DEGENERATE PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS: QUASILINEAR CASE

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ABSTRACT. In this paper, we study the Cauchy problem for a quasilinear degenerate parabolic stochastic partial differential equation driven by a cylindrical Wiener process. In particular, we adapt the notion of kinetic formulation and kinetic solution and develop a well-posedness theory that includes also an L^1-contraction property. In comparison to the previous works of the authors concerning stochastic hyperbolic conservation laws (Debussche and Vovelle, 2010) and semilinear degenerate parabolic SPDEs (Hofmanová, 2013), the present result contains two new ingredients that provide simpler and more effective method of the proof: a generalized Itô formula that permits a rigorous derivation of the kinetic formulation even in the case of weak solutions of certain nondegenerate approximations and a direct proof of strong convergence of these approximations to the desired kinetic solution of the degenerate problem.

1. INTRODUCTION

We consider the Cauchy problem for a quasilinear degenerate parabolic stochastic partial differential equation

$$\begin{aligned} &\mathrm{d}u + \operatorname{div}\big(B(u)\big)\,\mathrm{d}t = \operatorname{div}\big(A(u)\nabla u\big)\,\mathrm{d}t + \varPhi(u)\,\mathrm{d}W, \quad x\in\mathbb{T}^N,\ t\in(0,T),\\ &u(0)=u_0, \end{aligned} \tag{1.1}$$

where W is a cylindrical Wiener process. Equations of this type model the phenomenon of convection-diffusion of ideal fluids and therefore arise in a wide variety of important applications, including for instance two or three phase flows in porous media or sedimentation-consolidation processes (for a thorough exposition of this area given from a practical point of view we refer the reader to [10] and the references therein). The addition of a stochastic noise to this physical model is fully natural as it represents external perturbations or a lack of knowledge of certain physical parameters. Towards the applicability of the results, it is necessary to treat the problem (1.1) under very general hypotheses. Particularly, without the assumption of positive definiteness of the diffusion matrix A, the equation can be degenerate which brings the main difficulty in the problem solving. We assume the matrix A to be positive semidefinite and, as a consequence, it can for instance vanish completely which leads to a hyperbolic conservation law. We point out, that we do not intend to employ any form of regularization by the noise to solve (1.1) and thus the deterministic equation is included in our theory as well.

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In order to find a suitable concept of solution for our model problem (1.1), we observe that already in the case of deterministic hyperbolic conservation law it is possible to find simple examples supporting the two following claims (see e.g. [20]):

(i) classical C^1 solutions do not exist,
(ii) weak (distributional) solutions lack uniqueness.

The first claim is a consequence of the fact that any smooth solution has to be constant along characteristic lines, which can intersect in finite time (even in the case of smooth data) and shocks can be produced. The second claim demonstrates the inconvenience that often appears in the study of PDEs and SPDEs: the usual way of weakening the equation leads to the occurrence of nonphysical solutions and therefore additional assumptions need to be imposed in order to select the physically relevant ones and to ensure uniqueness. Hence one needs to find some balance that allows to establish existence of a unique (physically reasonable) solution.

Towards this end, we adapt the notion of kinetic formulation and kinetic solution. This concept was first introduced by Lions, Perthame, Tadmor [19] for deterministic hyperbolic conservation laws. In comparison to the notion of entropy solution introduced by Kružkov [16], kinetic solutions seem to be better suited particularly for degenerate parabolic problems since they allow us to keep the precise structure of the parabolic dissipative measure, whereas in the case of entropy solution part of this information is lost and has to be recovered at some stage. This technique also supplies a good technical framework to establish a well-posedness theory which is the main goal of the present paper.

Other references for kinetic or entropy solutions in the case of deterministic hyperbolic conservation laws include for instance [3], [14], [18], [22], [23]. Deterministic degenerate parabolic PDEs were studied by Carrillo [3] and Chen and Perthame [4] by means of both entropy and kinetic solutions. Also in the stochastic setting there are several papers concerned with entropy solutions for hyperbolic conservation laws, see [1], [8], [15], [24]. The first work dealing with kinetic solutions in the stochastic setting and also the first complete well-posedness result for hyperbolic conservation laws driven by a general multiplicative noise was given by Debussche and Vovelle [6]. Their concept was then further generalized to the case of semilinear degenerate parabolic SPDEs by Hofmanová [11]. To the best of our knowledge, stochastic equations of type (1.1) have not been studied yet, neither by means of kinetic formulation nor by any other approach.

In comparison to the previous works of the authors [6] and [11], the present proof of well-posedness contains two new ingredients: a generalized Itô formula that permits a rigorous derivation of the kinetic formulation even in the case of weak solutions of certain nondegenerate approximations (see Appendix A) and a direct proof of strong convergence of these approximations to the desired kinetic solution of the degenerate problem (see Subsection 6.2). In order to explain these recent developments more precisely, let us recall the basic ideas of the proofs in [6] and [11].

In the case of hyperbolic conservation laws [6], the authors defined a notion of generalized kinetic solution and obtained a comparison result showing that any generalized kinetic solution is actually a kinetic solution. Accordingly, the proof of existence simplified since only weak convergence of approximate viscous solutions was necessary. The situation was quite different in the case of semilinear degenerate parabolic equations [11], since this approach was no longer applicable. The proof of the comparison principle was much more delicate and, consequently, generalized kinetic solutions were not allowed and therefore strong convergence of approximate solutions was needed in order to prove existence. The limit argument was based on a compactness method: uniform estimates yielded tightness and consequently also strong convergence of the approximate sequence on another probability space and the existence of a martingale kinetic solution followed. The existence of a pathwise kinetic solution was then obtained by the Gyöngy-Krylov characterization of convergence in probability.

Due to the second order term in (1.1), we are for the moment not able to apply efficiently the method of generalized kinetic solutions. Let us explain why, by considering the Definition 2.2 of solution. We may adapt this definition to introduce a notion of generalized kinetic solution (in the spirit of [6] for example), and we would then easily obtain the equivalent of the kinetic equation (2.6) by passing to the limit on suitable approximate problems. This works well in the first-order case, provided uniqueness of generalized solutions can be shown. To prove such a result here, with second-order terms, we need the second important item in Definition 2.2, the chain-rule (2.5). We do not know how to relax this equality and we do not know how to obtain it by mere weak convergence of approximations: strong convergence seems to be necessary. Therefore, it would not bring any simplification here to consider generalized solutions. On the other hand, it would be possible to apply the compactness method as established in [11] to obtain strong convergence. However, as this is quite technical, we propose a simpler proof of the strong convergence based on the techniques developed in the proof of the comparison principle: comparing two (suitable) nondegenerate approximations, we obtain the strong convergence in L^1 directly. Note, that this approach does not apply to the semilinear case as no sufficient control of the second order term is known.

Another important issue here was the question of regularity of the approximate solutions. In both works [6] and [11], the authors derived the kinetic formulation for sufficiently regular approximations only. This obstacle was overcome by showing the existence of these regular approximations in [12], however, it does not apply to the quasilinear case where a suitable regularity result is still missing: even in the deterministic setting the proofs, which can be found in [17], are very difficult and technical while the stochastic case remains open. In the present paper, we propose a different way to solve this problem, namely, the generalized Itô formula (Proposition A.1) that leads to a clear-cut derivation of the kinetic formulation also for weak solutions and hence avoids the necessity of regular approximations.

The paper is organized as follows. In Section 2, we introduce the basic setting, define the notion of kinetic solution and state our main result, Theorem 2.7. Section 3 is devoted to the proof of uniqueness together with the L^1-comparison principle, Theorem 3.3. The remainder of the paper deals with the existence part of Theorem 2.7 which is divided into four parts. First, we prove existence under three additional hypotheses: we consider (1.1) with regular initial data, positive definite diffusion matrix A and Lipschitz continuous flux function B, Section 4. Second, we relax the hypothesis upon B and prove existence under the remaining two additional hypotheses in Section 5. In Section 6, we proceed to the proof of existence in the degenerate case while keeping the assumption upon the initial condition. The proof of Theorem 2.7 is then completed in Section 7. In Appendix A, we establish the above mentioned generalized Itô formula for weak solutions of a general class of SPDEs.

2. Hypotheses and the main result

2.1. **Hypotheses.** We now give the precise assumptions on each of the terms appearing in the above equation (1.1). We work on a finite-time interval $[0,T]$, $T>0$, and consider periodic boundary conditions: $x\in\mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. The flux function

$$B=(B_1,\dots,B_N):\mathbb{R}\longrightarrow\mathbb{R}^N$$

is supposed to be of class C^1 with a polynomial growth of its derivative, which is denoted by $b=(b_1,\dots,b_N)$. The diffusion matrix

$$A=(A_{ij})_{i,j=1}^N:\mathbb{R}\longrightarrow\mathbb{R}^{N\times N}$$

is symmetric and positive semidefinite. Its square-root matrix, which is also symmetric and positive semidefinite, is denoted by σ. We assume that σ is bounded and locally γ-Hölder continuous for some $\gamma>1/2$, i.e.

$$|\sigma(\xi)-\sigma(\zeta)|\leq C|\xi-\zeta|^\gamma\qquad\forall\xi,\zeta\in\mathbb{R},\ |\xi-\zeta|<1. \tag{2.1}$$

Regarding the stochastic term, let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq0},\mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. Let $\mathcal{P}$ denote the predictable σ-algebra on $\Omega\times[0,T]$ associated to $(\mathscr{F}_t)_{t\geq0}$. The initial datum may be random in general, i.e. $\mathscr{F}_0$-measurable, and we assume $u_0\in L^p(\Omega;L^p(\mathbb{T}^N))$ for all $p\in[1,\infty)$. The process W is a cylindrical Wiener process: $W(t)=\sum_{k\geq1}\beta_k(t)e_k$ with $(\beta_k)_{k\geq1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\geq0}$ and $(e_k)_{k\geq1}$ a complete orthonormal system in a separable Hilbert space $\mathfrak{U}$. In this setting we can assume without loss of generality that the σ-algebra $\mathscr{F}$ is countably generated and $(\mathscr{F}_t)_{t\geq0}$ is the filtration generated by the Wiener process and the initial condition. For each $z\in L^2(\mathbb{T}^N)$ we consider a mapping $\varPhi(z):\mathfrak{U}\to L^2(\mathbb{T}^N)$ defined by $\varPhi(z)e_k=g_k(\cdot,z(\cdot))$. In particular, we suppose that $g_k\in C(\mathbb{T}^N\times\mathbb{R})$ and the following conditions

$$G^2(x,\xi)=\sum_{k\geq1}|g_k(x,\xi)|^2\leq C\big(1+|\xi|^2\big), \tag{2.2}$$

$$\sum_{k\geq1}|g_k(x,\xi)-g_k(y,\zeta)|^2\leq C\big(|x-y|^2+|\xi-\zeta|h(|\xi-\zeta|)\big), \tag{2.3}$$

are fulfilled for every $x,y\in\mathbb{T}^N$, $\xi,\zeta\in\mathbb{R}$, where h is a continuous nondecreasing function on $\mathbb{R}_+$ satisfying, for some $\alpha>0$,

$$h(\delta)\leq C\delta^\alpha,\quad \delta<1. \tag{2.4}$$

The conditions imposed on $\varPhi$, particularly assumption (2.2), imply that

$$\varPhi:L^2(\mathbb{T}^N)\longrightarrow L_2(\mathfrak{U};L^2(\mathbb{T}^N)),$$

where $L_2(\mathfrak{U};L^2(\mathbb{T}^N))$ denotes the collection of Hilbert-Schmidt operators from $\mathfrak{U}$ to $L^2(\mathbb{T}^N)$. Thus, given a predictable process $u\in L^2(\Omega;L^2(0,T;L^2(\mathbb{T}^N)))$, the stochastic integral $t\mapsto\int_0^t\varPhi(u)\mathrm{d}W$ is a well defined process taking values in $L^2(\mathbb{T}^N)$ (see [5] for detailed construction).

Finally, we define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 = \bigg\{ v = \sum_{k\geq 1} \alpha_k e_k;\ \sum_{k\geq 1} \frac{\alpha_k^2}{k^2} < \infty \bigg\},$$

endowed with the norm

$$\|v\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{\alpha_k^2}{k^2}, \qquad v = \sum_{k\geq 1} \alpha_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt. Moreover, trajectories of W are $\mathbb{P}$-a.s. in $C([0,T];\mathfrak{U}_0)$ (see [5]).

In this paper, we use the brackets $\langle \cdot, \cdot \rangle$ to denote the duality between the space of distributions over $\mathbb{T}^N \times \mathbb{R}$ and $C_c^\infty(\mathbb{T}^N \times \mathbb{R})$ and the duality between $L^p(\mathbb{T}^N \times \mathbb{R})$ and $L^q(\mathbb{T}^N \times \mathbb{R})$. If there is no danger of confusion, the same brackets will also denote the duality between $L^p(\mathbb{T}^N)$ and $L^q(\mathbb{T}^N)$. The differential operators of gradient ∇, divergence div and Laplacian Δ are always understood with respect to the space variable x.

2.2. **Definitions.** As the next step, we introduce the kinetic formulation of (1.1) as well as the basic definitions concerning the notion of kinetic solution. The motivation for this approach is given by the nonexistence of a strong solution and, on the other hand, the nonuniqueness of weak solutions, even in simple cases. The idea is to establish an additional criterion – the kinetic formulation – which is automatically satisfied by any weak solution to (1.1) in the nondegenerate case and which permits to ensure the well-posedness.

Definition 2.1 (Kinetic measure)**.** A mapping m from Ω to $\mathcal{M}_b^+([0,T]\times\mathbb{T}^M\times\mathbb{R})$, the set of nonnegative bounded measures over $[0,T]\times\mathbb{T}^N\times\mathbb{R}$, is said to be a kinetic measure provided

(i) m is measurable in the following sense: for each $\psi \in C_0([0,T]\times\mathbb{T}^N\times\mathbb{R})$ the mapping $m(\psi):\Omega\to\mathbb{R}$ is measurable,

(ii) m vanishes for large ξ: if $B_R^c = \{\xi\in\mathbb{R};\ |\xi|\geq R\}$ then

$$\lim_{R\to\infty} \mathbb{E}\, m\big([0,T]\times\mathbb{T}^N\times B_R^c\big) = 0,$$

(iii) for any $\psi \in C_0(\mathbb{T}^N\times\mathbb{R})$

$$\int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} \psi(x,\xi)\,\mathrm{d}m(s,x,\xi) \in L^2(\Omega\times[0,T])$$

admits a predictable representative[1].

Definition 2.2 (Kinetic solution)**.** Assume that, for all $p\in[1,\infty)$,

$$u \in L^p(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^p(\mathbb{T}^N)) \cap L^p(\Omega;L^\infty(0,T;L^p(\mathbb{T}^N)))$$

is such that

(i) $\operatorname{div}\int_0^u \sigma(\zeta)\,\mathrm{d}\zeta \in L^2(\Omega\times[0,T]\times\mathbb{T}^N)$,

(ii) for any $\phi\in C_b(\mathbb{R})$ the following chain rule formula hods true

$$\operatorname{div}\int_0^u \phi(\zeta)\sigma(\zeta)\,\mathrm{d}\zeta = \phi(u)\operatorname{div}\int_0^u \sigma(\zeta)\,\mathrm{d}\zeta \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^N) \text{ a.e. } (\omega,t). \tag{2.5}$$

[1] Throughout the paper, the term *representative* stands for an element of a class of equivalence.

