)^2 ds|\mathcal{F}_\theta] \quad \text{a.s.}$$

This result holds for all Lipschitz drivers f^1 and f^2 satisfying Assumption 4.1.
Step 2: Note that (Y^2, Z^2, k^2) is the solution the DRBSDE associated with barriers ξ^2, ζ^2 and driver $f^1(t, y, z, k) + \delta f_t$. By applying the result of Step 1 to the driver $f^1(t, y, z, k)$ and the driver $f^1(t, y, z, k) + \delta f_t$ (instead of f^2), we get the desired result. □

Remark 5.12. The previous two corollaries show the relevance of the characterization of the solution of the (non-linear) DRBSDE with irregular obstacles as the value of an extended $\mathcal{E}^f$-Dynkin game, as established in Theorem 4.9. In particular, this characterization allows us to provide estimates with universal constants which, it seems, cannot be obtained by using Gal'chouk-Lenglart's formula. Indeed, up to now in the literature, Itô-type techniques have not proved useful for showing estimates with universal constants, even in the simplest case of continuous barriers and Brownian filtration (cf. Remark 4.5 in [8] for details).

6 Appendix

6.1 Reflected BSDE with driver 0 and irregular obstacle

Let $T > 0$ be a fixed terminal time. Let $\xi = (\xi_t)_{t\in[0,T]}$ be a left-limited process in $\mathcal{S}^2$. A process ξ satisfying the previous properties will be called a *barrier*, or an *obstacle*.

Remark 6.13. Let us note that in the following definitions and results we can relax the assumption of existence of left limits for the obstacle ξ. All the results still hold true provided we replace the process $(\xi_{t-})_{t\in]0,T]}$ by the process $(\limsup_{s\uparrow t, s<t} \xi_s)_{t\in]0,T]}$, known as the left-uppersemicontinuous envelope of ξ.

The following result has been proved in [19] (cf. Theorem 3.1):

Proposition 6.3. Let ξ be a left-limited process in $\mathcal{S}^2$. There exists a unique solution of the reflected BSDE with driver equal to 0 and obstacle ξ, that is a unique process $(X, \pi, l, A, C) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2 \times \mathcal{S}^2$ such that

$$X_t = \xi_T - \int_t^T \pi_s dW_s - \int_t^T \int_E l_s(e) \tilde{N}(ds, de) + A_T - A_t + C_{T-} - C_{t-} \text{ for all } t \in [0,T] \text{ a.s.,} \tag{6.1}$$

$X_t \geq \xi_t$ for all $t \in [0,T]$ a.s.,
A is a nondecreasing right-continuous predictable process with $A_0 = 0$ and such that

$$\int_0^T \mathbf{1}_{\{X_t > \xi_t\}} dA^c_t = 0 \text{ a.s. and } (X_{\tau-} - \xi_{\tau-})(A^d_\tau - A^d_{\tau-}) = 0 \text{ a.s. for all predictable } \tau \in \mathcal{T}_0,$$

C is a nondecreasing right-continuous adapted purely discontinuous process with $C_{0-} = 0$ and such that $(X_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0$ a.s. for all $\tau \in \mathcal{T}_0$.

.

We introduce the following operator:

Definition 6.1 (Operator induced by an RBSDE with driver 0). *For a process $(\xi_t) \in \mathcal{S}^2$, we denote by $\mathcal{R}ef[\xi]$ the first component of the solution to the Reflected BSDE with (lower) barrier ξ and with driver 0.*

Remark 6.14. Note that by Proposition 6.3, together with Remark 6.13 the operator $\mathcal{R}ef : \xi \mapsto \mathcal{R}ef[\xi]$ is well-defined on $\mathcal{S}^2$.

We give some useful properties of the operator $\mathcal{R}ef$ in the following two lemmas.

Lemma 6.2. *The operator $\mathcal{R}ef : S^2 \to S^2$ satisfies the following properties:*

1. *The operator $\mathcal{R}ef$ is nondecreasing, that is, for $\xi, \xi' \in S^2$ such that $\xi \leq \xi'$ we have $\mathcal{R}ef[\xi] \leq \mathcal{R}ef[\xi']$.*
2. *If $\xi \in S^2$ is a strong supermartingale, then $\mathcal{R}ef[\xi] = \xi$.*
3. *For each $\xi \in S^2$, $\mathcal{R}ef[\xi]$ is a strong supermartingale and satisfies $\mathcal{R}ef[\xi] \geq \xi$.*

Proof. By definition, we have $\mathcal{R}ef[\xi] = X$, where $X = (X_t)_{t\in[0,T]}$ is the first coordinate of the solution of the reflected BSDE (6.1). Now, by Theorem 3.1 in [19], the process X is equal to the value function of the optimal stopping problem with payoff ξ, that is for each stopping time θ, we have

$$X_\theta = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau \mid \mathcal{F}_\theta] \quad \text{a.s.}$$

Hence, by classical results of Optimal Stopping Theory, the process $\mathcal{R}ef[\xi] = X$ is equal to the *Snell envelope* of the process ξ, that is, the smallest strong supermartingale greater than or equal to ξ. Using this property, we easily derive the three assertions of the lemma. □

Remark 6.15. We recall that the nondecreasing limit of a sequence of strong supermartingales is a strong supermartingale (which can be easily shown by the Lebesgue theorem for conditional expectations).

