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# An Efficient and Strategy-Proof Double-Track Auction for Substitutes and Complements

Ning Sun and Zaifu Yang

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
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# An Efficient and Strategy-Proof Double-Track Auction for Substitutes and Complements[1]

Ning Sun[2] and Zaifu Yang[3]

**Abstract:** We propose a dynamic auction mechanism for efficiently allocating multiple heterogeneous indivisible goods. These goods can be split into two distinct sets so that items in each set are substitutes but complementary to items in the other set. The seller has a reserve value for each bundle of goods and is assumed to report her values truthfully. In each round of the auction, the auctioneer announces the current prices for all items, bidders respond by reporting their demands at these prices, and then the auctioneer adjusts simultaneously the prices of items in one set upwards but those of items in the other downwards. We prove that although bidders are not assumed to be price-takers and thus can strategically exercise their market power, this dynamic auction always induces the bidders to bid truthfully as price-takers, yields an efficient outcome and also has the merit of being a detail-free, transparent and privacy preserving mechanism.

**Keywords:** Dynamic auction, gross substitutes and complements, incentives, efficiency, indivisibility, incomplete information.

JEL classification: D44

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[2]N. Sun, School of Economics, Shanghai University of Finance and Economics, Shanghai, China; nsun@mail.shufe.edu.cn

[3]Z. Yang, Department of Economics and Related Studies, University of York, Heslington, York, YO10 5DD, UK; zaifu.yang@york.ac.uk.

# 1 Introduction

Our purpose is to provide a dynamic auction mechanism that can efficiently allocate multiple heterogeneous indivisible goods to many bidders and at the same time induces bidders to behave truthfully. An important feature of the auction is that it can handle a typical pattern of complementarity among the goods. Traditionally, research has focused on examining auctions for selling a single item. However, over the last twenty years auctions for selling multiple items have become popular and widespread use, see e.g., Klemperer (2004) and Milgrom (2004) on auctioning spectrum rights. The past study has improved our understanding of how the design of auction affects its outcome and also how a market environment influences its auction design.

In a seminal paper, Ausubel (2006) develops an ingenious dynamic auction mechanism for selling heterogeneous goods. His auction yields an efficient outcome, induces bidders to bid sincerely as price-takers, and at the same time protects bidders' private values from being fully exposed. Therefore this auction not only maintains the important strategy-proof property of the famous Vickrey-Clarke-Groves (VCG) mechanism but also overcomes the informational inefficiency problem facing the VCG mechanism.[4] More specifically, Ausubel (2006) examines two auction models: In his first model, the goods are perfectly divisible and bidders have strictly concave value functions, whereas in his second model, all goods are indivisible and are viewed as substitutes in the sense that every bidder's demand for the goods satisfies the gross substitutes (GS) condition of Kelso and Crawford (1982).[5] His analysis concentrates on the first model and is based on calculus and convex analysis.

This paper aims to show that we can extend and generalize Ausubel's auction from the setting with substitute goods to a more general and more practical setting that permits complementarities among goods. More precisely, we examine an auction market where a seller wishes to sell two disjoint sets $S_1$ and $S_2$ of heterogeneous items to many bidders and has a reservation value for every bundle of goods. The seller trades her products in order to maximize revenues. Generally, items in the same set $S_i$ are substitutes but are complementary to items in the other set $S_j$. This relation is introduced by Sun and

[4]See Rothkopf, Teisberg, and Kahn (1990), Ausubel (2004, 2006), Perry and Reny (2005), Milgrom (2007), and Rothkopf (2007) on the merits and demerits of the VCG mechanism in detail.

[5]In a seminal paper, Kelso and Crawford (1982) examine a job matching model and prove by a salary adjustment process that there exists an efficient matching between firms and workers which is supported by a system of competitive salaries, provided that every firm views all workers as substitutes. Gul and Stacchetti (2000) propose an ascending auction for discovering a Walrasian equilibrium price vector in a market where all goods for sale are substitutes. Milgrom (2000) introduces an ascending auction for selling substitute goods and discusses its application to the sale of spectrum licenses in the USA. The crucial difference between the first three processes and Ausubel's is that the latter one is not only efficient but also strategy-proof.

Yang (2006) and called *gross substitutes and complements* (*GSC*).[6] This fundamental pattern captures many familiar and important situations. For instance, in the view of manufacturing firms, workers and machines are typically complements, whereas workers are substitutes and so are machines. In our earlier analysis (Sun and Yang 2009), we propose a price adjustment process and show that this process always yields a Walrasian equilibrium if all bidders are assumed to be price-takers. However, the important strategic and incentive issues have not yet been addressed. In the current model, we assume instead that every bidder has a private value on each bundle of the goods and may have an incentive to economize on his private information. So in this setup, bidders are not assumed to behave naively as price-takers and could strategically exercise their market power. Now the central issue is how to devise a dynamic auction that can restore an incentive for bidders to act truthfully as price-takers and at the same time yields an efficient outcome in this complex environment where items for sale can create synergies.[7]

Built upon and improving the adjustment process of Sun and Yang (2009), we will develop a strategy-proof dynamic auction design for the environment described above. The auction works roughly as follows. Starting from an arbitrary price vector, the auctioneer calls out the current price vector, bidders submit their demands at these prices, and then the auctioneer adjusts the prices of over-demanded items in one set $S_1$ (or $S_2$) upwards but those of under-demanded items in the other set $S_2$ (or $S_1$) downwards. We call this *a double-*

[6]Ostrovsky (2008) independently presents an analogous condition for a vertical supply chain model with contracts where prices of goods are fixed and a non-Walrasian equilibrium solution is used. He proves constructively the existence of stable matching under the condition, which allows complementarity between upstream and downstream contracts. Hatfield and Milgrom (2005) introduce the notion of contracts to matching models and establish the existence of stable matching for substitutable contracts. Hatfield et al. (2013) examine a trading network which allows cycles. They show the existence of equilibrium under a variant of GSC condition in the sense that every agent in the network has quasi-linear utility and views his upstream (downstream) contracts as substitutes but upstream (i.e., buying in) and downstream (i.e., selling out) contracts together as complements. Their network is a directed graph in which agents are nodes and both upstream and downstream contracts are directed edges. Baldwin and Klemperer (2013, section 6), Sun and Yang (2011), and Teytelboym (2013, Chapter 3; 2014) independently study a related but different type of competitive trading network also permitting cycles and demonstrate the existence of equilibrium provided that the network does not contain any odd cycle and every agent's demand for his concerned goods satisfies the GSC condition. In contrast to Hatfield et al.'s network, this latter trading network is an undirected graph in which each set of goods is a node, and every agent is an undirected edge. See also Drexl (2013).

[7]Complementarities or synergies among items are known as a difficult issue in auction design and equilibrium models and well-documented in Milgrom (2000, 2004), Jehiel and Moldovanu (2003), Porter et al. (2003), Klemperer (2004), and Maskin (2005) among others. As pointed out by Kelso and Crawford (1982), complementarity can even cause problems with existence of competitive equilibrium in the presence of indivisibilities. Nonetheless, GS or GSC guarantees the existence of competitive equilibrium in economies with indivisibilities.

*track auction* because it simultaneously updates prices in two opposite directions (ascending and descending). We show that this allocation mechanism always induces bidders to bid sincerely and finds an efficient outcome in finitely many rounds. In particular, this auction exhibits a significant strategic property that sincere bidding by every bidder is an ex post strongly perfect equilibrium of the dynamic game of incomplete information induced by the auction. More specifically, this means that after the auction has run up to any time $t^*$, no matter what has happened up to $t^*$ and no matter whether it is now on or off an equilibrium path, sincere bidding is an optimal strategy for every bidder $i$, as long as from $t^*$ on, every his opponent $j$ bids sincerely according to a certain fixed GSC utility function $\tilde{u}^j$ which need not be his true GSC utility function $u^j$. The notion of ex post strongly perfect equilibrium is slightly stronger than the concept of ex post perfect equilibrium used in Ausubel (2004, 2006). In addiction, this new auction guarantees ex post a nonnegative payoff for every bidder no matter how his opponents bid in the auction.

This auction is also detail-free, robust against any regret and independent of the probability distribution of every bidder's valuations over the goods. Another attractive feature of this auction is that it is simple, transparent, and privacy-preserving in the sense of Hurwicz (1973) and Ausubel (2006); see Kearns et al. (2013) for a recent development on the last issue. This auction does not only subsume and generalize Ausubel's from the setting with substitutes to the setting with both substitutes and complements, but also improves Ausubel's itself.[8] Aside from the theoretical interest and general applicability of this dynamic auction, our analysis complements Ausubel's which focuses on the model of divisible goods and relies on calculus and convex analysis. In contrast to Ausubel's analysis, ours is quite different, elementary and intuitive, and can facilitate a better understanding of his results. Unlike Ausubel (2006), Gul and Stacchetti (2000), Milgrom (2000), Sun and Yang (2009), the current model permits the seller to have a reservation value for every bundle of goods and allows her to maximize revenues, and thus makes the model closer to reality. The proposed auction will also ensure a nonnegative benefit of trading for the seller.

The remainder of this paper goes as follows. Section 2 presents the auction model. Section 3 describes the price adjustment process. Section 4 provides the main results. Section 5 concludes with some practical applications.

[8] In each step of Ausubel's auction, the auctioneer needs to compute the smallest or largest solution of an optimization problem which typically has multiple solutions. We will show that this cumbersome computation is not needed. This improvement is very useful for practical auction design. See Section 3 in detail.

# 2 The Auction Model

A seller (denoted by 0) wishes to auction a set $N = \{\beta_1, \beta_2, \cdots, \beta_n\}$ of $n$ indivisible items to a finite group $I$ of bidders. The items may be heterogeneous and can be divided into two sets $S_1$ and $S_2$ (i.e., $N = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$). For example, one can think of $S_1$ as tables and of $S_2$ as chairs. Items in the same set can be also heterogeneous. Let $I_0 = I \cup \{0\}$ denote the set of all agents (bidders and seller) in the market. Every agent $i \in I_0$ has a value function $u^i : 2^N \to \mathbb{R}$ specifying his/her valuation $u^i(B)$ (in units of money) on each bundle $B$ with $u^i(\emptyset) = 0$, where $2^N$ denotes the family of all bundles of items.[9] It is standard to assume that $u^i$ is weakly increasing, and that every bidder (he) can pay up to his value, and every agent has quasi-linear utilities in money. The seller (she) is a revenue-maximizer while the bidders are profit-maximizers. Here we allow the seller to have a utility function $u^0$ and so the model can accommodate more practical situations than the usual situation of assuming $u^0$ to be always zero.[10]

A price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$ indicates a price $p_h$ for each item $\beta_h \in N$. Agent $i$'s demand correspondence $D^i(p)$, the net utility function $v^i(A, p)$, and the indirect utility function $V^i(p)$, are defined respectively by

$$\begin{array}{rcl} D^i(p) & = & \arg\max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h\}, \\ v^i(A, p) & = & u^i(A) - \sum_{\beta_h \in A} p_h, \text{ and} \\ V^i(p) & = & \max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h\}. \end{array} \tag{2.1}$$

Because the seller is a revenue-maximizer, the family of her retaining bundles at prices $p$ are given by

$$S(p) = \arg\max_{A \subseteq N}\{u^0(A) + \sum_{\beta_h \in N \setminus A} p_h\}.$$

We first have the following basic observation which will be used later. The proof of the next result, and Lemma 2.3 and Theorem 3.1 will be relegated to the Appendix.

