# The Role of Information in Markets with Quality Uncertainty

Dissertation zur Erlangung des Doktorgrades
der Wirtschaftswissenschaften (Dr. rer. pol.)
an der Fakultät für Wirtschaftswissenschaften
der Universität Bielefeld

vorgelegt von

M.Sc. Christopher Gertz

September 2014

## Gutachter

| | |
|---|---|
| Erster Gutachter | Prof. Christoph Kuzmics, PhD |
| Zweiter Gutachter | Prof. Dr. Herbert Dawid |

Adresse
Universität Bielefeld
Fakultät für Wirtschaftswissenschaften
Universitätsstraße 25
33615 Bielefeld
Germany

## Acknowledgements

Many people had a part in the creation of this thesis. Most of all, I want to thank my supervisor Christoph Kuzmics. Not only did he agree to the supervision, he also let me work in a field of my interests in which he, as he says, does not see himself as an expert. I want to thank him for the discussions about my work, always being positive about my ideas, asking helpful questions and pointing out problems, possible generalizations and related literature. According to him, he feels like he did not have a big role in the content of my thesis; I think he underestimates this by a lot.

I thank the German scientific foundation (DFG) for the funding over the first three years of my doctoral studies. I also thank the Center for Mathematical Economics (IMW) and especially its director Frank Riedel for funding me over the last nine months of this time. In addition, the IMW and its members and staff are responsible for a lot of great experiences, conversations and - of course - part of my academic accomplishments.

Many people gave me helpful advice for the content of my thesis, including Klaus Ritzberger, Andreas Blume and Manfred Nermuth. Additionally, helpful remarks came from conference participants in Bielefeld, Paris and Vienna.

I want to thank Nikoleta, Jörg and Pascal and all the others who were part of my great semester in Paris. The time there went by maybe even too fast.

During and before my time as a doctoral student I received support from many other people. Most of all from my parents who supported me my entire life, financed parts of my studies and my first semester in Paris. I feel very grateful in knowing that I can always count on them.

Many people had an influence on me and became good friends during my time in the university. I can not name all of them but want to explicitly thank Andrea, Bastian, Frederik, Janning, Jörg, Julia, Katha, Kathi, Lazar, Lisa, Matthieu, Nikoleta, Oliver as well as my QEM generation, especially Andrea, Enareta, Matthijs and Maxime. We managed to stay in contact over all these years and I hope we always will. As I said, there are many more and if you think you may be in the list, you are. I had an amazing time ever since I came to Bielefeld nine years ago and I am grateful to everyone who played a part in it.

Finally, I want to thank Karsten Frick for his contagious enthusiasm about math and for supporting me in recognizing and developing my mathematical talents. He was probably the main reason I studied mathematical economics.

# Contents

# 1 Introduction

Over the last decades most parts of the economic world has moved from the classical theory of supply and demand to models of interacting agents. Instead of taking demand functions as given, economists try to resolve the motives behind individual behavior and how this aggregates to market parameters. Decision theory advanced to a point where it is already quite predictive about what is possible and what is not possible with rational decision makers.

Game theory has made an important contribution to the way we look at these economic interactions. Instead of reacting to given, fixed market behavior, market participants need to optimize and make their decisions, having in mind the motives of their contract partners and competitors. Moving away from the classical theory, actors are aware of the impact of their actions on market parameters like prices. The concept of Nash equilibria and similar constructs more and more replaced the one of Walrasian equilibria in many fields of economic theory.

A big question to further develop this theory is the one of information. Not only may some elements of the environment not be known to decision makers, there may be information which is only available to some of them while others do not have access at all or for different conditions. These information asymmetries lead to even more complex phenomena. Private information of one agent can willingly or unwillingly be transferred to others, either by consciously transmitting the information or by behaving in a way which lets others (partly) deduce this knowledge from observed behavior. As it turns out, these model uncertainties and information asymmetries largely influence the outcome and structure of market behavior.

This thesis primarily focuses on the question of information, how it is obtained and - most of all - the impact of information asymmetries on the equilibria in a classical one-good market. In particular, we focus on quality information. Quality, in the sense used in our context, is a general term which determines the overall value of a product to its buyer. When faced with a take-it-or-leave-it offer like in a supermarket, the good's quality needs to at least even out the disutility of paying the price (*willingness-*

*to-pay*). Otherwise a consumer would not decide to purchase the good. While different potential buyers may prefer different characteristics of a product (so-called *horizontal* properties like color, taste), there are other features which are considered to be agreed upon to be either good or bad (*vertical* properties, like the durability of a car). We focus on the latter, speaking about quality as something which is universally true for everyone. These markets with quality uncertainty (so-called *lemon markets*) are well-known in the economic literature every since they were introduced by George Akerlof in 1970. They were analyzed under various assumptions and with different focuses.

In the course of the described transition from the classical economic analysis to the issue of information there appeared a very important question, known as the Grossman-Stiglitz paradox. Analyzing a (financial) market with publicly known information, standard results dictate that the market prices in equilibrium fully reflect the available information and hence, in fact, having the original information is not necessary for optimal behavior. But thinking this to the end, there is no incentive for anyone to actually acquire this knowledge, no matter how little effort it might cost. The assumption of public information seems absurd, having such results.

The first chapter of this thesis uses the idea that this is not only relevant in financial markets but basically in every context where public information is perfectly reflected in equilibrium behavior. In particular, if quality is assumed to be known in a classical one-good economy, prices in the market perfectly match this quality. Hence nobody would actually need to acquire quality information when they see the prices. Especially in this every-day context it is very important to resolve this paradox to get to the ground of what constitutes an equilibrium. The original analysis of Akerlof is not able to cover this approach in the market context and little has been done so far to explain the consequences of this oddity and how it can be resolved. The work in the first chapter provides a credible explanation how and to which extend this paradox can be settled.

For this, we use a quite standard model and introduce the possibility for the consumer to acquire information. She can decide on her own how much effort she wants to put into this process. The probability of receiving the correct information depends on this effort. By analyzing the equilibria of these markets we find that this ability is not always beneficial for the consumer. Although her possibility to reduce the information gap between her and the firm should intuitively improve her situation, the market equilibria can shift to a state in which this behavior is exploited by the producer of a high quality product. This happens in particular in situations where the information is relatively cheap. Transferred to real-life applications, the result implies that the modern possibilities of easily obtaining quality information may not empower customers in a way one would expect. Moreover, the analysis of this case - when information gets cheaper and approaches the limit of being costless - resolves the paradox described above, showing that a certain amount of information acquisition is always performed, no matter how cheap the information is and how close the market prices match the true quality values. We show that the case of perfect information can never be reached or even approached, no matter how easy it is to acquire information.

The second chapter tackles a technical question in the context of lemon markets with two-sided asymmetric information. It is very common in the literature to model quality uncertainty with two different possible quality values. Having such a high and a low value is the obvious minimum requirement for modeling lemon market situations but it is, of course, not very realistic. Many quality aspects like the expected time for which one can use a second hand car or the quality of a TV picture, can assume a continuum of different levels.

Two justifications prevail in the literature for this simplification. The technical one is that assuming more than two quality levels very often makes the model analytically unsolvable. This argument, which we also make in the first chapter, is hard to argue with. A model witch yields no results, although being more realistic than others, is still a bad model. The other reason, however, is that two quality levels are considered to be not only necessary but also sufficient to capture the relevant effects in these markets. The second chapter investigates this argument, first providing an example setting in which it is indeed true. The two main features of separating lemon market equilibria - adverse selection and a positive price-quality relationship - are preserved when extending the classical model to a setting with a continuum of quality types. In the spirit of the first chapter we then continue to analyze a similar market in the presence of an additional signal for the consumer. We find that, although the positive price-quality relation is still present, adverse selection is not a relevant phenomenon in the market equilibria.

We then proceed, as in the first chapter, to investigate the perfect information limit. Adverse to previous results and also defying the intuition, increasing signal precision does not imply convergence to the full information case and even the opposite is true in some sense. Instead of admitting full sales of all types, as it would be under perfect information, contagion effects between similar quality types let the maximal achievable sale probability go to zero; the market breaks down. Although this stunning result is surely not the ultimate truth and the model is certainly imperfect in many other ways, the observed behavior is a warning to revisit the two-quality assumption in lemon markets.

Both, the first and the second chapter, show that there is a discontinuity in the limit of perfect information. In both models, letting information be arbitrarily perfect or arbitrarily cheap does not lead to convergence of market behavior to the analysis under perfect information. This shows that the information structure in such markets is crucial and can not just be overlooked with the argument that information is sufficiently high or easy to obtain.

Finally, the last chapter tackles additional important questions in markets with quality uncertainty. For one, what is the difference between the static concept of Bayesian equilibria to a setting with multiple time periods? Most of the literature on lemon markets focuses on static equilibrium concepts. In fact, equilibria may be expected to be the steady state of a time-dynamic process and hence their analysis might be sufficient to capture the main effects of the market behavior. On the other hand, Bayesian equilibria with their belief definition are very different from the dynamic concept of experienced-based consumer beliefs. We find a substantial difference to the classical outcome of the lemon market literature, regarding this question. Although quality is not known in the classical lemon market models and the consumer has no possibility of acquiring any reliable information, every equilibrium is either pooling or admits a positive price-quality relationship, meaning that a high price roughly signals high quality. This is due to the nature of the used equilibrium concept and the rationality of the consumers who, when faced with a different situation, would rather buy with certainty for low and not at all for high prices. In many markets, however, consumers are neither informed nor very rational. In combination with the time-component there are incentives for a firm to price high quality goods with a low price to acquire customers and boost overall sales. We find that this effect is indeed quite strong in markets with not too many informed customers, i.e. when there is an experience good about which quality information is either not available or too costly (in comparison to the product price) to obtain. Supermarket wine is a prime example for such a product.

An important and active research area is the debate of how rational people react in situations in which there is no rational decision, so-called out-of-equilibrium play. When being in a stable, non-changing environment, there are some situations which are never observed because they are never rationally played by any of the agents. Although these situations in theory never occur, the hypothetical reaction of the market participants is a very important aspect for the existence and stability of equilibria. Multiple refinements on out-of-equilibrium beliefs and behavior have been introduced in the literature. This thesis adds to the theory by introducing two new concepts of deciding what is rational or realistic and hence which equilibria are more credible than others.

# 2 Markets with Quality Uncertainty and Imperfect Information Acquisition

## 2.1 Introduction

Along with the price, the quality of a product is one of the two major characteristics relevant to a purchase decision. While price information can be tricky to obtain in situations with negotiation possibilities or multiple retailers, it is usually even harder to get a good idea about the quality of a product. This phenomenon occurs not only in classical examples like second hand cars but is also present in almost every potential purchase for goods with which you do not have much experience. Whether you have not bought a TV for many years, want to buy wine in an unfamiliar supermarket, consider to buy an upgrade of the operating system on your computer or to invest in a financial product. You do not have full information about what you really get for your money if you are not an expert for these products or at least inform yourself prior to the purchase.

On the other hand, the seller of a product usually has much better information about the quality. A second hand car has been used by the owner for many years, giving him the knowledge of any accident, repair or defects that occurred over the last years and might still exist or be relevant. In the same way, a producer knows the characteristics and weaknesses of the product for sale. While positive qualities might be advertised or otherwise pointed out to the buyer, there is no incentive for the seller to do the same for bad qualities or missing features if not mandated by law.[1] If the consumer wants this information, she usually has to acquire it on her own. And even the advertised positive qualities may be exaggerated, untrue or not

[1] Dziuda (2011) shows that, in a game theoretic setting in which an expert might be biased or not, some bad properties may be disclosed even by a biased expert. In our market setting, however, there is no such ambiguity over the goal of the firm.

relevant to the consumer. In any case, it takes effort of the consumer to either gather information or check and filter the information provided.

This asymmetry of quality information has been present in the literature ever since George Akerlof's famous paper of 1970 and his previously mentioned second hand car example. The question of how such quality uncertainty influences the market outcome has since been discussed in many papers. There have been various attempts to mitigate between the extreme nature of Akerlof's model and the classical case of perfect information. For instance, Bagwell and Riordan (1991) introduce multiple periods in which firms can set different prices. High quality firms can then acquire customers by setting low prices in the first period and use this to charge highly afterwords. Milgrom and Roberts (1986) allow the firms to give an additional, costly advertising signal to distinguish oneself and keep low quality sellers from imitating the high quality sellers' behavior.

While these examples follow the idea of giving high quality firms certain additional possibilities to signal their excellence, it is only natural to also look at the other side of the market, namely the consumer. In the Akerlof model buyers do not have any other possibility than either to trust the firms or not, their posterior beliefs about quality depending on the price and being determined by Bayes' law in an equilibrium. Having the examples of the first paragraph in mind, the idea of the consumer having no further information is obviously a very strong assumption and not true in most purchasing situations. A second hand car can be tested before buying, a bottle of wine can be bought and tried out before you decide to serve it at your dinner party and for most products you can find multiple tests, ratings and reviews online. Especially with the Internet, the amount of quality information available to consumers has dramatically increased in recent years. With more and more people owning smartphones, this information is available and can be looked up even inside the shop.

Few papers have so far considered to relax this part of Akerlof's model. Bester and Ritzberger (2001) let the consumer decide about buying a perfect quality signal and base their decision on the additional information. In Voorneveld and Weibull (2011) the buyer receives an additional, costless but noisy signal which is correlated to the true quality. One can interpret this as an independent, non-perfect test review that is observed by the consumer in any case. These two models of additional quality information do not quite capture the real life situations, as it is usually not costless (in terms of money or time) to acquire information and this information does not have to be perfect. Even if there are free tests and reviews on the Internet, one has to take the time and put effort in finding and reading these sources. These tests, on the other hand, may not contain all relevant information and can be incorrect, biased or based on a faulty product. The same holds for the information from friends and experts. Testing a TV in the store can not fully simulate the home environment and how the TV works together with other devices, etc. All this adds some unreliability to the information acquisition procedure and suggests some probability of false information.[2]

This chapter simultaneously covers both ideas of these two works; the consumer is able to choose how much effort or costs she wants to invest in acquiring quality information and this will result in a quality signal. This signal, on the other hand, will be more precise when exerting more search effort to such an extend that one might even reach perfect information.

Our analysis shows that in the market with imperfect information acquisition, different kinds of equilibria can occur. Under reasonable refinements, the most important two categories of equilibria are one in which the consumer does not spend effort on quality information and one in which she does. Only in the first type of these equilibria she has positive utility. Her ability to search, although not executed, lowers the price to below the expected quality. In equilibria with search, on the other hand, the price of the product is relatively high and all the consumer's possibility of acquiring information provides market power for the firm, not for the consumer.

Starting from these insights, we investigate how the existence and outcomes of these equilibria develop when information is available more easily. We find that a higher ability of acquiring quality information stops the existence of the consumer-friendly equilibrium and thus takes away all consumer utility.

At last, we investigate the limit behavior of the model in the case when quality information is very expensive or very cheap. Surprisingly, making information acquisition arbitrarily easy does not lead to convergence to the full information case in that the error probability of the signal stays bounded away form zero and a non-vanishing share of high quality products is not traded. Although the possibility of imperfect information acquisition generally lowers adverse selection phenomena, it does not get rid of them even in the limit of perfect information. Making the information very expensive can lead to the outcomes of the classical models of quality uncertainty but often also, for a wide range of parameters, converges to an equilibrium that was

[2] It is worth noting that the work of Kihlstrom (1974) was motivated by the same ideas as this paper. His analysis, however, solely focuses on the consumer side (the market for information) and does not give any indication about the implications for market equilibria.

previously disregarded in the literature.

The paper is structured as follows. The first section describes the model of quality information acquisition and the market participants. We then proceed by analyzing the consumer behavior. This is embedded in a formally defined monopolistic market model and the rational Bayesian equilibria are analyzed. Finally, we compare outcomes of different levels of search efficiency to investigate the market impact of cheaper or more expensive quality information.

## 2.2 The Model

We consider a monopolistic market with one product and one potential consumer (or "buyer")[3]. The quality of the product can take two fixed values and is drawn by nature with a publicly know probability $\eta$ of high quality.[4] The realization is known only by the firm and will be denoted by its type $\theta \in \{H, L\}$.

A high quality firm faces production costs $c^H > 0$ while the low quality firm pays $c^L > 0$ for producing one unit of the product. This cost is only incurred when the good is actually sold. We assume $c^L < c^H$ so that high quality production is at least marginally more costly than for low quality. These costs can also be seen as losing an outside option. For example, the seller of a second hand car could also bring the vehicle to a professional dealer who would pay him the amount $c^\theta$. This option is lost in case of a successful sale.

The seller (or firm) makes a take-it-or-leave-it offer and is a risk-neutral payoff maximizer. Selling one unit of the good for a price $p$ with probability $\delta$ yields the payoff

$$\pi^\theta(p, \delta) = \delta \cdot (p - c^\theta), \quad \theta \in \{H, L\}\,.$$

Only observing the price $p$ set by the firm, not the quality of the product, the risk-neutral consumer maximizes her expected gains from trade. After a purchase she learns her valuation; her utility from having bought a product of quality $q$ for price $p$ then is $q - p$.

[3] The analysis would not change if we assumed multiple, identical buyers. For simplicity, we only speak of one consumer.

[4] The assumption that quality is not a strategic decision of the firm is crucial in lemon markets. While quality uncertainty may also exist in equilibria under different models, they allow for separation even at the quality level. See Shaked and Sutton (1982) for an example of such a result.

We denote $q^H > q^L$ the consumer's valuation for the high and the low quality product, respectively. To always ensure possible gains from trade, we assume $c^L < q^L$ and $c^H < q^H$.[5]

The buyer always has a certain, endogenous belief of the quality after observing a price. Fix a price $p$ and let $\hat{\mu} \in [0,1]$ be the conceived probability of facing a high type firm. Then the consumer's expected utility from buying the good is

$$u_b(p,\hat{\mu}) := \hat{\mu}q^H + (1-\hat{\mu})q^L - p$$

while the utility from not buying the good is $u_n := 0$.

She faces a third option, namely to pay a cost (or to exert effort) of a chosen level $k \geq 0$ to then obtain a binary signal $s \in \left\{s^H, s^L\right\}$ about the product quality. This signal might be incorrect with an error probability of $\varepsilon(k) \in [0, \frac{1}{2}]$. Mathematically this means[6]

$$Prob(s^H|\theta = L) = Prob(s^L|\theta = H) = \varepsilon(k).$$

The exogenously given error function $\varepsilon$ satisfies the following assumptions.

- $\varepsilon : \mathbb{R}_+ \to [0, \frac{1}{2}]$ is continuous and non-increasing.
- $\varepsilon(0) = \frac{1}{2}$
- Denote $\bar{k} = \inf\left\{k \in \mathbb{R}_+ | \varepsilon(k) = 0\right\}$ the costs for a perfect signal where $\inf \emptyset := \infty$. Then $\varepsilon$ is twice continuously differentiable on $(0, \bar{k})$.
- $\varepsilon'(k) < 0, \varepsilon''(k) > 0 \qquad \forall\ k \in (0, \bar{k})$

While the first and third points are of technical nature, the second assumption says that the signal does not contain any information if the consumer exerts no effort. The last point ensures that higher effort always leads to a strictly higher signal precision while the marginal precision gain is diminishing. This accounts for the fact that information acquisition, such as reading reviews, will often give redundant

[5] Adriani and Deidda (2009) focus on a case in which trade would not always be beneficial under full information. They show that this leads to market breakdown in their setting under the D1-Refinement of Cho and Kreps (1987).

[6] The assumption of both error types being the same is certainly somehow restrictive but is not believed to have a qualitative impact on the results. See Martin (2012) for the use of a more complex information structure. In his analysis, however, the firm can choose only between two exogenously given prices.

information and thus the amount of new information gained via a certain increase of search effort is decreasing. Due to the second point we can assume that the consumer also receives the (non-informative) signal when she chooses $k = 0$.

Note that we allow for obtaining a *perfect signal*, i.e. there may be a finite cost $\bar{k}$ for which the error probability is zero. Depending on the error function, this value might also be infinite so that perfect information would not be achievable. We do not restrict attention to any of these cases.

By the last assumption the expression

$$\varepsilon'(0) := \lim_{\substack{k \to 0 \\ k \in (0, \bar{k})}} \varepsilon'(k) = \inf_{k \in (0, \bar{k})} \varepsilon'(k) \in [-\infty, 0)$$

is well-defined. This value is important in the analysis. For illustrating results, we use the simple function $\varepsilon(k) = \max\left\{\frac{1}{2} - \sqrt{k}, 0\right\}$ which satisfies the assumptions above.

Naturally, agents on this market do not act simultaneously. At the time when the consumer makes her decision, the price was already set by the firm and this requires the quality level to already be realized. Figure 2.1 outlines the timing of the market.

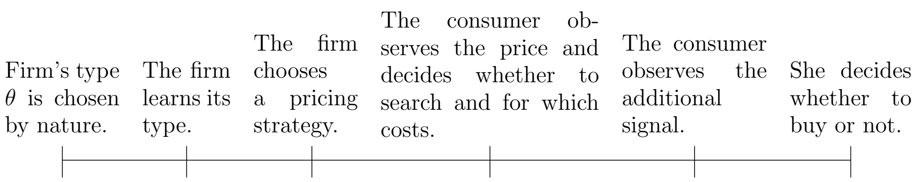


Figure 2.1: The timing of the market

The consumer holds a belief system $\mu : \mathbb{R}_+ \mapsto [0, 1]$, later determined by the equilibrium definition, which assigns to each possible price $p$ a belief $\mu(p)$ about the probability that the product is of high quality conditional on the observed price. In the analysis it is sometimes useful to consider a fixed price $p$ and a fixed corresponding posterior belief $\mu(p)$. In this case we abbreviate the latter by writing $\hat{\mu}$ instead of $\mu(p)$. We define the expected quality based on such beliefs by

$$\bar{q}_{\hat{\mu}} := \hat{\mu} q^H + \big(1 - \hat{\mu}\big) q^L.$$

In the same way, to avoid imprecisions, single values of other functions are denoted similarly. Note that the true a priori probability of high quality is denoted by $\eta$ while the letter $\mu$ is reserved for posterior belief values.

## 2.3 Consumer Behavior

Since we analyze a multi-stage game, we proceed by backward induction, thus first dealing with the buyer's decision problem. This problem itself has two stages. When observing the price $p \in \mathbb{R}_+$, she has to decide about the search amount $k \geq 0$. In the second step, she receives the signal and decides whether to buy the good or not. We allow for mixed strategies, so it is possible for the consumer to buy the good only with a certain probability. Remember that the two possible quality valuations $q^L, q^H$ as well as the a priori probability $\eta$ of facing a high type producer is known to the consumer.

### 2.3.1 After Receiving the Signal

Assume for now that $k$ has been chosen. Let $\hat{\varepsilon} := \varepsilon(k)$ be the corresponding error probability and $\mu(p) \in (0, 1)$ the posterior belief that a product with price $p$ has quality $q^H$. In this section, $p$ and $\mu(p)$ can be viewed as fixed so that we write $\hat{\mu}$ for the posterior belief.

Conditional on observing the high signal $s^H$, the probability of the quality being high is

$$\frac{(1-\hat{\varepsilon})\hat{\mu}}{(1-\hat{\varepsilon})\hat{\mu} + \hat{\varepsilon}(1-\hat{\mu})}$$

which follows from Bayes' law.

The expected utility from buying (not taking into account the sunk cost $k$), given this situation is then

$$\frac{(1-\hat{\varepsilon})\hat{\mu}}{(1-\hat{\varepsilon})\hat{\mu} + \hat{\varepsilon}(1-\hat{\mu})} q^H + \frac{\hat{\varepsilon}(1-\hat{\mu})}{(1-\hat{\varepsilon})\hat{\mu} + \hat{\varepsilon}(1-\hat{\mu})} q^L - p.$$

Note that with $\hat{\varepsilon} = \frac{1}{2}$ this is the original utility from buying without the additional signal.

The case on observing $s^L$ is computed similarly. Clearly, the consumer will buy the good if this utility is above zero and not buy it if it is strictly below this value. The interesting insight here is that the signal is only relevant to her if not for all signals $s^H$ and $s^L$ the utilities lie both above or both below zero. Remember that she made a rational choice to pay an amount $k > 0$ and thus she can intuitively not be indifferent between the signal outcomes.

**Lemma 2.3.1.** *Let a price $p$ with corresponding posterior belief $\hat{\mu}$ be given. If the consumer has optimally exerted positive search effort, she buys if and only if she receives the signal $s^H$.*

This result may not come as a surprise. If it was optimal to ignore a signal, it would be pointless to pay for its precision. The lemma is in the same spirit as the corresponding statement in Bester and Ritzberger (2001). It shows that the two pieces of information, namely the inherent information of the price given by the corresponding posterior belief $\hat{\mu}$ and the additional signal, are essentially not considered simultaneously. The former is used to decide about how much search effort to exert and if zero effort is chosen, it is used to determine whether to buy the good or not. Once the consumer decides to pay for signal precision, the buying decision only depends on the signal, not on the value $\hat{\mu}$ of the posterior belief. This, of course, does not occur in situations where an additional, informative signal is received regardless of the decision of the consumer as in Voorneveld and Weibull (2011).

While this effect also arises in Bester and Ritzberger (2001), consumers in their model observe a perfect signal and it is natural to dismiss prior information after learning the true state. In the situation at hand the reason is more subtle, basically lying in the backward induction argument. The probable implications of receiving various signals of a certain error probability are taken into account before the decision of costly acquiring the information is formed. Essentially, also the choice to buy only at a high signal is already made at that stage.

The proof of this lemma is straightforward. Like all others, it can be found in the appendix.

### 2.3.2 Choosing the Optimal Search Effort

We proceed by determining the optimal search costs $k$. Assume therefore that the consumer pays a cost $k > 0$ for search and that this level is optimal. We know by the previous lemma that the only possible behavior after receiving the signal is to buy if and only if the signal is $s^H$, i.e. if the quality is high and the signal is correct or if the quality is low and the signal is wrong. Then the expected utility, given price $p$ and posterior beliefs $\hat{\mu}$, is

$$u_s(p,\hat{\mu},k) := \underbrace{\hat{\mu}(1-\varepsilon(k))(q^H-p)}_{\text{correct high signal}} + \underbrace{(1-\hat{\mu})\varepsilon(k)(q^L-p)}_{\text{false high signal}} - k.$$

This formula consists of three terms. The (possibly subjective) probability of facing a high good is $\hat{\mu}$. The consumer then buys if she receives a correct signal which has the probability $1-\varepsilon(k)$. This yields the utility $q^H - p$. The second term of the formula reflects the possibility and consequences of buying a low quality product because of a false high signal. The search costs $k$ have to be paid regardless of the quality and the buying decision.

We want to stress that this is the expected utility after observing the price and before receiving the signal, and only if the optimal search cost is positive. Lemma 2.3.1 allows us to ignore the updated beliefs after observing the additional quality information.

Maximizing this utility with respect to search costs, we get the first order condition

$$\begin{aligned} & \hat{\mu}(-\varepsilon'(k))(q^H-p) + (1-\hat{\mu})\varepsilon'(k)(q^L-p) = 1 \\ \Leftrightarrow \qquad & \varepsilon'(k) = \frac{1}{-\hat{\mu}(q^H-p)+(1-\hat{\mu})(q^L-p)} \\ & \quad = \frac{-1}{\hat{\mu}(q^H-p)+(1-\hat{\mu})(p-q^L)} =: d(p,\hat{\mu}) =: \hat{d}. \end{aligned}$$

The parameter $\hat{d}$ depends both on the price $p$ and the posterior belief $\hat{\mu}$ and is always negative in the relevant range of prices $[q^L, q^H]$ and when $\hat{\mu} \in (0,1)$. Its value is roughly an indicator of whether the price fits the expected valuation given by the belief. If $p$ and $\hat{\mu}$ are both high or both low, $\varepsilon$ must have an extreme slope and thus the optimal $k$ is low. If there is a discrepancy between $p$ and $\hat{\mu}$, $\hat{d}$ is closer to zero and thus $k$ is higher. This shows that search is used more extensively if the consumer has reason not to trust the price. Figure 2.2 depicts this effect.

Note that we can rewrite the utility in the form

$$u_s(p,\hat{\mu},k) = \left[\hat{\mu}(q^H-p)+(1-\hat{\mu})(p-q^L)\right](-\varepsilon(k)) - k + \hat{\mu}(q^H-p)$$

so that the function is strictly concave in $k$ in the range $(0,\bar{k})$ for all values $\hat{\mu} \in (0,1)$ and $p \in [q^L, q^H]$. The first order condition thus provides the interior solution if and only if there is one.

It follows that the utility maximizing search cost for the consumer problem is

$$k^*(p,\hat{\mu}) := \begin{cases} 0 & \hat{d} \leq \varepsilon'(0) \\ (\varepsilon')^{-1}(\hat{d}) & \varepsilon'(0) < \hat{d} < \varepsilon'(\bar{k}) \\ \bar{k} & \hat{d} \geq \varepsilon'(\bar{k}). \end{cases} \tag{2.1}$$

This function is continuous and piecewise differentiable in both arguments. However, its form presents some problem for the analysis, namely that there is a saddle point at $(\frac{q^H - q^L}{2}, \frac{1}{2})$. Figure 2.2 shows an example of this function. It also shows the effect that search effort is high in the areas in which $\hat{\mu}$ and $p$ do not correspond to each other.

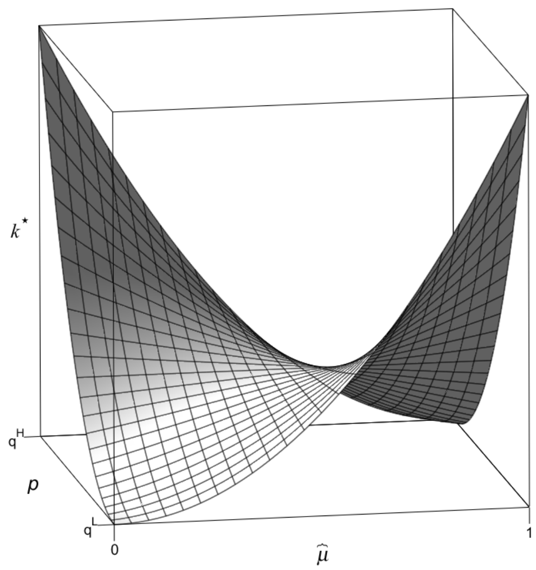


Figure 2.2: The function $k^*$ for $\varepsilon(k) = \max\left\{\frac{1}{2} - \sqrt{k}, 0\right\}$.

As mentioned above, this analysis is based on Lemma 2.3.1 and thus gives a *necessary* condition. If the consumer optimally pays a positive cost, it has to be given by the function $k^*$. To ensure that paying this cost and then acting in accordance to the signal (provided $k^*$ is positive) is optimal, the corresponding error probability must be low enough to yield positive utility when the signal is $s^H$ and negative utility in case of receiving $s^L$. We thus have to test whether $k^*$ meets this condition. In general, this is not the case for all pairs $(p,\hat{\mu}) \in [q^L, q^H] \times (0,1)$. The following lemma, however, shows that this is never an issue when utility implied by the optimal search behavior exceeds the one from not buying or from buying without extra information.

**Definition 2.3.2.** *Let*

$$u_s^*(p, \hat{\mu}) := u_s(p, \hat{\mu}, k^*(p, \hat{\mu}))$$

*denote the maximal achievable utility if the consumer was committed to buy if and only if she receives signal* $s^H$.

**Lemma 2.3.3.** *Let* $(p, \hat{\mu}) \in [q^L, q^H] \times (0, 1)$ *be given and denote* $\hat{k}^* := k^*(p, \hat{\mu})$. *Moreover, assume*

$$u_s^*(p, \hat{\mu}) > \max\{0, \bar{q}_{\hat{\mu}} - p\} = \max\{u_n, u_b(p, \hat{\mu})\}. \tag{2.2}$$

*Then we have* $\hat{k}^* > 0$ *and the error probability* $\varepsilon(\hat{k}^*)$ *is low enough so that the consumer buys the product if and only if she receives the signal* $s^H$.

We denote the optimal utility, given a price $p$ and a corresponding posterior belief $\hat{\mu}$ by

$$u^*(p, \hat{\mu}) := \max\left\{u_b(p, \hat{\mu}), u_n, u_s^*(p, \hat{\mu})\right\}.$$

Having the three options of *searching*, *not buying* and *buying without search*, the consumer acquires information if the condition (2.2) of the previous lemma is met (while there can be mixed strategies in case of equality). We continue by investigating when this is the case and when the consumer prefers either of the two other options, depending on the observed price $p$ and the corresponding posterior belief $\hat{\mu}$. Note that, due to the complicated behavior of the optimal search costs and hence the signal precision, the area in which positive search effort occurs is not trivially well-shaped.

**Lemma 2.3.4.** *For all* $\hat{\mu} \in (0, 1)$ *there are prices* $\underline{p}_{\hat{\mu}}, \overline{p}_{\hat{\mu}}$ *such that*

$$q^L < \underline{p}_{\hat{\mu}} \le \bar{q}_{\hat{\mu}} \le \overline{p}_{\hat{\mu}} < q^H$$

*and the consumer strictly prefers buying without search whenever the price* $p$ *is below* $\underline{p}_{\hat{\mu}}$, *she strictly prefers searching whenever* $p \in (\underline{p}_{\hat{\mu}}, \overline{p}_{\hat{\mu}})$ *and she strictly prefers not buying whenever* $p > \overline{p}_{\hat{\mu}}$, *provided that* $\hat{\mu}$ *is the corresponding posterior belief to* $p$.

Figure 2.3 gives a graphical intuition for how the utility of each of the three options depends on $p$ for a fixed value of $\hat{\mu}$. There is a counter-intuitive effect when $\hat{\mu} > \frac{1}{2}$.

The optimal search effort $k^*$ is then decreasing in the price and hence a price increase could have a positive effect for the consumer's utility. The proof of Lemma 2.3.4 shows that this effect is, however, negligible such that we indeed always have a decreasing behavior of the search payoff in the price variable. The thicker line in Figure 2.3 depicts the function $u^*$, the maximum utility value of all three options "search", "buy" and "don't buy". Note that $u_s^*$ is not a linear function but the proof shows that its slope is always below zero and above the slope of $u_b$ which leads to the result above.

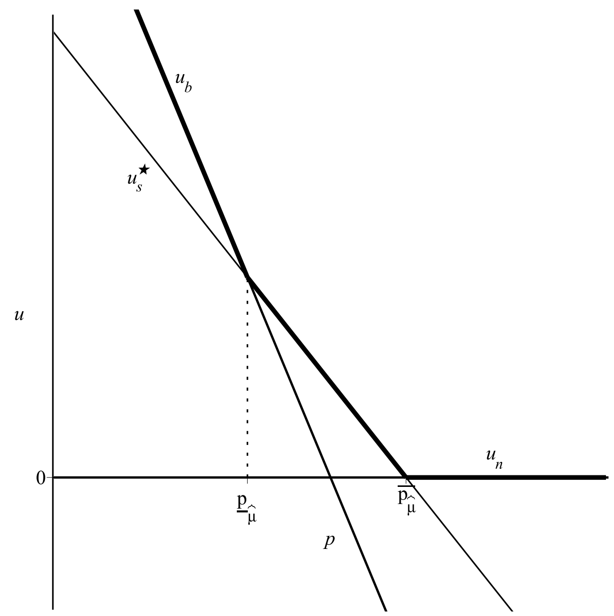


Figure 2.3: The utility development with $p$ and determination of the prices $\underline{p}_{\hat{\mu}}$ and $\overline{p}_{\hat{\mu}}$ for fixed $\hat{\mu}$. The function $u^*$ is given by the thick upper contour line.

Having this lemma, we are particularly interested in situations where the interval $(\underline{p}_{\hat{\mu}}, \overline{p}_{\hat{\mu}})$ is not empty. As it turns out, this is always the case as long as the marginal gain of signal precision from search effort is sufficiently high at zero.

**Lemma 2.3.5.** *For all $\hat{\mu} \in (0,1)$, the strict inequality $\underline{p}_{\hat{\mu}} < \overline{p}_{\hat{\mu}}$ holds if and only if*

$$\varepsilon'(0) < \frac{-1}{2\hat{\mu}(1-\hat{\mu})(q^H - q^L)}.$$

*In this case, one even has $\underline{p}_{\hat{\mu}} < \bar{q}_{\hat{\mu}} < \overline{p}_{\hat{\mu}}$.*

In other words: Every non-degenerate posterior belief can lead to search behavior if the marginal benefit from search is sufficiently high.

Note that search and hence a positive probability of trade exists even with prices above expected quality $\bar{q}_{\hat{\mu}}$. It is of importance for the later analysis that the statement of this lemma is always true if we have $\varepsilon'(0) = -\infty$.

A similar result to Lemma 2.3.4 is true for the dependence of consumer's behavior on the posterior belief $\hat{\mu}$. This follows from the following, stronger observation.

**Lemma 2.3.6.** *The values $\underline{p}_{\hat{\mu}}$ and $\overline{p}_{\hat{\mu}}$ are continuous and piecewise differentiable in $\hat{\mu}$. Moreover, we have*

$$\frac{\partial}{\partial \hat{\mu}} \underline{p}_{\hat{\mu}} > 0 \text{ and } \frac{\partial}{\partial \hat{\mu}} \overline{p}_{\hat{\mu}} > 0$$

*for each point in which the respective function is differentiable and*

$$\lim_{\hat{\mu} \to 0} \underline{p}_{\hat{\mu}} = \lim_{\hat{\mu} \to 0} \overline{p}_{\hat{\mu}} = q^L \qquad \lim_{\hat{\mu} \to 1} \underline{p}_{\hat{\mu}} = \lim_{\hat{\mu} \to 1} \overline{p}_{\hat{\mu}} = q^H.$$

This relatively nice behavior of the lower and upper bound for prices for which search is optimal comes as a surprise considering the shape of the optimal search effort function. It is needless to say that these properties facilitate the following equilibrium analysis.

To give a better feeling for how the three options of "search", "buy" and "don't buy" are distributed, we give a graphical example. Figure 2.4 shows the various areas for $q^H = 1, q^L = .5, \varepsilon(k) = \max\left\{\frac{1}{2} - \frac{3}{2}\sqrt{k}, 0\right\}$. Note that this error function satisfies $\varepsilon'(0) = -\infty$ and hence for every non-degenerate value of $\hat{\mu}$ there is a price for which search is strictly optimal.

## 2.4 The Market and Equilibrium Behavior

Having determined the behavior of the consumer, we investigate how this leads to various equilibria. We first need to formally define the game, i.e. the strategies and the equilibrium concept.

**Definition 2.4.1.** *A* consumer strategy *is a function $b : \mathbb{R}_+ \to \mathbb{R}_+ \times [0,1]^2$ where, for every price $p$, $b(p) = (k, \gamma^H, \gamma^L)$ denotes the amount of search effort $k$ and the probabilities $\gamma^H, \gamma^L$ of buying the product conditional on receiving the high or low signal.*

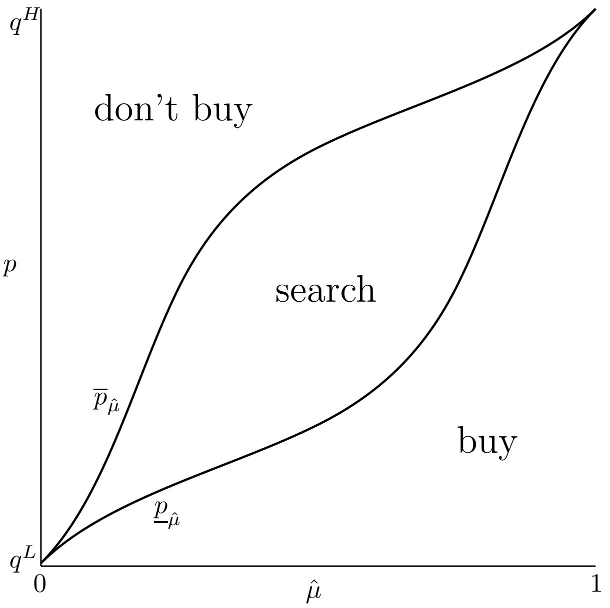


Figure 2.4: The areas of consumer behavior and the development of $\underline{p}_{\hat{\mu}}$ and $\overline{p}_{\hat{\mu}}$.

*A* firm's strategy $a : \{H, L\} \to \Delta(\mathbb{R}_+)$ *is a mapping that maps each type to a probability distribution over the price space* $\mathbb{R}_+$*.*

We write $a_H$ and $a_L$ instead of $a(H)$ or $a(L)$. Using Lemma 2.3.1 of the previous section, we know that the consumer optimally either pays a positive search cost and then buys if and only if a positive signal arises or she does not search and buys with a certain probability $\gamma \in [0, 1]$ independent of the signal that does not convey any information[7]. Based on this behavior, it is convenient to narrow down the set of possible consumer strategies.

**Definition 2.4.2.** *A* consistent consumer strategy *is a strategy where for all* $p \in \mathbb{R}_+$ *we have* $b(p) = (k, 1, 0)$ *or* $b(p) = (0, \gamma, \gamma)$ *with* $k > 0$ *and* $\gamma \in [0, 1]$*.*

Having this, we give the formal definition of an equilibrium in this setting.

**Definition 2.4.3.** *Let* $(a, \mu, b)$ *be a tuple where* $a$ *is the firm's strategy,* $\mu : \mathbb{R}_+ \to [0, 1]$ *is a posterior belief system of the consumer and* $b$ *is a consistent consumer strategy.*
*This tuple is an* equilibrium *if*

[7] Of course, still having different probabilities for each (meaningless) signal is possible. It is clear, however, that playing a strategy $b(p) = (0, \alpha, \beta)$ is equivalent to playing $b(p) = (0, \gamma, \gamma)$ with $\gamma = \frac{1}{2}\alpha + \frac{1}{2}\beta$.

- *Every price in the support of $a_H$ and $a_L$ maximizes the profit of the respective type*
- *$\mu$ is determined by Bayes' law whenever possible*[8]
- *$b$ maximizes the consumer's utility with respect to $\mu(p)$ for each prize $p$.*

Note that this is similar to the classical weak Perfect Bayesian Equilibrium as used in the text book by Mas-Colell et al. (1995) but adapted to the general strategy space of this model. An equilibrium in which $a_H = a_L$ is called a *pooling equilibrium* while a *separating equilibrium* is one in which the supports of $a_H$ and $a_L$ have an empty intersection. We call any other equilibrium a *hybrid equilibrium*.

For an equilibrium $EQ = (a, \mu, b)$, any price $p$ that is in at least one of the supports of $a_L$ or $a_H$ is called an *equilibrium price of $EQ$*. If additionally $b(p)$ has the form $(k, 1, 0)$, we call $p$ a *search price* of $EQ$, otherwise a *no-search price*. Abusing notation, we denote $u(p, \mu(p), b(p))$ the consumer's expected utility when observing a price $p$ with corresponding posterior belief $\mu(p)$ and playing strategy $b(p)$. For each firm type $\theta$ we define the equilibrium profit $\pi^\theta(EQ)$ as the expected profit when setting a price in the support of $a_\theta$. This value is well-defined by the first point in the equilibrium definition.

It is trivial to see that there can be a separating equilibrium in which the high type always sets price $q^H$, the low type sets the price $q^L$, the consumer has the belief system $\mu(p) = \mathbf{1}_{\{p=q^H\}}$ and only buys for prices smaller than or equal to $q^L$. For this to actually be an equilibrium, one must have $c^H \geq q^L$ so the high quality firm has no strict incentive to set the price $q^L$. This equilibrium would also occur if one did not allow for information acquisition and is present in many other models of markets with quality uncertainty. Note that high quality is not traded at all in this setting. We thus refer to this constellation as the *total adverse selection* (TAS) equilibrium.

To emphasize the relation to the classical model of quality uncertainty and the perfect information case, we briefly discuss these two cases.

With full information, the situation is quite obvious. Since the consumer always knows the type, the firm can always demand the true value $q^\theta$ and the buyer buys

[8]This point is often not precisely formulated in the literature. Formally, we apply the classic version of Bayes' law for every price $p$ where $a_H(\{p\}) + a_L(\{p\}) > 0$. For prices that are in the support of exactly one of the two distributions, we assume that the posterior belief is either 1 or 0, according to the type that uses $p$. No restriction is made for prices that are in both supports but have probability 0.

with probability one. Otherwise, any slightly lower price would lead to sure buying and thus causes the firm to deviate. There are no other equilibria.

If the consumer had no possibility of obtaining information about the product quality, the described situation corresponds to a lemon market model in the spirit of Akerlof that is similar but not quite equal to the analysis of Ellingsen (1997)[9]. It appears as a boundary case of our model if we set $\varepsilon(k) = \frac{1}{2}$ for all $k$ (which, of course, would not satisfy the assumptions). In that setting, if $c^H \leq \bar{q}_\eta$[10], pooling equilibria exist for a price in $[\max\{q^L, c^H\}, \bar{q}_\eta]$ while separating equilibria with prices $q^L$ and $q^H$ always exist in which the low quality firm sells with probability one and the high quality firm with a probability in

$$\left[\max\left\{0, \frac{q^L - c^H}{q^H - c^H}\right\}, \frac{q^L - c^L}{q^H - c^L}\right].$$

In particular, the total adverse selection equilibrium exists if and only if $q^L \leq c^H$ as was already observed in the setting of this paper. There are other, hybrid equilibria in Ellingsen's setting. While they are disregarded due to his refinements and although his analysis is not completely applicable to this setting, such equilibria also appear here.

