Front propagation in the non-local Fisher-KPP equation

Dissertation zur Erlangung des Doktorgrades

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der Universität Bielefeld

vorgelegt von

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Februar 2017

Gedruckt auf alterungsbeständigem Papier nach DIN-ISO 9706.

Contents

1 Introduction

There are two standpoints, from which one can investigate the dynamics of populations: the Lagrangian standpoint involves identifying each individual and following the consequent evolution; in the Eulerian standpoint characteristics of the whole population (e.g. density) are considered [67]. In the Lagrangian framework individual organisms are presented by points in space, so that demographic processes such as birth, death and dispersal can be presented by the appearance, disappearance and movement of points. In the Eulerian framework one consider the so-called correlation (factorial moment) functions ([9, 82, 83]), which satisfy an infinite system of equations that links to each other correlation functions of different order. In the case of Hamiltonian dynamics such system of equations is called BBGKY hierarchy [51]. Generally, the lower-order moments depend on the higher-order moments. Both the Lagrangian and Eulerian frameworks correspond to the microscopical level of description, where quantitative and qualitative analysis of the evolution of the population is a decisively difficult problem and an approximation scheme is required. A possible approximation may be constructed applying a mesoscopic limit [79] (e.g. a mean-field limit), which can be obtained by various kinds of scalings. Commonly, a mesoscopic approximation of a system of correlation functions brings to the finite closed system of equations called kinetic equation, which preserves some information about behavior of the microscopical system and may be easier to study.

A particular example of a population dynamics may be described on the microscopical level as follows: an evolving population of identical point entities, which are distributed over $\mathbb{R}^d$ and may produce themselves and die, also due to competition. Birth means that any point of the population may produce with a given rate a new one, which appears randomly in $\mathbb{R}^d$ according to a fixed distribution. Competition is a form of pairwise interaction which increases death rate of the particles according to a distribution. The model was originally introduced in [10] and subsequent papers [11, 30, 68, 74]; for father biological references see e.g. [75] and the recent review [78]. The rigorous microscopical description was done in [49] for the finite configurations in the Lagrangian framework. The resulting mesoscopic equation was derived for the integrable in space functions. The Eulerian framework was considered in [42] (see also [44, 46]) for the infinite configurations under additional assumptions. The resulting mesoscopic equation was derived for the bounded in space functions. In both cases the following kinetic equation was obtained,

$$\frac{\partial u}{\partial t}(x,t) = \varkappa^+ \big(a^+ * u\big)(x,t) - \varkappa^- u(x,t)\big(a^- * u\big)(x,t) - mu(x,t), \tag{1.1}$$

where $(a^\pm * u)(x,t)$ mean the convolutions (in x) between u and nonnegative integrable probability kernels $a^\pm = a^\pm(x) \geq 0$ on $\mathbb{R}^d$; namely,

$$(a^\pm * u)(x,t) = \int_{\mathbb{R}^d} a^\pm(x-y)u(y,t)dy, \qquad \int_{\mathbb{R}^d} a^\pm(x)\,dx = 1.$$

The meaning of $u(x,t)$ is the (approximate) value of the local density of a system in a point $x \in \mathbb{R}^d$ at a moment of time $t \geq 0$. A particle located at a point $y \in \mathbb{R}^d$ may produce a 'child' at a point $x \in \mathbb{R}^d$ with the intensity $\varkappa^+$ and according to the dispersion kernel $a^+(x-y)$. Next, any particle may die with the constant intensity m. And additionally, a particle located at x may die according to the competition with the rest of the particles; the intensity of the death because of a competitive particle located at y is equal to $\varkappa^-$ and the distribution of the competition is described by $a^-(x-y)$.

This equation may be considered as a spatial (inhomogeneous) version of the classical logistic

(Verhulst) equation

$$\frac{du}{dt} = (\varkappa^+ - m)u(t) - \varkappa^-(u(t))^2, \tag{1.2}$$

corresponding to $u(x,t) = u(t)$, $x \in \mathbb{R}^d$. Of course, in the logistic model one needs to assume that $\varkappa^+ > m$; then (1.2) has two stationary nonnegative solutions: unstable $u = 0$ and stable $u = \frac{\varkappa^+ - m}{\varkappa^-}$. For $\varkappa^+ \leq m$, (1.2) has the unique stationary stable solution $u = 0$.

The equation (1.1) appeared in [71, 72], for $\varkappa^+ a^+ = \varkappa^- a^-$ and $m = 0$, as a model of an epidemic. In [32], the same equation was derived for $\varkappa^+ a^+ = \varkappa^- a^-$ and $m \geq 0$ from a 'crabgrass model' of spatial ecology in $\mathbb{Z}^d$. In [10], it was proposed in the form of (1.1) as a deterministic analogue of the moment equations for ecological systems.

The equation (1.1) can be rewritten as follows:

$$\frac{\partial u}{\partial t}(x,t) = (L_{a^+}u)(x,t) + F(u, a^- * u)(x,t), \tag{1.3}$$

where, for a bounded function v on $\mathbb{R}^d$, the operator

$$(L_{a^+}v)(x) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x-y)[v(y) - v(x)]\, dy, \tag{1.4}$$

describes the so-called nonlocal diffusion (jumps), see e.g. [4] and references below, and F is a mapping on bounded functions, given by

$$F(v_1, v_2)(x) = \varkappa^- v_1(x)\bigl(\theta - v_2(x)\bigr), \qquad \theta = \frac{\varkappa^+ - m}{\varkappa^-}. \tag{1.5}$$

In such form behaviour of the solution in time will depend on the interplay between the nonlinear nonlocal interaction (or reaction) described by F and jumps in space described by L_{a^+}.

For the known results about (1.3), one can refer to [41, 42, 49], in the general case; to [77, 94, 100], in the case $\theta > 0$, i.e. $\varkappa^+ > m$, see also details below; and to [90, 91], for $\varkappa^+ = m$.

If F is a local operator, namely $a^-(x) = \delta(x)$, then one gets from (1.3) another nonlocal Fisher–KPP equation

$$\frac{\partial u}{\partial t} = L_{a^+}u + f(u). \tag{1.6}$$

For a general monostable f as above, this equation was considered in e.g. [2, 12, 19–22, 24, 25, 52, 62, 69, 81, 88, 99], see also some details below.