We now show a monotone convergence result for the operator $\mathcal{R}ef$.

Lemma 6.3. *Let (ξ^n) be a sequence of processes belonging to S^2, supposed to be nondecreasing, i.e., such that for each $n \in I\!N$, $\xi^n \leq \xi^{n+1}$. Let $\xi := \lim_{n\to+\infty} \xi^n$. If $\xi \in \mathcal{S}^2$, then $\mathcal{R}ef[\xi] = \lim_{n\to+\infty} \mathcal{R}ef[\xi^n]$.*

Proof. As the operator $\mathcal{R}ef$ is nondecreasing, the sequence $(\mathcal{R}ef[\xi^n])$ is nondecreasing. Let $X := \lim_{n\to+\infty} \mathcal{R}ef[\xi^n]$. Again, due to the nondecreasingness of the operator $\mathcal{R}ef$, we have $\mathcal{R}ef[\xi^0] \leq \mathcal{R}ef[\xi^n] \leq \mathcal{R}ef[\xi]$, for all $n \in I\!N$. By letting n go to $+\infty$, we get $\mathcal{R}ef[\xi^0] \leq X$ and

$$X \leq \mathcal{R}ef[\xi]. \tag{6.2}$$

In particular, we have $X \in \mathcal{S}^2$. Let us now show that $X \geq \mathcal{R}ef[\xi]$. By definition of $\mathcal{R}ef[\xi^n]$ as the solution of the reflected BSDE with obstacle ξ^n, we have $\mathcal{R}ef[\xi^n] \geq \xi^n$, for all $n \in I\!N$. By letting n go to $+\infty$, we get $X \geq \xi$. Hence,

$$\mathcal{R}ef[X] \geq \mathcal{R}ef[\xi]. \tag{6.3}$$

We note now that for each $n \in I\!N$, $\mathcal{R}ef[\xi^n]$ is a strong supermartingale (cf. Lemma 6.2). It follows that X is a strong supermartingale as the nondecreasing limit of a sequence of strong supermartingales (cf. Remark 6.15). Hence, $X = \mathcal{R}ef[X]$ (cf. Lemma 6.2, second assertion). By (6.3), we thus have $X \geq \mathcal{R}ef[\xi]$, which, using (6.2), implies $X = \mathcal{R}ef[\xi]$. □

6.2 Proofs

Proof of Proposition 3.2 The proof relies on Lemmas 6.2 and 6.3. We first show that $\mathcal{X}^n \geq 0$ and $\mathcal{X}'^n \geq 0$, for all $n \in I\!N$. By definition, $\mathcal{X}'^n_T = \mathcal{X}^n_T = 0$. Since $\tilde{\xi}^f_T = \tilde{\zeta}^f_T = 0$, it follows that $(\mathcal{X}'^n + \tilde{\xi}^f)\mathbf{1}_{[0,T)} = \mathcal{X}'^n + \tilde{\xi}^f$ and $(\mathcal{X}^n - \tilde{\zeta}^f)\mathbf{1}_{[0,T)} = \mathcal{X}^n - \tilde{\zeta}^f$. Moreover, since $\mathcal{X}^n$ is a strong supermartingale, we have $\mathcal{X}^n_\theta \geq E[\mathcal{X}^n_T|\mathcal{F}_\theta] = 0$ a.s. for all $\theta \in \mathcal{T}_0$, which implies that $\mathcal{X}^n \geq 0$. [5] Similarly, we see that $\mathcal{X}'^n \geq 0$.

We prove recursively that $(\mathcal{X}^n)_{n\in\mathbb{N}}$ and $(\mathcal{X}'^n)_{n\in\mathbb{N}}$ are nondecreasing sequences of processes. We have $\mathcal{X}^1 \geq 0 = \mathcal{X}^0$ and $\mathcal{X}'^1 \geq 0 = \mathcal{X}'^0$. Suppose that $\mathcal{X}^n \geq \mathcal{X}^{n-1}$ and $\mathcal{X}'^n \geq \mathcal{X}'^{n-1}$. The induction hypothesis and the nondecreasingness of the operator $\mathcal{R}ef$ (cf. Lemma 6.2) give

$$\mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^f] \geq \mathcal{R}ef[\mathcal{X}'^{n-1} + \tilde{\xi}^f] \quad ; \quad \mathcal{R}ef[\mathcal{X}^n - \tilde{\zeta}^f] \geq \mathcal{R}ef[\mathcal{X}^{n-1} - \tilde{\zeta}^f]. \tag{6.4}$$

Hence, $\mathcal{X}^{n+1} \geq \mathcal{X}^n$ and $\mathcal{X}'^{n+1} \geq \mathcal{X}'^n$, which is the desired result.