### 2.4.1 Equilibrium Analysis

We start with observing some rather obvious and intuitive features that are quite standard and can be found in similar form in other models. They are nevertheless important for the analysis of equilibria.

**Lemma 2.4.4.** *In every equilibrium, the following statements hold.*

*i) The support of $a_L$ is a subset of $[q^L, q^H]$, the support of $a_H$ is a subset of $[q^L, \infty)$.*

*ii) The low type does not set the price $q^H$ with positive probability.*

*iii) The low type's profit is weakly larger than $q^L - c^L$.*

*iv) Every price in $(q^L, q^H)$ is either in both supports of $a_L$ and $a_H$ or in neither.*

[9] Ellingsen assumes equal differences between valuation and production costs for each type, thus corresponding to the case $q^H - c^H = q^L - c^L$.

[10] Although Ellingsen excludes this case in his paper, the set of pooling equilibria is easy to derive. The separating equilibria are the same.

These points are not surprising considering the nature of an equilibrium. Any price below $q^L$ would induce sure buying and thus always yield a lower profit than a higher price with the same property. The low type thus always has the option to deviate to a price arbitrarily close to $q^L$ and to receive a profit close to $q^L - c^L$ which shows iii). For the low type, setting a price $q^H$ or higher with positive probability would lead to a posterior belief below 1 and hence the consumer does not buy. The resulting profit is zero and contradicts iii).

If a price is set by one type but not by the other, the equilibrium definition implies that the consumer knows the true quality. If it was a low type and the price was above $q^L$, this would result in not buying at all, making it irrational for the low type to set this price. On the other hand, a price below $q^H$ set by only the high firm would result in sure buying and this would attract the low quality firm to imitate that behavior. The formal versions of these arguments can be found in the appendix.

Since the consumer never buys a product for a price higher than $q^H$, every such strategy is at least weakly dominated by any price in $(c^H, q^H]$. We thus assume that also the high type does not set a price above $q^H$.

We now know that, apart from the prices $q^L$ and $q^H$, every price is either in both types' support or in neither of them. However, there could in principal still be a large number of such prices, making further analysis even more complicated by adding measure theoretic obstacles. We show that this is in fact not the case and that there cannot be more than two such non-boundary prices played in equilibrium.

**Lemma 2.4.5.** *In an equilibrium, there are no two prices that are in both supports of $a_L$ and $a_H$ and for which the consumer searches.*

**Lemma 2.4.6.** *In an equilibrium, there cannot exist two different prices that are in both supports and for which the consumer does not search.*

For both of these lemmas, the first property of the equilibrium definition implies that both types must be indifferent between the prices in the support of their price distribution. In the proof, we show that this can not be the case for two search prices or two no-search prices. It is, however, possible that both types are indifferent between a search price and a no-search price.

These observations already significantly reduce the set of possible equilibrium strategies. Although we put no a priori restrictions on the firm's price-setting behavior, in equilibrium, each type does not play more than two prices in the set $(q^L, q^H)$.

If the error function $\varepsilon$ satisfies an additional, Inada-like condition, we can rule out even more equilibria. As seen in Lemma 2.3.5, the value of $\varepsilon'(0)$ is of importance when it comes to determining the consumer reaction. It has to be low enough to ensure the existence of a search price for any given posterior belief $\hat{\mu} \in (0,1)$. The bound itself depends on this belief and hence may vary between different equilibria or even between different equilibrium prices. It is hence convenient to define the following property.

**Definition 2.4.7.** *An error function $\varepsilon$ satisfies the assumption* (I) *if*

$$\varepsilon'(0) = -\infty.$$

Having this, we can even go further in narrowing down the set of equilibria.

**Lemma 2.4.8.** *Assume that $\varepsilon$ satisfies assumption (I) and let $p \in [q^L, q^H)$ be a no-search equilibrium price. Then $b(p) = (0,1,1)$ so that the consumer buys with probability one.*

This statement follows from Lemma 2.3.5. If the consumer buys with a probability in $(0,1)$, she is indifferent between buying and not buying, hence $\overline{q}_{\mu(p)} = p$. This price, however, leads to search when $\varepsilon'(0)$ is low enough. Having $b(p) = (0,0,0)$ would give zero profit to both firms and thus violates Lemma 2.4.4 iii).

The previous lemmas now allow us to define quite precisely the form of possible equilibria in the model.

**Proposition 2.4.9.** *If assumption (I) is satisfied, in every equilibrium the inclusions*

$$\begin{aligned} & supp(a_L) \subset \{q^L, p_s\} \qquad supp(a_H) \subset \{p_s, q^H\} \\ or \quad & supp(a_L) = \{p_1\} \qquad supp(a_H) \subset \{p_1, q^H\} \end{aligned}$$

*hold where $p_s$ is a search price, $q^L$ and $p_1$ induce sure buying and if $q^H$ is played, we have $b(q^H) = (0, \gamma, \gamma)$ with $\gamma$ low enough to not attract the low type firm.*

Summarized, these are the different types of potential equilibria in the model

- Separating adverse selection equilibria
- Pooling equilibria without search in which both types set the same price $p_1 \leq \underline{p}_\eta$
- Pooling equilibria with search and a price $p_s \in [\underline{p}_\eta, \overline{p}_\eta]$

- Hybrid equilibria in which the high type firm demands a high search price $p_s$ and the low type plays $a_L(p_s) = \alpha$, $a_L(q^L) = 1 - \alpha$ for some $\alpha \in (0, 1)$.
- Other equilibria with $q^H \in \text{supp}(a_H)$ and $b(q^H) = (0, \gamma, \gamma)$, $\gamma > 0$.

All these equilibria exist provided the buyer and the high type firm make non-negative profit and the low type earns at least $q^L - c^L$. We denote the pooling search equilibrium with the highest possible price $\overline{p}_\eta$ as $PE_s$ and the pooling no-search equilibrium with the price $\underline{p}_\eta$ as $PE_b$. If at least one hybrid equilibrium exists, the one with the highest search price $p_s$ is denoted as $HE$. These are the important equilibria due to the following robustness check.

The set of potential equilibria is significantly narrowed down but still too large to draw qualitative conclusions from the model. In what follows, we argue in which way some of these equilibria, and in particular the belief systems by which they are supported, can be disregarded.

### 2.4.2 Selection of Equilibria

There are various, well established refinements to eliminate implausible equilibria in signaling games. Bester and Ritzberger (2001) use a modification of the well-known Intuitive Criterion introduced by Cho and Kreps (1987). In this model, as well as in theirs, the original version of the Intuitive Criterion is not sufficient. The modification used by Bester and Ritzberger, however, is not well defined in our setting since the firms' profit functions are not monotone in beliefs. We thus follow another approach of arguing which consumer beliefs are unconvincing and hence rule out the equilibria supported by these beliefs.

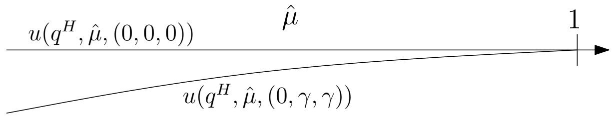


Figure 2.5: Buying for the price $q^H$ leads to negative utility if $\hat{\mu} < 1$

To illustrate the idea of the following refinement, consider an adverse selection equilibrium in which the high type firm makes positive profit, i.e. a separating equilibrium in which the low type sets price $q^L$, the high type price $q^H$ and the consumer buys the high quality product with some probability $\gamma > 0$. From the equilibrium property we must have $\mu(q^H) = 1$ so the consumer knows the quality when she sees the high price. Note that she is then indifferent between buying and not buying since the price matches her valuation. If she had any doubts about her posterior belief $\mu(q^H)$, i.e. if she admits that there is even the smallest possibility to be wrong about her belief, "not buying" would be strictly better than her strategy $b(q^H) = (0, \gamma, \gamma)$. Since "not buying" is optimal even for her rational belief $\hat{\mu} = 1$, her strategy is dominated in a certain sense. This idea is depicted in Figure 2.5 and formally written down in the following refinement.

**Definition 2.4.10.** *Let $p, \mu(p)$ be given. The action $b \in \mathbb{R}_+ \times [0,1]^2$ is* locally dominated in beliefs *if there exists another action $b^* \in \mathbb{R}_+ \times [0,1]^2$ and a $\delta > 0$ such that*

$$u(p, \hat{\mu}, b^*) \geq u(p, \hat{\mu}, b) \;\forall\; \hat{\mu} \in \big(\mu(p) - \delta, \mu(p) + \delta\big) \cap [0,1]$$

*and the inequality is strict for $\hat{\mu} \neq \mu(p)$.*

*An equilibrium $(a, \mu, b)$ has* belief-robust responses *if for no equilibrium price $p$ and corresponding belief $\mu(p)$ the action $b(p)$ is dominated in beliefs.*

This condition reflects some doubts about the posterior beliefs. A best response $b$ which violates this criterion is not a strict one, meaning that there is another best response $b^*$ to $(p, \mu(p))$ that yields the same payoff. Moreover, choosing $b$ over $b^*$ is not a robust behavior and only rational if the buyer is absolutely confident about the firm's strategy.

The criterion is one of local robustness of the strategy. Other criteria in the same spirit can be found in the literature, for example the *robust best reply* definition in Okada (1983).

Note that this condition does not in general rule out mixed strategies of the consumer. In this case, however, it leads to eliminating all equilibria in which the buyer plays a mixed strategy for the highest possible price $q^H$, including the classic adverse selection equilibria, mentioned above, in which the high type makes positive profits.

**Lemma 2.4.11.** *Let assumption (I) be satisfied. For an equilibrium, the following is equivalent.*

*i) The equilibrium has belief-robust responses.*

*ii) The price $q^H$ is not an equilibrium price or the equilibrium is the separating equilibrium with total adverse selection.*

The reason why most adverse selection equilibria are ruled out is not specific to this setting. In fact, a similar refinement excludes these equilibria e.g. in Ellingsen

(1997).[11] In that paper, he uses another refinement under which only the separating equilibrium with the highest possible high type trade probability survives. While this is a legitimate approach, the richness of equilibria in our setting allows us to exclude these equilibria and still obtain interesting results.[12]

As one can see, a lot of the equilibria survive this refinement. This gives us the opportunity to address another issue of implausible consumer behavior, namely the possibility of extreme belief changes.

Imagine two situations in which the consumer observes a price $p$ or a similar price that is very close to $p$. It does not seem intuitive that the posterior beliefs should differ too much, especially if we let the difference of the two prices be arbitrarily small. Even if one admits that real prices usually can not differ by less than one cent, posterior beliefs that assign $\mu(p) = 1$ and $\mu(p + 0.01) = 0$ seem quite extreme. In fact, marginal price changes are often due to retailer behavior and may not even be perfectly perceived by consumers.[13] It is thus more realistic that the consumer acknowledges the closeness of the prices by assigning a similar posterior belief. Formally, we postulate continuity of beliefs in those prices that actually occur in equilibrium.

**Definition 2.4.12.** *An equilibrium* $(a, \mu, b)$ *satisfies the* locally continuous beliefs *condition if for every equilibrium price* $p$ *the function* $\mu$ *is continuous in* $p$.

Local continuity is not a very strong assumption considering that it just excludes jumps in beliefs but still allows for arbitrarily strongly increasing or decreasing posteriors. The described behavior for the one-cent difference in the motivating example would actually still be possible under locally continuous beliefs. However, this slight step has a big impact on the number of equilibria.

Before we determine the consequences of this refinement, note that it usually[14] rules out the pooling equilibrium without search ($PE_b$) if we have $\varepsilon(k^*(\underline{p}_\eta, \eta)) = 0$ so that

[11] Compare Proposition 5 of Ellingsen (1997). Note that elimination of strategies that are locally dominated in beliefs could be substituted by elimination of weakly dominated strategies in this paper without changing the results.

[12] Interestingly enough, Ellingsen justifies using his other refinement by saying “in reality, a seller will typically not know exactly the buyer’s valuation” which is true. In the same spirit, however, the idea that the consumer might not be perfectly confident about her posterior beliefs should not be ignored. Ellingsen’s idea of “elastic demand” is incorporated in the next refinement and thus our approach covers both aspects of imperfections to some extend.

[13] See Zeithaml (1988) for an overview on perception of price and other product characteristics by consumers.

[14] There can be cases in which the pair $(\underline{p}_\eta, \eta)$ is exactly on the border defined by (2.1) so that there might be a continuous “path” $\mu(p)$ of posterior beliefs under which the $PE_b$ equilibrium

there is perfect search for the border case of a pooling equilibrium price in which "buy" and "search" yield the same outcome to the consumer. The reason for this is that, with continuous beliefs, the high quality firm would want to deviate to a slightly higher price than $\underline{p}_\eta$ which, because the consumer receives a perfect signal, also yields a selling probability of one for high quality products.

**Proposition 2.4.13.** *Let assumption (I) be satisfied. The strategies* $(a, b)$ *can form an equilibrium with a posterior belief system that satisfies* locally continuous beliefs *and such that it has* belief-robust responses *if and only if they are the strategies of one of the following equilibria:*

- *the pooling no-search equilibrium* $PE_b$ *with price* $\underline{p}_\eta$. *This equilibrium exists if and only if* $\underline{p}_\eta \geq c^H$ *and* $\varepsilon(k^*(\underline{p}_\eta, \eta)) > 0$
- *the pooling search equilibrium* $PE_s$ *with price* $\overline{p}_\eta$. *It exists if and only if* $\overline{p}_\eta \geq c^H$ *and* $\pi^L \geq q^L - c^L$
- *hybrid equilibria in which the high quality firm sets a price* $p = \overline{p}_{\hat{\mu}}$ *and the low quality firm sets this price with probability* $\alpha \in (0, 1)$ *while setting* $q^L$ *with probability* $1 - \alpha$ *and we have* $\hat{\mu} = \mu(p) = \frac{\eta}{\eta + \alpha(1-\eta)} > \eta$. *This equilibrium exists if and only if* $\overline{p}_{\hat{\mu}} \geq c^H$ *and* $\pi^L = q^L - c^L$.
- *the total adverse selection equilibrium (TAS). It exists if* $c^H \geq q^L$.

While three of these equilibria are unique within their class if they exist, there may be multiple hybrid equilibria. Every value $\hat{\mu} > \eta$ for which the equation

$$\varepsilon(k^*(\overline{p}_{\hat{\mu}}, \hat{\mu})) \cdot (\overline{p}_{\hat{\mu}} - c^L) = q^L - c^L$$

holds yields such an equilibrium if $c^H$ does not exceed the price $\overline{p}_{\hat{\mu}}$. The reason is that a low quality firm must be indifferent between the prices $\overline{p}_{\hat{\mu}}$ and $q^L$. Figure 2.6 shows such a constellation in which not only multiple hybrid equilibria but also the pooling search equilibrium $PE_s$ exist at the same time. It is useful to note that the existence of a hybrid equilibrium implies

$$\begin{aligned} \frac{1}{2}(q^H - c^L) > \varepsilon(k^*(\overline{p}_{\hat{\mu}}, \hat{\mu})) \cdot (\overline{p}_{\hat{\mu}} - c^L) &= q^L - c^L \\ \Rightarrow \qquad q^L - c^L &< q^H - q^L. \end{aligned}$$

can be sustained. Since this is a non-generic case, we omit the detailed analysis and just write $\varepsilon(k^*(\underline{p}_\eta, \eta)) > 0$ as condition for the existence of $PE_b$.

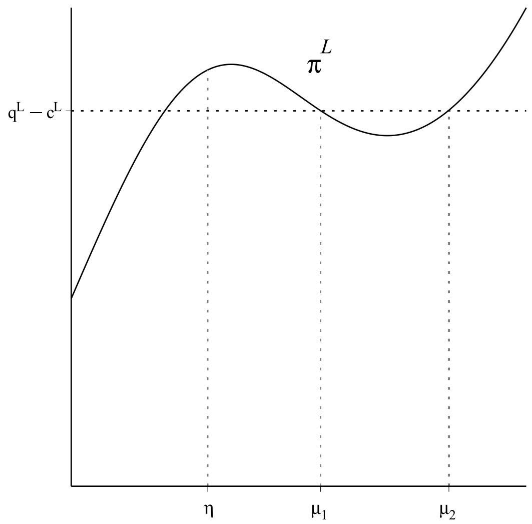


Figure 2.6: The coexistence of $PE_s$ and multiple hybrid equilibria. The function depicts the low type profit for each $\hat{\mu}$ when setting the price $\overline{p}_{\hat{\mu}}$. The values $\mu_1$ and $\mu_2$ are the posterior beliefs of search prices in hybrid equilibria.

It is a common result in lemon markets that all equilibria are not efficient (so that some goods are not traded with full probability) or the consumer has a chance of buying a good for a higher price than his valuation. We also observe this, here. Note that, although we focus on take-it-or-leave-it offers, the famous result of Myerson and Satterthwaite (1984) suggests that this can not be overcome when using a different mechanism.[15]

To give an overview over the qualitative implications of these equilibria, their properties are summarized in the following table. The "$\gtrsim$" symbol indicates generic strict inequalities, i.e. the set of parameters for which equality occurs is a Lebesgue null set in the parameter space.[16]

[15]Their formal result does not apply here. To give the connection, production costs $c$ can be seen as the seller's valuation, $q$ as the buyer's value. In contrast to the original result, they are not independent and not drawn from an interval $[c^L, c^H]$, $[q^L, q^H]$. The only efficient (unrefined) equilibrium that guarantees non-negative profits and consumer surplus in every outcome is the pooling equilibrium on the price $q^L$. It exists if and only if $c^H \leq q^L$ which directly translated to a violation of the assumption of Myerson and Satterthwaite that the intervals $[c^L, c^H]$ and $[q^L, q^H]$ overlap.

[16]For example, in the $PE_s$ equilibrium if $\overline{p}_\eta$ happens to be exactly $c^H$, the high type makes no profit. The value of $\overline{p}_\eta$ does not depend on $c^H$ so this is a Lebesgue null set.

| | $\pi^H$ | $\pi^L$ | cons. utility | existence condition |
|---|---|---|---|---|
| $PE_b$ | $\gtrsim 0$ | $> q^L - c^L$ | $> 0$ | $\underline{p}_\eta \geq c^H$ and $\varepsilon(\underline{p}_\eta) > 0$ |
| $PE_s$ | $\gtrsim 0$ | $\gtrsim q^L - c^L$ | 0 | $\overline{p}_\eta \geq c^H$ and $\pi^L \geq q^L - c^L$ |
| hybrid | $> 0$ | $q^L - c^L$ | 0 | $\exists \hat{\mu} > \eta : \overline{p}_{\hat{\mu}} \geq c^H$ and $\pi^L = q^L - c^L$ |
| $TAS$ | 0 | $q^L - c^L$ | 0 | $c^H \geq q^L$ |

Table 2.1: The properties of equilibria surviving the refinements

This table shows an interesting aspect especially about the consumer utility. There is only one equilibrium in which she has positive utility and this does not involve search. The possibility of search does not allow the $PE_b$ equilibrium to have a higher price than $\underline{p}_\eta$. Remember that in the classic lemon market this price would be equal to $\bar{q}_\eta$ so that the consumer had zero expected utility if we apply the same refinements to the pooling equilibria of the classical case. Introducing search can thus benefit the consumer but only if she does not use this new "ability". Naturally, this consumer friendly equilibrium only exists if the price is still high enough for a high quality firm to make positive profit. It also shows, however, that if the optimal search effort on the pooling price leads to perfect information, this equilibrium fails the refinements. In this case, the consumer's ability to search destroys her only chance of having positive utility. We elaborate on this effect in the next section.

Quality uncertainty situations being famous for their adverse selection effects, we can now investigate how the model behaves in this regard. The following shows that introducing search, as one would assume, indeed reduces the advantage of low quality goods over high quality goods in terms of traded amount.

**Observation 2.4.14.** In $PE_b$ and $PE_s$, a high quality firm has a weakly higher probability of selling the good than the low type. In any hybrid equilibrium of Proposition 2.4.13, the probability for a high firm of selling the good is higher than in any separating adverse selection equilibrium.

Note that in a hybrid equilibrium, the low type firm can have a higher chance of selling its good than the other type. This value is $1 - \alpha + \alpha\hat{\varepsilon}$ where $\alpha$ is the share with which it sets the high search price and $\hat{\varepsilon}$ is the error probability of that price.

To go even further, observe that there is a partial ranking in Pareto dominance between the existing equilibria.

**Definition 2.4.15.** *An equilibrium* $(a, \mu, b)$ Pareto dominates *another equilibrium* $(\tilde{a}, \tilde{\mu}, \tilde{b})$ *if the equilibrium payoffs satisfy*

$$\pi^H \geq \tilde{\pi}^H, \pi^L \geq \tilde{\pi}^L \text{ and } u^* \geq \tilde{u}^*$$

*and at least one of these inequalities is strict.*

This definition of Pareto dominance is taken after the quality of the firm is revealed, thus taking each type's profit into account separately. This gives a stricter version than an a priori Pareto dominance in which one would only consider the expected profit before the firm learns its type. However, an interesting dominance ranking holds even with this stronger condition.

**Lemma 2.4.16.** *Ignoring non-generic cases, the following items reflect the full Pareto dominance ranking between the equilibria of Proposition 2.4.13.*

- *If multiple hybrid equilibria exist, the one with the highest search price ($HE$) dominates the others.*
- *$TAS$ is dominated by $PE_b$, $PE_s$ and $HE$ whenever one of these equilibria exists.*
- *$PE_s$ and $HE$ are dominated by $PE_b$ if and only if $\pi^H(PE_s) \leq \pi^H(PE_b)$ or $\pi^H(HE) \leq \pi^H(PE_b)$, respectively.*

It is quite natural to observe that the equilibria $PE_s$ and $HE$ are somehow similar. In both equilibria, there is a search price on the upper border of the search area and the consumer has zero utility. Indeed, the coexistence of these equilibria is rare and does never occur if the probability of having high quality is sufficiently high.

**Lemma 2.4.17.** *Let $q^L - c^L < q^H - q^L$. There is a lower bound $\underline{\eta} < \frac{1}{2}$ such that whenever $\eta > \underline{\eta}$, there exists either $PE_s$ or $HE$ provided that the search price of one of these equilibria exceeds $c^H$.*

The reason for having this lower bound lies in the profit of the low quality type. It strictly increases when the posterior $\hat{\mu}$ goes from $\frac{1}{2}$ to 1 and the price is $\overline{p}_{\hat{\mu}}$. Thus, the $PE_s$ condition $\pi^L \geq q^L - c^L$ implies that for all higher beliefs the low type's profit is even larger. In $HE$, however, the profit must exactly attain this bound. The situation in Figure 2.6 corresponds to a case in which $\eta < \underline{\eta}$. In this figure, $\underline{\eta}$ can be chosen to be $\mu_1$.

Notice also that the actual value of $\underline{\eta}$ might be zero so that the negative profit effect of losing customers never outweighs the positive effect of a higher price for the low type firm. The condition $q^L - c^L < q^H - q^L$ of the lemma follows from the existence condition of a hybrid equilibrium. If this is violated, a hybrid equilibrium can never exist.

The so far established results already shed some light on how the market outcome is influenced by introducing information acquisition costs in the classical model of quality uncertainty. It shows that if the cost for high quality production is low, a pooling equilibrium without search exists. While this is also true in the classical model, there are qualitatively different aspects, namely that the actual price to pay in the pooling equilibrium is strictly below the average quality valuation and hence the consumer has strictly positive utility. Of course, this effect is caused by the same issue that rules out these equilibria for high quality costs between $\underline{p}_{\eta}$ and $\bar{q}_{\eta}$. In these cases, introducing the possibility of information acquisition leads to search behavior but does not help the consumer.

A different phenomenon can be observed in the $PE_s$ and $HE$ equilibria. They exist whenever $\bar{p}$ is high enough. Since these equilibria contain search prices, they do not occur in the classical model but dominate and thus eliminate the otherwise existing separating equilibria. Although also the payoffs are different, the main contribution of these equilibria is the weakening or complete elimination of adverse selection phenomena.

## 2.5 Search Efficiency

The previous section investigates a market in which the search possibility for the consumer is fixed by the function $\varepsilon$. As mentioned in the introduction, we are also interested in comparing situations in which consumer might have higher or lower costs for searching. Since the actual costs $k$ are endogenously chosen by the consumer, we have to clarify what "lower search costs" means in this setting. It is rather to be viewed as higher "search efficiency" which means that the consumer gets a more precise signal for the same search effort. Think about someone who wants to buy a TV in 2013 or someone in the 1980s. Getting information about a certain product is much easier now than it was back then, due to the Internet, multiple test magazines and websites. It is safe to say that it is both less time consuming and cheaper to get the same amount of information now than it was back then.

To capture this efficiency in the model we introduce a parameter $a$ to the function $\varepsilon$. The extended function satisfies the following properties.

- $\varepsilon : \mathbb{R}_+ \times \mathbb{R}_{++} \to [0, \frac{1}{2}], (k, a) \mapsto \varepsilon(k, a)$ is continuous.
- For every $a > 0$, $\varepsilon(\cdot, a)$ satisfies the assumptions from section 2.2 on page 13 and assumption $(I)$. Denote $\bar{k}(a)$ the perfect information cost for parameter $a$.
- The function is twice continuously differentiable on

$$K := \left\{(k, a)|0 < k < \bar{k}(a)\right\} = \left\{(k, a)|0 < \varepsilon(k, a) < \tfrac{1}{2}\right\},$$

  the area in which information is neither perfect nor meaningless.
- $\varepsilon_a(k, a) := \frac{\partial}{\partial a}\varepsilon(k, a) < 0$ for all $(k, a) \in K$.
- For all $k > 0$ we have $\lim_{a\to\infty} \varepsilon(k, a) = 0$ and $\lim_{a\to 0} \varepsilon(k, a) = \frac{1}{2}$.

Having $\varepsilon(\cdot, a)$ satisfy the same conditions as before, one can use the results of the previous model and perform comparative statics by varying parameter $a$. The fourth point ensures that for increasing $a$, the signal precision for the same search effort becomes higher. Finally, the last item ensures that in the pointwise limit, the error function reflects perfect information (for $a \to \infty$) or the classical lemon market without information acquisition (for $a \to 0$). Hence, it allows us to use the parameter as a mediator between these two widely acknowledged models. One simple example for such a function is

$$\varepsilon(k, a) = \max\left\{\tfrac{1}{2} - a\sqrt{k}, 0\right\}. \tag{2.3}$$

Most expressions of the previous sections now depend on the new parameter. We denote them in the same way but adding the value $a$ as the last argument of every function.

The aim of this section is to compare the various types of equilibria and their level of price, consumer utility, average quality etc. under a change of search efficiency. Is is also interesting to see whether the limit behavior of the error probability function, when taking $a \to \infty$ or $a \to 0$, also leads to market behavior that converges to the equilibria of the classical lemon market or the perfect information case as discussed above.

This analysis is necessarily different from the one of Bester and Ritzberger (2001) since their cost has an exogenously given value and could thus just be directly increased or decreased. Here, the effort level is chosen by a rational consumer. A direct change of the costs can thus not be done. We rather facilitate the access to information by giving more signal precision for the same effort.

### 2.5.1 Analytical Results

The first result analyzes the price behavior under pooling equilibria where no search occurs. Remember that the consumer is indifferent between *searching* and *buying without search* in the pooling price $\underline{p}_\eta$ of these equilibria.

**Proposition 2.5.1.** *The price $\underline{p}_\eta(a)$ of the $PE_b$ equilibrium is continuous, piecewise differentiable and non-increasing in $a$. Moreover,*

$$\lim_{a \to 0} \underline{p}_\eta(a) = \bar{q}_\eta \text{ and } \lim_{a \to \infty} \underline{p}_\eta(a) = q^L$$

*holds. If $q^L < c^H < \bar{q}_\eta$, there are values $0 < \underline{a} \leq \bar{a}$ such that the pooling no-search equilibrium $PE_b$ exists if $a \leq \underline{a}$ and it does not exist for $a \geq \bar{a}$.*

The reason why $\underline{a} \neq \bar{a}$ can not be excluded despite the monotonicity of the price in $PE_b$ is that the condition $\hat{\varepsilon} > 0$ for $PE_b$ to be an equilibrium might be violated for some lower $a$ but be true for a higher search efficiency. This, however, appears for rather special parameters and is not further investigated. A direct consequence of this proposition is that the profit of both firms decreases with increasing search efficiency while the consumer's utility rises in the pooling equilibrium without search. This can be seen by just observing that neither the average quality nor the amount of trade is different between each of these equilibria.

Remember that in the end of the last section we conclude that the existence of $PE_b$ is due to moderate production costs of high quality goods. The proposition provides a similar statement in terms of search costs. Only when search costs are high, pooling no-search equilibria can exist. However, as long as $a < \underline{a}$, making search more efficient lets the equilibrium price decrease and thus gives a higher utility to the consumer. This supports the first intuition that a more efficient way of searching should increase the consumer's power and thus increases her surplus. Note, however, that no search occurs in these equilibria. Instead, all products are sold for a price that decreases with better search efficiency. Here, the possibility of search is rather used as a threat than as a tool. If search gets too efficient, there is no low-price equilibrium and thus only those equilibrium can exist which provide zero consumer utility. In a sense, quality information is too cheap from the view of the consumer, the producer of a high quality product benefits from the higher information level.[17]

One can deduce a similar proposition for the upper bound $\overline{p}_\eta$ of the consumer's search area.

**Proposition 2.5.2.** *For every $\hat{\mu} \in (0,1)$ the function $\overline{p}_{\hat{\mu}}(a)$ is continuous, piecewise differentiable and non-decreasing in $a$. Moreover, we have*

$$\lim_{a\to 0} \overline{p}_{\hat{\mu}}(a) = \bar{q}_\eta \text{ and } \lim_{a\to\infty} \overline{p}_{\hat{\mu}}(a) = q^H.$$

Inferring to equilibrium behavior from this proposition is not as easy as it was before, since both $PE_s$ and $HE$ make use of search prices. It is not obvious which of these equilibria exist for a given $a$. However, noticing that the high price of $HE$ is always higher than $\overline{p}_\eta$, it follows that the equilibrium search price behavior for $a \to \infty$ is not influenced by this question.

**Corollary 2.5.3.** *Let $c^H > q^L$ and $q^L - c^L < q^H - q^L$. For $a \to \infty$, all undominated equilibria converge to a separating state in which the high type sells with probability $1 - \frac{q^L - c^L}{q^H - c^L}$.*

This result is quite interesting in the background of the Grossman-Stiglitz paradox which stems from their 1980 paper. Essentially they argue that in situations in which all arbitrage opportunities are eliminated because all available information is reflected in the prices, there is no incentive for any market participant to obtain this information. Hence the assumption of freely available information would not be justified anymore. While their reasoning originally applied to financial markets, it is in the same way questionable how (in the perfect information case) quality information can be public knowledge when prices perfectly signal the quality and thus the consumer does not need to obtain this information. Here, we see that even with arbitrarily easily accessible information, this paradox does not occur since there is always an incentive for the consumer to obtain information. However, this comes with the cost of the high type not being able to sell with full probability.

If the condition $c^H > q^L$ was violated, the pooling equilibrium without search $PE_b$ would exist for all $a$. The second condition ensures the existence of $HE$ even for high values of $a$. If this inequality is not true, neither of the equilibria $PE_b$, $PE_s$

[17] In a related but weaker result, Bar-Isaak, Caruana and Cu nat (2012)[5] show, in a setting with quality investments, fixed prices and an information acquisition procedure close to Bester and Ritzberger (2001), that consumer utility can be non-monotone in the acquisition costs. The reason for this effects are, however, quite different from our setting.

and $HE$ exist for high values of $a$. It follows that the $TAS$ equilibrium is the only equilibrium and hence efficient information would lead to an even worse outcome for the high quality firm. Since every equilibrium in this setting has only a finite number of prices and we are not interested in the out-of-equilibrium beliefs, convergence of equilibria is just taken as convergence of the firms' actions.

This result is striking in that even though the information costs approach zero, the consumer still makes errors and does not get perfect information. The firm's behavior converges to the separating state that would appear in the perfect information case but the probability of selling does not approach one. Note that there is an important difference to the similar result by Bester and Ritzberger (2001). They show that in their setting, letting the information costs $k$ approach zero, a non-vanishing share of consumers still acquires information but the high type still sells to all buyers that would by the product under perfect information. In contrast, the limit behavior at hand shows a sustainable loss in sales for the high quality type.

In the setting of Voorneveld and Weibull (2011), they show that for the limit of perfect information there exists a continuum of limit equilibria between the perfect information equilibrium and the one given in Corollary 2.5.3. Their result is thus similar but weaker. Moreover, they argue that between these equilibria, the perfect information equilibrium with full sales of high quality Pareto dominates all the other limit situations.

The analysis of equilibria in the direction of high search costs is quite tricky. When search becomes inefficient, the prices of $PE_b$ and $PE_s$ necessarily converge to $\bar{q}_\eta$ and the question if this is an equilibrium depends on whether the high quality production costs exceed this value. If they do, $PE_b$ and $PE_s$ do not exist for low values of $a$ but $TAS$ does. In other words, these equilibria converge to the two possible classic lemon market equilibria with the exact same existence conditions. One might expect that this price convergence is also true for $HE$ since $\overline{p}_{\hat{\mu}}(a)$ is the high quality price. The issue here is that in $HE$, the posterior $\hat{\mu}$ for the search price itself depends on $a$ so that the convergence result from Proposition 2.5.2 does not apply and may, in fact, not be true for the equilibrium price.

**Proposition 2.5.4.** *Let $\eta(q^H - q^L) < q^L - c^L < q^H - q^L$ and $c^H < 2q^L - c^L$. Then, for each value of $a$ which is close enough to $0$, $HE$ exists and for $a \to 0$ these equilibria converge to a semi-separating state with the high type setting $\hat{p} := 2q^L - c^L$ while the low type mixes between this price and $q^L$.*

The limit state of the $HE$ equilibria involves a high price $\hat{p} = 2q^L - c^L$ and a posterior belief $\mu(\hat{p}) = \frac{q^L - c^L}{q^H - q^L}$ while the probability of selling is $\frac{1}{2}$. If $q^H - c^H = q^L - c^L$, this situation corresponds to one of the *semi-mixed equilibria* computed by Ellingsen (1997) in his model without information acquisition. There, this equilibrium exists with many others of a similar type and fails the D1 refinement by Cho and Kreps (1987). However, the behavior here shows that this equilibrium, if it exists, is robust with respect to costly information acquisition.

### 2.5.2 A Numerical Example

To increase the understanding of what happens with various levels of search efficiency, we continue with a concrete example. Even with a relatively simple error function satisfying the assumptions, the model is too complex to solve for explicit expressions of the various equilibria. We thus rely on numerical calculations to illustrate the results on the development of the model outcomes.

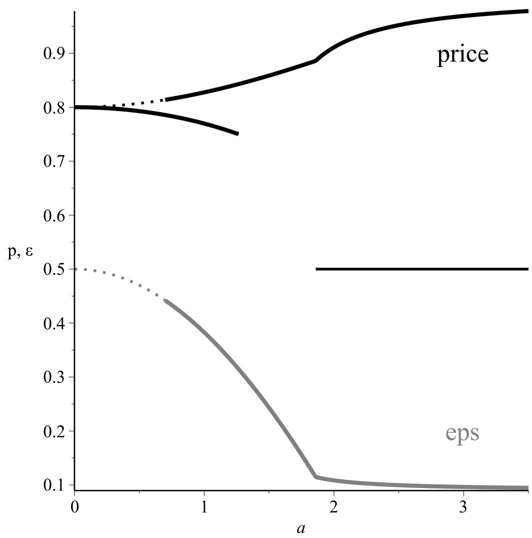


Figure 2.7: Price and error probability development for increasing search efficiency

For these calculations we choose the error function (2.3) and set $\eta = .6$, $q^H = 1$, $q^L = \frac{1}{2}$, $c^H = \frac{3}{4}$ and $c^L = .45$. Note that since $c^H < \bar{q}_\eta = .8$, the existence of pooling no-search equilibria is possible for small values of $a$. Choosing $\eta \geq \frac{1}{2}$ ensures, using Lemma 2.4.17, that $PE_s$ and $HE$ do not exist at the same time to obtain clearer pictures. Corollary 2.5.3 is applicable to this setting but Proposition 2.5.4 is not. We give a second example to illustrate its result.

Figure 2.7 shows the development of the price and the corresponding error probability for the various search prices. The thin black line is the price $q^L$ for the low quality in $HE$. To show the convergence even of the dominated pooling search equilibria, their values are displayed as dotted lines. One observes that for low values of $a$, the pooling no-search equilibrium exists and that its price is decreasing. While at first the other equilibrium is dominated, there is an interval of values of $a$ in which both pooling equilibria exist at the same time until the lower price falls under the production costs of the high type. As search gets more efficient, the error probability decreases and converges to $\frac{q^L - c^L}{q^H - c^L} \approx 0,091$ as predicted by Corollary 2.5.3.

Figure 2.8 illustrates the profit development of the different types in these equilibria.

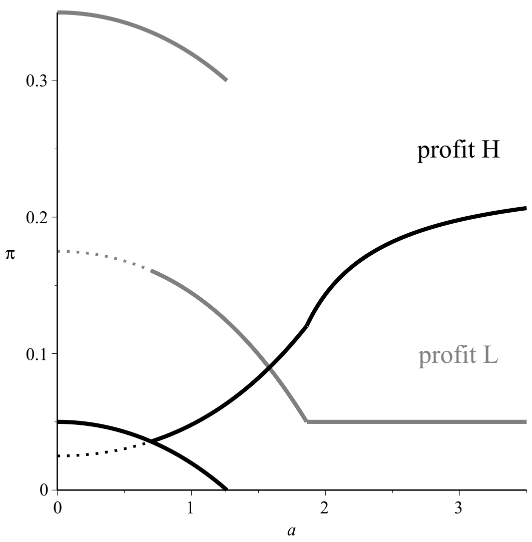


Figure 2.8: The profits of both types, depending on the search efficiency and the equilibrium. Dotted Lines show Pareto dominated equilibria.

One can see very well how profits decrease in $PE_b$ and how $PE_s$ stops being Pareto dominated when its high type's profit catches up with the one from $PE_b$. The low type's profit never falls below $q^L - c^L = .05$ and attains this value in the hybrid equilibrium.

We look at the consumer side of the market in Figure 2.9. The utility of the consumer behaves exactly as predicted, dropping to zero when $PE_b$ does not exist anymore. An interesting effect is observable for the search costs. Although search gets more efficient with increasing $a$, the absolute effort increases in the pooling

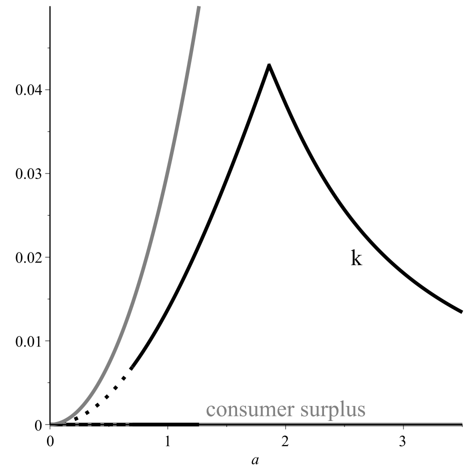


Figure 2.9: Utility and search effort of the consumer

search equilibrium. This effect has already been observed by Bester and Ritzberger (2001) and is reproduced here under a different model of search behavior. It shows how the power given to the consumer by allowing for information acquisition can be exploited by the high type to increase prices, search behavior and thus also the probability of selling the high quality product.

Finally, Figure 2.10 depicts the development of some market figures. Welfare here is simply computed as a sum of the firm's expected profit $\eta\pi^H + (1-\eta)\pi^L$ and the consumer utility. The high welfare value in $PE_b$ stems not only from the higher consumer utility but mostly from the fact that all products are traded with probability one and no utility is "wasted" on search effort. All possible gains from trade are thus exploited and distributed among the market participants.

In contrast to markets with quality uncertainty being famous for their adverse selection phenomena, average traded quality in this setting is even higher than the offered one. This is of course due to the higher trade probabilities for the high type on search prices and thus occurs in $PE_s$ and $HE$ (except for very high search efficiency). As search gets more efficient, the low type shifts its price distribution more to $q^L$ and hence sells with an overall higher probability which causes average traded quality to go down and, because of the not vanishing error probability, to go even below the a priori expected quality.

To also give an example of Proposition 2.5.4, we consider the case where $q^H = 1, q^H = 0.5, c^H = 0.7, c^L = 0.2$ and $\eta = .35$. Note that this also implies that $PE_b$

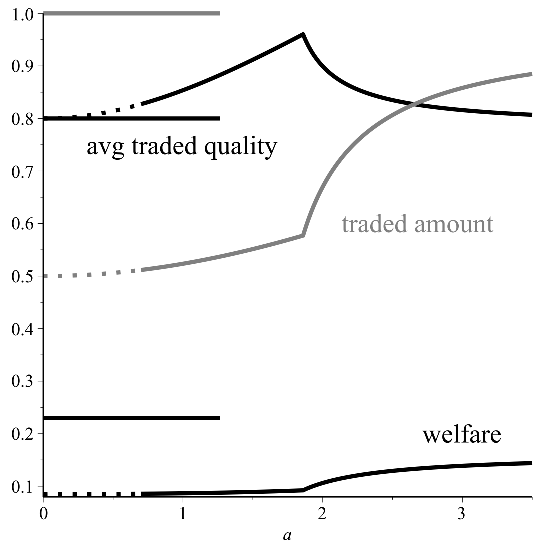


Figure 2.10: The development of various market characteristics

and $PE_s$ do not exist if $a$ is close to 0 because of their low price close to $\overline{q}_\eta = .675$. The values are explicitly chosen so that they fit the model of Ellingsen (1997) in which we have $q^H - c^H = q^L - c^L$. As mentioned before, his refinements forecast a separating equilibrium for this case without information acquisition. This result may serve as a hint that these equilibria might not be disregarded, after all.

## 2.6 Conclusion and Possible Extensions

The paper shows the outcome of a monopolistic market with quality uncertainty in which the consumer has the possibility to costly acquire information about the product quality. This information could be perfect or imperfect, the exact precision depending on the endogenous search effort exerted by the consumer.

Given the optimal consumer behavior, the market offers many possible equilibria, some of which are already present in the classical model without information acquisition. After eliminating implausible and Pareto dominated equilibria, we are left with three main categories of market behavior. In pooling equilibria without search, the consumer has a positive profit and the highest possible welfare was reached. Equilibria which comprise search leave no utility to the consumer but, except for when information is extremely cheap, have an average traded quality that is above the actual average and thus show the opposite effect to the classical adverse selection


Figure 2.11: The convergence of HE for vanishing search efficiency.

results on lemon markets. The third category of total adverse selection occurs when high quality production costs are high and search is very efficient. The analysis shows that information acquisition possibilities only benefit the consumer if she does not acquire any information. If she can use her search abilities as a threat rather than actually acquire costly information, she forces prices to be lower than the average quality and thus have positive utility. In contrast, actual quality search in equilibrium leads to a higher market power of the high type and thus to higher prices. The consumer welfare is zero in these equilibria.

An important contribution of this paper is the comparison of situations with different search efficiencies. We show that an increase in efficiency can benefit or hurt the consumer and that the consumer's utility will with certainty drop to zero after a certain threshold of search efficiency.

At last, the analysis shows that the case of perfect information is not the limit case of high search efficiency. Even when making information acquisition arbitrarily cheap, the probability of consumers receiving a false signal does not vanish. Moreover, this limit error level does not depend on the error function. Making information acquisition inefficient can lead to the same behavior as predicted in the classical models but for a substantial range of parameters the limit equilibrium is one that was previously disregarded.

Starting from this model, certain extensions come to mind to enrich the analysis and lead to a more realistic behavior.

As most other papers in this field, this work does not incorporate competition between multiple firms. The consumer is always confronted with exactly one good and her only choices are on how much to search and if to buy the product. In the same way the firm does not have to worry about actual competition. The type of "rival" it faces exists only theoretically in the head of the consumer who has to figure out which type she is facing. Extending this setting to an oligopolistic market will certainly proof to be interesting.

We only look at one consumer. As we have pointed out, this could be extended to multiple identical consumers without changing the analysis and thus the outcome. In reality, this is not realistic. Besides different valuations or outside options that people might have, the relative efficiency of today's quality information is mainly due to new technologies which in return can not be assumed to be accessible for everyone with the same efficiency, even when restricted to single countries. Some people are more adapt or have better access to these new technologies than others and this difference can be quite severe from one person to another. It is hence important to account for this in a more realistic model.

Other restrictive aspects of the model might be generalized such as the amount of quality levels and signals. Especially this former aspect is important to obtain meaningful results about the relationship between prices and quality in lemon markets with information acquisition.

## 2.A Appendix

*Proof of Lemma 2.3.1.* Fix a price $p$ and a corresponding posterior belief $\hat{\mu} \in (0,1)$, assume that the consumer has paid a cost $k \geq 0$ for the signal precision and denote $\hat{\varepsilon} := \varepsilon(k)$ the error probability. Receiving the high signal $s^H$, the updated posterior belief is

$$\hat{\mu}_H := Prob(q^H|s^H) = \frac{\hat{\mu}(1-\hat{\varepsilon})}{\hat{\mu}(1-\hat{\varepsilon}) + (1-\hat{\mu})\hat{\varepsilon}}$$

which is just Bayes' law applied.

