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The Shapley Value for Countably Many Players

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Abstract

Within this paper we are treating a version of the Shapley value (see [11]) for countably many players. However, our approach is in marked difference to the one favored by most authors, in particular by Shapley [12] and Artstein [2], who attacked the problem of constructing a Shapley value on a suitable class of games by a basically measure-theoretic approach (see also [1], [13], [7]). The basis to our version is rather provided by group-theoretical considerations, more precisely, we use the structure of the symmetric group of $I\!N$. This group, viewed as a subset of the orderings of $I\!N$, has measure 0. However, if we introduce a suitable metric, it admits nevertheless a normalized, nonatomic additive measure, which, on a sufficiently large subalgebra of the clopen subsets is invariant under left and right multiplication (and *a fortiori* on all group automorphisms). Integration of the marginal worth of a player with respect to this measure yields the Shapley value.

Key words: Shapley value, invariant measure.

AMS(MOS) Subject Classification: 52A20, 52A43, 20B27, 90D40.

1 Introduction, Notation

The *Shapley value* is by now one of the most widespread concepts in Game Theory and Mathematical Economics. In his seminal paper [11] L. S. Shapley introduced the value for finite sets of players, providing various equivalent characterizations. For a continuum of players, the Shapley value is also well established and widely used, the first comprehensive treatment was of course provided by Aumann and Shapley [1]; there is an abundance of literature concerning this subject.

defined on the set of all subsets of $I\!N$ (the _coalitions_) and satisfying $v(\emptyset) = 0$ is called a _game._ (Analogously for I instead of $I\!N$.)

The Shapley value is a mapping attaching a (signed, σ−additive) measure to any game (within a certain class - say games with bounded total variation). For the sake of completeness, let as shortly review the relevant definition within finite context.

To this end, let $I := \{1, ..., n\}$ denote the set of _players,_ and let

$$v : \mathcal{P}(I) \to I\!R$$

be a game. For any player $i \in I$ the _Shapley value_ is defined as:

$$\Phi_i(v) := \frac{1}{n!} \sum_{\pi \in \Sigma_n} (v(S^\pi_{\pi(i)}) - v(S^\pi_{\pi(i)-1})).$$

Here, the sum is taken over all permutations π of the symmetric group Σ_n and for $\pi \in \Sigma_n$ we define

$$S^\pi_k := \{j \in I \mid \pi(j) \le k\} = \pi^{-1}(\{1, ..., k\})$$

and thus

$$S^\pi_{\pi(i)} := \{j \in I \mid \pi(j) \le \pi(i)\}.$$

From the probabilistic point of view, the above formula for the Shapley value can be considered as an _"expected value of a random variable - the marginal worth of i",_ which is defined on the symetric group of $\{1, ..., n\}$, the expectation beeing taken with respect to the _"uniform distribution"_. More formally:

Let Σ_n denote the permutation group of the set $\{1, ..., n\}$. Given $I := \{1, ..., n\}$, and

$$v : \mathcal{P}(I) \to I\!R,$$

define for any player $i \in I$ a _random variable:_

$$f^v_i : \Sigma_n \to I\!R$$

by

$$f^v_i(\pi) := v(S^\pi_{\pi(i)}) - v(S^\pi_{\pi(i)-1}).$$

Next consider _"uniform distribution"_ given by:

$$p(\{\pi\}) := \frac{1}{n!} \quad for \quad \pi \in \Sigma_n \ ;$$

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Theorem 2.2

(a) *The group Σ is not countable*

(b) *The group Σ^* is a normal subgroup of Σ.*

(c) *The function Σ*

$$d : \Sigma \times \Sigma \to [0,1]$$

given by:

$$d(\pi, \sigma) := \sum_{k=1}^{\infty} \frac{1}{2^k} \left[\frac{|\pi(k) - \sigma(k)|}{1 + |\pi(k) - \sigma(k)|} + \frac{|\pi^{-1}(k) - \sigma^{-1}(k)|}{1 + |\pi^{-1}(k) - \sigma^{-1}(k)|} \right],$$

constitutes a metric on Σ such that (Σ, d) is a separable complete metric space with the dense subset Σ^*.