We now define two processes H^f and H'^f as follows:

$$H^f_t := H_t + E[\xi^-_T|\mathcal{F}_t] + E[\int_t^T f^-(s)ds|\mathcal{F}_t]; \quad H'^f_t := H'_t + E[\xi^+_T|\mathcal{F}_t] + E[\int_t^T f^+(s)ds|\mathcal{F}_t],$$

where H and H' come from Mokobodzki's condition for (ξ, ζ) (cf. Eq. (3.6)). We note that H^f and H'^f are nonnegative strong supermartingales in $\mathcal{S}^2$. From Mokobodzki's condition, we get

$$\tilde{\xi}^f \leq H^f - H'^f \leq \tilde{\zeta}^f. \tag{6.5}$$

We prove recursively that $\mathcal{X}^n \leq H^f$ and $\mathcal{X}'^n \leq H'^f$, for all $n \in \mathbb{N}$. Note first that $\mathcal{X}^0 = 0 \leq H^f$ and $\mathcal{X}'^0 = 0 \leq H'^f$. Suppose now that $\mathcal{X}^n \leq H^f$ and $\mathcal{X}'^n \leq H'^f$. From this, together with (6.5), we get $\mathcal{X}'^n \leq H'^f \leq H^f - \tilde{\xi}^f$, which implies $\mathcal{X}'^n + \tilde{\xi}^f \leq H^f$. Since the operator $\mathcal{R}ef$ is non decreasing, we derive $\mathcal{X}^{n+1} = \mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^f] \leq \mathcal{R}ef[H^f]$. Since H^f is a strong supermartingale, the second assertion of Lemma 6.2 gives $\mathcal{R}ef[H^f] = H^f$. Hence, $\mathcal{X}^{n+1} \leq H^f$. Similarly, we show $\mathcal{X}'^{n+1} \leq H'^f$. The desired conclusion follows.

By definition, we have $\mathcal{X}^f = \lim \uparrow \mathcal{X}^n$ and $\mathcal{X}'^f = \lim \uparrow \mathcal{X}'^n$. The processes $\mathcal{X}^f$ and $\mathcal{X}'^f$ are optional (valued in $[0, +\infty]$) as the limit of sequences of optional (nonnegative) processes. Since for all $n \in \mathbb{N}$, $\mathcal{X}^n_T = \mathcal{X}'^n_T = 0$ a.s., we have $\mathcal{X}^f_T = \mathcal{X}'^f_T = 0$ a.s. Moreover, since for all $n \in \mathbb{N}$, $0 \leq \mathcal{X}^n \leq H^f$ and $0 \leq \mathcal{X}'^n \leq H'^f$, we obtain $0 \leq \mathcal{X}^f \leq H^f$ and $0 \leq \mathcal{X}'^f \leq H'^f$. As $H^f, H'^f \in \mathcal{S}^2$, it follows that $\mathcal{X}^f$ and $\mathcal{X}'^f$ belong to $\mathcal{S}^2$.

Moreover, $\mathcal{X}^f$ and $\mathcal{X}'^f$ are strong supermartingales as limits of nondecreasing sequences of strong supermartingales (cf. Remark 6.15).

It remains to show that $\mathcal{X}^f$ and $\mathcal{X}'^f$ are solutions of the system (3.12). Recall that, since $\mathcal{X}'^n_T = \tilde{\xi}^f_T = 0$, by (3.14), we have for all $n \in I\!N$,

$$\mathcal{X}^{n+1} = \mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^f]. \tag{6.6}$$

Note that the sequence $(\mathcal{X}'^n + \tilde{\xi}^f)_{n\in I\!N}$ is non decreasing and converges to $\mathcal{X}'^f + \tilde{\xi}^f$. By Lemma 6.3, we thus derive that $\lim_{n\to+\infty} \mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^f] = \mathcal{R}ef[\mathcal{X}'^f + \tilde{\xi}^f]$. Hence, by letting n tend to $+\infty$ in (6.6), we get $\mathcal{X}^f = \mathcal{R}ef[\mathcal{X}'^f + \tilde{\xi}^f]$. Similarly, it can be shown that $\mathcal{X}'^f = \mathcal{R}ef[\mathcal{X}^f - \tilde{\zeta}^f]$. Since $\mathcal{X}^f_T = \mathcal{X}'^f_T = 0$ a.s., it follows that $\mathcal{X}^f$ and $\mathcal{X}'^f$ are solutions of the system (3.12).

Note now that $\mathcal{X}^f$, $\mathcal{X}'^f$ satisfy the inequalities $\tilde{\xi}^f \leq \mathcal{X}^f - \mathcal{X}'^f \leq \tilde{\zeta}^f$. Moreover, they are the minimal nonnegative strong supermartingales in $\mathcal{S}^2$ satisfying these inequalities.

