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Abstract

In this paper we analyze an inter-temporal optimization problem of a representative firm that invests in horizontal and vertical innovations and that faces a constraint with respect to total R&D spending. We find that there may exist two different steady-states of the economy when the amount of research spending falls short of an endogenously determined threshold: one with higher productivities and less new technologies being developed, and the other with more technologies being created and lower productivities. Thus, a lock-in effect may arise that, however, can be overcome by raising R&D spending sufficiently such that the steady-state becomes unique and the firm produces the whole spectrum of available technologies.

Keywords: Multiple steady-states, lock-in, innovations, R&D constraint, optimal control

JEL classification: C61, D92, O32

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1 Introduction

It is widely known that in the framework with co-existing vertical and horizontal innovations firms tend to invest more into existing products development rather than into the creation of new products. This is a typical situation in industries where large firms are multi-product monopolies due to patent regimes with examples being pharmaceutical and packaging (Tetra Pak) industries. At the same time the technology lock-in phenomena is recently described for endogenous growth models, which postpones the introduction of newer technologies because they are more risky/underdeveloped, like in (Zeppini and van den Bergh 2013) and (Acemoglu, Aghion, Bursztyn, and Hemous 2012).

In this paper we obtain the technology lock-in result for a single multi-product firm under conditions of scarce R&D funding. This funding may reflect the research subsidy fromt he government devoted to fostering new innovations or the fixed fraction of the firm's profit. It is argued that if multiple steady state levels of R&D investments may exist in such a setup, the way to overcome technology lock-in lies at the microeconomic level of firms incentives, rather than at the level of market regulations.

The problem of multi-product innovations has received attention in the IO literature starting with (Lambertini 2003). In these papers the optimal behaviour of a multi-product R&D firms is analysed. However it is assumed that there are no restrictions on R&D spending. At the same time in endogenous growth literature the multiplicity of steady states of the economy stemming from certain budget constraints is frequently discussed especially in the context of transition to renewable energy in environmental models, see for example (Greiner and Semmler 2008) for such type of models. The motivation for this paper is thus to consider whether the research budget constraint would lead to such a multiplicity of R&D levels for a single firm. This multiplicity involves different levels of diversity of technologies following (van den Bergh 2008)and of their achieved productivity. The difference with aforementioned model is that we allow for expansion of a continuous range of technologies through creation of new ones.

We employ the same methodology to R&D problem as it has been done for drug markets regulation in (Baveja, Feichtinger, Hartl, Haunschmied, and Kort 2000) where it si demonstrated that multiplicity of steady states may arise from the perimeter-type constraints on optimal dynamics . The formulation of the problem itself follows the setup with linear profit and cost reduction from innovations as in (Dawid, Greiner, and Zou 2010) allowing us to neglect the production and demand side effects.

The suggested model accounts for the role of heterogeneity of new products, as in (Hopenhayn and Mitchell 2001). The multi-product monopolist is modelled as a single agent in the industry (market). The process of innovative activity follows ideas of (Romer 1990) in horizontal innovations and of (Schumpeter 1942; Aghion and Howitt 1992) in vertical innovations trying to unify these approaches in partial equilibrium context.

The description of multi-product innovations follows the ideas of papers (Lambertini and Orsini 2001; Lin 2004; Belyakov, Tsachev, and Veliov 2011) and more closely (Bondarev 2012) where the single-agent dynamic optimization problem with infinite life cycles of technologies is treated.

The main contribution of the paper is the study of conditions under which the multiplicity of steady states in the R&D investment problem may arise. We find out that one of these states corresponds to the situation of lower range of technologies with high development of existing products while the second one describes the situation with higher range of technologies with all of them being less developed. This technology lock-in may be overcome if research spending is sufficiently high and two steady state collapse into one optimal one.

The rest of the paper is organized as follows. The next section presents the structure of our model. Section 3 derives the optimal solution and section 4 analyzes the question of whether the steady-state is unique or whether multiple steady-states can exist. Section 5, finally, concludes the paper.