Let $n_1 : \Omega \to \mathcal{M}_b^+([0,T]\times\mathbb{T}^M\times\mathbb{R})$ be defined as follows: for any $\varphi\in C_0([0,T]\times\mathbb{T}^N\times\mathbb{R})$

$$n_1(\varphi)=\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}\varphi(t,x,\xi)\bigg|\operatorname{div}\int_0^u\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\mathrm{d}\delta_{u(t,x)}(\xi)\,\mathrm{d}x\,\mathrm{d}t.$$

Then u is said to be a kinetic solution to (1.1) with initial datum u_0 provided there exists a kinetic measure $m\geq n_1$, $\mathbb{P}$-a.s., such that the pair $(f=\mathbf{1}_{u>\xi},m)$ satisfies, for all $\varphi\in C_c^\infty([0,T)\times\mathbb{T}^N\times\mathbb{R})$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\int_0^T\big\langle f(t),\partial_t\varphi(t)\big\rangle\mathrm{d}t+\big\langle f_0,\varphi(0)\big\rangle+\int_0^T\big\langle f(t),b\cdot\nabla\varphi(t)\big\rangle\mathrm{d}t&\\ +\int_0^T\big\langle f(t),A:\mathrm{D}^2\varphi(t)\big\rangle\mathrm{d}t&\\ =-\sum_{k\geq1}\int_0^T\int_{\mathbb{T}^N}g_k\big(x,u(t,x)\big)\varphi\big(t,x,u(t,x)\big)\mathrm{d}x\,\mathrm{d}\beta_k(t)&\\ -\frac12\int_0^T\int_{\mathbb{T}^N}G^2\big(x,u(t,x)\big)\partial_\xi\varphi\big(t,x,u(t,x)\big)\mathrm{d}x\,\mathrm{d}t+m(\partial_\xi\varphi).&\end{aligned}\tag{2.6}$$

We have used the notation $A:B=\sum_{i,j}a_{ij}b_{ij}$ for two matrices $A=(a_{ij})$, $B=(b_{ij})$ of the same size.

Remark 2.3. We emphasize that a kinetic solution is, in fact, a class of equivalence in $L^p(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^p(\mathbb{T}^N))$ so not necessarily a stochastic process in the usual sense. Nevertheless, it will be seen later (see Corollary 3.4) that, in this class of equivalence, there exists a representative with good continuity properties, namely, $u\in C([0,T];L^p(\mathbb{T}^N))$, $\mathbb{P}$-a.s., and therefore, it can be regarded as a stochastic process.

By $f=\mathbf{1}_{u>\xi}$ we understand a real function of four variables, where the additional variable ξ is called velocity. In the deterministic case, i.e. corresponding to the situation $\varPhi=0$, the equation (2.6) in the above definition is a weak form of the so-called kinetic formulation of (1.1)

$$\partial_t\mathbf{1}_{u>\xi}+b\cdot\nabla\mathbf{1}_{u>\xi}-A:\mathrm{D}^2\mathbf{1}_{u>\xi}=\partial_\xi m$$

where the unknown is the pair $(\mathbf{1}_{u>\xi},m)$ and it is solved in the sense of distributions over $[0,T)\times\mathbb{T}^N\times\mathbb{R}$. In the stochastic case, we write formally[2]

$$\partial_t\mathbf{1}_{u>\xi}+b\cdot\nabla\mathbf{1}_{u>\xi}-A:\mathrm{D}^2\mathbf{1}_{u>\xi}=\delta_{u=\xi}\,\varPhi(u)\dot W+\partial_\xi\Big(m-\frac12G^2\delta_{u=\xi}\Big).\tag{2.7}$$

It will be seen later that this choice is reasonable since for any u being a weak solution to (1.1) that belongs to $L^p(\Omega;C([0,T];L^p(\mathbb{T}^N)))\cap L^2(\Omega;L^2(0,T;H^1(\mathbb{T}^N)))$, $\forall p\in[2,\infty)$, the pair $(\mathbf{1}_{u>\xi},n_1)$ satisfies (2.6) and consequently u is a kinetic solution to (1.1). The measure n_1 relates to the diffusion term in (1.1) and so is called parabolic dissipative measure.

We proceed with two related definitions.

[2]Hereafter, we employ the notation which is commonly used in papers concerning the kinetic solutions to conservation laws and write $\delta_{u=\xi}$ for the Dirac measure centered at $u(t,x)$.

Definition 2.4 (Young measure)**.** Let (X,λ) be a finite measure space. A mapping ν from X to the set of probability measures on $\mathbb{R}$ is said to be a Young measure if, for all $\psi\in C_b(\mathbb{R})$, the map $z\mapsto\nu_z(\psi)$ from X into $\mathbb{R}$ is measurable. We say that a Young measure ν vanishes at infinity if, for all $p\geq 1$,

$$\int_X\int_{\mathbb{R}}|\xi|^p\mathrm{d}\nu_z(\xi)\,\mathrm{d}\lambda(z)<\infty.$$

Definition 2.5 (Kinetic function)**.** Let (X,λ) be a finite measure space. A measurable function $f:X\times\mathbb{R}\to[0,1]$ is said to be a kinetic function if there exists a Young measure ν on X vanishing at infinity such that, for λ-a.e. $z\in X$, for all $\xi\in\mathbb{R}$,

$$f(z,\xi)=\nu_z(\xi,\infty).$$

Remark 2.6. Note, that if f is a kinetic function then $\partial_\xi f=-\nu$ for λ-a.e. $z\in X$. Similarly, let u be a kinetic solution of (1.1) and consider $f=\mathbf{1}_{u>\xi}$. We have $\partial_\xi f=-\delta_{u=\xi}$, where $\nu=\delta_{u=\xi}$ is a Young measure on $\Omega\times[0,T]\times\mathbb{T}^N$. Therefore, (2.6) can be rewritten as follows: for all $\varphi\in C_c^\infty([0,T)\times\mathbb{T}^N\times\mathbb{R})$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\int_0^T\big\langle f(t),\partial_t\varphi(t)\big\rangle\mathrm{d}t+\big\langle f_0,\varphi(0)\big\rangle+\int_0^T\big\langle f(t),b\cdot\nabla\varphi(t)\big\rangle\mathrm{d}t\\ +\int_0^T\big\langle f(t),A:\mathrm{D}^2\varphi(t)\big\rangle\mathrm{d}t\\ =-\sum_{k\geq1}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}g_k(x,\xi)\varphi(t,x,\xi)\mathrm{d}\nu_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}\beta_k(t)\\ -\frac{1}{2}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}G^2(x,\xi)\partial_\xi\varphi(t,x,\xi)\mathrm{d}\nu_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t+m(\partial_\xi\varphi).\end{aligned}$$

For a general kinetic function f with corresponding Young measure ν, the above formulation leads to the notion of generalized kinetic solution as introduced in [6]. Although this concept is not established here, the notation will be used throughout the paper, i.e. we will often write $\nu_{t,x}(\xi)$ instead of $\delta_{u(t,x)=\xi}$.

2.3. **Derivation of the kinetic formulation.** Let us now clarify that the kinetic formulation (2.6) represents a reasonable way to weaken the original model problem (1.1). In particular, we show that if u is a weak solution to (1.1) such that $u\in L^p(\Omega;C([0,T];L^p(\mathbb{T}^N)))\cap L^2(\Omega;L^2(0,T;H^1(\mathbb{T}^N)))$, $\forall p\in[2,\infty)$, then $f=\mathbf{1}_{u>\xi}$ satisfies

$$\mathrm{d}f+b\cdot\nabla f\,\mathrm{d}t-A:\mathrm{D}^2f\,\mathrm{d}t=\delta_{u=\xi}\,\varPhi\,\mathrm{d}W+\partial_\xi\left(n_1-\frac{1}{2}G^2\delta_{u=\xi}\right)\mathrm{d}t$$

in the sense of $\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})$, where

$$\mathrm{d}n_1(t,x,\xi)=|\sigma(u)\nabla u|^2\,\mathrm{d}\delta_{u=\xi}\,\mathrm{d}x\,\mathrm{d}t.$$

Indeed, it follows from Proposition A.1, for $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{T}^N)$,

$$\begin{aligned}\big\langle\varphi(u(t)),\psi\big\rangle=\big\langle\varphi(u_0),\psi\big\rangle&-\int_0^t\big\langle\varphi'(u)\,\mathrm{div}\,\big(B(u)\big),\psi\big\rangle\,\mathrm{d}s\\&-\int_0^t\big\langle\varphi''(u)\nabla u\cdot\big(A(u)\nabla u\big),\psi\big\rangle\,\mathrm{d}s\\&+\int_0^t\big\langle\,\mathrm{div}\,\big(\varphi'(u)A(u)\nabla u\big),\psi\big\rangle\,\mathrm{d}s\\&+\sum_{k\geq1}\int_0^t\big\langle\varphi'(u)g_k(u),\psi\big\rangle\,\mathrm{d}\beta_k(s)\\&+\frac12\int_0^t\big\langle\varphi''(u(s))G^2(u),\psi\big\rangle\,\mathrm{d}s.\end{aligned}$$

Afterwards, we proceed term by term and employ the chain rule for functions from Sobolev spaces. We obtain the following equalities that hold true in $\mathcal{D}'(\mathbb{T}^N)$

$$\begin{aligned}\varphi'(u)\,\mathrm{div}\,\big(B(u)\big)&=\varphi'(u)b(u)\cdot\nabla u\\&=\mathrm{div}\left(\int_{-\infty}^u b(\xi)\varphi'(\xi)\mathrm{d}\xi\right)=\mathrm{div}\langle b\mathbf{1}_{u>\xi},\varphi'\rangle_\xi\\\varphi''(u)\nabla u\cdot\big(A(u)\nabla u\big)&=-\big\langle\partial_\xi n_1,\varphi'\big\rangle_\xi\\\mathrm{div}\,\big(\varphi'(u)A(u)\nabla u\big)&=\mathrm{D}^2:\left(\int_{-\infty}^u A(\xi)\varphi'(\xi)\mathrm{d}\xi\right)=\mathrm{D}^2:\langle A\mathbf{1}_{u>\xi},\varphi'\rangle_\xi\\\varphi'(u)g_k(u)&=\langle g_k\delta_{u=\xi},\varphi'\rangle_\xi\\\varphi''(u)G^2(u)&=\langle G^2\delta_{u=\xi},\varphi''\rangle_\xi=-\big\langle\partial_\xi(G^2\delta_{u=\xi}),\varphi'\big\rangle_\xi.\end{aligned}$$

Moreover,

$$\big\langle\varphi(u(t)),\psi\big\rangle=\big\langle\mathbf{1}_{u(t)>\xi},\varphi'\psi\big\rangle_{x,\xi}$$

hence setting $\varphi(\xi)=\int_{-\infty}^\xi\phi(\zeta)\,\mathrm{d}\zeta$ for some $\phi\in C_c^\infty(\mathbb{R})$ yields the claim.

2.4. **The main result.**

To conclude this section we state our main result.

Theorem 2.7. *Let $u_0\in L^p(\Omega;L^p(\mathbb{T}^N))$, for all $p\in[1,\infty)$. Under the above assumptions, there exists a unique kinetic solution to (1.1) and it has almost surely continuous trajectories in $L^p(\mathbb{T}^N)$, for all $p\in[1,\infty)$. Moreover, if u_1, u_2 are kinetic solutions to (1.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for all $t\in[0,T]$*

$$\mathbb{E}\|u_1(t)-u_2(t)\|_{L^1(\mathbb{T}^N)}\leq\mathbb{E}\|u_{1,0}-u_{2,0}\|_{L^1(\mathbb{T}^N)}.$$

3. Comparison principle

Let us start with the question of uniqueness. As the first step, we follow the approach of [6] and [11] and obtain an auxiliary property of kinetic solutions, which will be useful later on in the proof of the comparison principle in Theorem 3.3.

Proposition 3.1 (Left- and right-continuous representatives)**.** *Let u be a kinetic solution to (1.1). Then $f=\mathbf{1}_{u>\xi}$ admits representatives f^- and f^+ which are almost surely left- and right-continuous, respectively, at all points $t^*\in[0,T]$ in the sense of distributions over $\mathbb{T}^N\times\mathbb{R}$. More precisely, for all $t^*\in[0,T]$ there exist* *kinetic functions $f^{*,\pm}$ on $\Omega\times\mathbb{T}^N\times\mathbb{R}$ such that setting $f^\pm(t^*)=f^{*,\pm}$ yields $f^\pm=f$ almost everywhere and*

$$\big\langle f^\pm(t^*\pm\varepsilon),\psi\big\rangle\longrightarrow\big\langle f^\pm(t^*),\psi\big\rangle\quad \varepsilon\downarrow 0\quad \forall\psi\in C^2_c(\mathbb{T}^N\times\mathbb{R})\quad \mathbb{P}\text{-a.s.}.$$

Moreover, $f^+=f^-$ for all $t^*\in[0,T]$ except for some at most countable set.

Proof. A detailed proof of this result can be found in [11, Proposition 3.1]. □

From now on, we will work with these two fixed representatives of f and we can take any of them in an integral with respect to time or in a stochastic integral.

As the next step towards the proof of uniqueness, we need a technical proposition relating two kinetic solutions of (1.1). We will also use the following notation: if $f:X\times\mathbb{R}\to[0,1]$ is a kinetic function, we denote by $\bar f$ the conjugate function $\bar f=1-f$.

Proposition 3.2 (Doubling of variables)**.** *Let u_1,u_2 be kinetic solutions to (1.1) and denote $f_1=\mathbf{1}_{u_1>\xi}$, $f_2=\mathbf{1}_{u_2>\xi}$. Then for all $t\in[0,T]$ and any nonnegative functions $\varrho\in C^\infty(\mathbb{T}^N)$, $\psi\in C^\infty_c(\mathbb{R})$ we have*

$$\begin{aligned}&\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho(x-y)\psi(\xi-\zeta)f_1^\pm(x,t,\xi)\bar f_2^\pm(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\\&\quad\le\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho(x-y)\psi(\xi-\zeta)f_{1,0}(x,\xi)\bar f_{2,0}(y,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y+\mathrm{I}+\mathrm{J}+\mathrm{K},\end{aligned}$$

where

$$\mathrm{I}=\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar f_2\big(b(\xi)-b(\zeta)\big)\cdot\nabla_x\varrho(x-y)\psi(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,$$

$$\begin{aligned}\mathrm{J}=\;&\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar f_2\big(A(\xi)+A(\zeta)\big):\mathrm{D}^2_x\varrho(x-y)\psi(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho(x-y)\psi(\xi-\zeta)\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}n_{2,1}(y,s,\zeta)\\&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho(x-y)\psi(\xi-\zeta)\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}y\,\mathrm{d}n_{1,1}(x,s,\xi),\end{aligned}$$

$$\mathrm{K}=\frac12\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho(x-y)\psi(\xi-\zeta)\sum_{k\ge1}\big|g_k(x,\xi)-g_k(y,\zeta)\big|^2\mathrm{d}\nu^1_{x,s}(\xi)\mathrm{d}\nu^2_{y,s}(\zeta)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

Proof. The proof follows the ideas developed in [6, Proposition 9] and [11, Proposition 3.2] and is left to the reader. □

Theorem 3.3 (Comparison principle)**.** *Let u be a kinetic solution to (1.1). Then there exist u^+ and u^-, representatives of u, such that, for all $t\in[0,T]$, $f^\pm(t,x,\xi)=\mathbf{1}_{u^\pm(t,x)>\xi}$ for a.e. (ω,x,ξ). Moreover, if u_1,u_2 are kinetic solutions to (1.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for all $t\in[0,T]$ we have*

$$\mathbb{E}\|u_1^\pm(t)-u_2^\pm(t)\|_{L^1(\mathbb{T}^N)}\le\mathbb{E}\|u_{1,0}-u_{2,0}\|_{L^1(\mathbb{T}^N)}.\tag{3.1}$$

Proof. Let (ϱ_ε), (ψ_δ) be approximations to the identity on $\mathbb{T}^N$ and $\mathbb{R}$, respectively, i.e. let $\varrho\in C^\infty(\mathbb{T}^N)$, $\psi\in C^\infty_c(\mathbb{R})$ be symmetric nonnegative functions such as $\int_{\mathbb{T}^N}\varrho=1$, $\int_{\mathbb{R}}\psi=1$ and $\operatorname{supp}\psi\subset(-1,1)$. We define

$$\varrho_\varepsilon(x)=\frac{1}{\varepsilon^N}\,\varrho\Big(\frac{x}{\varepsilon}\Big),\qquad \psi_\delta(\xi)=\frac{1}{\delta}\,\psi\Big(\frac{\xi}{\delta}\Big).$$

Then

$$\begin{aligned}\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}} & f_1^\pm(x,t,\xi)\bar f_2^\pm(x,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\ &=\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f_1^\pm(x,t,\xi)\bar f_2^\pm(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y+\eta_t(\varepsilon,\delta),\end{aligned}$$

where $\lim_{\varepsilon,\delta\to0}\eta_t(\varepsilon,\delta)=0$. With regard to Proposition 3.2 we need to find suitable bounds for terms I, J, K.