[5] Recall that, by a result of the general theory of processes, if $\phi \in \mathcal{S}^2$ and $\phi' \in \mathcal{S}^2$ are such that $\phi_\theta \leq \phi'_\theta$ a.s. for all $\theta \in \mathcal{T}_0$, then $\phi \leq \phi'$.

Indeed, if $J, J^{'}$ are nonnegative strong supermartingales in $\mathcal{S}^2$ satisfying $\tilde{\xi}^f \leq J - J^{'} \leq \tilde{\zeta}^f$, then, using the same arguments as above, we derive that $\mathcal{X}^f \leq J$ and $\mathcal{X}^{'f} \leq J^{'}$.

From this minimality property, it follows that $(\mathcal{X}^f, \mathcal{X}^{'f})$ is also characterized as the minimal solution of the system (3.12) of coupled RBSDEs. $\square$

Proof of Lemma 3.5 Let $\beta > 0$ and $\varepsilon > 0$ be such that $\beta \geq \frac{1}{\varepsilon^2}$. We set $\tilde{Y} := Y - \bar{Y}$, $\tilde{Z} := Z - \bar{Z}$, $\tilde{A} := A - \bar{A}$, $\tilde{A}' := A' - \bar{A}'$, $\tilde{C} := C - \bar{C}$, $\tilde{C}' := C' - \bar{C}'$, $\tilde{k} := k - \bar{k}$, and $\tilde{f}(\omega, t) := f(\omega, t) - \bar{f}(\omega, t)$. We note that $\tilde{Y}_T = \xi_T - \xi_T = 0$; moreover, for all $\tau \in \mathcal{T}_0$,

$$\tilde{Y}_\tau = \int_\tau^T \tilde{f}_t dt - \int_\tau^T \tilde{Z}_t dW_t - \int_\tau^T \int_E \tilde{k}_t(e) \tilde{N}(dt, de) + \tilde{A}_T - \tilde{A}_\tau + \tilde{C}_{T-} - \tilde{C}_{\tau-} - (\tilde{A}'_T - \tilde{A}'_\tau) - \tilde{C}'_{T-} - \tilde{C}'_{\tau-} \text{ a.s.}$$

Thus we see that $\tilde{Y}$ is an *optional strong semimartingale* in the vocabulary of [17] with decomposition $\tilde{Y} = \tilde{Y}_0 + M + \alpha_t + \gamma_t$, where $M_t := \int_0^t \tilde{Z}_s dW_s + \int_0^t \int_E \tilde{k}_s(e) \tilde{N}(ds, de)$, $\alpha_t := -\int_0^t \tilde{f}_s ds - \tilde{A}_t + \tilde{A}'_t$ and $\gamma_t := -\tilde{C}_{t-} + \tilde{C}'_{t-}$ (cf., e.g., Theorem A.3. and Corollary A.2 in [18]). Applying Gal'chouk-Lenglart's formula (cf. Corollary A.2 in [18]) to $\mathrm{e}^{\beta t} \tilde{Y}_t^2$ gives: almost surely, for all $t \in [0, T]$,

$$\begin{aligned}
\mathrm{e}^{\beta T} \tilde{Y}_T^2 = {} & \mathrm{e}^{\beta t} \tilde{Y}_t^2 + \int_{]t,T]} \beta \, \mathrm{e}^{\beta s} (\tilde{Y}_s)^2 ds - 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_{s-} \tilde{f}_s ds - 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_{s-} d\tilde{A}_s \\
& + 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_{s-} d\tilde{A}'_s + 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_{s-} \tilde{Z}_s dW_s + 2 \int_{]t,T]} \mathrm{e}^{\beta s} \int_E \tilde{Y}_{s-} \tilde{k}_s(e) \tilde{N}(ds, de) \\
& + \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Z}_s^2 ds + \sum_{t < s \leq T} \mathrm{e}^{\beta s} (\tilde{Y}_s - \tilde{Y}_{s-})^2 - \int_{[t,T[} 2 \, \mathrm{e}^{\beta s} \tilde{Y}_s d(\tilde{C})_{s+} + \int_{[t,T[} 2 \, \mathrm{e}^{\beta s} \tilde{Y}_s d(\tilde{C}')_{s+} \\
& + \sum_{t \leq s < T} \mathrm{e}^{\beta s} (\tilde{Y}_{s+} - \tilde{Y}_s)^2.
\end{aligned}$$

Thus, we get (recall that $\tilde{Y}_T = 0$): almost surely, for all $t \in [0, T]$,

$$\begin{aligned}
\mathrm{e}^{\beta t} \tilde{Y}_t^2 + \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Z}_s^2 ds = {} & - \int_{]t,T]} \beta \, \mathrm{e}^{\beta s} (\tilde{Y}_s)^2 ds + 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_s \tilde{f}_s ds + 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_{s-} d\tilde{A}_s \\
& - 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_{s-} d\tilde{A}'_s - 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_{s-} \tilde{Z}_s dW_s \\
& - 2 \int_{]t,T]} \mathrm{e}^{\beta s} \int_E \tilde{Y}_{s-} \tilde{k}_s(e) \tilde{N}(ds, de) - \sum_{t < s \leq T} \mathrm{e}^{\beta s} (\tilde{Y}_s - \tilde{Y}_{s-})^2 \\
& + 2 \int_{[t,T[} \mathrm{e}^{\beta s} \tilde{Y}_s d\tilde{C}_s - 2 \int_{[t,T[} \mathrm{e}^{\beta s} \tilde{Y}_s d\tilde{C}'_s - \sum_{t \leq s < T} \mathrm{e}^{\beta s} (\tilde{Y}_{s+} - \tilde{Y}_s)^2.
\end{aligned} \tag{6.7}$$

We give hereafter an upper bound for some of the terms appearing on the right-hand side (r.h.s. for short) of the above equality.