Recall that the classical Fisher–KPP (Kolmogorov–Petrovski–Piskunov) equation in $\mathbb{R}^d$ goes back to [48, 63] and has the form

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + f\bigl(v(x,t)\bigr), \tag{1.7}$$

see the seminal paper [6]. This equation was considered by Komlogorov et al. as an approximation of (1.6). Here Δ is the Laplace operator on $\mathbb{R}^d$, and f is a nonlinear monostable function on $\mathbb{R}$: namely, let $\theta > 0$, cf. (1.5), then we assume that $f(0) = f(\theta) = 0$, $f'(0) > 0$, $f'(\theta) < 0$; for example,

$$f(s) = \varkappa^- s(\theta - s), \quad s \geq 0. \tag{1.8}$$

Of course, there are a lot of generalisations for the equations (1.3), (1.6): the monostable-type function f may depend on time and space variables (e.g. nonlocal reaction-diffusion equation in a periodic media), the mapping F may include a convolution in time or just a time-delay, and many others. For some recent generalisations, see e.g. [5, 23, 29, 60, 62, 69, 70, 76, 80, 84, 85, 87, 88, 92, 98, 102].

In order to combine both (1.1) and (1.6), we will replace in (1.1) $\varkappa^- a^-(x)$ by $\varkappa^- \tilde{a}^-(x) = \kappa_1 \delta(x) + \kappa_2 a^-(x)$, so we will deal with the following nonlinear nonlocal evolution equation

$$\frac{\partial u}{\partial t}(x,t) = \varkappa^+ \bigl(a^+ * u\bigr)(x,t) - \varkappa^- u(x,t) \bigl(\tilde{a}^- * u\bigr)(x,t) - mu(x,t), \tag{1.9}$$

with a bounded initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}^d$, $d \geq 1$. Constants $m, \varkappa^+$ are assumed to be positive, κ_1, κ_2 are non-negative, such that

$$\varkappa^- = \kappa_1 + \kappa_2 > 0.$$

The aim of the thesis is to study the following problems.

(P1) Existence and uniqueness of solutions in Banach spaces of functions $L^\infty(\mathbb{R}^d)$ and $C_{ub}(\mathbb{R}^d)$ (the space of uniformly continuous functions with sup-norm) and uniform in time bounds for the norms of the solutions in the Banach spaces.

(P2) Existence and stability of stationary solutions.

(P3) Existence, uniqueness and properties of the traveling waves: solutions of the special form $u(x,t) = \psi(x \cdot \xi - ct)$, where ψ is a function on $\mathbb{R}$ called the profile of a wave, ξ belongs to the unit sphere S^{d-1} in $\mathbb{R}^d$, $x \cdot \xi = (x, \xi)_{\mathbb{R}^d}$ is the scalar product on $\mathbb{R}^d$, and $c \in \mathbb{R}$ describes the speed of the wave. Depending on the class of functions ψ the question may be referred to decaying waves, bounded waves *etc.*

(P4) The largest part of the thesis is devoted to studying existence and time-behavior of the front of propagation, i.e. a set $\Gamma_t = \mathbb{R}^d \setminus (\mathscr{C}_t \cup \mathscr{O}_t)$, such that for any $x_t \in \mathscr{C}_t$, the values of $u(x_t, t)$ will converge (as $t \to \infty$) to the upper stationary solution (θ in the notations above), whereas, for any $y_t \in \mathscr{O}_t$, the values of $u(y_t, t)$ will converge to the low stationary solution (i.e. to 0). The problem will be divided into two cases:

   (a) constant speed of propagation

   (b) acceleration

1.1 Outline of the thesis

We present now an overview of our results concerning the problems (P1)–(P4) for the equation (1.9).

Problem (P1) We will study (1.9) in the spaces $C_{ub}(\mathbb{R}^d)$ of the bounded uniformly continuous functions and $L^\infty(\mathbb{R}^d)$. To get an answer on the problem (P1), one does not need any further assumptions on parameters $m, \varkappa^\pm > 0$ and probability kernels $0 \leq a^\pm \in L^1(\mathbb{R}^d)$ (see Theorem 2.2 and Remark 2.3). We use standard fixed point arguments, which take into account, however, the negative sign before a^- in (1.9). The solution hence may be constructed on a time-interval $[\tau, \tau + \Delta\tau]$, whereas the $\Delta\tau$ depends on the supremum of the solution at τ. Since the values at the moment $\tau + \Delta\tau$ might be bigger, the next time-interval appears, in general, shorter. The mentioned usage of the negative sign allows us to show that, however, the series of the time-intervals diverges, and thus one can construct solution on an arbitrary big time-interval.

In spite of the possible growth of solution's space-supremum in time, we show (Theorem 2.8) that the solution in $C_{ub}(\mathbb{R}^d)$ remains uniformly bounded in time on $[0,\infty)$ under very weak assumptions: one needs only that a^- would be separated from zero in a neighbourhood of the origin and that a^+ would have a regular behavior at infinity, e.g. $a^+(x) \le p(|x|)$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $p \in L^1(\mathbb{R})$ monotonically decays at $\pm\infty$. This result is an analog of [59, Theorem 1.2], where a combination of the Laplace operator and nonlocal reaction is considered.

The rest of our results requires additional hypotheses. For the brevity, some of them are presented here in a more restrictive form (compare them with the real assumptions (A1)-(A10) and (B.1)–(B.5) within the paper); and surely, a particular result requires a part of the assumptions only. Note also that (H3a) and (H3b) are mutually exclusive.

(H1) $0 \le a^\pm \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $\varkappa^+ > m$, i.e. $\theta = \frac{\varkappa^+ - m}{\varkappa^-} > 0$.

(H2) the function

$$J_\theta := \varkappa^+ a^+ - \theta \kappa_2 a^-,$$

is almost everywhere (a.e., in the sequel) non-negative and it is separated from 0 a.e. in a neighbourhood of the origin.

(H3a) For some $\lambda_0 > 0$ and for all $\lambda > 0$,

$$\int_{\mathbb{R}^d} a^+(x) e^{\lambda_0 |x|}\, dx < \infty \quad \textit{and} \quad \sup_{x\in\mathbb{R}^d} u_0(x) e^{\lambda |x|} < \infty.$$

(H3b) There exist decreasing $c(s), b(s) : \mathbb{R}_+ \to (0,\infty)$, such that $\log b$ and $\log c$ are convex (plus some technical assumptions on b, c.f. Definition 6.21); for any $h > 0$,

$$c(s+h) \sim c(s) \quad \textit{and} \quad (b * b)(s) \sim 2\int_0^\infty b(\tau) d\tau\, b(s), \quad s \to \infty,$$

and the following estimate holds

$$c(|x|) \le \bigl(a^+ * u_0\bigr)(x) \le b(|x|), \quad x \in \mathbb{R}^d.$$

Let us compare these hypotheses with existing in the literature. First, we are working in the multi-dimensional settings, cf. [41, 42, 77]. We show (Proposition 4.4) how the problem (P3) may be reduced to a one-dimensional equation, whose kernels, however, will depend on a direction $\xi \in S^{d-1}$. Regarding to this, it should be emphasised, that we do not assume that a^+ is symmetric and deal with the so-called anisotropic settings. Note that in the last section upper and lower estimates on the accelerating front coincide for $\log c \sim \log b$ (see (H3b)), in particular in the case of radially symmetric $a^+ * u_0$ (namely, $c = b$).