Since b has at most polynomial growth, there exist $C>0$, $p>1$ such that

$$\big|b(\xi)-b(\zeta)\big|\le\Gamma(\xi,\zeta)|\xi-\zeta|,\qquad \Gamma(\xi,\zeta)\le C\big(1+|\xi|^{p-1}+|\zeta|^{p-1}\big).$$

Hence

$$|\mathrm{I}|\le\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar f_2\Gamma(\xi,\zeta)|\xi-\zeta|\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\big|\nabla_x\varrho_\varepsilon(x-y)\big|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

As the next step, we apply integration by parts with respect to ζ, ξ. Focusing only on the relevant integrals we get

$$\begin{aligned}\int_{\mathbb{R}}f_1(\xi)\int_{\mathbb{R}}&\bar f_2(\zeta)\Gamma(\xi,\zeta)|\xi-\zeta|\psi_\delta(\xi-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\xi\\ &=\int_{\mathbb{R}}f_1(\xi)\int_{\mathbb{R}}\Gamma(\xi,\zeta')|\xi-\zeta'|\psi_\delta(\xi-\zeta')\,\mathrm{d}\zeta'\,\mathrm{d}\xi\\ &\qquad-\int_{\mathbb{R}^2}f_1(\xi)\int_{-\infty}^{\zeta}\Gamma(\xi,\zeta')|\xi-\zeta'|\psi_\delta(\xi-\zeta')\,\mathrm{d}\zeta'\,\mathrm{d}\xi\,\mathrm{d}\nu^2_{y,s}(\zeta)\\ &=\int_{\mathbb{R}^2}f_1(\xi)\int_{\zeta}^{\infty}\Gamma(\xi,\zeta')|\xi-\zeta'|\psi_\delta(\xi-\zeta')\,\mathrm{d}\zeta'\,\mathrm{d}\xi\,\mathrm{d}\nu^2_{y,s}(\zeta)\\ &=\int_{\mathbb{R}^2}\Upsilon(\xi,\zeta)\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}\nu^2_{y,s}(\zeta),\end{aligned}$$

where

$$\Upsilon(\xi,\zeta)=\int_{-\infty}^{\xi}\int_{\zeta}^{\infty}\Gamma(\xi',\zeta')|\xi'-\zeta'|\psi_\delta(\xi'-\zeta')\,\mathrm{d}\zeta'\,\mathrm{d}\xi'.$$

Therefore we get

$$|\mathrm{I}|\le\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\Upsilon(\xi,\zeta)\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\big|\nabla_x\varrho_\varepsilon(x-y)\big|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

The function Υ can be estimated using the substitution $\xi''=\xi'-\zeta'$

$$\begin{aligned}\Upsilon(\xi,\zeta)&=\int_\zeta^\infty\int_{|\xi''|<\delta,\,\xi''<\xi-\zeta'}\Gamma(\xi''+\zeta',\zeta')|\xi''|\psi_\delta(\xi'')\,\mathrm{d}\xi''\,\mathrm{d}\zeta'\\&\leq C\delta\int_\zeta^{\xi+\delta}\max_{|\xi''|<\delta,\,\xi''<\xi-\zeta'}\Gamma(\xi''+\zeta',\zeta')\,\mathrm{d}\zeta'\\&\leq C\delta\int_\zeta^{\xi+\delta}\big(1+|\xi|^{p-1}+|\zeta'|^{p-1}\big)\,\mathrm{d}\zeta'\\&\leq C\delta\big(1+|\xi|^p+|\zeta|^p\big)\end{aligned}$$

so

$$|\mathrm{I}|\leq Ct\delta\varepsilon^{-1}.$$

In order to estimate the term J, we observe that

$$\begin{aligned}\mathrm{J}=&\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar{f}_2\,\big(\sigma(\xi)-\sigma(\zeta)\big)^2:\mathrm{D}_x^2\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&+2\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar{f}_2\,\sigma(\xi)\sigma(\zeta):\mathrm{D}_x^2\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu_{x,s}^1(\xi)\,\mathrm{d}x\,\mathrm{d}n_{2,1}(y,s,\zeta)\\&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu_{y,s}^2(\zeta)\,\mathrm{d}y\,\mathrm{d}n_{1,1}(x,s,\xi)\\=&\,\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4.\end{aligned}$$

Since σ is locally γ-Hölder continuous due to (2.1), it holds

$$|\mathrm{J}_1|\leq Ct\delta^{2\gamma}\varepsilon^{-2}.$$

Next, we will show that $\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4\leq0$. From the definition of the parabolic dissipative measure in Definition 2.2, we have

$$\begin{aligned}\mathrm{J}_3+\mathrm{J}_4=&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u_1-u_2)\bigg|\operatorname{div}_y\int_0^{u_2}\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u_1-u_2)\bigg|\operatorname{div}_x\int_0^{u_1}\sigma(\xi)\,\mathrm{d}\xi\bigg|^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Moreover, due to the chain rule formula (2.5) we deduce

$$\begin{aligned}\operatorname{div}\int_{\mathbb{R}}f\phi(\xi)\sigma(\xi)\,\mathrm{d}\xi=\operatorname{div}\int_{\mathbb{R}}\chi_f\phi(\xi)\sigma(\xi)\,\mathrm{d}\xi&=\operatorname{div}\int_0^u\phi(\xi)\sigma(\xi)\,\mathrm{d}\xi\\&=\phi(u)\operatorname{div}\int_0^u\sigma(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where $\chi_f = \mathbf{1}_{u>\xi} - \mathbf{1}_{0>\xi}$. With this in hand, we obtain

$$\begin{aligned}
\mathrm{J}_2 &= 2\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(\nabla_x f_1)^*\sigma(\xi)\sigma(\zeta)(\nabla_y f_2)\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&= 2\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\,\mathrm{div}_y\int_0^{u_2}\sigma(\zeta)\cdot\mathrm{div}_x\int_0^{u_1}\sigma(\xi)\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&= 2\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u_1-u_2)\,\mathrm{div}_x\int_0^{u_1}\sigma(\xi)\,\mathrm{d}\xi\cdot\mathrm{div}_y\int_0^{u_2}\sigma(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

And therefore

$$\begin{aligned}
\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4 = -\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}&\varrho_\varepsilon(x-y)\psi_\delta(u_1-u_2)\\
&\times\left|\mathrm{div}_x\int_0^{u_1}\sigma(\xi)\,\mathrm{d}\xi-\mathrm{div}_y\int_0^{u_2}\sigma(\zeta)\,\mathrm{d}\zeta\right|^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\leq 0.
\end{aligned}$$

The last term is, due to (2.3), bounded as follows

$$\begin{aligned}
\mathrm{K} &\leq C\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)|x-y|^2\int_{\mathbb{R}^2}\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&+C\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\int_{\mathbb{R}^2}\psi_\delta(\xi-\zeta)|\xi-\zeta|h(|\xi-\zeta|)\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\leq Ct\delta^{-1}\varepsilon^2+Cth(\delta).
\end{aligned}$$

As a consequence, we deduce for all $t\in[0,T]$

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{T}^N}&\int_{\mathbb{R}}f_1^\pm(x,t,\xi)\bar{f}_2^\pm(x,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f_{1,0}(x,\xi)\bar{f}_{2,0}(y,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+Ct\delta\varepsilon^{-1}+Ct\delta^{2\gamma}\varepsilon^{-2}+Ct\delta^{-1}\varepsilon^2+Cth(\delta)+\eta_t(\varepsilon,\delta).
\end{aligned}$$

Taking $\delta=\varepsilon^\beta$ with $\beta\in(1/\gamma,2)$ and letting $\varepsilon\to0$ yields

$$\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f_1^\pm(t)\bar{f}_2^\pm(t)\,\mathrm{d}\xi\,\mathrm{d}x\leq\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f_{1,0}\bar{f}_{2,0}\,\mathrm{d}\xi\,\mathrm{d}x.$$

Let us now consider $f_1=f_2=f$. Since $f_0=\mathbf{1}_{u_0>\xi}$ we have the identity $f_0\bar{f}_0=0$ and therefore $f^\pm(1-f^\pm)=0$ a.e. (ω,x,ξ) and for all t. The fact that $f^\pm$ is a kinetic function and Fubini's theorem then imply that, for any $t\in[0,T]$, there exists a set $\Sigma_t\subset\Omega\times\mathbb{T}^N$ of full measure such that, for $(\omega,x)\in\Sigma_t$, $f^\pm(\omega,x,t,\xi)\in\{0,1\}$ for a.e. $\xi\in\mathbb{R}$. Therefore, there exist $u^\pm:\Omega\times\mathbb{T}^N\times[0,T]\to\mathbb{R}$ such that $f^\pm=\mathbf{1}_{u^\pm>\xi}$ for a.e (ω,x,ξ) and all t. In particular, $u^\pm=\int_{\mathbb{R}}(f^\pm-\mathbf{1}_{0>\xi})\,\mathrm{d}\xi$ for a.e. (ω,x) and all t. It follows now from Proposition 3.1 and the identity

$$|\alpha-\beta|=\int_{\mathbb{R}}|\mathbf{1}_{\alpha>\xi}-\mathbf{1}_{\beta>\xi}|\,\mathrm{d}\xi,\qquad\alpha,\,\beta\in\mathbb{R},$$

that $u^+=u^-=u$ for a.e. $t\in[0,T]$. Since

$$\int_{\mathbb{R}}\mathbf{1}_{u_1^\pm>\xi}\overline{\mathbf{1}_{u_2^\pm>\xi}}\,\mathrm{d}\xi=(u_1^\pm-u_2^\pm)^+$$

we obtain the comparison principle (3.1). □

As a consequence, we obtain the continuity of trajectories in $L^p(\mathbb{T}^N)$ whose proof is given in [11, Corollary 3.4].

Corollary 3.4 (Continuity in time)**.** *Let u be a kinetic solution to (1.1). Then there exists a representative of u which has almost surely continuous trajectories in $L^p(\mathbb{T}^N)$, for all $p\in[1,\infty)$.*

4. Nondegenerate case - B Lipschitz continuous

As the first step towards the existence part of Theorem 2.7, we prove existence of a weak solution to (1.1) under three additional hypotheses. Recall that once this claim is verified, Theorem 2.7 follows immediately as any weak solution to (1.1) is also a kinetic solution to (1.1), due to Subsection 2.3. Throughout this section, we suppose that

(H1) $u_0\in L^p(\Omega;C^5(\mathbb{T}^N))$, for all $p\in[1,\infty)$,
(H2) A is positive definite, i.e. $A\geq\tau\mathrm{I}$,
(H3) B is Lipschitz continuous hence it has linear growth $|B(\xi)|\leq L(1+|\xi|)$.

In the following sections, we will show how we may relax all these assumptions one after the other.

Let us approximate (1.1) by

$$\begin{aligned}\mathrm{d}u+\operatorname{div}\big(B^\eta(u)\big)\,\mathrm{d}t&=\operatorname{div}\big(A^\eta(u)\nabla u\big)\,\mathrm{d}t-\eta\Delta^2u\,\mathrm{d}t+\varPhi^\eta(u)\,\mathrm{d}W,\\ u(0)&=u_0,\end{aligned}\tag{4.1}$$

where B^η, A^η, $\varPhi^\eta$ are smooth approximations of B, A and $\varPhi$, respectively, with bounded derivatives. Then the following existence result holds true.

Theorem 4.1. *For any $\eta\in(0,1)$, there exists a unique strong solution to (4.1) that belongs to*

$$L^p(\Omega;C([0,T];C^{4,\lambda}(\mathbb{T}^N))),\qquad\forall\lambda\in(0,1),\,\forall p\in[1,\infty).$$

Proof. The second order term in (4.1) can be rewritten in the following way

$$\operatorname{div}\big(A^\eta(u)\nabla u\big)=\sum_{i,j=1}^N\partial^2_{x_ix_j}\bar{A}^\eta_{ij}(u),\qquad\bar{A}^\eta(\xi)=\int_0^\xi A^\eta(\zeta)\,\mathrm{d}\zeta,$$

hence [12, Corollary 2.2] applies. □

Remark 4.2. Due to the fourth order term $-\eta\Delta^2u$ there are no a priori estimates of the $L^p(\mathbb{T}^N)$-norm for solutions of the approximations (4.1) and that is the reason why we cannot deal directly with (1.1) if the coefficients have polynomial growth. To overcome this difficulty we proceed in two steps and avoid the additional assumption upon B in the next section. Note that the linear growth hypothesis is satisfied for the remaining coefficients, i.e. for $\bar{A}(\xi)=\int_0^\xi A(\zeta)\,\mathrm{d}\zeta$ since $A\in C_b(\mathbb{R})$ and for $\varPhi$ due to (2.2).

Proposition 4.3. *For any $p\in[2,\infty)$, the solution to (4.1) satisfies the following energy estimate*

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|u^\eta(t)\|^p_{L^2(\mathbb{T}^N)}+p\tau\,\mathbb{E}\int_0^T\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\|\nabla u^\eta\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s&\\ \leq C\big(1+\mathbb{E}\|u_0\|^p_{L^2(\mathbb{T}^N)}\big),&\end{aligned}\tag{4.2}$$

where the constant C does not depend on η,τ and L.