(d) *The pair (Σ, d) is a topological group and the mapping*

$$\Sigma \longrightarrow \Sigma \qquad with \qquad \pi \mapsto \pi^{-1}$$

is an isometry.

Proof: We will first show part a):

Assume that Σ is countable and let $(\pi_i)_{i \in \mathbb{N}}$ be a enumeration of the elements of Σ.

Then we arrange them as an infinite matrix of the following type:

$$\pi_1 \ : \ n_{1,1} \ , \ n_{1,2} \ ; \ n_{1,3} \ , \ n_{1,4}, \ \ldots \ , \ n_{1,j}, \ \ldots$$

$$\pi_2 \ : \ n_{2,1} \ , \ n_{2,2} \ , \ n_{2,3} \ , \ n_{2,4}, \ \ldots \ , \ n_{2,j}, \ \ldots$$

$$\pi_3 \ : \ n_{3,1} \ , \ n_{132} \ , \ n_{3,3} \ , \ n_{3,4}, \ \ldots \ , \ n_{3,j}, \ \ldots$$

$$\cdots\cdots\cdots$$

$$\pi_l \ : \ n_{l,1} \ , \ n_{l,2} \ , \ n_{l,3} \ , \ n_{l,4}, \ \ldots \ , \ n_{l,j}, \ \ldots$$

$$\cdots\cdots\cdots$$

Now we construct a permutation $\hat{\pi} \in \Sigma$ in the following way:

Put $\hat{\pi}(1) := n_{1,2}$ and let us assume that for $i \in \{1,,,,,,,k\}$ $\quad \hat{\pi}(i) \in \mathbb{N}$ is already defined. Then we define:

$$\hat{\pi}(k+1) := n_{k+1, j_0}$$

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Proof: We will first show part a):

By definition of the metric

$$d : \Sigma \times \Sigma \longrightarrow [0,1]$$

with:

$$d(\pi, \sigma) := \sum_{k=1}^{\infty} \frac{1}{2^k} \left[\frac{|\pi(k) - \sigma(k)|}{1 + |\pi(k) - \sigma(k)|} + \frac{|\pi^{-1}(k) - \sigma^{-1}(k)|}{1 + |\pi^{-1}(k) - \sigma^{-1}(k)|} \right],$$

we have for any two elements $\pi, \sigma \in \Sigma_n$ with $\pi \neq \sigma$ that $d(\pi, \sigma) \geq \frac{1}{2^{n+1}}$.

This can be seen as follows: Since $\pi \neq \sigma$ there exists an element $k_0 \in \{1, ..., n\}$ such that $\pi(k_0) \neq \sigma(k_0)$. Hence

$$\frac{1}{2^{k_0}} \left[\frac{|\pi(k_0) - \sigma(k_0)|}{1 + |\pi(k_0) - \sigma(k_0)|} \right] \geq \frac{1}{2^{k_0+1}},$$

from which $d(\pi, \sigma) \geq \frac{1}{2^{n+1}}$ follows.

This means that Σ_n is a scattered subset in the metric space (Σ, d).
Now to part b):
First of all, observe that for an element $\pi \in B(\pi_0, 2^{-l})$ we have by definition of the metric, that for all $i \in \{1, ..., l-1\}$ $\pi(i) = \pi_0(i)$. For, assume that for some $i_0 \in \{1, ..., l-1\}$ we have $\pi(i_0) \neq \pi_0(i_0)$. Then:

$$d(\pi, \pi_0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left[\frac{|\pi(k) - \pi_0(k)|}{1 + |\pi(k) - \pi_0|} + \frac{|\pi^{-1}(k) - \pi_0^{-1}(k)|}{1 + |\pi^{-1}(k) - \pi_0^{-1}(k)|} \right]$$

$$\geq \frac{1}{2^{i_0}} \left[\frac{|\pi(i_0) - \pi_0(i_0)|}{1 + |\pi(i_0) - \pi_0(i_0)|} \right] \geq \frac{1}{2^{i_0+1}},$$

$$\geq \frac{1}{2} \cdot \frac{1}{2^{i_0}} \geq \frac{1}{2^{i_0+1}} \geq \frac{1}{2^l}.$$