The hypothesis (H3a) is sufficient for a front propagation with a constant speed. It was shown by Mollison in one-dimensional case under more restrictive assumptions on the initial condition (see [71,72]), that a weaker hypothesis (H3a$_\xi$) is necessary and sufficient to have a constant speed of propagation

(H3a$_\xi$) There exists $\lambda > 0$, such that

$$\mathfrak{a}_\xi(\lambda) := \int_{\mathbb{R}^d} a^+(x) e^{\lambda\, x\cdot\xi}\, dx < \infty \quad \textit{and} \quad \sup_{x\in\mathbb{R}^d} u_0(x) e^{\lambda\, x\cdot\xi} < \infty.$$

The equation (1.6), under (H3a) or its weaker form (H3a$_\xi$) was considered in [2, 12, 22, 25]. The corresponding results in [94, 100] about our equation (1.1) required, however, symmetric and quickly decaying a^+; the latter meant that (H3a) must hold for all $\lambda > 0$. Note that [94] dealt with a system of equations for a multi-type epidemic model, which is reduced in the one-type case to (1.1) with $\varkappa^+ a^+ \equiv \varkappa^- a^-$. It is worth noting also that we do not need a continuity of a^+ as well.

The hypothesis (H3b) is opposite to (H3a). Although it looks complicated the hypothesis is not very restrictive (see examples in Subsection 6.3 and 6.7). Informally, it means that either a^+ or u_0 decays slower than exponentially and does not oscillate rapidly.

The most restrictive, in some sense, hypothesis is (H2). It implies the comparison principle for the equation (1.9), cf. Theorem 3.1, Proposition 3.5. In particular, the latter states that the solution will be inside the strip $0 \leq u(x,t) \leq \theta$, for all $t > 0$, provided that the initial condition $u(x,0)$ was inside this strip. On the other hand, we show that (H2) is, in some sense, a necessary condition to have a comparison principle at all (Remark 3.7).

Problem (P2) In Subsection 3.1, we show also that $u \equiv \theta$ is a uniformly and asymptotically stable solution, whereas $u \equiv 0$ is an unstable one. The assumption (H2) ensures the absence of non-constant stationary solutions (see Proposition 5.12 and Problem (P4) below).

The maximum principle is considered in Subsection 3.2, cf. Theorem 3.10. In particular, we prove that the solution to (1.1) is strictly positive, even for a compactly supported initial condition $u_0(x) := u(x,0)$, and lies strictly less than θ, for any $u_0 \not\equiv \theta$ (Proposition 3.9, Corollary 3.11).

It is worth noting that the luck of the comparison principle, provided to (1.9) by (H2), leads for a similar equation (with the Laplace operator instead of the jump-generator L_{a^+}) to a non-trivial behavior: the upper stationary solution $u \equiv \theta$ may not be stable, moreover, a stationary inhomogeneous solution may exist (see [3, 5, 8, 40, 54, 59, 73]).

Problem (P3) We study monotonically non-increasing traveling waves only (i.e. the profile ψ is a non-increasing function on $\mathbb{R}$). To ensure the existence of a traveling wave solution to (1.1) in a direction $\xi \in S^{d-1}$ it suffices to suppose that there exists $\lambda > 0$ such that $\mathfrak{a}_\xi(\lambda) < \infty$ (c.f. (H3a$_\xi$)). Namely, we prove that there exists a minimal traveling wave speed $c_*(\xi) \in \mathbb{R}$, such that, for any $c \geq c_*(\xi)$, there exists a traveling wave in the direction ξ with the speed c; and, for any $c < c_*(\xi)$, such a traveling wave does not exist (Theorem 4.9). We use here an abstract result from [99] and apply it to (1.1) similarly to how it was done in [99] for (1.6). This allow us to prove the existence of such finite $c_*(\xi)$ without an assumption about a quick decaying of a^+ in the direction ξ; i.e. that we do not need that $\mathfrak{a}_\xi(\lambda) < \infty$ holds, for all $\lambda > 0$, in contrast to [94, 100]. It is worth noting that the hypothesis (H2) evidently holds under the assumptions from [94], where $\varkappa^+ = \varkappa^-$, $a^+ = a^-$, as well as it holds under the assumptions from [100], where one of the considered cases may be rewritten in the form $\frac{\partial}{\partial t} u = J_\theta * u - mu + \varkappa^-(\theta - u)(a^- * u)$, which is equivalent to (1.1).

A specific feature of the equation (1.9) is that any monotonic traveling wave with a non-zero speed $c \geq c_*(\xi)$ has a smooth profile $\psi_c \in C^\infty(\mathbb{R})$, whereas, for the traveling wave with the zero speed (which does exist, if only $c_*(\xi) \leq 0$), one can only prove that its profile $\psi_0 \in C(\mathbb{R})$ (Proposition 4.11, Corollary 4.12), in contrast to the equation (1.6), cf. [22], where a weaker smoothness was shown. This allow us to consider the equation for traveling waves point-wise, for $s \in \mathbb{R}$:

$$c\psi'(s) + \varkappa^+(\check{a}^+ * \psi)(s) - m\psi(s) - \kappa_1\psi^2(s) + \kappa_2\psi(s)(\check{a}^- * \psi)(s) = 0, \tag{1.10}$$

where the kernels $\check{a}^\pm$ are obtained by the integration of $a^\pm$ over the orthogonal complement $\{\xi\}^\perp$, see (4.6) below. Moreover, in Proposition 4.13, we show that ψ is a strictly decaying function.

We study properties of the solutions to (1.10) using a bilateral-type Laplace transform: $(\mathfrak{L}\psi)(z) = \int_{\mathbb{R}} \psi(s)e^{zs}\,ds$, $\operatorname{Re} z > 0$. To do this, we prove that any solution (1.10) has a positive abscissa $\lambda_0(\psi)$ of this Laplace transform, i.e. that $(\mathfrak{L}\psi)(\lambda) < \infty$, for some $\lambda > 0$ (Proposition 4.14). Moreover, in Theorem 4.23, we prove, in particular, that $\lambda_0(\psi)$ is finite and bounded by $\lambda_0(\check{a}^+)$; note that the latter abscissa will be infinite in the case of quickly decaying kernel a^+, i.e. when (H3a$_\xi$) holds, for any $\lambda > 0$. We also find in Theorem 4.23 the explicit formula for $c_*(\xi)$:

$$c_*(\xi) = \inf_{\lambda>0} \frac{\varkappa^+ \mathfrak{a}_\xi(\lambda) - m}{\lambda},$$

where $\mathfrak{a}_\xi$ is defined in (H3a$_\xi$); and we show that the dependence of the abscissa $\lambda_0(\psi_c)$ for a traveling wave profile ψ_c corresponding to a speed c is strictly decreasing in $c \geq c_*(\xi)$. Note that this expression for the minimal traveling wave speed coincides with the known one for the equation (1.6), see e.g. [22].