2 The inter-temporal optimization problem with horizonal and vertical innovations

We assume a representative firm that wants to maximize the discounted stream of profits, given by the productivities of all new technologies minus the cost from investments in horizontal and vertical innovations, and that faces a resource constraint with respect to

R&D spending. The model follows the lines of (Belyakov, Tsachev, and Veliov 2011), where the mathematical foundations for this type of models are discussed, but it is more closely to the stylized model in (Bondarev 2012). Investments into horizontal and vertical innovations are determined from an inter-temporal optimization problem subject to the laws of motion describing the productivity increase for each technology, subject to the process of variety expansion of the technologies and subject to the resource constraint. Denoting the discount rate of the firm by r, the objective functional to be maximized can be written as:

$$J \stackrel{\text{def}}{=} \max_{u(\cdot),g(\cdot)} \int_0^\infty \mathrm{e}^{-rt} \left(\int_0^{n(t)} \left[q(i,t) - \frac{1}{2} g(i,t)^2 \right] di - \frac{1}{2} u(t)^2 \right) dt. \tag{1}$$

with:

- $u(t)$: investments into variety expansion;
- $g(i,t)$: investments into the productivity growth of technology i at time t.

The firm continuously develops new technologies, $i \in \mathbf{I}$, from the potential spectrum of technologies. The process of acquiring new technologies follows a simple linear process. At the same time, the firm develops the productivity of all these new technologies. The dynamics for the variety of technologies, $n(t)$, and for the productivities of technologies, $q(t)$, are given by:

$$\begin{aligned} &\dot{n}(t) = \xi u(t), \\ &\dot{q}(i,t) = \psi(i) g(i,t) - \beta q(i,t), \ \forall i \in [0;1] = \mathbf{I} \subset \mathbb{R}, \end{aligned} \tag{2}$$

with:

- $\xi > 0$: efficiency of investments in the expansion of variety of technologies;
- $\psi(i) > 0$: efficiency of investments in the productivity growth of technology i;
- $\beta > 0$: rate of decay of productivity of technology i, identical across technologies.

In addition, there is a constraint on the total amount of R&D spending given by,

$$u(t) + \int_0^{n(t)} g(i,t) di = R, \tag{3}$$

stating that total R&D spending cannot exceed the exogenously determined value R.

The dynamic problem is then of the perimeter-type constrained one. We use the standard technique of the augmented Hamiltonian function to obtain a characterization of the solution, see (Fattorini 1999) for details of such a treatment.

One also has a number of static constraints on controls and states:

$$q(i,t)|_{i=n(t)} = 0,\ 0 \leq n(t) \leq \bar{n} \equiv 1,\ q(i,t) \geq 0,\ u(t) \geq 0,\ g(i,t) \geq 0. \tag{4}$$

From (2), (4) it can be seen, that the productivity of each technology can decline over time ($q(i,t)$ may decrease), but the technology itself, once invented, cannot be forgotten ($n(t)$ cannot decrease). The spectrum of technologies is bounded by some positive value $\bar{n}$ which is normalized to one. In such a framework, the number of technologies grows over time, but there is no structural change since older ones do not disappear from the system. It should also be noted that each new technology has zero productivity at the time it is invented,

$$q(i, t_i(0)) = 0, \tag{5}$$

which makes sense from an economic point of view and where $t_i(0)$ denotes the time of invention of the technology i. The time of invention of the technology, $t_i(0)$, is the inverse function of the process of variety expansion, $n(t)$:

$$t_i(0) = f^{-1}(n(t))|_{n(t)=i}. \tag{6}$$

It should be noted that the efficiency of investments, $\psi(i)$, plays a crucial role in the determination of the dynamics of productivities. If this is an increasing function of i, every next technology eventually becomes more productive than all the preceding ones, if this is a decreasing function, new technologies are harder to develop and have lower productivity in the end. If the efficiency is the same for all technologies, $\psi(i) = \psi_c$, all technologies evolve in the same fashion and the optimal control problem (1) subject to (2), (4) is equivalent to a problem with two states and without a resource constraint, as it is discussed in (Bondarev 2012).

To make the problem interesting we assume the following properties of investments efficiency function:

- The function $\psi(i)$ is continuous;
- It is invertible;
- It is decreasing in i.

Two first properties are necessary for the problem to have solution and the last one is added to include the trade-off between staying within the existing products range and expanding it further.