Hence at most $(n - l + 1)$ elements can be permutated, and this give:

$$\#(B(\pi_0, 2^{-l}) \cap \Sigma_n) \leq (n - l + 1)!.$$

for $l > 1$ and $n \geq l + 2$. □

Given any permutation $\pi \in \Sigma$ let us introduce the following *ordering* induced by π on $\mathbb{N}$ via the following convention:

$$\text{for } i, j \in \mathbb{N} \quad \text{we write } i \prec_\pi j \quad \longleftrightarrow \quad \pi(i) \leq \pi(j)$$

(a) a sequence $(\pi_k)_{k \in I\!N}$, $\pi_k \in \Sigma$ converges to an element $\pi^* \in \Sigma$ if and only if for every $m \in I\!N$ there exists an index $l(m) \in I\!N$, such that for all $k > l(m)$ and all $n < m : \pi_k(n) = \pi^*(n)$ holds. This implies, that for every $i, j \in I\!N$ the set $\delta_i^{-1}(j)$ is closed.

(b) similar as in the proof of part a) of theorem 2.3 it follows, that with every $\pi_0 \in \delta_i^{-1}(j)$ the open ball

$$B(\pi_0, r) := \{\pi \in \Sigma \mid d(\pi_0, \pi) < r\}$$

with center π_0 and radius $r = \frac{1}{2^{i+2}}$ is contained in $\delta_i^{-1}(j)$.

Combining and completing our remarks we now have:

Lemma 2.4

Let $\mathcal{B}(\Sigma)$ be the Boolean σ-algebra of Borel subsets of Σ. Then the follwing holds true:

(a) for every r-tupel $(i_1, ..., i_r)$ the block

$$F_{i_1, ..., i_r} := \{\pi \in \Sigma \mid i_1 \prec_\pi i_2 \prec_\pi i_3 .... \prec_\pi i_r\} \in \mathcal{B}(\Sigma)$$

is open and closed, hence $\mathcal{B}$-measurable.

(b) The blocks

$$F_{i_1, ..., i_r} := \{\pi \in \Sigma \mid i_1 \prec_\pi i_2 \prec_\pi i_3 .... \prec_\pi i_r\} \in \mathcal{B}(\Sigma)$$

generate $\mathcal{B}(\Sigma)$.

Proof: As to part (a), this is a direct consequence of our above remark concerning the type of convergence that takes place with respect to the metric d.

Concerning part (b), we proceed as follows:

Let $\mathcal{B}_F$ tentatively denote the ('Boolean') $\sigma-$ field generated by the blocks; given $\sigma \in \Sigma$, and $\varepsilon > 0$ we show that the closed $\varepsilon-$ sphere $B(\sigma, \varepsilon)$ is an element of $\mathcal{B}_F$. To this end, define for any $K \in I\!N$

$$B_K := \{\pi \in \Sigma \mid \sum_{k=1}^K \frac{1}{2^k} \left[\frac{|\pi(k) - \sigma(k)|}{1 + |\pi(k) - \sigma(k)|} + \frac{|\pi^{-1}(k) - \sigma^{-1}(k)|}{1 + |\pi^{-1}(k) - \sigma^{-1}(k)|} \right] < \varepsilon\};$$

then clearly $B(\sigma, \varepsilon) = \bigcap_{k=1}^\infty B_K$, thus it suffices to show that $B_K \in \mathcal{B}_F$. First of all it is seen at once that for $i, k \in I\!N$ we have

$$\{\pi \mid \pi(i) \geq k\} = \bigcup_{j_1, ..., j_{k-1}=1}^\infty \{\pi \mid j_1 \prec_\pi ... \prec_\pi j_{k-1} \prec_\pi i\}.$$

More generally we realize that sets of the type

$$\{\pi \mid \pi(i) = k, ..., \pi(l) = r\}$$

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(b) *For any $\pi \in \Sigma$ the field $\mathcal{F}_o$ is $R_\pi -$ invariant while $\mathcal{G}_o$ is $L_\pi -$ invariant.*

(c) *For any $\pi \in \Sigma$ the automorphisms L_π and R_π are measurable with respect to the Borelian field $\mathcal{B}(\Sigma)$.*

Proof: The first statement is directly veryfied and the others follow immediately. □

It is now our aim to construct a nonatomic finetely additive measure on the σ-algebra $\mathcal{B}(\Sigma)$ of Borel subsets of Σ,i.e.,

$$\mu : \mathcal{B}(\Sigma) \rightarrow [0,1],$$

with $\mu(\{\Sigma\}) = 1$, which is invariant under left and right multiplication (as well as under automorphisms of the group) on a suitable 'sufficiently large' field of clopen subsets of Σ.