Proof. Let us apply the Itô formula to the function $f(v)=\|v\|^p_{L^2(\mathbb{T}^N)}$. We obtain

$$\begin{aligned}\|u^\eta(t)\|^p_{L^2(\mathbb{T}^N)}&=\|u_0\|^p_{L^2(\mathbb{T}^N)}-p\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\big\langle u^\eta,\operatorname{div}\big(B^\eta(u^\eta)\big)\big\rangle\,\mathrm{d}s\\&\quad+p\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\big\langle u^\eta,\operatorname{div}\big(A^\eta(u^\eta)\nabla u^\eta\big)\big\rangle\,\mathrm{d}s\\&\quad-p\eta\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\big\langle u^\eta,\Delta^2u^\eta\big\rangle\,\mathrm{d}s\\&\quad+p\sum_{k\geq1}\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\big\langle u^\eta,g_k^\eta(u^\eta)\big\rangle\,\mathrm{d}\beta_k(s)\\&\quad+\frac{p}{2}\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\|G_\eta(u^\eta)\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s\\&\quad+\frac{p(p-2)}{2}\sum_{k\geq1}\int_0^t\|u^\eta\|^{p-4}_{L^2(\mathbb{T}^N)}\big\langle u^\eta,g_k^\eta(u^\eta)\big\rangle^2\mathrm{d}s\\&=\mathrm{J}_1+\cdots+\mathrm{J}_7.\end{aligned}$$

Setting $H(\xi)=\int_0^\xi B^\eta(\zeta)\mathrm{d}\zeta$, we conclude that the second term on the right hand side vanishes, the third one as well as the fourth one is nonpositive

$$\mathrm{J}_3+\mathrm{J}_4\leq-p\tau\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\|\nabla u^\eta\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s-p\eta\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\|\Delta u^\eta\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s,$$

the sixth and seventh term are estimated as follows

$$\mathrm{J}_6+\mathrm{J}_7\leq C\bigg(1+\int_0^t\|u^\eta\|^p_{L^2(\mathbb{T}^N)}\mathrm{d}s\bigg),$$

and since expectation of J_5 is zero, we get

$$\begin{aligned}\mathbb{E}\|u^\eta(t)\|^p_{L^2(\mathbb{T}^N)}&+p\tau\,\mathbb{E}\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\|\nabla u^\eta\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s\\&\leq\mathbb{E}\|u_0\|^p_{L^2(\mathbb{T}^N)}+C\bigg(1+\int_0^t\mathbb{E}\|u^\eta(s)\|^p_{L^2(\mathbb{T}^N)}\mathrm{d}s\bigg).\end{aligned}$$

Application of the Gronwall lemma now yields

$$\mathbb{E}\|u^\eta(t)\|^p_{L^2(\mathbb{T}^N)}+p\tau\,\mathbb{E}\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\|\nabla u^\eta\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s\leq C\big(1+\mathbb{E}\|u_0\|^p_{L^2(\mathbb{T}^N)}\big).$$

In order to obtain an estimate of $\mathbb{E}\sup_{0\leq t\leq T}\|u^\eta(t)\|^p_{L^2(\mathbb{T}^N)}$ we proceed similarly as above to get

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|u^\eta(t)\|^p_{L^2(\mathbb{T}^N)}&\leq\mathbb{E}\|u_0\|^p_{L^2(\mathbb{T}^N)}+C\bigg(1+\int_0^T\mathbb{E}\|u^\eta\|^p_{L^2(\mathbb{T}^N)}\mathrm{d}s\bigg)\\&\quad+p\,\mathbb{E}\sup_{0\leq t\leq T}\bigg|\sum_{k\geq1}\int_0^t\|u^\eta\|^{p-2}_{L^2(\mathbb{T}^N)}\big\langle u^\eta,g_k^\eta(u^\eta)\big\rangle\,\mathrm{d}\beta_k(s)\bigg|\end{aligned}$$

and for the stochastic integral we employ the Burkholder-Davis-Gundy and the Schwartz inequality, the assumption (2.2) and the weighted Young inequality

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\bigg|\sum_{k\ge1}\int_0^t\|u^\eta\|_{L^2(\mathbb{T}^N)}^{p-2}\big\langle u^\eta,g_k^\eta(u^\eta)\big\rangle\,\mathrm{d}\beta_k(s)\bigg|\\
&\qquad\le C\,\mathbb{E}\bigg(\int_0^T\|u^\eta\|_{L^2(\mathbb{T}^N)}^{2p-4}\sum_{k\ge1}\big\langle u^\eta,g_k^\eta(u^\eta)\big\rangle^2\mathrm{d}s\bigg)^{\frac12}\\
&\qquad\le C\,\mathbb{E}\bigg(\int_0^T\|u^\eta\|_{L^2(\mathbb{T}^N)}^{2p-2}\sum_{k\ge1}\|g_k^\eta(u^\eta)\|_{L^2(\mathbb{T}^N)}^2\,\mathrm{d}s\bigg)^{\frac12}\\
&\qquad\le C\,\mathbb{E}\bigg(\sup_{0\le t\le T}\|u^\eta(t)\|_{L^2(\mathbb{T}^N)}^p\bigg)^{\frac12}\bigg(1+\int_0^T\|u^\eta\|_{L^2(\mathbb{T}^N)}^p\,\mathrm{d}s\bigg)^{\frac12}\\
&\qquad\le\frac12\,\mathbb{E}\sup_{0\le t\le T}\|u^\eta(t)\|_{L^2(\mathbb{T}^N)}^p+C\bigg(1+\int_0^T\mathbb{E}\|u^\eta\|_{L^2(\mathbb{T}^N)}^p\,\mathrm{d}s\bigg).
\end{aligned}
$$

This gives (4.2). □

Proposition 4.4. *For all $\lambda\in(0,1/2)$, there exists a constant $C>0$ such that for all $\eta\in(0,1)$*

$$\mathbb{E}\|u^\eta\|_{C^\lambda([0,T];H^{-3}(\mathbb{T}^N))}\le C.$$

Proof. Recall that due to Proposition 4.3, the set $\{u^\eta;\,\eta\in(0,1)\}$ is bounded in $L^2(\Omega;L^2(0,T;H^1(\mathbb{T}^N)))$. Since the coefficients B^η, A^η have linear growth uniformly in η we conclude, in particular, that

$$\{\operatorname{div}(B^\eta(u^\eta))\},\quad\{\operatorname{div}(A^\eta(u^\eta)\nabla u^\eta)\},\quad\{\eta\Delta^2u^\eta\}$$

are bounded in $L^2(\Omega;L^2(0,T;H^{-3}(\mathbb{T}^N)))$ and consequently

$$\mathbb{E}\bigg\|u^\eta-\int_0^{\cdot}\varPhi^\eta(u^\eta)\,\mathrm{d}W\bigg\|_{C^{1/2}([0,T];H^{-3}(\mathbb{T}^N))}\le C.$$

Moreover, for all $\lambda\in(0,1/2)$, paths of the above stochastic integral are λ-Hölder continuous $L^2(\mathbb{T}^N)$-valued functions and

$$\mathbb{E}\bigg\|\int_0^{\cdot}\varPhi^\eta(u^\eta)\,\mathrm{d}W\bigg\|_{C^\lambda([0,T];L^2(\mathbb{T}^N))}\le C.$$

Indeed, it is a consequence of the Kolmogorov continuity theorem (see [5, Theorem 3.3]) since the following uniform estimate holds true. Let $a>2$, $s,t\in[0,T]$, then

$$
\begin{aligned}
\mathbb{E}\bigg\|\int_s^t\varPhi^\eta(u^\eta)\,\mathrm{d}W\bigg\|^a&\le C\,\mathbb{E}\bigg(\int_s^t\|\varPhi^\eta(u^\eta)\|_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}^2\mathrm{d}r\bigg)^{\frac a2}\\
&\le C\,|t-s|^{\frac a2-1}\mathbb{E}\int_s^t\bigg(\sum_{k\ge1}\|g_k^\eta(u^\eta)\|_{L^2(\mathbb{T}^N)}^2\bigg)^{\frac a2}\mathrm{d}r\\
&\le C\,|t-s|^{\frac a2}\Big(1+\mathbb{E}\sup_{0\le t\le T}\|u^\eta(t)\|_{L^2(\mathbb{T}^N)}^a\Big)\\
&\le C\,|t-s|^{\frac a2}\big(1+\mathbb{E}\|u_0\|_{L^2(\mathbb{T}^N)}^a\big),
\end{aligned}
$$

where we made use of the Burkholder-Davis-Gundy inequality, (2.2) and Proposition 4.3. □

4.1. **Compactness argument.** Let us define the path space $\mathcal{X} = \mathcal{X}_u \times \mathcal{X}_W$, where

$$\mathcal{X}_u = L^2\big(0,T;L^2(\mathbb{T}^N)\big)\cap C\big([0,T];H^{-4}(\mathbb{T}^N)\big),\qquad \mathcal{X}_W = C\big([0,T];\mathfrak{U}_0\big).$$

Let us denote by μ_{u^η} the law of u^η on $\mathcal{X}_u$, $\eta\in(0,1)$, and by μ_W the law of W on $\mathcal{X}_W$. Their joint law on $\mathcal{X}$ is then denoted by μ^η.

Proposition 4.5. *The set $\{\mu^\eta;\,\eta\in(0,1)\}$ is tight and therefore relatively weakly compact in $\mathcal{X}$.*

Proof. First, we prove tightness of $\{\mu_{u^\eta};\,\eta\in(0,1)\}$ which follows directly from Proposition 4.3 and 4.4 by making use of the embeddings

$$\begin{aligned} C^\lambda([0,T];H^{-3}(\mathbb{T}^N)) &\hookrightarrow H^\alpha(0,T;H^{-3}(\mathbb{T}^N)),\qquad \alpha<\lambda,\\ C^\lambda([0,T];H^{-3}(\mathbb{T}^N)) &\overset{c}{\hookrightarrow} C([0,T];H^{-4}(\mathbb{T}^N)),\\ L^2(0,T;H^1(\mathbb{T}^N))\cap H^\alpha(0,T;H^{-3}(\mathbb{T}^N)) &\overset{c}{\hookrightarrow} L^2(0,T;L^2(\mathbb{T}^N)). \end{aligned}$$

Indeed, for $R>0$ we define the set

$$B_R = \big\{u\in L^2(0,T;H^1(\mathbb{T}^N))\cap C^\lambda([0,T];H^{-3}(\mathbb{T}^N));\\ \|u\|_{L^2(0,T;H^1(\mathbb{T}^N))}+\|u\|_{C^\lambda([0,T];H^{-3}(\mathbb{T}^N))}\leq R\big\}$$

which is thus relatively compact in $\mathcal{X}_u$. Moreover, by Proposition 4.3 and 4.4

$$\begin{aligned}\mu_{u^\eta}\big(B_R^C\big) &\leq \mathbb{P}\bigg(\|u^\eta\|_{L^2(0,T;H^1(\mathbb{T}^N))}>\frac{R}{2}\bigg)+\mathbb{P}\bigg(\|u^\eta\|_{C^\lambda([0,T];H^{-3}(\mathbb{T}^N))}>\frac{R}{2}\bigg)\\ &\leq \frac{2}{R}\Big(\mathbb{E}\|u^\eta\|_{L^2(0,T;H^1(\mathbb{T}^N))}+\mathbb{E}\|u^\eta\|_{C^\lambda([0,T];H^{-3}(\mathbb{T}^N))}\Big)\leq\frac{C}{R}\end{aligned}$$

hence given $\vartheta>0$ there exists $R>0$ such that

$$\mu_{u^\eta}(B_R)\geq 1-\vartheta.$$

Besides, since the law μ_W is tight as being a Radon measure on the Polish space $\mathcal{X}_W$, we conclude that also the set of their joint laws $\{\mu^\eta;\,\eta\in(0,1)\}$ is tight and Prokhorov's theorem therefore implies that it is relatively weakly compact. □

Passing to a weakly convergent subsequence $\mu^n=\mu^{\eta_n}$ (and denoting by μ the limit law) we now apply the Skorokhod embedding theorem to infer the following result.

Proposition 4.6. *There exists a probability space $(\tilde\Omega,\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ with a sequence of $\mathcal{X}$-valued random variables $(\tilde u^n,\tilde W^n)$, $n\in\mathbb{N}$, and $(\tilde u,\tilde W)$ such that*

- *(i) the laws of $(\tilde u^n,\tilde W^n)$ and $(\tilde u,\tilde W)$ under $\tilde{\mathbb{P}}$ coincide with μ^n and μ, respectively,*
- *(ii) $(\tilde u^n,\tilde W^n)$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde u,\tilde W)$ in the topology of $\mathcal{X}$,*

Finally, let $(\tilde{\mathscr{F}}_t)$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration of the process $(\tilde u,\tilde W)$, that is

$$\tilde{\mathscr{F}}_t=\sigma\big(\sigma\big(\varrho_t\tilde u,\varrho_t\tilde W\big)\cup\big\{N\in\tilde{\mathscr{F}};\;\tilde{\mathbb{P}}(N)=0\big\}\big),\quad t\in[0,T],$$

where ϱ_t is the operator of restriction to the interval $[0,t]$, i.e. if E is a Banach space and $t\in[0,T]$, we define

$$\begin{aligned}\varrho_t: C([0,T];E)&\longrightarrow C([0,t];E)\\ k&\longmapsto k|_{[0,t]}.\end{aligned}$$

Clearly, ϱ_t is a continuous mapping.

4.2. **Identification of the limit.** The aim of this subsection is to prove the following.

Proposition 4.7. *$\big((\tilde\Omega,\tilde{\mathscr{F}},(\tilde{\mathscr{F}}_t),\tilde{\mathbb{P}}),\tilde W,\tilde u\big)$ is a weak martingale solution to (1.1) provided* (H1), (H2) *and* (H3) *are fulfilled.*

The proof is based on a new general method of constructing martingale solutions of SPDEs, that does not rely on any kind of martingale representation theorem and therefore holds independent interest especially in situations where these representation theorems are no longer available. For other applications of this method we refer the reader to [2], [11], [13], [21].

Let us define for all $t\in[0,T]$ and a test function $\varphi\in C^\infty(\mathbb{T}^N)$

$$\begin{aligned}
M^n(t)&=\big\langle u^n(t),\varphi\big\rangle-\big\langle u_0,\varphi\big\rangle+\int_0^t\big\langle\operatorname{div}\big(B^n(u^n)\big),\varphi\big\rangle\,\mathrm{d}s\\
&\qquad-\int_0^t\big\langle\operatorname{div}\big(A^n(u^n)\nabla u^n\big),\varphi\big\rangle\,\mathrm{d}s+\eta_n\int_0^t\big\langle\Delta^2u^n,\varphi\big\rangle\,\mathrm{d}s,\quad n\in\mathbb{N},\\
\tilde M^n(t)&=\big\langle \tilde u^n(t),\varphi\big\rangle-\big\langle u_0,\varphi\big\rangle+\int_0^t\big\langle\operatorname{div}\big(B^n(\tilde u^n)\big),\varphi\big\rangle\,\mathrm{d}s\\
&\qquad-\int_0^t\big\langle\operatorname{div}\big(A^n(\tilde u^n)\nabla \tilde u^n\big),\varphi\big\rangle\,\mathrm{d}s+\eta_n\int_0^t\big\langle\Delta^2\tilde u^n,\varphi\big\rangle\,\mathrm{d}s,\quad n\in\mathbb{N},\\
\tilde M(t)&=\big\langle \tilde u(t),\varphi\big\rangle-\big\langle u_0,\varphi\big\rangle+\int_0^t\big\langle\operatorname{div}\big(B(\tilde u)\big),\varphi\big\rangle\,\mathrm{d}s-\int_0^t\big\langle\operatorname{div}\big(A(\tilde u)\nabla\tilde u\big),\varphi\big\rangle\,\mathrm{d}s.
\end{aligned}$$

Hereafter, times $s,t\in[0,T]$, $s\le t$, and a continuous function

$$\gamma:C\big([0,s];H^{-4}(\mathbb{T}^N)\big)\times C\big([0,s];\mathfrak{U}_0\big)\longrightarrow[0,1]$$

will be fixed but otherwise arbitrary. The proof is an immediate consequence of the following two lemmas.