Thus, for 'exponentially decaying' a^+ (i.e. if there exists a finite supremum of λ's for which $\mathfrak{a}_\xi(\lambda) = \infty$), it is possible the situation in which the abscissa $\lambda_* = \lambda_0(\psi_{c_*(\xi)})$ of the traveling wave with the minimal possible speed coincides with $\lambda_0(\check{a}^+)$. This case is traditionally more difficult for an analysis of profiles' properties, cf. e.g. [2, Theorem 3, Remark 8]. We consider this special case in details and describe it in terms of the function a^+ and the parameters $m, \varkappa^\pm$, cf. Definition 4.20, Theorem 4.23.

The variety of possible situations demonstrates the following natural example, cf. Example 4.22. Let

$$a^+(x) = \frac{\alpha e^{-\mu|x|}}{1 + |x|^q}, \quad q \geq 0,\ \mu > 0, \tag{1.11}$$

where $\alpha > 0$ is a normalising constant. Then, for any $\xi \in S^{d-1}$, the abscissa $\lambda_0(\check{a}^+) = \mu$ is finite. We show that the strict inequality $\lambda_* < \mu$ always hold, for $q \in [0, 2]$. Next, there exist critical values $\mu_* > 0$ and $m_* \in (0, \varkappa^+)$, such that, for $q > 2$, one has $\lambda_* < \mu$ if $\mu > \mu_*$ or if $\mu \in (0, \mu_*]$ and $m \in (m_*, \varkappa^+)$. Respectively, for $q > 2$, $\mu \in (0, \mu_*]$, and $m \in (0, m_*]$, we show the equality $\lambda_* = \mu$, see Theorem 4.23.

To study the uniqueness of traveling waves, we find also the exact asymptotic at ∞ of the profiles of traveling waves with non-zero speeds. Namely, we show in Proposition 4.25, that, for a profile ψ corresponding to the speed $c \neq 0$,

$$\psi(t) \sim De^{-\lambda_0(\psi)t}, \quad c > c_*(\xi), \qquad \psi(t) \sim Dt\, e^{-\lambda_0(\psi)t}, \quad c = c_*(\xi), \tag{1.12}$$

as $t \to \infty$. Here $D > 0$ is a constant which may be chosen equal to 1 by a shift of ψ (see Remark 4.32). To get (1.12), one needs an additional assumption in the critical case for the speed $c = c_*(\xi)$; for example, in terms of the function (1.11), this assumption does not hold for the case $q \in (2, 3]$, $\mu \in (0, \mu_*]$, $m = m_*$ only (Remark 4.27).

The asymptotic (1.12) yields that (H3a$_\xi$) holds for $u_0 = \psi$ and $\lambda < \lambda_*$. The result was known for the equation (1.6), cf. e.g. [2, 15, 22]. In the two latter references, there was used a version of the Ikehara theorem which belongs to Delange [28]. However, we have met here with the following problem.

Both the classical Ikehara theorem (see e.g. [96]) and the Ikeraha–Delange theorem [28] (see also [34]) dealt with functions growing at infinity to ∞. In [15, 22], the corresponding results were postulated for functions (decreasing or increasing) which tend to 0 (on ∞ or $-\infty$, respectively). We did not find any arguments why we could apply or how one could modify the proofs of Ikehara-type theorems for such functions without proper additional assumptions. The natural assumption under which it can be realized is that the decreasing function $\psi(s)$ (a traveling wave in our context) must become an increasing one, being multiplied on an exponent $e^{\nu s}$, for a big enough $\nu > 0$.

Under such an assumption the Ikehara-type theorems might hold true, however, one needs more to cover the aforementioned case $\lambda_* = \mu$. In this case, the Laplace transform of $\check{a}^+$ is not analytic at its abscissa, that was a requirement for the mentioned theorems. Therefore, we used an another modification of the Ikehara theorem, the so-called Ikehara–Ingham theorem [89]. Under the assumption that a constant ν as above exists, we prove in Proposition 4.28 a version of the Ikehara–Ingham theorem for such decreasing functions. Next, using the ideas from [101], we show that, for any solution to (1.10) with $c \neq 0$, such a ν does exist.

Note also that the technique from [2] did not require the usage of Ikehara-type theorem, however, even for the local nonlinearity like in (1.6) it did not work in the critical case above.

The asymptotic (1.12) allows us to prove the uniqueness of the profiles for a traveling wave with a non-zero speed (Theorem 4.33). We follow there the technique proposed in [15].

Problem (P4) The results of Mollison (see [72]) motivate us to devide the problem into two cases: (H3a) and (H3b).

If (H3a) holds then one of the traditional way for the study of the front of propagation for integro-differential equations is the usage of abstract Weinberger's results from [93] (which are going back to [6], for the Fisher–KPP equation (1.7)). The information we obtained for the traveling waves allow us to describe in more details the behavior of $u(tx,t)$ 'out of the front'; here u is the solution to (1.3). Namely, in Theorem 5.9, we prove that, for a proper compact convex set Υ_1, the function $u(tx,t)$ decays exponentially in time, uniformly in $x \in \mathbb{R}^d \setminus \mathscr{O}$, for any open $\mathscr{O} \supset \Upsilon_1$, provided that the initial condition decays in space quicker than any exponent (in particular, we do not require a compactly supported initial condition).

To describe the behavior of $u(tx,t)$, for $x \in \Upsilon_1$, we start with an adaption of the results from [93] to our case. However, that abstract technique required that the initial condition should be separated from 0 on a set which can not be described explicitly (the existence of such a set was shown only, cf. Lemma 5.14 and Proposition 5.18 below). To avoid this restriction, we find, in Proposition 5.19, an explicit sub-solution to (1.3), and, moreover, we prove, in Proposition 5.20, that this sub-solution indeed becomes a minorant for the solution, after a finite time. This arguments allow us to show that $u(tx,t)$ converges to θ uniformly in $x \in \mathscr{C}$, for any compact $\mathscr{C} \subset \Upsilon_1$ (Theorem 5.10, Corollary 5.11). In notations of Problem (P4), it means informally that $\Gamma_t \approx t\,\partial\Upsilon_1$.