To be more precise let us formulate

Definition 2.7
Let $\mathcal{B}_o$ be the field generated by

$$\{L_\pi(F) \mid F \in \mathcal{F}_o, \ \pi \in \Sigma \ \} \cup \{R_\pi(G) \mid G \in \mathcal{G}_o, \ \pi \in \Sigma \ \}.$$

That is, $\mathcal{B}_o$ is the smallest algebra containing the algebra $\mathcal{F}_o$, the algbra $\mathcal{G}_o$ as well as all left and right transforms of blocks of both the generating algebras.

Theorem 2.8
There exists a finitely additive measure

$$\mu : \mathcal{B}(\Sigma) \rightarrow [0,1],$$

which enjoys the following properties:

(a) $\mu(\{\Sigma\}) = 1$,

(b) μ *is nonatomic.*

(c) μ *restricted to $\mathcal{B}_o$ is invariant under left and right multiplication, i.e.,*

$$L_\pi \mu|_{\mathcal{B}_o} = \mu|_{\mathcal{B}_o} = R_\pi \mu|_{\mathcal{B}_o}.$$

(d) μ *restricted to $\mathcal{B}_o$ is invariant under all automorphisms of the group.*

Proof: We embed the metric space (Σ, d) into its Stone-Čech compactification $\beta(\Sigma)$.

This is possible, since every uniformizable space is a dense subset of a compact space (see [10], Theorem 14.1.2, p.240).
Now we consider the Banach space

$$\mathcal{C}_0(\beta(\Sigma)) := \{x \mid x : \beta(\Sigma) \rightarrow I\!\!R\}$$

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Now

$$f_m(x^*) = \frac{1}{m!} \sum_{\pi \in \Sigma_m} x^*(\pi)$$

$$= \frac{1}{m!} \sum_{\pi \in \Sigma_m \cap B(\pi_0, 2^{-l})} x^*(\pi)$$

$$\leq \frac{1}{m!} \cdot \#(B(\pi_0, 2^{-l}) \cap \Sigma_m)$$

$$\leq \frac{(m - l + 1)!}{m!}$$

and this quotient tends to 0 for fixed $l \in \mathbb{N}$ and $m \to \infty$.

Hence we see, that for every $\varepsilon > 0$ there exists an $k \in \mathbb{N}$ such that

$$0 \leq \mu(B(\pi_0, 2^{-k}) \leq \varepsilon,$$

which means, that the measure is nonatomic.

Next we have to prove the invariance property of μ as claimed in our theorem.

To this end, observe that invariance on the *(Boolean)* fields $\mathcal{F}_o$ with respect to right multiplication and on $\mathcal{G}_o$ with respect to left multiplication is rather straightforward. For, choose $\pi \in \Sigma$ and any $F_{i_1,...,i_r} \in \mathcal{F}_o$. We know that, if m is such that $i_1, ..., i_k \leq m$, the measure μ_m satisfies $\mu_m(F) = \frac{1}{m!} \# F \cap \Sigma_m$. Using the definition of μ_m and specifying m such that $i_1, ..., i_k \leq m$ as well as $\pi^{-1}(i_1), ..., \pi^{-1}(i_k) \leq m$ we find therefore that

$$\mu_m(F_{i_1,...,i_k}) = \frac{1}{m!} \#(\{\rho \mid \rho \in F_{i_1,...,i_k}, \rho \in \Sigma_m\}) = \frac{1}{k!}$$

$$= \frac{1}{m!} \#(\{\rho \mid \rho \in F_{\pi^{-1}(i_1),...,\pi^{-1}(i_k)}, \rho \in \Sigma_m\}) = \mu_m(R_\pi(F_{i_1,...,i_k}).$$

This procedure is easy to perform for $F \in \mathcal{F}_o$ and π acting from the right. A similar consideration with respect to $\mathcal{G}_o$ and π acting from the left is obvious.