Lemma 4.8. *The process $\tilde W$ is a $(\tilde{\mathscr{F}}_t)$-cylindrical Wiener process, i.e. there exists a collection of mutually independent real-valued $(\tilde{\mathscr{F}}_t)$-Wiener processes $\{\tilde\beta_k\}_{k\ge1}$ such that $\tilde W=\sum_{k\ge1}\tilde\beta_ke_k$.*

Proof. Obviously, $\tilde W$ is a $\mathfrak{U}_0$-valued cylindrical Wiener process and is $(\tilde{\mathscr{F}}_t)$-adapted. According to the Lévy martingale characterization theorem, it remains to show that it is also a $(\tilde{\mathscr{F}}_t)$-martingale. It holds true

$$\tilde{\mathbb{E}}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\big[\tilde W^n(t)-\tilde W^n(s)\big]=\mathbb{E}\,\gamma\big(\varrho_su^n,\varrho_sW\big)\big[W(t)-W(s)\big]=0$$

since W is a martingale and the laws of $(\tilde u^n,\tilde W^n)$ and (u^n,W) coincide. Next, the uniform estimate

$$\sup_{n\in\mathbb{N}}\tilde{\mathbb{E}}\|\tilde W^n(t)\|^2_{\mathfrak{U}_0}=\sup_{n\in\mathbb{N}}\mathbb{E}\|W(t)\|^2_{\mathfrak{U}_0}<\infty$$

and the Vitali convergence theorem yields

$$\tilde{\mathbb{E}}\,\gamma\big(\varrho_s\tilde u,\varrho_s\tilde W\big)\big[\tilde W(t)-\tilde W(s)\big]=0$$

which finishes the proof. □

Lemma 4.9. *The processes*

$$\tilde M,\qquad \tilde M^2-\sum_{k\geq 1}\int_0^{\cdot}\big\langle g_k(\tilde u),\varphi\big\rangle^2\,\mathrm{d}r,\qquad \tilde M\tilde\beta_k-\int_0^{\cdot}\big\langle g_k(\tilde u),\varphi\big\rangle\,\mathrm{d}r$$

are $(\tilde{\mathscr F}_t)$-martingales.

Proof. Here, we use the same approach as in the previous lemma. Let us denote by $\tilde\beta^n_k,\ k\geq 1$ the real-valued Wiener processes corresponding to $\tilde W^n$, that is $\tilde W^n=\sum_{k\geq 1}\tilde\beta^n_k e_k$. For all $n\in\mathbb N$, the process

$$M^n=\int_0^{\cdot}\big\langle \varPhi^n(u^n)\,\mathrm{d}W(r),\varphi\big\rangle=\sum_{k\geq 1}\int_0^{\cdot}\big\langle g^n_k(u^n),\varphi\big\rangle\,\mathrm{d}\beta_k(r)$$

is a square integrable $(\mathscr F_t)$-martingale by (2.2) and (4.2) and therefore

$$(M^n)^2-\sum_{k\geq 1}\int_0^{\cdot}\big\langle g^n_k(u^n),\varphi\big\rangle^2\,\mathrm{d}r,\qquad M^n\beta_k-\int_0^{\cdot}\big\langle g^n_k(u^n),\varphi\big\rangle\,\mathrm{d}r$$

are $(\mathscr F_t)$-martingales. Besides, it follows from the equality of laws that

$$\begin{aligned}&\tilde{\mathbb E}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\big[\tilde M^n(t)-\tilde M^n(s)\big]\\&\quad=\mathbb E\,\gamma\big(\varrho_s u^n,\varrho_s W\big)\big[M^n(t)-M^n(s)\big]=0,\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\tilde{\mathbb E}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\bigg[(\tilde M^n)^2(t)-(\tilde M^n)^2(s)-\sum_{k\geq 1}\int_s^t\big\langle g^n_k(\tilde u^n),\varphi\big\rangle^2\,\mathrm{d}r\bigg]\\&=\mathbb E\,\gamma\big(\varrho_s u^n,\varrho_s W\big)\bigg[(M^n)^2(t)-(M^n)^2(s)-\sum_{k\geq 1}\int_s^t\big\langle g^n_k(u^n),\varphi\big\rangle^2\,\mathrm{d}r\bigg]=0,\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\tilde{\mathbb E}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\bigg[\tilde M^n(t)\tilde\beta^n_k(t)-\tilde M^n(s)\tilde\beta^n_k(s)-\int_s^t\big\langle g^n_k(\tilde u^n),\varphi\big\rangle\,\mathrm{d}r\bigg]\\&=\mathbb E\,\gamma\big(\varrho_s u^n,\varrho_s W\big)\bigg[M^n(t)\beta_k(t)-M^n(s)\beta_k(s)-\int_s^t\big\langle g^n_k(u^n),\varphi\big\rangle\,\mathrm{d}r\bigg]=0.\end{aligned}\tag{4.5}$$

Moreover, since the coefficients B, $\bar A$, $\sum_{k\geq 1}g_k$ have linear growth, we can pass to the limit in (4.3)-(4.5) due to (4.2) and the Vitali convergence theorem. We obtain

$$\begin{aligned}&\tilde{\mathbb E}\,\gamma\big(\varrho_s\tilde u,\varrho_s\tilde W\big)\big[\tilde M(t)-\tilde M(s)\big]=0,\\&\tilde{\mathbb E}\,\gamma\big(\varrho_s\tilde u,\varrho_s\tilde W\big)\bigg[\tilde M^2(t)-\tilde M^2(s)-\sum_{k\geq 1}\int_s^t\big\langle g_k(\tilde u),\varphi\big\rangle^2\mathrm{d}r\bigg]=0,\\&\tilde{\mathbb E}\,\gamma\big(\varrho_s\tilde u,\varrho_s\tilde W\big)\bigg[\tilde M(t)\tilde\beta_k(t)-\tilde M(s)\tilde\beta_k(s)-\int_s^t\big\langle g_k(\tilde u),\varphi\big\rangle\mathrm{d}r\bigg]=0,\end{aligned}$$

which gives the $(\tilde{\mathscr F}_t)$-martingale property. ∎

Proof of Proposition 4.7. Once the above lemmas established, we infer that

$$\bigg\langle\!\!\bigg\langle\tilde M-\int_0^{\cdot}\big\langle\varPhi(\tilde u)\,\mathrm{d}\tilde W,\varphi\big\rangle\bigg\rangle\!\!\bigg\rangle=0,$$

where $\langle\!\langle\cdot\rangle\!\rangle$ denotes the quadratic variation process. Accordingly,

$$\big\langle\tilde u(t),\varphi\big\rangle=\big\langle u_0,\varphi\big\rangle-\int_0^t\big\langle\operatorname{div}\big(B(\tilde u)\big),\varphi\big\rangle\,\mathrm{d}s+\int_0^t\big\langle\operatorname{div}\big(A(\tilde u)\nabla\tilde u\big),\varphi\big\rangle\,\mathrm{d}s+$$
$$+\int_0^t\big\langle\varPhi(\tilde u)\,\mathrm{d}\tilde W,\varphi\big\rangle,\qquad t\in[0,T],\quad\tilde{\mathbb{P}}\text{-a.s.},$$

and the proof is complete. □

4.3. **Pathwise solutions.** As a consequence of pathwise uniqueness established in Section 3 and existence of a martingale solution that follows from the previous subsection, we conclude from the Gyöngy-Krylov characterization of convergence in probability that the original sequence u^n defined on the initial probability space $(\Omega,\mathscr{F},\mathbb{P})$ converges in probability in the topology of $\mathcal{X}_u$ to a random variable u which is a weak solution to (1.1) provided (H1), (H2) and (H3) are fulfilled. For further details on this method we refer the reader to [11, Section 4.5].

Moreover, it follows from Proposition 4.3 that

$$u\in L^2(\Omega;L^\infty(0,T;L^2(\mathbb{T}^N)))\cap L^2(\Omega;L^2(0,T;H^1(\mathbb{T}^N)))$$

and one can also establish continuity of its trajectories in $L^2(\mathbb{T}^N)$. Towards this end, we observe that the solution to

$$\begin{aligned}\mathrm{d}z&=\Delta z\,\mathrm{d}t+\varPhi(u)\,\mathrm{d}W,\\ z(0)&=u_0,\end{aligned}$$

belongs to $L^2(\Omega;C([0,T];L^2(\mathbb{T}^N)))$. Setting $r=u-z$, we obtain

$$\begin{aligned}\partial_t r&=\Delta r-\operatorname{div}\big(B(u)\big)+\operatorname{div}\big((A(u)-\mathrm{I})\nabla u\big),\\ r(0)&=0,\end{aligned}$$

hence it follows by semigroup arguments that $r\in C([0,T];L^2(\mathbb{T}^N))$ a.s. and therefore

$$u\in L^2(\Omega;C([0,T];L^2(\mathbb{T}^N)))\cap L^2(\Omega;L^2(0,T;H^1(\mathbb{T}^N))).$$

5. Nondegenerate case - polynomial growth of B

In this section, we relax the additional hypothesis upon B and prove existence of a weak solution to (1.1) under the remaining two additional hypotheses of Section 4, i.e. (H1) and (H2).

First, we approximate (1.1) by

$$\begin{aligned}\mathrm{d}u+\operatorname{div}\big(B^R(u)\big)\,\mathrm{d}t&=\operatorname{div}\big(A(u)\nabla u\big)\,\mathrm{d}t+\varPhi(u)\,\mathrm{d}W,\\ u(0)&=u_0,\end{aligned}\tag{5.1}$$

where B^R is a truncation of B. According to the previous section, for all $R\in\mathbb{N}$ there exists a unique weak solution to (5.1) such that, for all $p\in[2,\infty)$,

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^R(t)\|^p_{L^2(\mathbb{T}^N)}+2\tau\,\mathbb{E}\int_0^T\|\nabla u^R\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s\leq C\big(1+\mathbb{E}\|u_0\|^p_{L^2(\mathbb{T}^N)}\big),$$

where the constant C is independent of R and τ. Furthermore, we can also obtain a uniform estimate of the $L^p(\mathbb{T}^N)$-norm that is necessary in order to deal with coefficients having polynomial growth.

Proposition 5.1. *For all $p\in[2,\infty)$, the solution to (5.1) satisfies the following estimate*

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^R(t)\|^p_{L^p(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^p_{L^p(\mathbb{T}^N)}\big), \tag{5.2}$$

where the constant C does not depend on R and τ.

Proof. As the generalized Itô formula (A.2) cannot be applied directly to $\varphi(\xi)=|\xi|^p$, $p\in[2,\infty)$, and $\psi(x)=1$, we follow the approach of [7] and introduce functions $\varphi_n\in C^2(\mathbb{R})$ that approximate φ and have quadratic growth at infinity as required by Proposition A.1. Namely, let

$$\varphi_n(\xi)=\begin{cases}|\xi|^p, & |\xi|\leq n,\\ n^{p-2}\Big[\frac{p(p-1)}{2}\xi^2-p(p-2)n|\xi|+\frac{(p-1)(p-2)}{2}n^2\Big], & |\xi|>n.\end{cases}$$

It is now easy to see that

$$\begin{aligned}|\xi\varphi_n'(\xi)|&\leq p\,\varphi_n(\xi),\\ |\varphi_n'(\xi)|&\leq p\,(1+\varphi_n(\xi)),\\ |\varphi_n'(\xi)|&\leq|\xi|\varphi_n''(\xi),\\ \xi^2\varphi_n''(\xi)&\leq p(p-1)\varphi_n(\xi),\\ \varphi''(\xi)&\leq p(p-1)(1+\varphi_n(\xi))\end{aligned} \tag{5.3}$$

hold true for all $\xi\in\mathbb{R}$, $n\in\mathbb{N}$, $p\in[2,\infty)$. Then by Proposition A.1

$$\begin{aligned}\int_{\mathbb{T}^N}\varphi_n(u^R(t))\,\mathrm{d}x&=\int_{\mathbb{T}^N}\varphi_n(u_0)\,\mathrm{d}x-\int_0^t\int_{\mathbb{T}^N}\varphi_n'(u^R)\,\mathrm{div}\,\big(B^R(u^R)\big)\,\mathrm{d}x\,\mathrm{d}s\\&\quad+\int_0^t\int_{\mathbb{T}^N}\varphi_n'(u^R)\,\mathrm{div}\,\big(A(u^R)\nabla u^R\big)\,\mathrm{d}x\,\mathrm{d}s\\&\quad+\sum_{k\geq1}\int_0^t\int_{\mathbb{T}^N}\varphi_n'(u^R)g_k(u^R)\,\mathrm{d}x\,\mathrm{d}\beta_k(s)\\&\quad+\frac12\int_0^t\int_{\mathbb{T}^N}\varphi_n''(u^R)G^2(u^R)\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

Setting $H(\xi)=\int_0^\xi\varphi_n''(\zeta)B^R(\zeta)\,\mathrm{d}\zeta$ it can be seen that the second term on the right hand side vanishes due to the boundary conditions. The third term is nonpositive as the matrix A is positive definite

$$\int_0^t\int_{\mathbb{T}^N}\varphi_n'(u^R)\,\mathrm{div}\,\big(A(u^R)\nabla u^R\big)\,\mathrm{d}x\,\mathrm{d}s=-\int_0^t\int_{\mathbb{T}^N}\varphi_n''(u^R)|\sigma(u^R)\nabla u^R|^2\,\mathrm{d}x\,\mathrm{d}s.$$

The last term is estimated by (5.3)

$$\begin{aligned}\frac12\int_0^t\int_{\mathbb{T}^N}\varphi_n''(u^R)G^2(u^R)\,\mathrm{d}x\,\mathrm{d}s&\leq\frac C2\int_0^t\int_{\mathbb{T}^N}\varphi_n''(u^R)(1+|u^R|^2)\,\mathrm{d}x\,\mathrm{d}s\\&\leq\frac{Cp(p-1)}{2}\int_0^t\int_{\mathbb{T}^N}(1+\varphi_n(u^R))\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

and therefore by Gronwall's lemma we obtain

$$\mathbb{E}\int_{\mathbb{T}^N}\varphi_n(u^R(t))\,\mathrm{d}x\leq C\bigg(1+\mathbb{E}\int_{\mathbb{T}^N}\varphi(u_0)\,\mathrm{d}x\bigg). \tag{5.4}$$

As a consequence, a uniform estimate of $\mathbb{E}\sup_{0\leq t\leq T}\|u^R(t)\|^p_{L^p(\mathbb{T}^N)}$ follows. Indeed, we proceed similarly as before only for the stochastic term we apply the Burkholder-Davis-Gundy and the Schwartz inequality, (5.3) and the weighted Young inequality

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\bigg|\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^N}\varphi_n'(u^R)g_k(u^R)\,\mathrm{d}x\,\mathrm{d}\beta_k(s)\bigg|\\
&\leq C\,\mathbb{E}\bigg(\int_0^T\sum_{k\geq 1}\bigg(\int_{\mathbb{T}^N}|\varphi_n'(u^R)|\,|g_k(u^R)|\,\mathrm{d}x\bigg)^2\mathrm{d}s\bigg)^{\frac{1}{2}}\\
&\leq C\,\mathbb{E}\bigg(\int_0^T\big\|\,|\varphi_n'(u^R)|^{\frac12}|u^R|^{\frac12}\big\|^2_{L^2(\mathbb{T}^N)}\sum_{k\geq 1}\big\|\,|\varphi_n'(u^R)|^{\frac12}|u^R|^{-\frac12}|g_k(u^R)|\big\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s\bigg)^{\frac12}\\
&\leq C\,\mathbb{E}\bigg(\int_0^T\int_{\mathbb{T}^N}\varphi_n(u^R)\,\mathrm{d}x\bigg(1+\int_{\mathbb{T}^N}\varphi_n(u^R)\,\mathrm{d}x\bigg)\,\mathrm{d}s\bigg)^{\frac12}\\
&\leq C\,\mathbb{E}\bigg(\sup_{0\leq t\leq T}\int_{\mathbb{T}^N}\varphi_n(u^R)\,\mathrm{d}x\bigg)^{\frac12}\bigg(1+\int_0^T\int_{\mathbb{T}^N}\varphi_n(u^R)\,\mathrm{d}x\,\mathrm{d}s\bigg)^{\frac12}\\
&\leq\frac12\,\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathbb{T}^N}\varphi_n(u^R)\,\mathrm{d}x+C\bigg(1+\int_0^T\mathbb{E}\int_{\mathbb{T}^N}\varphi_n(u^R)\,\mathrm{d}x\,\mathrm{d}s\bigg)
\end{aligned}$$

which together with (5.4) and Fatou's lemma yields (5.2). ∎

Having Proposition 5.1 in hand, the proof of Propositions 4.4 as well as all the proofs in Subsections 4.1, 4.2, 4.3 can be repeated with only minor modifications and, consequently, the following result deduced.