As a consequence, we prove that, under additional technical assumptions, there are not other non-negative time-stationary solutions to (1.3) except constant solutions 0 and θ (Proposition 5.12).

The condition (H3a$_\xi$) is crucial: we show in Theorem 5.21 and Corollary 5.22 that the absence of a λ and a $\xi \in S^{d-1}$ which ensure (H3a$_\xi$) leads to an infinite speed of propagation (i.e. the compact set Υ_1 above may be chosen arbitrary big) and hence to the absence of traveling waves at all. The corresponding result for (1.6) was received in [52] and it is goes back to [71, 72] mentioned above. The results of [77] cover Theorems 5.9, 5.10, and 5.21, for the equation (1.3) with $\varkappa^+ a^+ = \varkappa^- a^-$; however, a lot of details of the proofs (which used completely another technique) were omitted.

Informally, to obtain a propagation, which is faster than linear, one has to have that $a^+ * u_0$ is heavy-tailed. However, in order to estimate the propagation we require a class of probability densities with regular tails. Therefore, we consider so called long-tailed and sub-exponential densities. The classes of sub-exponential and long-tailed probability distributions which correspond to the non-negative random variables (and, therefore, are supported on $\mathbb{R}^+$) where considered by Chistyakov [17] to study the renewal equation. The corresponding classes of probability distributions on $\mathbb{R}$ and $\mathbb{Z}$ were considered in [18], [35]. To study integrable initial conditions and dimensions higher than one we need to consider densities on $\mathbb{R}$ instead of distributions (see [7], [33]). The corresponding technique is described in Subsection 6.1.1. The description of the level sets of solutions is done in Subsection 6.2.

It is proved in Theorem 6.67 that there exists a domain $\Lambda_\varepsilon^-(t) \subset \mathbb{R}^d$, which expands in space for large time, and where the solution tends uniformly to the constant θ. Theorem 6.85 shows that there exists another domain $\Lambda_\varepsilon^+(t) \subset \mathbb{R}^d$, where the solution is close to zero. The level sets of the solution are located between this domains in the set $\Delta_\varepsilon(t) := \mathbb{R}^d \setminus \bigl(\Lambda_\varepsilon^+(t) \cup \Lambda_\varepsilon^-(t)\bigr)$, for large time, and the set $\Delta_\varepsilon(t)$ will expand in space (see [53]). In Subsection 6.7 we consider different examples. Up to our knowledge, the first result of this type was obtained in [52] for (1.6), which was shown in one-dimensional case for compactly supported initial conditions. In [52] estimates from above on the solution are not close to the estimates from below. Consideration of the long-tailed and sub-exponential densities is a possible way to cover this gap, as we show for radially symmetric a^+ and u_0. The paper of Garnier was inspired by another remarkable result for the classical F–KPP equation [58], where it was shown that slowly decaying initial conditions lead to the acceleration of the propagation. In some sence we combine both of their assumptions in the form of (H3b).

To summarize, the structure of the paper is the following. In Section 2, we study Problem (P1); Section 3 is devoted to comparison and maximum principles, and, partially, to Problem (P2). Traveling waves, Problem (P3), are considered in Section 4. The long-time behavior, i.e. Problem (P4), and the rest of Problem (P2) are the topics of Sections 5 and 6.

2 Existence, uniqueness, and boundedness

Let $u = u(x,t)$ describe the local density of a system at the point $x \in \mathbb{R}^d$, $d \geq 1$, at the moment of time $t \in I$, where I is either a finite interval $[0,T]$, for some $T > 0$, or the whole $\mathbb{R}_+ := [0,\infty)$. The time evolution of u is given by the following initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \varkappa^+ (a^+ * u)(x,t) - u(x,t)\bigl(Gu\bigr)(x,t), & t \in I \setminus \{0\}, \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^d$ and $\bigl(Gu\bigr)(x,t) = m + \kappa_1 u(x,t) + \kappa_2 \bigl(a^- * u\bigr)(x,t)$. We will study the equation in a class of bounded in x nonnegative functions.

Here $m > 0$, $\varkappa^+ > 0$, $\kappa_1 \geq 0$, $\kappa_2 \geq 0$, $\varkappa^- = \kappa_1 + \kappa_2 > 0$ are constants, and functions $0 \leq a^\pm \in L^1(\mathbb{R}^d)$ are probability densities:

$$\int_{\mathbb{R}^d} a^+(y)dy = \int_{\mathbb{R}^d} a^-(y)dy = 1. \tag{2.2}$$

Here and below, for a function $u = u(y,t)$, which is (essentially) bounded in $y \in \mathbb{R}^d$, and a function (a kernel) $a \in L^1(\mathbb{R}^d)$, we denote

$$(a * u)(x,t) := \int_{\mathbb{R}^d} a(x-y)u(y,t)dy. \tag{2.3}$$

We assume that u_0 is a bounded function on $\mathbb{R}^d$. For technical reasons, we will consider two Banach spaces of bounded real-valued functions on $\mathbb{R}^d$: the space $C_{ub}(\mathbb{R}^d)$ of bounded uniformly continuous functions on $\mathbb{R}^d$ with sup-norm and the space $L^\infty(\mathbb{R}^d)$ of essentially bounded (with respect to the Lebesgue measure) functions on $\mathbb{R}^d$ with esssup-norm. Let also $C_b(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$

denote the spaces of continuous functions on $\mathbb{R}^d$ which are bounded and have compact supports, correspondingly.

Let E be either $C_{ub}(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$. Consider the equation (2.1) in E; in particular, u must be continuously differentiable in t, for $t > 0$, in the sense of the norm in E. Moreover, we consider u as an element from the space $C_b(I \to E)$ of continuous bounded functions on I (including 0) with values in E and with the following norm

$$\|u\|_{C_b(I\to E)} = \sup_{t\in I} \|u(\cdot,t)\|_E.$$

Such a solution is said to be a classical solution to (2.1); in particular, u will continuously (in the sense of the norm in E) depend on the initial condition u_0.