The expected quality with respect to this information is then

$$\bar{q}_{\hat{\mu}_H} := \hat{\mu}_H q^H + \big(1-\hat{\mu}_H\big) q^L.$$

With similar calculations, let $\bar{q}_{\hat{\mu}_L}$ be the expected quality on receiving a low signal. We have

$$\begin{aligned}\hat{\mu}_H - \hat{\mu}_L =& \frac{\hat{\mu}(1-\hat{\varepsilon})}{\hat{\mu}(1-\hat{\varepsilon}) + (1-\hat{\mu})\hat{\varepsilon}} - \frac{\hat{\mu}\hat{\varepsilon}}{\hat{\mu}\hat{\varepsilon} + (1-\hat{\mu})(1-\hat{\varepsilon})} \\ =& \frac{\hat{\mu}(1-\hat{\varepsilon})\left(\hat{\mu}\hat{\varepsilon} + (1-\hat{\mu})(1-\hat{\varepsilon})\right) - \hat{\mu}\hat{\varepsilon}\left(\hat{\mu}(1-\hat{\varepsilon}) + (1-\hat{\mu})\hat{\varepsilon}\right)}{\left(\hat{\mu}\hat{\varepsilon} + (1-\hat{\mu})(1-\hat{\varepsilon})\right)\left(\hat{\mu}(1-\hat{\varepsilon}) + (1-\hat{\mu})\hat{\varepsilon}\right)} \\ =& \frac{\hat{\mu}(1-\hat{\mu})(1-2\hat{\varepsilon})}{\left(\hat{\mu}\hat{\varepsilon} + (1-\hat{\mu})(1-\hat{\varepsilon})\right)\left(\hat{\mu}(1-\hat{\varepsilon}) + (1-\hat{\mu})\hat{\varepsilon}\right)} \geq 0.\end{aligned}$$

The inequality follows from $\hat{\varepsilon} \in [0, \frac{1}{2}]$. Thus

$$\bar{q}_{\hat{\mu}_L} \leq \bar{q}_{\hat{\mu}_H}$$

where equality holds if and only if $\hat{\mu}_H = \hat{\mu}_L$ which is equivalent to $\hat{\varepsilon} = \frac{1}{2}$.

There are now three cases that can occur, regarding the level of the price $p$.

First case: $\bar{q}_{\hat{\mu}_L} < p < \bar{q}_{\hat{\mu}_H}$

This implies that $\hat{\varepsilon} < \frac{1}{2}$, $k > 0$ and that the consumer only buys if she receives the high signal.

Second case: $p \leq \bar{q}_{\hat{\mu}_L}$

The consumer would either by with each signal or mix between "buying" and "not buying" in the case where $p = \bar{q}_{\hat{\mu}_L}$ and the low signal appears. In both cases, the payoff (with search costs and before observing $s$) is

$$\hat{\mu}q^H + \big(1-\hat{\mu}\big)q^L - p - k$$

which clearly has a maximum at $k = 0$.

Third case: $p \geq \bar{q}_{\hat{\mu}_H}$

The consumer would not buy on any signal (while with equality she may buy on $s^H$ but gets utility $\bar{q}_{\hat{\mu}_H} - p - k = -k$) so also here optimality implies $k = 0$.

$\Rightarrow$ If $k > 0$, the consumer buys if and only if the signal is $s^H$. $\square$

*Proof of Lemma 2.3.3.* We write $\hat{k}^* := k^*(p, \hat{\mu})$ and $\hat{\varepsilon} := \varepsilon(\hat{k}^*)$. Note first that with $\hat{k}^* = 0$ we would have $\hat{\varepsilon} = \frac{1}{2}$ and thus

$$u_s^*(p, \hat{\mu}) = \tfrac{1}{2}\hat{\mu}(q^H - p) + \tfrac{1}{2}(1-\hat{\mu})(q^L - p) = \tfrac{1}{2}u_b(p, \hat{\mu}) \leq \max\left\{u_b(p, \hat{\mu}), u_n\right\}$$

which contradicts the assumptions. Define

$$E_H := \frac{\hat{\mu}(1-\hat{\varepsilon})}{\hat{\mu}(1-\hat{\varepsilon})+(1-\hat{\mu})\hat{\varepsilon}}q^H + \frac{(1-\hat{\mu})\hat{\varepsilon}}{\hat{\mu}(1-\hat{\varepsilon})+(1-\hat{\mu})\hat{\varepsilon}}q^L - p \tag{2.4}$$

$$E_L := \frac{\hat{\mu}\hat{\varepsilon}}{\hat{\mu}\hat{\varepsilon}+(1-\hat{\mu})(1-\hat{\varepsilon})}q^H + \frac{(1-\hat{\mu})(1-\hat{\varepsilon})}{\hat{\mu}\hat{\varepsilon}+(1-\hat{\mu})(1-\hat{\varepsilon})}q^L - p \tag{2.5}$$

the expected utility from buying, disregarding the sunk search costs, when receiving a high signal or a low signal, respectively.

The following two important relations are immediate from these formulas.

$$\begin{aligned} u_s^*(p,\hat{\mu}) =& \hat{\mu}(1-\hat{\varepsilon})(q^H-p)+(1-\hat{\mu})\hat{\varepsilon}(q^L-p)-\hat{k}^* \\ =& \hat{\mu}(1-\hat{\varepsilon})q^H+(1-\hat{\mu})\hat{\varepsilon}q^L-(\hat{\mu}(1-\hat{\varepsilon})+(1-\hat{\mu})\hat{\varepsilon})p-\hat{k}^* \\ \overset{(2.4)}{=}& (\hat{\mu}(1-\hat{\varepsilon})+(1-\hat{\mu})\hat{\varepsilon})E_H-\hat{k}^* \end{aligned} \tag{2.6}$$

$$\begin{aligned} u_s^*(p,\hat{\mu}) =& \hat{\mu}(1-\hat{\varepsilon})(q^H-p)+(1-\hat{\mu})\hat{\varepsilon}(q^L-p)-\hat{k}^* \\ =& \hat{\mu}(q^H-p)+(1-\hat{\mu})(q^L-p)-\hat{\mu}\hat{\varepsilon}(q^H-p)-(1-\hat{\mu})(1-\hat{\varepsilon})(q^L-p)-\hat{k}^* \\ =& \bar{q}_{\hat{\mu}}-p-\big[\hat{\mu}\hat{\varepsilon}q^H+(1-\hat{\mu})(1-\hat{\varepsilon})q^L-\big(\hat{\mu}\hat{\varepsilon}+(1-\hat{\mu})(1-\hat{\varepsilon})\big)p\big]-\hat{k}^* \\ \overset{(2.5)}{=}& \bar{q}_{\hat{\mu}}-p-\big(\hat{\mu}\hat{\varepsilon}+(1-\hat{\mu})(1-\hat{\varepsilon})\big)E_L-\hat{k}^* \end{aligned} \tag{2.7}$$

Assume that the signal is not precise enough in the sense of the lemma. This can have two reasons. Either the expected value from buying is below zero even when receiving a high signal ($E_H \leq 0$) or it is above zero even on receiving a low signal ($E_L \geq 0$).

In the first case, equation (2.6) implies $u_s^*(p,\hat{\mu}) < 0 = u_n$ which contradicts the conditions of the lemma.

In the second case, equation (2.7) implies

$$u_s^*(p,\hat{\mu}) = \bar{q}_{\hat{\mu}}-p-\underbrace{\Big(\big(\hat{\mu}\hat{\varepsilon}+(1-\hat{\mu})(1-\hat{\varepsilon})\big)E_L+\hat{k}^*\Big)}_{>0} < \bar{q}_{\hat{\mu}}-p = u_b(p,\hat{\mu})$$

which is a contradiction for the same reason.

These contradictions prove that $E_L < 0 < E_H$. Hence the consumer buys if and only if she receives the high signal. □

*Proof of Lemma 2.3.4.* We begin by finding values $\underline{p}_{\hat{\mu}} \leq \overline{p}_{\hat{\mu}}$ such that the strict inequalities hold.

$$\begin{aligned} u_b(p_1,\hat{\mu}) &> \max\{u_n, u_s^*(p_1,\hat{\mu})\} && \forall\, p_1 \in [q^L, \underline{p}_{\hat{\mu}}) \\ u_s^*(p_2,\hat{\mu}) &> \max\{u_b(p_2,\hat{\mu}), u_n\} && \forall\, p_2 \in (\underline{p}_{\hat{\mu}}, \overline{p}_{\hat{\mu}}) \\ u_n &> \max\{u_b(p_3,\hat{\mu}), u_s^*(p_3,\hat{\mu})\} && \forall\, p_3 \in (\overline{p}_{\hat{\mu}}, q^H]. \end{aligned} \tag{2.8}$$

It is obvious from $\hat{\mu} \in (0,1)$ that the first inequality is satisfied for $p_1 = q^L$ and the last inequality for $p_3 = q^H$. It is thus only left to show that there is monotonic behavior in $p$ in the pairwise differences between $u_b(p,\hat{\mu}), u_s^*(p,\hat{\mu})$ and $u_n$. We show that the inequalities

$$\tfrac{\partial}{\partial p} u_b(p,\hat{\mu}) < \tfrac{\partial}{\partial p} u_s^*(p,\hat{\mu}) < \tfrac{\partial}{\partial p} u_n$$

hold on the interval $(q^L, q^H)$ wherever $k^*$ and thus $u_s^*$ is differentiable in $p$. The left and right component of this expression are obvious from their definitions.

$$\tfrac{\partial}{\partial p} u_b(p,\hat{\mu}) = \tfrac{\partial}{\partial p}(\hat{\mu} q^H + (1-\hat{\mu})q^L - p) = -1 \qquad \tfrac{\partial}{\partial p} u_n = 0$$

By the shape of the function $k^*$ (on page 18), we know that $u_s^*(p,\hat{\mu})$ is continuous and piecewise differentiable in $p$ on some (possibly empty) intervals $(q^L, p')$, $(p', p'')$, $(p'', q^H)$ and we have

$$u_s^*(p,\hat{\mu}) = \begin{cases} u_s(p,\hat{\mu},0) = \hat{\mu}\tfrac{1}{2}(q^H - p) + (1-\hat{\mu})\tfrac{1}{2}(q^L - p) & \text{if } p \in I_1 \\ u_s\Big(p,\hat{\mu},\big(\varepsilon'\big)^{-1}(\hat{d})\Big) & \text{if } p \in I_2 \\ u_s(p,\hat{\mu},\bar{k}) = \hat{\mu}(q^H - p) - \bar{k} & \text{if } p \in I_3. \end{cases}$$

In this expression we have $I_2 = (p', p'')$ and $I_1$, $I_3$ are the sets $[q^L, p')$ and $(p'', q^H]$.[18]

While the first and third components are easy to differentiate, the middle one becomes

$$\begin{aligned} &\tfrac{\partial}{\partial p} u_s\Big(p,\hat{\mu},\big(\varepsilon'\big)^{-1}(\hat{d})\Big) \\ =&\tfrac{\partial}{\partial p}\Big(\hat{\mu}\big(1-\varepsilon(k^*)\big)(q^H - p) + (1-\hat{\mu})\varepsilon(k^*)(q^L - p) - k^*\Big) \\ =&-\hat{\mu}(1-\hat{\varepsilon}) - \hat{\mu}(q^H - p)\tfrac{\partial}{\partial p}\varepsilon(k^*) - (1-\hat{\mu})\hat{\varepsilon} + (1-\hat{\mu})(q^L - p)\tfrac{\partial}{\partial p}\varepsilon(k^*) - \tfrac{\partial}{\partial p}k^* \end{aligned}$$

[18] Their order depends on the value of $\hat{\mu}$ which determines whether search effort increases or decreases in $p$. For $\hat{\mu} = \frac{1}{2}$, search effort is constant and two of the intervals are empty.

$$
\begin{aligned}
&= -\hat{\mu}(1-\hat{\varepsilon}) - (1-\hat{\mu})\hat{\varepsilon} + \tfrac{1}{d}\tfrac{\partial}{\partial p}\varepsilon(k^*) - \tfrac{\partial}{\partial p}k^* \\
&= -\hat{\mu}(1-\hat{\varepsilon}) - (1-\hat{\mu})\hat{\varepsilon} + \tfrac{1}{d}\varepsilon'(k^*)\tfrac{\partial}{\partial p}k^* - \tfrac{\partial}{\partial p}k^* \\
&= -\hat{\mu}(1-\hat{\varepsilon}) - (1-\hat{\mu})\hat{\varepsilon} + \tfrac{1}{d}\hat{d}\tfrac{\partial}{\partial p}k^* - \tfrac{\partial}{\partial p}k^* \\
&= -\hat{\mu}(1-\hat{\varepsilon}) - (1-\hat{\mu})\hat{\varepsilon} \in (\hat{\varepsilon}-1, -\hat{\varepsilon}),
\end{aligned}
$$

using the chain rule, omitting the arguments of $k^*$ and writing $\hat{\varepsilon} := \varepsilon(k^*)$.
Summarized, we end up with the following expression

$$
\frac{\partial}{\partial p}u_s^*(p,\hat{\mu}) = \begin{cases} -\frac{1}{2} & \text{if } p \in I_1 \\ -\hat{\mu}(1-\hat{\varepsilon}) - (1-\hat{\mu})\hat{\varepsilon} & \text{if } p \in I_2 \\ -\hat{\mu} & \text{if } p \in I_3. \end{cases}
$$

which is always strictly between $-1$ and $0$ and even continuous. This proves the inequalities (2.8).

It follows from Lemma 2.3.3 that the consumer really searches in the region $(\underline{p}_{\hat{\mu}}, \overline{p}_{\hat{\mu}})$. Finally, the price $p = \bar{q}_{\hat{\mu}}$ implies

$$
u_n = u_b(p,\hat{\mu})
$$

so that we must have $\underline{p}_{\hat{\mu}} \leq \bar{q}_{\hat{\mu}} \leq \overline{p}_{\hat{\mu}}$ for (2.8) to be true. □

*Proof of Lemma 2.3.5.* Set $\hat{p} = \bar{q}_{\hat{\mu}}$. Clearly $u_b(\hat{p},\hat{\mu}) := \bar{q}_{\hat{\mu}} - \hat{p} = 0 = u_s(\hat{p},\hat{\mu},0)$ holds. Moreover, we have

$$
\frac{\partial}{\partial k}u_s(\hat{p},\hat{\mu},k) = -\varepsilon'(k)\hat{\mu}(q^H - \hat{p}) + \varepsilon'(k)(1-\hat{\mu})(q^L - \hat{p}) - 1
$$

for all $k \in (0, \overline{k})$ and thus, taking the limit $k \to 0$

$$
\begin{aligned}
\lim_{k\to 0}\frac{\partial}{\partial k}u_s(\hat{p},\hat{\mu},k) &= -\varepsilon'(0)\left[\hat{\mu}(q^H - \hat{p}) + (1-\hat{\mu})(\hat{p} - q^L)\right] - 1 \\
&\overset{\hat{p}=\bar{q}_{\hat{\mu}}}{=} -\varepsilon'(0)\big[\hat{\mu}\left(q^H - (\hat{\mu}q^H + (1-\hat{\mu})q^L)\right) \\
&\qquad +(1-\hat{\mu})\left((\hat{\mu}q^H + (1-\hat{\mu})q^L) - q^L\right)\big] - 1 \\
&= -\varepsilon'(0)(2\hat{\mu}(1-\hat{\mu})(q^H - q^L)) - 1.
\end{aligned}
$$

This is positive by the assumption. Hence, $u_s^*(\hat{p},\hat{\mu}) > u_s(\hat{p},\hat{\mu},0) = u_b(\hat{p},\hat{\mu}) = u_n$ which proofs $q_{\hat{\mu}} = \hat{p} \in (\underline{p}_{\hat{\mu}}, \overline{p}_{\hat{\mu}})$.

Assume now that the inequality stated in the lemma is not true. Then, by the calculations above and the strict concavity of $\varepsilon$, $u_s(\hat{p}, \hat{\mu}, k)$ is decreasing in $k$ so that $k = 0$ is the optimal choice of search effort. Thus

$$u_s^*(\hat{p}, \hat{\mu}) = u_s(\hat{p}, \hat{\mu}, 0) = u_b(\hat{p}, \hat{\mu}) = u_n.$$

From the proof of Lemma 2.3.4, we know that

$$\frac{\partial}{\partial p} u_b(p, \hat{\mu}) < \frac{\partial}{\partial p} u_s^*(p, \hat{\mu}) < \frac{\partial}{\partial p} u_n$$

for all $p \in (q^L, q^H)$ and thus $u_b(p, \hat{\mu}) > u_s^*(p, \hat{\mu})$ for all $p < \bar{q}_{\hat{\mu}}$ and $u_n > u_s^*(p, \hat{\mu})$ for all $p > \bar{q}_{\hat{\mu}}$. Hence we have $\underline{p}_{\hat{\mu}} = \overline{p}_{\hat{\mu}} = \bar{q}_{\hat{\mu}}$. $\square$

*Proof of Lemma 2.3.6.* We only show the claim for $\underline{p}_{\hat{\mu}}$ since the other part is basically the same proof with even simpler arguments. Note that, given $\hat{\mu} \in (0, 1)$, $\underline{p}_{\hat{\mu}}$ is uniquely determined[19] by solving

$$\begin{aligned} u_b(p, \hat{\mu}) &= u_s^*(p, \hat{\mu}) \\ \Leftrightarrow \quad \hat{\mu} q^H + (1 - \hat{\mu}) q^L - p &= \hat{\mu}(1 - \varepsilon)(q^H - p) + (1 - \hat{\mu})\varepsilon(q^L - p) - k^* \end{aligned} \tag{2.9}$$

and since these expressions are continuous and piecewise differentiable, the function $\underline{p}_{\hat{\mu}}$ also has these properties. [20]

In the areas of differentiability, we either have $k^*(p, \hat{\mu}) = 0$ (implying $\varepsilon = \frac{1}{2}$), $k^*(p, \hat{\mu}) = \bar{k}$ (with $\varepsilon = 0$) or $k^*(p, \hat{\mu}) = (\varepsilon')^{-1}\big(d(p, \hat{\mu})\big)$. In the first two cases, (2.9) yields

$$\underline{p}_{\hat{\mu}} = \bar{q}_{\hat{\mu}} = \hat{\mu} q^H + (1 - \hat{\mu}) q^L \quad \text{or} \quad \underline{p}_{\hat{\mu}} = q^L + \frac{\bar{k}}{1 - \hat{\mu}}$$

which both induce a strictly positive derivative in $\hat{\mu}$.

[19] This is also true if $\underline{p}_{\hat{\mu}} = \bar{q}_{\hat{\mu}}$. From Lemma 2.3.4 it follows that in this case $k^*(p, \hat{\mu}) = 0, \varepsilon(k^*(p, \hat{\mu})) = \frac{1}{2}$ and the equation (2.9) holds.

[20] For the differentiability, the only problem occurs on the set

$$\{\hat{\mu} \in [0, 1] \mid d(p_{\hat{\mu}}, \hat{\mu}) = \varepsilon'(0) \text{ or } d(p_{\hat{\mu}}, \hat{\mu}) = \varepsilon'(\bar{k})\}$$

in which the possible non-differentiable points of $k^*$ are touched. By continuity, this set is closed and hence compact. It can thus be written as the union of finitely many open intervals and finitely many singletons. Within these intervals, the differentiation for $k^*(p, \hat{\mu}) = \bar{k}$ or $k^*(p, \hat{\mu}) = 0$ applies. The singletons are the only candidates in which $p_{\hat{\mu}}$ may not be differentiable in $\hat{\mu}$.

In the third case, differentiating (2.9) with respect to $\hat{\mu}$ yields

$$\begin{aligned}
q^H - q^L - \underline{p}'_{\hat{\mu}} =&(1-\hat{\varepsilon})(q^H - \hat{p}) + \hat{\varepsilon}(\hat{p} - q^L) - \hat{\mu}(\tfrac{\partial}{\partial \hat{\mu}}\varepsilon)(q^H - \hat{p}) \\
&+ (1-\hat{\mu})(\tfrac{\partial}{\partial \hat{\mu}}\varepsilon)(q^L - \hat{p}) - \underline{p}'_{\hat{\mu}}(\hat{\mu}(1-\hat{\varepsilon}) + (1-\hat{\mu})\hat{\varepsilon}) - \tfrac{\partial}{\partial \hat{\mu}}k^* \\
=&(1-\hat{\varepsilon})(q^H - \hat{p}) + \hat{\varepsilon}(\hat{p} - q^L) + \tfrac{1}{\hat{d}}(\tfrac{\partial}{\partial \hat{\mu}}\varepsilon) \\
&- \underline{p}'_{\hat{\mu}}(\hat{\mu}(1-\hat{\varepsilon}) + (1-\hat{\mu})\hat{\varepsilon}) - \tfrac{\partial}{\partial \hat{\mu}}k^*
\end{aligned}$$

where we left out the arguments for $d, \varepsilon$ and $k^*$ and wrote $\hat{p} = \underline{p}_{\hat{\mu}}, \hat{\varepsilon} = \varepsilon(k^*(\hat{p}, \hat{\mu}))$.

Reordering this equation and using $\frac{\partial}{\partial \hat{\mu}}\varepsilon = \varepsilon'(\hat{k}^*)\frac{\partial}{\partial \hat{\mu}}k^* = \hat{d} \cdot \frac{\partial}{\partial \hat{\mu}}k^*$, we get

$$\underline{p}'_{\hat{\mu}} = \frac{\hat{\varepsilon}(q^H - \hat{p}) + (1-\hat{\varepsilon})(\hat{p} - q^L)}{1 - \hat{\mu}(1-\hat{\varepsilon}) - (1-\hat{\mu})\hat{\varepsilon}} > 0.$$

The limit behavior $\lim_{\hat{\mu}\to 0} \underline{p}_{\hat{\mu}} = q^L$ is clear, since we have $q^L < \underline{p}_{\hat{\mu}} \leq \overline{q}_{\hat{\mu}}$ for all values $\hat{\mu} \in (0,1)$.

For $\hat{\mu}$ going to one, note that the convergence of $\underline{p}_{\hat{\mu}}$ is guaranteed by the strict monotonicity. Since $k^*$ can not be higher than $q^H$ for the equation (2.9) to be true, it is bounded and hence there is an increasing sequence $(\hat{\mu}_n)$, converging to 1, for which $k^*(\underline{p}_{\hat{\mu}_n}, \hat{\mu}_n)$ converges to a value $\kappa \geq 0$. Take such a sequence and the limit $n \to \infty$ in (2.9). We then obtain

$$\begin{aligned}
q^H - \lim_{\hat{\mu}\to 1} \underline{p}_{\hat{\mu}} &= \big(1 - \varepsilon(\kappa)\big)\left(q^H - \lim_{\hat{\mu}\to 1} \underline{p}_{\hat{\mu}}\right) - \kappa \\
\Leftrightarrow \qquad \varepsilon(\kappa)\left(q^H - \lim_{\hat{\mu}\to 1} \underline{p}_{\hat{\mu}}\right) &= -\kappa
\end{aligned}$$

which, since the left hand side is weakly positive, implies $\kappa = 0$, thus $\varepsilon(\kappa) = \frac{1}{2}$ and finally

$$\lim_{\hat{\mu}\to 1} \underline{p}_{\hat{\mu}} = q^H.$$

□

*Proof of Lemma 2.4.4.* As seen from section 2.3, the consumer would buy for any price below $q^L$. This shows that no such price $p < q^L$ can be part of an equilibrium because a deviation to any price in $(p, q^L)$ would yield a higher payoff. This shows the lower bound of i).

Assume that the equilibrium profit $\pi^L$ for the low type is strictly below $q^L - c^L$. A deviation to the price $q^L - \frac{q^L - c^L - \pi^L}{2}$ would then yield the profit

$$q^L - \frac{q^L - c^L - \pi^L}{2} - c^L = \frac{q^L - c^L}{2} + \frac{\pi^L}{2} > 2\frac{\pi^L}{2} = \pi^L.$$

This concludes the proof of statement iii). Having the low type set the price $q^H$ with positive probability, the equilibrium definition implies $\mu(q^H) < 1$ and thus it is optimal for the consumer to not buy the product, implying zero profit for the low type. This contradicts iii). Using an obvious similar argument, we conclude that the low type can not have any price above $q^H$ in its support. This shows ii) and the rest of i).

It is left to show statement iv). Let $p \in (q^L, q^H)$ be a price that is in the support of $a_L$ but not of $a_H$. By the equilibrium definition we must have $\mu(p) = 0$ so the consumer would know the true quality when observing price $p$. Hence, he would not buy the product and the low type would make no profit which contradicts the previous point.

Assume now that $p \in (q^L, q^H)$ is a price in the support of $a_H$ but not of $a_L$. Since $\mu(p) = 1$ and $p < q^H$, the consumer buys with probability 1. So there must be at least one price $p_L > p, p_L \in \text{supp}(a_L)$, otherwise the low type would deviate from any price to $p$. Since only consistent strategies are played in equilibrium, we are left with two cases.

First case: $b\,(p_L) = (0, \gamma, \gamma), \gamma \in [0, 1]$

By optimality of the low type's strategy, this price satisfies

$$p - c^L \leq (p_L - c^L)\gamma. \tag{2.10}$$

That is, the low type must make at least as much profit with setting price $p_L$ than with price $p$. By the previous part of the proof, $p_L$ is in the support of $a_H$. Hence the high type must be indifferent between setting prices $p$ or $p_L$.

$$\begin{aligned} & & p - c^H &= (p_L - c^H)\gamma \\ \overset{(2.10)}{\Rightarrow} & & (p_L - c^L)\gamma + c^L - c^H &\geq (p_L - c^H)\gamma \\ \Leftrightarrow & & (1-\gamma)c^L &\geq (1-\gamma)c^H \end{aligned}$$

which, since the first equation also implies $\gamma \neq 1$, is not compatible with the assumption $c^L < c^H$.

Second case: $b(p_L) = (k, 1, 0), k > 0$
Let $\hat{\varepsilon} := \varepsilon(k)$ be the probability of a false signal. This equals the chance that the low type will sell her product for the price $p_L$. Again by optimality of the low type's choice the following inequality holds.

$$p - c^L \leq (p_L - c^L)\hat{\varepsilon} \tag{2.11}$$

As before, the price $p_L$ must also be in the support of the high type. This yields

$$\begin{aligned} p - c^H &= (p_L - c^H)(1 - \hat{\varepsilon}) \\ \overset{(2.11)}{\Rightarrow} \quad (p_L - c^L)\hat{\varepsilon} + c^L - c^H &\geq (p_L - c^H)(1 - \hat{\varepsilon}) \\ \Leftrightarrow \quad \hat{\varepsilon}(p_L - c^H) &\geq (1 - \hat{\varepsilon})(p_L - c^L) \end{aligned}$$

which gives a contradiction for the same reason as before and using $\hat{\varepsilon} < 1 - \hat{\varepsilon}$. □

*Proof of Lemma 2.4.5.* Let $p < p'$ be two such prices, $\hat{\varepsilon}, \hat{\varepsilon}'$ the corresponding error probabilities. For search to be possible, the prices have to be in both supports and hence, by the indifference principle for both types,

$$\begin{gathered} (p - c^H)(1 - \hat{\varepsilon}) = (p' - c^H)(1 - \hat{\varepsilon}') \Rightarrow (1 - \hat{\varepsilon}) > (1 - \hat{\varepsilon}') \Leftrightarrow \hat{\varepsilon} < \hat{\varepsilon}' \\ (p - c^L)\hat{\varepsilon} = (p' - c^L)\hat{\varepsilon}' \Rightarrow \hat{\varepsilon} > \hat{\varepsilon}' \end{gathered}$$

which gives a contradiction. □

*Proof of Lemma 2.4.6.* Without search, there are two probabilities $\gamma, \gamma' \in [0, 1]$ of the consumer buying for the prices $p < p'$. By the optimality of the firm's strategy we have

$$\begin{aligned} (p - c^H)\gamma &= (p' - c^H)\gamma' \\ (p - c^L)\gamma &= (p' - c^L)\gamma'. \end{aligned}$$

Note that, since the low type always has positive profit, all of these factors must be strictly above zero. Reassembling these equations gives two different values for the ratio $\frac{\gamma}{\gamma'}$ since $p \neq p'$ and $c^L \neq c^H$.[21] This is a contradiction. □

[21] Here, we use that the fraction $\frac{p'-c}{p-c}$ for $p \neq p'$ is strictly monotone in $c$ for $c < p, p'$. This statement is easy to check via differentiation.

*Proof of Lemma 2.4.8.* Assume such a price $p \in (q^L, q^H)$ exists for which the consumer buys with probability lower than one. Then, because of Lemma 2.4.4 iv) and iii), *not buying* is not possible so she buys with probability $\gamma \in (0,1)$. She is thus indifferent between buying without search and not buying. In this case $p = \bar{q}_{\mu(p)}$ holds. Since $\varepsilon'(0) = -\infty$, she would search on that price by Lemma 2.3.5.

The case $q = q^L$ follows from the same argument as the proof of Lemma 2.4.4 iii). If sales had less than full probability, a slightly lower price would yield full sales and hence a higher profit. □

*Proof of Proposition 2.4.9.* The previous lemmas already show that not more than two equilibrium prices can exist in $(q^L, q^H)$. We then show that there can not be a search price $p_s$ and a no-search price $p_1$ played by both types. If this was the case, we must have $p_1 < p_s$ since otherwise there would be incentives to deviate from $p_s$ to $p_1$. For this, remember that Lemma 2.4.8 shows that the price $p_1$ induces sure buying.

Let now be $p_1$ and $p_s$ be played in an equilibrium by both types. Applying the indifference principle for both firms we get

$$\begin{aligned} p_1 - c^H &= (1-\hat{\varepsilon})(p_s - c^H) \\ p_1 - c^L &= \hat{\varepsilon}(p_s - c^L) \end{aligned}$$

and thus

$$1 - \hat{\varepsilon} = \frac{p_1 - c^H}{p_s - c^H} \overset{\substack{p_1 < p_s \\ c^H > c^L}}{<} \frac{p_1 - c^L}{p_s - c^L} = \hat{\varepsilon}$$

which contradicts $\hat{\varepsilon} \in [0, \frac{1}{2}]$.

Finally, note that if the price $p_1$ is played, the consumer buys with certainty and $p_1$ is thus the lowest price in the equilibrium. Thus, $q^L \notin \text{supp}(a_L)$.

The statement about the low value of $\gamma$ for $b(q^H) = (0, \gamma, \gamma)$ is obvious. If we denote the low type's equilibrium profit by $\pi^L$, one upper bound for $\gamma$ is $\frac{\pi^L}{q^H - c^L}$ which is strictly smaller than one. Note that there might also be a lower bound for this value, e.g. if $c^H < q^L$. See for example the existence condition for the total adverse selection equilibrium. □

*Proof of Lemma 2.4.11.* Let $(a, \mu, b)$ be an equilibrium in which $q^H$ is played by the high firm and the consumer has strategy $b\left(q^H\right) = (0, \gamma, \gamma)$, $\gamma > 0$. This is a best response because of $\hat{\mu} := \mu\left(q^H\right) = 1$ and thus the consumer's utility

$$\gamma \cdot u_b(q^H, \hat{\mu}) + (1-\gamma) \cdot \underbrace{u_n}_{=0} = \gamma(\hat{\mu} q^H + (1-\hat{\mu})q^L - q^H) = \gamma(1-\hat{\mu})(q^L - q^H)$$

attains its maximum in all values of $\gamma$. For slightly lower $\hat{\mu}$, however, this value has a unique maximum in $\gamma = 0$ and thus the original strategy $b\left(q^H\right)$ is locally dominated by $(0,0,0)$. On the other hand, the total adverse selection equilibrium does not have this problem since $(0,0,0)$ is the equilibrium strategy for $q^H$.

Every other equilibrium price is a search price $p_s$ or a non-search price $p_1$ being in both supports of the firm's strategy or the price $q^L$, set by only the low type. We show that non of these prices is locally dominated in beliefs.

Let $p_s$ be an equilibrium search price (implying that $\mu(p_s) \in (0,1)$) with consumer behavior $b(p_s) = (k, 1, 0), k > 0$. By the analysis of section 2.3, this strategy is the unique maximum over all search behaviors. The remaining candidates for domination are thus $(0,0,0)$ ("don't buy") and $(0,1,1)$ ("buy")[22]. If "don't buy" had the same utility, we had

$$\begin{aligned} u_s(p_s, \mu(p_s), k) &= \mu(p_s)\big(1-\varepsilon(k)\big)(q^H - p_s) + \big(1-\mu(p_s)\big)\varepsilon(k)(q^L - p_s) - k \\ &= 0 = u_n. \end{aligned}$$

Differentiating this with respect to the posterior belief, one sees that

$$\frac{\partial}{\partial m} u_s(p_s, m, k) = \big(1-\varepsilon(k)\big)(q^H - p_s) + \varepsilon(k)(p_s - q^L) > 0$$

such that we have $u_s(p_s, m, k) > u_n$ for all $m > \mu(p_s)$. The strategy $b(p_s)$ is thus not locally dominated by the strategy $(0,0,0)$.

If "buy" had the same utility as $b$, we had

$$\begin{aligned} u_s(p_s, \mu(p_s), k) &= \mu(p_s)\big(1-\varepsilon(k)\big)(q^H - p_s) + \big(1-\mu(p_s)\big)\varepsilon(k)(q^L - p_s) - k \\ &= \mu(p_s)q^H + (1-\mu(p_s))q^L - p_s = u_b(p_s, \mu(p_s)). \end{aligned}$$

[22]Note that the proof shows that also their convex combinations can not be local best responses in this case and thus are no candidates for dominating strategies.

and the derivatives

$$\frac{\partial}{\partial m}u_s(p_s,m,k) = \big(1-\varepsilon(k)\big)(q^H - p_s) + \varepsilon(k)(p_s - q^L) \qquad (2.12)$$
$$\leq \max\left\{q^H - p_s, p_s - q^L\right\} < q^H - q^L = \frac{\partial}{\partial m}u_b(p_s,m).$$

This shows that $b(p_s)$ is strictly better than $(0,1,1)$ for any $m < \mu(p_s)$. The strategy $b(p_s)$ is thus not locally dominated.

Now let $p_1 < q^H$ be an equilibrium price. If $b(p_1) = (0,1,1)$ is not a unique best response, there is a search strategy $(k,1,0)$ with the same payoff[23]. This also implies $\mu(p_1) \in (0,1)$ since search being optimal is not possible for degenerate posteriors. Using (2.12), we know that for a marginally higher posterior belief, this search strategy is worse than "buy". Strategy $b(p_1)$ is thus not locally dominated.

The last price to check is $q^L$ for which the equilibrium behavior $(0,1,1)$ is clearly the unique best response for any posterior belief $m > 0$ so that local domination is also excluded here. This also concludes the proof for showing that the total adverse selection equilibrium has belief-robust responses. □

*Proof of Proposition 2.4.13.* Let $(a,\mu,b)$ be an equilibrium. We first show that there are no equilibrium prices $p$ which differ from $q^L, q^H, \underline{p}_{\hat{\mu}}$ and $\overline{p}_{\hat{\mu}}$ where $\hat{\mu}$ is determined according to Bayes' law.

First, assume that there is an equilibrium no-search price $p$ in $(q^L, q^H)$ which is not equal to $\underline{p}_{\mu(p)}$. In this case, Lemma 2.3.4 shows that $u_b(p,\mu(p)) > u_s^*(p,\mu(p))$. Assuming that $\mu$ is continuous in $p$, the continuity of $u_b$ and $u_s^*$ implies that "buying" will still be better than "searching" for a marginal increase of the price. This is an incentive for both types to deviate which contradicts the equilibrium property. If $p = q^L$ but played by both types (thus $\mu(p) > 0$), the non-emptiness of $(q^L, \underline{p}_{\mu(p)})$ shows that the same argument holds.

Second, assume the existence of an equilibrium search price $p \neq \overline{p}_{\mu(p)}$. It follows again from Lemma 2.3.4 that we must have $u_s^*(p,\mu(p)) > u_n$. Again the continuity of these expressions implies that the consumer will also search for a marginal higher price, although the search effort and thus the error probability $\hat{\varepsilon}$ might change. Note that the profit of the firm, depending on the type, is $(p - c^H)(1-\hat{\varepsilon})$ or $(p - c^L)\hat{\varepsilon}$ such that, for a higher price, at least one of these values will increase. This gives an incentive for at least one type to deviate.

[23] Lemma 2.3.5 rules out no-search strategies giving the same payoff.

These two arguments together with Lemma 2.4.11 rule out all the equilibria from Proposition 2.4.9 that are not mentioned in Proposition 2.4.13. We thus only have to show that the rest of the equilibria can be supported by a locally continuous belief system.

Since all equilibria have a finite number of prices, these prices can be considered independently from each other by looking at non-intersecting environments of them. The price $q^L$, if only played by the low quality firm, can obviously be supported by $\mu(p) = 0$ in any environment of $q^L$. For the price $q^H$ note that for all values of $\hat{\mu} = \mu(q^H)$ the interval $(\overline{p}_{\hat{\mu}}, q^H)$ is non-empty and the lower bound is continuously increasing in $\hat{\mu}$ (see Lemma 2.3.6) so that there is an invertible, continuous function $p(\hat{\mu})$ with $p(\hat{\mu}) \in (\overline{p}_{\hat{\mu}}, q^H)$ for all $\hat{\mu} \in (0,1)$. By the definition of $\overline{p}_{\hat{\mu}}$, the inverse $\mu(p)$ of this function satisfies

$$u_n > \max\bigl\{u_b(p,\mu(p)), u_s^*(p,\mu(p))\bigr\} \quad \forall p \in \bigl(p(\tfrac{1}{2}), q^H\bigr)$$

so that the consumer would not buy with that belief system for any other price and hence there is no incentive for a deviation by any firm.

Let now $p_s$ be an equilibrium search price on the upper line of the search area. That is

$$p_s = \overline{p}_{\mu(p_s)}.$$

Keeping the function $\mu$ constant above $p_s$ leads to the consumer not buying for higher prices. This was shown in the proof of Lemma 2.4.11. For lower prices, we can use the same argument as before of $\overline{p}_{\hat{\mu}}$ being strictly increasing in $\hat{\mu}$ to show that there exists a continuous and even increasing belief system $\mu(p)$ for $p < p_s$ for which the consumer does not buy on lower prices.

At last, assume $p_1 = \underline{p}_{\mu(p)}$ for a no-search price, implying that the consumer buys with probability one. Moreover, assume $\varepsilon(k^*(p_1, \mu(p_1))) < 1$. Independent of the belief system, no firm would deviate to a lower price since the probability of selling can not grow ($p_1$ induces sure buying). Keeping the posterior belief constant to $\mu(p_1)$ for higher prices leads to search behavior for these prices. By continuity of $\varepsilon(k^*)$, we have

$$\lim_{p \downarrow p_1} (1 - \varepsilon(k^*(p, \mu(p))))(p - c^H) = (1 - \varepsilon(k^*(p_1, \mu(p_1))))(p_1 - c^H) \overset{\text{ass.}}{<} p_1 - c^H$$

so that a high quality firm has a lower profit for slightly higher prices. It is straightforward to show the same for the low type.

The existence conditions are obvious from the proof, the minimum payoffs of both types and the indifference principle for all prices in a type's price support. □

*Proof of Observation 2.4.14.* In $PE_b$, both types trade for sure while in $PE_s$ the high type firm has some trading probability $1-\hat{\varepsilon} > \hat{\varepsilon}$. It suffices to show the observation for any hybrid equilibrium.

Remember that the existence of a hybrid equilibrium implies $q^L - c^L < q^H - q^L$. In this case, the selling probability for the price $q^H$ in a separating equilibrium can not exceed

$$\frac{q^L - c^L}{q^H - c^L} = \frac{q^L - c^L}{q^H - q^L + q^L - c^L} < \frac{q^L - c^L}{q^L - c^L + q^L - c^L} = \frac{1}{2}.$$

to not make the low firm deviate to the higher price. The selling probability for a high quality firm in a hybrid equilibrium, however, is $1 - \hat{\varepsilon} > \frac{1}{2}$ which proofs the observation. □

*Proof of Lemma 2.4.16.* If two or more hybrid equilibria exist, the consumer surplus and the low type's profit are both at their minimum. Observe that the latter implies that the search precision $\varepsilon(k^*(\overline{p}_{\hat{\mu}}, \hat{\mu}))$ in these equilibria is higher if the price $\overline{p}_{\hat{\mu}}$ is higher. This also implies a higher chance of selling high quality goods and thus an overall strictly higher profit for the high type. Hence, $HE$ dominates all other hybrid equilibria.

Let either of $PE_b$, $PE_s$ or $HE$ exist. We know that in all these equilibria we have $\pi^L \geq q^L - c^L$, $\pi^H \geq 0$ and $u^* \geq 0$ which are the payoffs of the $TAS$ equilibrium. In $PE_b$, the consumer has positive utility while in $PE_s$ and $HE$, the high type firm has positive profit. Hence $TAS$ is dominated.

Now let $EQ$ either denote $PE_s$ or $HE$ and denote $p_s$ and $\hat{\varepsilon}$ the search price and its corresponding signal imprecision in $EQ$. We know that the consumer is strictly better off in $PE_b$. Assume $\pi^H(PE_b) \geq \pi^H(EQ)$. It then follows, since $p_s \geq \underline{p}_\eta$

$$\begin{aligned} & \frac{\underline{p}_\eta - c^L}{p_s - c^L} \geq \frac{\underline{p}_\eta - c^H}{p_s - c^H} = \frac{\pi^H(PE_b)}{\frac{1}{1-\hat{\varepsilon}}\pi^H(EQ)} \geq 1 - \hat{\varepsilon} > \hat{\varepsilon} \\ \Rightarrow \qquad & \pi^L(PE_b) = \underline{p}_\eta - c^L > \hat{\varepsilon} \cdot (p_s - c^L) = \pi^L(EQ). \end{aligned}$$

This shows that $PE_b$ dominates $EQ$.

It is straightforward to show that $HE$ and $PE_s$ do not dominate each other. The low type profit in $PE_s$ is higher than in $HE$ but this, together with the higher search price in $HE$, implies a lower sale probability for the low type and hence a higher probability for the high type. Hence the profit of the high type is higher in $HE$. $\square$

*Proof of Lemma 2.4.17.* Note first that the existence of $PE_s$ implies that the low type's profit $\hat{\varepsilon} \cdot (\overline{p}_\eta - c^L)$ is at least $q^L - c^L$. In the hybrid equilibrium, this bound must be attained for the low type to justify playing both prices. We show that this condition can not be met in both equilibria.

Taking the derivative of $\pi_L(\overline{p}_{\hat{\mu}}, \hat{\mu})$ with respect to the posterior belief $\hat{\mu}$ (for values in which this is differentiable) yields

$$\begin{aligned}
&\frac{\partial}{\partial \hat{\mu}} \pi_L(\overline{p}_{\hat{\mu}}, \hat{\mu}) \\
=&\frac{\partial}{\partial \hat{\mu}} \varepsilon(k^*(\overline{p}_{\hat{\mu}}, \hat{\mu})) \cdot (\overline{p}_{\hat{\mu}} - c^L) \\
=&\left(\frac{\partial}{\partial \hat{\mu}} \varepsilon(k^*(\overline{p}_{\hat{\mu}}, \hat{\mu}))\right)(\overline{p}_{\hat{\mu}} - c^L) + \varepsilon \overline{p}'_{\hat{\mu}} \\
=&\begin{cases} 0 & \text{if } \varepsilon = 0 \\ \varepsilon'(k^*)\left(\left[(\varepsilon')^{-1}\right]'(\hat{d})\frac{\partial}{\partial \hat{\mu}} d(\overline{p}_{\hat{\mu}}, \hat{\mu})\right)(\overline{p}_{\hat{\mu}} - c^L) + \varepsilon \overline{p}'_{\hat{\mu}} & \text{if } \varepsilon > 0 \end{cases} \\
=&\begin{cases} 0 & \text{if } \varepsilon = 0 \\ \varepsilon'(k^*)\frac{1}{\varepsilon''(k^*)}\hat{d}^2\left(q^H - q^L - 2\overline{p}_{\hat{\mu}} + \overline{p}'_{\hat{\mu}}(1 - 2\hat{\mu})\right)(\overline{p}_{\hat{\mu}} - c^L) + \varepsilon \overline{p}'_{\hat{\mu}} & \text{if } \varepsilon > 0 \end{cases}.
\end{aligned}$$

For this, we had to use standard results for the derivative of the inverse function $\left[(\varepsilon')^{-1}\right]' = \frac{1}{\varepsilon''}$ and the quotient differentiation theorem for the derivative of $d$.