Theorem 5.2. *Under the additional hypotheses* (H1), (H2)*, there exists a unique weak solution to* (1.1) *such that, for all* $p\in[2,\infty)$*,*

(5.5)
$$\mathbb{E}\sup_{0\leq t\leq T}\|u(t)\|^p_{L^p(\mathbb{T}^N)}+p(p-1)\mathbb{E}\int_0^T\int_{\mathbb{T}^N}|u|^{p-2}|\sigma(u)\nabla u|^2\leq C\big(1+\mathbb{E}\|u_0\|^p_{L^p(\mathbb{T}^N)}\big)$$

and the constant C is independent of τ.

Sketch of the proof. Following the approach of the previous section, we obtain

(i) For all $\lambda\in(0,1/2)$ there exists $C>0$ such that for all $R\in\mathbb{N}$
$$\mathbb{E}\|u^R\|_{C^\lambda([0,T];H^{-1}(\mathbb{T}^N))}\leq C.$$
(ii) The laws of $\{u^R;\,R\in\mathbb{N}\}$ form a tight sequence on
$$L^2(0,T;L^2(\mathbb{T}^N))\cap C([0,T];H^{-2}(\mathbb{T}^N)).$$
(iii) There exists $\big((\tilde\Omega,\tilde{\mathscr{F}},(\tilde{\mathscr{F}}_t),\tilde{\mathbb{P}}),\tilde W,\tilde u\big)$ that is a weak martingale solution to (1.1).
(iv) There exists $u\in L^2(\Omega;C([0,T];L^2(\mathbb{T}^N)))\cap L^p(\Omega;L^\infty(0,T;L^p(\mathbb{T}^N)))\cap L^2(\Omega;L^2(0,T;H^1(\mathbb{T}^N)))$ that is a weak solution to (1.1).
(v) By the approach of Proposition 5.1 we obtain (5.5).

∎

6. Degenerate case

As the next step in the existence proof of Theorem 2.7, we can finally proceed to the degenerate case. Throughout this section, we only assume the additional hypothesis upon the initial condition, i.e. (H1).

Consider the following nondegenerate approximations of (1.1)

$$\begin{aligned} \mathrm{d}u + \operatorname{div}\big(B(u)\big)\,\mathrm{d}t &= \operatorname{div}\big(A(u)\nabla u\big)\,\mathrm{d}t + \tau\Delta u\,\mathrm{d}t + \Phi(u)\,\mathrm{d}W, \\ u(0) &= u_0. \end{aligned} \tag{6.1}$$

According to the results of Section 5, we have for any fixed $\tau > 0$ the existence of $u^\tau \in L^2(\Omega; C([0,T]; L^2(\mathbb{T}^N))) \cap L^2(\Omega; L^2(0,T; H^1(\mathbb{T}^N)))$ which is a weak solution to (6.1) and satisfies (cf. (5.5))

(6.2)

$$\begin{aligned} \mathbb{E}\sup_{0\le t\le T}\|u^\tau(t)\|^p_{L^p(\mathbb{T}^N)} + p(p-1)\,\mathbb{E}\int_0^T\int_{\mathbb{T}^N}|u^\tau|^{p-2}\big(|\sigma(u^\tau)\nabla u^\tau|^2 + \tau|\nabla u^\tau|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\ \le C\big(1 + \mathbb{E}\|u_0\|^p_{L^p(\mathbb{T}^N)}\big) \end{aligned}$$

with a constant that does not depend on τ. As the next step, we employ the technique of Subsection 2.3 to derive the kinetic formulation that is satisfied by $f^\tau = \mathbf{1}_{u^\tau > \xi}$ in the sense of $\mathcal{D}'(\mathbb{T}^N \times \mathbb{R})$. It reads as follows

$$\begin{aligned} \mathrm{d}f^\tau + b\cdot\nabla f^\tau\mathrm{d}t - A : \mathrm{D}^2 f^\tau\mathrm{d}t - \tau\Delta f^\tau\mathrm{d}t \\ = \delta_{u^\tau=\xi}\,\Phi\,\mathrm{d}W + \partial_\xi\left(n_1^\tau + n_2^\tau - \frac{1}{2}G^2\delta_{u^\tau=\xi}\right)\mathrm{d}t, \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} \mathrm{d}n_1^\tau(t,x,\xi) &= |\sigma\nabla u^\tau|^2\,\mathrm{d}\delta_{u^\tau=\xi}\,\mathrm{d}x\,\mathrm{d}t, \\ \mathrm{d}n_2^\tau(t,x,\xi) &= \tau|\nabla u^\tau|^2\,\mathrm{d}\delta_{u^\tau=\xi}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

6.1. **Uniform estimates.** Next, we prove a uniform $W^{\lambda,1}(\mathbb{T}^N)$-regularity of the approximate solutions u^τ. Towards this end, we make use of two seminorms describing the $W^{\lambda,1}$-regularity of a function $u \in L^1(\mathbb{T}^N)$ (see[6, Subsection 3.4] for further details). Let $\lambda \in (0,1)$ and define

$$p^\lambda(u) = \int_{\mathbb{T}^N}\int_{\mathbb{T}^N}\frac{|u(x)-u(y)|}{|x-y|^{N+\lambda}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$p^\lambda_\varrho(u) = \sup_{0<\varepsilon<2D_N}\frac{1}{\varepsilon^\lambda}\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}|u(x)-u(y)|\varrho_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

where (ϱ_ε) is the approximation to the identity on $\mathbb{T}^N$ that is radial, i.e. $\varrho_\varepsilon(x) = 1/\varepsilon^N\varrho(|x|/\varepsilon)$; and by D_N we denote the diameter of $[0,1]^N$. The fractional Sobolev space $W^{\lambda,1}(\mathbb{T}^N)$ is defined as a subspace of $L^1(\mathbb{T}^N)$ with finite norm

$$\|u\|_{W^{\lambda,1}(\mathbb{T}^N)} = \|u\|_{L^1(\mathbb{T}^N)} + p^\lambda(u).$$

According to [6], the following relations holds true between these seminorms. Let $s \in (0,\lambda)$, there exists a constant $C = C_{\lambda,\varrho,N}$ such that for all $u \in L^1(\mathbb{T}^N)$

$$p^\lambda_\varrho(u) \le C p^\lambda(u), \qquad p^s(u) \le \frac{C}{\lambda - s}\,p^\lambda_\varrho(u).$$

Proposition 6.1 ($W^{\varsigma,1}$-regularity). *Set $\varsigma=\min\big\{\frac{2\gamma-1}{\gamma+1},\frac{2\alpha}{\alpha+1}\big\}$, where γ was defined in (2.1) and α in (2.4). Then for all $s\in(0,\varsigma)$ there exists a constant $C_s>0$ such that for all $t\in[0,T]$ and all $\tau\in(0,1)$*

$$\mathbb{E}\,p^s\big(u^\tau(t)\big)\leq C_s\big(1+\mathbb{E}\,p^\varsigma(u_0)\big).$$

In particular, there exists a constant $C_s>0$ such that for all $t\in[0,T]$

$$\mathbb{E}\|u^\tau(t)\|_{W^{s,1}(\mathbb{T}^N)}\leq C_s\big(1+\mathbb{E}\|u_0\|_{W^{\varsigma,1}(\mathbb{T}^N)}\big).$$

Proof. Proof of this statement is based on Proposition 3.2. We have

$$\begin{aligned}
&\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\varepsilon(x-y)f^\tau(x,t,\xi)\bar{f}^\tau(y,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f^\tau(x,t,\xi)\bar{f}^\tau(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y+\delta\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f_0(x,\xi)\bar{f}_0(y,\zeta)\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}x\mathrm{d}y+\delta+\mathrm{I}+\mathrm{J}+\mathrm{J}^\tau+\mathrm{K}\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\varepsilon(x-y)f_0(x,\xi)\bar{f}_0(y,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y+2\delta+\mathrm{I}+\mathrm{J}+\mathrm{J}^\tau+\mathrm{K},
\end{aligned}$$

where I, J, K are defined similarly to Proposition 3.2, J^τ corresponds to the second order term $\tau\Delta u^\tau$:

$$\begin{aligned}
\mathrm{J}^\tau&=2\tau\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f^\tau\bar{f}^\tau\Delta_x\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu^\tau_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}n^\tau_2(y,s,\zeta)\\
&\qquad-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu^\tau_{y,s}(\zeta)\,\mathrm{d}y\,\mathrm{d}n^\tau_2(x,s,\xi)\\
&=-\tau\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta\big(u^\tau(x)-u^\tau(y)\big)\big|\nabla_x u^\tau-\nabla_y u^\tau\big|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\leq 0.
\end{aligned}$$

and the error term δ was obtained as follows

(6.4)

$$\begin{aligned}
&\bigg|\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\varepsilon(x-y)f^\tau(x,t,\xi)\bar{f}^\tau(y,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad-\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f^\tau(x,t,\xi)\bar{f}^\tau(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\bigg|\\
&\quad=\bigg|\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\int_{\mathbb{R}}\mathbf{1}_{u^\tau(x)>\xi}\int_{\mathbb{R}}\psi_\delta(\xi-\zeta)\Big[\mathbf{1}_{u^\tau(y)\leq\xi}-\mathbf{1}_{u^\tau(y)\leq\zeta}\Big]\mathrm{d}\zeta\mathrm{d}\xi\mathrm{d}x\mathrm{d}y\bigg|\\
&\quad\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\varepsilon(x-y)\,\mathbf{1}_{u^\tau(x)>\xi}\int_{\xi-\delta}^{\xi}\psi_\delta(\xi-\zeta)\,\mathbf{1}_{\zeta<u^\tau(y)\leq\xi}\,\mathrm{d}\zeta\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\varepsilon(x-y)\,\mathbf{1}_{u^\tau(x)>\xi}\int_{\xi}^{\xi+\delta}\psi_\delta(\xi-\zeta)\,\mathbf{1}_{\xi<u^\tau(y)\leq\zeta}\,\mathrm{d}\zeta\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq\frac12\,\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\int_{u^\tau(y)}^{\min\{u^\tau(x),u^\tau(y)+\delta\}}\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\frac12\,\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\int_{u^\tau(y)-\delta}^{\min\{u^\tau(x),u^\tau(y)\}}\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\leq\delta.
\end{aligned}$$

Hence by the proof of Theorem 3.3

$$\begin{aligned}
&\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\big|u^\tau(x,t)-u^\tau(y,t)\big|\,\mathrm{d}x\,\mathrm{d}y\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\big|u_0(x)-u_0(y)\big|\,\mathrm{d}x\,\mathrm{d}y+C_T\big(\delta+\delta\varepsilon^{-1}+\delta^2\varepsilon^{-2}+\delta^{-1}\varepsilon^2+\delta^\alpha\big).
\end{aligned}$$

By optimization in δ, i.e. setting $\delta=\varepsilon^\beta$, we obtain

$$\sup_{0<\tau<2D_N}\frac{C_T\big(\delta+\delta\varepsilon^{-1}+\delta^{2\gamma}\varepsilon^{-2}+\delta^{-1}\varepsilon^2+\delta^\alpha\big)}{\varepsilon^\varsigma}\leq C_T,$$

where the maximal choice of the parameter ς is $\min\big\{\frac{2\gamma-1}{\gamma+1},\frac{2\alpha}{\alpha+1}\big\}$. As a consequence,

$$\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\big|u^\tau(x,t)-u^\tau(y,t)\big|\,\mathrm{d}x\,\mathrm{d}y\leq C_T\varepsilon^\varsigma\,\big(1+\mathbb{E}\,p^\varsigma(u_0)\big).\tag{6.5}$$

Finally, multiplying the above by ε^{-1-s}, $s\in(0,\varsigma)$, and integrating with respect to $\varepsilon\in(0,2D_N)$ gives the claim. □

6.2. **Strong convergence.** According to (6.2), the set $\{u^\tau;\,\tau\in(0,1)\}$ is bounded in $L^p(\Omega;L^p(0,T;L^p(\mathbb{T}^N)))$ and therefore possesses a weakly convergent subsequence. The aim of this subsection is to show that even strong convergence holds true. Towards this end, we make use of the ideas developed in Section 3.