We will also use the space $C_b(I \to E)$ with $I = [T_1, T_2]$, $T_1 > 0$. For simplicity of notations, we denote

$$\mathcal{X}_{T_1,T_2} := C_b\big([T_1,T_2] \to C_{ub}(\mathbb{R}^d)\big), \qquad T_2 > T_1 \geq 0,$$

and the corresponding norm will be denoted by $\|\cdot\|_{T_1,T_2}$. We set also $\mathcal{X}_T := \mathcal{X}_{0,T}$, $\|\cdot\|_T := \|\cdot\|_{0,T}$, and

$$\mathcal{X}_\infty := C_b\big(\mathbb{R}_+ \to C_{ub}(\mathbb{R}^d)\big),$$

with the corresponding norm $\|\cdot\|_\infty$. The upper index '+' will denote the cone of nonnegative functions in the corresponding space, namely,

$$\mathcal{X}_\sharp^+ := \{u \in \mathcal{X}_\sharp \mid u \geq 0\},$$

where $\sharp$ is one of the sub-indexes above. Finally, the corresponding sets of functions with values in $L^\infty(\mathbb{R}^d)$ will be denoted by the tilde above, e.g.

$$\begin{aligned}
\tilde{\mathcal{X}}_T &:= C_b([0,T] \to L^\infty(\mathbb{R}^d)),\\
\tilde{\mathcal{X}}_T^+ &:= \big\{u \in \tilde{\mathcal{X}}_T \mid u(\cdot,t) \geq 0,\ t \in [0,T],\ \text{a.a. } x \in \mathbb{R}^d\big\}.
\end{aligned}$$

We will also omit the sub-index for the norm $\|\cdot\|_E$ in E, if it is clear whether we are working with sup- or esssup-norm.

We start with a simple lemma.

Lemma 2.1. *Let $a \in L^1(\mathbb{R}^d)$, $f \in L^\infty(\mathbb{R}^d)$. Then $a * f \in C_{ub}(\mathbb{R}^d)$. Moreover, if $v \in C_b(I \to E)$, $I \subset \mathbb{R}_+$, then $a * v \in C_b(I \to C_{ub}(\mathbb{R}^d))$.*

Proof. The convolution is a bounded function, as

$$|(a * f)(x)| \leq \|f\|_E \, \|a\|_{L^1(\mathbb{R}^d)}, \qquad a \in L^1(\mathbb{R}^d), f \in E. \tag{2.4}$$

Next, let $a_n \in C_0(\mathbb{R}^d)$, $n \in \mathbb{N}$, be such that $\|a - a_n\|_{L^1(\mathbb{R}^d)} \to 0$, $n \to \infty$. For any $n \geq 1$, the proof of that $a_n * f \in C_{ub}(\mathbb{R}^d)$ is straightforward. Next, by (2.4), $\|a * f - a_n * f\| \to 0$, $n \to \infty$. Hence $a * u$ is a uniform limit of uniformly continuous functions that fulfills the proof of the first statement. The second statement is followed from the first one and the inequality (2.4). □

The following theorem yields existence and uniqueness of a solution to (2.1) on a finite time-intervals $[0, T]$.

Theorem 2.2. *Let $u_0 \in C_{ub}(\mathbb{R}^d)$ and $u_0(x) \geq 0$, $x \in \mathbb{R}^d$. Then, for any $T > 0$, there exists a unique nonnegative solution u to the equation (2.1) in $C_{ub}(\mathbb{R}^d)$, such that $u \in \mathcal{X}_T$.*

Proof. Let $T>0$ be arbitrary. Take any $0\leq v\in\mathcal{X}_T$. For any $\tau\in[0,T)$, consider the following linear equation in the space $C_{ub}(\mathbb{R}^d)$ on the interval $[\tau,T]$:

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t)=\varkappa^+(a^+*v)(x,t)-u(x,t)\big(Gv\big)(x,t) & t\in(\tau,T],\\ u(x,\tau)=u_\tau(x),\end{cases}\tag{2.5}$$

where $0\leq u_s\in C_{ub}(\mathbb{R}^d)$, $s>0$, are some functions, and u_0 is the same as in (2.1). By Lemma 2.1, in the right hand side (r.h.s. in the sequel) of (2.5), there is a time-dependent linear bounded operator (acting in u) in the space $C_{ub}(\mathbb{R}^d)$ whose coefficients are continuous on $[\tau,T]$. Therefore, there exists a unique solution to (2.5) in $C_{ub}(\mathbb{R}^d)$ on $[\tau,T]$, given by $u=\Phi_\tau v$ with

$$(\Phi_\tau v)(x,t):=(Bv)(x,\tau,t)u_\tau(x)+\int_\tau^t(Bv)(x,s,t)\varkappa^+(a^+*v)(x,s)\,ds,\tag{2.6}$$

for $x\in\mathbb{R}^d$, $t\in[\tau,T]$, where we set

$$(Bv)(x,s,t):=\exp\biggl(-\int_s^t\big(Gv\big)(x,p)\,dp\biggr),\tag{2.7}$$

for $x\in\mathbb{R}^d$, $t,s\in[\tau,T]$. Note that, in particular, $(\Phi_\tau v)(\cdot,t),(Bv)(\cdot,s,t)\in C_{ub}(\mathbb{R}^d)$. Clearly, $(\Phi_\tau v)(x,t)\geq 0$ and, for any $\Upsilon\in(\tau,T]$,

$$\|\Phi_\tau v(\cdot,t)\|\leq\|u_\tau\|+\varkappa^+(\Upsilon-\tau)\|v\|_{\tau,\Upsilon},\qquad t\in[\tau,\Upsilon],\tag{2.8}$$

where we used (2.4). Therefore, Φ_τ maps $\mathcal{X}^+_{\tau,\Upsilon}$ into itself, $\Upsilon\in(\tau,T]$.

Let now $0\leq\tau<\Upsilon\leq T$, and take any $v,w\in\mathcal{X}^+_{\tau,\Upsilon}$. By (2.6), one has, for any $x\in\mathbb{R}^d$, $t\in[\tau,\Upsilon]$,

$$\big|(\Phi_\tau v)(x,t)-(\Phi_\tau w)(x,t)\big|\leq J_1+J_2,\tag{2.9}$$

where

$$\begin{aligned}J_1&:=\big|(Bv)(x,\tau,t)-(Bw)(x,\tau,t)\big|u_\tau(x),\\ J_2&:=\varkappa^+\int_\tau^t\big|(Bv)(x,s,t)(a^+*v)(x,s)-(Bw)(x,s,t)(a^+*w)(x,s)\big|\,ds.\end{aligned}$$

Since $|e^{-a}-e^{-b}|\leq|a-b|$, for any constants $a,b\geq 0$, one has, by (2.7), (2.4),

$$J_1\leq\varkappa^-(\Upsilon-\tau)\|u_\tau\|\|v-w\|_{\tau,\Upsilon}.\tag{2.10}$$

Next, for any constants $a,b,p,q\geq 0$,

$$\big|pe^{-a}-qe^{-b}\big|\leq e^{-a}|p-q|+q\max\big\{e^{-a},e^{-b}\big\}|a-b|,$$

therefore, by (2.7), (2.4),

$$\begin{aligned}
J_2 &\leq \varkappa^+ \int_\tau^t (Bv)(x,s,t)\bigl(a^+ * |v-w|\bigr)(x,s)\,ds \\
&\quad + \varkappa^+ \int_\tau^t \max\bigl\{(Bv)(x,s,t),(Bw)(x,s,t)\bigr\}(a^+ * w)(x,s) \\
&\quad\quad \times \int_s^t \kappa_2\bigl(a^- * |v-w|\bigr)(x,r) + \kappa_1 |v-w|(x,r)\,dr\,ds \\
&\leq \varkappa^+(\Upsilon-\tau)\|v-w\|_{\tau,\Upsilon} + \varkappa^+\varkappa^-\|w\|_{\tau,\Upsilon}\|v-w\|_{\tau,\Upsilon}\int_\tau^t e^{-m(t-s)}(t-s)\,ds \\
&\leq \varkappa^+\Bigl(1+\frac{\varkappa^-}{me}\|w\|_{\tau,\Upsilon}\Bigr)(\Upsilon-\tau)\|v-w\|_{\tau,\Upsilon},
\end{aligned} \tag{2.11}$$

as $re^{-r} \leq e^{-1}$, $r \geq 0$.