However, when π is acting from the left on $F_{i_1,...,i_k}$ the procedure turns out to require a little bit more effort. We are going to construct for suitable large m :

- an injective mapping

$$F_{i_1,...,i_k} \cap \Sigma_m \longrightarrow L_\pi(F_{i_1,...,i_k}) \cap \Sigma_m$$

which will imply that

$$\mu_m(F_{i_1,...,i_k}) \leq \mu_m(L_\pi(F_{i_1,...,i_k})$$

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with $\xi = \pi_o \cdot \tau \in \Sigma_m$. Since we assume that $i_1, ..., i_k \in \{1, ..., m\}$, a permutation $\kappa \in \Sigma^*$ can be defined as follows:

$$\kappa := \begin{pmatrix} \tau(1) & , & \tau(2) & , & ... & , & \tau(i_1) & , & ... & , & \tau(i_2) & , & ... & , & \tau(i_k) & , & ... & , & \tau(m) \\ \tau(1) & , & \tau(2) & , & ... & , & \tau(j_1) & , & ... & , & \tau(j_2) & , & ... & , & \tau(j_k) & , & ... & , & \tau(m) \end{pmatrix},$$

where the indices $j_1, ..., j_k$ are chosen such that

$$\pi_o(\tau(j_1)) < \pi_o(\tau(j_2)) < ... < \pi_o(\tau(j_m))$$

and

$$\{j_1, ..., j_k\} = \{i_1, ..., i_k\}$$

Therefore

$$\pi_0 \cdot \kappa \cdot \tau \in F_{i_1, ..., i_k}$$

since

$$\pi_o(\tau(j_1)) = \pi_0 \cdot \kappa \cdot \tau(i_1), ..., \pi_o(\tau(j_k)) = \pi_0 \cdot \kappa \cdot \tau(i_k).$$

In addition we have

$$\pi_0 \cdot \kappa \cdot \tau \in \Sigma_m$$

since $\pi_0 \tau \in \Sigma_m$ and $\kappa : \{\tau(1), ..., \tau(m)\} \rightarrow \{\tau(1), ..., \tau(m)\}$. Now, this means

$$\pi_0 \cdot \kappa \cdot \pi_o^{-1} \cdot \xi \in F_{i_1, ..., i_k}$$

and

$$\pi_0 \cdot \kappa \cdot \pi_o^{-1} \cdot \xi \in \Sigma_m.$$

Hence

$$\xi \mapsto \pi_0 \cdot \kappa \cdot \pi_o^{-1} \cdot \xi$$

constitutes the second injective mapping. Following our above reasoning we have now established that

$$\mu_m(F_{i_1, ..., i_k}) = \mu_m(L_{\pi_o}(F_{i_1, ..., i_k})$$

holds true.

This is the slightly more difficult version; the same formula for R_{π_o} was easy as we have seen above.

It remains to carry the procedure through the limit.

To this end let $\varepsilon > 0$ and $n \in I\!\!N$ be given (n sufficiently large). Then there exists an $m \in I\!\!N$, $m > n$ such that

$$|\mu(F_{i_1, ..., i_k}) - \mu_m(F_{i_1, ..., i_k})| \leq \varepsilon$$

(Note that the indicator function of $F \in \mathcal{F}_o$ is continuous). Now for the *same* index m we have in view of our above formula

$$|\mu(F_{i_1, ..., i_k}) - \mu_m(L_{\pi_o}(F_{i_1, ..., i_k})| \leq \varepsilon$$

It is nice to observe that the group Σ allows for (at least) two further representations. Thus our construction of an invariant measure may also be regarded in a different context.