**Lemma 2.1** *For the seller, it holds that $S(p) = D^0(p)$.*

An allocation of items in $N$ is a *partition* $\pi = (\pi(i), i \in I_0)$ of items among all agents in $I_0$, i.e., $\pi(i) \cap \pi(j) = \emptyset$ for all $i \neq j$ and $\cup_{i \in I_0} \pi(i) = N$. Note that $\pi(i) = \emptyset$ is allowed. At allocation $\pi$, agent $i$ receives bundle $\pi(i)$. $\pi(0) \neq \emptyset$ is the bundle of unsold items and will be retained by the seller. An allocation $\pi$ is *efficient* if $\sum_{i \in I_0} u^i(\pi(i)) \geq \sum_{i \in I_0} u^i(\rho(i))$ for every allocation $\rho$. Given an efficient allocation $\pi$, let $R(N) = \sum_{i \in I_0} u^i(\pi(i))$. We call $R(N)$ *the market value* of the items which is the same for all efficient allocations.

[9] The seller's value function $u^0$ actually denotes her reservation price function and can be quite general.

[10] For instance, Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2006), Sun and Yang (2009) assume that the seller's reservation value is zero.

Let $\mathcal{M}$ denote the market with the set $I_0$ of agents and the set $N$ of items, and for each bidder $i \in I$, let $\mathcal{M}_{-i}$ denote the market $\mathcal{M}$ *without bidder i*. Let $I_{-i} = I_0 \setminus \{i\}$ for every bidder $i \in I$, and for convenience also let $\mathcal{M}_{-0} = \mathcal{M}$ and $I_{-0} = I_0$.

Next, we introduce two fundamental solution concepts for this auction model: the Walraisian equilibrium and the Vickrey-Clarke-Groves (VCG) outcome.

**Definition 2.2** *A Walrasian equilibrium* $(p, \pi)$ *consists of a price vector* $p \in \mathbb{R}^n_+$ *and an allocation* $\pi$ *such that* $\pi(i) \in D^i(p)$ *for every bidder* $i \in I$ *and* $\pi(0) \in S(p)$ *for the seller.*

In equilibrium $(p, \pi)$, the seller retains the bundle $\pi(0)$ of goods and collects the payment $\sum_{j \in I} \sum_{\beta_h \in \pi(j)} p_h$ from her sold goods and thus her equilibrium revenue is $u^0(\pi(0)) + \sum_{j \in I} \sum_{\beta_h \in \pi(j)} p_h$. Notice that in Gul and Stacchetti (1999, 2000), Milgrom (2000), Ausubel (2006), Sun and Yang (2006, 2009) it is assumed the seller values every bundle of goods at zero and consequently in equilibrium all goods will be sold to bidders. In the current model, because the seller has reservation value for every bundle, we need to slightly modify the notion of equilibrium. The following lemma shows that the modification is appropriate.

**Lemma 2.3** *Let* $(p, \pi)$ *be a Walrasian equilibrium. Then* $\pi$ *is an efficient allocation.*

The following defines the Vickrey-Clarke-Groves mechanism. The definition is slightly more general than its standard one because here we permit the seller to have her own utility function. The standard one assumes that the seller values everything at zero.

**Definition 2.4** *The VCG outcome is the outcome of the following procedure: every agent* $i \in I_0$ *reports his/her value function* $u^i$. *Then the auctioneer computes an efficient allocation* $\pi$ *with respect to all reported* $u^i$ *and assigns bundle* $\pi(i)$ *to bidder* $i \in I$ *and charges him a payment of* $q^*_i = u^i(\pi(i)) - R(N) + R_{-i}(N)$, *where* $R(N)$ *and* $R_{-i}(N)$ *are the market values of the markets* $\mathcal{M}$ *and* $\mathcal{M}_{-i}$ *based on* $u^i$ $(i \in I_0)$, *respectively. Bidder i's VCG payoff equals* $R(N) - R_{-i}(N)$, $i \in I$.

To ensure the existence of a Walrasian equilibrium, it will be necessary for us to impose some conditions. The most important one is known as *gross substitutes and complements condition*, which is introduced and used in Sun and Yang (2006, 2009), and defined as follows.[11]

**Definition 2.5** *The value function* $u^i$ *of agent i satisfies the* gross substitutes and complements (GSC) *condition if for any price vector* $p \in \mathbb{R}^n$, *any item* $\beta_k \in S_j$ *for* $j = 1$ *or* 2, *any* $\delta \geq 0$, *and any* $A \in D^i(p)$, *there exists* $B \in D^i(p + \delta e(k))$ *such that* $(A \cap S_j) \setminus \{\beta_k\} \subseteq B$ *and* $(A^c \cap S^c_j) \subseteq B^c$.

[11]The following piece of notation will be used. For any positive integer $k \leq n$, $e(k)$ denotes the $k$th unit vector in $\mathbb{R}^n$. Let $\mathbb{Z}^n$ stand for the integer lattice in $\mathbb{R}^n$ and 0 the $n$-vector of 0's. For any subset $A$ of $N$, let $e(A) = \sum_{\beta_k \in A} e(k)$. When $A = \{\beta_k\}$, we also write $e(A)$ as $e(k)$. For any subset $A$ of $N$, let $A^c$ denote its complement, i.e., $A^c = N \setminus A$. For any finite set $A$, $|A|$ denotes the number of elements in $A$.

GSC says that agent $i$ views items in each set $S_j$ as substitutes, but items across the two sets $S_1$ and $S_2$ as complements. In particular, when either $S_1 = \emptyset$ or $S_2 = \emptyset$, GSC reduces to the gross substitutes (GS) condition of Kelso and Crawford (1982). GS requires that all the items be substitutes, and thus excludes any complementarity among items. The GS case has been studied extensively in the literature; see e.g., Kelso and Crawford (1982), Gul and Stacchetti (1999, 2000), Milgrom (2000, 2004), and Ausubel (2006). Milgrom and Strulovici (2009) examine substitute goods in a more general setting. Sun and Yang (2014) investigate a related but different model in which all items for sale are complementary.[12]

The following three assumptions will be maintained throughout:

(A1) *Integer Private Values for Bidders*: Every bidder $i$'s value function $u^i : 2^N \to \mathbb{Z}_+$ takes integer values and is his private information.

(A2) *Integer Public Values for Seller*: The seller's value function $u^0 : 2^N \to \mathbb{Z}_+$ takes integer values and is public information, taking the form of $u^0(S) = u^0_1(S \cap S_1) + u^0_2(S \cap S_2)$ for any $S \subseteq N$, where $u^0_h : 2^{S_h} \to \mathbb{Z}_+$, $h = 1, 2$.

(A3) *Gross Substitutes and Complements*: The value function $u^i$ of every agent $i \in I_0$ satisfies the GSC condition with respect to the two sets $S_1$ and $S_2$.

In the literature, the value of the seller over each bundle is usually assumed to be zero and this information is made public. Here A2 is more general and can accommodate more realistic situations where the seller's reservation value over her goods for sale need not be zero and may vary from one bundle to another. The seller's utility over goods from the two sets is separable but the utility over goods from the same set $S_h$ need not be separable and can be very general. Although the seller has a reservation value function, she is assumed to reveal this function truthfully. This is a natural assumption from the well-known impossibility result of Myerson and Satterthwaite (1983), saying that it is impossible to design a trading mechanism even just for one buyer and one seller with one

[12]The major differences between Sun and Yang (2014) and the current one are (1) while in the current model there are two sets of items, items of each set are substitutable and can be heterogeneous but are complementary to items in the other set, goods in the model of Sun and Yang (2014) are all complementary; (2) while the current model has a Walrasian equilibrium (Sun and Yang 2006) in which the pricing rule is anonymous and linear, the model of Sun and Yang (2014) can only guarantee the existence of a nonlinear pricing Walrasian equilibrium in which the pricing rule is anonymous but nonlinear; (3) while the current auction is a blend of ascending and descending formats where prices are specified on individual items, the auction of Sun and Yang (2014) is and must be a package auction in which prices are specified on bundles of items; (4) there does not exist any transformation between the current model and Sun and Yang (2014) and in fact the structure of equilibrium price vectors for the two models is inherently different; see Sun and Yang (2009, Theorem 3, p. 937; 2014, Theorem 1, p. 432). These two models describe two typical, basic, and closely related yet intrinsically different economic environments.

item in such a way that it is optimal for the seller and the buyer to reveal their values honestly; budgets are balanced; and the final allocation is efficient. Because of this, in the literature on auction design it is often implicitly or explicitly assumed that the seller (or auctioneer) acts honestly, while bidders may behave strategically; see e.g., Gul and Stacchetti (2000), and Ausubel (2006).

# 3 The Price Adjustment Process

## 3.1 An Illustration

It is helpful to use a simple example to illustrate how an ascending (or descending) auction might be plagued by the exposure problem and how the double-track auction overcomes the problem to succeed in finding a Walrasian equilibrium. Consider now a market where a seller wishes to sell two volumes $A$ and $B$ of a book to two buyers. *Each buyer knows his values privately and the seller does not know those values.* Buyers' values are given in the Table 1, and the seller values every bundle at zero. Observe that every buyer views $A$ and $B$ as complements.

Table 1: Buyers' values over items.

| | $\emptyset$ | $A$ | $B$ | $AB$ |
|---|---|---|---|---|
| Buyer 1 | 0 | 2 | 2 | 5 |
| Buyer 2 | 0 | 2 | 2 | 5 |

**The ascending auction:** In an ascending auction, the seller initially announces a low price vector of $p(0) = (p_A(0), p_B(0)) = (0, 0)$ so that every buyer demands both $A$ and $B$. Buyers respond by reporting their demand sets at $p(0)$: $D^1(p(0)) = D^2(p(0)) = \{AB\}$. According to the reported demand sets, the seller subsequently adjusts the price vector $p(0)$ to the next one $p(1) = p(0) + \delta(0) = (1, 1)$ by increasing the price of every good by 1, because both goods are over-demanded at $p(0)$. The seller faces a similar situation at $p(1)$ and $p(2)$. The auction ends up with the price vector $p(3) = (3, 3)$ at which no bidder wants to demand the items anymore, and thus gets stuck in disequilibrium. We summarize the entire process in the Table 2. The reader can also verify that starting with a high price vector $p(0) = (p_A(0), p_B(0)) = (q, q)$ for any integer $q \geq 6$ so that no buyer demands any item, a descending auction will terminate with the price vector $\bar{p} = (2, 2)$ at which both buyers demand both items, and thus get stuck in disequilibrium, too. We remind the reader that prices in auction processes are adjusted in integer or fixed quantities, which are common in practice.

Table 2: The data created by the ascending auction for the example.

| Price vector | Buyer 1 | Buyer 2 | Price variation |
|---|---|---|---|
| $p(0) = (0, 0)$ | $\{AB\}$ | $\{AB\}$ | $\delta(0) = (1, 1)$ |
| $p(1) = (1, 1)$ | $\{AB\}$ | $\{AB\}$ | $\delta(1) = (1, 1)$ |
| $p(2) = (2, 2)$ | $\{AB\}$ | $\{AB\}$ | $\delta(2) = (1, 1)$ |
| $p(3) = (3, 3)$ | $\{\emptyset\}$ | $\{\emptyset\}$ | $\delta(3) = (0, 0)$ |

**The double-track auction:** Unlike the previous two cases, in the current double-track auction, the seller initially announces a price vector of $p(0) = (p_A(0), p_B(0)) = (0, 6)$ (a low price for item $A$ but a high price for item $B$) so that every buyer demands only item $A$ and not item $B$. Buyers respond by reporting their demand sets at $p(0)$: $D^1(p(0)) = D^2(p(0)) = \{A\}$. Using the reported demands, the seller subsequently adjusts the price vector $p(0)$ to the next one $p(1) = p(0) + \delta(0) = (1, 5)$ by increasing the price of $A$ by 1 but decreasing the price of $B$ by 1, because $A$ is over-demanded but $B$ is under-demanded at $p(0)$. At $p(1)$, the seller faces a similar situation. An interesting moment occurs when $p(1)$ advances to $p(2) = (2, 4)$ at which $B$ is clearly still under-demanded, but $A$ can be seen as either over-demanded or balanced. According to the rule of the double-track auction to be discussed soon in detail, the seller treats $A$ as balanced and so she adjusts $p(2)$ to $p(3) = (2, 3)$ by decreasing the price of $B$ by 1 and holding the price of $A$ constant. At $p(3)$, the market reaches an equilibrium in which the seller can assign items $A$ and $B$ to buyer 1 and asks him to pay 5, while buyer 2 gets nothing and pays nothing. We can summarize the entire process in the Table 3. Observe that in this process, the seller increases the price of item $A$ (since it is over-demanded) but decreases the price of item $B$ (since it is under-demanded) until the market is clear.