For any $T_2 > T_1 \geq 0$, we define

$$\mathcal{X}^+_{T_1,T_2}(r) := \bigl\{ v \in \mathcal{X}^+_{T_1,T_2} \bigm| \|v\|_{T_1,T_2} \leq r \bigr\}, \quad r > 0.$$

Take any $\mu \geq \|u_\tau\|$. By (2.8)–(2.11), one has, for any $v, w \in \mathcal{X}^+_{\tau,\Upsilon}(r)$, $r > 0$,

$$\begin{gathered}
\bigl|(\Phi_\tau v)(x,t) - (\Phi_\tau w)(x,t)\bigr| \leq \Bigl(\mu\varkappa^- + \varkappa^+ + \frac{\varkappa^+\varkappa^-}{me} r\Bigr)(\Upsilon-\tau)\|v-w\|_{\tau,\Upsilon}, \\
\bigl|(\Phi_\tau v)(x,t)\bigr| \leq \mu + \varkappa^+ r(\Upsilon-\tau).
\end{gathered}$$

Therefore, Φ_τ will be a contraction mapping on the set $\mathcal{X}^+_{\tau,\Upsilon}(r)$ if only

$$\Bigl(\mu\varkappa^- + \varkappa^+ + \frac{\varkappa^+\varkappa^-}{me} r\Bigr)(\Upsilon-\tau) < 1 \quad \text{and} \quad \mu + \varkappa^+ r(\Upsilon-\tau) \leq r. \tag{2.12}$$

Take any $\alpha \in (0,1)$ and set

$$\begin{gathered}
C := \varkappa^-\Bigl(1+\frac{\varkappa^+}{me}\Bigr), \qquad r := \mu + \frac{\alpha\varkappa^+}{C}, \\
\Upsilon := \tau + \frac{\alpha}{Cr} = \tau + \frac{\alpha}{C\mu + \alpha\varkappa^+}.
\end{gathered} \tag{2.13}$$

Then, the second inequality in (2.12) evidently holds (and it is just an equality), and the first one may be rewritten as follow

$$\Bigl(C\mu + \varkappa^+ + \frac{\varkappa^+\varkappa^-}{me}\frac{\alpha\varkappa^+}{C}\Bigr)\frac{\alpha}{Cr} < 1,$$

or, equivalently,

$$\alpha C\mu + \alpha^2 \frac{\varkappa^+\varkappa^-}{me}\frac{\varkappa^+}{C} < C\mu. \tag{2.14}$$

To fulfill (2.14), one should choose $\alpha \in (0,1)$ such that

$$\frac{\alpha^2}{1-\alpha} < \frac{C^2\mu me}{(\varkappa^+)^2\varkappa^-}. \tag{2.15}$$

Since function $f(\alpha)=\frac{\alpha^2}{1-\alpha}$ is strictly increasing on $[0,1)$ and $f(0)=0$, one can always choose $\alpha\in(0,1)$ that satisfies (2.15).

As a result, choosing $\mu=\mu(\tau)>\|u_\tau\|$ (to include the case $u_\tau\equiv 0$) and α that satisfies (2.15), one gets that Φ_τ will be a contraction on the set $\mathcal{X}^+_{\tau,\Upsilon}(r)$ with Υ and r given by (2.13); the latter set naturally forms a complete metric space. Therefore, there exists a unique $u\in\mathcal{X}^+_{\tau,\Upsilon}(r)$ such that $\Phi_\tau u=u$. This u will be a solution to (2.1) on $[\tau,\Upsilon]$.

To fulfill the proof of the statement, one can do the following. Set $\tau:=0$, choose any $\mu_1>\|u_0\|$ and fix an α that satisfies (2.15) with $\mu=\mu_1$. One gets a solution u to (2.1) on $[0,\Upsilon_1]$ with $\Upsilon_1=\frac{\alpha}{C\mu_1+\alpha\varkappa^+}$, $\|u\|_{\Upsilon_1}\leq\mu_1+\frac{\alpha\varkappa^+}{C}$.

Iterating this scheme, take sequentially, for each $n\in\mathbb{N}$, $\tau:=\Upsilon_n$, $u_{\Upsilon_n}(x):=u(x,\Upsilon_n)$, $x\in\mathbb{R}^d$,

$$\mu_{n+1}:=\mu_n+\frac{\alpha\varkappa^+}{C}\geq\|u_{\Upsilon_n}\|.$$

Since $\mu_{n+1}>\mu_n$, the same α as before will satisfy (2.15) with $\mu=\mu_{n+1}$ as well. Then, one gets a solution u to (2.1) on $[\Upsilon_n,\Upsilon_{n+1}]$ with initial condition u_{Υ_n}, where

$$\Upsilon_{n+1}:=\Upsilon_n+\frac{\alpha}{C\mu_{n+1}+\alpha\varkappa^+},$$

and

$$\|u\|_{\Upsilon_n,\Upsilon_{n+1}}\leq\mu_{n+1}+\frac{\alpha\varkappa^+}{C}=\mu_{n+2}.$$

As a result, we will have a solution u to (2.1) on intervals $[0,\Upsilon_1]$, $[\Upsilon_1,\Upsilon_2]$, ..., $[\Upsilon_n,\Upsilon_{n+1}]$, $n\in\mathbb{N}$, where $\mu_{n+1}=\mu_1+n\frac{\alpha\varkappa^+}{C}$, and, thus,

$$\Upsilon_{n+1}:=\Upsilon_n+\frac{\alpha}{C\mu_1+(n+1)\alpha\varkappa^+}. \tag{2.16}$$

By Lemma 2.1, the r.h.s. of (2.1), will be continuous on each of constructed time-intervals, therefore, one has that u is continuously differentiable on $(0,\Upsilon_{n+1}]$ and solves (2.1) there. By (2.16),

$$\Upsilon_{n+1}:=\sum_{j=1}^{n+1}\frac{\alpha}{C\mu_1+j\alpha\varkappa^+}\to\infty,\quad n\to\infty,$$

therefore, one has a solution to (2.1) on any $[0,T]$, $T>0$.