While the right hand side of the lower term is always positive, the left hand side is positive if $\hat{\mu} \geq \frac{1}{2}$ and $\overline{p}_{\hat{\mu}} \geq \frac{q^H + q^L}{2}$. Note that the former condition implies the latter as we always have $\overline{p}_{\hat{\mu}} \geq \overline{q}_{\hat{\mu}}$. It is thus sufficient to have $\hat{\mu} \geq \frac{1}{2}$. Moreover, in an equilibrium where the high type always sets the search price, Bayes' law implies $\hat{\mu} \geq \eta$ on that price. The derivative of the low type's profit it thus always non-negative for $\hat{\mu} \in [\frac{1}{2}, 1)$. It is even strictly positive for all beliefs in which $\varepsilon(\overline{p}_{\hat{\mu}}, \hat{\mu}) > 0$ which must be the case in equilibrium. If $\eta \geq \frac{1}{2}$, this monotonicity results hold for all posterior beliefs that can occur on equilibrium search prices.

By Lemma 2.3.6, this profit converges to $\frac{1}{2}(q^H - c^L)$ when taking $\hat{\mu} \to 1$. Using the condition $q^L - c^L < q^H - q^L$, we know

$$\tfrac{1}{2}(q^H - c^L) = \tfrac{1}{2}(q^H - q^L + q^L - c^L) > q^L - c^L.$$

Let $\eta \geq \frac{1}{2}$ and $\varepsilon(k^*(\overline{p}_\eta, \eta))(\overline{p}_\eta - c^L) < q^L - c^L$. It then follows from the strict monotonicity and the convergence that there exists exactly one $\hat{\mu} \in (\eta, 1)$ with $\varepsilon(k^*(\overline{p}_{\hat{\mu}}, \hat{\mu}))(\overline{p}_{\hat{\mu}} - c^L) = q^L - c^L$. On the other hand, if we had $\varepsilon(k^*(\overline{p}_\eta, \eta))(\overline{p}_\eta - c^L) \geq q^L - c^L$, we have no such value for $\hat{\mu}$.

We see that the conditions of the low type having a higher profit than $q^L - c^L$ in one equilibrium and exactly this profit in the other are mutually exclusive. By continuity, this extends to an open interval of values of $\eta$ below $\frac{1}{2}$ which proofs the existence of $\underline{\eta}$. □

*Proof of Proposition 2.5.1.* In this proof, we often write $p(a)$ instead of $\underline{p}_{\hat{\mu}}(a)$ for expositional reasons. Define

$$P := \big\{(p, a) | d(p, \eta) < \varepsilon_k(\bar{k}(a), a)\big\}$$

the open area for which $\varepsilon(k^*(p, \eta, a), a)$ is positive.

Note that $k^*(p, \eta, a)$ as defined by (2.1) is continuous[24] and piecewise differentiable in $(p, a)$ in $P$ and its complement $P^c$. It thus also holds for the composition $\varepsilon(k^*, a)$. Hence $\underline{p}_\eta(a)$ has the same properties[25] , being the unique implicit solution of the equation

$$\begin{aligned} u_b(p(a), \eta, a) &= u_s^*(p(a), \eta, a) \\ \bar{q}_\eta - p(a) &= \eta\big(1 - \varepsilon(k^*, a))\big)\big(q^H - p(a)\big) + (1 - \eta)\varepsilon(k^*, a)\big(q^L - p(a)\big) - k^* \end{aligned} \tag{2.13}$$

[24]While $\bar{k}$ can take the value $\infty$, $k^*$ can not. Since $\bar{k}$ is continuous when restricted to the open set on which it is finite, $k^*$ is continuous.

[25]A similar differentiability argument as in footnote 20 on page 49 applies here. The set

$$\{a | d(p(a), \eta) = \varepsilon_k(\bar{k}(a), a)\}$$

might not be bounded and hence not compact. But every intersection with $[0, A], A > 0$ is compact. The previous argument applies to these sets and hence there are countably many, ordered potential discontinuities $a_1 < a_2 < \dots$ .

where we left out the arguments of $k^*$. Differentiating this expression with respect to $a$, in the areas where this is differentiable, and writing $p_a := \frac{\partial}{\partial a} p(a)$ yields

$$\begin{aligned}
&- p_a \\
&= \eta \left[ (-\tfrac{\partial}{\partial a}\varepsilon)(q^H - p) + (1-\varepsilon)(-p_a) \right] + (1-\eta) \left[ (\tfrac{\partial}{\partial a}\varepsilon)(q^L - p) + \varepsilon(-p_a) \right] - \tfrac{\partial}{\partial a} k^* \\
&= -\Big( \eta(1-\varepsilon) + (1-\eta)\varepsilon \Big) p_a - \Big( \eta(q^H - p) + (1-\eta)(p - q^L) \Big) \tfrac{\partial}{\partial a}\varepsilon - \tfrac{\partial}{\partial a} k^* \\
&= -\Big( \eta(1-\varepsilon) + (1-\eta)\varepsilon \Big) p_a + \tfrac{1}{d(p,\eta)} \tfrac{\partial}{\partial a}\varepsilon - \tfrac{\partial}{\partial a} k^*
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{\partial}{\partial a}\varepsilon = \tfrac{\partial}{\partial a}\varepsilon(k^*, a) &= \varepsilon_k(k^*, a) \tfrac{\partial}{\partial a} k^*(p(a), a) + \varepsilon_a(k^*, a) \\
&= \varepsilon_k \frac{\partial}{\partial a} k^* + \varepsilon_a \\
&= d(p, \eta) \tfrac{\partial}{\partial a} k^* + \varepsilon_a.
\end{aligned}$$

whenever $(p(a), a) \in P$ and

$$\tfrac{\partial}{\partial a}\varepsilon = \tfrac{\partial}{\partial a} 0 = 0$$

in every open subset of $P^c$. Combining these expressions, we either get

$$\begin{aligned}
- p_a &= -\Big( \eta(1-\varepsilon) + (1-\eta)\varepsilon \Big) p_a + \frac{\varepsilon_a}{d} \\
\Leftrightarrow \quad p_a &= \frac{\eta(q^H - p) + (1-\eta)(p - q^L)}{1 - \eta(1-\varepsilon) - (1-\eta)\varepsilon} \varepsilon_a < 0
\end{aligned}$$

or

$$p_a = \frac{1}{1 - \eta(1-\varepsilon) - (1-\eta)\varepsilon} \tfrac{\partial}{\partial a} \bar{k}(a) \leq 0.$$

It is left to show the limit of $p(a)$ when we let $a$ go to 0 or $\infty$. We begin with the latter case.

<u>First claim:</u> $\lim_{a \to \infty} \varepsilon(k^*(\underline{p}_\eta(a), \eta, a), a) = 0$

We write $k^*(a) := k^*(\underline{p}_\eta(a), \eta, a)$. Assume that $\limsup_{a \to \infty} \varepsilon(k^*(a), a) =: e > 0$. This also implies

$$\varepsilon_k(k^*(a_n), a_n) = d(\underline{p}_\eta(a_n), \eta) \text{ and } \varepsilon(k^*(a_n), a_n) > \tfrac{e}{2} \quad \forall\, n \in \mathbb{N}$$

for some sequence $(a_n)$ going to infinity and having $\lim_{n\to\infty} \varepsilon(k^*(a_n), a_n) = e$. Since $\underline{p}_\eta(a_n)$ converges (due to the monotonicity), so does $d(\underline{p}_\eta(a_n), \eta)$ and we have

$$\lim_{n\to\infty} \varepsilon_k(k^*(a_n), a_n) = \lim_{n\to\infty} d(\underline{p}_\eta(a_n), \eta) =: \delta \le \frac{-1}{\max\{\eta, 1-\eta\}(q^H - q^L)} < 0$$

which implies $\varepsilon_k(k^*(a_n), a_n) > 2\delta$ for large $n$. Choose $k = -\frac{e}{8\delta} > 0$ and $n$ large enough such that this inequality holds. We then have for all these $n$ either

$$\varepsilon(k, a_n) \ge \varepsilon(k^*(a_n), a_n) > \frac{e}{2}$$

if $k \le k^*(a_n)$ or

$$\begin{aligned} \varepsilon(k, a_n) &= \varepsilon(k^*(a_n), a_n) + \int_{k^*(a_n)}^{k} \underbrace{\varepsilon_k(l, a_n)}_{\ge \varepsilon_k(k^*(a_n), a_n) > 2\delta} dl > \frac{e}{2} + (k - k^*(a_n))2\delta \\ &\ge \frac{e}{2} + k2\delta = \frac{e}{2} - \frac{e}{4} = \frac{e}{4} \end{aligned}$$

otherwise. This is a contradiction to $\lim_{n\to\infty} \varepsilon(k, a_n) = 0$.

Second claim: $\lim_{a\to\infty} k^*(\underline{p}_\eta(a), \eta, a) = 0$

The argument here is almost the same. Assume $\limsup_{a\to\infty} k^*(\underline{p}_\eta(a), \eta, a) =: \kappa > 0$. Take a sequence $a_n$, $\lim_{n\to\infty} a_n = \infty$ with $\lim_{n\to\infty} k^*(a_n) = \kappa$ and $k^*(a_n) > \frac{\kappa}{2}$ for all $n$. Let $\delta$ be as before, and let $n$ be large enough so that the inequality $\varepsilon_k(k^*(a_n), a_n) > 2\delta$ holds. We then have

$$\begin{aligned} \varepsilon(\tfrac{\kappa}{2}, a_n) =& \varepsilon(k^*(a_n), a_n) - \int_{\frac{\kappa}{2}}^{k^*(a_n)} \varepsilon_k(l, a_n) dl \ge \varepsilon(k^*(a_n), a_n) - \left(k^*(a_n) - \tfrac{\kappa}{2}\right) \tfrac{\delta}{2} \\ \ge& - \left(k^*(a_n) - \tfrac{\kappa}{2}\right) \tfrac{\delta}{2} \to -\tfrac{\kappa}{2}\tfrac{\delta}{2} > 0, \quad n \to \infty, \end{aligned}$$

contradicting $\lim_{n\to\infty} \varepsilon(\frac{\kappa}{2}, a_n) = 0$.

From the two claims, it now follows easily by equation (2.13) that

$$\begin{aligned} \bar{q}_\eta - \lim_{a\to\infty} \underline{p}_\eta(a) &= \eta(q^H - \lim_{a\to\infty} \underline{p}_\eta(a)) \\ \Rightarrow \qquad \lim_{a\to\infty} \underline{p}_\eta(a) &= q^L. \end{aligned}$$

This concludes the proof for the case $a \to \infty$.

The proof for $a \to 0$ is quite similar. Using basically the same arguments, we show that $\lim_{a\to 0}\varepsilon(k^*(p(a),a),a) = \frac{1}{2}$ and that $\lim_{a\to 0} k^*(p(a),a) = 0$. Hence the limit of equation (2.13) yields

$$\begin{aligned}
& \bar{q}_\eta - \lim_{a\to 0} \underline{p}_\eta(a) = \frac{1}{2}\eta(q^H - \lim_{a\to 0} \underline{p}_\eta(a)) + \frac{1}{2}(1-\eta)(q^L - \lim_{a\to 0} \underline{p}_\eta(a)) \\
\Rightarrow\quad & \bar{q}_\eta - \lim_{a\to 0} \underline{p}_\eta(a) = \frac{1}{2}\bar{q}_\eta - \frac{1}{2}\lim_{a\to 0} \underline{p}_\eta(a) \\
\Rightarrow\quad & \qquad \lim_{a\to 0} \underline{p}_\eta(a) = \bar{q}_\eta.
\end{aligned}$$

The last part of the proposition follows from this convergence of $\underline{p}_\eta$ (so that for low values, the price of $PE_b$ is above $c^H$ and for high $a$ it is not) and the previously shown

$$\lim_{a\to 0} \varepsilon(k^*(p(a),a),a) = \tfrac{1}{2} > 0$$

which ensures that $\varepsilon > 0$ for low values of $a$. This is part of the existence condition for $PE_b$. □

*Proof of Proposition 2.5.2.* The arguments here are basically the same as in the previous proof, using that $\overline{p}_\eta(a)$ is implicitly defined by the equation

$$\begin{gathered}
u_n = u_s^*(p(a),\eta,a) \\
0 = \eta\big(1-\varepsilon(k^*,a))\big)\big(q^H - p(a)\big) + (1-\eta)\varepsilon(k^*,a)\big(q^L - p(a)\big) - k^*.
\end{gathered}$$

The derivative is thus either

$$\frac{\eta(q^H-p)+(1-\eta)(p-q^L)}{-\eta(1-\varepsilon)-(1-\eta)\varepsilon}\varepsilon_a > 0 \text{ or } \frac{1}{-\eta(1-\varepsilon)-(1-\eta)\varepsilon}\frac{\partial}{\partial a}\bar{k}(a) \geq 0.$$

The arguments for the convergence to $\bar{q}_\eta$ and $q^H$ are again very similar to the previous proof and are thus omitted. □

*Proof of Corollary 2.5.3.* It follows from the convergence of $\underline{p}_\eta(a)$ that $PE_b$ does not exist for high values of $a$, since its price would be lower than the high quality production costs $c^H$ from some point on. The search prices of both, $PE_s$ and $HE$, converge to $q^H$. The proof of Proposition 2.5.2 shows that for $PE_s$, the corresponding signal error $\varepsilon\left(k^*(\overline{p}_\eta(a),\eta),a\right)$ converges to 0 when $a$ goes to infinity. Thus for high values of $a$ we have

$$\varepsilon\left(k^*(\overline{p}_\eta(a),\eta),a\right)(\overline{p}_\eta(a)-c^L) < q^L - c^L \tag{2.14}$$

so that $PE_s$ does not exist. The convergence of $\overline{p}_\eta(a)$ to $q^H$ also shows that for high values of $a$ we must have $\overline{p}_\eta(a) > c^H$. For each of these values of $a$, since the left hand side of (2.14) converges to $\frac{1}{2}(q^H - c^L) > q^L - c^L$ when $\eta$ goes to one, there exists $\hat{\mu} \in (\eta, 1)$ so that

$$\varepsilon\left(k^*(\overline{p}_{\hat{\mu}}(a),\hat{\mu}),a\right)(\overline{p}_{\hat{\mu}}(a)-c^L) = q^L - c^L.$$

This constitutes the existence of a hybrid equilibrium and thus of $HE$.

The above equality combined with the limit behavior

$$q^H \geq \overline{p}_{\hat{\mu}}(a) \geq \overline{p}_\eta(a) \to q^H, \quad a \to \infty$$

implies the convergence of $\varepsilon\left(k^*(\overline{p}_{\hat{\mu}}(a),\hat{\mu})\right)$ to $\frac{q^L-c^L}{q^H-c^L}$.

□

*Proof of Proposition 2.5.4.* From the proof of Proposition 2.5.2 we know that

$$\lim_{a\to 0}\overline{p}_\eta(a) = \overline{q}_\eta,\ \lim_{a\to 0}k^*(\overline{p}_\eta(a),\eta,a) = 0 \text{ and } \lim_{a\to 0}\varepsilon(k^*(\overline{p}_\eta(a),\eta,a),a) = \frac{1}{2}.$$

The low type profit in $PE_s$ thus converges to

$$\begin{aligned}\lim_{a\to 0}\varepsilon\left(k^*(\underline{p}_\eta(a),\eta,a),a\right)(\overline{p}_\eta(a)-c^L) &= \tfrac{1}{2}(\overline{q}_\eta - c^L)\\ &= \tfrac{1}{2}(\eta q^H + (1-\eta)q^L - c^L)\\ &= \tfrac{1}{2}(\underbrace{\eta(q^H - q^L)}_{<q^L-c^L} + q^L - c^L)\\ &< q^L - c^L\end{aligned}$$

which shows that for low $a$ this equilibrium type does not exist. It follows from the proof of Lemma 2.4.17 that for each such $a$ there is at least one belief $\hat{\mu}(a) > \eta$ for which the low type exactly attains the profit $q^L - c^L$ at price $\hat{p}(a) := \overline{p}_{\hat{\mu}(a)}(a)$ if the consumer behaves optimally. Writing $\hat{k}^*(a) := k^*(\hat{p}(a), \hat{\mu}(a), a)$, this means

$$\underbrace{\varepsilon(\hat{k}^*(a), a)}_{=:\hat{\varepsilon}(a)} \cdot (\hat{p}(a) - c^L) = q^L - c^L.$$

We can use a similar argument to the one in the proof of Proposition 2.5.1 to show that

$$\lim_{a \to 0} \hat{k}^*(a) = 0 \text{ and } \lim_{a \to 0} \hat{\varepsilon}(a) = \frac{1}{2}.$$

The above equality then dictates that $\lim_{a \to 0} \hat{p}(a) = 2q^L - c^L$ and, since the posterior belief satisfies

$$\hat{\mu}(a)(1 - \hat{\varepsilon}(a))(q^H - \hat{p}(a)) + (1 - \hat{\mu}(a))\hat{\varepsilon}(a)(q^L - \hat{p}(a)) - \hat{k}^*(a) = 0$$

for all $a$, taking the limit and applying the result yields $\lim_{a \to 0} \hat{\mu}(a) = \frac{q^L - c^L}{q^H - q^L}$.

Finally, the condition $c^H < 2q^L - c^L = \lim_{a \to 0} \hat{p}(a)$ ensures that these prices indeed form hybrid equilibria for low values of $a$. $\square$

# 3 A Model of Quality Uncertainty with a Continuum of Quality Levels

## 3.1 Introduction

Markets with quality uncertainty have been well discussed in the recent decades, starting from the famous paper by George Akerlof (1970). Since then, many articles have formalized the idea in different ways, most of which focused on a particular market feature to implement into the classical model. Some works like Bagwell and Riordan (1991) enriched the market by introducing multiple periods and thus letting the market price not only be determined by equilibrium posterior beliefs but also by past experience of the consumers. Others focused on advertising possibilities in terms of wasteful spending and thus costly signaling (Milgrom and Roberts (1986)) or on the possibility of the consumers to receive additional information before the purchase (Bester and Ritzberger (2001), Voorneveld and Weibull (2011), Martin (2012) and the previous chapter). Some efforts were made in transfering the monopolistic setting into a model with multiple sellers. See Adriani and Deidda (2011) for a case with finitely many sellers and buyers. Daughety and Reinganum (2007) consider a duopolistic setting in which the good differs in a "safety" aspect. Wilson (1980) introduced a setting with a continuum of sellers and buyers.

Most of the literature has an assumption in common which seems innocuous. While quality is modeled to be unknown to the consumer, it can only have finitely many different values in the real numbers. In most cases, there is only a "good" and a "bad" quality level. Two objections directly arise to this assumption. For one, when we think about the quality of a car, we think of many different aspects which are relevant and enter the computation. *Performance, safety, handling, comfort* are only some broad categories, each of which could be split into multiple characteristics of a car. Quality should thus intuitively be something multidimensional. However, it is widely known that under relatively mild assumptions, preferences over multidimensional objects can be expressed by a von Neumann utility function and thus the comparison can be made in the real numbers. One sure has to be careful of whether even these weak assumptions apply to all real-life situations but in this chapter we do not focus on relaxing this assumption.

The other objection, which is the more severe one, is the assumption of finitely many quality levels. Certainly, some characteristics, like the resolution of a TV screen, only have finitely many possible values but others, like its life period or the quality of its colors, would better be modeled on a continuous scale. Most of the literature ignores this aspect, the predominant reasons being the mathematical simplicity, expositional benefits and the idea that two quality levels are enough to capture the relevant market effects.

This chapter takes a closer view at the last point. Is it really the case that having a continuum of quality levels does not lead to qualitatively different phenomena compared to only two possible values? Is this true in every model or could some positive answers to this question hide other issues which occur only when the setting is enhanced?

We present a model with quality uncertainty and a continuum of quality levels that resembles the classical monopolistic model of quality uncertainty as similarly stated in Ellingsen (1997). We show two examples in which under "regular" assumptions, having many quality values either leads to undetermined behavior or does not add interesting phenomena to the comparable model with only two quality levels.

We then continue modifying the model by adding private information to the consumer. When receiving a free signal which is correlated to the true quality, there naturally arise mathematical problems when trying to update beliefs about the quality distribution in a mathematically correct way. The form of the objective function of a firm bears the problem that the type space can not directly be split into convex subsets, all in which types set the same price. Consequently, Bayesian updating can be impossible or at best highly complex for the consumer to realize.

To overcome this issue, we introduce an elegant generalization of building an expected quality level, demanding Bayesian updating only in the easiest cases and otherwise allowing for non-perfectness or (to some degree) irrationality of the consumer while at the same time preserving the possibility of full rationality.

Analyzing the structure of equilibria, we characterize their pricing function and find that there is always a positive prize-quality relationship in every equilibrium. Moreover, adverse selection phenomena do in general not occur. Since profits are non-decreasing in the quality, only low quality types can completely be excluded from trade. We further investigate the limit behavior when the consumer's information becomes perfect, i.e. the signal precision approaches perfect information. We show that in this case, the market breaks down uniformly over all existing equilibria. Furthermore, the proof shows that this effect is mainly caused by the interval structure of available quality levels.

The paper is structured as follows. We shortly present the model before we show two cases with a continuous quality set but with only one-sided asymmetric quality information. We show that these models do not provide interesting or previously not known behavior. We then proceed by analyzing the model with two-sided asymmetric information. After defining a generalization of expected quality with respect to Bayesian updating, we analyze the equilibria of the market. Interesting aspects of equilibria can be found already at this stage. Applying a refinement to these equilibria, we finally find that approaching the perfect information case drives low quality firms out of the market and leads to market breakdown in equilibrium.

## 3.2 The Model

We consider a minimalistic market with one firm and one consumer. The firm (or *seller*) produces and offers an indivisible good of random quality $q \in [0,1]$, unobserved by the consumer. The consumer (or *buyer*) can buy this good for a certain price which is set by the firm as a take-it-or-leave it offer. For each quality, the buyer has a certain, publicly known utility $u(q)$. For simplicity, we normalize $u(q) = q$ and speak equivalently of the firm's *quality* or *type*.

This type $q$ is drawn by nature by a distribution on $[0,1]$ with a continuous, everywhere-positive density function $f$. This distribution is known by the consumer, while the realized quality is not. The price $p$ is set by the firm after observing the quality $q$. The action set of a firm is the set of all price functions

$$\begin{aligned} \pi : [0,1] &\to [0,1] \\ q &\mapsto \pi(q). \end{aligned}$$

The consumer buys at most one unit of the good. In addition to the price, she observes a signal $s$ before the purchase decision. This signal is costless and can be interpreted as the private observation of a test result or of the result of an own quality information acquisition process with a fixed cost.[1] Having the realized quality level $q$, the signal is uniformly distributed on the interval $[q-\kappa, q+\kappa]$ and hence depends on the true quality $q$. The error variable $\kappa$ is fixed, strictly positive and known to the seller and the buyer. Denote $S := [-\kappa, 1+\kappa]$ the set of possible signal realizations.

The buyer is a risk-neutral utility maximizer. Observing the price and the signal and having built an expectation $E(p,s)$ of the realized type, her expected utility is

$$E(p,s) - p$$

from buying the good and 0 otherwise. Whenever these values are equal, she buys with some indifference probability $\alpha \in [0,1]$, chosen by her. The strategy of the consumer can thus be characterized by this value.

We need some notation for the analysis. We denote the complete Lebesgue measure on $\mathbb{R}$ by $\lambda$. In Particular, a set $A \subset \mathbb{R}$ is called a null set if and only if there exists a Borel set $B$ with $\lambda(B) = 0$ and $A \subset B$. Having two sets $A$ and $B$, we denote $A \triangle B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference of these two sets. If $A, B \neq \emptyset$, we use the notation

$$A < B \quad \Leftrightarrow \quad a < b \;\; \forall\, a \in A, b \in B.$$

An element $a$ is a limit point of the set $A$ if there exists a sequence $(a_n)$ in $A$ with $\lim_{n\to\infty} a_n = a$.

## 3.3 One-sided Asymmetric Information

Before we deal with the model, we consider the simpler case in which the consumer does not get the additional signal but only observes the price before making the buying decision. This would be the natural extension of the standard lemon market models. Two types with the same pricing strategy then have the same chance of selling since the buyer receives the identical information and hence behaves the same. From the optimality in an equilibrium[2], each type's pricing strategy must maximize

[1] For example, if you always do a test drive before buying a second hand car, the resulting information is available to you and the (fixed) cost of the test drive does not enter your utility maximization considerations.

[2] In this section, we speak of Bayesian equilibria without giving the formal definition. Updating behavior is rather easy in these cases (as long as the price function is well-behaved) and the optimality conditions of seller's and buyer's behavior is obvious. Since all the results in this section state *necessary* properties of equilibria and do not deal with existence, we do not have to worry about out-of-equilibrium beliefs.

the payoffs. Since there are no payoff differences between types, every strategy which is used by some type yields the same payoff. Note that for each price and without further information, the consumer reaction can only be “not buying”, “buying” or “buying with probability $\alpha$” where $\alpha$ can not differ between prices. Since every price of every pricing strategy must yield the same payoff, this leaves only two possible prices for each equilibrium.

**Proposition 3.3.1.** *Let there be no extra signal for the consumer. Then in every equilibrium in which some type makes positive profit, there are at most two prices* $p = \alpha p'$ *where* $\alpha \in (0, 1)$ *is the consumer's indifference strategy.*

It is interesting to notice that the order of types setting these two prices is not clearly determined. From the consumer reaction it is clear that the set of types setting the high price $p'$ must yield the expected quality $p'$ because the buyer uses its indifference strategy $\alpha$. In the same way, the expected quality from the set of types setting price $p$ must be strictly above $p$. Each constellation which satisfies these assumptions constitutes an equilibrium. This, however, is not very restrictive and allows for many types of behavior, all of which only involve two prices but can have positive or negative price correlation. One example of such a setting is shown in Figure 3.1.

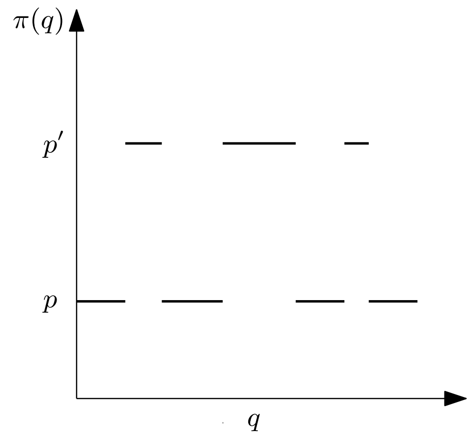


Figure 3.1: An example of a possible price function in the case without additional signal.

This behavior might actually stem from some of the other restrictions we make about the market. In particular, we assume one value $\alpha$ for all consumer reactions in which she is indifferent. Instead, one might think about allowing a different reaction for each price in which neither buying nor the absence from the purchase is the unique best reply. The result of only having two prices certainly stems from this restriction.

In the same spirit, quality-depending production costs (or outside options) could be present in the market which implies that the same price yields not only the same chance of selling but not the same profit for all types setting the price. This is what drives the high indeterminacy of the pricing function which was observed above. However, although getting rid of these restrictions does indeed help to overcome this behavior, it does not lead to new insights.

**Proposition 3.3.2.** *Let $c : [0,1] \to \mathbb{R}_+$ be a strictly increasing cost function and let the consumer strategy have the more general form $\gamma : [0,1] \to [0,1]$. Then in every equilibrium, if one exists, the price function is monotonically increasing and $\gamma$ is strictly decreasing when being restricted to the equilibrium prices $\pi^{-1}([0,1])$.*

Knowing the results in the classical two-quality case, this statement is not very surprising and does not provide anything new to the matter. The monotonicity of the price function admits a positive price-quality relationship. This, in combination with the decreasing willingness of the consumer to buy with higher prices, also implies an adverse selection effect. Higher quality has a higher price and thus a lower chance of selling.

We could generalize this even more and allow the firm to have a mixed strategy. One can see in the proof that this modification would not change the result.

This detour shows that generalizing the standard model in a way *just* to incorporate a continuum of quality levels does not enrich the results in any way. Our model component of having the extra signal $s$ is thus crucial for the following analysis and results. We now go back to the model presented in the previous section.

## 3.4 The Consumer

The notion of consumer's utility involves the building of an expectation based on the observed price and signal. The question, of course, is how this expectation is formed. If we followed classical Bayesian theory, a buyer would observe her information, in this case the price $p$ and the signal $s$, and then hold a posterior belief $\mu(p,s) \in \Delta[0,1]$ about the actual product's quality. In an equilibrium, this probability distribution would be derived by Bayes' law whenever $p$ and $s$ correspond to at least one possible quality realization, given the signal distribution and the equilibrium price function $\pi$. While this works well in settings with finitely many quality levels, there are issues in our model that can not easily be overcome when sticking to this posterior belief assumption. In particular, the relatively unrestricted shape of the function $\pi$ in the equilibrium definition below causes problems which are not easy to overcome.

Bayesian equilibria have of course been analyzed before, also in settings with continuous state spaces. A famous example is the signaling paper by Crawford and Sobel (1982). They analyze a sender-receiver setting in which the sender is biased and tries to induce a receiver's action which is not optimal for the receiver. In their setting, however, they show that no matter what the receiver's strategy, the optimal behavior of the sender is to divide the state space into (almost surely) convex sets and send messages depending on the set the state space is in. It is easy to show that Bayesian updating is always well-defined on these convex sets.[3] Similar arguments apply for extensions of this model to the multi-dimensional case (Metzger, Jäger, Riedel (2011)) and for uncertainty about language competence (Blume, Board (2013)).

To approach this issue in our setting, imagine that the function $\pi$ is fixed and known to the consumer and she observes a price $p$ and a signal $s$. From the price $p$, and knowing the price function $\pi$, she infers that the true quality must be in the set

$$Q_p^\pi := \pi^{-1}(\{p\}) = \{q \in [0,1] | \pi(q) = p\}.$$

She also knows that the quality level is not more than $\kappa$ away from the observed signal which yields

$$q \in Q_s := [s-\kappa, s+\kappa] \cap [0,1] = \{q \in [0,1] | s \in [q-\kappa, q+\kappa]\}.$$

If the quality level was outside of this set, the received signal would not be in the support of the signal distribution and could thus not be received. Altogether, she can infer that the true quality level must lie in the preimage

$$Q_{p,s}^\pi = Q_p^\pi \cap Q_s = \pi^{-1}(\{p\}) \cap [s-\kappa, s+\kappa].$$

[3] Their definition of the posterior belief (the function $p$ in point (2) on page 1434), is not well-defined if the integral $\int_0^1 q(n|t)f(t)dt$ is zero. The points (5),(6) and (7) in the proof of Lemma 1 show that they do not have to tackle this problem.

If $Q^\pi_{p,s}$ is a Borel set with positive Lebesgue measure and with non-empty interior, a posterior distribution $\mu$ is given by the density function

$$g_\mu(q|p,s) = \begin{cases} \frac{f(q)}{\int_{Q^\pi_{p,s}} f(x)dx} & q \in Q^\pi_{p,s} \\ 0 & \text{else} \end{cases} \tag{3.1}$$

which is the normalized restriction of the original density function $f$ to the set $Q^\pi_{p,s}$.[4] A similar expression is possible for the case in which this set is finite.[5]

In general, however, the set $Q^\pi_{p,s}$ can not be assumed to have this form and does not even have to be measurable. Even when assuming measurability, $Q^\pi_{p,s}$ could in theory be an infinite null set. Even if we excluded all these cases and agree on updating on finite sets, we would still be forced to distinguish situations in which we face a finite set or one of positive measure. We thus take a different, more general approach that allows us to keep the basic idea of a posterior distribution without having to further restrict the set of possible price functions $\pi$.

Note that if we had a posterior belief $\mu(p,s)$, the consumer would buy the product if the expected quality exceeds the price $p$, while there can be mixed behavior in the case of equality. In particular, the buying decision does not depend on the distribution $\mu$ itself but on the expected quality derived from this belief. Using this, we restrict ourselves to only consider expected quality instead of posterior beliefs.

**Definition 3.4.1.** *Let a price function $\pi$ be given. An expectation system with respect to $\pi$ is a function $E : [0,1] \times S \to [0,1]$ such that*

*(i) For every pair $(p,s)$ for which $Q^\pi_{p,s}$ is not empty we have*

$$E(p,s) \in \left[\inf Q^\pi_{p,s}, \sup Q^\pi_{p,s}\right].$$

*(ii) The function is non-decreasing in $s$.*

*(iii) For each two pairs $(p,s),(p,s')$ with $Q^\pi_{p,s} = Q^\pi_{p,s'} \neq \emptyset$, we have $E(p,s) = E(p,s')$. If $Q^\pi_{p,s} = Q^\pi_{p,s'} = \emptyset$ and $s < s'$, $E(p,s) < E(p,s')$.*

*(iv) For two signals $s < s'$, if $Q^\pi_{p,s} \triangle Q^\pi_{p,s'}$ is not a null set, then $E(p,s) < E(p,s')$.*

[4] Updating only $f$ - and not the joint distribution of the type and the signal - is possible due to the uniform distribution of the signal.

[5] Voorneveld and Weibull (2011) use a version for the finite case in which the distribution over the set is just the normalized values of the density function. This can be justified as approximation from conditioning on environments around each point and letting these environments go to zero. In the strict sense, however, conditioning on null sets is problematic.

(v) *Whenever* $Q^{\pi}_{p,s}$ *is a non-empty interval,* $E(p, s)$ *is the expectation of the distribution given in* (3.1).

*We say that* $E$ *is an expectation system if there exists a price function* $\tilde{\pi}$ *so that* $E$ *is an expectation system with respect to* $\tilde{\pi}$*.*

Property (v) ensures that Bayesian updating is used at least in the simple case when we have an interval. The other items translate properties of this Bayesian updating to situations in which it can not be applied. Item (i) ensures that the consumer rationally does not assume a value outside the extremes of the set of possible quality levels. Property (ii) captures the fact that the induced quality distribution of a signal $s$, namely the uniform distribution on the interval $[s - \kappa, s + \kappa]$ first-order-stochastically dominated any other such distribution induced by any lower signal. Moreover, the signal is objective and not influenced by the firm. It is easy to check that when $Q^{\pi}_{p,s}$ is a Borel set with positive measure for two signals, Bayesian updating leads to this monotonic behavior in the signal. This effect is captured in an even stricter form by (iv). Whenever a signal increase removes or adds a set of qualities which is not a null set, the expectation must strictly increase, as it would in a Bayesian setting.

Property (iii) already contains an important refinement about the rationality of the consumer. On the one hand, having the same (non-empty) set of possible types for the same price should lead to the same expectation. Even if the signal $s'$ is higher than $s$, the consumer rationally infers that there is no difference in the quality and thus the expectation is the same. This is different if $Q^{\pi}_{p,s}$ is empty. In this case it is clear that there was a deviation from the price function $\pi$. Although the definition is not very restrictive on these cases, we do need that a higher signal leads to a higher expectation when two of these deviations are observed for the same price. After all, the set of quality levels who could send the signal $s$ is *strictly lower* (in an obvious sense) than the set for $s'$. While the information is proof for out-of-equilibrium behavior, the signal is the only objective, non-strategic information available to the consumer.

Overall, the concept of an expectation system not only allows to overcome measurability and Bayesian updating issues but also relaxes assumptions on the rationality of the consumer. She could be completely rational, using Bayesian updating whenever she can, or she can behave differently if the problem of updating is too complex. Heuristics or other forms of bounded rationality could be applied here.

Having introduced this new mathematical construct, one might wonder whether such an expectation system always exists or if one has to put assumptions on the price function.

**Lemma 3.4.2.** *For each price function $\pi$, there exists an expectation system.*

In particular, the concept of an expectation system does not impose a further restriction on the pricing function.

The proof is constructive, the first insight being that the definition of an expectation system does not contain restrictions across prices. We can thus define the value $E(p, s)$ for a fixed price. This is done by first using property (v) when it applies and then extend it to all signals for which $Q^{\pi}_{p,s}$ is not empty. The extension to the empty cases is then always possible.

Having this structure, there are some interesting consequences for the behavior of the consumer.

**Lemma 3.4.3.** *Let $E$ be an expectation system and let $p$ be a price. Then there exist unique values $\underline{s} \leq \overline{s}$ in $S$ with*

$$E(p, s) \begin{cases} < p & s < \underline{s} \\ = p & \underline{s} < s < \overline{s} \\ > p & s > \overline{s}. \end{cases}$$

*Moreover, we have $\overline{s} - \underline{s} \leq 2\kappa$.*

In the situation of the lemma, define $\underline{q} = \overline{s} - \kappa$, $\overline{q} = \underline{s} + \kappa$, $\underline{\underline{q}} = \underline{s} - \kappa$, $\overline{\overline{q}} = \overline{s} + \kappa$, the quality levels which can just "reach" the signals $\underline{s}$ or $\overline{s}$. From $\overline{s} - \underline{s} \leq 2\kappa$ it also follows that we have $\overline{q} - \underline{q} \leq 2\kappa$. We say that the interval $[\underline{q}, \overline{q}]$ has *full length* if $\overline{q} - \underline{q} = 2\kappa$. This describes the special case $\underline{s} = \overline{s}$ so that the consumer is almost surely never indifferent between buying and not buying. Note that the order $\underline{\underline{q}} \leq \underline{q} \leq \overline{q} \leq \overline{\overline{q}}$ is always satisfied.

To illustrate these values, assume that $Q^{\pi}_{p}$ is an interval $[a, b]$ of length smaller than $2\kappa$ and that the expected quality, restricted to that interval, matches the price $p$. This situation occurs regularly in equilibria as is shown in the equilibrium analysis below. If a signal is higher than the value $a + \kappa$, it can only have come from a certain fraction of the right side of the interval, which yield a higher expectation and thus must lead to sure buying. In the same way, a signal lower than $b - \kappa$ causes the buyer to not spend anything. Any signal between $b - \kappa$ and $a + \kappa$ would, on the other hand, give no further information to the consumer and she would thus stay indifferent. These boundary signals are the values of $\underline{s}$ and $\overline{s}$ from the lemma above.

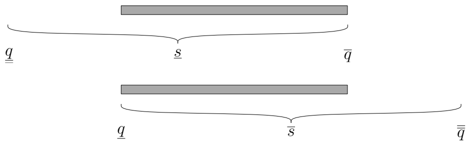


Figure 3.2: The different values in the case of an interval

It is worth mentioning that all these values are completely characterized by only knowing the pair $(\underline{s}, \overline{s})$ or the pair $(\underline{q}, \overline{q})$. Note also that in the example of the interval, $\underline{q}$ and $\overline{q}$ are the interval's end points $a$ and $b$.

The values depend on the price $p$ so we would have to write $\underline{s}(p), \overline{s}(p), \ldots$. For readability, we introduce a notation to leave out these arguments. A price denoted by $p_q$ implies that the values $\underline{\underline{q}}, \underline{q}, \overline{q}$ and $\overline{\overline{q}}$ are determined with respect to this price. In the same way, prices $p_r$ and $p_t$ have the corresponding values $\underline{\underline{r}}, \ldots$ and $\underline{\underline{t}}, \ldots$, respectively. If only one price is considered at a certain point, the values $\underline{s}$ and $\overline{s}$ are taken with respect to that price.

Using the concept on an expectation system, we can analyze a basic property of what will later be an equilibrium. If we fix such an expectation system and assume that the firm knows it as well as the consumer indifference reaction $\alpha$, every firm type should set a price that yields the highest profit of all prices.

**Lemma 3.4.4.** *Let $E$ be an expectation system and $\alpha \in [0,1]$ be an indifference strategy. Define*

$$\phi(q, p; E, \alpha) := p \frac{1}{2\kappa} \int_{q-\kappa}^{q+\kappa} \alpha \mathbf{1}_{E(p,s)=p}(s) + \mathbf{1}_{E(p,s)>p}(s) ds$$

*the profit of type $q$ when setting price $p$. Moreover, let $\pi$ be an optimal price system*[6] *to the buyer's behavior. Then the function*

$$\phi_\pi(q; E, \alpha) := \phi(q, \pi(q); E, \alpha)$$

*is continuous and non-decreasing.*

[6] A price system is optimal if for every type $q$ the price $\pi(q)$ maximizes the type's profit, given the consumer reaction.

Whenever $E$ and $\alpha$ are given, we just write $\phi(q,p)$ instead of $\phi(q,p;E,\alpha)$. A short way of writing the profit function is by defining the probability $\gamma$ of selling a product of quality $q$ for a certain price $p$

$$
\begin{aligned}
\gamma(q,p) :=& \frac{1}{2\kappa}\left(\int_{q-\kappa}^{q+\kappa} \alpha \mathbf{1}_{E(p,s)=p} + \mathbf{1}_{E(p,s)>p} ds\right)\\
=& \frac{1}{2\kappa}\left(\alpha\lambda\big([q-\kappa,q+\kappa]\cap[\underline{s},\overline{s}]\big) + \lambda\big([q-\kappa,q+\kappa]\cap(\overline{s},\infty)\big)\right)\\
=& \begin{cases} 0 & q+\kappa \leq \underline{s}\\ \frac{1}{2\kappa}\alpha(q+\kappa-\underline{s}) & q+\kappa \in (\underline{s},\overline{s})\\ \frac{1}{2\kappa}(\alpha(\overline{s}-\underline{s})+(q+\kappa-\overline{s})) & q-\kappa \leq \underline{s}, \overline{s} \leq q+\kappa\\ \frac{1}{2\kappa}(\alpha(\overline{s}-(q-\kappa))+(q+\kappa-\overline{s})) & q-\kappa \in (\underline{s},\overline{s})\\ 1 & q-\kappa \geq \overline{s} \end{cases}\\
=& \begin{cases} 0 & q \leq \underline{\underline{q}}\\ \frac{1}{2\kappa}\alpha(2\kappa-(\overline{q}-q)) & q \in (\underline{\underline{q}},\underline{q})\\ \frac{1}{2\kappa}(\alpha(2\kappa-(\overline{q}-\underline{q}))+(q-\underline{q})) & q \in [\underline{q},\overline{q}]\\ \frac{1}{2\kappa}(\alpha(2\kappa-(q-\underline{q}))+(q-\underline{q})) & q \in (\overline{q},\overline{\overline{q}})\\ 1 & q \geq \overline{\overline{q}} \end{cases}
\end{aligned}
\tag{3.2}
$$

and writing $\phi(q,p) = p \cdot \gamma(q,p)$.

Given an expectation system $E$, an indifference strategie $\alpha$ and some price $p$, the form and slope of the profit function $\phi(q,p)$ is of high importance for the understanding of the proofs in the analysis. Note that we can have $E(p,s) < p$ for every signal, e.g. if no type is associated to the price $p$, so $Q^{\pi}_{p} = \emptyset$.[7]. If this happens, the profit of the firm is always zero whenever it sets the price $p$, regardless of its quality. In the other cases, however, the function looks as shown in Figure 3.3.[8]

This form of the profit function is why the classical concept of a Bayesian equilibrium is problematic in our setting and why the standard approach does not work. For two different prices $p, p'$, it is possible to have types $q' < q < q''$ with $q'$ and $q''$ preferring the price $p'$ while the optimal price for $q$ is $p$. This, given a fixed consumer reaction,

[7] An example of such a construction is given in the proof of Lemma 3.4.2.

[8] Technically, this is not a special case but is equivalent to $\underline{s} = \overline{s} = 1 + \kappa$.

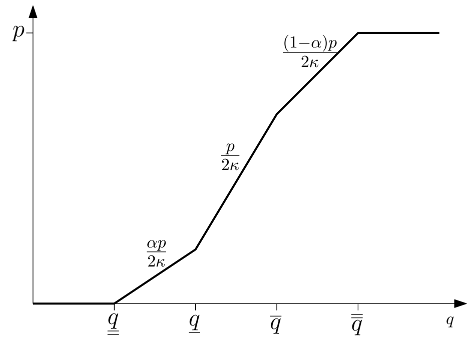


Figure 3.3: The typical form of $\phi(\cdot, p)$ and its slope for a non-trivial price.

allows for non-convexity of type regions $Q_p^\pi$ setting the same price, even when the firm's behavior is optimal.[9]

**Definition 3.4.5.** *A tuple $(\pi, E, \alpha)$, consisting of a price function $\pi$, an expectation system $E$ and an indifference strategy $\alpha$ is called an* equilibrium *if the price function $\pi$ assumes finitely many values, $E$ is an expectation system with respect to $\pi$ and for every type $q \in [0, 1]$ the price $\pi(q)$ maximizes the firm's profit, given $E$ and $\alpha$.*

This definition is the natural adaptation of a Bayesian equilibrium, using the notion of expectation systems. The usual assumption of correct updating is replaced by the property of $E$ being an expectation system for $\pi$. The optimality of the consumer's behavior is implicitly assumed, leaving her only $\alpha$ as choice variable. We assume that this price function can only take finitely many values, as is the case in most markets.[10]

**Definition 3.4.6.** *Let an equilibrium $(\pi, E, \alpha)$ be given. We call a price $p$ an* equilibrium price *if there exists a type $q \in Q_p^\pi$ which makes positive profit in the equilibrium.*

For an equilibrium price $p$, denote $Q_p^* := Q_p \cap \{q \in [0, 1] | \phi_\pi(q) > 0\}$ the set of types setting this price and making positive profits in equilibrium. In this notation, we drop the superscript $\pi$ for expositional reasons. We call a type $q$ *profitable* if $q \in Q_{\pi(q)}^*$. Types that are in $Q_p^\pi$ but have zero gains from the market are not

[9]To see this in Figure 3.3, take some $p_r > \alpha p$ with $[\underline{r}, \overline{r}]$ having full length (so that $\overline{r} - \underline{r} = 2\kappa$ and the graph has only one increasing line, going from 0 to $p_r$) and $\underline{r} = \underline{\underline{r}} \in (\underline{\underline{q}}, \underline{q})$. One can calibrate this so that the new graph is above the existing one in $\underline{q}$ while it is below this graph in a point on the left and a point on the right side of $\underline{q}$.