Table 3: The data created by the double-track auction for the example.

| Price vector | Buyer 1 | Buyer 2 | Price variation |
|---|---|---|---|
| $p(0) = (0, 6)$ | $\{A\}$ | $\{A\}$ | $\delta(0) = (1, -1)$ |
| $p(1) = (1, 5)$ | $\{A\}$ | $\{A\}$ | $\delta(1) = (1, -1)$ |
| $p(2) = (2, 4)$ | $\{\emptyset, A\}$ | $\{\emptyset, A\}$ | $\delta(2) = (0, -1)$ |
| $p(3) = (2, 3)$ | $\{\emptyset, A, AB\}$ | $\{\emptyset, A, AB\}$ | $\delta(3) = (0, 0)$ |

## 3.2 The Formal Price Adjustment Process

In this subsection we give a detailed description of the double-track auction. In a dynamic auction, at each time $t \in \mathbb{Z}_+$ and with respect to a price vector $p(t) \in \mathbb{R}^n$, each bidder $i$ selects a bid $C^i(t)$, a subset of $2^N$. We say that bidder $i$ *bids sincerely* relative to value function $u^i$ if his bid always equals his true demand correspondence, i.e., $C^i(t) = D^i(p(t)) = \arg\max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h(t)\}$.

In this section we assume that bidders are price-takers and thus bid sincerely. We will present a modified version of the double-track adjustment process introduced by Sun and Yang (2009). This process always yields an equilibrium and provides a key ingredient for the auction design in Section 4 where bidders $i \in I$ are not assumed to behave as price-takers and thus may act strategically. Throughout the paper, in the price adjustment process and in the auction mechanism, at the beginning the seller reports her reserve price function $u^0$ to the auctioneer who then uses $u^0$ to calculate the seller's demand correspondence $D^0(p(t))$ at prices $p(t)$ in every round $t$. Thus, the auctioneer (she) acts as a proxy bidder for the seller. Recall that since by Lemma 2.1, $D^0(p(t)) = S(p(t))$, the seller can act as a bidder. In the sequel, the seller may be also called a bidder. Nevertheless, remember that this proxy bidder always acts sincerely.

The price adjustment process makes use of the Lyapunov function $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\mathcal{L}(p) = \sum_{\beta_h \in N} p_h + \sum_{i \in I_0} V^i(p) \tag{3.2}$$

where $V^i$ is the indirect utility function of agent $i \in I_0$. Although the use of Lyapunov function is well-known in the literature on economies with divisible goods (see Arrow and Hahn (1971), and Varian (1981)), it was only recently explored by Ausubel (2005, 2006) in a striking manner to deal with discrete economies with substitutes and extended by Sun and Yang (2009) to the case including both substitutes and complements. Observe that the Lyapunov function introduced above includes also the seller's indirect utility function $V^0$ and is more general than those previously used in the literature.

By Sun and Yang (2009, Lemma 1 and Theorem 3), the Lyapunov function $\mathcal{L}$ defined above is a convex function and has its minimizers, which correspond to equilibrium price vectors under certain mild conditions, which are satisfied by (A3). The double-track auction explores this link to discover an equilibrium price vector. To use this idea, we need to clear two major hurdles: one is how to connect observable information such as prices and demands with the unobservable Lyapunov function $\mathcal{L}$, and the other is how to resolve the exposure problem. As shown in the previous subsection, while neither an ascending nor descending auction mechanism will work, the double-track auction does work in this case.

To describe the double-track auction, we introduce the following $n$-dimensional cube for price adjustment

$$\Phi = \{\delta \in \mathbb{R}^n \mid 0 \le \delta_k \le 1, \forall \beta_k \in S_1,\ -1 \le \delta_l \le 0, \forall \beta_l \in S_2\}.$$

Let $\Delta = \Phi \cap \mathbb{Z}^n$ be the discrete set and $\Phi^* = -\Phi$, $\Delta^* = -\Delta$. Through $\Phi$ ($\Delta$), we lower prices of items in $S_2$ but raise prices of items in $S_1$, while through $\Phi^*$ ($\Delta^*$), we lower prices of items in $S_1$ but raise prices of items in $S_2$. The auction works as follows: Given an integer price vector $p(t) \in \mathbb{Z}^n$ at time $t \in \mathbb{Z}_+$, the auctioneer asks every bidder $i$ to report his demand $D^i(p(t))$. Then she uses every bidder's reported demand $D^i(p(t))$ to search for a price adjustment $\delta \in \Phi$ so as to reduce the value of the Lyapunov function $\mathcal{L}(p(t)+\delta)$ as much as possible, in the hope that the minimum of the Lyapunov function will be reached. Formally, this amounts to solving the continuous maximization problem with the unknown objective function $\mathcal{L}$

$$\max_{\delta \in \Phi}\{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} \tag{3.3}$$

Sun and Yang (2009, pp. 940-942) derive the following crucial relationship in detail:[13]

$$\max_{\delta \in \Phi}\{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta \in \Delta}\Big\{\sum_{i \in I_0}\Big(\min_{S \in D^i(p(t))} \sum_{\beta_h \in S} \delta_h\Big) - \sum_{\beta_h \in N} \delta_h\Big\} \tag{3.4}$$

Observe that the left hand continuous maximization problem over the entire cube $\Phi$ reduces to the right hand discrete maximization problem over a finite set $\Delta$ of integer price vectors, and that the relation shows a dramatic change from the unobservable Lyapunov function $\mathcal{L}$ to the observable reported demands of bidders and integer price adjustment $\delta$. In the right hand formula, the price of each item in $S_1$ increases either one unit or nothing, whereas the price of each item in $S_2$ decreases either one unit or nothing. Furthermore, the right hand max-min formula has an intuitive and meaningful economic interpretation: when the auctioneer adjusts the prices from $p(t)$ to $p(t+1) = p(t) + \delta(t)$, she acts in an elaborate manner so that the seller can extract a maximal gain whereas every bidder can achieve a minimal loss in indirect utility. Observe that the auctioneer is responsible for executing the computation of (3.4) based on bidders' reported demands $D^i(p(t))$. It is fairly easy to calculate the value $(\min_{S \in D^i(p(t))} \sum_{\beta_h \in S} \delta_h)$ for each given $\delta \in \Delta$ or $\Delta^*$ and bidder $i$. We can now present the detailed steps of the adjustment process as follows:

**The improved double-track (IDT) adjustment process**

Step 1: The seller reports her reserve price function $u^0$ to the auctioneer, who announces the initial price vector $p(0) \in \mathbb{Z}^n_+$. Let $t := 0$ and go to Step 2.

Step 2: The auctioneer asks every bidder $i \in I_0$ (this also includes the proxy bidder 0) to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer computes a solution $\delta(t)$ to the problem (3.4). If $\delta(t) = 0$, go to Step 3. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t+1$. Return to Step 2.

[13] A brief self-contained explanation is given in the appendix.

Step 3: The auctioneer asks every bidder $i \in I_0$ to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer computes a solution $\delta(t)$ to the problem (3.4) where $\Delta$ is replaced by $\Delta^*$. If $\delta(t) = 0$, then the auction stops. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to Step 3.

Observe that in both Step 2 and Step 3 the auctioneer needs only an arbitrary solution to the problem (3.4) with respect to $\Delta$ or $\Delta^*$. This improves considerably the original process of Sun and Yang (2009) which requires to take the smallest or largest solution to the same problem if there are several solutions. (In fact, the set of solutions to the problem (3.4) is a nonempty lattice and typically has multiple solutions.) This improvement is very useful and important for practical auction design and makes the implementation easy and fast. Consequently, it also improves the auction of Ausubel (2006). Recall that in his auction model with indivisible goods, all goods are assumed to be substitutes, i.e., $S_1 = \emptyset$ or $S_2 = \emptyset$ in the current model. In each step of his auction, the auctioneer must compute the smallest or largest solution of an optimization problem which typically has multiple solutions. The above process shows that this cumbersome computation is no longer needed. Observe that the IDT process may go to Step 3 from Step 2 but will never return to Step 2 from Step 3. This is another attractive property and means that we can improve Ausubel's (2006) global auction which requires repeated implementation of his ascending auction and his descending auction one after another. Now that requirement can be dropped. His global auction just needs to execute his ascending auction and descending auction each at most once.

The following theorem shows the global convergence of the IDT adjustment process.

**Theorem 3.1** *For the market model under Assumptions* (A1), (A2) *and* (A3)*, starting with any integer price vector, the IDT adjustment process converges to an equilibrium price vector in a finite number of rounds.*

# 4 The Strategy-Proof Dynamic Auction Mechanism

In the previous section we have assumed that every bidder acts honestly as a price-taker. In this section we totally drop that assumption by allowing bidders $i \in I$ to strategically exercise their market power. In this environment, we need to address two basic questions. First, is it possible to design an auction mechanism that induces bidders to act honestly as price-takers? Second, is it possible to devise an auction that requires just enough but not excessive information from bidders so that bidders' privacy can be preserved? To answer these questions in the affirmative, we propose a dynamic auction that not only possesses the appealing strategy-proof property but also has the merit of privacy-preservation, transparency and detail-freeness.

## 4.1 The Auction Mechanism Design

We now present the dynamic auction mechanism. The mechanism runs the IDT adjustment process for all markets $\mathcal{M}_{-m}$ ($m \in I_0$) simultaneously in parallel and in coordination. The IDT adjustment process works for every market $\mathcal{M}_{-m}$ exactly as described in Section 3 but needs the following modifications: Consider any market $\mathcal{M}_{-m}$. At $t \in \mathbb{Z}_+$ and $p^{-m}(t) \in \mathbb{Z}^n_+$, every bidder $i \in I_{-m}$ reports a bid $C^i_{-m}(t) \subseteq 2^N$ (which need not be his demand set $D^i(p^{-m}(t))$[14]) and the problem (3.4) becomes the next one for $\Delta$ or $\Delta^*$ respectively,

$$\max_{\delta\in\Delta(\text{ or }\Delta^*)}\Big\{\sum_{i\in I_{-m}}\Big(\min_{S\in C^i_{-m}(t)}\sum_{\beta_h\in S}\delta_h\Big)-\sum_{\beta_h\in N}\delta_h\Big\} \tag{4.5}$$

If the auctioneer finds a solution $\sigma^{-m}(t)$ of (4.5) for $\Delta$ ($\Delta^*$), she obtains the next price vector $p^{-m}(t+1) = p^{-m}(t) + \delta^{-m}(t)$ whenever $\delta^{-m}(t) \neq 0$. We say the IDT adjustment process *finds an allocation* $\pi^{-m}$ in $\mathcal{M}_{-m}$ if $\delta^{-m}(t) = 0$ for $\Delta^*$ (i.e., in Step 3 of the auction) and $\pi^{-m}(i) \in C^i_{-m}(t)$ for all $i \in I_{-m}$. The IDT adjustment process needs to go back to Step 2 from Step 3 if $\delta^{-m}(t) = 0$ for $\Delta^*$ but it finds no allocation $\pi^{-m}$ in $\mathcal{M}_{-m}$ such that $\pi^{-m}(i) \in C^i_{-m}(t)$ for all $i \in I_{-m}$—this modification is meant to tolerate minor mistakes or manipulations committed by bidders. The IDT adjustment process *detects serious manipulation* if it never finds an allocation in $\mathcal{M}_{-m}$ in which case the auction is said to *stop at time* $\infty$. Now we have

**The strategy-proof double-track (SPDT) auction**

Step 1: Run the IDT adjustment process simultaneously in parallel for every market $\mathcal{M}_{-m}$ ($m \in I_0$) by starting with a common initial price vector $p^{-m}(0) = p(0) \in \mathbb{Z}^n_+$. At $t \in \mathbb{Z}_+$ and $p^{-m}(t) \in \mathbb{Z}^n$, every bidder $i \in I_{-m} \setminus \{0\} = I \setminus \{m\}$ reports a bid $C^i_{-m}(t) \subseteq 2^N$, the proxy bidder 0 bids truthfully by reporting $C^0_{-m}(t) = D^0(p^{-m}(t))$, and the auctioneer finds the next price vector $p^{-m}(t+1) = p^{-m}(t) + \delta^{-m}(t)$. If the IDT adjustment process detects serious manipulations in any market, go to Step 3. Otherwise, the IDT adjustment process continues until it finds an allocation $\pi^{-m}$ in every market $\mathcal{M}_{-m}$ ($m \in I_0$) at $p^{-m}(T^{-m}) \in \mathbb{Z}^n_+$, and $T^{-m} \in \mathbb{Z}_+$. Go to Step 2.