Let us first consider the sum of **the first and the second term** on the r.h.s. of equality (6.7). By applying the inequality $2ab \leq (\frac{a}{\varepsilon})^2 + \varepsilon^2 b^2$, valid for all $(a, b) \in \mathbf{R}^2$, we get: a.s. for all $t \in [0, T]$,

$$-\int_{]t,T]} \beta \, \mathrm{e}^{\beta s} (\tilde{Y}_s)^2 ds + 2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{Y}_s \tilde{f}_s ds \leq (\frac{1}{\varepsilon^2} - \beta) \int_{]t,T]} \mathrm{e}^{\beta s} (\tilde{Y}_s)^2 ds + \varepsilon^2 \int_{]t,T]} \mathrm{e}^{\beta s} \tilde{f}^2(s) ds.$$

As $\beta \geq \frac{1}{\varepsilon^2}$, we have $(\frac{1}{\varepsilon^2} - \beta) \int_{]t,T]} \mathrm{e}^{\beta s} (\tilde{Y}_s)^2 ds \leq 0$, for all $t \in [0, T]$ a.s.

.

For **the third term** (resp. **the fourth term**) on the r.h.s. of (6.7) it can be shown that, a.s. for all $t \in [0,T]$, $+2\int_{]t,T]} e^{\beta s}\,\tilde{Y}_{s-} d\tilde{A}_s \leq 0$ (resp. $-2\int_{]t,T]} e^{\beta s}\,\tilde{Y}_{s-} d\tilde{A}'_s \leq 0$) The proof uses property (3.3) of the definition of the DRBSDE and the properties $Y \geq \xi$, $\bar{Y} \geq \xi$ (resp. $Y \leq \zeta$, $\bar{Y} \leq \zeta$) ; the details are similar to those in the case of RBSDE (with one lower obstacle) (cf., for instance, the proof of Lemma 3.2 in [18]).

For the **eighth** and the **ninth** terms on the r.h.s. of (6.7) we show that, a.s. for all $t \in [0,T]$, $+2\int_{[t,T[} e^{\beta s}\,\tilde{Y}_s d\tilde{C}_s \leq 0$ and $-2\int_{[t,T[} e^{\beta s}\,\tilde{Y}_s d\tilde{C}'_s \leq 0$. These inequalities are based on property (3.4) of the DRBSDE, on the non-decreasingness of (almost all trajectories of) C, $\bar{C}$, C' and $\bar{C}'$, and on the inequalities $Y \geq \xi$, $\bar{Y} \geq \xi$, $Y \leq \zeta$, $\bar{Y} \leq \zeta$. The details, which are similar to those of the proof of Lemma 3.2 in [18], are left to the reader. The above observations, together with equation (6.7), lead to the following inequality: a.s., for all $t \in [0,T]$,

$$
\begin{aligned}
e^{\beta t}\,\tilde{Y}_t^2 + \int_{]t,T]} e^{\beta s}\,\tilde{Z}_s^2 ds \leq \varepsilon^2 \int_{]t,T]} e^{\beta s}\,\tilde{f}^2(s) ds - 2\int_{]t,T]} e^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s \\
- 2\int_{]t,T]} e^{\beta s} \int_E \tilde{Y}_{s-}\tilde{k}_s(e)\tilde{N}(ds,de) - \sum_{t<s\leq T} e^{\beta s}(\tilde{Y}_s - \tilde{Y}_{s-})^2.
\end{aligned}
\tag{6.8}
$$

Note that we have:

$$
\begin{aligned}
\int_{]t,T]} e^{\beta s}\,\|\tilde{k}_s\|_\nu^2 ds - \sum_{t<s\leq T} e^{\beta s}(\tilde{Y}_s - \tilde{Y}_{s-})^2 &= \int_{]t,T]} e^{\beta s}\,\|\tilde{k}_s\|_\nu^2 ds - \int_{]t,T]} e^{\beta s} \int_E \tilde{k}_s^2(e) N(ds,de) \\
&\quad - \sum_{t<s\leq T} e^{\beta s}(\Delta\tilde{A}'_s - \Delta\tilde{A}_s)^2 \\
&\leq - \int_{]t,T]} e^{\beta s} \int_E \tilde{k}_s^2(e)\tilde{N}(ds,de).
\end{aligned}
$$