[10]The most obvious example would be product prices in a supermarket. But it also applies to goods which can have even finer pricing like petrol at a gas station. Since the good we have is indivisible, it is also natural to assume a finite number of values for the pricing strategy.

bounded by incentive constraints and thus their behavior is quite arbitrary. Many statements about equilibrium behavior have to be restricted to profitable types.

## 3.5 Equilibrium Analysis

The obvious next step is to determine under which conditions a market equilibrium exists and what its main features are. The following result shows the structure of equilibrium price behavior.

**Theorem 3.5.1.** *An equilibrium exists. Let* $(\pi, E, \alpha)$ *be an equilibrium and let* $q_{\min}$ *be the infimum of all profitable types. Then* $\pi$ *restricted to* $(q_{\min}, 1]$ *is almost surely a non-decreasing step function.*

In terms of price-quality relation, this is a strong statement, at least for the profitable types. One can argue that firms with a product of quality lower than $q_{\min}$ would not survive in the market and eventually drop out. Prices then monotonically increase with quality which implies that a higher price corresponds to higher quality. Although the relation is not one-to-one (so some ambiguity is left to the consumer for every price), prices roughly signal the right quality.

This result, does not come natural. The formal proof involves a series of technical lemmas and is given in an extra section. Note that this statement also holds if $\kappa$ is large, so that the additional signal does not convey much information. It is thus implied that even in the case of a rather uninformative signal, the indeterminate behavior which was shown in section 3.3 for the absence of a signal is prevented.

Having arrived at this result, our definition of an equilibrium and the construct of an expectation system may seem like overkill, considering that now the sets on which to update are well shaped. Nevertheless, we need the expectation system concept to reach this point of having convex sets of types setting the same price. This step was not easily given to us as it would be in other models, e.g. the classical signaling game of Crawford and Sobel (1982).

To give an intuition on the proof, we continue to state the informal version of the needed steps. The most important observation, fixing an equilibrium price $p_q$ and having in mind the points $\underline{\underline{q}}, \underline{q}, \overline{q}$ and $\overline{\overline{q}}$, is to see that one of the types $\underline{q}$ and $\overline{\overline{q}}$ must have the price $p_q$ as its optimal choice. They are the types which can just reach the signal $\overline{s}$ as upper or lower bound of the corresponding signal range. By the definition of $\overline{s}$, the expectation of the consumer must differ when receiving signals slightly above or below this value. In an equilibrium, this means that the information, i.e.

the set of quality levels assigned to a signal, must differ between these signals. But the only difference in types can occur in environments of $\underline{q}$ and $\overline{\overline{q}}$. Applying a limit argument, we see that at least one of the points $\underline{q}$ and $\overline{\overline{q}}$ is a limit point of the set $Q^{\pi}_{p_q}$. Using continuity, setting price $p_q$ must yield the optimal profit for this limit type. In the same way, this holds for the points $\overline{q}$ and $\underline{\underline{q}}$.

This observation is then extended to further statements. We show that $\overline{q}$ and $\overline{\overline{q}}$, if they are different, can not both be limit points at the same time. Moreover, in this case, there must be a type in an environment of $[\underline{q}, \overline{q}]$ actually setting the price $p_q$. Finally, we show that essentially no type in the sets $(\underline{\underline{q}}, \underline{q})$ and $(\overline{q}, \overline{\overline{q}})$ sets the price $p_q$. While the first points require rather technical arguments, the last property stems from item (iv) of the definition of an expectation system. If more than a null set of types in the two sets set the price $p_q$, it would contradict the definition of the signals $\underline{s}$ and $\overline{s}$.

Having these observations, we compare each two equilibrium prices $p_q > p_r$ for all different possible orders of the points $\underline{q}, \overline{q}, \underline{r}$ and $\overline{r}$. In each case we find that the situation is either impossible or the order $Q^*_{p_r} < Q^*_{p_q}$ holds almost surely which shows the monotonicity and thus the step function form of the equilibrium pricing behavior.

Existence of an equilibrium is shown quite easily by just noting that every single-price setting can be an equilibrium.

This equilibrium existence proof reveals a flaw of our so-far used equilibrium concept. Setting $E(p, s)$ low for all non-equilibrium prices, deviation is never profitable for the firm and thus every constant price function can be an equilibrium, independent of whether the market price is high or low. This phenomenon is not new and essentially the same as in regular Bayesian equilibria. To resolve these issues, we look closer at an equilibrium with a particularly low price. Consider the price function $\pi(q) = .1$ for all $q \in [0, 1]$ in a setting with $\kappa = \frac{1}{10}$. The type $q = .8$ then sells for this very low price but with probability 1. The consumer, when facing such a type, observes the price $p$ and a signal $s \in [.7, .9]$, indicating a far higher quality than the price would suggest. While it is not counter-intuitive that the consumer does not hesitate to buy the product for the price .1, it is harder to believe that for any slightly higher price $p'$ she would assign a much lower expectation to any (also high) signal and never buy. Our next refinement captures this idea.

**Definition 3.5.1** (Locally continuous equilibrium). *An equilibrium $(\pi, E, \alpha)$ is called* locally continuous *if for every signal $s$ the function $E(\cdot, s)$ is continuous in every equilibrium price.*

This refinement is in the same spirit as in the first chapter. It ensures that marginal price deviations do not cause a jump in equilibrium beliefs (and thus expected values). In the example above, the lowest possible signal coming from a type of quality .8 is $.8 - \kappa = .7$. Receiving this low signal, the consumer knows that the quality must be at least .6. Hence the value $E(.1, s)$ is at least .6 for every signal that could come from type .8. The local continuity of $E(\cdot, s)$ at the price $p = .1$ shows that for some marginally higher price the expectation must still be above $p$ for every signal possibly induced by the quality level. The firm would thus still sell with probability 1 and this makes a deviation profitable. The constant-price equilibrium would then not be possible, at least for such low prices.

**Lemma 3.5.2.** *A locally continuous equilibrium exists. Let $(\pi, E, \alpha)$ be a* locally continuous *equilibrium. Then for every equilibrium price $p_q$ - except for the lowest one - $Q^*_{p_q}$ is an interval with endpoints $\underline{q}$ and $\overline{q}$. For each of these intervals, the expected quality matches the price, i.e.*

$$p_q = Exp(q|q \in [\underline{q}, \overline{q}]) = \tfrac{1}{F(\overline{q})-F(\underline{q})} \int_{\underline{q}}^{\overline{q}} qf(q)dq$$

This result shows how step function behavior is further enforced by the refinement. Although single-priced equilibria are still possible, the corresponding price can not be too far away from the highest possible quality level.[11] Moreover, the unrefined equilibrium definition in general allows for types that sell for sure in a way that every of their possible signals induces a consumer expectation strictly above the price. With local continuity, this "high reputation" can be used by the firm to demand a higher price, as described above. Note that even with this refinement, it is possible for a firm to sell with probability one but only in equilibria with $\alpha = 1$.

To illustrate the market outcome, we can now look at such an equilibrium. We choose $\kappa = .25$ and a uniform quality distribution. From this, it follows that for each step of the price function (except the lowest one), the price is the middle point of the quality interval. Choosing the first discontinuity to be at .99, we get the following equilibrium price function. The value of $q_{\min}$ is positive in this example, as one can

[11] This can be seen in the proof of Lemma 3.5.2.

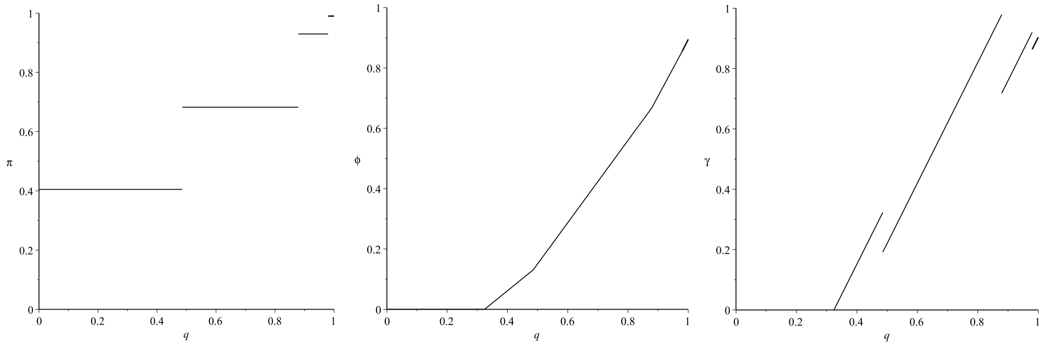


Figure 3.4: The equilibrium price, profit and selling probabilities in our example

.

see in the profit function. Note that the price setting of types below $q_{\min}$ could be chosen differently to some extent. For expositional reasons, it is chosen to match the lowest price. The selling probabilities are increasing within the areas of same prices but are overall not continuous and not monotonic. One can hardly speak of an adverse selection effect in this equilibrium.

Adverse selection is thus not a big issue, anymore. Unlike in the classical model of Ellingsen (1997), high quality is in general not traded with a lower probability than low quality. Selling probabilities can go down but this is always compensated by a higher price so that profits still increase with quality. This result is partly driven by the missing production costs in this model. With such costs, this part of the result may be different. Note, however, that the existence of the lower bound $q_{\min}$ is not mainly caused by this assumption.

Regarding this cutoff value of profitable types, we did not yet say anything about its exact value and its dependence on the parameters. In particular, the signal precision variable $\kappa$ does not appear in the so far established results. The example does not show the upper and lower bound of possible values of $q_{\min}$ over all equilibria. Clearly, choosing a different location for the last discontinuity (instead of .99) would change the point from which profits start to be positive.

Before we present the next result, we briefly think about the case of perfect information. With $\kappa = 0$, quality information would be public and hence the only equilibrium in such a market is that every type $q$ sells its product for the "fair" price $p = q$ with probability one. The product would always be sold regardless of its quality. Of course, our assumption of only having a finite number of equilibrium price rules out this behavior. Nevertheless, looking at the previous result, one may expect the lower bound $q_{\min}$ to approach zero in a comparative static analysis when $\kappa$ becomes small. Otherwise convergence to the full information case would not be possible in any sense.

The next result shows, however, that even the opposite phenomenon occurs. The result is stated for the special case in which the type's distribution is uniform.

**Theorem 3.5.2** (Market breakdown on perfect information)**.** *Let the firm type $q$ be uniformly distributed over $[0, 1]$. With signal precision approaching perfect information ($\kappa \to 0$), the maximal*[12] *expected amount of sold goods over all locally continuous equilibria converges to zero.*

The following proof of this theorem shows very nicely that the market breakdown is caused by the interplay of quality types who are close to each other. The incentive compatibility constraints for types on adjacent steps of the price function dictated that the length of these steps can not get arbitrarily large. This effect gets more extreme in a way that even the sum of these length is bounded with the bound going to zero as $\kappa$ becomes small.

*Proof.* For fixed $\kappa > 0$, let $(\pi, E, \alpha)$ be a locally continuous equilibrium. Proposition 3.5.2 implies that for all equilibrium prices $p_q$ the set $Q^*_{p_q}$ is an interval with endpoints $\underline{q}$ and $\overline{q}$ or $p_q$ is the lowest equilibrium price. Using this, we have $\underline{q} - \overline{q} = 2\kappa$ or $E([\underline{q}, \overline{q}]) = p_q$. The former case of having full length is only possible for the lowest price. Otherwise, the profit of type $\underline{q}$ would be zero which is impossible for types strictly above $q_{\min}$.

Theorem 3.5.1 shows that $\pi$ is almost surely a step function. Because of the profit's continuity, each type that lies on a discontinuity of the price function must be indifferent between setting either of the two adjacent prices.

In the case where $\pi$ is a constant function above $q_{\min}$, note that we have[13] $q_{\min} \geq 1 - 2\kappa$ which converges to one with $\kappa \to 0$. In the same way, convergence of all price functions with two steps can be shown. In fact, for every fixed number of steps, the corresponding equilibria must yield uniform convergence of $q_{\min}$ to 1. But there is still an infinite number of possible steps and thus the convergence result

[12] Technically, the existence of a maximum is not guaranteed and we should speak of a supremum, here.

[13] This is shown in the existence proof for locally continuous equilibria. Intuitively, having steps of a size larger than $2\kappa$, some types always send signals above $\overline{s}$. This is not compatible with locally continuous equilibria. The proof for any finite number of steps follows with the same argument.

does not follow from these thoughts. However, it shows that for the following proof we can assume the price function to have at least three different prices. This also implies $\alpha > 0$, otherwise the lowest type of each step would get zero profit which is a contradiction.

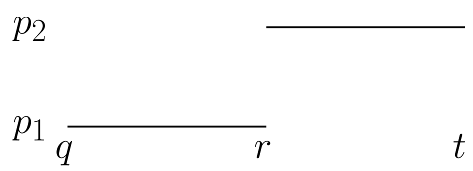


Figure 3.5: The situation of $q$,$r$ and $t$ in the proof

Let $q < r < t$ be three types that lay on adjacent discontinuities and denote $p_1 < p_2$ the corresponding prices as depicted in Figure 3.5. Assume that $p_1$ is not the lowest equilibrium price. For $\kappa$ low enough we can choose these values so that $r$ is above $\frac{1}{2}+\kappa$. The prices must be equal to the expected qualities over the intervals $[q, r]$ and $[r, t]$, respectively. From the uniform type distribution it follows that $p_1 = \frac{q+r}{2}$ and $p_2 = \frac{r+t}{2}$. Because of the continuity of the profit function, the type $r$ is indifferent between setting price $p_1$ or $p_2$. Hence the following equation holds.

$$\begin{aligned}
& & \phi(r, p_1) &= \phi(r, p_2) \\
&\overset{(3.2)}{\Leftrightarrow} & p_1 \tfrac{1}{2\kappa}(r - q + \alpha(2\kappa - (r - q))) &= p_2 \tfrac{1}{2\kappa}(\alpha(2\kappa - (t - r))) \\
&\Leftrightarrow & \tfrac{q+r}{2}(r - q + \alpha(2\kappa - (r - q))) &= \tfrac{r+t}{2}(\alpha(2\kappa - (t - r))) \\
&\Leftrightarrow & r^2 - q^2 + \alpha(2\kappa(r + q) - (r^2 - q^2)) &= \alpha(2\kappa(t + r) - (t^2 - r^2))
\end{aligned}$$

Reordering this, one gets

$$\begin{aligned}
\alpha t^2 - 2\kappa\alpha t + (1 - 2\alpha)r^2 - (1 - \alpha)q^2 + 2\alpha\kappa q &= 0 \\
t^2 - 2\kappa t + \tfrac{1-2\alpha}{\alpha} r^2 - \tfrac{1-\alpha}{\alpha} q^2 + 2\kappa q &= 0
\end{aligned}$$

and solving this for $t$ yields

$$\begin{aligned}
t =& \kappa \pm \sqrt{\kappa^2 - \tfrac{1-2\alpha}{\alpha} r^2 + \tfrac{1-\alpha}{\alpha} q^2 - 2\kappa q} \\
=& \kappa \pm \sqrt{(\kappa - r)^2 - \tfrac{1-\alpha}{\alpha} \underbrace{(r^2 - q^2)}_{>0} + 2\kappa(r - q)} \\
\overset{\alpha\in(0,1]}{\leq}& \kappa + \sqrt{(\kappa - r)^2 + 2\kappa(r - q)}.
\end{aligned}$$

In other words, for each pair $q, r$ we get an upper bound for the next discontinuity $t$ which is independent of the parameter $\alpha$.
For expositional purposes, we introduce the notation $t' := t - \kappa$ which we use similarly for the other variables. The inequality then becomes

$$\begin{aligned} t' \leq & \sqrt{r'^2 + 2\kappa(r' - q')} \\ = & \sqrt{r'^2} + \int_{r'^2}^{r'^2+2\kappa(r'-q')} \frac{1}{2\sqrt{z}} dz \\ \leq & r' + \int_{r'^2}^{r'^2+2\kappa(r'-q')} \frac{1}{2\sqrt{r'^2}} dz \\ = & r' + \frac{1}{2r'} 2\kappa(r' - q') \\ \overset{r' \geq \frac{1}{2}}{\leq} & r' + 2\kappa(r' - q') \end{aligned}$$

which shows that for the adjacent values $q, r$ and $t$ we have

$$t - r = t' - r' \leq 2\kappa(r' - q') = 2\kappa(r - q).$$

Take $q_0$ the smallest (satisfying $q_0 > \frac{1}{2} + \kappa$) such type that lays on a discontinuity of the price function and let $q_1, q_2, \ldots$ be the following discontinuities. It follows that for all $n \in \mathbb{N}$ we get

$$\begin{aligned} q_n = & q_0 + (q_n - q_0) = q_0 + \sum_{i=1}^{n} (q_i - q_{i-1}) \leq q_0 + \sum_{i=1}^{n} (2\kappa)^{i-1} \underbrace{(q_1 - q_0)}_{\leq 2\kappa} \\ \leq & q_0 + \sum_{i=1}^{n} (2\kappa)^i \leq q_0 + \frac{2\kappa}{1 - 2\kappa}. \end{aligned}$$

Remember that $q_n$ must be equal to 1 for some $n$. Letting $\kappa$ go to zero forces $q_0$ to go to 1 uniformly for all equilibria.
Since all types below $q_{\min} \geq q_0 - 2\kappa$ are not able to sell their product, overall sales necessarily converge to zero uniformly over all equilibria when $\kappa$ goes to zero and $q_0$ approaches one. □

## 3.6 The Proof of Theorem 3.5.1

This section presents lemmas and their proofs necessary for establishing the result in Theorem 3.5.1. They show how to use the properties of an expectation system and the optimality of the firm's behavior to determine the structure of an equilibrium price function.

As is shown below, the definition of an expectation system carries some properties similar to Bayesian updating, thus allowing for a similar analysis without assuming - but not excluding - perfect rationality on the consumer side.

The proofs of this section are presented directly after their corresponding statements. We use the shape of the profit function for a given equilibrium price, as depicted in Figure 3.3, very often. It is important to be familiar with the different areas of its slope to perfectly understand the proofs.

One of the main points we need to know about expectation systems in equilibria is formulated in the following lemma which generalizes a property from Bayesian updating.

**Lemma 3.6.1.** *In any equilibrium $(\pi, E, \alpha)$ and for each equilibrium price $p_q$, at least one of the points $\underline{q}$ and $\overline{\overline{q}}$ and at least one of the points $\overline{q}$ and $\underline{\underline{q}}$ are limit points of $Q^{\pi}_{p_q}$.*

The connection to the Bayesian case becomes clear if we remember the interval example. The points $\underline{q}$ and $\overline{q}$ are then the endpoints of the interval. The lemma shows this property in a weaker sense, only using the equilibrium system definition. Note that even in the case of regular Bayesian updating, it can happen that not $\underline{q}$ but $\overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$, e.g. if we have two intervals $[a, b] < [c, d]$ with $c - a < 2\kappa$ and $Exp(q|q \in [a, b]) = p_q$. Then the point[14] $c = \overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$ but $\underline{q} = c - 2\kappa < a$ is not.

*Proof.* We only show that $\underline{q}$ or $\overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$. If any of these two points are in $Q^{\pi}_{p_q}$, we are done. Assume now that this is not the case. We construct a sequence of types in $Q^{\pi}_{p_q}$, converging to either $\underline{q}$ or $\overline{\overline{q}}$.

Start with any $\varepsilon_0 > 0$ and observe that by definition of $\overline{s}$ the values $E(p_q, \overline{s} - \varepsilon_0)$ and $E(p_q, \overline{s} + \varepsilon_0)$ are not equal.[15]

Consider the corresponding sets $Q^{\pi}_{p_q,\overline{s}-\varepsilon_0}$ and $Q^{\pi}_{p_q,\overline{s}+\varepsilon_0}$. If one of them is not empty, they can not be equal due to Definition 3.4.1 (iii). This leaves two cases to consider.

First case: $Q^{\pi}_{p,\overline{s}-\varepsilon_0} = Q^{\pi}_{p,\overline{s}+\varepsilon_0} = \emptyset$

[14] To see that we have $c = \overline{\overline{q}}$, note that for a signal $s$ slightly below $c - \kappa$, we have $Q^{\pi}_{p_q,s} = [a, b]$ so that the consumer is indifferent. For signals above $c - \kappa$, we must have $E(p_q, s) > p_q$. This is dictated by property (iv) of an expectation system. Hence $c - \kappa = \overline{s}$ and thus $c = \overline{\overline{q}}$.

[15] Since $p_q$ is an equilibrium price, $\overline{s}$ can not be on the limit of $S = [-\kappa, 1 + \kappa]$. With $\varepsilon_0$ small enough, the expressions are well-defined.

This situation is depicted in Figure 3.6. Because both sets are empty, we have $Q^{\pi}_{p_q,\overline{s}} \subset Q^{\pi}_{p_q,\overline{s}-\varepsilon_0} \cup Q^{\pi}_{p_q,\overline{s}+\varepsilon_0} = \emptyset$ and this is true for all smaller choices of $\varepsilon_0 > 0$. From 3.4.1 (iii), we know that $E(p_q, s)$ is strictly increasing in the signal within some interval around $\overline{s}$. Hence it follows that we have $\underline{s} = \overline{s}$ and thus $\overline{q} = \overline{\overline{q}}$. Since $p_q$ is an equilibrium price, there must be some profitable type $q$ with $\pi(q) = p_q$. The only way to make positive profit is if this type is above $\underline{s} - \kappa$ and thus above $\overline{q} + \varepsilon_0$. Hence the type $q$ sells with probability one and we have $\phi_\pi(q) = p_q$. By the monotonicity of $\phi_\pi$ and since every type in the interval $(\overline{q}, q)$ can attain this profit, we know that $\phi_\pi(q') = p_q$ for all $q' \in (\overline{q}, q)$. Any two types $q', q''$ in this interval, not setting the price $p_q$, must have a selling probability in $(0,1)$ and the same profit $\phi_\pi(q') = \phi_\pi(q'') = p_q$. But $\phi(q', \pi(q')) = \phi(q'', \pi(q'')) = p_q$ is not possible if $\pi(q') = \pi(q'') \neq p_q > 0$ (see Figure 3.3, the same profit for the same price implies that this profit is either zero or matches the price). It follows that each type in the interval $(\overline{q}, \overline{q}+\varepsilon_0)$ sets a different price. Since there are only finitely many equilibrium prices, this is a contradiction. Hence only the following, second case can occur.

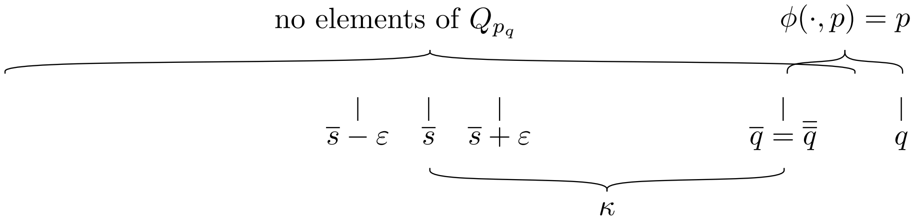


Figure 3.6: The situation of the first case

Second case: $Q^{\pi}_{p_q,\overline{s}-\varepsilon_0} \neq Q^{\pi}_{p_q,\overline{s}+\varepsilon_0}$

Choose $q_0$ in the (non-empty) symmetric difference of these two sets and note that we have

$$\begin{aligned} q_0 &\in [\overline{s} - \kappa - \varepsilon_0, \overline{s} - \kappa + \varepsilon_0] \cup [\overline{s} + \kappa - \varepsilon_0, \overline{s} + \kappa + \varepsilon_0] \\ &= [\underline{q} - \varepsilon_0, \underline{q} + \varepsilon_0] \cup [\overline{\overline{q}} - \varepsilon_0, \overline{\overline{q}} + \varepsilon_0] \end{aligned}$$

By construction we have $q_0 \in Q^{\pi}_{p_q}$. Choose $\varepsilon_1 = \frac{1}{2}\min(|q_0 - \underline{q}|, |q_0 - \overline{\overline{q}}|) \in (0, \frac{\varepsilon_0}{2})$. Repeating these arguments[16], using the values $\varepsilon_1, \varepsilon_2, \ldots$, we obtain a sequence $(q_n)$ in $Q^{\pi}_{p_q}$ whose elements satisfy

$$|q_n - \underline{q}| < \varepsilon_n \text{ or } |q_n - \overline{\overline{q}}| < \varepsilon_n$$

[16]Since the first case leads to a contradiction, we always end up with the second case.

for all $n \in \mathbb{N}$. At least one of these two conditions is true for an infinite number of indices and hence there exists a subsequence of $(q_n)$ such that either the left or right inequality is true for all of its elements. Since $(\varepsilon_n)$ converges to zero, this subsequence converges to either $\underline{q}$ or $\overline{\overline{q}}$. This limit is thus a limit point of $Q^\pi_{p_q}$.

The proof for $\overline{q}$ or $\underline{\underline{q}}$ being a limit point uses the same arguments, starting with $\underline{s}$ instead of $\overline{s}$. We omit this part of the proof. $\square$

Acknowledging this lemma, we say that a type is a $p_q$-limit point if it is a limit point of $Q^\pi_{p_q}$.

While this intermediate result may seem innocuous, it is very important for the analysis of the structure of equilibrium price systems. Knowing that these points are limit points, the continuity of the profit function $\phi_\pi$ implies that the corresponding profit of these types must attain its maximum in the price $p_q$. No other price can yield strictly higher profits to a firm with these quality levels. Hence we have[17]

$$\phi_\pi(\overline{q}) = \phi(\overline{q}, p_q) \text{ or } \phi_\pi(\underline{\underline{q}}) = \phi(\underline{\underline{q}}, p_q)$$

and

$$\phi_\pi(\underline{q}) = \phi(\underline{q}, p_q) \text{ or } \phi_\pi(\overline{\overline{q}}) = \phi(\overline{\overline{q}}, p_q),$$

depending on which of these types has the limit point property described above.

The next result is the first direct step to determining the equilibrium price function. It excludes two possible combinations of ordering $p_q$- and $p_r$-limit points when the order of these two prices is known. Its proof is a direct application of the previous lemma.

**Lemma 3.6.2.** *In an equilibrium, let $p_r < p_q$ be two equilibrium prices and assume $\underline{r} \geq \underline{q}$. Then we have $\overline{r} < \overline{q}$.*

*Proof.* Assume $\underline{r} \geq \underline{q}$ and $\overline{r} \geq \overline{q}$ as shown in Figure 3.7. This implies

$$\gamma(\underline{r}, p_q) \overset{(3.2)}{=} \begin{cases} \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{q} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in [\underline{q}, \overline{q}] \\ \frac{1}{2\kappa}\left(\alpha(2\kappa - (\underline{r} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in (\overline{q}, \overline{\overline{q}}) \end{cases}$$

[17] Note, however, that for example the inequality $\pi(\overline{q}) = p_q$ does not follow from $\phi_\pi(\overline{q}) = \phi(\overline{q}, p_q)$. The type $p_q$ may set a different price. However, there are arbitrarily close types which set the price $p_q$.


Figure 3.7: The two situations excluded by lemma 3.6.2

$$
\begin{aligned}
&= \begin{cases} \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \underline{r} - (\overline{q} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in [\underline{q}, \overline{q}] \\ \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \underline{r} - (\underline{r} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in (\overline{q}, \overline{\overline{q}}) \end{cases} \\
&= \begin{cases} \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \overline{q}) + (1-\alpha)(\underline{r} - \underline{q})\right) & \underline{r} \in [\underline{q}, \overline{q}] \\ \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \underline{r}) + (1-\alpha)(\underline{r} - \underline{q})\right) & \underline{r} \in (\overline{q}, \overline{\overline{q}}) \end{cases} \\
&\geq \frac{1}{2\kappa}\ \alpha(2\kappa - (\overline{r} - \underline{r})) \\
&\overset{(3.2)}{=} \gamma(\underline{r}, p_r)
\end{aligned}
$$

in the case where $\underline{r} < \overline{\overline{q}}$. If $\underline{r} \geq \overline{\overline{q}}$, this inequality is simple to show.

$$\gamma(\underline{r}, p_q) \geq \gamma(\overline{\overline{q}}, p_q) = 1 \geq \gamma(\underline{r}, p_r)$$

The type $\underline{r}$ thus has a weakly higher change of selling for the high price $p_q$ than for the price $p_r$. Note that only in the case where $\gamma(\underline{r}, p_q) = \gamma(\underline{r}, p_r) = 0$ this does not lead to a strictly higher profit when setting the high price. This case, however, would imply[18] that $\underline{q} = \underline{r} = \underline{\underline{q}} = \underline{\underline{r}}$. Setting $p_r$ would thus be dominated by setting $p_q$ in the sense that $\phi(q, p_q) > \phi(q, p_r)$ whenever $\phi(q, p_r) > 0$ for any type $q$. No profitable type could optimally set $p_r$; it would not be an equilibrium price.

Having $\phi(\underline{r}, p_q) > \phi(\underline{q}, p_r)$ shows that $\underline{r}$ is not a limit point of $Q^{\pi}_{p_r}$. It follows from $\underline{r} \geq \underline{q}$ that $\overline{\overline{r}} \geq \overline{\overline{q}}$ and thus $\gamma(\overline{\overline{r}}, p_r) = \gamma(\overline{\overline{r}}, p_q) = 1$. Since $p_q$ is the higher price, we have $\phi(\overline{\overline{r}}, p_r) < \phi(\overline{\overline{r}}, p_q)$ so that $\overline{\overline{r}}$ is also not a $p_r$-limit point. This contradicts Lemma 3.6.1. □

This lemma excludes the most extreme cases of negative price-quality relation. The pairs $(\underline{q}, \underline{r})$ and $(\overline{q}, \overline{\overline{r}})$ can not both be ordered opposite to the corresponding prices. Thinking about the interval example, this implies that there can not be two intervals $Q^{\pi}_{p_q} < Q^{\pi}_{p_r}$ so that the higher price is only set by lower types.

[18]It is easy to see that a zero selling probability of $\underline{r}$ implies $\underline{r} = \underline{\underline{r}}$. The equality $\phi(\underline{r}, p_q) = 0$ implies the first inequality of $\underline{r} \leq \underline{\underline{q}} \leq \underline{q} \leq \underline{r}$.

We continue to use this lemma to show two further equilibrium properties which help us to determine the form of equilibrium price functions.

**Lemma 3.6.3.** *In every equilibrium, for every equilibrium price $p_q$ and corresponding values $\underline{\underline{q}}, \underline{q}, \overline{q}, \overline{\overline{q}}$, we have:*

*(1) The set*

$$Q^{\pi}_{p_q} \cap \Big( (\underline{\underline{q}}, \underline{q}) \cup (\overline{q}, \overline{\overline{q}}) \Big)$$

*is a null set.*

*(2) If $\overline{q} - \underline{q} < 2\kappa$ and thus $\overline{q} \neq \overline{\overline{q}}$, the points $\overline{q}$ and $\overline{\overline{q}}$ are not both $p_q$-limit points.*

*(3) If there exists $\varepsilon > 0$ such that $Q^{\pi}_{p_q} \cap [\underline{q} - \varepsilon, \overline{q} + \varepsilon] = \emptyset$, the interval $[\underline{q}, \overline{q}]$ has full length.*

The last point may be a little surprising in that you may expect the set $[\underline{q} - \varepsilon, \overline{q} + \varepsilon]$ to always contain a type of $Q^{\pi}_{p_q}$. To see that this needs not always to be the case, imagine $p_q = .5, \kappa = .1$ and $Q^{\pi}_{p_q} = [.2, .3] \cup [.7, .8]$. We then have $E(p_q, .4) = .3 < p_q < .7 = E(p_q, .6)$. In what follows, it is possible to have $\overline{s} = \underline{s} = .5$ so that $[\underline{q}, \overline{q}] = [.4, .6]$ which has full length. A narrow environment of this interval contains no element of $Q^{\pi}_{p_q}$.

*Proof.* Proof of (1)

Note that this is trivial if $\underline{\underline{q}} = \underline{q}$ and thus also $\overline{q} = \overline{\overline{q}}$. If $\underline{\underline{q}} < \underline{q}$, we also have $\underline{s} < \overline{s}$ and thus $Q^{\pi}_{p_q,s}$ and $Q^{\pi}_{p_q,s'}$ are non-empty[19] and we have $E(p_q, s) = E(p_q, s')$ for every pair $s, s' \in (\underline{s}, \overline{s})$. By property (iv) of an expectation system, this implies that

$$Q^{\pi}_{p_q} \cap \Big( (\underline{\underline{q}}, \underline{q}) \cup (\overline{q}, \overline{\overline{q}}) \Big) \subset \bigcup_{s,s' \in (\underline{s}, \overline{s}) \cap \mathbb{Q}} Q^{\pi}_{p,s} \triangle Q^{\pi}_{p,s'}$$

is a null set.

Proof of (2)

From $\overline{q} - \underline{q} < 2\kappa$ we know that $\overline{q} \neq \overline{\overline{q}} = \underline{q} + 2\kappa$. Assume that $\overline{q}$ and $\overline{\overline{q}}$ are $p_q$-limit points. Pick any type $r \in (\overline{q}, \overline{\overline{q}})$ with corresponding prize $p_r = \pi(r) \neq p_q$. This is possible due to the first point of this lemma. Note that because $\overline{q}$ and $\overline{\overline{q}}$ are limit points for $p_q$, we must have $\phi(\overline{q}, p_r) \leq \phi(\overline{q}, p_q)$ and $\phi(\overline{\overline{q}}, p_r) \leq \phi(\overline{\overline{q}}, p_q)$ while in $r$, the

[19] Formally, there can not be two such empty sets over all possible values of $s$ and $s'$ (see by Definition 3.4.1 (iii)). It is trivial that, if at most one of these sets is empty, none of them are.

opposite is true: $\phi(r, p_r) \geq \phi(r, p_q)$. Since the slope of $\phi(\cdot, p_q)$ has the constant value $\frac{1-\alpha}{2\kappa} p_q$ in the whole interval $(\overline{q}, \overline{\overline{q}})$, it follows that the slope of $\phi(\cdot, p_r)$ must be weakly above this value in some point between $\overline{q}$ and $r$ while it is weakly smaller than this value in $(r, \overline{\overline{q}})$.

If the slope of $\phi(\cdot, p_r)$ also had the constant value $\frac{1-\alpha}{2\kappa} p_q$ in the whole interval $(\overline{q}, \overline{\overline{q}})$ there are two options, either having $\alpha p_r = (1-\alpha) p_q$ or $p_r = (1-\alpha) p_q$. Refer to Figure 3.3 to see this.

In the first case, we had $\underline{\underline{r}} \leq \overline{q} < \overline{\overline{q}} \leq \underline{r}$ which implies via (1) that the set of types setting $p_r$ in the interval $(\overline{q}, \overline{\overline{q}})$ is a null set and there is a different price that we could have chosen in the beginning. We assume without loss of generality that this is the case.[20]

The second possibility $p_r = (1-\alpha) p_q$ implies $p_r < p_q$ and $\overline{r} \geq \overline{\overline{q}}$ which is excluded by Lemma 3.6.2.

The slope of $\phi(\cdot, p_r)$ is hence weakly decreasing and not constant over the whole interval $(\overline{q}, \overline{\overline{q}})$. Again referring to Figure 3.3, we deduce that $\overline{\overline{r}} \in (\overline{q}, \overline{\overline{q}})$. To see this, note that $\overline{r}$ and $\overline{\overline{r}}$ are the only points at which the profit $\phi(\cdot, p_r)$ from setting the price $p_r$ strictly decreases. One of these values thus has to be in the interval $(\overline{q}, \overline{\overline{q}})$. If this is not true for $\overline{\overline{r}}$, we had $\overline{q} < \overline{r} < \overline{\overline{q}} \leq \overline{\overline{r}}$ which also implies $\underline{q} \leq \underline{r}$. Moreover, comparing the slopes in the interval $(\overline{r}, \overline{\overline{q}})$ it yields $\frac{1-\alpha}{2\kappa} p_r < \frac{1-\alpha}{2\kappa} p_q$ and hence $p_r < p_q$. This constitutes a situation which is, again, excluded by Lemma 3.6.2.

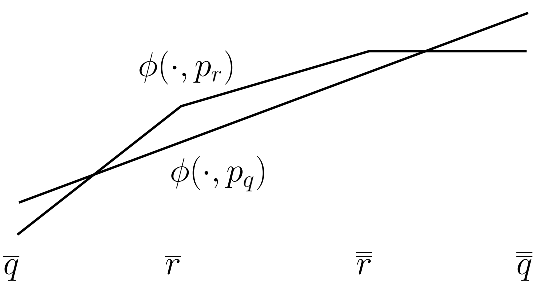


Figure 3.8: The situation of the proof, and the development of the different profit functions

We now know that $\overline{\overline{r}} < \overline{\overline{q}}$ and hence $p_r = \phi(\overline{\overline{q}}, p_r) \leq \phi(\overline{\overline{q}}, p_q) = p_q$. Since the prices are not equal, even the strict inequality is true. This shows that $\overline{q} < \overline{r}$, otherwise the slope of $\phi(\cdot, p_r)$ would never be below the one of $\phi(\cdot, p_q)$ in the interval $(\overline{q}, \overline{\overline{q}})$.

[20] The new price $p_{r'}$ can not have the same property since then we would have $\alpha p_r = (1-\alpha) p_q = \alpha p_{r'}$. This contradicts $p_r \neq p_{r'}$.

From the continuity and monotonicity of the profit functions, we know that there must be an interval (close to $\overline{\overline{q}}$) contained in $(\overline{\overline{r}}, \overline{\overline{q}})$ in which the profit from setting $p_q$ is strictly higher than from setting $p_r$. Figure 3.8 shows the situation. Again by the first part of this lemma, we can find a type $t$ in this interval that does not set the price $p_q$ (and does not set $q_r$, as well, since it does not yield the highest profit). Using the same arguments as before, we end up with another equilibrium price $p_t$ which is strictly below $p_q$ (for the same arguments) but must be strictly above $p_r$ since type $t > \overline{\overline{r}}$ sets this price and thus

$$p_r = \phi(t, p_r) < \phi(t, p_t) \leq p_t.$$

The relation of the functions $\phi(\cdot, p_t)$ and $\phi(\cdot, p_q)$ follow as before, using the same reasoning.

By further repeating these arguments, we end up with an infinite and strictly increasing sequence of equilibrium prices which are all below $p_q$. This contradicts the assumption that there can only be finitely many prices in an equilibrium.

Proof of (3)

Assume that $[\underline{q}, \overline{q}]$ does not have full length, i.e. $\underline{s} < \overline{s}$. The situation is given in Figure 3.9.

If there are $s < s'$ in $(\underline{s}, \overline{s})$ with $Q^{\pi}_{p_q,s} = \emptyset = Q^{\pi}_{p_q,s'}$, we have $E(p_q, s) < E(p_q, s')$ by property (iii) of an expectation system.

If there are no two such signals, define $\varepsilon' = \min\{\varepsilon, \frac{\overline{s}-\underline{s}}{4}\}$ (which is strictly greater than zero by the assumptions) and pick two points $s \in (\underline{s}, \underline{s}+\varepsilon')$ and $s' \in (\overline{s}-\varepsilon', \overline{s})$. Figure 3.9 shows the situation. By construction we now have $\underline{s} < s < s' < \overline{s}$. Note that $Q^{\pi}_{p_q,s}$ contains no element above $\underline{q} - \varepsilon$ while $Q^{\pi}_{p_q,s'}$ contains no element below $\overline{q} + \varepsilon$. By the assumption of this paragraph, we can choose $s$ and $s'$ so that these sets are not empty. Then we have

$$\sup Q^{\pi}_{p,s} < \underline{q} < \overline{q} < \inf Q^{\pi}_{p,s'}$$

which, by property (i), also implies $E(p_q, s) < E(p_q, s')$.


Figure 3.9: The situation in (3).

In both cases, the resulting inequality $E(p_q, s) < E(p_q, s')$, is a contradiction to $s, s' \in (\underline{s}, \overline{s})$. $\square$

Having the monotonicity result of Lemma 3.6.2, one might think that this relation is even more extreme and that the ordering $\underline{q} \leq \underline{r}$ could never occur with $p_r < p_q$. The following lemma indeed shows that, although the case itself is not excluded, the implication for the order of profitable types setting the two prices is preserved.

**Lemma 3.6.4.** *In an equilibrium, let $p_r < p_q$ be two equilibrium prices. If $\underline{r} \geq \underline{q}$ and $\overline{r} > \overline{q}$ (the case of Lemma 3.6.2), we have*

$$Q^*_{p_r} < Q^*_{p_q}.$$

*If additionally $\underline{r} > \underline{q}$, the interval $[\underline{q}, \overline{q}]$ has full length.*

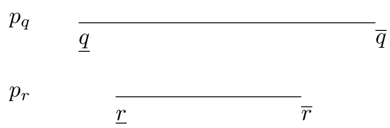


Figure 3.10: The situation of Lemma 3.6.4.

*Proof.* A picture of the situation at hand is given in Figure 3.10. First, consider the strict case $\underline{r} > \underline{q}$. For all $q \geq \overline{q}$ we have $\phi(q, p_q) > \phi(q, p_r)$ since

$$\begin{aligned}
\gamma(\overline{q}, p_q) &= \tfrac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{q} - \underline{q})) + \overline{q} - \underline{q}\right) \\
&\geq \tfrac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{q} - \underline{r})) + \overline{q} - \underline{r}\right) \\
&= \gamma(\overline{q}, p_r) > 0
\end{aligned}$$

so that we have $\phi(\overline{q}, p_q) = p_q \cdot \gamma(\overline{q}, p_q) > p_r \cdot \gamma(\overline{q}, p_r) = \phi(\overline{q}, p_r)$. With higher types than $\overline{q}$, the left hand side of this inequality grows faster than the right hand side until the value $\overline{\overline{q}}$ from where we have $\phi(q, p_q) = p_q > p_r \geq \phi(q, p_r)$. This shows that $\overline{\overline{r}}$ (which is above $\overline{q}$) is not a $p_r$-limit point. We thus know that $\underline{r}$ is a $p_r$-limit point and hence $\phi(\underline{r}, p_r) \geq \phi(\underline{r}, p_q)$.

It follows that

$$\phi(\underline{q}, p_r) = \phi(\underline{r}, p_r) - \int_{\underline{q}}^{\underline{r}} \phi'(t, p_r)dt$$

$$
\begin{aligned}
&=\phi(\underline{r}, p_r) - \int_{\underline{q}}^{\underline{r}} \frac{\alpha p_r}{2\kappa} dt \\
&>\phi(\underline{r}, p_q) - \int_{\underline{q}}^{\underline{r}} \frac{p_q}{2\kappa} dt \\
&=\phi(\underline{r}, p_q) - \int_{\underline{q}}^{\underline{r}} \phi'(t, p_q) dt \\
&=\phi(\underline{q}, p_q),
\end{aligned}
$$

using that both of these functions are differentiable in the non-empty interval $(\underline{q}, \underline{r})$. Finally, since $\overline{r} < \overline{q}$, we have $\underline{\underline{r}} < \underline{\underline{q}}$, thus $\phi(\underline{\underline{q}}, p_q) = 0 < \phi(\underline{\underline{q}}, p_r)$. These inequalities imply that neither $\underline{q}$ nor $\underline{\underline{q}}$ is a limit point of $Q^{\pi}_{p_q}$. Using Lemma 3.6.3 (2), we see that $[\underline{q}, \overline{q}]$ has full length.

Observe now that the function $\phi(\cdot, p_q)$ always has a strictly higher slope than $\phi(\cdot, p_r)$ in the interval $(\underline{\underline{q}}, \overline{\overline{q}})$. Together with the inequalities

$$\phi(\underline{\underline{q}}, p_q) = 0 < \phi(\underline{\underline{q}}, p_r) \text{ and } \phi(\overline{\overline{q}}, p_q) = p_q > \phi(\overline{\overline{q}}, p_r)$$

this proves the existence of a "critical type" $q_c$ with $\phi(q, p_q) < \phi(q, p_r)$ whenever $\underline{\underline{q}} < q < q_c$ and $\phi(q, p_q) > \phi(q, p_r)$ if $q > q_c$. Note that also no profitable type below $\underline{\underline{q}}$ sets the price $p_q$ since then the profit would be zero. This proves $Q^*_{p_r} \leq q_c \leq Q^*_{p_q}$ and thus $Q^*_{p_r} < Q^*_{p_q}$ since the sets are disjoint.

The special case $\underline{r} = \underline{q}$ needs a different treatment. As before, having $\overline{r} < \overline{q}$ implies that $\overline{q}$ is a limit point of $Q^{\pi}_{p_q}$. If $\overline{\overline{q}}$ also was such a limit point, the proof above works and we are done. Hence, we consider the case in which $\underline{q}$ is a limit point and thus $\phi(\underline{q}, p_q) \geq \phi(\underline{q}, p_r)$. If this inequality was strict, $\underline{r} = \underline{q}$ could not be a limit point of $Q^{\pi}_{p_r}$. The same obviously holds for $\overline{\overline{r}} = \overline{\overline{q}}$ which gives a contradiction to Lemma 3.6.1.