For the control problem to make sense we also require compactness of the state space and this requires function $\psi(i)$ to be bounded. We choose the following specific form for this efficiency function, allowing it to vary across technologies:

$$\psi(i) = \psi_c \sqrt{1-i}, \ \psi_c > 0. \tag{7}$$

Such a function permits a closed-form solution of the variety expansion problem, while assuming that it is more complicate to raise the efficiency of new technologies than that of older ones. A different function that is also decreasing in i would be ψ_c/i. However, we choose the specification $\psi_c\sqrt{1-i}$ in order to assure the boundedness of the potential technologies space and in order to get the closed-form solution of the model. But, the qualitative analysis would be the same for any monotonic and decreasing function.

3 Solution of the optimization problem

To solve the dynamic optimization problem given by (1) subject to (2) and (3), we construct the augmented Hamiltonian:

$$\mathcal{H} = \int_0^{n(t)} \Big[q(i,t) - \frac{1}{2}g(i,t)^2\Big] di - \frac{1}{2}u(t)^2 + \lambda_n(t) \cdot (\xi u(t)) +$$
$$+ \int_0^{n(t)} \lambda_q(i,t) \cdot (\psi(i) g(i,t) - \beta q(i,t)) di + l(t) \cdot \Big(R - u(t) - \int_0^{n(t)} g(i,t) di\Big), \tag{8}$$

where λ_n, λ_q are the shadow prices or co-state variables of the variety expansion and of the productivities of technologies, respectively, and $l(t)$ is the time-varying Lagrange multiplier for the resource constraint.

The first order conditions for this problem are given by,

$$u(t) = \xi \lambda_n(t) - l(t); \tag{9}$$

$$g(i,t) = \psi(i)\lambda_q(i,t) - l(t); \tag{10}$$

$$R - u(t) - \int_0^{n(t)} g(i,t)di = 0, \tag{11}$$

and the differential equation system for the co-state variables is:

$$\begin{aligned}\dot{\lambda}_n(t) &= r\lambda_n(t) - \frac{\partial \mathcal{H}}{\partial n} = \\ &= r\lambda_n(t) + \frac{1}{2}g^2(n(t),t) - \lambda_q(n(t),t)\,\psi(n(t))\,g(n(t),t) + l(t)\,g(n(t),t),\end{aligned} \tag{12}$$

$$\forall i \leq n(t) : \dot{\lambda}_q(i,t) = r\lambda_q(t) - \frac{\partial \mathcal{H}}{\partial q} = (r+\beta)\lambda_q(i,t) - 1. \tag{13}$$

In deriving that system, we made use of the following:

- in the first equation we make use of the condition $q(i,t)|_{i=n(t)} = 0$,
- $g(n(t),t) = g(i,t)|_{i=n(t)}$ is the value of investments into the productivity of the next technology to be invented,
- $\lambda_q(n(t),t) = \lambda_q(i,t)|_{i=n(t)}$ is the shadow price of investments into the boundary technology productivity,
- $\psi(n(t)) = \psi(i)|_{i=n(t)}$ is the value of the efficiency function at the boundary of variety expansion at the moment t.

We can summarize our results as concerns the optimality conditions in the following Proposition 1.

Proposition 1 (Characterization of the optimality conditions)
For the inter-temporal optimization problem of the firm, given by the maximization of (1) *subject to* (2) *and* (3), *the optimal solution is characterized as follows:*

1. *The optimal controls for the problem are given by* (9), (10) *and* (11);

2. The dynamics of the shadow prices for the variety expansion and for the productivities of technologies are given by (12) *and by* (13)*, respectively.*

In addition, the limiting transversality conditions

$$\lim_{t\to\infty} e^{-rt}\lambda_n(t) = 0, \lim_{t\to\infty} e^{-rt}\lambda_q(i,t) = 0, \forall i \leq n(t), \tag{14}$$

must hold.

4 Uniqueness and multiplicity of steady-states

The closer analysis of the dynamic system describing the optimal R&D investment reveals that the presence of the constraint on R&D spending leads to the possibility of multiplicity of steady-states for this system. Before we present the analysis in detail, we first define a steady-state for our model and then analyze the conditions for existence and multiplicity of these states.