Proposition 2.11

(a) *The group Σ is homeomorphic to the group*

$$aut(\beta N) := \{\varphi \mid \varphi : \beta N \to \beta N, \quad \varphi \text{ is a homeomorphism }\}$$

(b) *The group Σ is homeomorphic to the group*

$$hom(l^\infty) := \{T \mid T : l^\infty \to l^\infty, \quad T \text{ is a Banach-algebra isomorphism }\}$$

Proof: To prove part (a) let us first observe, that every permutation $\pi : N \to N$ can be extended to a continuous map $\varphi_\pi : \beta N \to \beta N$ by definition of the Stone-Čech compactification. Namely, we embed N into βN and consider $\pi : N \to N \subset \beta N$. Then by the commutative diagram in [10], part 14.1.1, there exists a continuous extension of π to $\varphi_\pi : \beta N \to \beta N$, i.e.
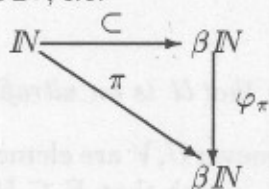
Since $N \subset \beta N$ is dense in βN and $\varphi_\pi|N = \pi$, it follows, that $\varphi_\pi : \beta N \to \beta N$ is bijective, and by the theorem of Hausdorff a homeomorphism (see [10], corollary 7.1.7).

Now let us consider a homeomorphism $\varphi : \beta N \to \beta N$. Since every homeomorphism transforms isolated points into isolated point, and the subset $N \subset \beta N$ is isolated, we see that $\varphi|N = \pi$ is a permutation. Hence part (a) is proved.

To prove part (b) we observe that the Banach-algebra

$$l^\infty := \{x \mid x := (x_n)_{n \in N} \quad \text{is a bounded real sequence }\}$$

endowed with the supremum-norm

$$\|x\| = \max_{n \in N} |x_n|,$$

is isomorphic as a Banach-algebra to $C_0(\beta N)$, i.e. the Banach-algebra of the real-valued continuous functions defined on βN, and that every Banach-algebra isomorphisms of $C_0(\beta N)$ is induced by a uniquely determined homeomorphism $\varphi : \beta N \to \beta N$ (see [10]

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for a suitable $\hat{i}$ which is ordered according to $\prec$. As $F_\kappa \cap F_\lambda = \emptyset (\kappa \in J)$, we conclude that $F_i = \bigcap_{\kappa \in J} F_\lambda$ and $B = \bigcup_{\kappa \in J} F_\kappa$ have a void intersection; hence $\Sigma - B \supseteq F_i$ holds true, meaning that $\Sigma - B \in \mathcal{U}$. □

Let us furthermore remark, that the structure of the groups Σ and Σ/Σ^* has been extensively studied in the literature. In especially it turns out, that both groups are the product of four resp. three conjugation classes of a permutation $\pi \in \Sigma \setminus \Sigma^*$, which is denoted by the notation *groups with small covering numbers*. For details we refer to [4] and [5].

3   The Definition of the Shapley value

Our previous results provide us with a possibility to extend the Shapley value to games with countably infinitely many players, using the invariant measure approach. Of course this approach is similar to the procedure employede by Shapley and Shapiro (see [11]),however our measure space is a much smaller one. In addition, we are able to show, that the value constructed this way coincides with the one introduced by Artstein [2] on a suitable Banach space of functions v with bounded variation.

Definition 3.1
Let
$$v : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$$
be a game and let $i \in \mathbb{N}$ be a player.

(a) *The* <u>marginal contribution</u> *of player i given v is the function*
$$f_i^v : \Sigma \to \mathbb{R}$$
given by
$$f_i^v(\pi) := v(S_{\pi(i)}^\pi) - v(S_{\pi(i)-1}^\pi) \text{ and } S_{\pi(i)}^\pi := \{j \in I \mid \pi(j) \leq \pi(i)\}.$$

(b) *The* <u>Shapley value</u> *of v for player i is given by*
$$\Phi_i(v) := \int_\Sigma f_i^v(\pi) d\mu(\pi),$$
provided the integral exists.

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Proof: For $i \in I\!N$ we have

$$\Phi_i(\sigma \cdot v) := \int_\Sigma f_i^{\sigma \cdot v}(\pi) \mu(d\pi) = \int_\Sigma f_{\sigma^{-1}(i)}^v[(\sigma \cdot \mu)(d\pi)]$$

$$= \int_\Sigma f_{\sigma^{-1}(i)}^v(\mu)(d\pi) = \Phi_{\sigma^{-1}(i)}(v) = (\sigma \cdot \Phi)_i(v).$$

Here we have used the formula for transformation of the variable as well as the invariance of μ under permutations of Σ^*. $\square$

Lemma 3.3
Let
$$v : \mathcal{P}(I\!N) \to I\!R$$
be a game.
For every $i \in I\!N$ the marginal contribution
$$f_i^v : \Sigma \to I\!R$$
is continuous.