Step 2: In this case all markets are clear. For every $m \in I_0$, every agent $i \in I_{-m}$ and every $t = 0, 1, \cdots, T^{-m} - 1$, let $\Delta^{-m}_i(t)$ denote the "indirect utility change" of agent

[14]However, the proxy bidder 0 (the seller) always bids honestly by reporting her demand set $C^0_{-m}(t) = D^0(p^{-m}(t))$.

$i$ in $I_{-m}$ when prices move from $p^{-m}(t)$ to $p^{-m}(t+1)$, where

$$\Delta_i^{-m}(t) = \min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta_h^{-m}(t) \tag{4.6}$$

Every bidder $i \in I$ will be assigned the bundle $\pi^{-0}(i)$ of the allocation $\pi^{-0}$ found in the market $\mathcal{M}_{-0} = \mathcal{M}$ and required to pay $q_i$, with the option to decline when his payoff becomes negative, where

$$q_i = \sum_{j \in I_{-i}} \Big( \sum_{t=0}^{T^{-0}-1} \Delta_j^{-0}(t) - \sum_{t=0}^{T^{-i}-1} \Delta_j^{-i}(t) \Big) + \sum_{\beta_h \in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \tag{4.7}$$

The auction stops.

Step 3: In this case every bidder $i \in I$ receives no item and pays nothing. The auction stops.

The payment $q_i$ of bidder $i \in I$ has an intuitive interpretation: $q_i$ is equal to the accumulation of "indirect utility changes" of his opponents $l \in I_{-i}$ (also including the proxy bidder 0) along the path from $p^{-i}(T^{-i})$ to $p(0)$ (in the market $\mathcal{M}_{-i}$) and the path from $p(0)$ to $p^{-0}(T^{-0})$ (in the market $\mathcal{M}$ ) by subtracting $\sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}$–the equilibrium payments by bidder $i$'s opponents in the market $\mathcal{M}$, and adding $\sum_{\beta_h \in N} p_h^{-i}(T^{-i})$–the equilibrium payments by bidder $i$'s opponents in the market $\mathcal{M}_{-i}$. Notice that Ausubel's auction (2006) and his payment rule are not symmetric, whereas the current auction and payment rule are symmetric and simpler.[15]

Notice that the option of rejection in Step 2 is a new auction rule in contrast to Ausubel (2004, 2006) which do not have such rules. This rule means that if the assignment of bidder $i$ gives him a negative payoff $u^i(\pi^{-0}(i)) - q_i < 0$, he can reject the assignment and leave the auction empty handed without any cost. In Step 3, in contrast to Ausubel's penalty of infinity, we adopt the lenient policy of no punishment, which is common in practice. This is possible because we use the convention that if honesty for an agent is one of his optimal policies, he will only adopt the honesty policy. Finally, it is simple but important to observe that the SPDT auction tolerates any mistakes or manipulations committed by bidders and allows them to correct so that for any time $t^* \in \mathbb{Z}_+$, no matter what has

[15] More precisely, the current auction starts with the same initial price vector $p(0)$ for all markets $\mathcal{M}$ and $\mathcal{M}_{-i}$, $i \in I$, whereas Ausubel's (Ausubel 2006, pp.615-616) starts with the same initial price vector $p(0)$ only for the markets $\mathcal{M}_{-i}$, $i \in I$, but for the market $\mathcal{M}$ his auction starts with the equilibrium price vector $p^{-k^*}$ of any chosen market $\mathcal{M}_{-k^*}$. In Ausubel's auction, the payment of bidder $k^*$ is given by Equation (7) (Ausubel 2006, p.611) using the price vectors along the path from $p^{-k^*}$ to $p^*$. The VCG payment of bidder $i$ ($i \in I_{-k^*}$) is also given by Equation (7) but using the price vectors along the path from $p^{-i}$ to $p^0$; the path from $p^0$ to $p^{-k^*}$; and the path from $p^{-k^*}$ to $p^*$.

happened before $t^*$, as long as from $t^*$ on every bidder $i$ bids according to his GSC value function $u^i$, the auction will find a Walrasian equilibrium in every market in finitely many rounds and thus terminates in Step 2, because the IDT adjustment process converges to a Walrasian equilibrium from any integer price vector.

## 4.2 Incentive and Strategic Issues

To study the incentive and strategic properties of the SPDT auction mechanism, we will formulate this auction as an extensive-form dynamic game of incomplete information in which bidders are players. Prior to the start of the (auction) game, nature reveals to every player $i \in I$ only his own value function $u^i \in \mathcal{U}$ of private information and a joint probability distribution $F(\cdot)$ from which the profile $\{u^i\}_{i\in I}$ is drawn, where $\mathcal{U}$ denotes the family of all value functions $u : 2^N \to \mathbb{Z}_+$ satisfying Assumptions (A1) and (A2). Let $H_i^t$ be the part of the information (or history) of play that player $i$ has observed just before he submits his choice sets at time $t \in \mathbb{Z}_+$. A natural and sensible specification is that $H_i^t$ comprises the complete set of all observable price vectors and all players' choice sets, i.e.,

$$H_i^t = \{p^{-m}(t), p^{-m}(s), C_{-m}^j(s) \mid m \in I_0, j \in I, 0 \leq s < t, m \neq j\}$$

Note that $H_i^t = H_j^t$ for all $i, j \in I$, namely, all bidders share a common history just like in an English auction. Let $T^*$ be the time when the SPDT auction stops at Steps 2 or 3. If the auction has found an allocation in any $\mathcal{M}_{-m}$, for consistency and convenience, we define $C_{-m}^i(t) = C_{-m}^i(T^{-m})$ and $p^{-m}(t) = p^{-m}(T^{-m})$ for any $i \in I_{-m}$ and any $t \in \mathbb{Z}_+$ between $T^{-m}$ and $T^*$. After any history $H_i^t$ and at any time $t \in \mathbb{Z}_+$, each player $i$ updates his posterior beliefs $\mu_i(\cdot \mid t, H_i^t, u^i)$ over opponents' value functions; see also Ausubel (2006). We stress that even after the auction is finished, player $i$ may not know his opponents' value functions precisely.

A (dynamic) *strategy* $\sigma_i$ of player $i(i \in I)$ is a set-valued function $\{(t, m, H_i^t, u^i) \mid t \in \mathbb{Z}_+, m \in I_{-i}, u^i \in \mathcal{U}\} \to 2^N$, which tells him to bid $\sigma_i(t, m, H_i^t, u^i) \subseteq 2^N$ for every market $\mathcal{M}_{-m}(m \in I_{-i})$ at each time $t \in \mathbb{Z}_+$ when he observes $H_i^t$. Let $\Sigma_i$ denote player $i's$ strategy space of all such strategies $\sigma_i$. We say that $\sigma_i$ is *a regular bidding strategy* for player $i$ if irrespective of his true utility function $u^i$, he always reports his choice set $C_{-m}^i(t)$ according to some utility function $\tilde{u}^i \in \mathcal{U}$ for any $m \in I_{-i}$, $t \in \mathbb{Z}_+$, $p^{-m}(t) \in \mathbb{Z}^n$, and $H_i^t$, i.e.,

$$\sigma_i(t, m, H_i^t, u^i) = C_{-m}^i(t) = \arg\max_{A \subseteq N}\{\tilde{u}^i(A) - \sum_{\beta_h \in A} p_h^{-m}(t)\}$$

Note that $\tilde{u}^i$ may or may not be his true utility function $u^i$. We denote such a regular bidding strategy by $\sigma_i^{\tilde{u}^i}$. Thus, every GSC utility function $\tilde{u}(\tilde{u} \in \mathcal{U})$ determines a regular bidding strategy for each player. For simplicity, we also use $\mathcal{U}$ to denote the family of all such strategies. Clearly, $\mathcal{U} \subseteq \Sigma_i$. A regular bidding strategy $\sigma_i^{\tilde{u}^i}$ is *sincere bidding* (strategy) for player $i$ if $\tilde{u}^i$ is equal to his true utility function $u^i$, namely, if he always reports his demand set $D^i(p^{-m}(t))$ as defined by (2.1) with respect to his true utility function $u^i$, i.e., $\sigma_i(t, m, H_i^t, u^i) = C_{-m}^i(t) = D^i(p^{-m}(t)) = \arg\max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h^{-m}(t)\}$ for all $t \in \mathbb{Z}_+$, $m \in I_{-i}$ and $p^{-m}(t) \in \mathbb{Z}^n$. The strategy space $\Sigma_i$ of player $i$ contains regular bidding strategies, sincere bidding strategies and also various other strategies.

Given the auction rules, the outcome of this auction game depends entirely upon the realization of utility functions and the strategies the bidders take. When every bidder $i \in I$ takes a strategy $\sigma_i$ and the SPDT auction terminates in Step 2, then bidder $i \in I$ receives bundle $\pi^{-0}(i)$ and pays $q_i$ given by (4.7), or gets nothing and pays nothing. When every bidder $i \in I$ takes a strategy $\sigma_i$ and the SPDT auction stops in Step 3, every bidder gets nothing and pays nothing. In summary, every player $i's$ payoff function $W_i(\cdot, \cdot)$ is given by

$$W_i\Big(\{\sigma_j\}_{j\in I}, \{u^j\}_{j\in I}\Big) = \begin{cases} \max\{0, u^i(\pi^{-0}(i)) - q_i\} & \text{if the auction stops in Step 2,} \\ 0 & \text{if the auction stops in Step 3.} \end{cases}$$

We now recall the notion of ex post perfect equilibrium used by Ausubel (2004, 2006) to dynamic auction games of incomplete information. For such a game, the $\sharp(I)$-tuple $\{\sigma_i\}_{i\in I}$ is said to be *an ex post perfect equilibrium*[16] if for any time $t \in \mathbb{Z}_+$, any history profile $\{H_i^t\}_{i\in I}$, and any realization $\{u^i\}_{i\in I}$ of profile of utility functions of private information, the continuation strategy $\sigma_i(\cdot \mid t, H_i^t, u^i)$ of every player $i \in I$ (i.e., $\sigma_i(s, m, H_i^s \mid t, H_i^t, u^i) \subseteq 2^N$ for all $s \geq t$, $m \in I_{-i}$ and $H_i^s$) constitutes his best response against the continuation strategies $\{\sigma_j(\cdot \mid t, H_j^t, u^j)\}_{j\in I_{-i}}$ of player $i$'s opponents of the game even if the realization $\{u^i\}_{i\in I}$ becomes common knowledge.