Definition 1 (Steady-state)
The steady-state of the model is characterized by the following conditions:

$$\forall i \in [0,1]: \; \dot{q}(i,t) = 0, \; \dot{\lambda}_q(i,t) = 0, \; \dot{n}(t) = 0, \; \dot{\lambda}_n(t) = 0. \tag{15}$$

Due to the form of the dynamic constraints, given by (2), the levels of productivities of all existing technologies reach their respective steady-state values, too, as long as the system giving the evolution of the variety is in steady-state. The steady-state levels of productivities depend on the level of the variety at the steady-state, $\tilde{n}$, and on the value of the shadow prices, $\tilde{\lambda}_n$, for every given level of the research budget R, with the tilde $\tilde{}$ denoting steady-state values. Since $\dot{\lambda}_n$ and $\dot{n}$ only depend on λ_n and on n (see Appendix A), the overall steady-state of the model depends on the steady-state of the dynamic system (A.6), (A.7), given in Appendix A.

The inspection of the equation (A.6) shows that the steady-state condition for the shadow price λ_n from (15) is a polynomial of second order in this variable. From fundamental algebra we know that such a polynomial has exactly two roots. Thus, for every value of $n(t)$ there are two steady-state values of the shadow price. At the same time, the equation (A.7) is linear in the shadow price so that there is only one steady-state of $n(t)$ for every value of λ_n. These considerations demonstrate that the system (A.6), (A.7) can have at most two different steady-states. The isocline $\dot{\lambda}_n = 0$ generates two lines with one origin at $n \leq 1$, and the isocline $\dot{n}(t) = 0$ is an initially rising concave function that becomes vertical at $n = 1$. Two steady-states arise when the $\dot{n}(t) = 0$ isocline intersects the $\dot{\lambda}_n = 0$ isocline for values of $n < 1$ and a unique steady-state is obtained when the isoclines intersect at $n = 1$.

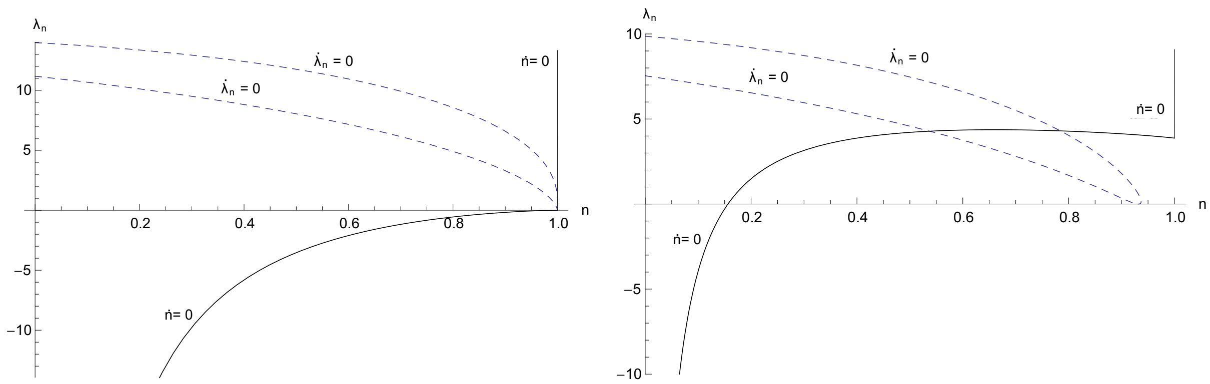
Figure 1 illustrates the case of a unique steady-state and of two steady-states for this system depending on the level of R. To draw Figure 1 we resorted to the parameter values given in table 1. The research budget is set to $R = 0.9$ for the multiple steady-states case

Table 1: Parameters values used in Figures 1 and 2.

Parameter	Value
n_0	0
r	0.05
ψ_c	0.9
ξ	0.8
β	0.1

and to $R = 4$ for the unique steady-state case. It should be noted that for values of $R > 4$, the isocline $\dot{n}(t) = 0$ shifts downward and the kink of that isocline occurs for values of $\lambda_n < 0$ and not excatly at $\lambda_n = 0$, as it is the case for $R = 4$. Of course, the steady-state is also unique for all $R > 4$.