Proof: Let $\pi \in \Sigma$ and $\frac{1}{2^{i+2}} > \delta > 0$ be given. Define $k := \pi(i)$. Then for every $\sigma \in \Sigma$ with $d(\sigma, \pi) < \delta$ we have by definition of the metric:

$$\pi(i) = \sigma(i) \quad \text{and} \quad \pi^{-1}(l) = \sigma^{-1}(l) \quad \text{for all} \quad l \in \{1, ..., k\}$$

Hence it follows that

$$S_{\sigma(i)}^\sigma = \sigma^{-1}(\{1, ..., \sigma(i)\}) = \sigma^{-1}(\{1, ..., k\})$$

$$= \pi^{-1}(\{1, ..., k\}) = \pi^{-1}(\{1, ..., \pi(i)\}) = S_{\pi(i)}^\pi.$$

Hence for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\sigma \in \Sigma$, with $d(\sigma, \pi) < \delta$, it follows that

$$|f_i^v(\sigma) - f_i^v(\pi)| = 0 < \varepsilon,$$

which means that $f_i^v : \Sigma \to I\!R$ is continuous. $\square$

Lemma 3.4
Let
$$v : \mathcal{P}(I\!N) \to I\!R$$
be a monotone game.
Then for every $i \in I\!N$ it follows that $\Phi(v) \geq 0$ and

$$\sum_{i \in I\!N} \Phi_i(v) \leq v(I\!N).$$

holds true.

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mapping:

Proposition 3.5
The map
$$\Phi : \mathbf{BV}(I\!N) \to \mathcal{A}_\sigma(I\!N)$$

defined by
$$\Phi(v)(S) := \sum_{i \in S} \Phi_i(v)$$

is linear and satisfies $\|\Phi\| \leq 1$.

Proof: By definition it is clear that for every $v \in \mathbf{BV}(I\!N)$ the set function $\Phi(v) : \mathcal{P}(I\!N) \to I\!R$ is of bounded variation.

Moreover, it is obvious that the map $\Phi : \mathbf{BV}(I\!N) \to \mathcal{A}_\sigma(I\!N)$ is linear.

Hence, it remains to show that $\|\Phi\| \leq 1$ holds true.
Now, since every element $v \in \mathbf{BV}(I\!N)$ has a Jordan decomposition, say $v = v^+ - v^-$, it follows that $\Phi(v) = \Phi(v^+) - \Phi(v^-)$ holds true. Moreover for every $v \in \mathbf{BV}(I\!N)$ we have $\|v\| = v^+(I\!N) + v^-(I\!N)$. In view of lemma 3.4 we deduce for $v \in \mathbf{BV}(I\!N)$
$$\|\Phi(v)\| \leq \|\Phi(v^+)\| + \|\Phi(v^-)\| \leq v^+(I\!N) + v^-(I\!N) = \|v\|.$$

This means that $\|\Phi\| \leq 1$. □

Collecting the pieces we obtain the following result which links the present version of the Shapley value to Artsteins version ([2].)

Theorem 3.6
Let $\mathbf{E}(I\!N) := \{v \in \mathbf{BV}(I\!N) \mid \Phi(v)(I\!N) = v(I\!N) \}$

Then the mapping:
$$\Phi : \mathbf{E}(I\!N) \to \mathcal{A}_\sigma(I\!N)$$

defined by
$$\Phi(v)(S) := \sum_{i \in S} \Phi_i(v)$$

satisfies the following conditions:

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This follows from the work of Schreier and Ulam (see [9]) since all automorphisms of the group Σ are inner automorphisms , i.e: for every group automorphism $\chi : \Sigma \to \Sigma$ there exists an element $\rho \in \Sigma$ such that

$$\chi(\pi) = \rho^{-1} \cdot \pi \cdot \rho \text{ for all } \pi \in \Sigma$$

holds true.