Theorem 6.2. *There exists $u\in L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N))$ such that*

$$u^\tau\longrightarrow u\qquad in\qquad L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N)).$$

Proof. By similar techniques as in the proofs of Proposition 3.2 and Theorem 3.3, we obtain for any two approximate solutions u^τ, u^σ

(6.6)

$$\begin{aligned}\mathbb{E}\int_{\mathbb{T}^N}\big(u^\tau(t)-u^\sigma(t)\big)^+\,\mathrm{d}x&=\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f^\tau(x,t,\xi)\bar f^\sigma(x,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\&=\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f^\tau(x,t,\xi)\bar f^\sigma(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y+\eta_t(\tau,\sigma,\varepsilon,\delta).\end{aligned}$$

(Here ε and δ are chosen arbitrarily and their value will be fixed later.) The idea now is to show that the error term $\eta_t(\tau,\sigma,\varepsilon,\delta)$ is in fact independent of τ, σ. Indeed, we have

$$\begin{aligned}\eta_t(\tau,\sigma,\varepsilon,\delta)&=\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f^\tau(x,t,\xi)\bar f^\sigma(x,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\&\quad-\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f^\tau(x,t,\xi)\bar f^\sigma(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\\&=\bigg(\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f^\tau(x,t,\xi)\bar f^\sigma(x,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\&\quad-\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\varepsilon(x-y)f^\tau(x,t,\xi)\bar f^\sigma(y,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\bigg)\\&\quad+\bigg(\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\varepsilon(x-y)f^\tau(x,t,\xi)\bar f^\sigma(y,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\&\quad-\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)f^\tau(x,t,\xi)\bar f^\sigma(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\bigg)\\&=\mathrm{H}_1+\mathrm{H}_2,\end{aligned}$$

where

$$\begin{aligned}|\mathrm{H}_1|&=\bigg|\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\int_{\mathbb{R}}\mathbf{1}_{u^\tau(x)>\xi}\Big[\mathbf{1}_{u^\sigma(x)\le\xi}-\mathbf{1}_{u^\sigma(y)\le\xi}\Big]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\bigg|\\&=\bigg|\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\big(u^\sigma(y)-u^\sigma(x)\big)\,\mathrm{d}x\,\mathrm{d}y\bigg|\le C\varepsilon^\varsigma\end{aligned}$$

due to (6.5) and $|H_2|\le\delta$ due to (6.4). Therefore the claim follows, that is $|\eta_t(\tau,\sigma,\varepsilon,\delta)|\le C\varepsilon^\varsigma+\delta$. Heading back to (6.6) and using the same calculations as in Proposition 3.2, we deduce

$$\mathbb{E}\int_{\mathbb{T}^N}\big(u^\tau(t)-u^\sigma(t)\big)^+\,\mathrm{d}x\le 2C\varepsilon^\varsigma+2\delta+\mathrm{I}+\mathrm{J}+\mathrm{J}^\#+\mathrm{K}.$$

The terms I, J, K are defined and can be dealt with exactly as in Proposition 3.2 and Theorem 3.3. The term $\mathrm{J}^\#$ is defined as

$$\begin{aligned}\mathrm{J}^\#&=(\tau+\sigma)\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f^\tau\bar f^\sigma\Delta_x\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&\quad-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu^\tau_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}n_2^\sigma(y,s,\zeta)\\&\quad-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\varepsilon(x-y)\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu^\sigma_{y,s}(\zeta)\,\mathrm{d}y\,\mathrm{d}n_2^\tau(x,s,\xi)\end{aligned}$$

so

$$\begin{aligned}
\mathrm{J}^{\#} &= (\tau+\sigma)\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u^\tau-u^\sigma)\nabla_x u^\tau\cdot\nabla_y u^\sigma\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad-\tau\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u^\tau-u^\sigma)|\nabla_x u^\tau|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad-\sigma\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u^\tau-u^\sigma)|\nabla_y u^\sigma|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&=-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u^\tau-u^\sigma)\big|\sqrt{\tau}\,\nabla_x u^\tau-\sqrt{\sigma}\,\nabla_y u^\sigma\big|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad+(\sqrt{\tau}-\sqrt{\sigma}\,)^2\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\varepsilon(x-y)\psi_\delta(u^\tau-u^\sigma)\nabla_x u^\tau\cdot\nabla_y u^\sigma\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&=\mathrm{J}_1^{\#}+\mathrm{J}_2^{\#}.
\end{aligned}$$

The first term on the right hand side is nonpositive and can be thus forgotten, for the second one we have

$$\big|\mathrm{J}_2^{\#}\big|\leq(\sqrt{\tau}-\sqrt{\sigma}\,)^2\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f^\tau\bar{f}^\sigma\psi_\delta(\xi-\zeta)\big|\Delta_x\varrho_\varepsilon(x-y)\big|\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

and proceeding similarly as in the case of I we get

$$\begin{aligned}
\big|\mathrm{J}_2^{\#}\big|&\leq(\sqrt{\tau}-\sqrt{\sigma}\,)^2\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}|\xi-\zeta+\delta|\,\mathrm{d}\nu^\tau_{x,s}(\xi)\,\mathrm{d}\nu^\sigma_{y,s}(\zeta)\big|\Delta_x\varrho_\varepsilon(x-y)\big|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\leq C(\sqrt{\tau}-\sqrt{\sigma}\,)^2\varepsilon^{-2},
\end{aligned}$$

where the last inequality follows from (6.2). Consequently, we see that

$$\begin{aligned}
\mathbb{E}\int_0^T\int_{\mathbb{T}^N}\big(u^\tau(t)-u^\sigma(t)\big)^+\,\mathrm{d}x\,\mathrm{d}t&\leq C\big(\varepsilon^\varsigma+\delta+\delta\varepsilon^{-1}+\delta^{2\gamma}\varepsilon^{-2}+\delta^{-1}\varepsilon^2+\delta^\alpha\big)\\
&\qquad+C(\tau+\sigma)\,\varepsilon^{-2}
\end{aligned}$$

and therefore, given $\vartheta>0$ one can fix ε and δ small enough so that the first term on the right hand side is estimated by $\vartheta/2$ and then find $\iota>0$ such that also the second term is estimated by $\vartheta/2$ for any $\tau,\sigma<\iota$. Thus, we have shown that the set of approximate solutions $\{u^\tau\}$ is Cauchy in $L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N))$, as $\tau\to0$. □

Corollary 6.3. *For all $p\in[1,\infty)$,*

$$u^\tau\longrightarrow u\qquad in\qquad L^p(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^p(\mathbb{T}^N))$$

and the following estimate holds true

$$\mathbb{E}\operatorname*{ess\,sup}_{0\leq t\leq T}\|u(t)\|^p_{L^p(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^p_{L^p(\mathbb{T}^N)}\big).$$

Proof. The claim follows directly from Theorem 6.2 and the estimate (6.2). □

Theorem 6.4. *The process u constructed in Theorem 6.2 is the unique kinetic solution to (1.1) under the additional hypothesis* (H1).

Proof. Let $t \in [0,T]$. According to Corollary 6.3, there exists a set $\Sigma \subset \Omega \times [0,T] \times \mathbb{T}^N$ of full measure and a subsequence still denoted by $\{u^n;\, n \in \mathbb{N}\}$ such that $u^n(\omega,t,x) \to u(\omega,t,x)$ for all $(\omega,t,x) \in \Sigma$. We infer that

$$\mathbf{1}_{u^n(\omega,t,x)>\xi} \longrightarrow \mathbf{1}_{u(\omega,t,x)>\xi} \tag{6.7}$$

whenever

$$\Big(\mathbb{P}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big\{(\omega,x)\in\Sigma;\, u(\omega,t,x)=\xi\big\}=0,$$

where by $\mathcal{L}_{\mathbb{T}^N}$ and $\mathcal{L}_{[0,T]}$ we denoted the Lebesque measure on $\mathbb{T}^N$ and $[0,T]$, respectively. However, the set

$$D=\Big\{\xi\in\mathbb{R};\, \Big(\mathbb{P}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(u=\xi\big)>0\Big\}$$

is at most countable since we deal with finite measures. To obtain a contradiction, suppose that D is uncountable and denote

$$D_k=\Big\{\xi\in\mathbb{R};\, \Big(\mathbb{P}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(u=\xi\big)>\frac{1}{k}\Big\},\qquad k\in\mathbb{N}.$$

Then $D=\cup_{k\in\mathbb{N}}D_k$ is a countable union so there exists $k_0\in\mathbb{N}$ such that D_{k_0} is uncountable. Hence

$$\begin{aligned}\Big(\mathbb{P}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(u\in D\big)&\geq\Big(\mathbb{P}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(u\in D_{k_0}\big)\\&=\sum_{\xi\in D_{k_0}}\Big(\mathbb{P}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(u=\xi\big)>\sum_{\xi\in D_{k_0}}\frac{1}{k_0}=\infty\end{aligned}$$

and the desired contradiction follows. We conclude that the convergence in (6.7) holds true for a.e. (ω,t,x,ξ) and obtain by the dominated convergence theorem

$$f^n\overset{w^*}{\longrightarrow}f\qquad\text{in}\qquad L^\infty(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R}).$$

As a consequence, we can pass to the limit in all the terms on the left hand side of the weak form of (6.3) and obtain the left hand side of (2.6). Convergence of the stochastic integral as well as the last term in the weak form (6.3) to the corresponding terms in (2.6) can be verified easily using Corollary 6.3 and the energy estimate (6.2).

In order to obtain the convergence of the remaining term $\partial_\xi m^\tau=\partial_\xi n_1^\tau+\partial_\xi n_2^\tau$ to a kinetic measure, we observe that due to the computations used in the proof of (6.2), it holds

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^N}|\sigma(u^\tau)\nabla u^\tau|^2\mathrm{d}x\,\mathrm{d}t+\tau|\nabla u^\tau|^2\mathrm{d}x\,\mathrm{d}t&\leq C\|u_0\|^2_{L^2(\mathbb{T}^N)}\\&+C\sum_{k\geq1}\int_0^T\int_{\mathbb{T}^N}u^\tau g_k(u^\tau)\,\mathrm{d}x\,\mathrm{d}\beta_k(t)+C\int_0^T\int_{\mathbb{T}^N}G^2(u^\tau)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Taking square and expectation and finally by the Itô isometry, we deduce

$$\mathbb{E}\big|m^\tau([0,T]\times\mathbb{T}^N\times\mathbb{R})\big|^2=\mathbb{E}\bigg|\int_0^T\int_{\mathbb{T}^N}|\sigma(u^\tau)\nabla u^\tau|^2\mathrm{d}x\,\mathrm{d}t+\tau|\nabla u^\tau|^2\mathrm{d}x\,\mathrm{d}t\bigg|^2\leq C.$$

Hence the set $\{m^\tau;\,\tau\in(0,1)\}$ is bounded in $L^2_w(\Omega;\mathcal{M}_b([0,T]\times\mathbb{T}^N\times\mathbb{R}))$ and, according to the Banach-Alaoglu theorem, it possesses a weak* convergent subsequence, denoted by $\{m^n;\,n\in\mathbb{N}\}$. Now, it only remains to show that its weak* limit m is actually a kinetic measure. The first point of Definition 2.1 is straightforward as it corresponds to the weak*-measurability of m. The second one giving the behavior for large ξ is a consequence of the uniform estimate (6.2). Indeed, let (χ_δ) be a truncation on $\mathbb{R}$, then it holds, for $p\in[2,\infty)$,

$$\begin{aligned}\mathbb{E}\int_{[0,T]\times\mathbb{T}^N\times\mathbb{R}}|\xi|^{p-2}\,\mathrm{d}m(t,x,\xi)&\leq\liminf_{\delta\to0}\mathbb{E}\int_{[0,T]\times\mathbb{T}^N\times\mathbb{R}}|\xi|^{p-2}\chi_\delta(\xi)\,\mathrm{d}m(t,x,\xi)\\&=\liminf_{\delta\to0}\lim_{n\to\infty}\mathbb{E}\int_{[0,T]\times\mathbb{T}^N\times\mathbb{R}}|\xi|^{p-2}\chi_\delta(\xi)\,\mathrm{d}m^n(t,x,\xi)\leq C,\end{aligned}$$

where the last inequality follows from (6.2). Accordingly, m vanishes for large ξ. In order to verify the remaining requirement of Definition 2.1, let us define

$$x^n(t)=\int_{[0,t]\times\mathbb{T}^N\times\mathbb{R}}\psi(x,\xi)\,\mathrm{d}m^n(s,x,\xi)$$

and take the limit as $n\to\infty$. These processes are predictable due to the definition of measures m^n. Let $\alpha\in L^2(\Omega)$, $\gamma\in L^2(0,T)$, then, by the Fubini theorem,

$$\mathbb{E}\bigg(\alpha\int_0^T\gamma(t)x^n(t)\,\mathrm{d}t\bigg)=\mathbb{E}\bigg(\alpha\int_{[0,T]\times\mathbb{T}^N\times\mathbb{R}}\psi(x,\xi)\Gamma(s)\,\mathrm{d}m^n(s,x,\xi)\bigg)$$

where $\Gamma(s)=\int_s^T\gamma(t)\,\mathrm{d}t$. Hence, since Γ is continuous, we obtain by the weak convergence of m^n to m

$$\mathbb{E}\bigg(\alpha\int_0^T\gamma(t)x^n(t)\,\mathrm{d}t\bigg)\longrightarrow\mathbb{E}\bigg(\alpha\int_0^T\gamma(t)x(t)\,\mathrm{d}t\bigg),$$

where

$$x(t)=\int_{[0,t]\times\mathbb{T}^N\times\mathbb{R}}\psi(x,\xi)\,\mathrm{d}m(s,x,\xi).$$

Consequently, x^n converges to x weakly in $L^2(\Omega\times[0,T])$ and, in particular, since the space of predictable L^2-integrable functions is weakly closed, the claim follows.

Finally, by the same approach as above, we deduce that there exist kinetic measures o_1, o_2 such that

$$n_1^n\xrightarrow{w^*}o_1,\quad n_2^n\xrightarrow{w^*}o_2\qquad\text{in}\qquad L^2_w(\Omega;\mathcal{M}_b([0,T]\times\mathbb{T}^N\times\mathbb{R})).$$

Then from (6.2) we obtain

$$\mathbb{E}\int_0^T\int_{\mathbb{T}^N}\bigg|\operatorname{div}\int_0^{u^n}\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\mathrm{d}x\,\mathrm{d}t\leq C$$

hence application of the Banach-Alaoglu theorem yields that, up to subsequence, $\operatorname{div}\int_0^{u^n}\sigma(\zeta)\,\mathrm{d}\zeta$ converges weakly in $L^2(\Omega\times[0,T]\times\mathbb{T}^N)$. On the other hand, from the strong convergence given by Corollary 6.3 and the fact that $\sigma\in C_b(\mathbb{R})$, we conclude using integration by parts, for all $\psi\in C^1([0,T]\times\mathbb{T}^N)$, $\mathbb{P}$-a.s.,

$$\int_0^T\int_{\mathbb{T}^N}\bigg(\operatorname{div}\int_0^{u^n}\sigma(\zeta)\,\mathrm{d}\zeta\bigg)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\longrightarrow\int_0^T\int_{\mathbb{T}^N}\bigg(\operatorname{div}\int_0^{u}\sigma(\zeta)\,\mathrm{d}\zeta\bigg)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

and therefore

$$\operatorname{div}\int_0^{u^n}\sigma(\zeta)\,\mathrm{d}\zeta\xrightarrow{w}\operatorname{div}\int_0^{u}\sigma(\zeta)\,\mathrm{d}\zeta\qquad\text{in}\qquad L^2([0,T]\times\mathbb{T}^N),\quad\mathbb{P}\text{-a.s..}\tag{6.8}$$

Since any norm is weakly sequentially lower semicontinuous, it follows for all $\varphi \in C_0([0,T]\times\mathbb{T}^N\times\mathbb{R})$ and fixed $\xi\in\mathbb{R}$, $\mathbb{P}$-a.s.,

$$\int_0^T\int_{\mathbb{T}^N}\bigg|\operatorname{div}\int_0^u\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\varphi^2(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{n\to\infty}\int_0^T\int_{\mathbb{T}^N}\bigg|\operatorname{div}\int_0^{u^n}\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\varphi^2(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}t$$

and by the Fatou lemma

$$\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}\bigg|\operatorname{div}\int_0^u\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\varphi^2(t,x,\xi)\,\mathrm{d}\delta_{u=\xi}\,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{n\to\infty}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}\bigg|\operatorname{div}\int_0^{u^n}\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\varphi^2(t,x,\xi)\,\mathrm{d}\delta_{u^n=\xi}\,\mathrm{d}x\,\mathrm{d}t,\qquad\mathbb{P}\text{-a.s..}$$

In other words, this yields that $n_1\leq o_1$, $\mathbb{P}$-a.s., hence $n_2=o_2+(o_1-n_1)$ is a.s. a nonnegative measure.