For the current model, we introduce and use the following stronger equilibrium solution than the previous one. A strategy $\sigma_i$ of player $i$ constitutes *an ex post strongly perfect strategy* for him if for any time $t \in \mathbb{Z}_+$, any history profile $\{H_j^t\}_{j\in I}$, and any realization $\{u^j\}_{j\in I}$ of profile of utility functions of private information, the continuation strategy $\sigma_i(\cdot \mid t, H_i^t, u^i)$ of player $i$ is his best response against *all continuation regular bidding strategies* $\{\sigma_j^{\tilde{u}^j}(\cdot \mid t, H_j^t, u^j)\}_{j\in I_{-i}}$ of player $i$'s opponents, even if the realization $\{u^i\}_{i\in I}$ becomes common knowledge. The $\sharp(I)$-tuple $\{\sigma_i\}_{i\in I}$ of regular bidding strategies comprises *an ex post strongly perfect (Nash) equilibrium* if for every player $i \in I$, his regular bidding strategy $\sigma_i$ is an ex post strongly perfect strategy. Clearly, every ex post strongly perfect equilibrium is an ex post perfect equilibrium but the reverse may not be true. Stronger than Bayesian equilibrium or perfect Bayesian equilibrium, ex post (strongly) perfect equilibria have a number of additional desirable properties, i.e., they are not only robust against any regret but also independent of any probability distribution. Furthermore, in the complete

[16]In (static or sealed-bid) auction games of incomplete information, the ex post equilibrium is used by Crémer and McLean (1985) and Krishna (2002).

information case, ex post perfect equilibrium simply coincides with the familiar notion of subgame perfect equilibrium.

In the current auction game, although the auctioneer knows that every bidder $i \in I$ possesses a GSC utility function $u^i$, she has no precise knowledge of $u^i$. This implies that as long as a bidder reports his demand according to some fixed GSC utility function $\tilde{u}^i$ not necessarily being his true utility function, it is extremely hard if not impossible to prove whether he bids truthfully or not. According to Hurwicz (1973, p.23) on mechanism design, "it is conceivable that the participants would cheat without openly violating the rules." This is why we focus on "all regular bidding strategies" instead of "all dynamic strategies" of all opponents of every bidder $i \in I$ in the definition of the proposed solution. Regular bidding strategies are safe, whereas irregular ones are unsafe in the sense that they have a high probability of being detected for open violation of the auction rules.

Finally, we introduce one more desirable property, which we believe is also important for any practical auction design. An auction mechanism is said to be *ex post individually rational*, if, for every bidder, no matter how his opponents bid in the auction, as long as he is *sufficiently rational* in the sense that he can judge whether his payoff is negative or nonnegative, he will never end up with a negative payoff. It will be shown that the proposed auction also possesses this appealing property. It might be worth mentioning that Ausubel's auction (2006) does not have this property.

Now we are prepared to establish our major theorem.

**Theorem 4.1** *Suppose that the market $\mathcal{M}$ satisfies Assumptions* (A1), (A2) and (A3).
(i) *When every bidder bids sincerely, the SPDT auction converges to a Walrasian equilibrium, yields a Vickrey-Clarke-Groves outcome for the market $\mathcal{M}$ in a finite number of rounds, and the seller receives a nonnegative benefit of trading.*
(ii) *Sincere bidding by every bidder is an ex post strongly perfect equilibrium in the SPDT auction.*
(iii) *The SPDT auction is ex post individually rational.*

Proof: We first prove (i). By the argument in Section 3, we see that when every bidder $i$ bids sincerely according to his true GSC function $u^i$ , the auction terminates at Step 2 and finds a Walrasian equilibrium $(p^{-m}(T^{-m}), \pi^{-m})$ in every market $\mathcal{M}_{-m}$, $m \in I_0$. By the rules, every bidder $i$ receives bundle $\pi^{-0}(i)$ and pays $q_i$ of (4.7). It follows from (5.9) in the Appendix that

$$\Delta_i^{-m}(t) = \min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta_h^{-m}(t) = V^i(p^{-m}(t)) - V^i(p^{-m}(t+1))$$

for all $i \in I$ and $m \in I_0$ $(i \neq m)$, where $C^i_{-m}(t) = D^i(p^{-m}(t))$. Using these equations, we will show that $q_i$ coincides with the VCG payment $q_i^* = u^i(\pi^{-0}(i)) - R(N) + R_{-i}(N)$, where $R(N) = \sum_{j\in I} u^j(\pi^{-0}(j))$ and $R_{-i}(N) = \sum_{j\in I_{-i}} u^j(\pi^{-i}(j))$. Observe that payment $q_i$ of (4.7) satisfies

$$\begin{array}{rcl} q_i & = & \sum_{j\in I_{-i}}\Big(\sum_{t=0}^{T^{-0}-1}(V^j(p^{-0}(t)) - V^j(p^{-0}(t+1))) \\ & & \quad -\sum_{t=0}^{T^{-i}-1}(V^j(p^{-i}(t)) - V^j(p^{-i}(t+1)))\Big) \\ & & \quad +\sum_{\beta_h\in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h\in N\setminus\pi^{-0}(i)} p_h^{-0}(T^{-0}) \\ & = & \sum_{j\in I_{-i}}\Big((V^j(p^{-0}(0)) - V^j(p^{-0}(T^{-0}))) - (V^j(p^{-i}(0)) - V^j(p^{-i}(T^{-0})))\Big) \\ & & \quad +\sum_{\beta_h\in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h\in N\setminus\pi^{-0}(i)} p_h^{-0}(T^{-0}) \\ & = & \Big(\sum_{j\in I_{-i}} V^j(p^{-i}(T^{-0})) + \sum_{\beta_h\in N} p_h^{-i}(T^{-i})\Big) \\ & & \quad -\Big(\sum_{j\in I_{-i}} V^j(p^{-0}(T^{-0})) + \sum_{\beta_h\in N\setminus\pi^{-0}(i)} p_h^{-0}(T^{-0})\Big) \\ & = & \sum_{j\in I_{-i}} u^j(\pi^{-i}(j)) - \sum_{j\in I_{-i}} u^j(\pi^{-0}(j)) \\ & = & u^i(\pi^{-0}(i)) - R(N) + R_{-i}(N) \\ & = & q_i^*. \end{array}$$

Bidder $i's$ payoff $u^i(\pi^{-0}(i)) - q_i$ equals his VCG payoff $R(N) - R_{-i}(N)$.

We next prove that the seller receives a nonnegative benefit. First note that for every buyer $i \in I$, it satisfies that

$$R_{-i}(N) \geq u^0(\pi^{-0}(0) \cup \pi^{-0}(i)) + \sum_{j\in I\setminus\{i\}} u^j(\pi^{-0}(j)).$$

Thus, for the final payoff $\tilde{W}_0$ of the seller, we have

$$\begin{array}{rcl} \tilde{W}_0 & = & u^0(\pi^{-0}(0)) + \sum_{i\in I} q_i^* \\ & = & u^0(\pi^{-0}(0)) + \sum_{i\in I}\Big(u^i(\pi^{-0}(i)) - R(N) + R_{-i}(N)\Big) \\ & = & \sum_{i\in I} R_{-i}(N) - (m-1)R(N) \\ & \geq & \sum_{i\in I}\Big(u^0(\pi^{-0}(0)\cup\pi^{-0}(i)) + \sum_{j\in I\setminus\{i\}} u^j(\pi^{-0}(j))\Big) - (m-1)R(N) \\ & = & \sum_{i\in I}\Big([u^0(\pi^{-0}(0)\cup\pi^{-0}(i)) - u^0(\pi^{-0}(0)] + R(N) - u^i(\pi^{-0}(i))\Big) - (m-1)R(N) \\ & = & u^0(\pi^{-0}(0)) + \sum_{i\in I}[u^0(\pi^{-0}(0)\cup\pi^{-0}(i)) - u^0(\pi^{-0}(0)] \\ & = & \sum_{i\in I} u^0(\pi^{-0}(0)\cup\pi^{-0}(i)) - (m-1)u^0(\pi^{-0}(0). \end{array}$$

By Assumptions (A2) and (A3) on the seller's utility function $u^0$, for every $k = 1, 2, \cdots, m-1$, we have

$$u^0(\cup_{i=0}^{k}\pi^{-0}(i)) + u^0(\pi^{-0}(0)\cup\pi^{-0}(k+1)) \geq u^0(\cup_{i=0}^{k+1}\pi^{-0}(i)) + u^0(\pi^{-0}(0).$$

Thus, we can iteratively show that

$$\begin{array}{rcl} \tilde{W}_0 & = & \sum_{i\in I} u^0(\pi^{-0}(0)\cup\pi^{-0}(i)) - (m-1)u^0(\pi^{-0}(0) \\ & \geq & u^0(\cup_{i=0}^{m}\pi^{-0}(i)) = u^0(N). \end{array}$$

Consequently, the seller's benefit $\tilde{W}_0 - u^0(N)$ is nonnegative.

Now we prove (ii). It suffices to show that sincere bidding is every player $i's$ ex post strongly perfect strategy. Consider any time $t^* \in \mathbb{Z}_+$, any history profile $\{H_j^{t^*}\}_{j\in I}$ (which may be on or off the equilibrium path), and any realization $\{u^j\}_{j\in I}$ of profile of utility functions in $\mathcal{U}^I$ of private information.[17] Suppose that from this time $t^*$ on every opponent $j(j \in I_{-i})$ will report his bids according to a regular bidding strategy $\sigma_j^{\tilde{u}^j}$. That is, every player $j(j \in I_{-i})$ according to some $\tilde{u}^j \in \mathcal{U}$ reports his $C_{-m}^j(t)$ at every round $t(t \geq t^*)$, namely,

$$\sigma_j^{\tilde{u}^j}(t, m, H_j^t, u^j) = C_{-m}^j(t) = \arg\max_{A\subseteq N}\{\tilde{u}^j(A) - \sum_{\beta_h\in A} p_h^{-m}(t)\}$$

for every $m \in I_{-j}$. Of course, it is possible that $\tilde{u}^j \neq u^j$. Clearly, in this continuation game from time $t^*$, when all opponents of player $i$ choose regular bidding strategies, because of the option rule of rejection in Step 2, bidder $i$ prefers a strategy which results in the auction terminating at Step 2 and a nonnegative payoff, to any other strategies which result in the auction stopping at Step 3 and a zero payoff. Therefore, it sufficient to compare the sincere bidding strategy with any other strategies which also result in the auction finishing at Step 2. Suppose that $\sigma_i'(\cdot \mid t^*, H_i^{t^*}, u^i)$ ($\sigma_i'$ in short) is such a continuation strategy of player $i$ resulting in an allocation $\rho$ for $\mathcal{M}$, and that bidder $i$'s (continuation) sincere bidding strategy results in an allocation $\pi$ for $\mathcal{M}$. Without any loss of generality, we assume that by the time $t^*$, the auction for the markets $\mathcal{M}$ and $\mathcal{M}_{-i}$ has not yet finished, i.e., $t^* < T^{-0}$ and $t^* < T^{-i}$. When player $i$ chooses the strategy $\sigma_i'$, his payment $q_i'$ given by (4.7) is

$$\begin{aligned}
q_i' &= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}\Delta_j^{-0}(t) + \sum_{t=t^*}^{T^{-0}-1}[\tilde{V}^j(p^{-0}(t)) - \tilde{V}^j(p^{-0}(t+1))] \\
&\qquad - \textstyle\sum_{t=0}^{t^*-1}\Delta_j^{-i}(t) - \sum_{t=t^*}^{T^{-i}-1}[\tilde{V}^j(p^{-i}(t)) - \tilde{V}^j(p^{-i}(t+1))]\Big) \\
&\quad + \textstyle\sum_{\beta_h\in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h\in N\setminus\rho(i)} p_h^{-0}(T^{-0}) \\
&= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}[\Delta_j^{-0}(t) - \Delta_j^{-i}(t)] + \tilde{V}^j(p^{-0}(t^*)) + \tilde{V}^j(p^{-i}(T^{-i})) - \tilde{V}^j(p^{-i}(t^*))\Big) \\
&\qquad + \textstyle\sum_{\beta_h\in N} p_h^{-i}(T^{-i}) \\
&\quad - \Big(\textstyle\sum_{j\in I_{-i}}\tilde{V}^j(p^{-0}(T^{-0})) + \sum_{\beta_h\in N\setminus\rho(i)} p_h^{-0}(T^{-0})\Big) \\
&= \mathit{constant} - \textstyle\sum_{j\in I_{-i}}\tilde{u}^j(\rho(j)),
\end{aligned}$$

where $\tilde{V}^j$ is bidder $j$'s indirect utility function based on $\tilde{u}^j$ and *constant* is given by

$$\begin{aligned}
\mathit{constant} &= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}[\Delta_j^{-0}(t) - \Delta_j^{-i}(t)]\Big) \\
&\quad + \textstyle\sum_{j\in I_{-i}}\Big(\tilde{V}^j(p^{-0}(t^*)) + \tilde{V}^j(p^{-i}(T^{-i})) - \tilde{V}^j(p^{-i}(t^*))\Big) + \sum_{\beta_h\in N} p_h^{-i}(T^{-i})
\end{aligned}$$