Having $\phi(\underline{q}, p_q) = \phi(\underline{q}, p_r)$, it follows that $\phi(t, p_q) > \phi(t, p_r)$ for all $t > \underline{q} = \underline{r}$ and $\phi(t, p_q) < \phi(t, p_r)$ for $t \in (\underline{r}, \underline{q})$. This again can be seen by comparing the slopes of the profit functions. Hence $Q^*_{p_r} < Q^*_{p_q}$.

□

The previous lemmas deal with the counter-intuitive cases in which, although the price $p_r$ is lower than $p_q$, the order $\underline{q} \leq \underline{r}$ holds and thus there could be a negative quality-price relation. In what follows, we show what happens if this relation has the "natural" order $\underline{r} < \underline{q}$.

**Lemma 3.6.5.** *Let $p_r < p_q$ be two equilibrium prices with $\underline{r} < \underline{q}$. Then we have $\overline{r} \leq \overline{q}$ and $Q^*_{p_r} \leq \underline{q} \leq Q^*_{p_q}$ a.s.*[21].

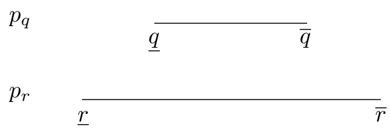


Figure 3.11: The situation excluded by Lemma 3.6.5

.

To prove this statement, we use the following intermediary result.

**Lemma 3.6.6.** *Let $p_r < p_q$ be two equilibrium prices with $\underline{r} < \underline{q}$ and $\overline{q} < \overline{r}$. Then there exists another equilibrium price $p_t$ such that either*

$$p_q < p_t, \overline{t} < \overline{r} \text{ and } \underline{t} > \underline{r}$$

*or*

$$p_t < p_r, \underline{t} < \underline{q} \text{ and } \overline{t} > \overline{q}$$

*holds.*

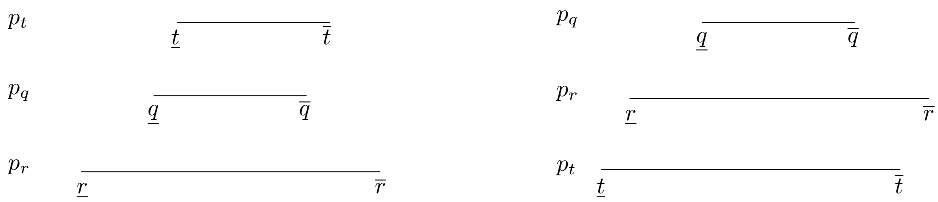


Figure 3.12: The two situations with the new equilibrium price $p_t$

*Proof.* We have

$$\gamma(\underline{r}, p_q) = \frac{1}{2\kappa}\alpha(2\kappa - (\overline{q} - \underline{r}))$$

[21] The "almost surely" notation is only necessary in a very special case, as one can see in the proof.

$$\begin{aligned}&\geq \tfrac{1}{2\kappa}\alpha(2\kappa - (\overline{r} - \underline{r}))\\&=\gamma(\underline{r}, p_r).\end{aligned}$$

Note that we always have $\gamma(\underline{r}, p_q) > 0$ since $\overline{q} < \overline{r}$ and thus $\underline{\underline{q}} < \underline{\underline{r}} \leq \underline{r}$. Hence the inequality above implies

$$\phi(\underline{r}, p_q) = p_q \cdot \gamma(\underline{r}, p_q) > p_r \cdot \gamma(\underline{r}, p_r) = \phi(\underline{r}, p_r)$$

so that $\underline{r}$ is not a $p_r$-limit point.

The condition $\overline{q} < \overline{r}$ implies $\phi(\underline{\underline{r}}, p_q) > \phi(\underline{\underline{r}}, p_r) = 0$. This also proves that $\overline{r}$ and $\overline{\overline{r}}$ must be limit points for $p_r$. Using Lemma 3.6.3 (2), this shows that the interval $[\underline{r}, \overline{r}]$ has full length, hence $\overline{r} = \overline{\overline{r}}$.

Assume now that $\phi(\overline{q}, p_q) \leq \phi(\overline{q}, p_r)$. From this it follows that $\phi(q, p_q) < \phi(q, p_r)$ for all $q \in [\underline{q}, \overline{q})$, implying that not $\underline{q}$ but $\overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$. But since $\overline{q} - \underline{q} < \overline{r} - \underline{r} \leq 2\kappa$, Lemma 3.6.3 (2) then implies that $\overline{q}$ is not such a limit point. This proves the existence of $\varepsilon > 0$ such that $[\underline{q} - \varepsilon, \overline{q} + \varepsilon] \cap Q^{\varepsilon}_{p_q}$ is empty[22], implying by Lemma 3.6.3 (3) that $[\underline{q}, \overline{q}]$ has full length. This case is excluded in the situation at hand.

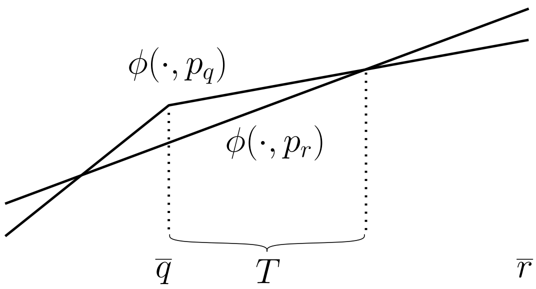


Figure 3.13: The type $t$ is chosen from the open interval $T$.

We now know that $\phi(\overline{q}, p_q) > \phi(\overline{q}, p_r)$. By continuity, the same is true for an interval $T$ of types above $\overline{q}$ (See Figure 3.13). Take any type $t$ in this interval with $\pi(t) \neq p_q$. It exists by Lemma 3.6.3 (1) and since $\overline{q} < \overline{\overline{q}}$. We know that the corresponding price $p_t := \pi(t)$ is also not equal to $p_r$ since it is not optimal for $t$ to set $p_r$. We can assume from Lemma 3.6.3 (1) that $t$ has been chosen with $t \notin (\overline{t}, \overline{\overline{t}}) \cup (\underline{\underline{t}}, \underline{t})$[23]. There are two

[22] This is true even if we had, $\phi(\overline{q}, p_q) = \phi(\overline{q}, p_r)$. The non-existence of such an $\varepsilon$ would make $\overline{q}$ a $p_q$-limit point, thus causing a contradiction.

[23] Otherwise, take a different $t$ from the interval $T$. Since there are only finitely many prices and thus finitely many sets of the form $(\overline{t}, \overline{\overline{t}}) \cup (\underline{\underline{t}}, \underline{t}))$ which are all null sets, there exists a $t \in T$ which is not in any of these sets.

cases left to consider.

First case: $t \in [\underline{t}, \overline{t}]$

Then we also have $\overline{t} \geq t > \overline{q}$.

First, assume $p_t < p_q$. We know from Lemma 3.6.2 that $\underline{t} < \underline{q}$ which is the situation of the lemma. Hence with the same reasoning, we can show that $[\underline{t}, \overline{t}]$ has full length. Moreover, we have $p_t < p_r$. Otherwise, Lemma 3.6.2 shows $\overline{r} < \overline{q}$ and hence the inequality $\phi(t, p_t) > \phi(t, p_r)$ implies $\phi(\overline{r}, p_t) > \phi(\overline{r}, p_r)$ which is a contradiction to $\overline{r}$ being a $p_r$-limit point. We thus have $p_t < p_r$ and (via Lemma 3.6.2) $\underline{t} < \underline{r} < \underline{q}$ and $\overline{t} > \overline{q}$. This is the second case stated in the lemma.

Second, assume $p_t > p_q$ and thus also $p_t > p_r$. Because of the higher slope of $\phi(\cdot, p_t)$ compared to $\phi(\cdot, p_r)$ in the interval $(t, \overline{t})$, we have $\overline{t} < \overline{r} = \overline{\overline{r}}$. Otherwise $\overline{r}$ could not be a $p_r$-limit point. Lemma 3.6.2 now implies $\underline{t} > \underline{r}$. This satisfies the first of the two cases stated in the lemma.

Second case: $t \geq \overline{\overline{t}}$

From $\overline{\overline{t}} \leq t < \overline{r}$ we know

$$p_t = \phi(\overline{r}, p_t) \overset{\overline{r} \text{ is } p_r\text{-limit point}}{\leq} \phi(\overline{r}, p_r) \leq p_r$$

so that $p_t < p_r < p_q$. Note that we must have $\overline{t} > \overline{q}$. This can be seen by observing that otherwise the slope of $\phi(\cdot, p_q)$ is always higher than the slope of $\phi(\cdot, p_t)$ in the interval $(\underline{q}, t)$. Since $\phi(t, p_t) \geq \phi(t, p_q)$, the strict inequality would hold within this interval, making it impossible for any type in $[\underline{q} - \varepsilon, \overline{q} + \varepsilon]$ (for some $\varepsilon > 0$) to optimally set the price $p_q$ which, again by Lemma 3.6.3 (3), gives a contradiction.

Knowing $\overline{t} > \overline{q}$ and $p_q > p_t$, Lemma 3.6.2 dictates $\underline{t} < \underline{q}$.

We thus have $\overline{t} > \overline{q}$, $\underline{t} < \underline{q}$ and $p_t < p_q$. This is the second of the two possibilities stated in the lemma. □

With this result we continue to prove the original statement.

*Proof of Lemma 3.6.5.* Assume that we had $\overline{q} < \overline{r}$. Denote $p_{\min} = p_r$, $p_{\max} = p_q$. Applying Lemma 3.6.6, the resulting price $p_t$ is either higher or lower than both, $p_{\min}$ and $p_{\max}$. Redefine these values so that $p_{\min}$ and $p_{\max}$ are the most extreme of these three prices, note that the new values of $p_{\min}$ and $p_{\max}$ satisfy the assumptions of Lemma 3.6.6. We can thus repeat these arguments over and over, ending up with an infinite number of equilibrium prices. This contradicts the assumption of finitely many equilibrium prices and proves $\overline{q} \geq \overline{r}$.

It is left to show that the two sets $Q^*_{p_r}$ and $Q^*_{p_q}$ can be strictly separated as stated in the lemma. This again has to be done considering multiple cases.

<u>First Case:</u> $\overline{r} < \overline{q}$
It then follows that $\overline{q}$ is a limit point of $Q^\pi_{p_q}$ (since $\underline{\underline{r}} < \underline{\underline{q}}$, $\underline{\underline{q}}$ can not be a $p_q$-limit point). From Lemma 3.6.3 we know that $\overline{\overline{q}}$ is not a limit point but $\underline{q}$ is. We thus have $\phi(\underline{q}, p_q) \geq \phi(\underline{q}, p_r)$. In the whole interval $[\underline{q}, \overline{\overline{q}}]$, the slope of $\phi(\cdot, p_q)$ is greater than the one of $\phi(\cdot, p_r)$. Hence $\phi(q, p_q) > \phi(q, p_r)$ for all $q > \underline{q}$. In the other direction, note that $\overline{r} < \overline{q}$ implies $\underline{\underline{r}} < \underline{\underline{q}}$ so that $0 = \phi(\underline{\underline{q}}, p_q) < \phi(\underline{\underline{q}}, p_r)$. The slope of $\phi(\cdot, p_q)$ is constant while the slope of $\phi(\cdot, p_r)$ is non-decreasing in $[\underline{\underline{q}}, \underline{q}]$. This proves the existence of some $t \in (\underline{\underline{q}}, \underline{q})$ with $\phi(q, p_q) > \phi(q, p_r)$ if $q > t$ and $\phi(q, p_q) < \phi(q, p_r)$ if $\underline{\underline{q}} \leq q < t$. Thus $Q^*_{p_r} \leq t \leq Q^*_{p_q}$.

<u>Second Case:</u> $\overline{r} = \overline{q}, \underline{r} > \underline{\underline{q}}$
We show that this situation is not possible and leads to a contradiction. We have, since $\underline{r} < \underline{q}$,

$$\phi(\underline{r}, p_r) = \frac{\alpha p_r}{2\kappa}(\underline{r} - \underline{\underline{r}}) < \frac{\alpha p_q}{2\kappa}(\underline{r} - \underline{\underline{r}}) = \frac{\alpha p_q}{2\kappa}(\underline{r} - \underline{\underline{q}}) = \phi(\underline{r}, p_q).$$

From this, it follows that $\overline{\overline{r}}$ is a limit point of $Q^\pi_{p_r}$. Note that, since $\overline{q} = \overline{r}$, the slope of $\phi(\cdot, p_q)$ is higher than the one of $\phi(\cdot, p_r)$ in the whole interval $[\underline{q}, \overline{\overline{r}}]$. For $\phi(\overline{\overline{r}}, p_r) \geq \phi(\overline{\overline{r}}, p_q)$ to be possible, we thus have $\phi(\overline{q}, p_q) < \phi(\overline{q}, p_r)$ and $\phi(\underline{q}, p_q) < \phi(\underline{q}, p_r)$. By continuity, this shows that

$$Q^\pi_{p_q} \cap [\underline{q} - \varepsilon, \overline{q} + \varepsilon] = \emptyset$$

for some $\varepsilon > 0$. Lemma 3.6.3 (3) then implies that $[\underline{q}, \overline{q}]$ has full length. But the assumptions of the second case imply

$$\overline{q} - \underline{q} < \overline{q} - \underline{r} < \overline{q} - \underline{\underline{q}} = 2\kappa.$$

<u>Third case:</u> $\overline{r} = \overline{q}, \underline{r} = \underline{\underline{q}}$
It follows that the interval $[\underline{r}, \overline{r}]$ has full length. If $p_r > \alpha p_q$, the claim $Q^*_{p_r} < Q^*_{p_q}$ automatically follows from observing that the slope of $\phi(\cdot, p_r)$ is strictly higher than $\phi(\cdot, p_q)$ before the point $\underline{q}$ and strictly lower afterwards. A similar argument holds if

$p_r < \alpha p_q$, the slope always being lower and thus contradicting $p_r$ being an equilibrium price. No type can make positive profit when setting this price. Both of these cases are shown in Figure 3.14.

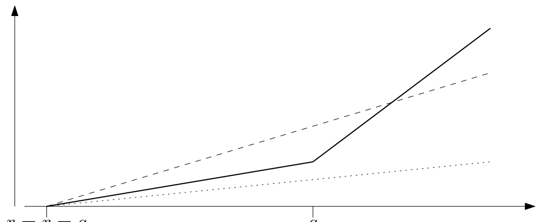


Figure 3.14: The situation of the third case when the price is high (dashed) or low (dotted).

A special case appears when $p_r = \alpha p_q$. All types in the interval $[\underline{\underline{q}}, \underline{q}]$ are then indifferent between setting $p_q$ or $p_r$. While we know that only a null set of types in this interval can actually set $p_q$, this would still be enough for the inequality $Q^*_{p_r} < Q^*_{p_q}$ not to be true. However, it is enough to observe that this inequality holds almost surely. $\square$

**Lemma 3.6.7.** *Let $p_r < p_q$ be two equilibrium prices with $\overline{r} \leq \underline{q}$. Then we have $Q^*_{p_r} < Q^*_{p_q} a.s.$.*

This lemma covers the intuitive case in which the intervals $[\underline{r}, \overline{r}]$ and $[\underline{q}, \overline{q}]$ are ordered according to their prices. The proof is rather easy, compared to the previous lemmas.

*Proof.* As before, $\overline{r} < \overline{q}$ implies that $\overline{q}$ is a $p_q$-limit point.

If $[\underline{q}, \overline{q}]$ has full length, the slope of $\phi(\cdot, p_q)$ is higher than the one of $\phi(\cdot, p_r)$ in the whole interval $[\underline{q}, \overline{q}]$. For values $q > \overline{q} = \overline{\overline{q}}$ we then have $\phi(q, p_q) = p_q > \phi(q, p_r)$. This proves the existence of $t \in (\underline{q}, \overline{q})$ so that

$$\phi(q, p_q) > \phi(q, p_r) \text{ if } q > t, \phi(q, p_q) < \phi(q, p_r) \text{ if } q < t,$$

proving the inequality $Q^*_{p_r} < Q^*_{p_q}$.

If $[\underline{q}, \overline{q}]$ does not have full length, Lemma 3.6.3 (2) shows that $\overline{\overline{q}}$ is not a $p_q$-limit points, so $\underline{q}$ is one. We thus have $\phi(\underline{q}, p_q) \geq \phi(\underline{q}, p_r)$ and $\phi(q, p_q) > \phi(q, p_r)$ for all types $q > \underline{q}$ (using the usual argument of $\phi(\cdot, p_q)$ growing faster than $\phi(\cdot, p_r)$). Hence $Q^{\pi}_{p_r} \leq \underline{q}$. Lemma 3.6.3 (1) tells us that only a null set of profitable types below $\underline{q}$ can set the price $p_q$ so that we have $Q^*_{p_r} \leq \underline{q} \leq Q^*_{p_q} a.s.$ which concludes the proof. $\square$

Finally, having these lemmas as preparation, we are able to proof our main theorem.

*Proof of Theorem 3.5.1.* First assume the existence of an equilibrium. Let $p_1 < \ldots < p_n$ be the equilibrium prices. The previous lemmas show that for each two indices $i < j$, the order of corresponding types setting the prices $p_i < p_j$ almost surely satisfy $Q^*_{p_i} < Q^*_{p_j}$.Using this, we have $Q^*_{p_1} < \ldots < Q^*_{p_n}$ *a.s.*. Every type $t$ in the non-empty set $Q^*_{p_1}$ is profitable by definition. Since the profit function is monotone in the type, all higher types also make positive profit and must hence set an equilibrium price. This shows that $\bigcup_{i=1}^n Q^*_{p_i} \supset (\inf Q^*_{p_1}, 1]$ so that all types above $q_{\min} := \inf Q^*_{p_1}$ set one of the prices $p_1, \ldots, p_n$. Thus $\pi$ is almost surely a non-decreasing step function when being restricted to types above $q_{\min}$.

The existence of an equilibrium is easy to show, noting that every constant price function $\pi(q) = p$ constitutes an equilibrium, independent of the indifference strategy $\alpha$. This can easily be seen by noting that $E(p, s)$ is uniquely determined by regular Bayesian updating and that for every other price, $E(\cdot, s)$ can be set low enough like in the existence proof of Lemma 3.4.2 to not allow beneficial deviations. The construction of an expectation system in the proof of Lemma 3.4.2 is done in this way. By this construction, the price $p$ always maximizes the firm's profit and we have an equilibrium. Note that the parameter $\alpha$ can be chosen arbitrarily. $\square$

## 3.7 Conclusion and Discussion

We studied a model of quality uncertainty, modified in such a way to admit a continuum of possible quality types and a costless extra quality signal for the consumer. The analysis shows that the price of the good depends on the quality in a positively correlated way in that a firm with a certain quality level never sets a higher price than if it would with any higher quality product. Hence in every equilibrium, the price behavior is a step function.

An interesting aspect of the model is the result of having a clear equilibrium pricing structure which is not unique but always takes the form of a step function, at least in those regions where actual trade takes place. A result obtained in a context which does not require - but allows - full rationality and high computational capabilities on the consumer side. Instead, our concept of an expectation system, skipping the step of Bayesian updating in most settings and thus generalizing the concept of Bayesian equilibria, gives an answer to the criticism on the "homo economicus" assumption present in the vast majority of economic literature. At the same time, the class of consumers for which this result holds contains the completely rational behavior.

Of course, once the monotonicity of price behavior is established, the form of the price function follows from our assumption of having only finitely many equilibrium prices. As explained in the text, this assumption is not unrealistic in many settings. There are, for example, only finitely many prices that you can encounter in a supermarket (assuming there is an upper bound for how much an item can cost). But it is worth mentioning that even without this assumption, pricing behavior must leave some ambiguity. If the pricing function $\pi$ was one-to-one, prices would perfectly signal the quality and thus the extra signal does (at least in equilibrium) not convey any information. If the signal had a marginal cost, the consumer would not choose to acquire it, thus only rely on price information and give lower quality levels an incentive to deviate. Similar arguments to the ones in Section 3.3 would apply to this situation. Moreover, in classical signaling games, the result of imperfect signaling even in the case where there are “enough” signals for perfect signaling, is common in the presence of a sender and a receiver with different objectives. The first to show this were Crawford and Sobel (1982). It is thus not at all clear whether the step function price behavior disappears if we relax our assumptions.

One of the most remarkable features of this model is certainly the fact that in the limit of perfect information, the maximum trade amount over all possible equilibria uniformly converges to zero, thus admitting an entirely different limit behavior than in the limit case of perfect information where the only equilibrium admits perfect trade for all quality types. Moreover, the proof of this phenomenon shows that it is indeed the continuum of types which causes this result.

While having more than two and even a whole interval of possible types is certainly more realistic than in many of other discussed models, our model is farther from the true situation in different aspects. We do not even claim that the case discussed here is closer to reality than certain other models with two quality levels (as for example given in the first chapter). However, this work should serve as a warning that in lemon markets, some simplification assumptions may not at all be innocuous.

## 3.A Appendix

*Proof of Proposition 3.3.2.* For two equilibrium prices $p > p'$, we must have $\gamma(p) < \gamma(p')$, otherwise the price $p'$ would never be set by any type which makes positive profit, since the higher price $p$ has a higher chance of selling and thus dominates setting $p'$. Hence the consumer strategy $\gamma$ must be strictly decreasing in the equilibrium prices.

Take any firm type $q$ and let $p$ be the price set by that type. For any other equilibrium price $p'$, we thus have

$$\gamma(p)(p - c(q)) \geq \gamma(p')(p' - c(q)).$$

Take such a price $p'$ with $p' < p$ and let $q'$ be a higher quality level than $q$ with a strictly higher production cost. We have $\gamma(p) < \gamma(p')$ and thus

$$\begin{aligned}\gamma(p)(p - c(q')) =& \gamma(p)(p - c(q)) + \gamma(p)(c(q) - c(q')) \\ >& \gamma(p')(p' - c(q)) + \gamma(p')(c(q) - c(q')) \\ =& \gamma(p')(p' - c(q'))\end{aligned}$$

so that higher quality level than $q$ never sets a lower price than $p$. Hence we have monotonicity in the price function. □

*Proof of Lemma 3.4.2.* We perform the proof by construction of a function $E$, given any price function $\pi$. It is enough to define the function value for a fixed price $p$ since all of the properties only involve one price. If $Q^\pi_p = \emptyset$, we can just choose $E(p, s) = p\frac{s+\kappa}{2(1+2\kappa)} \in [0, \frac{p}{2}]$. This obviously satisfies the definition and ensures that the consumer always strictly prefers to abstain from buying which later becomes important for the out-of-equilibrium consumer reaction in equilibria. In the case of $p = 0$, choose any increasing function. If $Q^\pi_p \neq \emptyset$, the property (v) of the definition determines the value of $E(p, s)$ for every signal in the set

$$\mathcal{S}_I := \left\{s \in S | Q^\pi_{p,s} \text{ is a non-empty interval (including singletons)}\right\}.$$

It is clear, since $Q^\pi_{p,s}$ is "increasing" (in an obvious sense) in $s$ and the Bayesian posterior depends only on the set $Q^\pi_{p,s}$ (not on $p$ and $s$ itself), that these values do not violate the other properties of the definition.

In what follows, we extend the function $E(p, \cdot)$ to the whole space $S$. Define the non-empty set

$$\mathcal{S}_{\neq\emptyset} := \left\{s \in S | Q^\pi_{p,s} \neq \emptyset\right\}.$$

On $\mathcal{S}_{\neq\emptyset}$, define the non-decreasing functions

$$\overline{E}(s) := \sup Q^{\pi}_{p,s}$$
$$\underline{E}(s) := \inf Q^{\pi}_{p,s}.$$

We can easily extend this function to the set $S_{\emptyset} := S \setminus s_{\neq\emptyset}$ in a way that ensures $\underline{E}(s) < \overline{E}(s)$ for all $s \notin S_I$.[24]

Take any $s \in \mathcal{S}_{\neq\emptyset} \setminus \mathcal{S}_I$. Define

$$\underline{\sigma} := \sup\left\{s' \in \mathcal{S}_I, s' < s\right\}, \overline{\sigma} := \inf\left\{s' \in \mathcal{S}_I, s' > s' \geq s\right\}$$
$$\text{and } \underline{e} := \sup\left\{E(p,s')|s' \in \mathcal{S}_I, s' < s\right\}, \overline{e} := \inf\left\{E(p,s')|s' \in \mathcal{S}_I, s' > s\right\},$$

the endpoints of the maximum interval of types around $s$ for which $E$ is not yet defined and the maximal and minimal value until these points. If one or both of these sets are empty, set $(\underline{\sigma}, \underline{e}) = (-\kappa, 0)$ or $(\overline{\sigma}, \overline{e}) = (1+\kappa, 1)$, respectively. Figure 3.15 shows the situation.

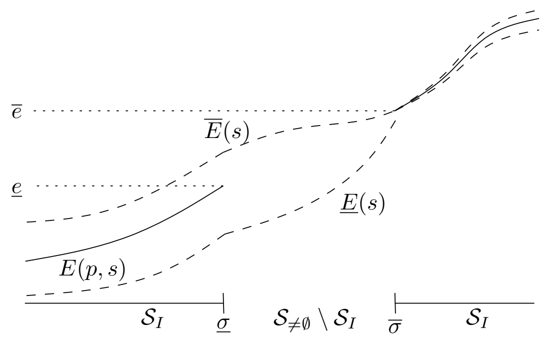


Figure 3.15: The situation of the proof after defining $E(p,s)$ on the set $\mathcal{S}_I$. The situation at $\overline{\sigma}$ shows the special case in which $E(p,\overline{\sigma}) = \overline{E}(\overline{\sigma}) = \underline{E}(\overline{\sigma})$.

To find values for the expectation system in the interval $(\underline{\sigma}, \overline{\sigma})$[25], it suffices to show that we have $\overline{E}(s) > \underline{e}$ and $\underline{E}(s) < \overline{e}$ for all signals $s$ in this interval. We can then choose values for $E(p,s)$ in the never-empty corridor $\left(\max\{\underline{e}, \underline{E}(s)\}, \min\left\{\overline{e}, \overline{E}(s)\right\}\right)$

[24] If $s \in \mathcal{S}_{\neq\emptyset}$, we know that $Q^{\pi}_{p,s}$ contains at least two elements. For the extension to $s \in S_{\emptyset}$, note that

$$\sup Q^{\pi}_{p,s'} < s - \kappa < s + \kappa \inf Q^{\pi}_{p,s''} \quad \forall\, s', s'' \in \mathcal{S}_{\neq\emptyset}, s' < s < s''.$$

Any two strictly increasing extensions with values in $[s-\kappa, s+\kappa] \cap [0,1]$ does the job.

[25] This interval could be empty in special cases. For single points like $\underline{\sigma}$ and $\overline{\sigma}$ one can just choose an appropriate value in $[\underline{e}, \overline{e}]$, keeping in mind the monotonicity assumptions of Definition 3.4.1.

(for $s \in (\underline{\sigma}, \overline{\sigma})$) in accordance to the expectation system definition (Letting it be constant when the sets $Q^{\pi}_{p,s}$ do not change and strictly increasing when they are empty). We only show the left inequality, the other direction using the same argument.

Note that we have $\underline{e} \leq \overline{E}(\underline{\sigma}) \leq \overline{E}(s)$ for all signals in the interval. The first inequality is not strict if and only if $\overline{E}(\underline{\sigma}) = \underline{E}(\underline{\sigma})$[26]. But in this case, $\underline{\sigma}$ is in $\mathcal{S}_I$, while signals $s'$ slightly above $\underline{\sigma}$ are not. This implies $\underline{e} = \overline{E}(\underline{\sigma}) = \underline{E}(\underline{\sigma}) \leq \underline{E}(s') < \overline{E}(s')$, the strict inequality using the property from our extension of $\underline{E}, \overline{E}$ to $S_{\emptyset}$.

□

*Proof of Lemma 3.4.3.* The existence and uniqueness follows just from item (ii) of an expectation system. Note that we can have $\underline{s} = \overline{s} \in \{-\kappa, 1+\kappa\}$ in the case where the expectation is never or always higher than the price.

Let now $\pi$ be a price function to which $E$ is an expectation system. Assume that $\overline{s} - \underline{s} > 2\kappa$. Then there exist $\underline{s} < s < s' < \overline{s}$ with $s' - s > 2\kappa$ and $E(p,s) = E(p,s') = p$. Definition 3.4.1 (iii) implies that one of the sets $Q^{\pi}_{p,s}$ and $Q^{\pi}_{p,s'}$ is not empty. If none of them is empty, we have

$$E(p,s) \leq \sup Q^{\pi}_{p,s} < \inf Q^{\pi}_{p,s'} \leq E(p,s')$$

which is a contradiction.

If $Q^{\pi}_{p,s}$ is empty, choose some $s''$ with $s < s'' < s' - 2\kappa < s'$. Then either $Q^{\pi}_{p,s''}$ is not empty (hence the argument above applies) or it is empty and we have by Definition 3.4.1 (iii) and (ii)

$$E(p,s) < E(p,s'') \leq E(p,s'),$$

again contradicting the equality of the left and the right expression.

The case of $Q^{\pi}_{p,s'} = \emptyset$ uses the same arguments. □

*Proof of Lemma 3.4.4.* Let $q < q'$ be two types. It then follows that

$$\begin{aligned}\phi_{\pi}(q') &\geq \phi(q', \pi(q)) \\ &= \pi(q)\frac{1}{2\kappa}\left(\int_{q'-\kappa}^{q'+\kappa} \underbrace{\alpha \mathbf{1}_{E(\pi(q),s)=\pi(q)}(s) + \mathbf{1}_{E(\pi(q),s)>\pi(q)}(s)}_{=:\beta(s)}\, ds\right)\end{aligned}$$

[26] To see this, note that the density $f$ has a positive minimum value so that there is $\beta \in (0,1)$ with $E(p,s) < \beta\overline{E}(s) + (1-\beta)\underline{E}(s)$. A picture of such a special case (for $\overline{\sigma}$) is given in Figure 3.15. The statement is trivially true in the "border" cases when $\underline{\sigma} = -\kappa$ or $\overline{\sigma} = 1 + \kappa$.

$$= \phi_\pi(q) + \frac{\pi(q)}{2\kappa}\left(\int_{q+\kappa}^{q'+\kappa} \beta(s)ds - \int_{q-\kappa}^{q'-\kappa} \beta(s)ds\right)$$
$$\geq \phi_\pi(q)$$

where the first inequality comes from optimality. To see the last inequality, let $s < s'$ be two signals. We then have the implications

$$E(\pi(q), s) > \pi(q) \Rightarrow E(\pi(q), s') > \pi(q)$$
$$E(\pi(q), s) = \pi(q) \Rightarrow E(\pi(q), s') \geq \pi(q)$$

by using the monotonicity of $E$. In what follows, since $\alpha \leq 1$, $\beta(s) \leq \beta(s')$. The left of the two integrals is thus larger since the integration area contains higher signals.

We now prove the continuity of $\phi^\pi$. Although this looks like a standard envelope theorem application, the function $\phi(q,p)$ is not continuous in the price component.

Let $q \in (0,1]$ be some type and let $(q_n)$ be a sequence of types below $q$, converging to $q$. We have

$$\phi^\pi(q) \overset{\text{Monot.}}{\geq} \phi^\pi(q_n) \overset{\text{Optimality}}{\geq} \phi(q_n, \pi(q)) \;\forall\; n.$$

Since the right hand side converges to $\phi(q, \pi(q)) = \phi^\pi(q)$ ($\phi(q,p)$ is continuous in $q$), $\phi^\pi$ is left-continuous.

For $q \in [0,1)$, let $(q_n)$ now be a sequence converging to $q \in [0,1)$ from below. For all $n \in \mathbb{N}$ we have

$$\phi_\pi(q_n) \overset{\text{Monot.}}{\geq} \phi_\pi(q) \overset{\text{Optimality}}{\geq} \phi(q, \pi(q_n)) \geq \phi_\pi(q_n) - C \cdot (q_n - q)$$

where $C$ is an upper bound for the slope of $\phi(\cdot, p), p \in [0,1]$.[27] Taking the limit shows $\lim_{n\to\infty} \phi_\pi(q_n) = \phi_\pi(q)$ and thus right-continuity. □

*Proof of Lemma 3.5.2 (The Existence Part).* Fix a price $\hat{p} \in (1-\kappa, 1)$. We claim that the constant price function

$$\pi(q) = \hat{p} \;\forall\; q \in [0,1]$$

[27] This upper bound can be chosen to be $\frac{1}{2\kappa}$, see Figure 3.3.

can be part of a locally continuous equilibrium. For this, we have to construct an expectation system. Note that without the local continuity assumption, out-of-equilibrium beliefs can just be taken low enough so that the buyer would never buy for any price other than $\hat{p}$. Now, we have to define the values for $E(p,s)$ for all signals $s$ in an environment of $\hat{p}$ in a continuous way. In what follows, the construction of out-of-equilibrium beliefs is taken not only in a locally but even in a globally continuous way, without the need to restrict ourselfs to an environment of $\hat{p}$.

Because $Q_{\hat{p},s}$ is always an interval, $E(\hat{p},s)$ is given by Bayesian updating and is thus strictly increasing in $s$. So there exists a pivotal signal $\hat{s} \in (1-2\kappa, 1+\kappa)$ with

$$E(\hat{p},s) < \hat{p} \ \forall s < \hat{s} \quad E(\hat{p},s) > \hat{p} \ \forall s > \hat{s}.$$

The existence and range of the signal comes from noting that signals close to $1+\kappa$ prove a quality above $\hat{p}$ and that signals below $1-2\kappa$ induce an expectation below $1-\kappa$ which is below the price $\hat{p}$.

For lower prices than $\hat{p}$, we set

$$E(p,s) = E(\hat{p},s) \cdot \frac{p}{\hat{p}}, s \in S, p < \hat{p}.$$

This construction preserves the strict monotonicity (demanded by definition 3.4.1 (iii)) to the lower prices and ensures $E(p,s) > p \Leftrightarrow E(\hat{p},s) > \hat{p}$ so that for all prices $p$ no signal can give a higher sale probability than the price $\hat{p}$. Hence, deviation to a lower price is not profitable.

The case of higher prices is a bit trickier. Not only do we have to ensure that sale probabilities do not increase when setting a higher price, they have to fall fast enough to nullify the positive price effect.

Claim: There exists $C > 0$ such that $E(\hat{p},s) \leq E(\hat{p},\hat{s}) + C \cdot (s-\hat{s})$ for all $s \in (\hat{s}, 1+\kappa)$. To proof this statement, note that we have

$$E(\hat{p},s) = \mathrm{Exp}(q | q \geq s-\kappa) = \frac{\int_{s-\kappa}^{1} q f(q) dq}{\int_{s-\kappa}^{1} f(q) dq}$$

by Baye's law, using Definition 3.4.1 (v). Differentiating this expression with respect to $s$, we get

$$\frac{\partial}{\partial s} E(\hat{p},s) = \frac{1}{\left(\int_{s-\kappa}^{1} f(q) dq\right)^2} \cdot \left( -(s-\kappa) f(s-\kappa) \int_{s-\kappa}^{1} f(q) dq + f(s-\kappa) \int_{s-\kappa}^{1} q f(q) dq \right)$$

$$
\begin{aligned}
&= \tfrac{f(s-\kappa)}{\left(\int_{s-\kappa}^{1} f(q)dq\right)^2} \cdot \int_{s-\kappa}^{1} (q-(s-\kappa))f(q)dq \\
&= \tfrac{f(s-\kappa)}{\int_{s-\kappa}^{1} f(q)dq} \cdot \operatorname{Exp}(q-(s-\kappa)|q \geq s-\kappa) \\
&\leq \tfrac{f_{\max}}{f_{\min}\cdot(1-(s-\kappa))} \cdot (1-(s-\kappa)) = \tfrac{f_{\max}}{f_{\min}} =: C.
\end{aligned}
$$

The values $f_{\max}$ and $f_{\min}$ refer to the maximum and minimum values of $f$. They exist and are positive due to our assumptions. We now have

$$E(\hat{p}, s) = E(\hat{p}, \hat{s}) + \int_{\hat{s}}^{s} \frac{\partial}{\partial t} E(\hat{p}, t)dt \leq E(\hat{p}, \hat{s}) + C \cdot (s-\hat{s})$$

which proves the claim.

Having this parameter $C$, we define the expectation for higher prices than $\hat{p}$ as follows.

$$E(p, s) = E(\hat{p}, s) \frac{p}{\hat{p} + C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)}$$

This is a continuous expression in $p$ and preserves the strict monotonicity in $s$ for every price. Now, we have

$$
\begin{aligned}
E(p, \hat{s} + \frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})) &= E\left(\hat{p}, \hat{s} + \frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right) \frac{p}{\hat{p} + C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)} \\
&\leq \left(E(\hat{p}, \hat{s}) + C\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right) \frac{p}{\hat{p} + C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)} \\
&= \left(\hat{p} + C\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right) \frac{p}{\hat{p} + C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)} \\
&= p
\end{aligned}
$$

which, because of the strict monotonicity, implies that a firm can only sell for a price $p$ if the signal is above $\hat{s} + \frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})$. Hence, for every quality type $q$

$$
\begin{aligned}
\phi(q, p) = p \cdot \gamma(q, p) &\leq p\left(q + \kappa - \left(\hat{s} + \frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)\right) \\
&\leq p\left(q + \kappa - \left(\hat{s} + \frac{1+\kappa-\hat{s}}{p}(p-\hat{p})\right)\right) \\
&= p(q-1) + \hat{p}(1+\kappa-\hat{s}) \leq \hat{p}(q-1) + \hat{p}(1+\kappa-\hat{s})
\end{aligned}
$$

$$= \hat{p}(q + \kappa - \hat{s}) = \hat{p} \cdot \gamma(q, \hat{p}) = \phi(q, \hat{p})$$

in the case where $\gamma(q,p) > 0$. Otherwise we trivially have $\gamma(q,p) = 0 \leq \phi(q,\hat{p})$. This shows that a deviation to a higher price is not profitable and we have an equilibrium. $\square$

*The rest of the proof of Lemma 3.5.2.* We first apply Theorem 3.5.1. Knowing that prices are set due to a step function, the set $Q^*_{p_q}$ is an interval for all equilibrium prices $p_q$. Lemma 3.6.3 (1) shows that in this case the interval $[\underline{q}, \overline{q}]$ has full length or we have $Q^*_{p_q} \subset [\underline{q}, \overline{q}]$.

In the latter case, Lemma 3.6.1 implies that $Q^*_{p_q}$ must be an interval with endpoints $\underline{q}$ and $\overline{q}$.

We thus only have to show that the former case can not occur. Note that, if $p_q$ is not the lowest equilibrium price, the case of $[\underline{q}, \overline{q}]$ having full length implies that, $\overline{q} = \overline{\overline{q}}$ is a $p_q$-limit point but $\underline{q} = \underline{\underline{q}}$ is not. Otherwise $\phi_\pi(\underline{\underline{q}}) = \phi(\underline{\underline{q}}, p_q) = 0$ so that there can be no lower profitable type. From this it follows that there must be a type $q > \overline{\overline{q}}$ which sets the price $p_q$. Otherwise let

$$t := \inf Q^*_{p_q} > \underline{\underline{q}}.$$

All signals in $(\overline{\overline{q}} - \kappa, t + \kappa) \neq \emptyset$ yield the same expectation due to property (iii) of an expectation system. This expectation must be equal to the price. If it was lower, profits would be zero. If it was higher, the local continuity condition implies that type $\overline{\overline{q}}$ could set a marginally higher price and still sell with full probability, making it profitable to deviate. Having $E(p_q, s) = p_q$ for all signals $s \in (\overline{\overline{q}} - \kappa, t + \kappa) \neq \emptyset$ is a contradiction to $[\underline{q}, \overline{q}]$ having full length.

The existence of the type $q > \overline{\overline{q}}$ setting the price $p_q$ implies that $E(q,s) > p_q$ for all signals $s \in [q - \kappa, q + \kappa]$ so that for the same reason as before a higher price could be demanded by type $q$ under the local continuity condition. $\square$

# 4 Reputation Concerns with Repeated Purchase

## 4.1 Introduction

While trying to emulate the main features of markets with quality uncertainty, models of lemon markets necessarily make strong assumptions on the market structure and leave out many aspects which may be crucial to mirror the real world markets and consumer reaction to quality uncertainty. The mathematical treatments of the previous two chapters tackle at least two of these aspects. Consumers in real life can acquire pre-purchase information and the quality of a product can usually have more than two possible values.

Not only do these chapters show interesting results within their respective model, their difference phenomena of the standard literature also emphasizes that results in lemon market models highly depend on the assumptions and shortcomings of the model. The reason for these drawbacks of mathematical treatments is clear. With each aspect which becomes more realistic, the model is harder to solve and very soon nothing can be said about its implications. While this of course is a problem which is present everywhere in theoretical economics, the literature on quality uncertainty especially seems to yield different results with different assumptions. There is a reason why the implication of adverse selection and even the existence of pure lemon markets was and is highly debated by economists[1]. To make things worse, the predicted effects are difficult to check with empirical methods.

Certainly, nobody would argue that any mathematical model in economics wants to perfectly reflect the real world. Their purpose, especially in microeconomics, is to show effects which can be present in particular settings and their potential cause. The adverse selection effect in lemon markets is a very good example. Nevertheless,

[1] Gans and Shepherd (1984) write that, although the rejection of Akerlof's paper by the Journal of Political Economy stated *triviality* as the main reason, a referee report suggested that if the paper was true, no market would exist in the presence of quality differences.

it always remains the question to which extend the results of the stylized world translate to actual market phenomena.

One of these phenomena was shown by a study by Schnabel and Storchmann (2010). They show that in markets for wine, the relationship between price and quality is quite weak when considering retail prices. On the other hand, the same analysis for wholesale prices shows a significant positive price-quality relationship. A remarkable feature of their empirical data is that in a certain quality region, retail prices are even going down when the quality is rising. Such a behavior is not covered by the classical lemon market models, although supermarket wine is one of the prime examples for experience goods and thus quality uncertainty. We have already seen that pooling equilibria are common in lemon market models but this is the most extreme case admitted by the standard method of Bayesian equilibria.

This chapter serves two main purposes. For one, we present an expandable tool to numerically compute long-run behavior in a lemon market with repeated purchase, non-rational heterogeneous[2] consumers, quite rational firms and multiple quality levels. Although a lot of benefits from the mathematical equilibrium treatments are lost using such a numerical method, this tool can be used to understand lemon markets from a descriptive point of view, implementing empirically credible non-rational consumer behavior and letting the firm be an almost rational revolving planner which looks multiple stages ahead. The second purpose is to use said tool to show that reputation considerations with repeated purchase can be the cause for the pricing behavior observed by Schnabel and Storchmann (2010).

One widely used approach to overcome problems of too rational market participants is the simulation of markets with agent-based numerical computations, allowing for a high degree of heterogeneity of agents and using heuristics to determine the agents' behavior. On the one hand, this idea permits more realistic settings and can hence claim to be closer to reality than most mathematical models. On the other hand, while certain effects may be observed in the results, their cause is usually hidden in the complexity of the setup. An agent-based approach does in general not tell you as much about implications of parameter changes than mathematical models. Certainly, outcomes can be compared under different parameters values, but this does not yield precise predictions for other settings or the conclusion whether the

[2]Goeree and Holt (2001) investigate empirical behavior in games. They find that, also in signaling games, the unique equilibrium behavior derived under rationality assumptions is not always observed.

new configuration implies a different equilibrium regime.[3] This completely heuristic simulation approach has been tried on lemon markets in the papers of Kim and Lee (2005) and Murata and Nisuo (2012). They use their models to show under which condition adverse selection can occur in an agent-based setting and the possibilities of quality-related signals.

This chapter follows a similar idea, trying to deal with shortcomings of the mathematical models like the limited number of quality levels (of the first chapter) and the homogeneity of buyers. As written before, buyers in general are numerous and their perception of quality or reputation may differ. An agent-based approach can cover this very well. The main issue to tackle here, however, is the absence of repetition in most lemon market models. While equilibrium analysis in markets is a static approach, one can easily argue that, in reality, most buyers will purchase a good not only once but multiple times in their live. Even if the quality of this good is not known beforehand, past experiences play a role if they are correlated to future quality levels.[4] Finally, our approach gets rid of the predominant assumption that the distribution of quality levels is known by the consumers. In course of this, perfectly rational updating is not possible which justifies the use of heuristics rather than Bayesian theory.

Bagwell and Riordan (1991) created an important work, introducing a time component into a lemon market. In their setting, a firm sells a product over multiple time periods. They show that under certain conditions the producer of a high quality product starts with a high price which declines over time. The reason for this lies in the nature of separating lemon market equilibria in which high quality has a high price but sells with a low probability. With time passing, more consumers get informed about the good's quality so the firm does not have to use a very high price to signal its quality in a separating equilibrium.

Bagwell and Riordan consider quality to be chosen once by nature and then staying constant over time, resulting in consumers who know the quality once and for all if they ever bought the product. With many kinds of products, however, this assumption is problematic since the quality of a producer's good may change between periods. A wine producer can be struck by a bad harvest and a person providing a service can have a bad day. Most technological products are constantly improved and their technology may completely change. There are now cars with non-traditional

[3]Markose, Arifovic and Sunder (2007) call this "wind tunnel tests" which is a pretty good analogy.