Concerning the chain rule formula (2.5), we observe that it holds true for all u^n due to their regularity, i.e. for any $\phi\in C_b(\mathbb{R})$

$$\operatorname{div}\int_0^{u^n}\phi(\zeta)\sigma(\zeta)\,\mathrm{d}\zeta=\phi(u^n)\operatorname{div}\int_0^{u^n}\sigma(\zeta)\,\mathrm{d}\zeta\quad\text{in}\quad\mathcal{D}'(\mathbb{T}^N),\ \text{a.e. }(\omega,t).\tag{6.9}$$

Furthermore, as we can easily obtain (6.8) with the integrant σ replaced by $\phi\sigma$, we can pass to the limit on the left hand side and, making use of the strong-weak convergence, also on the right hand side of (6.9). The proof is complete. □

7. General initial data

In this final section we complete the proof of Theorem 2.7. In particular, we show existence of a kinetic solution to (1.1) for a general initial data $u_0\in L^p(\Omega;L^p(\mathbb{T}^N))$, $\forall p\in[1,\infty)$. It is a straightforward consequence of the previous section. Indeed, let us approximate the initial condition by a sequence $\{u_0^\varepsilon\}\subset L^p(\Omega;C^\infty(\mathbb{T}^N))$, $\forall p\in[1,\infty)$, such that $u_0^\varepsilon\to u_0$ in $L^1(\Omega;L^1(\mathbb{T}^N))$ and

$$\|u_0^\varepsilon\|_{L^p(\Omega;L^p(\mathbb{T}^N))}\leq\|u_0\|_{L^p(\Omega;L^p(\mathbb{T}^N))},\qquad\varepsilon\in(0,1),\ p\in[1,\infty).\tag{7.1}$$

According to Theorem 6.4, for each $\varepsilon\in(0,1)$, there exists a unique kinetic solution u^ε to (1.1) with initial condition u_0^ε. Besides, by the comparison principle (3.1),

$$\mathbb{E}\int_0^T\|u^{\varepsilon_1}(t)-u^{\varepsilon_2}(t)\|_{L^1(\mathbb{T}^N)}\,\mathrm{d}t\leq T\,\mathbb{E}\|u_0^{\varepsilon_1}-u_0^{\varepsilon_2}\|_{L^1(\mathbb{T}^N)},\qquad\varepsilon_1,\varepsilon_2\in(0,1),$$

hence $\{u^\varepsilon;\,\varepsilon\in(0,1)\}$ is a Cauchy sequence in $L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N))$. Consequently, there exists $u\in L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N))$ such that

$$u^\varepsilon\longrightarrow u\qquad\text{in}\qquad L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N)).$$

By (7.1), we have the uniform energy estimate, $p\in[1,\infty)$,

$$\mathbb{E}\operatorname*{ess\,sup}_{0\leq t\leq T}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}\leq C,$$

as well as

$$\mathbb{E}\big|m^\varepsilon([0,T]\times\mathbb{T}^N\times\mathbb{R})\big|^2\leq C.$$

Thus, using this observations as in Theorem 6.4, one finds that there exists a subsequence $\{u^n;\,n\in\mathbb{N}\}$ such that

(i) $f^n \overset{w^*}{\longrightarrow} f \qquad \text{in} \qquad L^\infty(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$,

(ii) there exists a kinetic measure m such that

$$m^n \overset{w^*}{\longrightarrow} m \qquad \text{in} \qquad L^2_w(\Omega;\mathcal{M}_b([0,T]\times\mathbb{T}^N\times\mathbb{R}))$$

and $m=n_1+n_2$, where

$$\mathrm{d}n_1(t,x,\xi)=\bigg|\operatorname{div}\int_0^u\sigma(\zeta)\,\mathrm{d}\zeta\bigg|^2\mathrm{d}\delta_{u(t,x)}(\xi)\,\mathrm{d}x\,\mathrm{d}t$$

and n_2 is a.s. a nonnegative measure over $[0,T]\times\mathbb{T}^N\times\mathbb{R}$.

With these facts in hand, we are ready to pass to the limit in (2.6) and conclude that u is the unique kinetic solution to (1.1). The proof of Theorem 2.7 is thus complete.

Appendix A. Generalized Itô's formula

In this section, we establish a generalized Itô formula for weak solutions of a very general class of SPDEs of the form

$$\begin{aligned}\mathrm{d}u&=F(t)\,\mathrm{d}t+\operatorname{div}G(t)\,\mathrm{d}t+H(t)\,\mathrm{d}W,\\ u(0)&=u_0,\end{aligned}\tag{A.1}$$

where W is the cylindrical Wiener process defined in Section 2. In the present context, the result is applied in the derivation of the kinetic formulation in Subsection 2.3 as well as in the proof of a priori $L^p(\mathbb{T}^N)$-estimates in Proposition 5.1. The result reads as follows.

Proposition A.1. *Let $\psi\in C^1(\mathbb{T}^N)$ and $\varphi\in C^2(\mathbb{R})$ with bounded second order derivative. Assume that the coefficients F,G_i, $i=1,\dots,N$, belong to $L^2(\Omega;L^2(0,T;L^2(\mathbb{T}^N)))$ and $H\in L^2(\Omega;L^2(0,T;L_2(\mathfrak{U};L^2(\mathbb{T}^N))))$, we denote $H_k=He_k$, $k\in\mathbb{N}$. Let the equation (A.1) be satisfied in $H^{-1}(\mathbb{T}^N)$ for some*

$$u\in L^2(\Omega;C([0,T];L^2(\mathbb{T}^N)))\cap L^2(\Omega;L^2(0,T;H^1(\mathbb{T}^N))).$$

Then almost surely, for all $t\in[0,T]$,

$$\begin{aligned}\big\langle\varphi(u(t)),\psi\big\rangle&=\big\langle\varphi(u_0),\psi\big\rangle+\int_0^t\big\langle\varphi'(u(s))F(s),\psi\big\rangle\,\mathrm{d}s\\&\quad-\int_0^t\big\langle\varphi''(u(s))\nabla u\cdot G(s),\psi\big\rangle\,\mathrm{d}s\\&\quad+\int_0^t\big\langle\operatorname{div}\big(\varphi'(u(s))G(s)\big),\psi\big\rangle\,\mathrm{d}s\\&\quad+\int_0^t\big\langle\varphi'(u(s))H(s)\,\mathrm{d}W(s),\psi\big\rangle\\&\quad+\frac12\sum_{k\geq1}\int_0^t\big\langle\varphi''(u(s))H_k^2(s),\psi\big\rangle\,\mathrm{d}s.\end{aligned}\tag{A.2}$$

Proof. In order to prove the claim, we use regularization by convolutions. Let (ϱ_δ) be an approximation to the identity on $\mathbb{T}^N$. For a function f on $\mathbb{T}^N$, we denote by f^δ the convolution $f*\varrho_\delta$. Recall, that if $f\in L^2(\mathbb{T}^N)$ then

$$\|f^\delta\|_{L^2(\mathbb{T}^N)}\leq\|f\|_{L^2(\mathbb{T}^N)},\qquad\|f^\delta-f\|_{L^2(\mathbb{T}^N)}\longrightarrow0.$$

Using $\varrho_\delta(x-\cdot)$ as a test function in (A.1), we obtain that

$$u^\delta(t)=u_0^\delta+\int_0^t F^\delta(s)\,\mathrm{d}s+\int_0^t \operatorname{div} G^\delta(s)\,\mathrm{d}s+\sum_{k\geq1}\int_0^t H_k^\delta(s)\,\mathrm{d}\beta_k(s)$$

holds true for every $x\in\mathbb{T}^N$. Hence we can apply the classical 1-dimensional Itô formula to the function $u(x)\mapsto\varphi(u(x))\psi(x)$ and integrate with respect to x

$$\begin{aligned}\big\langle\varphi(u^\delta(t)),\psi\big\rangle&=\big\langle\varphi(u_0^\delta),\psi\big\rangle+\int_0^t\big\langle\varphi'(u^\delta(s))F^\delta(s),\psi\big\rangle\,\mathrm{d}s\\&\quad-\int_0^t\big\langle\varphi''(u^\delta(s))\nabla u^\delta(s)\cdot G^\delta(s),\psi\big\rangle\,\mathrm{d}s\\&\quad+\int_0^t\big\langle\operatorname{div}\big(\varphi'(u^\delta(s))G^\delta(s)\big),\psi\big\rangle\,\mathrm{d}s\\&\quad+\sum_{k\geq1}\int_0^t\big\langle\varphi'(u^\delta(s))H_k^\delta(s),\psi\big\rangle\,\mathrm{d}\beta_k(s)\\&\quad+\frac12\sum_{k\geq1}\int_0^t\big\langle\varphi''(u^\delta(s))[H_k^\delta(s)]^2,\psi\big\rangle\,\mathrm{d}s=\mathrm{J}_1+\cdots+\mathrm{J}_6.\end{aligned}\tag{A.3}$$

We will now show that each term in (A.3) converge a.s. to the corresponding term in (A.2). For the stochastic term, we apply the Burkholder-Davis-Gundy inequality

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T}\bigg|\sum_{k\geq1}\int_0^t\big\langle\varphi'(u^\delta)H_k^\delta-\varphi'(u)H_k,\psi\big\rangle\,\mathrm{d}\beta_k(s)\bigg|\\&\leq C\,\mathbb{E}\bigg(\int_0^T\sum_{k\geq1}\big|\big\langle\varphi'(u^\delta)H_k^\delta-\varphi'(u)H_k,\psi\big\rangle\big|^2\mathrm{d}s\bigg)^{\frac12}\\&\leq C\,\mathbb{E}\bigg(\int_0^T\big\|\varphi'(u^\delta)-\varphi'(u)\big\|^2_{L^2(\mathbb{T}^N)}\big\|H^\delta\big\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}\mathrm{d}s\bigg)^{\frac12}\\&\quad+C\,\mathbb{E}\bigg(\int_0^T\big\|\varphi'(u)\big\|^2_{L^2(\mathbb{T}^N)}\big\|H^\delta-H\big\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}\mathrm{d}s\bigg)^{\frac12}.\end{aligned}\tag{A.4}$$

Since φ' is Lipschitz we have $\|\varphi'(u^\delta)-\varphi'(u)\|_{L^2(\mathbb{T}^N)}\to0$ a.e. in ω,t and

$$\begin{aligned}&\mathbb{E}\bigg(\int_0^T\big\|\varphi'(u^\delta)-\varphi'(u)\big\|^2_{L^2(\mathbb{T}^N)}\big\|H^\delta\big\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}\mathrm{d}s\bigg)^{\frac12}\\&\leq C\,\mathbb{E}\bigg(\int_0^T\|u\|^2_{L^2(\mathbb{T}^N)}\|H\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}\mathrm{d}s\bigg)^{\frac12}\\&\leq C\,\mathbb{E}\sup_{0\leq t\leq T}\|u\|^2_{L^2(\mathbb{T}^N)}+C\,\mathbb{E}\int_0^T\|H\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}\mathrm{d}s\end{aligned}$$

hence the first term on the right hand side of (A.4) converges to zero by dominated convergence theorem. The second one can be dealt with similarly as $\|H^\delta-H\|_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}\to0$ a.e. in ω,t. As a consequence, we obtain (up to subsequences) the almost sure convergence of J_5.

All the other terms can be dealt with similarly using the dominated convergence theorem. Let us now verify the convergence of J_3. It holds

$$\begin{aligned}\Big|\int_0^t \big\langle \varphi''(u^\delta)\nabla u^\delta\cdot G^\delta-\varphi''(u)\nabla u\cdot G,\psi\big\rangle\,\mathrm{d}s\Big|\\ \leq \int_0^t \big|\big\langle \varphi''(u^\delta)\nabla u^\delta\cdot\big(G^\delta-G\big),\psi\big\rangle\big|\,\mathrm{d}s\\ +\int_0^t \big|\big\langle \varphi''(u^\delta)\big(\nabla u^\delta-\nabla u\big)G,\psi\big\rangle\big|\,\mathrm{d}s\\ +\int_0^t \big|\big\langle \big(\varphi''(u^\delta)-\varphi''(u)\big)\nabla u\cdot G,\psi\big\rangle\big|\,\mathrm{d}s.\end{aligned}$$

Since φ'' is bounded and $\|G^\delta-G\|_{L^2(\mathbb{T}^N)}\to 0$, $\|\nabla u^\delta-\nabla u\|_{L^2(\mathbb{T}^N)}\to 0$ a.e. in ω,t we deduce by dominated convergence that the first two terms converge to zero. For the remaining term we shall use the fact that $\varphi''(u^\delta)-\varphi''(u)\to 0$ a.e. in ω,t,x and dominated convergence again.

In the case of J_4, we have

$$\begin{aligned}\Big|\int_0^t \big\langle \varphi'(u^\delta)G^\delta-\varphi'(u)G,\nabla\psi\big\rangle\,\mathrm{d}s\Big|\\ \leq \int_0^t \big|\big\langle \varphi'(u^\delta)\big(G^\delta-G\big),\nabla\psi\big\rangle\big|\,\mathrm{d}s\\ +\int_0^t \big|\big\langle \big(\varphi'(u^\delta)-\varphi'(u)\big)G,\nabla\psi\big\rangle\big|\,\mathrm{d}s\end{aligned}$$

hence $\|G^\delta-G\|_{L^2(\mathbb{T}^N)}\to 0$, $\|\varphi'(u^\delta)-\varphi'(u)\|_{L^2(\mathbb{T}^N)}\to 0$ a.e. in ω,t yield the conclusion. Similarly for J_2.

Concerning J_6, it holds

$$\begin{aligned}\Big|\sum_{k\geq 1}\int_0^t \big\langle \varphi''(u^\delta)[H_k^\delta]^2-\varphi''(u)H_k^2,\psi\big\rangle\,\mathrm{d}s\Big|\\ \leq \sum_{k\geq 1}\int_0^t \big|\big\langle \varphi''(u^\delta)\big([H_k^\delta]^2-H_k^2\big),\psi\big\rangle\big|\,\mathrm{d}s\\ +\sum_{k\geq 1}\int_0^t \big|\big\langle \big(\varphi''(u^\delta)-\varphi''(u)\big)H_k^2,\psi\big\rangle\big|\,\mathrm{d}s\end{aligned}$$

where for the first term we make use of boundedness of φ'', the fact that

$$\big\|[H_k^\delta]^2-H_k^2\big\|_{L^1(\mathbb{T}^N)}\leq \big\|H_k^\delta-H_k\big\|_{L^2(\mathbb{T}^N)}\big\|H_k^\delta+H_k\big\|_{L^2(\mathbb{T}^N)}\longrightarrow 0$$

a.e. in ω,t and dominated convergence. For the second one we employ that $\varphi''(u^\delta)-\varphi(u)\to 0$ a.e. in ω,x,t together with boundedness of φ''.

Since φ' has a linear growth, we obtain the convergence of J_1 as well as the term on the left hand side of (A.3). Indeed, for all $t\in[0,T]$ we have

$$\big|\big\langle\varphi(u^\delta(t))-\varphi(u(t)),\psi\big\rangle\big|\leq C\big(1+\|u(t)\|_{L^2(\mathbb{T}^N)}\big)\|u^\delta(t)-u(t)\|_{L^2(\mathbb{T}^N)}\longrightarrow 0$$

and the proof is complete. ∎

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