where, in order to obtain the equality, we have used the fact that the processes A, $\bar{A}$, A', and $\bar{A}'$ jump only at predictable stopping times, and $N(\cdot,de)$ jumps only at totally inaccessible stopping times (cf. also Remark 3.3).
By adding the term $\int_{]t,T]} e^{\beta s}\,\|\tilde{k}_s\|_\nu^2 ds$ on both sides of inequality (6.8) and by using the above computation, we derive that almost surely, for all $t \in [0,T]$,

$$
\begin{aligned}
e^{\beta t}\,\tilde{Y}_t^2 + \int_{]t,T]} e^{\beta s}\,\tilde{Z}_s^2 ds + \int_{]t,T]} e^{\beta s}\,\|\tilde{k}_s\|_\nu^2 ds \leq \varepsilon^2 \int_{]t,T]} e^{\beta s}\,\tilde{f}^2(s) ds - 2\int_{]t,T]} e^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s \\
- \int_{]t,T]} e^{\beta s} \int_E (2\tilde{Y}_{s-}\tilde{k}_s(e) + \tilde{k}_s^2(e))\tilde{N}(ds,de).
\end{aligned}
\tag{6.9}
$$

From here, using (6.8) and (6.9), we conclude by following exactly the same arguments as in the proof of Lemma 3.2. in [18]. □

Proof of Theorem 3.8 For each $\beta > 0$, we denote by $\mathcal{B}_\beta^2$ the space $\mathcal{S}^2 \times I\!H^2 \times I\!H_\nu^2$ which we equip with the norm $\|(\cdot,\cdot,\cdot)\|_{\mathcal{B}_\beta^2}$ defined by $\|(Y,Z,k)\|_{\mathcal{B}_\beta^2}^2 := |\!|\!|Y|\!|\!|_\beta^2 + \|Z\|_\beta^2 + \|k\|_{\nu,\beta}^2$, for $(Y,Z,k) \in \mathcal{S}^2 \times I\!H^2 \times I\!H_\nu^2$. Since $(I\!H^2, \|\cdot\|_\beta)$, $(I\!H_\nu^2, \|\cdot\|_{\nu,\beta})$, and $(\mathcal{S}^2, |\!|\!|\cdot|\!|\!|_\beta)$ are Banach spaces, it follows that $(\mathcal{B}_\beta^2, \|\cdot\|_{\mathcal{B}_\beta})$ is a Banach space.

We define a mapping Φ from $\mathcal{B}_\beta^2$ into itself as follows: for a given $(y,z,l) \in \mathcal{B}_\beta^2$, we set $\Phi(y,z,l) := (Y,Z,k)$, where Y, Z, k are the first three components of the solution (Y,Z,k,A,C,A',C') to the DRBSDE associated with driver $f_s := f(s, y_s, z_s, l_s)$ and with the pair of admissible barriers (ξ,ζ). The mapping Φ is well-defined by Theorem 3.6.

Using the estimates provided in Lemma 3.5 and following the same arguments as in the proof of Theorem 3.4 in [18], we derive that for a suitable choice of the parameter $\beta > 0$ the mapping Φ is a contraction from the Banach space $\mathcal{B}^2_\beta$ into itself.

By the Banach fixed-point theorem, we get that Φ has a unique fixed point in $\mathcal{B}^2_\beta$, denoted by (Y, Z, k), that is, such that $(Y, Z, k) = \phi(Y, Z, k)$. By definition of the mapping Φ, the process (Y, Z, k) is thus equal to the first three components of the solution (Y, Z, k, A, C, A', C') to the DRBSDE associated with the driver process $g(\omega, t) := f(\omega, t, Y_t(\omega), Z_t(\omega), k_t(\omega))$ and with the pair of barriers (ξ, ζ). This property first implies that (Y, Z, k, A, C, A', C') is the unique solution to the DRBSDE with parameters (f, ξ, ζ). □

Proof of the statement of Proposition 3.7 Let (A, C) (resp. (A', C')) be the *Mertens process* associated with the strong supermartingale $\mathcal{X}^f$ (resp. $\mathcal{X}'^f$), that is satisfying

$$\mathcal{X}^f_t = E[A_T - A_t + C_{T-} - C_{t-} \,|\, \mathcal{F}_t] \text{ (resp. } \mathcal{X}'^f_t = E[A'_T - A'_t + C'_{T-} - C'_{t-} \,|\, \mathcal{F}_t]), \text{ for all } t \in [0, T].$$