Observe that *constant* is totally determined by the history profile $\{H_j^{t^*}\}_{j\in I}$ and the market $\mathcal{M}_{-i}$ without bidder $i$, and does not depend on player $i$'s strategy $\sigma_i'$, (and that $\Delta_j^{-0}(t)$ and $\Delta_j^{-i}(t)$ for $t < t^*$ cannot be expressed by $\tilde{V}^j$, because player $j$ may not have bid according

[17] In this case, the outcome of the game depends on the histories $H_j^{t^*}$ and the strategies that all bidders will take in the continuation game starting from $t^*$. Bidders cannot change histories but can influence the path of the future from $t^*$ on.

to $\tilde{u}^j$ before $t^*$). Analogously we can show that when bidder $i$ uses the (continuation) sincere bidding strategy, his payment $\tilde{q}_i$ will be $\tilde{q}_i = constant - \sum_{j\in I_{-i}} \tilde{u}^j(\pi(j))$, where *constant* is the same as the previous one. Furthermore, we know from the argument in Section 3 that (in the continuation game) when bidder $i$ bids sincerely according to his utility function $u^i$ and every his opponent $j(j \in I_{-i})$ bids according to a regular bidding strategy $\sigma_j^{\tilde{u}^j}$ (i.e., according to a GSC utility function $\tilde{u}^j \in \mathcal{U}$), the resulted allocation $\pi$ must be efficient for $\mathcal{M}$ w.r.t. $u^i$ and $\tilde{u}^j, j \in I_{-i}$. This implies that

$$u^i(\pi(i)) + \sum_{j\in I_{-i}} \tilde{u}^j(\pi(j)) \geq u^i(\rho(i)) + \sum_{j\in I_{-i}} \tilde{u}^j(\rho(j)).$$

Thus, for bidder $i's$ payoff $W_i$ of the assignment resulting from the sincere bidding strategy and his payoff $W_i'$ of the assignment resulting from the strategy $\sigma_i'$, we have

$$\begin{aligned} W_i &= u^i(\pi(i)) - \tilde{q}_i = u^i(\pi(i)) - (constant - \textstyle\sum_{j\in I_{-i}} \tilde{u}^j(\pi(j))) \\ &= u^i(\pi(i)) + \textstyle\sum_{j\in I_{-i}} \tilde{u}^j(\pi(j)) - constant \\ &\geq u^i(\rho(i)) + \textstyle\sum_{j\in I_{-i}} \tilde{u}^j(\rho(j)) - constant = u^i(\rho(i)) - q_i' \\ &= W_i'. \end{aligned}$$

Consequently, for bidder $i's$ final payoff $\tilde{W}_i$ with the sincere bidding strategy and his final payoff $\tilde{W}_i'$ with the strategy $\sigma_i'$, we have

$$\tilde{W}_i = \max\{W_i, 0\} \geq \max\{W_i', 0\} = \tilde{W}_i'.$$

Therefore, every player's sincere bidding strategy is his ex post strongly perfect strategy, so sincere bidding by every bidder is an ex post strongly perfect equilibrium.

Finally, we prove (iii). Since for every bidder there is the option of rejection in Step 2 and no punishment in Step 3, his final payoff cannot be negative if he is sufficiently rational, not necessarily optimizing his actions. Clearly, the SPDT auction is ex post individually rational. □

Observe that Ausubel's analysis (2006) on his auction's strategic property focuses on economies with divisible goods and relies on calculus and Theorem 1 of Krishna and Maenner (2001), whereas the current analysis is quite different, elementary and intuitive.

# 5 Applications

In many practical economic environments, substitutes and complements are jointly observed. We name, but a few of basic instances, tables and chairs, left and right shoes, keys and lockers, software and hardware packages, landing and taking-off slots, machines and workers. The GSC condition captures the key feature of such environments. That is, there are two different kinds of goods in which goods of the same kind are substitutes and can be heterogeneous but are complementary to goods of the other kind. The GSC condition is defined with respect to two disjoint sets $S_1$ and $S_2$. When one of the two sets is empty, the GSC condition coincides with the famous GS condition of Kelso and Crawford (1982). Either of the two conditions ensures the existence of a Walrasian equilibrium. It is also known from e.g., Sun and Yang (2014, Example 1, p. 429) that if the GSC condition applies to three or more different kinds of goods in which goods of the same kind are substitutes but complementary to goods of any other kind, the existence of a Walarasian equilibrium is not guaranteed anymore. A major open question here is whether the GSC condition defined for three or more different kinds of goods guarantees the existence of a nonlinear pricing Walrasian equilibrium or not.

In the following we discuss two practical and common situations in which the GSC condition is naturally satisfied. The double-track auction can be easily applied to these environments, while neither an ascending nor descending auction can work, as the simple example in Section 3.1 indicates.

**Example 1:** Many goods are made up of two basic components. For example, the Bible consists of Old Testament and New Testament. Let $S_1$ denote the set of identical items of component 1, and $S_2$ the set of identical items of component 2. Identical items are labeled differently. The set $N$ stands for the set of all items, i.e., $N = S_1 \cup S_2$. Here goods from each component $S_h$, $h = 1, 2$, may be called intermediate goods, and a pair of one item from each component $S_h$, $h = 1, 2$, forms a unit of the final good. Consider now a market in which a seller wishes to sell all goods in $N$ to a group $I$ of buyers. To each buyer $i \in I$, units of each intermediate good (and the final good) are perfect substitutes and can be represented by a weakly increasing and concave function $f^i_h : \mathbb{R}_+ \to \mathbb{R}_+$ with $f^i_h(0) = 0$, where $h = 1, 2$ indicate for the intermediate goods and $h = 0$ stands for the final good. When facing up a set $A$ of goods, each buyer $i \in I$ has to pick up best choices among all possible combinations, which determine agent $i's$ value function $u^i : 2^N \to \mathbb{R}$ by

$$u^i(A) = \max_{k\in\{j\in\mathbb{Z}_+ \mid j\leq\min\{k_1,k_2\}\}} \Big(f^i_0(k) + f^i_1(k_1 - k) + f^i_2(k_2 - k)\Big)$$

where $k_1 = |A \cap S_1|$ and $k_2 = |A \cap S_2|$.

Given such a market, it is natural and important to ask (i) whether there exists any competitive equilibrium, and (ii) in the yes case, which items should be assigned to whom at what prices. We will prove that every agent $i$'s utility function $u^i$ satisfies the GSC condition, and thus the market has a competitive equilibrium. It is then clear that the double-track auction automatically answers the second question.

**Theorem 5.1** *In the market, the value function $u^i \in I$ as defined above satisfies the* gross substitutes and complements (GSC) *condition. The market has at least one competitive* *equilibrium.*

Proof: In the proof, we ignore the index $i$ of each agent $i \in I$. Take any price vector $p \in \mathbb{R}^n$, any $A \in D(p)$, any item $\beta_{i^*} \in S_1$ (or $S_2$), and any $\delta \geq 0$. Define $\hat{D}(p + \delta e(i^*)) = D(p+\delta e(i^*)) \cap \{B \subseteq N \mid (A \cap S_1) \setminus \{\beta_{i^*}\} \subseteq B\}$. To show that $u$ satisfies the GSC condition, we just need to prove that there exists a set $B \in \hat{D}(p + \delta e(i^*))$ such that $B \cap S_2 \subseteq A$.

Let $a_k = f_0(k) - f_0(k-1)$, $b_k = f_1(k) - f_1(k-1)$, and $c_k = f_2(k) - f_2(k-1)$ for all $k = 1, 2, \cdots$. It follows that $a_k \geq a_{k+1}$, $b_k \geq b_{k+1}$, and $c_k \geq c_{k+1}$ for all $k = 1, 2, \cdots$. For convenience, we also set $a_0 = b_0 = c_0 = 0$. Suppose that $u(A) = f_0(r) + f_1(s) + f_2(t)$, where $r, s$ and $t$ are all non-negative integers satisfying $r+s = |A \cap S_1|$ and $r+t = |A \cap S_2|$. Then for every $\beta_i \in A \cap S_1$, $\beta_j \in A \cap S_2$, $\beta_{i'} \in S_1 \setminus A$ (if $\neq \emptyset$), and $\beta_{j'} \in S_2 \setminus A$ (if $\neq \emptyset$), we have the following observations from the function $u$ and the demand set $D(p)$ that

(1) $p_i \leq p_{i'}$, $p_j \leq p_{j'}$, $b_{s+1} \leq p_{i'}$, $c_{t+1} \leq p_{j'}$, and $a_{r+1} \leq p_{i'} + p_{j'}$;

(2) if $s \geq 1$, then $b_s \geq \max\{p_i, a_{r+1} - p_{j'}\}$;

(3) if $t \geq 1$, then $c_t \geq \max\{p_j, a_{r+1} - p_{i'}\}$;

(4) if $r \geq 1$, then $a_r \geq \max\{p_i + p_j, c_{t+1} + p_i, b_{s+1} + p_j\}$.

We first prove that $\hat{D}(p + \delta e(i^*)) \neq \emptyset$. For this purpose, it is sufficient to show that there exists $B \in D(p + \delta e(i^*))$ such that $|B \cap S_1| \geq |A \cap S_1| - 1$, because all items in $S_1$ are identical and $p_i \leq p_{i'}$ for all $\beta_i \in A \cap S_1$ and $\beta_{i'} \in S_1 \setminus A$. Suppose to the contrary that $\hat{D}(p + \delta e(i^*)) = \emptyset$, i.e., $|B \cap S_1| < |A \cap S_1| - 1$ for every $B \in D(p + \delta e(i^*))$. Pick up any $\bar{B} \in D(p + \delta e(i^*))$ satisfying $|B \cap S_1| \leq |\bar{B} \cap S_1| < |A \cap S_1| - 1$ for all $B \in D(p + \delta e(i^*))$. Then, there must be some item $\beta_{\bar{i}} \in (A \cap S_1) \setminus \{\beta_{i^*}\}$ such that $\beta_{\bar{i}} \notin \bar{B}$. Suppose that $u(\bar{B}) = f_0(k) + f_1(k_1) + f_2(k_2)$, where $k, k_1$ and $k_2$ are all nonnegative integers satisfying $k + k_1 = |\bar{B} \cap S_1| < r + s - 1$ and $k + k_2 = |\bar{B} \cap S_2|$. Then, we have $p_{\bar{i}} > b_{k_1+1}$, or else $\bar{B} \cup \{\beta_{\bar{i}}\} \in D(p + \delta e(i^*))$. It follows from $b_{k_1+1} < p_{\bar{i}} \leq b_s$ that $k_1 \geq s$. Thus, it is only possible that $k < r - 1$. However, we can show that $A \cap S_2 \subseteq \bar{B}$ and so $k_2 > t + 1$. This is because if not, take any $\beta_{\bar{j}} \in (A \cap S_2) \setminus \bar{B} \neq \emptyset$, then $a_{k+1} \geq a_r \geq p_{\bar{i}} + p_{\bar{j}}$. And so, $\bar{B} \cup \{\beta_{\bar{i}}, \beta_{\bar{i}}\} \in D(p + \delta e(i^*))$, yielding a contradiction. Moreover, it follows from property (4) that $a_{k+1} \geq a_r \geq c_{t+1} - p_{\bar{i}} \geq c_{k_2} - p_{\bar{i}}$. This implies $\bar{B} \cup \{\beta_{\bar{i}}\} \in D(p + \delta e(i^*))$, leading to a contradiction. Consequently, $\hat{D}(p + \delta e(i^*)) \neq \emptyset$.