The lower the level of the R&D budget is, the higher is the chance for a multiplicity of steady-states where one of the steady-states corresponds to a lower level of variety of technologies and higher shadow prices of investments than the other. That holds because a decline in R&D shifts the $\dot{n}(t) = 0$ isocline upwards. The following proposition summarizes the result.



(a) Unique steady-state, $R = 4$ (b) Multiple steady-states, $R = 0.9$

Figure 1: Uniqueness and multiplicity of R&D steady-states

Proposition 2 (Uniqueness and multiplicity of steady-states)
The dynamic system $\dot{\lambda}_n$, $\dot{n}$, given by (A.6), (A.7), *has at most two steady-states. Other things equal the R&D budget R defines the number of steady-states of the system. With $R < R^\star$ there exist two steady-states with a low and a high level of variety of technologies. With $R \geq R^\star$ there exists only one steady-state with $\tilde{n} = 1$.*

The proof of Proposition 2 amounts to the calculation of the derivatives of (A.6) and (A.7) with respect to R and taking into consideration the discussion above.

Now, it should be noted that to each of the steady-states of the system (A.6), (A.7) corresponds a steady-state level of productivity for each of the developed technologies $q(i)|_{i \leq \tilde{n}}$. Since the research budget is limited by R, one can realize from system (2) that this leads to a stop of the development of productivities due to increasing support costs for each technology.[1] The steady-state level of productivity for technology i is given by,

$$\tilde{q}(i) = \frac{\psi_c^2(1-i)}{\beta(r+\beta)} + \frac{2}{3}\psi_c^2\sqrt{1-i}\frac{(1-\tilde{n})^{3/2}}{(1+\tilde{n})} + \psi_c\frac{\sqrt{1-i}}{(1+\tilde{n})}(R(r+\beta) - \frac{2}{3}\psi_c - \xi(r+\beta)\tilde{\lambda}_n). \tag{16}$$

[1]Formally, this is seen from (A.5) as follows: $\dot{n} = 0$ implies $u = 0$ leading to $l = \xi\tilde{\lambda}_n$, giving $\dot{q} = \psi_c\sqrt{1-i}\left(\frac{\psi_c\sqrt{1-i}}{r+\beta} - \xi\tilde{\lambda}_n\right) - \beta q$. Since the first term is constant, q converges to its steady-state value.

Solving $\dot{n}(t) = 0$ with respect to λ and substituting the result for $\tilde{\lambda}_n$ in (16), $\tilde{q}(i)$ becomes a function of n and of other parameters. The derivative of the steady-state productivity with respect to the variety level n is always negative:

$$\frac{\partial \tilde{q}(i)}{\partial n} = -\frac{1}{3}\frac{\psi_c\sqrt{1-i}}{\sqrt{1-n}n^2} \cdot \left(\psi_c\left(2(1-\sqrt{1-n}) - n(n-1)\right) + (3\sqrt{1-n}(r+\beta)R)\right) < 0. \tag{17}$$

Thus, we have the following proposition:

Proposition 3 (Productivities with multiple steady-states)
With a constraint on R&D spending giving rise to multiple steady-states for the dynamic system $\dot{\lambda}_n$, $\dot{n}$, given by (A.6), (A.7)*, there also exist two steady-states for the productivities of all technologies. They are characterized by,*

$$\tilde{q}^H(i) < \tilde{q}^L(i), \tag{18}$$

where H (L) denotes the steady-state with the higher (lower) number of variety. At the same time, each newer technology has a lower steady-state level:

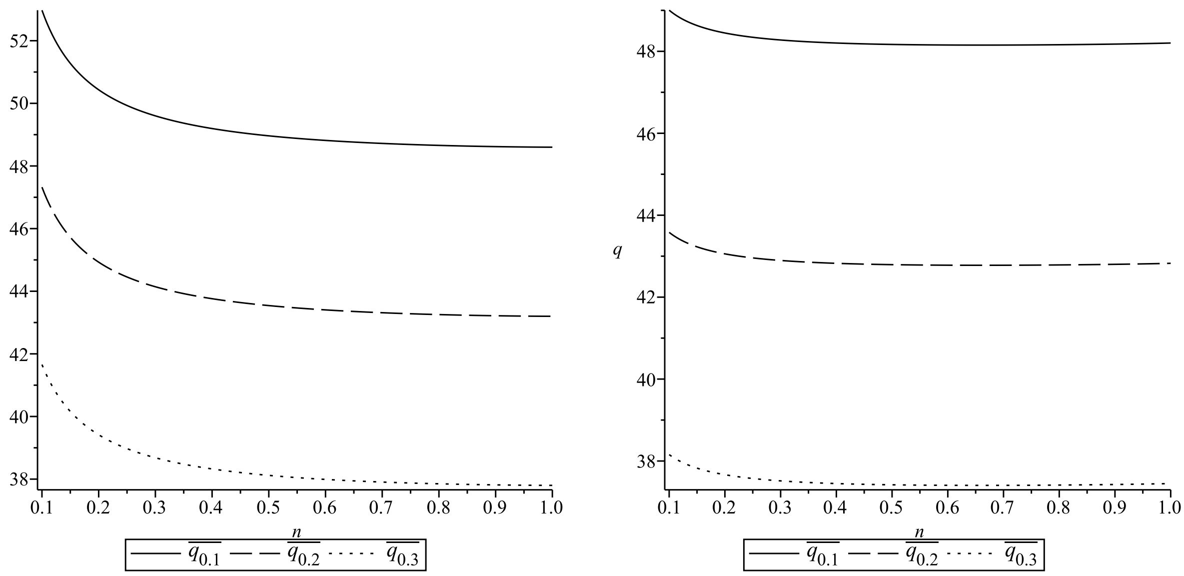
$$\tilde{q}^H(i+\delta) < \tilde{q}^H(i),\ \tilde{q}^L(i+\delta) < \tilde{q}^L(i),\ \delta \to 0. \tag{19}$$

The last result follows from the assumed form for the efficiency function, (7).

Proposition 3 shows that in the case of two steady-states, the one with the higher variety of technologies implies a lower productivity of these technologies. Hence, there is a trade-off between variety and productivity in case of two steady-states. But it must be underlined that this trade-off only exists for a given level of R&D in the situation of two steady-states. Thus, raising R&D may lead to a unique steady-state and to a situation with a higher variety and a higher productivity, compared to the case with two steady-states as our considerations below will show.

Figure 2 illustrates the difference between the average productivities of three technologies at the steady-state for the cases of a unique steady-state and for multiple steady-states with the parameters set to the values given in Table 1. In the case of multiple steady-states we have taken the one with the higher productivity of technologies. It can be realized that in the unique steady-state case each technology has a higher productivity.

That is due to the higher amount of R&D spending in the case of a unique steady-state compared to the case of two steady-states.



(a) Unique steady-state, $R = 4$ (b) Multiple steady-states, $R = 0.9$

Figure 2: Average productivities of technologies in steady-states

Finally, we analyse the effect of changes in the R&D budget on the number of steady-states in general and we determine the threshold level R^*. First, note that a higher R leads to the collapse of the two steady-states into a unique one for the system (A.6), (A.7). The unique steady-state implies $\tilde{n} = 1$ and $\tilde{\lambda}_n = 0$. Using this, one can easily compute the threshold level of the research budget, R^*, from $\dot{n}(t) = 0$ as:

$$R^* = \frac{2}{3}\frac{\psi_c}{r + \beta}. \tag{20}$$

With the parameters given in table 1 one obtains $R^* = 4$. It should be pointed out that the quantity $\psi_c/(r + \beta)$ gives the optimal investments into overall productivity growth. Thus, a unique steady-state can be obtained only if the research budget is at least $2/3$ of optimal investments into the growth of productivites of existing technologies. It should also be pointed out that in the case of multiple steady-states both steady-states are inefficient in the sense that they do not allow for the introduction of all potential technologies into the economy, i.e. $\tilde{n} < 1$ holds for both steady-states.