[4]Even if quality realizations are i.i.d. (and thus uncorrelated) but their distribution is unknown, past quality levels contain valuable information for a consumer.

fuel, TVs changed from CRT technology to LCD or plasma flatscreens, accompanied by further technological advancements like LED lighting and 3D support. A company that used to produce good TVs is not guaranteed to produce the same quality with a new technology. Our analysis encompasses this feature of changing quality levels.

Another approach of repeated purchase in markets with quality uncertainty was done by Riordan (1986). Also in that paper, quality stays constant over time but is determined by a strategic choice of the firms which in turn are rivals in a monopolistic competition setting.

This analysis is also different from the growing literature of Reputation Dynamics, as for example in Mailath and Samuelson (2001) and Liu (2011). In these models, firms costly influence the quality of their good, creating a moral hazard problem which differs over time with the amount of reputation. In contrast, we concentrate on Lemon markets with their indispensable assumption that the quality is not a strategic choice.[5] As mentioned in previous comments about this assumption, it can be seen as the stochastic part of a strategic choice like R & D expenditures.

Models of reputation are best known from the string of literature following Selten's chain store example from his 1978 paper. Although there are modifications with changing types (see for example Wiseman (2008)), non-perfectly-rational behavior (Liu and Skrzypacz (2014)) and imperfect information (Kreps and Wilson (1982)), the structure of the game, in which each competitor plays exactly once against the incumbent, does not apply to our market setting.

Part of the technical side of this chapter is quite similar to the ones of some of the examples given in the book by Stokey, Lucas and Prescott (1996). In their work, they introduce applications and technical aspects of Bellmann's *principle of optimality* in deterministic and stochastic models. Our construction of a Markov chain in discrete time and with a finite state space is similar to their approach. Their applications, however, do not cover quality uncertainty and consumer learning.

The chapter is divided as follows. We present the model, the consumers' belief adaptation process and the firm's objective with different levels of rationality. In what follows, the firm's behavior is determined and we describe how the market's Markov chain behavior can be numerically computed. Furthermore, we present a

[5]Shapiro (1982) argues that "models with exogenous quality supply are of limited usefulness in product markets [...] the market will be overrun by minimal quality items. The same result occurs in a dynamic model if consumers do not learn about the quality of individual firms over time." While the first two chapters certainly clarify the first point and put it into perspective, the following analysis - in a sense - picks up the last remark.

method to determine the convergence speed of the market distribution and estimates of expectations. We then implement these theoretical results and show that not only a negative price-quality relation is present in the market, its existence is caused by reputation concerns and is very robust to changes in the market parameters. We also use these robustness checks to determine key market aspects for this result of negative price-quality relation.

## 4.2 The Market

We model a lemon market situation with one firm, a finite number of customers and repeated purchase with discrete time. The firm produces a single product, the quality of which changes between periods in an independent way[6]. At each point in time, the quality is drawn from a finite set $Q \subset [0, 1]$, following some distribution $\eta$ on this set. Draws of different periods are independent. This quality distribution as well as the realized quality at each point in time is known by the firm only.

Consumers behave adaptive in the sense that they do not have access to the current or past quality realizations but they are endowed with a certain belief about the quality of the firm which is only changed when they buy a product and observe its quality.

We emphasize that in contrast to most of the literature on markets with quality uncertainty, consumers do not know the quality distribution $\eta$[7] and adjust their beliefs using heuristics rather than Bayesian updating.[8]

For numerical analysis purposes, the consumer state can not be too complex. Otherwise we would not be able to compute transitional matrices of Markov chains in which the state is the market situation, i.e. the combination of consumer beliefs. In accordance to the previous work, consumers would each hold a certain expected quality value $\mu \in [0, 1]$. Even if this belief would not depend on the observed price,

[6]This assumption, although mostly made for technical reasons, seems natural for wine producers who depend on the whether. It may be less true for electronic products. However, our example of the changing technologies in TVs suggests an imperfect correlation.

[7]The assumption of a public quality distribution is not crucial in settings without repeated purchase. The firm only needs to consider the current quality, the realization of which it knows. The quality distribution is only used as a prior of the consumer and as an important parameter in the equilibrium. However, it is not important that this prior coincides with the actual distribution. With repeated purchase, the firm needs to take into account further quality realizations and thus the correct distribution.

[8]There is a vast literature on behavior and updating without a prior distribution. Ellsberg (1961) showed in contrast to Savage (1954) that observed preferences are often not compatible with expected utility with one prior. Gilboa and Marinacci (2013) provide a survey of this literature.

the possible states of the market would be the set $[0,1]^N$. A finite Markov chain can thus not be constructed. To overcome this problem we work with a finite number of *categories* of posterior beliefs[9]. Let $C \geq 2$ be the number of categories. Then each posterior category $c \in \{1, \ldots, C\}$ is an interval $[a_c, b_c] \subset [0,1]$. Right before the purchase decision of an individual with category $c$ the actual expectation relevant for the decision manifests via a uniform distribution over this interval. This reflects non-observable elements of and influences on the purchase decision like the mood of the consumer while in the store. Using this procedure simplifies the state space of the dynamics while preserving a consumer individuality similar to the case with an infinite expectation space. We assume that $\{[a_c, b_c)\}_{c=1\ldots C}$ is a partition of $[0,1)$ and that the intervals are ordered according to their index $c$, i.e.

$$0 = a_1 < b_1 = a_2 < b_2 \ldots < b_C.$$

A *consumer state* $s$ then describes the number of adaptive consumers for each posterior category. Formally, it is a vector $(s_1, \ldots, s_C) \in \mathbb{N}^C$ with $\sum s_c = N$. This state is observed by the firm in any point of time. Whenever necessary, to avoid confusion, the time index $\tau$ is given as a superscript of the state in the form $s^\tau$. We reserve the letters $s$ and $t$ for states and $\tau$ for time periods.

After each period, there is a certain probability $\delta$ for each consumer to die and new adaptive consumers are reborn. For simplicity, we assume that the overall number $N$ of adaptive consumers is constant over time and the probability of consumers dying is independent of their age.[10] The initial category for each consumer is uniformly distributed over all categories. Moreover, all these random birth and death probabilities are independent between individuals and time periods.

The firm is a risk-neutral utility maximizer. We do not restrict ourselves to heuristics for the firm behavior but instead model it to be profit maximizing over a certain period of time, executing a *revolving planning* strategy with a rather high foresight level and thus behaving in a manner close to complete rationality. The mathematical details of this behavior are given below in the firm's analysis. We belief that the

[9] Many papers on decision making exist which use a form of categorization. One example, among many, is Martignon, Katsikopoulos and Woike (2008). See Gigerenzer and Gaissmaier (2011) for a discussion about various decision making processes.

[10] One can also look at this process as "forgetting the past experience" in which case the constant number of consumers is very well justified.

assumption of having a quite rational firm and heuristic-based consumers is natural to make and in accordance to the literature on boundedly rational behavior.[11]

Overall, the timeline of the market is as follows. In each period $\tau$, the state $s^\tau$ is known to the firm. The construction of the new state follows from the timing given in Figure 4.1.

State $s^\tau$ → Quality $q$ realizes → Firm sets price $p$ depending on $q$ → Consumers decide about buying → Consumers update their beliefs if they bought the product → Some consumers die, others are born → New State $s^{\tau+1}$

Figure 4.1: The timing of the market in one period.

## 4.3 Markov Chain Analysis

We show that the setting described in the model section describes a time-homogeneous Markov chain with a finite state space. It is well known that the distribution of such a Markov chain can be described by its initial distribution in the first period and the Matrix of transition probabilities between each two states. In what follows, we lay the mathematical foundation for computing this matrix. From the timeline in Figure 4.1 we can see that the transitional probabilities can be split into two main processes. The first contains the pricing of the firm, the buying decisions of the consumers and the adjustment of quality beliefs. The second one is the birth/death process which we consider first. Because of their timing, we can compute stochastic transition matrices for each of those two cases. Their product gives us the transition matrix for the overall process.

### 4.3.1 The Birth-Death Process

Denote $(X_\tau)_{\tau\in\mathbb{N}}$ our Markov chain over the state space $S$. We need to compute the stochastic matrix $B$ with $B(s,t) = Prob(X_{\tau+1} = t|X_\tau = s)$ for all states $s$ and $t$. If we have $s_c > t_c$ for some category $c$, at least $s_c - t_c$ people in the population have

[11] Hoyer (1984) finds that consumers apply very simple heuristics on repeated purchase decisions when the decision is not too important (laundry detergent, supermarket wine). They empirically analyze the choice between different options only at one point in time but find that the buyers' main motivation was the price and their experience with the good.

to die to reach state $t$. Thus define $m_c := (s_c - t_c)^+$ this minimal value and observe that the death vectors for which reaching $t$ is possible are in

$$\Delta(s,t) := \times_{c=1}^{C} \{m_c, \dots, s_c\}.$$

For each time index $\tau \in \mathbb{N}$, let $D_\tau$ and $B_\tau$ be the random variables for the vector of dying and reborn buyers, respectively. We have $X_{\tau+1} = X_\tau + B_\tau - D_\tau$. The probability of going from $s$ to $t$ can then be split up by summing over the death vectors.

$$\begin{aligned} B(s,t) &= \sum_{d \in \Delta(s,t)} Prob(D_\tau = d | X_\tau = s) \cdot Prob(X_{\tau+1} = t | D_\tau = d, X_\tau = s) \\ &= \sum_{d \in \Delta(s,t)} Prob(D_\tau = d | X_\tau = s) \cdot Prob(B_\tau = t - (s-d) | D_\tau = d) \end{aligned}$$

For each of these vectors $d$, the left factor is given by

$$Prob(D_\tau = d | X_\tau = s) = \prod_{c=1}^{C} \binom{s_c}{d_c} \delta^{d_c} (1-\delta)^{s_c - d_c}$$

since the probability of dying is independent between individuals. Hence for each posterior category $c$, the number of dying consumers is binomially distributed with the parameters $\delta$ and $s_c$.

For each such a death vector $d$, the state $t$ is reached if and only if the birth vector $b$ is $b = t - (s - d)$ since $s - d$ is the remaining population. For any real vector $v$, denote $|v|$ its sum. Using that each new born individual is independently drawn from a uniform distribution over the posterior categories, the probability of obtaining some non-negative birth vector $b$ is

$$Prob(B_\tau = b | D_\tau = d) = \begin{cases} \frac{|b|!}{b_1! \cdot \ldots \cdot b_C!} \frac{1}{C^{|b|}} = \binom{|b|}{b_1, \ldots, b_C} \frac{1}{C^{|b|}} & |b| = |d| \\ 0 & |b| \neq |d| \end{cases}$$

which is the value of a multinomial distribution. Note that the condition $D_\tau = d$ only ensures $|b| = |d|$ and does not have any other role in the formula.

Putting things together, the stochastic birth/death matrix has the following form.

$$B(s,t) = \sum_{d \in \Delta(s,t)} \binom{|t-s+d|}{(t-s+d)_1, \ldots, (t-s+d)_C} \frac{1}{C^{|(t-s+d)|}} \prod_{c=1}^{C} \binom{s_c}{d_c} \delta^{d_c} (1-\delta)^{s_c - d_c} \tag{4.1}$$

### 4.3.2 The Consumer

As described above, each consumer $n$ holds a posterior category $c$ which roughly determines her belief about the product's quality. Between observing the price $p$ and deciding about whether to buy the good or not, her quality expectation $\mu$ is drawn with a uniform distribution from the interval $[a_c, b_c]$. She then buys the product if and only if this expectation is higher than $p$. The buying amount is thus given by

$$\gamma(\mu, p) = \mathbf{1}_{\mu \geq p}$$

where $\mu$ is uniformly distributed over $[a_c, b_c]$. In what follows, a consumer will always buy if the price is lower than $a_c$. Thus, for every price $p \in (0,1]$, define $c(p)$ the category with $p \in (a_c, b_c]$[12]. Then we set

$$\sigma(s,p) = \sum_{c=c(p)+1}^{C} s_c \quad \forall\ s \in S, p \in (0,1]$$

the number of *sure buyers*, i.e. the consumers who buy with probability 1.

Only if a buyer purchases the good, her category can change for the next period. For now, we model the consumer to have inertia, i.e. the category never changes more than one step. Let $q$ be the actual quality for the good which is observed after the purchase. The category change is then given by the function

$$c_{\text{new}}(q,c) = \begin{cases} c-1 & q < a_c & \textit{(disappointment)} \\ c & q \in [a_c, b_c] & \textit{(correct expectations)} \\ c+1 & q > b_c & \textit{(positive surprise).} \end{cases}$$

Below, this assumption is checked for robustness. The combination of adaptive consumers and the birth-death process is similar to Schmalensee (1978). There, in

[12]The assignment of the boundaries to either of the adjacent categories has no influence on the results.

a setting with multiple sellers, dissatisfied consumers have a certain probability of buying from different sellers in the next period.

For a given price $p$ and any two states $s, t$, define $F_p(s,t)$ the probability for going from state $s$ to state $t$ when the price $p$ is set under the above process. The matrix $M_p := F_p \cdot B$ then describes the transition probabilities for this process and a subsequent birth-death procedure. A firm, setting price $p$, can thus determine the distribution of states which it faces in the next period over this matrix.

### 4.3.3 The Firm

While the consumer reaction follows the simple, rational rule to buy if and only if the price is below the expected value, the price setting behavior of the firm is more complicated. A rational firm maximizes its expected discounted revenue, knowing the consumer reaction and the distribution of future quality levels. We denote the inter-period discount rate by $r$. This optimization process is not easy to compute, especially the infinite horizon case. However, instead of resorting to simple heuristics, we approximate the optimal behavior, applying a dynamic, inductive optimization process described below.

#### The Myopic Firm

We call a firm *myopic* if it optimizes the revenue of only the current point of time, i.e. after observing the realized quality it maximizes the expectation of the expression

$$\sum_{n=1}^{N} p \cdot \gamma(\mu_n, p) \tag{4.2}$$

over the realized posterior beliefs $\mu_n$. It is obvious that this optimization problem does not in any way depend on the actual quality but only on the current state of the world, i.e. the posterior categories of the consumers. Knowing the consumers' buying behavior, we have

$$Prob\big(\gamma(\mu_n, p) = 1\big) = Prob\big(\mu_n > p\big) = \begin{cases} 1 & p < a_{c_n} \\ \frac{b_{c_n} - p}{b_{c_n} - a_{c_n}} & p \in [a_{c_n}, b_{c_n}] \\ 0 & p > b_{c_n} \end{cases}.$$

For a fixed $p$, denote $\bar{c} := c(p)$ the posterior category which contains $p$. This implies that expression (4.2) has the same distribution as

$$p \cdot \left( \sum_{c'=\bar{c}+1}^{C} s_{c'} + \sum_{i=1}^{s_{\bar{c}}} B_i \right) \tag{4.3}$$

where $B_i$ are iid random variables with Bernoulli distribution

$$P(B_i = 1) = \frac{b_{\bar{c}} - p}{b_{\bar{c}} - a_{\bar{c}}} = 1 - P(B_i = 0).$$

Expression (4.3) hence becomes

$$p \cdot (\sigma(s,p) + Z)$$

where $Z$ is binomially distributed with parameters $s_{\bar{c}}$ and $\frac{b_{\bar{c}}-p}{b_{\bar{c}}-a_{\bar{c}}}$. It is now easy to determine the expectation of this expression:

$$\phi(s,p) := p \cdot \left( \sigma(s,p) + s_{\bar{c}} \cdot \frac{b_{\bar{c}} - p}{b_{\bar{c}} - a_{\bar{c}}} \right). \tag{4.4}$$

Note that the value of $\bar{c}$ itself depends on $p$ but that it is constant in every posterior interval. It is thus sufficient to maximize expression (4.4) with respect to $p$ within every interval $[a_c, b_c], c = 1 \ldots C$ and compare the maximal values.

$$\max_{p\in[0,1]} \phi(s,p) = \max_{c=1\ldots C} \max_{p\in[a_c,b_c]} \phi(s,p)$$

Fix such an interval $[a,b]$ and observe that $\phi$ as given in (4.4) is a quadratic function in $p$ with negative coefficient of the quadratic term[13]. The first order condition gives

$$\frac{d}{dp} p \cdot \left( \sigma(s,p) + s_{\bar{c}} \cdot \frac{b-p}{b-a} \right) = \sigma(s,p) + s_{\bar{c}} \frac{b-2p}{b-a} \stackrel{!}{=} 0$$

[13] The case $s_c = 0$ is trivial and not explicitly treated. The value $\sigma(s,p)$ depends on $c(p)$ and is thus constant for $p \in [a,b]$.

which resolves to $p = \frac{1}{2}\left(\frac{b-a}{s_{\bar{c}}}\sigma(s,p) + b\right)$. The optimal price in the interval $[a,b]$ thus is unique and given by

$$p^* = \operatorname*{argmax}_{p\in[a,b]} \phi(s,p) = \begin{cases} a & \frac{1}{2}\left(\frac{b-a}{s_{\bar{c}}}\sigma(s,p) + b\right) < a \\ b & \frac{1}{2}\left(\frac{b-a}{s_{\bar{c}}}\sigma(s,p) + b\right) > b \\ \frac{1}{2}\left(\frac{b-a}{s_{\bar{c}}}\sigma(s,p) + b\right) & \text{else.} \end{cases}$$

In terms of computational complexity, this could hardly be easier. Computing the optimal price for every posterior category and comparing the optimal levels yields the optimal behavior of the myopic firm. Of course, this simplicity is not surprising if neither the realized quality, nor the profits of future periods are taken into account[14]. In what follows, we make the firm more rational.

### The T-foresight Firm

Let $T$ be a natural number. We call a firm a *T-foresight firm* if in every period its prizing behavior

$$p_\tau : S \times Q \to [0,1] \quad \tau = 0,\dots,T$$

optimizes the expected discounted profit

$$\sum_{\substack{s^1,\dots,s^T\in S\\ q_1,\dots,q_T\in Q}} Prob(s^1,\dots s^T, q_1,\dots q_T|(p_\tau), q_0, s^0) \sum_{\tau=0}^{T} (1-r)^\tau \cdot p_\tau \cdot \gamma(s^\tau, p_\tau)$$

where $q_0$ is the realization of the current quality, $q_1,\dots,q_T$ are the random future quality levels and the same holds for the state variables. Note that we left out the arguments for $p_\tau = p_\tau(q_\tau, s^\tau)$. The exact expression for the probability of the various state and quality realizations, given the current state and quality and the pricing strategy, is determined in the proof of the next result. This definition is consistent with the previous analysis, in the sense that a myopic firm is a 0-foresight firm.

[14]The use of this "foresight" rationality is a substantial difference to the analysis of Bagwell and Riordan (1991). They consider equilibria in which the firm does not take into account future periods. Only the market parameters, i.e. the share of informed consumers, change and lead to different equilibrium outcomes.

The next lemma provides the basis for an inductive maximization argument of a $T$-foresight firm. It is similar to the examples in the book by Stokey, Lucas and Prescott (1996) .

**Lemma 4.3.1.** *For $T \geq 1$ let $\Pi_{T-1}$ be the optimal profit for a $T-1$-foresight firm and let $\tilde{p}_\tau$ be one of its optimal pricing strategy. Then an optimal pricing strategy for the $T$-foresight firm is*

$$p_\tau = \tilde{p}_{\tau-1}, \tau \geq 1$$

*and for each $q_0$, $s^0$ the price $p_0(q_0, s^0)$ maximizes*

$$p_0 \cdot \gamma(s_0, p_0) + (1-r) \sum_{s^1 \in S, q_1 \in Q} \eta(q_1) \cdot \Pi_{T-1}(q_1, s^1) \cdot M_{p_0}(s^0, s^1).$$

*Moreover, this maximum is equal to $\Pi_T(q_0, s^0)$.*

The proof of this lemma can be found in the appendix. We see that if the optimal strategy of a $T-1$-foresight firm is known (and thus the optimal profit, depending on the realized quality), the profit of a $T$-foresight firm can be computed without taking into account the whole time horizon. We use this to inductively increase and compare the rationality of a firm, using the myopic firm as a basis. Since the firm uses a revolving planning strategy and thus "resets" its behavior in every period, the interesting part of the strategy is the pricing $p_0$ of the zero-period. Although the optimal pricing strategy is not necessarily unique, the optimal profit $\Pi_T$ is. Since $p_0$ only depends on $\Pi_{T-1}$ and not on $p_1, \dots, p_{t-1}$, this non-uniqueness problem is not amplified with higher foresight levels.

The equation of the lemma resembles a Bellman equation in discrete time which is used in control problems. Indeed, $\Pi_T$ can be seen as a value function of the current state and quality realization, while $p_\tau$ is the control strategy.

The maximizing pricing strategy of a $T$-foresight firm can now be computed as follows. Assume for $T \geq 1$ that the expected profit $\phi_{s'} := E(\Pi_{T-1}(q, s'))$ of a $T-1$-foresight firm is given for every state $s'$. Given the current state $s$ and the realized quality $q$, we fix a category $c$ and denote $[a, b] = [a_c, b_c]$ the posterior interval of that category. Let $s_{\text{new}}(i)$ be the new state which occurs in the next time period after exactly $i \in \{0 \dots s_c\}$ buyers from category $c$ (and all buyers with higher categories) obtain the good and update their beliefs (given the quality $q$). For any price $p \in [a, b]$, the expected profit is then given by

$$
\begin{aligned}
& p \cdot \gamma(s,p) + (1-r) \sum_{s' \in S} \underbrace{\sum_{q \in Q} \Pi_{T-1}(q,s') \cdot \eta(q)}_{=E(\Pi_{T-1}(q,s'))=\phi_{s'}} \cdot M_p(s,s') \\
& \overset{(4.4)}{=} p(\sigma(s,p) + s_c \cdot \tfrac{b-p}{b-a}) + (1-r) \sum_{s'} \phi_{s'} \cdot M_p(s,s') \\
& = p(\sigma(s,p) + s_c \cdot \tfrac{b-p}{b-a}) + (1-r) \sum_{s'} \phi_{s'} \cdot (F_p \cdot B)(s,s') \\
& = p(\sigma(s,p) + s_c \cdot \tfrac{b-p}{b-a}) + (1-r) \sum_{s'} \phi_{s'} \cdot \sum_{s''} \underbrace{F_p(s,s'')}_{\neq 0 \text{ only if } s''=s_{\text{new}}(i) \text{ for some } i} \cdot B(s'',s') \\
& = p(\sigma(s,p) + s_c \cdot \tfrac{b-p}{b-a}) + (1-r) \sum_{s'} \phi_{s'} \cdot \sum_{i=0}^{s_c} \underbrace{\binom{s_c}{i} \left(\tfrac{b-p}{b-a}\right)^i \left(\tfrac{p-a}{b-a}\right)^{s_c-i}}_{\text{Chance of } i \text{ buyers from cat. } c} \cdot B(s_{\text{new}}(i),s') \\
& = p(\sigma(s,p) + s_c \cdot \tfrac{b-p}{b-a}) + (1-r) \sum_{i=0}^{s_c} \binom{s_c}{i} \left(\tfrac{b-p}{b-a}\right)^i \left(\tfrac{p-a}{b-a}\right)^{s_c-i} \sum_{s'} \phi_{s'} \cdot B(s_{\text{new}}(i),s') \\
& = p(\sigma(s,p) + s_c \cdot \tfrac{b-p}{b-a}) + (1-r) \sum_{i=0}^{s_c} \binom{s_c}{i} \left(\tfrac{b-p}{b-a}\right)^i \left(\tfrac{p-a}{b-a}\right)^{s_c-i} \langle \phi, B(s_{\text{new}}(i),\cdot) \rangle.
\end{aligned}
$$

The notation $\langle \ldots \rangle$ denotes the Euclidean scalar product. Note that the first order condition of this expression does not have a closed solution for $p$ but the maximal value can be computed numerically. Again, we have to consider each posterior category $c$ and find the optimal price within the interval $[a_c, b_c]$. We then compare the profit levels for these optimal prices to obtain the best price, given the state $s$ and the realized quality $q$.

### 4.3.4 The Markov Chain

Having the matrix $B$, the matrices $F_p$ and an optimal[15] first-period pricing behavior $p_0$ of the $T$-foresight firm, the stochastic process can be described with the following transition probabilities from one state $s^\tau$ to another state $s^{\tau+1}$.

$$\begin{aligned}
Prob(X_{\tau+1} = s^{\tau+1} | X_\tau = s^\tau) &= \sum_{q \in Q} \eta(q) M_{p_0(s^\tau, q)}(s^\tau, s^{\tau+1}) \\
&= \sum_{q \in Q} \eta(q) \sum_{s' \in S} F_{p_0(s^\tau, q)}(s^\tau, s') B(s', s^{\tau+1}) \\
&= \sum_{s' \in S} \underbrace{\sum_{q \in Q} \eta(q) F_{p_0(s^\tau, q)}(s^\tau, s')}_{=: F(s^\tau, s')} B(s', s^{\tau+1}) \\
&= (F \cdot B)(s^\tau, s^{\tau+1}) =: M(s^\tau, s^{\tau+1})
\end{aligned}$$

The states $s'$ are the ones which can be reached in the stage after the buying procedure and before the birth/death process. This lets us compute the stochastic transition matrix $M$ of our Markov process which incorporates the optimal behavior, belief adaptations and substitution of the market participants.

### 4.3.5 Convergence to the Limit Distribution

The established results show that the market yields a finite Markov chain with state space $S$ and transitional probability matrix $M = F \cdot B$ where $F$ is the state adjustment from the realization of quality, the price setting of the firm and the consumer reaction when buying a product while $B$ is the transitional matrix from the birth-death process. Since $B$ has only strictly positive entries, the same follows for $M$ so that the Markov chain in particular is ergodic. It hence possesses a unique invariant distribution $\gamma \in \Delta(S)$[16] with[17]

$$\lim_{n \to \infty} d_{TV}(\zeta \cdot M^n, \gamma) := \lim_{n \to \infty} \sup_{A \subset S} |(\zeta \cdot M^n)(A) - \gamma(A)| = 0.$$

[15] We know that this $p_0$ does not have to be unique and hence our definition of the transition matrix $M$ depends on this $p_0$. However, the non-uniqueness only occurs if the expression in Lemma 4.3.1 has a non-unique solution. This appears for rather special parameters and is not a concern, here. In the implementation, we take the one price behavior given by the numerical optimization function. The rather smooth resulting graphs below suggest that this is very robust.

[16] We call $\gamma \in \Delta(S)$ an invariant distribution of $M$ if $\gamma \cdot M = \gamma$.

[17] Notation: For a vector $x \in \mathbb{R}^S$ and a set $A \subset S$ we write $x(A) := \sum_{a \in A} x(a)$.

This metric $d_{TV}$ is the distance in total variation. To control this distance, we use the positivity of $M$ to apply a standard result[18] and obtain the inequality

$$d_{TV}(\zeta \cdot M^n, \gamma) \leq (1 - \underbrace{|S| \min_{k,l} M_{k,l}}_{=:q \in (0,1]})^n \ \forall \ n \in \mathbb{N}$$

Although this does not yet give us direct control over convergence to the expectations of random variables, we can apply it to obtain a convergence of an expected value estimator.

**Lemma 4.3.2.** *Let $M$ be an everywhere-positive stochastic matrix over a finite set $S$ and $\gamma$ its invariant distribution. Moreover, let $Y : S \mapsto \mathbb{R}$ and let $B$ be an upper bound for $|Y(s)|, s \in S$. Then for every probability distribution $\zeta$ on $S$, we have*

$$|\sum_{s \in S} (\zeta \cdot M^n)(s) \cdot Y(s) - Exp_\gamma(Y)| \leq |S| \cdot B \cdot d_{TV}(\zeta \cdot M^n, \gamma)$$

*where $Exp_\gamma(Y)$ is the expectation of the random variable $Y$ under the limit distribution $\gamma$ on $S$.*

Together with the estimate of the total variation, this lets us now control the convergence of expectations, e.g. of expected market prices under the limit distribution.

## 4.4 Implementation

In the following, we use the theoretical thoughts to numerically compute the transition matrix of the Markov chain, using the optimal behavior of the firm and the consumer. In the end, to show an example and an interesting result, we compute the average price of high and low quality under the invariant distribution.

The structure of the model, using only categories of quality beliefs, not single values for each consumer, reduces the number of possible states. However, the state space $S = \{s \in \mathbb{N}^C | \sum_{c=1}^C s_c = N\}$ has $\binom{N+C-1}{C-1}$ elements[19], a number which grows

[18] This result is implicitly given in Häggström (2008). For completeness, an explicit proof is given in the appendix.

[19] To see this, note that each vector $s$ can be uniquely described by choosing $C - 1$ "bounds" on a linear grid of length $N + C - 1$.

$$N = 4, C = 3: \qquad \circ\circ|\circ|\circ \leftrightarrow (2,1,1) \qquad |\circ\circ\circ|\circ \leftrightarrow (0,3,1)$$

The number of states is then the same as the number of possible bound combinations, which amounts to selecting $C - 1$ elements from a set of $N + C - 1$ elements.

very fast with the number of consumers or categories. This is a challenge for the computations. For the numerical results presented in this section, we use $N = 20$ and $C = 4$ which gives a number of 1771 possible states.

Let $K = |S|$.[20] When computing the convergence speed to the limit distribution, we use the formulas from section 4.3.5 and the identity

$$a^{2^n} = \underbrace{\left((a^2)^{\cdots}\right)^2}_{n \text{ times}}$$

in which $a$ can be a real number or a square matrix. We thus only have to square the elements $n$ times to get an exponent of $2^n$.

The formula poses a different problem to numerical programs. The small number $q := |S| \min_{k,m} M_{k,m}$ is easily captured by the numerical representation which stores the first non-zero digits and the exponent in the way $q \sim a \cdot 10^b$. While this is very precise, the number $1 - q$ is not “small” and becomes equal to one in computer calculations. We thus use the equivalency

$$\begin{aligned}
& & K \cdot (1-q)^{2^n} &\leq \varepsilon & &\mid \log_{1-q}(\ldots) \\
&\overset{1-q<1}{\Leftrightarrow} & \frac{\ln(K)}{\ln(1-q)} + 2^n &\geq \frac{\ln(\varepsilon)}{\ln(1-q)} & & \\
&\Leftrightarrow & 2^n &\geq \frac{\ln(\varepsilon) - \ln(K)}{\ln(1-q)} & &\mid \log_2(\ldots) \\
&\Leftrightarrow & n &\geq \ln\left(\frac{\ln(\varepsilon) - \ln(K)}{\ln(1-q)}\right) / \ln(2) & &
\end{aligned}$$

for all $\varepsilon \in (0, K)$ to establish a lower bound of the number of squares we have to perform to reach a certain precision. As said before, the value $1 - q$ will evaluate to 1 in the numerical program so that the expression above is not well defined. We use the well-known inequality $\ln(x) \leq x - 1$ and the monotonicity of the logarithm to get the approximation

$$n \geq \ln\left(\frac{\ln(\varepsilon) - \ln(K)}{\ln(1-q)}\right) / \ln(2) \geq \ln\left(\frac{\ln(K) - \ln(\varepsilon)}{q}\right) / \ln(2)$$

which can be computed.

[20]In the estimation of Lemma 4.3.2 we set $B = 1$ because we want to compute the expectation of prices. They can not be higher than 1.

By squaring the matrix $M$ $n$ times, we then know by section 4.3.5 that every line of the resulting matrix is sufficiently close to the invariant distribution.[21] Additional numerical problems can occur here from the multiple matrix multiplications. In particular, we experience that the sum of the entries of the multiplied matrix can go down over time so that the matrix is not anymore stochastic. To minimize these issues, we take the best approximation $\vartheta$ of the invariant distribution, for which the expression $\|\vartheta \cdot M - \vartheta\|_\infty$ is the lowest. The value of this difference was never larger than $10^{-15}$ in our tests, suggesting a very high precision.

## 4.5 Main Result

We apply the Markov chain analysis to different sets of parameters. As mentioned above, we use 20 buyers and four posterior categories which amounts to 1771 different states. We fix the death rate $\delta$ to five percent and, for now, choose two possible quality levels, $q \in \{0, 1\}$ with a sixty percent chance of high quality in each period.

First, we use a discount rate of zero and check the impact of different foresight levels $T$ of the $T$-foresight firm. Figure 4.2 shows, as predicted, that there is no difference in pricing high and low quality products if the firm is myopic. Improving the foresight level, the firm tends to price high quality products lower than low quality products. This may come as a surprise but can easily be explained, having in mind that in each state the myopic profits are not depending on the quality but the future sales depend on how many people buy the product. This effect is adverse to the actual quality. If quality is high, there is an incentive to let more people buy the product to increase future sales. With a low quality product, the price is set higher so that fewer sales are made and thus less potential buyers reduce their posterior category.

The observed price difference is quite large. Some of it might come from the lack of discounting future sales. This naturally amplifies reputation concerns. To check the impact of this factor, we investigate the market behavior under different discount rates. The results are shown in Figure 4.3. Although the price difference for high foresight values is indeed lower with higher discount, it stays on a quite high level even for a high discount of ten percent. This suggests that the reputation effect is

[21] As Häggström points out, it is not common to actually compute the required exponent but instead choose a number and argue why it is sufficient. Indeed, the resulting number of these formal derivations is usually very large and not very practical. Many steps of this procedure use estimations so that the actual convergence is faster, anyway. In our case, however, the resulting number is not too high so it stays part of the analysis.

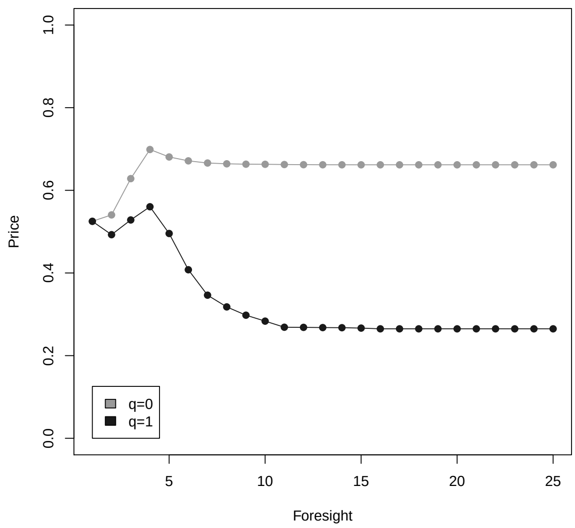


Figure 4.2: The average prices for high and low quality for different foresight levels, no discount

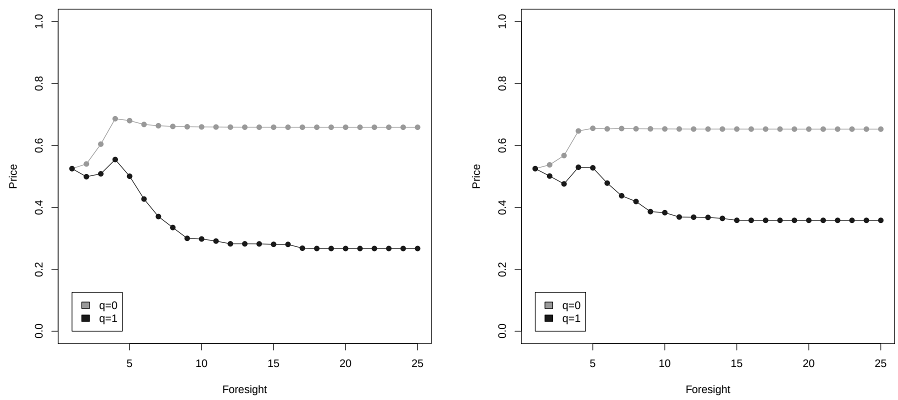


Figure 4.3: The average prices for high and low quality for 2% and 10% discount.

very strong in this market with repeated purchase. For the rest of this chapter we fix the discount rate at five percent.

These two graphics show a dent in the price of high quality products for lower foresight levels (4 or 5), prices first going up and then converging to a lower level. This is no computation error and also appears in other computations in even more extreme form when we change the model below. It suggests that short-term pricing effects are quite complex and the exact foresight value is important for the market behavior except in the range when we have quite large foresight and the firm is very rational. This is the case we are interested in.

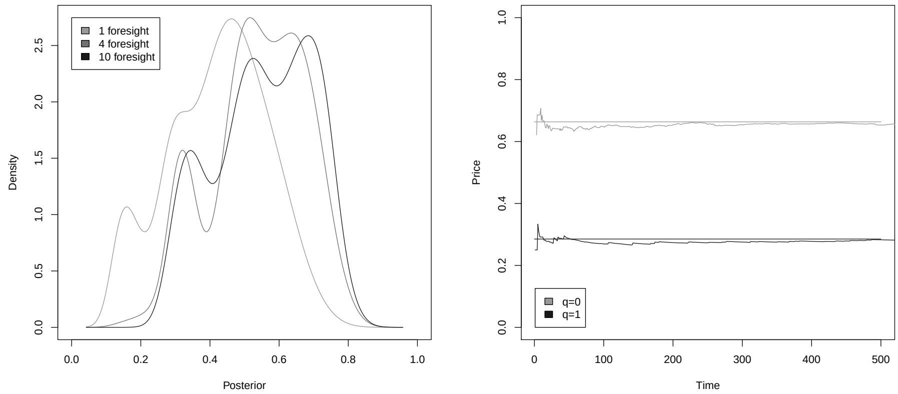


Figure 4.4: Left: The density (kernel estimation) of average expected quality under the invariant distribution. Right: A sample run of the Markov chain. The running average of prices for high and low quality goods.

To further increase intuition about the background of these result and the complexity of the market behavior, Figure 4.4 shows the long-run distribution of posterior beliefs for different foresight values and one example run of the Markov chain. We see that the distribution of perceived quality shifts to higher values with more foresight because of the reputation effect. The average prices for high and low quality levels are very close to the computed average value right from the beginning. This is representative and appears in all our trial runs. While actual convergence might be slower and is hampered by random clusters of sequential high or low quality values, the main effect is present from the beginning.

## 4.6 Introducing Experts

The previous result is quite strong in the sense that if the firm is only a bit rational and cares about the next period, high quality products are priced lower than those with low quality and this price difference becomes rather big with more foresight. Since there is no way for the consumers to know the current quality, the realization only enters the firm's optimization problem in terms of future sales, not in the revenue of the current period. One can easily argue that in many markets this assumption is too extreme. There are people knowing the quality of a product either because they acquired this information or because they are experts and can judge the product's quality before the purchase. For example, a technically affine person might know from the data sheet whether a TV will suit her needs while someone else only notices at home if the screen resolution is sufficient or the technology produces a good picture.

To acknowledge this, we introduce the presence of experts in the market.[22] In addition to the adaptive consumers, there is a fixed number $e$ of experts, always knowing the realized quality and only buying if the price is not higher than the quality $q$. For simplicity we assume that experts are not part of the birth/death process, i.e. their number is constant and the state space still only contains the adaptive buyers. This way we are also able to compare the results.

Most of the analysis above can easily be adapted to the new situation. The non-expert consumers' behavior stays the same, the firm's objective changes in an obvious way. The pricing decision must include the expert's choice, giving additional incentive to highly price the good products. A myopic firm now maximizes the expectation of the expression

$$p \cdot \mathbf{1}_{p \leq q} e + \sum_{n=1}^{N} p \cdot \gamma(\mu_n, p) = p \cdot \left(\mathbf{1}_{p \leq q} e + \sigma(s, p) + Z\right),$$

using the notation from the previous analysis. One can see directly that a myopic firm optimally charges a higher price than before if the quality is very high (and thus $\mathbf{1}_{p \leq q} = 1$ for a certain range of prices). In the old model, this price did not depend on the quality.

The myopic-firm analysis of the case with experts is very similar to the previous one and is omitted.

[22]The introduction of experts is common in the literature and used, for example, by Linnemer (2002) in the context of lemon markets.

A $T$-Foresight firm uses a similar optimization problem, inductively knowing the next-period profit for each state and taking into account the experts' purchases in the current period. Formally, the objective becomes to maximize the expression

$$p(\sigma(s,p) + s_c \cdot \tfrac{b-p}{b-a} + e \cdot \mathbf{1}_{q_0 \geq p}) + \delta \sum_{i=0}^{s_c} \binom{s_c}{i} \left(\tfrac{b-p}{b-a}\right)^i \left(\tfrac{p-a}{b-a}\right)^{s_c - i} \langle \phi, B(s_{\text{new}}(i), \cdot)\rangle$$

which, again, only differs in one term from the previous formula. Note, however, that the value of $\phi$ also differs in the computations and that, if $q \in (a_c, b_c)$ for some category, optimization is separately needed in the intervals $[a_c, q]$ and $[q, b_c]$ instead of the interval $[a_c, b_c]$.

Having this, we can do the same Markov chain analysis to see whether the negative price-quality relationship holds in the presence of experts. Figure 4.5 shows graphs for the prices of high/low quality with various numbers of experts. We see that even when more than half of the consumers are experts, a very rational firm still sets a low price for good quality.

We also see that there is a limit number of experts after which even a foresight of 25 periods (and probably any other value) is not enough to create a sufficient effect for a negative price-quality relationship. But it is remarkable that even with 30 experts (and only 20 adaptive consumers) the effects is strong enough to be observed with a quite rational firm.

## 4.7 Further Robustness Checks

There are other aspects of the model which we have not yet discussed. So far, we are using a consumer behavior which is quite moderate. If a buyer acquires a product and its quality does not fit the category, she switches to the next category closer to the experienced quality. This was cautiously chosen so that the consumer reaction is slow and reputation is both, hard to obtain and hard to lose, thus even dampening reputation effects. Intuitively, having more extreme consumers will lead to an even more severe reputation effect, letting the gap in quality pricing be even higher. To check this intuition, we consider *extreme consumers* which directly jump to the posterior category to which the experienced quality belongs. Additionally to this, we check *unforgiving consumers* who behave like extreme consumers when the quality is lower than expected but behave like in our original computations when quality is higher. In other words, reputation is very hard to obtain but very easy

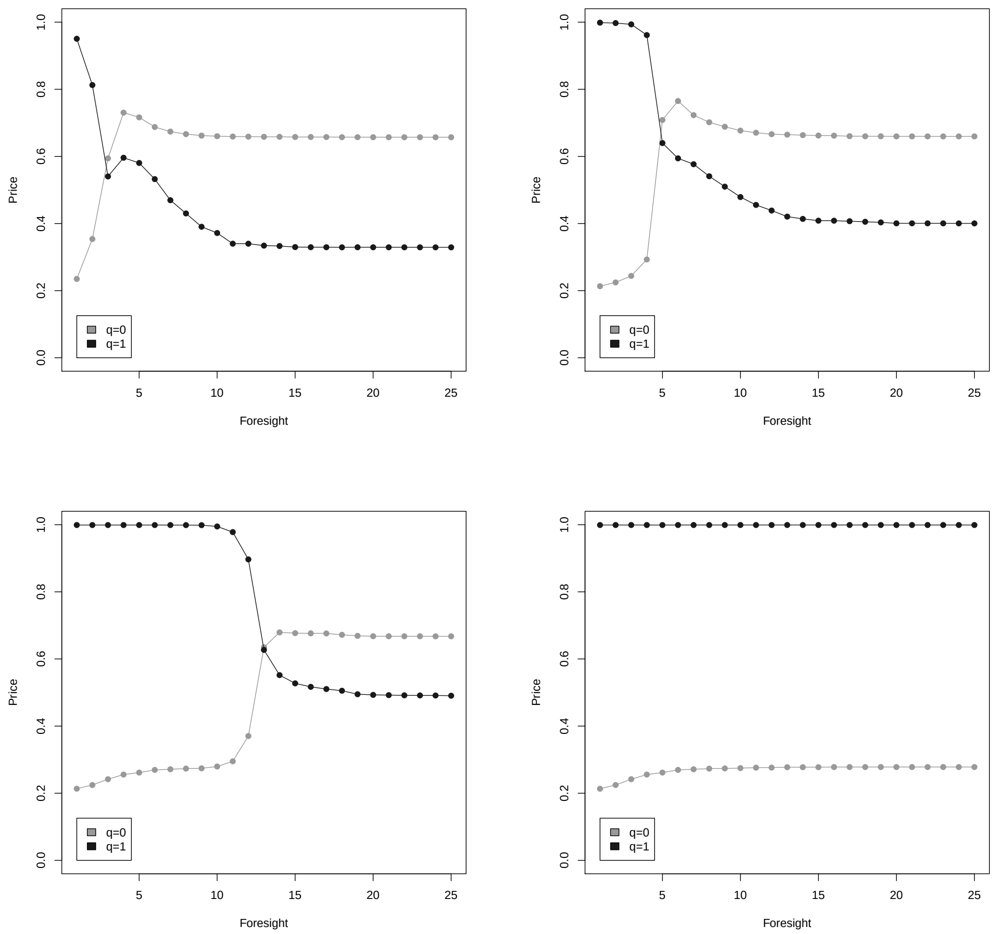


Figure 4.5: The pricing of low quality and high quality goods with different foresight levels and number of experts. From top left to bottom right, there are 5, 15, 30 and 50 experts present in the market.

to loose with unforgiving buyers. Finally, we test the opposite behavior, called *enthusiastic consumers*, who jump to high quality very quickly but go down only single steps.