We have to show that A, C, A' and C' are equal to the four last coordinates of the solution of the DRBSDE associated with parameters (ξ, ζ, f). To this purpose, we apply the same arguments as those used in the proof of Proposition 3.3 to $X = \mathcal{X}^f$ and $X' = \mathcal{X}'^f$. Let B, D, B' and D' be defined as in this proof. Set $H_t := E[B_T - B_t + D_{T-} - D_{t-} \,|\, \mathcal{F}_t]$ and $H'_t := E[B'_T - B'_t + D'_{T-} - D'_{t-} \,|\, \mathcal{F}_t]$. Since $dB_t << dA_t$, $dB'_t << dA'_t$, $dD_t << dC_t$ and $dD'_t << dC'_t$, we have $H \leq \mathcal{X}^f$ and $H' \leq \mathcal{X}'^f$. Moreover, $H - H' = \mathcal{X}^f - \mathcal{X}'^f$, which yields that $\tilde{\xi}^f \leq H - H' \leq \tilde{\zeta}^f$. By the minimality property of $(\mathcal{X}^f, \mathcal{X}'^f)$ (cf. the last assertion of Proposition 3.2), we derive that $H = \mathcal{X}^f$ and $H' = \mathcal{X}'^f$. Hence, $B = A$, $B' = A'$, $D = C$ and $D' = C'$. By the properties of B, B', D, and D', we thus get $dA_t \perp dA'_t$ and $dC_t \perp dC'_t$. Let now Y be defined by (3.13) with $X = \mathcal{X}^f$ and $X' = \mathcal{X}'^f$, and let (Z, k) be defined as in the proof of Proposition 3.3. The process (Y, Z, k, A, C, A', C') is then the solution of the doubly reflected BSDE with parameters (f, ξ, ζ). The proof is thus complete. □

Proof of Lemma 4.3 Let us first prove that the process $(Y_t,\, \theta \leq t \leq \tau^\varepsilon_\theta)$ is a strong $\boldsymbol{\mathcal{E}}^f$-submartingale. By definition of τ^ε_θ, we have $Y_t > \xi_t + \varepsilon$ on $[\theta, \tau^\varepsilon_\theta[$ a.s. Hence, A^c is constant on $[\theta, \tau^\varepsilon_\theta[$ a.s. (cf. Skorokhod conditions); by continuity of the process A^c, A^c is constant on the closed interval $[\theta, \tau^\varepsilon_\theta]$, a.s. Also, the process A^d is constant on $[\theta, \tau^\varepsilon_\theta[$, a.s. (cf. Skorokhod conditions). Moreover, $Y_{(\tau^\varepsilon_\theta)^-} \geq \xi_{(\tau^\varepsilon_\theta)^-} + \varepsilon$ a.s., which implies that $\Delta A^d_{\tau^\varepsilon_\theta} = 0$ a.s. Finally, for a.e. $\omega \in \Omega$, for all $t \in [\theta(\omega), \tau^\varepsilon_\theta(\omega)[$, $\Delta C_t(\omega) = C_t(\omega) - C_{t-}(\omega) = 0$; we deduce that for a.e. $\omega \in \Omega$, $(C_{t-}(\omega))$ is constant on $[\theta(\omega), \tau^\varepsilon_\theta(\omega)[$, and even on the closed interval $[\theta(\omega), \tau^\varepsilon_\theta(\omega)]$, since the trajectories of (C_{t-}) are left-continuous. Thus, the process $(A_t + C_{t-})$ is constant on $[\theta, \tau^\varepsilon_\theta]$ a.s. By Proposition A.4 in [18], we derive that the process $(Y_t,\, \theta \leq t \leq \tau^\varepsilon_\theta)$ is a strong $\boldsymbol{\mathcal{E}}^f$-submartingale. By similar arguments, one can show that $(Y_t,\, \theta \leq t \leq \sigma^\varepsilon_\theta)$ is a strong $\boldsymbol{\mathcal{E}}^f$-supermartingale, which ends the proof of the lemma. □

Proof of Theorem 4.6 The proof of Theorem 4.6 relies on the following lemma.

Lemma 6.4. *Let f be a driver satisfying Assumption* (4.1). *Let (ξ, ζ) be an admissible pair of barriers satisfying Mokobodzki's condition and such that ξ is right-u.s.c and and ζ is right l.s.c. Let (Y, Z, k, A, A', C, C') be the solution of the DRBSDE (3.1). Assume moreover that A (resp. A') is* continuous. *For each $\theta \in \mathcal{T}_0$, the following assertions hold:*

1.

$$Y_{\tau_\theta^*} = \xi_{\tau_\theta^*} \quad (\text{resp.} \quad Y_{\sigma_\theta^*} = \zeta_{\sigma_\theta^*}) \quad \textit{a.s.} \tag{6.10}$$

Moreover, the process $(Y_t,\, \theta \leq t \leq \tau_\theta^*)$ *is a strong* $\boldsymbol{\mathcal{E}}^f$*-submartingale (resp.* $(Y_t,\, \theta \leq t \leq \sigma_\theta^*)$ *is a strong* $\boldsymbol{\mathcal{E}}^f$*-supermartingale).*

2.

$$Y_{\overline{\tau}_\theta} = \xi_{\overline{\tau}_\theta} \quad (\text{resp.} \quad Y_{\overline{\sigma}_\theta} = \zeta_{\overline{\sigma}_\theta}) \quad \textit{a.s.} \tag{6.11}$$