It remains to prove that there exists some $B \in \hat{D}(p + \delta e(i^*))$ such that $B \cap S_2 \subseteq A$. It suffices to show that there exists $B \in \hat{D}(p+\delta e(i^*))$ satisfying $|B \cap S_2| \leq |A \cap S_2|$, because all items in $S_2$ are homogeneous and $p_j \leq p_{j'}$ for all $\beta_j \in A \cap S_2$ and $\beta_{j'} \in S_2 \setminus A$. Notice that if there is $B \in D(p+\delta e(i^*))$ with $\beta_{i^*} \in B$, then $A \in \hat{D}(p+\delta e(i^*))$ and $A \cap S_2 \subseteq A$, and so the proof is finished. We assume now that there is not $B \in D(p+\delta e(i^*))$ so that $\beta_{i^*} \in B$. Pick up any $\bar{B} \in \hat{D}(p+\delta e(i^*))$ satisfying $\beta_{i^*} \notin \bar{B}$ and $|\bar{B}\cap S_2| \leq |B\cap S_2|$ for all $B \in \hat{D}(p+\delta e(i^*))$. We will show $|\bar{B}\cap S_2| \leq |A\cap S_2|$. Assume by way of contradiction that $|\bar{B}\cap S_2| > |A\cap S_2|$, and pick up any $\beta_{j'} \in (\bar{B}\cap S_2)\setminus A$. Suppose that $u(\bar{B}) = f_0(k)+f_1(k_1)+f_2(k_2)$, where $k, k_1$ and $k_2$ are all nonnegative integers satisfying $k+k_1 = |\bar{B}\cap S_1|$ and $k+k_2 = |\bar{B}\cap S_2| > r+t$. If $k_2 > t$, then $c_{k_2} \leq c_{t+1} \leq p_{j'}$. This implies a contradiction that $\bar{B}\setminus\{\beta_{j'}\} \in \hat{D}(p+\delta e(i^*))$. If $k_2 \leq t$ and $k + k_1 \geq r + s = |A \cap S_1|$, then $k > r$ and $(S_1 \setminus A) \cap \bar{B} \neq \emptyset$. Take any $\beta_{i'} \in (S_1 \setminus A) \cap \bar{B}$. Then we have $a_k \leq a_{r+1} \leq p_{i'} + p_{j'}$ and $\bar{B} \setminus \{\beta_{i'}, \beta_{j'}\} \in \hat{D}(p + \delta e(i^*))$, leading to a contradiction. Otherwise, we have $k_2 \leq t$ and $k+k_1 < |A\cap S_1| = r+s$, which implies $k > r$ and $k_1 < s$. It follows from property (2) that $a_k \leq a_{r+1} \leq b_s+p_{j'} \leq b_{k_1+1}+p_{j'}$. This implies a contradiction that $\bar{B} \setminus \{\beta_{j'}\} \in \hat{D}(p + \delta e(i^*))$.

This concludes that the value function $u$ satisfies the GSC-condition. By Theorem 3.1 of Sun and Yang (2006), the market has an equilibrium. □

The following example is due to Sun and Yang (2006) and reflects a typical and fundamental case in the manufacturing industry.

**Example 2:** Consider a manufacturing industry which consists of finitely many firms, workers and machines. Let $I$ denote the set of manufacturing firms, $S_1 = \{w_1, w_2, \cdots, w_K\}$ the set of workers and $S_2 = \{m_1, m_2, \cdots, m_L\}$ the set of machines. Firms need not be identical and can be heterogeneous, so do workers and machines. Every firm can hire as many workers and buy as many machines as it wishes, under its budget constraint, for any given salaries and prices. At each moment in time each worker can work for at most one firm and each machine can be used by at most one firm. When worker $w_j$ operates machine $m_k$ in firm $i$, this yields a revenue to the firm, denoted by $r_i(j,k)$. As a modeling convention, we assume that no machine or worker does harm to any firm if they stay idle. When firm $i \in I$ uses a set $A$ of workers and machines, the revenue $u^i(A)$ of these workers and machines to the firm is completely determined by the pairwise combinations of worker and machine that the members in $A$ can generate, and is given by

$$u^i(A) = \max\{0, r_i(j_1,k_1) + r_i(j_2,k_2) + \cdots + r_i(j_l,k_l)\}$$

with the maximum to be taken over all sets $\{(w_{j_1}, m_{k_1}), (w_{j_2}, m_{k_2}), \cdots, (w_{j_l}, m_{k_l})\}$ of $l$ distinct worker-machine pairs in $A$. In other words, when facing a set $A$ of workers and machines, every firm $i \in I$ need to solve an optimal worker-machine assignment problem. The whole industry, however, faces a larger and more complex problem of whether there exists a system of competitive salaries and prices through which all workers and machines can be efficiently allocated to the firms. Sun and Yang (2006, Theorem 4.1) prove that the revenue function $u^i$ of each firm $i \in I$ satisfies the GSC condition and the industry has a competitive equilibrium. The double-track auction proposed in the current paper can actually discover competitive equilibrium prices of workers and machines by which firms can be efficiently assigned with their optimal choices of workers and machines.

As mentioned earlier, Hatfield et al. (2013), Baldwin and Klemperer (2013), Drexl (2013), Sun and Yang (2011), and Teytelboym (2013, 2014) have found other important environments from which the GSC pattern arises naturally. The interested reader can also refer to Scarf (1960), Shapley (1962), Samuelson (1974), Rassenti et al. (1982) for earlier venerable studies on complementarity.

# Appendix

**Proof of Lemma 2.1** Because, at any given prices $p$,

$$\begin{aligned}\max_{A\subseteq N}\{u^0(A)+\textstyle\sum_{\beta_h\in N\setminus A}p_h\} &= \max_{A\subseteq N}\{u^0(A)-\textstyle\sum_{\beta_h\in A}p_h+\\ &\qquad +\textstyle\sum_{\beta_h\in A}p_h+\sum_{\beta_h\in N\setminus A}p_h\}\\ &= \max_{A\subseteq N}\{u^0(A)-\textstyle\sum_{\beta_h\in A}p_h\}+\sum_{\beta_h\in N}p_h,\end{aligned}$$

clearly we have $S(p)=D^0(p)$. □

**Proof of Lemma 2.3** Take any Walrasian equilibrium $(p,\pi)$ and any allocation $\rho$. By definition, we have for any bidder $i\in I$

$$u^i(\pi(i))-\sum_{\beta_h\in\pi(i)}p_h\geq u^i(\rho(i))-\sum_{\beta_h\in\rho(i)}p_h$$

and for the seller

$$u^0(\pi(0))+\sum_{\beta_h\in N\setminus\pi(0)}p_h\geq u^0(\rho(0))+\sum_{\beta_h\in N\setminus\rho(0)}p_h$$

Summing up the two inequalities yields

$$\sum_{i\in I_0}u^i(\pi(i))\geq\sum_{i\in I_0}u^i(\rho(i)).$$

This shows that $\pi$ is efficient. □

Here we give a brief self-contained explanation about the relationship (3.4), i.e.,

$$\max_{\delta\in\Phi}\{\mathcal{L}(p(t))-\mathcal{L}(p(t)+\delta)\}=\max_{\delta\in\Delta}\Big\{\sum_{i\in I_0}\Big(\min_{S\in D^i(p(t))}\sum_{\beta_h\in S}\delta_h\Big)-\sum_{\beta_h\in N}\delta_h\Big\}$$

We sketch how to derive the above relationship from the left to the right. The interested reader can refer to Sun and Yang (2009) in detail. Write down the Lyapunov function $\mathcal{L}$ as

$$\mathcal{L}(p(t))-\mathcal{L}(p(t)+\delta)=\sum_{i\in I_0}(V^i(p(t))-V^i(p(t)+\delta))-\sum_{\beta_h\in N}\delta_h \tag{5.8}$$

Observe that the above formula involves every bidder's valuation of every bundle of goods, so it involves private information. Apparently, it is impossible for the auctioneer to know such information unless the bidders tell her. Fortunately, she can fully infer the difference between $\mathcal{L}(p(t))$ and $\mathcal{L}(p(t)+\delta)$ just from the reported demands $D^i(p(t))$ and the price variation $\delta$. To see this, we know from Sun and Yang (2009) that when prices move from $p(t)$ to $p(t)+\delta$, the change in indirect utility for every bidder $i$ is unique and is given by

$$V^i(p(t)) - V^i(p(t)+\delta) = \min_{S\in D^i(p(t))} \sum_{\beta_h\in S} \delta_h \tag{5.9}$$

Consequently, the equation (5.8) becomes the following simple formula whose right side involves only price variation $\delta$ and optimal choices at $p(t)$:

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{i\in I_0}\Big(\min_{S\in D^i(p(t))} \sum_{\beta_h\in S} \delta_h\Big) - \sum_{\beta_h\in N} \delta_h$$

It follows immediately that

$$\max_{\delta\in\Phi}\{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta\in\Delta}\Big\{\sum_{i\in I_0}\Big(\min_{S\in D^i(p(t))} \sum_{\beta_h\in S} \delta_h\Big) - \sum_{\beta_h\in N} \delta_h\Big\}.$$

This completes a brief discussion of the important formula of (3.4).

In the rest we will prove Theorem 3.1. To do so, we first introduce several notations. Let $p, q \in \mathbb{R}^n$ be any vectors. With respect to the two given sets $S_1$ and $S_2$, we define their generalized *meet* $s = (s_1, \cdots, s_n) = p \wedge_g q$ *and join* $t = (t_1, \cdots, t_n) = p \vee_g q$ by

$$\begin{aligned} s_k &= \min\{p_k, q_k\}, \quad \beta_k \in S_1, \quad s_k = \max\{p_k, q_k\}, \quad \beta_k \in S_2; \\ t_k &= \max\{p_k, q_k\}, \quad \beta_k \in S_1, \quad t_k = \min\{p_k, q_k\}, \quad \beta_k \in S_2. \end{aligned}$$

Notice that the two operations are different from the standard meet and join operations. For $p, q \in \mathbb{R}^n$, we introduce a new order by defining $p \leq_g q$ if and only if $p_h \leq q_h$ for all $\beta_h \in S_1$ and $p_h \geq q_h$ for all $\beta_h \in S_2$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *a generalized submodular function* if $f(p \wedge_g q) + f(p \vee_g q) \leq f(p) + f(q)$ for all $p, q \in \mathbb{R}^n$.

**Proof of Theorem 3.1** By Theorem 3.1 of Sun and Yang (2006) the market has a Walrasian equilibrium and by Lemma 1 of Sun and Yang (2009) the Lyapunov function $\mathcal{L}(\cdot)$ attains its minimum value at any equilibrium price vector and is bounded from below. Since the prices and value functions take only integer values, the Lyapunov function is an integer valued function and it lowers by a positive integer value in each round of the IDT adjustment process. This guarantees that the auction terminates in finitely many rounds, i.e., $\delta(t^*) = 0$ in Step 3 for some $t^* \in \mathbb{Z}_+$.