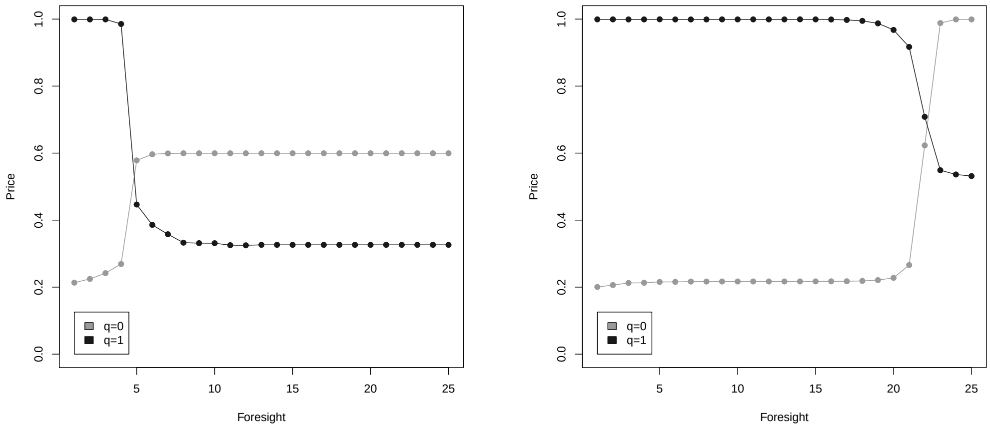


Figure 4.6: The case of enthusiastic consumers (left) and unforgiving consumers (right) in the presence of 30 Experts and a discount rate of 0.05.

Figure 4.6 shows that changing the nature of the consumer reaction does not change the qualitative aspects of the model outcomes, the reputation effects still being high enough to outweigh the presence of the experts. Interestingly, the reputation effect (in terms of the price difference) is higher with unforgiving consumers, as predicted, once the critical foresight level is reached. However, this critical foresight level is much higher than in the case of unforgiving consumers or consumers with inertia. The firm tends to ignore buyers who are not easy to satisfy.

Basically the same effect is observable with extreme consumers who directly jump to the experience category and thus have no inertia in any direction. With 30 experts (Figure 4.7), the level of foresight needed to observe a negative price-quality relationship is about the same as for the enthusiastic consumers but the price level is much lower than before. Since lost reputation can be re-obtained quite fast, incentives for high pricing of low quality are not as high, resulting in an overall lower price difference.

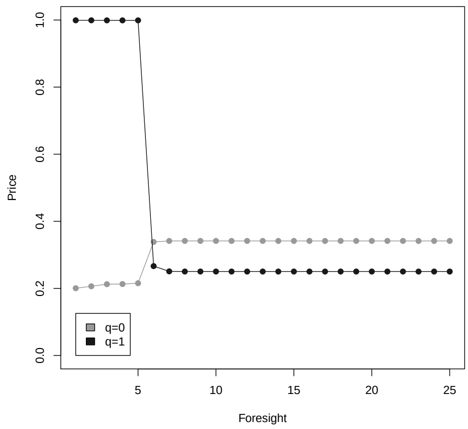


Figure 4.7: The case of extreme consumers with 30 experts.

### 4.7.1 More Quality Levels

The previous parts of the chapter uses only two quality levels to not make the graphics too confusing. Having the previous chapter in mind, it is worth checking whether this restriction obfuscates important phenomena. It would be possible that the pricing behavior opposed to the quality is only valid for the lowest or the highest price while all other prices are ordered according to their quality. To check this, we let the computations run with four different quality levels[23], each in the middle of an interval of the equal-sized, four-part partition of $[0, 1]$. We let all of these quality levels be equally likely.

We set the number of experts back to zero and the consumers to the regular adaption behavior with inertia. As we see in Figure 4.8, this leads to an interesting observation.

With low but non-zero foresight, the market indeed shows the described behavior. Only the very low quality is priced highly to prevent consumers from buying while the prices of all other quality levels preserve their order. With increasing foresight, however, the market converges quickly to a perfect inverse price-quality relation. It has to be noted, though, that the price difference of high quality goods is significantly lower than the one for the low quality goods.

[23]Note that, because of the structure of the model, quality levels in the same category are equivalent in the absence of experts. Hence, with four categories, the case with four quality levels yields the maximum quality diversity.

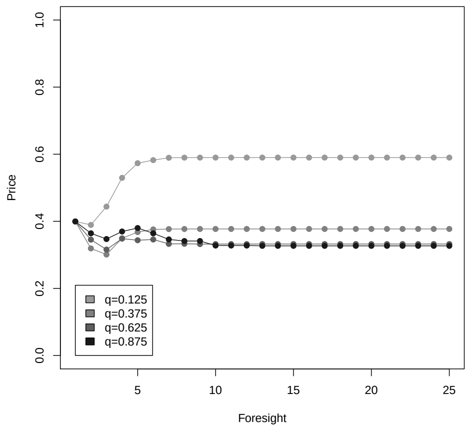


Figure 4.8: The average price development for the four different quality levels.

## 4.8 Conclusion and Discussion

The chapter presents a method to compute market behaviour under quality uncertainty and repeated purchase in a numerical way but with mathematical justification and correct estimation of the convergence to behavior under the invariant distribution of the market's Markov chain. It applies this method to investigate the relation between price and quality in such a market, suggesting that there is a strong reputation effect which causes high quality products to be sold for a relatively low price while at the same time pricing low quality products very high to prevent consumers from bad experiences and thus losing reputation.

The robustness check shows that this effect even occurs in the presence of relatively many informed consumers (experts) if the seller maximizes over a longer horizon. Moreover, even when consumers are very forgiving (so that reputation is easily built but not easy to lose), this effect occurs. Overall, the results suggest that a negative price-quality relationship can be caused by reputation considerations, a non-negligible share of uninformed consumers and farsighted firms.

The observed effect is surely not present in most markets. Many aspects are very specific and do not occur in this extreme manner. With most products, people get informed before the purchase, at least after the first time they have bad experiences. The more expensive a product, the more the consumer will try to inform herself before the purchase, thus probably creating a level of expertise which is strong enough to outweigh the reputation effects. As the expert section shows, this is possible. Moreover, we ignored advertising and attention effects. In some markets, however, the assumptions do not seem too far away. Especially with wine, the high share of "non-experts" who buy by the trial-and-error principle is quite high and empirical findings show in the direction of our results. The paper thus shows possible reasons for those experimental findings.

The tool presented here is explicitly designed to be easily extensible and is hence able to incorporate other aspects which one might investigate. For example, network opinion effects in the style of DeGroot (1974) or population-dependent reproduction (e.g. a Moran process with mutations, see Ewens (2004), p.104-109) can be introduced by just changing the birth-death matrix, leaving everything else constant in the analysis and the code.[24] The firm will automatically adapt its optimal behavior to the different situation. In the same way, the data provided can be used to analyze more aspects of the market outcome. It will be interesting to see this tool applied with focus on other phenomena and incorporating different behavioral assumptions.

## 4.A Appendix

*Proof of Lemma 4.3.1.* As stated in the text, the expected profits to maximize is

$$\sum_{\substack{s^1,\dots,s^T \in S\\ q_1,\dots,q_T \in Q}} Prob(s^1,\dots s^T, q_1,\dots q_T|(p_t), q_0, s^0) \sum_{\tau=0}^{T} (1-r)^\tau \cdot p_\tau \cdot \gamma(s^\tau, p_\tau).$$

Using the Markov chain structure and the fact that quality realizations are independent (from each other and from the states), this expression is equivalent to

$$\begin{aligned}
&\sum_{\substack{s^1,\dots,s^T\\ q_1,\dots,q_T}} \left(\prod_{\tau=1}^{T} \eta(q_\tau)\right) \left(\prod_{\tau=0}^{T-1} M_{p_\tau}(s^\tau, s^{\tau+1})\right) \sum_{\tau=0}^{T} (1-r)^\tau \cdot p_\tau \cdot \gamma(s^\tau, p_\tau) \\
=&(1-r)^0 \cdot p_0 \cdot \gamma(s^0, p_0) \\
&\quad + (1-r) \sum_{\substack{s^1,\dots,s^T\\ q_1,\dots,q_T}} \left(\prod_{\tau=1}^{T} \eta(q_\tau)\right) \left(\prod_{\tau=0}^{T-1} M_{p_\tau}(s^\tau, s^{\tau+1})\right) \sum_{\tau=1}^{T} (1-r)^{\tau-1} \cdot p_\tau \cdot \gamma(s^\tau, p_\tau) \\
=&p_0 \cdot \gamma(s^0, p_0) + (1-r) \sum_{s^1, q_1} \eta(q_1) M_{p_0}(s^0, s^1)
\end{aligned}$$

[24]The important part here is to choose a process which either keeps the matrix $B$ positive in all entries or to find a different argument why the Markov chain is ergodic under the new process.

$$\cdot \sum_{\substack{s^2,\ldots,s^T \\ q_2,\ldots,q_T}} \left(\prod_{\tau=2}^{T} \eta(q_\tau)\right) \left(\prod_{\tau=1}^{T-1} M_{p_\tau}(s^\tau, s^{\tau+1})\right) \sum_{\tau=1}^{T} (1-r)^{\tau-1} \cdot p_\tau \cdot \gamma(s^\tau, p_\tau).$$

Maximizing this with respect to $p_\tau(q_\tau, s^\tau)$ for all values of $\tau, q_\tau$ and $s^\tau$, we can use that the first terms do not depend on $p_1, \ldots, p_T$ to obtain

$$\max_{p_0} \left\{ p_0 \cdot \gamma(s^0, p_0) + (1-r) \sum_{s^1, q_1} \eta(q_1) \cdot M_{p_0}(s^0, s^1) \right.$$

$$\left. \cdot \underbrace{\max_{p_1,\ldots,p_T} \left\{ \sum_{\substack{s^2,\ldots s^T \in S \\ q_2,\ldots,q_T \in Q}} \prod_{\tau=2}^{T} \eta(q_\tau) \prod_{\tau=1}^{T-1} M_{p_\tau}(s^\tau, s^{\tau+1}) \sum_{\tau=1}^{T} (1-r)^{\tau-1} \cdot p_\tau \cdot \gamma(s^\tau, p_\tau) \right\}}_{\text{Optimization problem for a } T-1\text{-foresight firm at time 1, for fixed } q_1 \text{ and } s^1.} \right\}$$

$$= \max_{p_0} \left\{ p_0 \cdot \gamma(s^0, p_0) + (1-r) \cdot \sum_{s^1, q_1} \eta(q_1) \cdot M_{p_0}(s^0, s^1) \cdot \Pi_{T-1}(q_1, s^1) \right\}.$$

$\square$

*Proof of the convergence result of section 4.3.5.* Let $m := \min_{k,l} M_{k,l}$. We then have

$$\begin{aligned}
d_{TV}(\zeta M, \gamma) =& \frac{1}{2}\|\zeta M - \gamma\|_1 = \frac{1}{2}\|\zeta M - \gamma M\|_1 = \frac{1}{2}\|(\zeta - \gamma)M\|_1 \\
=& \frac{1}{2}\sum_i |\sum_j (\zeta_j - \gamma_j) M(j,i) - \underbrace{m \sum_j (\zeta_j - \gamma_j)}_{\text{added term},=0} | \\
=& \frac{1}{2}\sum_i |\sum_j (\zeta_j - \gamma_j)(\underbrace{M(j,i) - m}_{\geq 0})| \\
\leq& \frac{1}{2}\sum_j |(\zeta_j - \gamma_j)| \underbrace{\sum_i (M(j,i) - m)}_{=1-m\cdot|S|} \\
=& \frac{1}{2}\|\zeta - \gamma\|_1 (1 - |S| \cdot m)
\end{aligned}$$

where the first equality ist well known for finite state spaces. It hence follows inductively that we have

$$\begin{aligned} d_{TV}(\zeta M^n, \gamma) &\leq \frac{1}{2}\|(\zeta M^{n-1} - \gamma)\|_1 (1 - |S| \cdot m) \leq \dots \\ &\leq \frac{1}{2} \underbrace{\|\zeta - \gamma\|_1}_{\leq \|\zeta\|_1 + \|\gamma\|_1 = 2} (1 - |S| \cdot m)^n \leq (1 - |S| \cdot m)^n \end{aligned}$$

□

*Proof of Lemma 4.3.2.*

$$\begin{aligned} |\sum_{s\in S}(\zeta M^n)(s) \cdot Y(s) - Exp_\gamma(Y)| &= |\sum_{s\in S}(\zeta M^n)(s) \cdot Y(s) - \sum_{s\in S}\gamma(s) \cdot Y(s)| \\ &= |\sum_{s\in S}\big[(\zeta M^n)(s) \cdot Y(s) - \gamma(s) \cdot Y(s)\big]| \\ &\leq \sum_{s\in S} |Y(s)| \cdot |(\zeta M^n)(s) - \gamma(s)| \\ &\leq \sum_{s\in S} B \cdot d_{TV}(\zeta M^n, \gamma) \\ &= |S| \cdot B \cdot d_{TV}(\zeta M^n, \gamma) \end{aligned}$$

The last inequality stems directly from the definition of $d_{TV}$. □

## 4.B The Code

The following file contains the parameter definition and the main function which is used to compute the invariant market distribution. It implements other important code which is given further below.

main.r

```
## Create the list of parameters. Change this for different settings
params = list();
params$num_people <- 20; # The number of non-experts
params$num_categories <- 4; # Number of posterior categories
params$dying_prob <- .05; # For the birth/death process
params$discount <- .05; # Firm's discount of future income
params$num_experts <- 0; # Number of experts
params$qualities <- c(0,1); # The possible quality levels
params$qual_dist<- c(.4,.6); # The distribution of quality levels

## uncomment this for the four-quality case
```

```
#params$qualities <- c(.125,.375,.625,.875);
#params$qual_dist<- c(.25,.25,.25,.25);

## derived parameters:
num_states = choose(params$num_people + params$num_categories - 1,
     params$num_categories - 1);
num_qualities = length(params$qualities);

source(paste(getwd(), "/utils.r", sep="")); # Also creates the matrix of
     possible_states
source(paste(getwd(), "/birth_death_matrix.r", sep="")); # creates or loads the
     matrix
source(paste(getwd(), "/firm_behavior.r", sep="")); # optimal firm behavior
source(paste(getwd(), "/consumer_behavior.r", sep="")); # optimal consumer
     behavior

## The following is used for minimizing numerical errors.
# All computations of the paper are done with the following configuration
normalize_matrix = FALSE; # disabled. Seems to increase numerical errors
normalize_inv_dist = TRUE;
optimize_with_original = TRUE;

num_stages = 25; # = Foresight

## For different consumer behavior: uncomment one of the next lines
#consumer_transition_when_buying = consumer_transition_when_buying_extreme;
#consumer_transition_when_buying = consumer_transition_when_buying_unforgiving;
#consumer_transition_when_buying = consumer_transition_when_buying_enthusiastic;

create_data <- function() {
  load_data(); # See if we already have data

  if(!is.list(data)) {
    # prepare the data structure
    data = list();
    data$stages = list();
    # Save the matrix and the params for later reference
    data$B = B;
    data$params = params;
  }

  # The following is useful when picking up earlier computations
  start_stage = length(data$stages) + 1;
  if(start_stage <= num_stages) {
    # At this point there is actually something to be computed!

    for(stage in start_stage:num_stages) {
      # Keep track of time
      print(paste("Starting stage ", stage));
      time.start = as.numeric(Sys.time());

```

```
    next_stage_profit = c();
    if(stage > 1) {
      next_stage_profit = data$stages[[stage - 1]]$F$profit;
    }

    ## Get the F-matrix from the firm's behavior
    F = get_price_setting_behavior(next_stage_profit);

    data$stages[[stage]] = list();
    data$stages[[stage]]$F = F;

    ## Create the transitional matrix
    M = data$stages[[stage]]$F$matrix %*% data$B;
    M_original = M; # Store for later use

    ## Compute the number of square operations
    b <- 1 # Upper bound for price values
    precision_const = nrow(M) * b; # By the lemma. The factor before the q^n
    eps <- .001 # the precision to reach
    num_squares = log( (log(precision_const) - log(eps)) / ( nrow(M) * min(M) )
        ) / log(2);
    num_squares = ceiling(num_squares); # to get an integer number

    ## Now square the matrix according to the computed number
    inv_dist = rep(1/num_states, num_states); # start_value
    min_diff = 1; # start_value
    for(i in 1:num_squares) {
      M = M%*%M;
      if(normalize_matrix) {
        M = apply(M, 1, utils.normalize); # normalize
      }
      inv_dist_tmp = apply(M,2,mean);

      if(normalize_inv_dist) {
        inv_dist_tmp = utils.normalize(inv_dist_tmp);
      }

      diff = max(abs(inv_dist_tmp - inv_dist_tmp %*% M_original))
      if(diff <= min_diff) {
        # new optimum. Store the values.
        inv_dist = inv_dist_tmp;
        min_diff = diff;
      }

      if(abs(sum(M) - ncol(possible_states)) > 1 ) {
        # At this point, numerical errors from matrix multiplication got out of
            hand. No point of continuing.
        break;
      }

    }
```

```

## To minimize numerical errors: multiply the computed distribution with the orininal matrix
if(optimize_with_original) {
  for(i in 1:1000) {
    inv_dist_tmp = as.vector(inv_dist_tmp %*% M_original);
    if(normalize_inv_dist) {
      inv_dist_tmp = utils.normalize(inv_dist_tmp);
    }
    diff = max(abs(inv_dist_tmp - inv_dist_tmp %*% M_original));
    if(diff < min_diff) {
      inv_dist = inv_dist_tmp;
      min_diff = diff;
      print(paste("new inv distr from optim. Diff:", diff));
    }
  }
}

# Store the distribution in the result list
data$stages[[stage]]$inv_dist = inv_dist;

# Create the average price statistics
prices = matrix(0,nrow = length(params$qualities), ncol = ncol(M));
avg_prices = rep(0, length(params$qualities));
for(col in 1:ncol(M)) {
  state = possible_states[,col];
  for(qual_index in 1:length(params$qualities)) {
    qual = params$qualities[qual_index];
    price = choice(state, qual, next_stage_profit)$p;
    prices[qual_index,col] = price;
    avg_prices[qual_index] = avg_prices[qual_index] + inv_dist[col] * price;
  }
}
data$stages[[stage]]$prices = prices;
data$stages[[stage]]$avg_prices = avg_prices;

# Create average posterior category
data$stages[[stage]]$avg_posterior = (posteriors[1,] %*% possible_states / params$num_people) %*% inv_dist;

# Create the number of sales for each
num_people_buying = matrix(0, nrow = length(params$qualities), ncol = num_states)
for(i in 1:num_states) {
  state = possible_states[,i];
  prices = data$stages[[stage]]$price[,i];
  for(q in 1:length(params$qualities)) {
    p = prices[q];
    sure_buying_categories = posteriors[2,] >= p
    num_people_buying[q,i] = sum(state[sure_buying_categories])

```

```
          crit = params$num_categories - sum(sure_buying_categories) # the critical category
          if(crit > 0) {
            a = posteriors[2,crit];
            b = posteriors[3,crit];
            num_people_buying[q,i] = num_people_buying[q,i] + state[crit] * (b-p)/(b-a)
          }
        }
      }
      data$stages[[stage]]$num_people_buying = num_people_buying; # by state
      data$stages[[stage]]$avg_num_people_buying = num_people_buying %*% inv_dist;

      # Save the data for future reference
      filename = get_filename(); # see utils
      print(paste("saving file to ", filename));
      save("data", file= filename);

      # Print the time for orientation
      time.end = as.numeric(Sys.time());
      print(paste("Finishing stage ", stage, " Seconds: ", time.end - time.start));
    }
  } # end if start_stage < num_stages

  return(data);
}

## compute once
data = create_data();

## This is an example of how to use the create_data() function with different parameters
create_data_with_discount <- function(discount) {
  # store the old value
  old_discount = params$discount;
  params$discount <<- discount;

  create_data();

  # restore the old value
  params$discount <<- old_discount;
}

## compute for different discount parameters
d = c(.02, .05, .1, .25);
library(parallel)
result = mclapply(d, create_data_with_discount, mc.silent=FALSE, mc.cores = 4) # only for linux
#result = mclapply(d, create_data_with_discount, mc.silent=FALSE) # for windows.
```

The birth-death matrix is computed in the following file right when it is implemented.

birth_death_matrix.r

```
## Returns the distributions over states by just computing dying and rebirth. The result will be one line of the birth/death matrix
probabilities_one_line <- function(state) {
  probs = rep(0, ncol(possible_states));
  num_people = sum(state);

  for(col in 1:ncol(possible_states)) {
    new_state = possible_states[,col];

    # Create the minimal dying state
    min_dying = state - new_state;
    min_dying = min_dying * (min_dying > 0);
    l = list();
    for(i in 1:params$num_categories) {
      l[[i]] = (min_dying[i]) : (state[i]);
    }
    dying_combinations = data.matrix(expand.grid(l));
    # The death vectors are now the row entries of the matrix

    prob = 0;
    # go over all death vectors
    for(row in 1:nrow(dying_combinations)) {
      d = dying_combinations[row,];

      # The value of THIS dying probability
      p1 = dbinom(d, state, params$dying_prob);
      p1 = prod(p1);

      # and now the part of the birth process
      b = new_state - (state - d);
      p2 = factorial(sum(b)) / (prod(factorial(b)) * params$num_categories^(sum(b)));

      prob = prob + (p1 * p2);
    }

    # save the value
    probs[col] = prob;
  }

  return(probs);
}

## Return the whole matrix
get_dying_prob_matrix <- function() {
  n <- ncol(possible_states);
```

```
  B = matrix(0, nrow = n, ncol = n); # prepare
  # compute the transition probabilities for each line and enter them into the
      matrix
  for(row in 1:n) {
    state = possible_states[,row];
    B[row,] = probabilities_one_line(state);
  }

  return(B);
}

# The following code is executed with the include() command in main.r

# Create the filename where the matrix will be stored / loaded from
filename=sprintf("B_N=%s,C=%s,death=%s.rdata", params$num_people,
    params$num_categories, params$dying_prob);
# Check if the matrix file exists
if(file.exists(filename)) {
  load(filename); # just load from the file
} else {
  print("Creating the birth/death matrix. This can take a while.")

  B = get_dying_prob_matrix(); # Create
  save("B", file = filename); # store for next time

  print("Done creating the birth/death matrix.")
}
```

The following code takes care of the optimal consumer behavior as described in section 4.3.2. It allows for the different consumer reactions which are used for the robustness checks.

consumer_behavior.r

```
##  contains functions for the consumer behavior. Different functions are provided
     for different behavior types
consumer_transition_when_buying_inertia <- function(c,q) {
  # Go one up or down:
  if(q < posteriors[2,c])
    return(c-1);

  if(q > posteriors[3,c])
    return(c+1);

  # If we are here, q is in category c. Nothing changes.
  return(c);
}

consumer_transition_when_buying_extreme <- function(c,q) {
  # this gives the category in which q lies:
  cat = sum(posteriors[2,] <= q);
```

```

  return(cat);
}

consumer_transition_when_buying_unforgiving <- function(c,q) {
  # For positive experience: go one up
  if(q > posteriors[3,c])
    return(c+1);

  # Otherwise the same as the "extreme" case
  cat = sum(posteriors[2,] <= q);
  return(cat)
}

consumer_transition_when_buying_enthusiastic <- function(c,q) {
  # For negative experiences: go one down
  if(q < posteriors[2,c])
    return(c-1);

  # Otherwise the same as the "extreme" case
  cat = sum(posteriors[2,] <= q);
  return(cat)
}

consumer_transition_when_buying <- consumer_transition_when_buying_inertia # This
    is the default. Uncomment the corresponding line in main.r to change the
    behavior.

consumer_transition <- function(state, p, q) {
  # Find the posterior interval in which the price is
  critical_posterior = which.max(posteriors[3,] >= p);

  # sure_move will contain the category adaptation of those consumers
  # who buy independently of their posterior realization
  sure_move = rep(0,params$num_categories);
  if(critical_posterior < params$num_categories) {
    # There are "sure buyers"
    # move all the positions of the high posteriors that buy for sure
    for(post in (critical_posterior + 1):params$num_categories) {
      sure_move[post] = sure_move[post] - state[post];
      new_posterior = consumer_transition_when_buying(post, q);
      sure_move[new_posterior] = sure_move[new_posterior] + state[post]; # move
          down
    }
  }

  a = posteriors[2,critical_posterior]; # left boundary
  b = posteriors[3,critical_posterior]; # right boundary

  new_post = consumer_transition_when_buying(critical_posterior, q);
  if(new_post == critical_posterior) {
```

```
    # nothing changes. q is in the correct interval...
    max_num_moving = 0;
  }
  else {
    max_num_moving = state[critical_posterior];
  }

  # The individual chance of buying the good:
  chance_buying = (b-p) / (b-a);

  chances = rep(0, ncol(possible_states)); # prepare

  # For the critical category, we have to check each case of the number of buyers
  # who have a realized posterior above p.
  for(k in 0:max_num_moving) {
    # k is the number of people buying
    new_state = state + sure_move;
    new_state[critical_posterior] = new_state[critical_posterior] - k
    new_state[new_post] = new_state[new_post] + k

    # Binomial distribution. See the results of the chapter.
    chance = dbinom(k, max_num_moving, chance_buying);
    chances[index_of_state(new_state)] = chance;
  }

  return(chances);
}

## given the state, price and true quality, this function returns all states
## with a positive probability of occuring after buying.
consumer_transition_possible_states <- function(state, p, q) {
  result = consumer_transition(state, p, q);

  return(which(result > 0))
}
```

The firm behavior is given by the following lines in accordance to the theoretical part.

firm_behavior.r

```
## Returns the choice of a myopic firm, given the current state and the quality level q
choice_myopic <- function(state, q) {
  maxprofit = 0;
  popt = 0;
  # Check all the posterior categories:
  for(i in 1:params$num_categories) {
    a = posteriors[2,i]; # left interval bound
    b = posteriors[3,i]; # right interval bound
    n = state[i]; # number of people having this posterior
    s = 0
```

```
    e = params$num_experts;
    if(i< params$num_categories) {
      s = sum(state[(i+1):params$num_categories]); # number of sure buying people
    }
    if(e > 0 && q == 1) {
      s = s + e; # The experts buy as well
      # NOTE: This only works for q = 0 or q=1. Not for q in (a,b) if experts are present. This case is not yet implemented but also not needed for the examples in the paper. An implementation would require a further interval to be considerd, splitting [a,b] and separately maximizing over [a,q] and [q,b].
    }

    # first order solution for p:
    if(n == 0) {
      p = b;
    }
    else {
      p = ((b-a)*s/n + b) / 2;
    }

    # Check the boundary cases:
    if(p <= a) {
      p = a;
      profit = p * (s + n); # We sell to all people in [a,b] and the experts
    }
    else if(p >= b) {
      p=b;
      profit = p * s;
    }
    else {
      # interior solution
      profit = p * (s + n * (b-p)/(b-a));
    }

    # If we have a higher profit, we store the value
    if(profit > maxprofit) {
      maxprofit = profit;
      popt = p;
    }
  }

  return(list(p = popt, profit = maxprofit));
}

## Returns the choice of a foresight firm, given the state, realized quality and the next_stage_profit. Due to the recursive structure, this function is independent of the actual foresight level.
choice_foresight <- function(state, q, next_stage_profit = c()) {
  if(length(next_stage_profit) == 0) {
    # No foresight. return myopic
```

```
    return(choice_myopic(state, q));
  }

  maxprofit = 0;
  popt = 0;
  for(i in 1:params$num_categories) {
    a = posteriors[2,i]; # left
    b = posteriors[3,i]; # right
    n = state[i]; # number of people having this posterior
    s = 0;
    if(i < params$num_categories)
      s = sum(state[(i+1):params$num_categories]); # number of sure buying people

    # Compute the best behavior
    res = max_foresight_profit(state, q, a, b, s, n, next_stage_profit);
    if(res$profit > maxprofit) {
      maxprofit = res$profit;
      popt = res$price;
    }
  }

  return(list(p = popt, profit = maxprofit));
}

## Computes the expected profit when setting the price p, given the next-stage-profits
foresight_profit <- function(p, state, q, a, b, s, n, next_stage_profits) {
  next_states = const.next_states;
  num_states = length(next_states);

  e = params$num_experts;
  if(e > 0 && q >= p) {
    s = s + e; # The experts buy
  }
  # NOTE: This only works for q = 0 or q=1. See comment in choice_myopic.

  # cover the boundary cases for efficiency and higher precision
  if(p == b) {
    # None of the n buyers in [a,b] buys
    return(p * s + (1-params$discount) * next_stage_profits[next_states[1]]);
  }
  if(p == a) {
    # All n buyers in [a,b] buy
    return(p * (s + n) + (1-params$discount) * next_stage_profits[next_states[num_states]]);
  }

  # If we are here, we have p in (a,b)

  # probability of buying:
  prob = (b-p)/(b-a);
```

```

  # The profit from sales in this period
  profit = p * (s + n * prob);

  # Compute the expected profit, using the profits vector
  if(num_states == 1) {
    # Easy case
    next_distribution = B[next_states[1],];
    next_profit = next_stage_profits %*% next_distribution;
    return(profit + (1-params$discount) * next_profit);
  }

  # Here, each state has the probability dbinom(k,n,prob) since the number of
      buying customers is binomially distributed.
  for(k in 0:n) {
    # Get the distribution for this state after the birth-death process is
    next_distribution = B[next_states[k+1],];
    # And the profit associated to this
    next_profit = next_stage_profits %*% next_distribution;

    profit = profit + (1-params$discount) * dbinom(k,n,prob) * next_profit;
  }
  return(profit);
}

## Comutes the maximal achievable profit when setting a price in the interval [a,b
    ], given the state and the next-period profits.
max_foresight_profit <- function (state,q,a,b,s,n, profits) {

  # Save everything in constants
  max_foresight_profit_state <<- state;
  max_foresight_profit_q <<- q;
  max_foresight_profit_a <<- a;
  max_foresight_profit_b <<- b;
  max_foresight_profit_s <<- s;
  max_foresight_profit_n <<- n;
  max_foresight_profit_profits <<- profits;
  # next_states. This is constant for all prices in [a,b].
    const.next_states <<- consumer_transition_possible_states(state, .5 * a + .5 *
        b, q);

  result = optimize(f = foresight_profit, interval=c(a,b), state, q, a, b, s, n,
      profits, maximum = TRUE, tol=.0000001);
  # rename
  result = list(profit = result$objective, price = result$maximum);
  # now check the boundaries. This reduces numerical issues with the "optimize"
      function
  profit_b = foresight_profit(b, state, q, a, b, s, n, profits);
  if(profit_b >= result$profit) {
    result = list(profit = profit_b, price = b);
  }
```

```
  profit_a = foresight_profit(a, state, q, a, b, s, n, profits);
  if(profit_a >= result$profit) {
    result = list(profit = profit_a, price = a);
  }

  return(result);
}

## For the current state "state", given the next_stage_profit, returns the optimal
     behavior of the firm.
get_price_transition_probs <- function(state, next_stage_profit) {
  profit = 0;
  prices = rep(0,num_qualities);
  transition = rep(0,num_states);

  # For each quality: compute optimal behavior
  for(i in 1 : num_qualities) {
    # The behavior:
    current_choice = choice(state, params$qualities[i], next_stage_profit);
    prices[i] = current_choice$p;
    # update the expectation values:
    profit = profit + params$qual_dist[i] * current_choice$profit;
    transition = transition + params$qual_dist[i] * consumer_transition(state,
        current_choice$p, params$qualities[i]);
  }

  return(list(profit = profit, transition = transition, prices = prices));
}

choice = choice_foresight;

## Returns a list, containing a matrix "matrix" and a profit vector "profit"
get_price_setting_behavior <- function(next_stage_profit = c()) {
  num_states = ncol(possible_states)
  m <- matrix(0,ncol = num_states, nrow = num_states);
  profit = rep(0,num_states);
  prices = matrix(0, nrow = num_qualities, ncol = num_states);
  for(s in 1:num_states) {
    probs = get_price_transition_probs(possible_states[,s], next_stage_profit);
    m[s,] = probs$transition;
    profit[s] = probs$profit;
    prices[,s] = probs$prices;
  }

  return(list(matrix = m, profit = profit, prices = prices));
}

## Returns the optimal expected profit of the firm.
create_next_stage_profit <- function(next_stage_profit = c()) {
  num_states = ncol(possible_states);
  for(s in 1:num_states) {
```

```
    # Get the possible states which can occur
    possible_states = c();
    for(c in 1:params$num_categories) {
      possible_states = c(possible_states, consumer_transition_possible_states(s, posteriors[1,c], q));
    }
    # delete duplicates
    possible_states = unique(possible_states);

    # For each of these states, compute the expected revenue
    for(ps in possible_states) {
      probs = probabilities(possible_states[,ps])
      expectations = probs * next_stage_profit;
    }
  }
}
```

Finally, the following code contains utility functions used for managing the state space and saving/loading data.

utils.r

```
## This function returns a matrix where each column is a state vector.
get_possible_states <- function(num_people, num_categories) {
  possible_state_indices <<- array(0,dim = rep(num_people + 1, num_categories-1));

  # Distribute the "gaps"
  comb = combn(num_people + num_categories - 1, num_categories - 1);
  states = matrix(0,nrow = num_categories, ncol = ncol(comb));

  nrows = nrow(comb);

  # Now the data is in the form oo|o|ooo|o||o where the number of o's determines the number of owners of a posterior.
  for(col in 1:ncol(comb)) {
    gaps = comb[,col];
    states[,col] = diff(c(0,gaps, (num_people+num_categories))) - 1;
    possible_state_indices[matrix(states[1:(num_categories-1),col] + 1, nrow = 1)] <<- col;
  }

  return(states);
}

## Create the possible-states matrix
# executed when this file is loaded from main.r
possible_states <-get_possible_states(params$num_people, params$num_categories);

## Create a matrix with the posterior category. Each column contains the middle of the interval, as well as the left and the right bound.
# with four categories:
# avg | .125    .375    .625    .875
```

```
# a   |    0     .25     .50     .75
# b   |  .25     .50     .75       1
posteriors <- matrix(0,nrow = 3, ncol = params$num_categories);
for(i in 1:params$num_categories) {
  left = (i-1) / params$num_categories;
  right = i / params$num_categories;
  middle = (left + right) / 2;
  posteriors[,i] = c(middle, left, right);
}

## returns the column number of a state vector in the possible_states matrix
index_of_state <- function(state) {
  return( possible_state_indices[matrix(state[1:(params$num_categories-1)] + 1,
      nrow = 1)] );
}

## returns the filename used to store the data-list
get_filename <- function() {
  consumer = "";
  if(identical(consumer_transition_when_buying,
      consumer_transition_when_buying_inertia))
    consumer = "inertia";
  if(identical(consumer_transition_when_buying,
      consumer_transition_when_buying_extreme))
    consumer = "extreme";
  if(identical(consumer_transition_when_buying,
      consumer_transition_when_buying_enthusiastic))
    consumer = "enthusiastic";
  if(identical(consumer_transition_when_buying,
      consumer_transition_when_buying_unforgiving))
    consumer = "unforgiving";

  name = sprintf("data_fs_N=%i,p=%i,death=%s,discount=%s,experts=%s,c=%s",
      params$num_people, params$num_categories, params$dying_prob,
      params$discount, params$num_experts, consumer);

  if(length(params$qualities) > 2) {
    name = paste(name, "q", length(params$qualities), sep="");
  }

  filename = paste(name, ".rdata", sep="");

  return(filename);
}

## loads the data from a file, given the current params. Returns TRUE iff the data
     can be loaded from a file and the stored params are identical to the global
    ones.
load_data <- function(directory = "") {
  # Cleanup. Just to be sure.
  suppressWarnings(rm("data", pos = ".GlobalEnv"));
```

```
  suppressWarnings(rm("data"));

  filename = get_filename();

  if(length(directory) > 0) {
    directory = paste(directory, "/", sep="");
  }

  if(file.exists(filename)) {
    load(filename, .GlobalEnv);
    if(identical(params, data$params)) {
      return(TRUE);
    }
    warning("The data params of the loaded data are not identical to the global
        params");
    return (FALSE);
  }

  return(FALSE);
}

## normalization function
utils.normalize <- function(v) {
  s = sum(v);
  return(v / s);
}
```

# References

Adriani, Fabrizio and Luca G. Deidda: *Price signaling and the strategic benefits of price rigidities.* Games and Economic Behavior, 67(2):335–350, November 2009.

Adriani, Fabrizio and Luca G Deidda: *Competition and the signaling role of prices.* International Journal of Industrial Organization, 29(4):412–425, 2011.

Akerlof, George A.: *The Market for "Lemons": Quality Uncertainty and the Market Mechanism.* The quarterly journal of economics, 84(3):488–500, 1970.

Bagwell, Kyle and Michael H. Riordan: *High and Declining Prices Signal Product Quality.* The American Economic Review, 81(1):224–239, 1991.

Bar-Isaac, Heski, Guillermo Caruana and Vicente Cuñat: *Information Gathering Externalities for a Multi-Attribute Good.* The Journal of Industrial Economics, 60(1):162–185, 2012.

Bester, Helmut and Klaus Ritzberger: *Strategic pricing, signalling, and costly information acquisition.* International Journal of Industrial Organization, 19(9):1347–1361, 2001.

Blume, Andreas and Oliver Board: *Language barriers.* Econometrica, 81(2):781–812, 2013.

Chan, Yuk-Shee and Hayne Leland: *Prices and qualities in markets with costly information.* The Review of Economic Studies, 49(4):499–516, 1982.

Cho, In-Koo and David M Kreps: *Signaling Games and Stable Equilibria.* The Quarterly Journal of Economics, 102(2):179–221, January 1987.

Crawford, Vincent P and Joel Sobel: *Strategic information transmission.* Econometrica: Journal of the Econometric Society, pages 1431–1451, 1982.

Daughety, Andrew F and Jennifer F Reinganum: *Competition and confidentiality: Signaling quality in a duopoly when there is universal private information.* Games and Economic Behavior, 58(1):94–120, 2007.

DeGroot, Morris H: *Reaching a consensus.* Journal of the American Statistical Association, 69(345):118–121, 1974.

Dziuda, Wioletta: *Strategic argumentation.* Journal of Economic Theory, 146(4):1362–1397, 2011.

Ellingsen, Tore: *Price signals quality: The case of perfectly inelastic demand.* International Journal of Industrial Organization, 16(1):43–61, 1997.

Ellsberg, Daniel: *Risk, ambiguity, and the Savage axioms.* The quarterly journal of economics, pages 643–669, 1961.

Ewens, Warren J: *Mathematical Population Genetics 1: I. Theoretical Introduction*, volume 27. Springer, 2004.

Gans, Joshua S. and George B. Shepherd: *How Are the Mighty Fallen: Rejected Classic Articles by Leading Economists.* The Journal of Economic Perspectives, 8(1):pp. 165–179, 1994.

Gertz, Christopher: *Quality Uncertainty with Imperfect Information Acquisition.* Center of Mathematical Economics Working Paper Nr. 487, Bielefeld University, September 17, 2013.

Gigerenzer, Gerd and Wolfgang Gaissmaier: *Heuristic decision making.* Annual review of psychology, 62:451–482, 2011.

Gilboa, Itzhak and Massimo Marinacci: *Ambiguity and the Bayesian Paradigm.* Advances in Economics and Econometrics, Tenth World Congress of the Econometric Society: Economic Theory, 1, 2013.

Goeree, Jacob K and Charles A Holt: *Ten little treasures of game theory and ten intuitive contradictions.* American Economic Review, pages 1402–1422, 2001.

Grossman, Sanford J and Joseph E Stiglitz: *On the impossibility of informationally efficient markets.* The American Economic Review, 70(3):393–408, 1980.

Häggström, Olle: *Finite Markov chains and algorithmic applications*, volume 52 of *London Mathematical Society student texts ; 52*. Cambridge Univ. Press, 5. print. edition, 2008.

Hoyer, Wayne D: *An examination of consumer decision making for a common repeat purchase product*. Journal of consumer research, pages 822–829, 1984.

Jäger, Gerhard, Lars P Metzger and Frank Riedel: *Voronoi languages: Equilibria in cheap-talk games with high-dimensional types and few signals*. Games and economic behavior, 73(2):517–537, 2011.

Judd, Kenneth L and Michael H Riordan: *Price and quality in a new product monopoly*. The Review of Economic Studies, 61(4):773–789, 1994.

Kihlstrom, Richard: *A Bayesian model of demand for information about product quality*. International Economic Review, 15(1):99–118, 1974.

Kim, Jae-Cheol and Byungtae Lee: *When a lemon market emerges?* Proceedings of The Korean Economic Association, 2:1–2, 2005.

Kreps, David M and Robert Wilson: *Reputation and imperfect information*. Journal of economic theory, 27(2):253–279, 1982.

Linnemer, Laurent: *Price and advertising as signals of quality when some consumers are informed*. International Journal of Industrial Organization, 20(7):931–947, 2002.

Liu, Qingmin: *Information acquisition and reputation dynamics*. The Review of Economic Studies, 78(4):1400–1425, 2011.

Liu, Qingmin and Andrzej Skrzypacz: *Limited records and reputation bubbles*. Journal of Economic Theory, 151:2–29, 2014.

Mailath, George J and Larry Samuelson: *Who wants a good reputation?* The Review of Economic Studies, 68(2):415–441, 2001.

Markose, Sheri, Jasmina Arifovic and Shyam Sunder: *Advances in experimental and agent-based modelling: Asset markets, economic networks, computational mechanism design and evolutionary game dynamics*. Journal of Economic Dynamics and Control, 31(6):1801 – 1807, 2007.

Martignon, Laura, Konstantinos V Katsikopoulos and Jan K Woike: *Categorization with limited resources: A family of simple heuristics.* Journal of Mathematical Psychology, 52(6):352–361, 2008.

Martin, Daniel: *Strategic Pricing with Rational Inattention to Quality.* Job Market Paper, New York University, October 21, 2012.

Mas-Colell, Andreu, Michael Dennis Whinston, Jerry R Green et al.: *Microeconomic theory*, volume 1. Oxford university press New York, 1995.

Milgrom, Paul and John Roberts: *Price and Advertising Signals of Product Quality.* Journal of Political Economy, 94(4):796–821, 1986.

Murata, Tadahiko and Du Nisuo: *Agent-based evaluation for effects of signaling in markets with asymmetric information.* In *Soft Computing and Intelligent Systems (SCIS) and 13th International Symposium on Advanced Intelligent Systems (ISIS), 2012 Joint 6th International Conference on*, pages 2134–2137. IEEE, 2012.

Myerson, Roger B and Mark A Satterthwaite: *Efficient mechanisms for bilateral trading.* Journal of economic theory, 29(2):265–281, 1983.

Nelson, Phillip: *Information and consumer behavior.* The Journal of Political Economy, pages 311–329, 1970.

Okada, Akira: *Robustness of equilibrium points in strategic games.* Inst. of Technology, Department of Information Sciences, 1983.

Savage, Leonard J: *The foundations of statistics.* John Wiley & Sons, Inc., 1954.

Schmalensee, Richard: *A model of advertising and product quality.* The Journal of Political Economy, pages 485–503, 1978.

Schnabel, Hubert and Karl Storchmann: *Prices as quality signals: evidence from the wine market.* Journal of Agricultural & Food Industrial Organization, 8(1), 2010.

Selten, Reinhard: *The chain store paradox.* Theory and decision, 9(2):127–159, 1978.

Shaked, Avner and John Sutton: *Relaxing price competition through product differentiation.* The review of economic studies, pages 3–13, 1982.

SHAPIRO, CARL: *Consumer information, product quality, and seller reputation.* The Bell Journal of Economics, pages 20–35, 1982.

STOKEY, NANCY L., ROBERT E. LUCAS and EDWARD C. PRESCOTT: *Recursive methods in economic dynamics.* Harvard Univ. Press, Cambridge, Mass. [u.a.], 4. print. edition, 1996.

VOORNEVELD, MARK and JÖRGEN W WEIBULL: *A Scent of Lemon - Seller Meets Buyer with a Noisy Quality Observation.* Games, 2(1):163–186, 2011.

WILSON, CHARLES: *The nature of equilibrium in markets with adverse selection.* The Bell Journal of Economics, pages 108–130, 1980.

WISEMAN, THOMAS: *Reputation and impermanent types.* Games and Economic Behavior, 62(1):190–210, 2008.

WOLINSKY, ASHER: *Prices as signals of product quality.* The Review of Economic Studies, 50(4):647–658, 1983.

ZEITHAML, VALARIE A.: *Consumer Perceptions of Price, Quality, and Value: A Means-End Model and Synthesis of Evidence.* Journal of Marketing, 52(3):2–22, 1988.

# Short Curriculum Vitae

## Christopher Gertz

| | |
|---|---|
| 2010 - 2014 | Member of the International Research Training Group *Economic Behavior and Interaction Models* (EBIM) at Bielefeld University, Germany in collaboration with Université Paris 1 Panthéon Sorbonne, France<br>Member of the Bielefeld international Graduate School in Economics and Management (BiGSEM) at Bielefeld University, Germany |
| 2008 - 2010 | Master of Science in Mathematical Economics at Bielefeld University, Germany<br>Member of the master program Quantitative Economics Models (QEM) at Bielefeld University and Université Paris 1 Panthéon Sorbonne, France |
| 2005 - 2008 | Bachelor of Science in Mathematical Economics at Bielefeld University, Germany |
| 2005 | Abitur at the Eichenschule Scheeßel, Germany |