Moreover, the process $(Y_t,\, \theta \leq t \leq \overline{\tau}_\theta)$ *is a strong* $\boldsymbol{\mathcal{E}}^f$*-submartingale (resp.* $(Y_t,\, \theta \leq t \leq \overline{\sigma}_\theta)$ *is a strong* $\boldsymbol{\mathcal{E}}^f$*-supermartingale).*

Proof. We suppose A is continuous (the case where A' is continuous can be treated by similar arguments). To prove the first statement we note that $Y_{\tau_\theta^*} \geq \xi_{\tau_\theta^*}$ a.s., since Y is (the first component of) the solution to the DRBSDE with barriers ξ and ζ. We show that $Y_{\tau_\theta^*} \leq \xi_{\tau_\theta^*}$ a.s. by using the assumption of right-upper semicontinuity on the process ξ; the arguments are similar to those used in the proof of Lemma 4.4 and are left to the reader. Moreover, by definition of τ_θ^*, we have $Y_t > \xi_t$ on $[\theta, \tau_\theta^*[$ a.s.; hence, the process A is constant on $[\theta, \tau_\theta^*[$ and even on the closed interval $[\theta, \tau_\theta^*]$ due to the continuity. We show that C_{t-} is constant on $[\theta, \tau_\theta^*]$ by the same arguments as those of the proof of Lemma 4.3. Thus, the process $(A_t + C_{t-})$ is constant on $[\theta, \tau_\theta^*]$. By Proposition A.4 in [18], we derive that the process $(Y_t,\, \theta \leq t \leq \tau_\theta^*)$ is a strong $\boldsymbol{\mathcal{E}}^f$-submartingale, which completes the proof of the first statement.

Let us prove the second statement. By definition of $\overline{\tau}_\theta$, we have $A_{\overline{\tau}_\theta} = A_\theta$ a.s. and $C_{\overline{\tau}_{\theta-}} = C_{\theta^-}$ a.s. because (A_t) and (C_{t^-}) are left-continuous. By Proposition A.4 in [18], the process $(Y_t,\, \theta \leq t \leq \overline{\tau}_\theta)$ is thus a strong $\boldsymbol{\mathcal{E}}^f$-submartingale. Moreover, since the continuous process A increases only on $\{Y_t = \xi_t\}$ and $\Delta C_t = \mathbf{1}_{\{Y_t = \xi_t\}} \Delta C_t$, we have $Y_{\overline{\tau}_\theta} = \xi_{\overline{\tau}_\theta}$ a.s., which ends the proof of the second assertion. □

Using the above lemma, we prove Theorem 4.6.

Proof of Theorem 4.6. Let $\theta \in \mathcal{T}_0$. By Theorem 4.5, we have $Y_\theta = \overline{V}(\theta) = \underline{V}(\theta)$ a.s. Moreover, by Proposition 3.1, since ξ is left u.s.c.along stopping times and ζ is left l.s.c.along stopping times, it follows that the non decreasing processes A and A' are continuous. Since by Lemma 6.4 (first assertion), the process $(Y_t,\, \theta \leq t \leq \tau \wedge \sigma_\theta^*)$ is a strong $\boldsymbol{\mathcal{E}}^f$-supermartingale, we get

$$Y_\theta \geq \boldsymbol{\mathcal{E}}^f_{\theta, \tau \wedge \sigma_\theta^*}[Y_{\tau \wedge \sigma_\theta^*}] \quad \text{a.s.} \tag{6.12}$$

Since $Y \geq \xi$ and $Y_{\sigma_\theta^*} = \zeta_{\sigma_\theta^*}$ a.s. (by Lemma 6.4), we also have

$$Y_{\tau \wedge \sigma_\theta^*} = Y_\tau \mathbf{1}_{\tau \leq \sigma_\theta^*} + Y_{\sigma_\theta^*} \mathbf{1}_{\sigma_\theta^* < \tau} \geq \xi_\tau \mathbf{1}_{\tau \leq \sigma_\theta^*} + \zeta_{\sigma_\theta^*} \mathbf{1}_{\sigma_\theta^* < \tau} = I(\tau, \sigma_\theta^*) \quad \text{a.s.}$$

By inequality (6.12) and the non decreasing property of $\boldsymbol{\mathcal{E}}^f$, we get $Y_\theta \geq \boldsymbol{\mathcal{E}}^f_{\theta, \tau \wedge \sigma_\theta^*}[I(\tau, \sigma_\theta^*)]$ a.s. Similarly, one can show that for each $\sigma \in \mathcal{T}_\theta$, we have: $Y_\theta \leq \boldsymbol{\mathcal{E}}^f_{\theta, \tau_\theta^* \wedge \sigma}[I(\tau_\theta^*, \sigma)]$ a.s. It follows that $(\tau_\theta^*, \sigma_\theta^*)$ is a saddle point at time θ. Similarly, using Lemma 6.4 (second assertion), it can be shown that $(\overline{\tau}_\theta, \overline{\sigma}_\theta)$ is a saddle point at time θ, which ends the proof. □

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