Further, setting $R > R^\star$ also implies a unique steady-state. However, that would be inefficient since the same result can be obtained for $R = R^\star$ because this also gives the maximum value $\tilde{n} = 1$ so that any additional R&D spending would be a waste of resources. We state our result in the following proposition:

Proposition 4 (Technology lock-in with a constraint on R&D) *In the R&D sector described by the dynamic system $\dot{q}$, $\dot{\lambda}_n$, $\dot{n}$, given by* (A.5), (A.6), (A.7)*, the resource constraint R is crucial as concerns the emergence of a technology lock-in.*

1. *For $R < \frac{2}{3}\frac{\psi_c}{r+\beta}$, the economy is characterized by a technology lock-in with two inefficient steady-states;*

2. *For $R \geq \frac{2}{3}\frac{\psi_c}{r+\beta}$, no lock-in effect arises and the variety of technologies reaches its maximum steady-state level $\tilde{n} = 1$;*

3. *Optimal R&D expenditures are given by $R^\star = \frac{2}{3}\frac{\psi_c}{r+\beta}$.*

It is important to observe that the existence of multiplicity of steady states relies heavily on the heterogeneity of the technological space being represented by the efficiency function $\psi(i)$. If the efficiency of investments is the same for all the technologies, no technology lock-in will arise. To see that just consider the underlying control problem with $\psi(i) = \psi_c$. It follows that there will be no interplay between vertical and horizontal innovations and thus no multiplicity of steady states may arise. This result follows the same lines as in the benchmark model in (Bondarev 2012).

5 Conclusion

In this paper we have analyzed the inter-temporal optimization problem of a firm that invests in R&D to generate both horizontal and vertical innovations and that faces a constraint with respect to total R&D spending. We have found that the model may be characterized by multiple steady-states which is the result of an uneven distribution of investments between the introduction of new technologies and the development of older ones. With limited research expenditures it is likely that the majority of resources will be spent on the development of existing technologies, rather than on the introduction of new ones. This will lead to the technology lock-in phenomenon, as described in the literature, when newer technologies are underdeveloped or even non-existent. However, due to the structure of the R&D process considered in our model, this lock-in can be overcome by an increase in research spending above a certain threshold that depends on the structural parameters determining the R&D process and on the discount rate of the firm. Thus the policy implication of our result is as follows. The regulator should subsidize R&D in those industries where the technologies are sufficiently different and each next is harder to develop. In the industries where technologies are similar to each other in their investment characteristics no subsidies are necessary, since the only equilibrium dynamics is the optimal one.

A Derivation of the dynamic system

In (14) we posit that the standard transversality conditions for the finite time horizon T also holds for the infinite time horizon, with $\lim\limits_{t\to\infty}$ replacing $\lim\limits_{t\to T}$. For λ_q this implies:

$$\forall i \leq n(t) : \lim_{t\to\infty} e^{-rt}\lambda_q(i,t) = 0. \tag{A.1}$$

Then, the co-state for each technology productivity can be obtained from (13) as:

$$\forall i \leq n(t) : \lambda_q(i,t) = \frac{1 - e^{(r+\beta)(t-T)}}{(r+\beta)}, \tag{A.2}$$

which yields a constant shadow price in time for each technology for the infinite horizon case, i.e. for $T \to \infty$,

$$\forall i \leq n(t) : \lambda_q(i,t)|_{T\to\infty} = \frac{1}{(r+\beta)}. \tag{A.3}$$

The form of $l(t)$ results from substituting (9), (10) into (11) taking into account specification (7) and the form of the co-state variable $\lambda_q(i,t) = 1/(r+\beta)$:

$$l(t) = \frac{2\psi_c - 2\psi_c(1-n(t))^{3/2} + 3(r+\beta)(\xi\lambda_n(t) - R)}{3\,(r+\beta)(1+n(t))}, \tag{A.4}$$

which is a function of the variety expansion and of its co-state variable. The differential equations for all system variables, then, are obtained by substituting the controls in (9), (10) with $l(t)$ defined in (A.4). The resulting productivities, $q(i,t)$, are functions of the variety of technologies, $n(t)$, and of the resource constraint R which enters optimal investments through the Lagrange multiplier:

$$\dot{q}(i,t) = \psi_c\sqrt{1-i}\left(\frac{\psi_c\sqrt{1-i}}{r+\beta} - \frac{2\psi_c - 2\psi_c(1-n(t))^{3/2} + 3(r+\beta)(\xi\lambda_n(t) - R)}{3\,(r+\beta)(1+n(t))}\right) - \beta q(i,t) \tag{A.5}$$