4 Weighted Majority Games

Weighted majority games are among the first that were treated by Shapiro and Shapley in [12] and as frequently occurs in our context, the main problem is to establish Pareto efficiency of the Shapley value. Since weighted majority games are of bounded variation but not necessarily absolutely continuous, the general **AC**-theory does not apply. The fact that "all games are regular" with respect to the Shapiro-Shapley measure as defined on the orderings of $I\!N$ has been established in all generality by Berbee ([3]). This result rests on a theory about upcrossings of certain stochastic processes and is quite involved.

We would like to show that in our context Berbee's Theorem is not needed in order to establish Pareto efficiency for weighted majority games. As on the other hand we can follow some considerations employed by Artstein ([2]), parts of this section will only appear as a sketch.

Let $m = (m_1, m_2, ...)$ be a nonnegative sequence generating an (absolutely) converging series which is normalized to

$$\sum_{i \in I\!N} m_i = 1.$$

As usual m is tantamount to a probability measure on $\mathcal{P}(I\!N)$ via the convention

$$m(S) := \sum_{i \in S} m_i \ (S \in \mathcal{P}(I\!N)).$$

A game v is said to be _representable_ or a _wheigted majority game_ if there exists a probability m and a real number $\alpha \in (0, 1)$ such that $v(S) = 1$ if $m(S) \geq \alpha$ and $v(S) = 0$ if $m(S) < \alpha$. In this case we shall sometimes call (m, α) a _representation_ of v and write $v := v_\alpha^m$. Next we shall introduce the notion of pivoting. Given (m, α) we shall say that player $i \in I\!N$ _pivots_ π if

$$m(S_{\pi(i)}^\pi - i) < \alpha \leq m(S_{\pi(i)}^\pi)$$

holds true. It is seen at once that the marginal contribution of player i w.r.t. the game $v = v_\alpha^m$ at π, that is $f_i^v(\pi)$, equals 1 or 0 according to whether player i pivots π or not. From this it follows at once that the Shapley value of v for player i is given by the probability that i pivots, that is,

$$\Phi_i(v) = \mu(\{\pi \mid i \text{ pivots } \pi\}).$$

Definition 4.4
Let

$$\mathbf{F}^{(i)} := \bigcup_{(S,T) \in \mathcal{F}_i} F_{S;T};$$

Then the following corollary follows immediately.

Corollary 4.5

$$\mathbf{F}^{(i)} = \{\pi \mid i \text{ pivots } \pi\}.$$

Proof:
$$\mathbf{F}^{(i)} \subseteq \{\pi \mid i \text{ pivots } \pi\}.$$

is obvious and the equation follows at once from theorem 4.3 by taking the disjoint union.
□

Remark 4.6
Let us tentatively denote by Ψ the Shapley value as defined by Artstein ([2]) via the approximating procedure. It has been shown by this author that, for any weighted majority game $v = v_\alpha^m$ and any $i \in \mathbb{N}$, it follows that Ψ satisfies

$$\Psi_i(v) = \sum_{(S,T) \in \mathcal{F}_i} \frac{s!t!}{(s+t+1)!},$$

moreover, Ψ yields

$$\sum_{i \in \mathbb{N}} \Psi_i(v) = 1.$$

Now we have finally

Theorem 4.7
Let m be a probability on $\mathcal{P}(\mathbb{N})$ and let α be a real number with $0 < \alpha < 1$. Then $v_\alpha^m \in \mathbf{E}$ holds true .

Proof: In view of corollary 2.10 we know that we have

$$\mu(F_{S;T}) = \frac{s!t!}{(s+t+1)!},$$

and by corollary 4.5 and the definition of Φ as the probability that i pivots, it follows that

$$\mu(\mathbf{F}^{(i)}) = \Phi_i(v).$$

Now, on one hand we have

$$\sum_{i \in \mathbb{N}} \sum_{(S,T) \in \mathcal{F}_i} \frac{s!t!}{(s+t+1)!} = \sum_{i \in \mathbb{N}} \Psi_i(v) = 1,$$

while on the other hand it follows that

$$\Phi(v)(\mathbb{N}) = \sum_{i \in \mathbb{N}} \Phi_i(v) = \sum_{i \in \mathbb{N}} \mu(\{\pi \mid i \text{ pivots } \pi\})$$

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