Let $p(0), p(1), \cdots, p(t^*)$ be the generated finite sequence of price vectors. Let $\bar{t} \in \mathbb{Z}_+$ be the time when the IDT adjustment process finds $\delta(\bar{t}) = 0$ at Step 2. We claim that

$\mathcal{L}(p) \geq \mathcal{L}(p(\bar{t}))$ for all $p \geq_g p(\bar{t})$. Suppose to the contrary that there exists some $p \geq_g p(\bar{t})$ such that $\mathcal{L}(p) < \mathcal{L}(p(\bar{t}))$. By the convexity of $\mathcal{L}(\cdot)$ via Theorem 3 (i) of Sun and Yang (2009), there is a strict convex combination $p'$ of $p$ and $p(\bar{t})$ such that $p' \in p(\bar{t}) + \Phi$ and $\mathcal{L}(p') < \mathcal{L}(p(\bar{t}))$. From equation (3.4) we know that $\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) < \mathcal{L}(p(\bar{t}))$, and so $\delta(\bar{t}) \neq 0$ in Step 2 of the IDT adjustment process, yielding a contradiction. Therefore, we have $\mathcal{L}(p \vee_g p(\bar{t})) \geq \mathcal{L}(p(\bar{t}))$ for all $p \in \mathbb{R}^n$, because $p \vee_g p(\bar{t}) \geq_g p(\bar{t})$ for all $p \in \mathbb{R}^n$. We will further show that $\mathcal{L}(p \vee_g p(t)) \geq \mathcal{L}(p(t))$ for all $t = \bar{t}+1, \bar{t}+2, \cdots, t^*$ and $p \in \mathbb{R}^n$. By induction, it suffices to prove the case of $t = \bar{t}+1$. Notice that $p(\bar{t}+1) = p(\bar{t}) + \delta(\bar{t})$, where $\delta(\bar{t}) \in \Delta^*$ is determined in Step 3 of the IDT adjustment process. Assume by way of contradiction that there is some $p \in \mathbb{R}^n$ such that $\mathcal{L}(p \vee_g p(\bar{t}+1)) < \mathcal{L}(p(\bar{t}+1))$. Then if we start the IDT adjustment process from $p(\bar{t}+1)$, we can by the same previous argument find a $\delta(\neq 0) \in \Delta$ in Step 2 such that $\mathcal{L}(p(\bar{t}+1)+\delta) < \mathcal{L}(p(\bar{t}+1))$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function by Theorem 3 (i) of Sun and Yang (2009), we have $\mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1)+\delta)) + \mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1)+\delta)) \leq \mathcal{L}(p(\bar{t}) + \mathcal{L}(p(\bar{t}+1)+\delta)$. Recall that $\mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1)+\delta)) \geq \mathcal{L}(p(\bar{t}))$. It follows that $\mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1)+\delta)) \leq \mathcal{L}(p(\bar{t}+1)+\delta) < \mathcal{L}(p(\bar{t}+1))$. Observe that $\delta' = 0 \wedge_g (\delta(\bar{t})+\delta) \in \Delta^*$ and $p(\bar{t}) \wedge_g (p(\bar{t}+1)+\delta) = p(\bar{t}) + \delta'$. This yields $\mathcal{L}(p(\bar{t}) + \delta') < \mathcal{L}(p(\bar{t}) + \delta(\bar{t}))$ and so $\delta' \neq \delta(\bar{t})$, contradicting the definition of $\delta(\bar{t}) \in \Delta^*$ by which $\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) = \min_{\delta \in \Delta^*} \mathcal{L}(p(\bar{t}) + \delta)$.

Next we prove that $\mathcal{L}(p \wedge_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. To see this, we first show that $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq_g p(t^*)$. Suppose to the contrary that there exists some $p \leq_g p(t^*)$ such that $\mathcal{L}(p) < \mathcal{L}(p(t^*))$. By the convexity of $\mathcal{L}(\cdot)$ via Theorem 3 (i) of Sun and Yang (2009), there is a strict convex combination $p'$ of $p$ and $p(t^*)$ such that $p' \in \{p(t^*)\} - \Phi$ and $\mathcal{L}(p') < \mathcal{L}(p(t^*))$. Because of the symmetry between Step 2 and Step 3, Lemma 3 (where $\Phi$ is replaced by $\Phi^* = -\Phi$) and Step 3 of the GDDT procedure imply that $\mathcal{L}(p(t^*) + \delta(t^*)) = \min_{\delta \in \Phi^*} \mathcal{L}(p(t^*) + \delta) = \min_{\delta \in \Delta^*} \mathcal{L}(p(t^*) + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p(t^*))$ and so $\delta(t^*) \neq 0$, contradicting the fact that the GDDT procedure stops in Step 3 with $\delta(t^*) = 0$. So we have $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq_g p(t^*)$. Because $p \wedge_g p(t^*) \leq_g p(t^*)$ for all $p \in \mathbb{R}^n$, it follows that $\mathcal{L}(p \wedge_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$.

We also proved above that $\mathcal{L}(p \vee_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function by Theorem 3 (i) of Sun and Yang (2009), we have $\mathcal{L}(p) + \mathcal{L}(p(t^*)) \geq \mathcal{L}(p \vee_g p(t^*)) + \mathcal{L}(p \wedge_g p(t^*)) \geq 2\mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. This shows that $\mathcal{L}(p(t^*)) \leq \mathcal{L}(p)$ holds for all $p \in \mathbb{R}^n$ and by Lemma 1 of Sun and Yang (2009), $p(t^*)$ is an equilibrium price vector. $\square$

## REFERENCES

Arrow, Kenneth J. and Frank Frank (1971), *General Competitive Analysis.* Holden-

Day, San Francisco.

Ausubel, Lawrence (2004), "An efficient ascending-bid auction for multiple objects." *American Economic Review*, 94, 1452-1475.

Ausubel, Lawrence (2005), "Walrasian tatonnement for discrete goods." Unpublished paper.

Ausubel, Lawrence (2006), "An efficient dynamic auction for heterogeneous commodities." *American Economic Review*, 96, 602-629.

Baldwin, Elizabeth and Klemperer Paul (2013), "Tropical geometry to analyse demand." Unpublished paper, Oxford University.

Brewer, Paul and Charles Plott (1996), "A binary conflict ascending price (BICAP) mechanism for the decentralized allocation of the right to use railroad tracks." *International Journal of Industrial Organization*, 14, 857-886.

Cremér, Jacques and Richard Mclean (1985), "Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent." *Econometrica*, 53, 345-362.

Demange, Gabrielle, David Gale, and Marida Sotomayor (1986), "Multi-item auctions." *Journal of Political Economy*, 94, 863-872.

Drexl, Moritz (2013), "Substitutes and complements in trading networks," Unpublished manuscript.

Fujishige, Satoru and Zaifu Yang (2003), "A note on Kelso and Crawford's gross substitutes condition." *Mathematics of Operations Research*, 28, 463-469.

Gul, Faruk and Ennio Stacchetti (1999), "Walrasian equilibrium with gross substitutes." *Journal of Economic Theory*, 87, 95-124.

Gul, Faruk and Ennio Stacchetti (2000), "The English auction with differentiated commodities." *Journal of Economic Theory*, 92, 66-95.

Hatfield, John, Scott Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp (2013), "Stability and competitive equilibrium in trading networks." *Journal of Political Economy*, 121, 966-1005.

Hatfield, John, and Paul Milgrom (2005), "Matching with Contracts." *American Economic Review*, 95, 913-935.

Hurwicz, Leonid (1973), "The design of mechanisms for resource allocation." *American Economic Review*, 63, 1-39.

Jehiel, Philippe and Benny Moldovanu (2003), "An economic perspective on auctions." *Economic Policy*, April, 269-308.

Kearns Michael, Mallesh Pai, Aaron Roth, and Jonathan Ullman (2013), "Mechanism design in large games: incentives and privacy." Unpublished paper.

Kelso, Alexander and Vincent Crawford (1982), "Job matching, coalition formation, and gross substitutes." *Econometrica*, 50, 1483-1504.

Klemperer, Paul (2004), *Auctions: Theory and Practice.* Princeton University Press, Princeton, New Jersey.

Krishna, Vijay (2002), *Auction Theory.* Academic Press, New York.

Krishna, Vijay and Eliot Maenner (2001), "Convex potentials with an application to mechanism design." *Econometrica*, 69, 1113-1119.

Maskin, Eric (2005), "Recent contributions to mechanism design: a highly selective review." Unpublished paper, Department of Economics, Northwestern University.

Milgrom, Paul (2000), "Putting auction theory to work: the simultaneous ascending auction." *Journal of Political Economy*, 108, 245-272.

Milgrom, Paul (2004), *Putting Auction Theory to Work.* Cambridge University Press, New York.

Milgrom, Paul (2007), "Package auctions and exchanges." *Econometrica*, 75, 935-965.

Milgrom, Paul and Bruno Strulovici (2009), "Substitute goods, auctions, and equilibrium." *Journal of Economic Theory*, 144, 212-247.

Myerson, Roger and Mark Satterthwaite (1983), "Efficient mechanisms for bilateral trading." *Journal of Economic Theory*, 28, 265-281.

Ostrovsky, Michael (2008), "Stability in supply chain network." *American Economic Review*, 98, 897-923.

Perry, Motty and Philp Reny (2005), "An efficient multi-unit ascending auction." *Review of Economic Studies*, 72, 567-592.

Porter, David, Stephen Rassenti, Anil Roopnarine, and Vernon Smith (2003), "Combinatorial auction design." *Proceedings of the National Academy of Sciences*, 100, 11153-11157.

Rassenti, Stephen, Vernon Smith, and Robert Bulfin (1982), "A combinatorial auction mechanism for airport time slot allocation." *The Bell Journal of Economics*, 13, 402-417.

Rothkopf, Michael (2007), "Thirteen reasons why the Vickrey-Clarke-Groves process is not practical." *Operations Research*, 55, 191-197.

Rothkopf, Michael, Thomas Teisberg, and Edward Kahn (1990), "Why are Vickrey auctions rare?" *Journal of Political Economy*, 98, 94-109.

Samuelson, Paul (1974), "Complementarity." *Journal of Economic Literature*, 12, 1255-1289.

Scarf, Herbert (1960), "Some examples of global instability of the competitive equilibrium." *International Economic Review*, 1, 157-172.

Shapley, Lloyd (1962), "Complements and substitutes in the optimal assignment problem." *Naval Research Logistics Quarterly*, 9, 45-48.

Sun, Ning, and Zaifu Yang (2006), "Equilibria and indivisibilities: gross substitutes and complements." *Econometrica*, 74, 1385-1402.

Sun, Ning, and Zaifu Yang (2008), "A double-track auction for substitutes and complements." DP No. 656, Institute of Economic Research, Kyoto University.

Sun, Ning, and Zaifu Yang (2009), "A double-track adjustment process for discrete markets with substitutes and complements." *Econometrica*, 77, 933-952.

Sun, Ning, and Zaifu Yang (2011), "A general competitive trading network with indivisibilities and gross substitutes and complements." Unpublished note.

Sun, Ning, and Zaifu Yang (2014), "An efficient and incentive compatible dynamic auction for multiple complements." *Journal of Political Economy*, 122, 422-466.

Teytelboym, Alexander (2013), *Essays in Networks and Market Design*, PhD Dissertation, Oxford University.

Teytelboym, Alexander (2014), "Gross substitutes and complements: a simple generalization." *Economics Letters*, 123, 135-138.

Varian, Hal (1981), "Dynamic systems with applications to economics," in K.J. Arrow and M.D. Intriligator (eds.) *Handbook of Mathematical Economics*, vol. 1, North-Holland, Amsterdam.