Using the expressions for the controls, (9), (10), the Lagrange multiplier, (A.4), and the efficiency of investments into the boundary technology, $\psi(n) = \psi_c\sqrt{1-n(t)}$, one can obtain the explicit expression for the change of the co-state variable as a function of $n(t)$ and of the budget constraint R:

$$\begin{aligned}
\dot{\lambda}_n(t) = {} & \left(-\frac{\beta\,\xi^2 r}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - 1/2\,\frac{r^2\xi^2}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - 1/2\,\frac{\beta^2\xi^2}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2}\right)\cdot \\
& \cdot\left(\lambda_n\left(t\right)\right)^2 + \frac{r^2\xi\,R - 2/3\,\psi_c\,r\xi + \beta^2\xi\,R + 2\,\beta\,\xi\,rR - 2/3\,\psi_c\,\beta\,\xi + 2/3\,\psi_c\,\left(1-n\left(t\right)\right)^{3/2}\beta\,\xi}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2}\cdot \\
& \cdot\lambda_n(t) + \left(\frac{\psi_c\,\sqrt{1-n\left(t\right)}\beta\,\xi + \psi_c\,\sqrt{1-n\left(t\right)}r\xi}{\left(r+\beta\right)^2\left(n\left(t\right)+1\right)} + r + 2/3\,\frac{\psi_c\,\left(1-n\left(t\right)\right)^{3/2}r\xi}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2}\right)\lambda_n\left(t\right) - \\
& - 7/6\,\frac{{\psi_c}^2\left(n\left(t\right)\right)^2}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - \frac{5}{18}\,\frac{{\psi_c}^2\left(n\left(t\right)\right)^3}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - 2/3\,\frac{\psi_c\,\left(1-n\left(t\right)\right)^{3/2}\beta\,R}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - \\
& - 2/3\,\frac{\psi_c\,\left(1-n\left(t\right)\right)^{3/2}rR}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} + 1/3\,\frac{{\psi_c}^2\left(n\left(t\right)\right)^2}{\left(r+\beta\right)^2\left(n\left(t\right)+1\right)} + 2/3\,\frac{{\psi_c}^2\sqrt{1-n\left(t\right)}}{\left(r+\beta\right)^2\left(n\left(t\right)+1\right)} - \\
& - 1/2\,\frac{R^2\beta^2}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - \frac{\psi_c\,\sqrt{1-n\left(t\right)}\beta\,R}{\left(r+\beta\right)^2\left(n\left(t\right)+1\right)} - \frac{\psi_c\,\sqrt{1-n\left(t\right)}rR}{\left(r+\beta\right)^2\left(n\left(t\right)+1\right)} + \\
& + 2/3\,\frac{\psi_c\,Rr}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - \frac{R^2\beta\,r}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - 1/2\,\frac{R^2r^2}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} + \\
& + 7/6\,\frac{{\psi_c}^2n\left(t\right)}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} - 5/3\,\frac{{\psi_c}^2}{\left(r+\beta\right)^2\left(n\left(t\right)+1\right)} + 4/9\,\frac{{\psi_c}^2\left(1-n\left(t\right)\right)^{3/2}}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} + \\
& + 4/3\,\frac{{\psi_c}^2n\left(t\right)}{\left(r+\beta\right)^2\left(n\left(t\right)+1\right)} + 2/3\,\frac{\psi_c\,R\beta}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2} + 1/18\,\frac{{\psi_c}^2}{\left(n\left(t\right)+1\right)^2\left(r+\beta\right)^2}
\end{aligned} \tag{A.6}$$

Substitution of (9) with $l(t)$, given by (A.4), into the dynamics of variety expansion from (2), yields the following differential equation for $n(t)$:

$$\dot{n}(t) = \xi^2 \lambda_n(t) \frac{n(t)}{1+n(t)} - \frac{\xi \left(2\psi_c - 2\psi_c(1-n(t)^{3/2} - 3R(r+\beta)\right)}{3\,(r+\beta)(1+n(t))} \tag{A.7}$$

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