To prove uniqueness, suppose that $v\in\mathcal{X}_T$ is a solution to (2.1) on $[0,T]$, with $v(x,0)\equiv u_0(x)$, $x\in\mathbb{R}^d$. Choose $\mu_1>\|v\|_T\geq\|u_0\|$. Since $\{\mu_n\}_{n\in\mathbb{N}}$ above is an increasing sequence, v will belong to each of sets $\mathcal{X}^+_{\Upsilon_n,\Upsilon_{n+1}}(\mu_{n+1})$, $n\geq 0$, $\Upsilon_0:=0$, considered above. Then, being solution to (2.1) on each $[\Upsilon_n,\Upsilon_{n+1}]$, v will be a fixed point for Φ_{Υ_n}. By the uniqueness of such a point, v coincides with u on each $[\Upsilon_n,\Upsilon_{n+1}]$ and, thus, on the whole $[0,T]$. □

Remark 2.3. The statement of Theorem 2.2 holds true for solutions in $L^\infty(\mathbb{R}^d)$ with $u\in\tilde{X}_T$: the proof will be mainly identical. See also [41, Theorem 4.1].

Consider the following quantity

$$\theta:=\frac{\varkappa^+-m}{\varkappa^-}\in\mathbb{R}. \tag{2.17}$$

Theorem 2.2 has a simple corollary:

Corollary 2.4. *Let $t_0 \geq 0$ be such that the solution u to* (2.1) *is a constant in space at the moment of time t_0, namely, $u(x,t_0) \equiv u(t_0) \geq 0$, $x \in \mathbb{R}^d$. Then this solution will be a constant in space for all further moments of time, more precisely,*

$$u(x,t) = u(t) = \frac{u(t_0)}{u(t_0)g_\theta(t) + \exp(-\varkappa^- \theta t)} \geq 0, \qquad x \in \mathbb{R}^d, \ t \geq t_0, \tag{2.18}$$

where

$$g_\theta(t) = \begin{cases} \dfrac{1 - \exp(-\varkappa^- \theta t)}{\theta}, & \theta \neq 0, \\ \varkappa^- t, & \theta = 0, \end{cases} \qquad t \geq t_0.$$

In particular, $u(t) \to \max\{0, \theta\}$, $t \to \infty$.

Proof. First of all, we note that in the proof of Theorem 2.2 we proved that the problem (2.1) has a unique solution. Next, straightforward calculations show that (2.18) solves (2.1) for $\tau = t_0$, that implies the first statement. The last statement is also straightforward then. □

Remark 2.5. Note that (2.18) solves the classical logistic equation, cf. (1.2):

$$\frac{d}{dt} u(t) = \varkappa^- u(t)(\theta - u(t)), \quad t > t_0, \quad u(t_0) \geq 0. \tag{2.19}$$

By Lemma 2.1, the mapping $A^+ v = \varkappa^+ a^+ * v$ defines a linear operator on $C_{ub}(\mathbb{R}^d)$, which is evidently bounded: by (2.4) and $A^+ 1 = \varkappa^+$, one has $\|A^+\| = \varkappa^+$. Then a solution u to (2.1) satisfies the following equation

$$u(x,t) = e^{-tm} e^{tA^+} u_0(x) - \int_0^t e^{-(t-s)m} e^{(t-s)A^+} \varkappa^- u(x,s)(a^- * u)(x,s)\, ds.$$

Therefore, $u(x,t) \geq 0$ implies $u(x,t) \leq e^{-tm} e^{tA^+} u_0(x)$, $x \in \mathbb{R}^d$, $t \geq 0$; and hence, by Theorem 2.2, $0 \leq u_0 \in C_{ub}(\mathbb{R}^d)$ yields

$$\|u(\cdot,t)\| \leq e^{(\varkappa^+ - m)t} \|u_0\|, \qquad t \geq 0. \tag{2.20}$$

In particular, for $m > \varkappa^+$, the solution $u(x,t)$ to (2.1) exponentially quickly in t tends to 0, uniformly in $x \in \mathbb{R}^d$.

We proceed now to show that, in fact, the solution to (2.1) is uniformly bounded in time on the whole $\mathbb{R}_+$, provided that the kernel a^- does not degenerate in a neighborhood of the origin and a^+ has an integrable decay at ∞.

Definition 2.6. Let $1\!\!1_A$ denote the indicator function of a measurable set $A \subset \mathbb{R}^d$. Recall that a sequence $f_n \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ is said to be locally uniformly convergent to an $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, if $1\!\!1_\Lambda f_n \to 1\!\!1_\Lambda f$ in $L^\infty(\mathbb{R}^d)$, $n \to \infty$, for any compact $\Lambda \subset \mathbb{R}^d$. We denote this convergence by $f_n \overset{\mathrm{loc}}{\Longrightarrow} f$. We will use the same notation to say that, for some $T > 0$ and $v_n, v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \times [0,T])$, one has $1\!\!1_\Lambda v_n \to 1\!\!1_\Lambda v$ in $L^\infty(\mathbb{R}^d \times [0,T])$, for any compact $\Lambda \subset \mathbb{R}^d$.

We start with a simple statement useful for the sequel.

Lemma 2.7. *Let $a \in L^1(\mathbb{R}^d)$, $\{f_n, f\} \subset L^\infty(\mathbb{R}^d)$, $\|f_n\| \leq C$, for some $C > 0$, and $f_n \overset{\mathrm{loc}}{\Longrightarrow} f$. Then $a * f_n \overset{\mathrm{loc}}{\Longrightarrow} a * f$.*

Proof. Let $\{a_m\} \subset C_0(\mathbb{R}^d)$ be such that $\|a_m - a\|_{L^1(\mathbb{R}^d)} \to 0$, $m \to \infty$, and denote $A_m := \operatorname{supp} a_m$. Note that, there exists $D > 0$, such that $\|a_m\|_{L^1(\mathbb{R}^d)} \leq D$, $m \in \mathbb{N}$. Next, for any compact $\Lambda \subset \mathbb{R}^d$,

$$\begin{aligned}|1\!\!1_\Lambda(x)(a_m * (f_n - f))(x)| &\leq \int_{\mathbb{R}^d} 1\!\!1_{A_m}(y) 1\!\!1_\Lambda(x)|a_m(y)||f_n(x-y) - f(x-y)|\,dy \\ &\leq \|a_m\|_{L^1(\mathbb{R}^d)} \|1\!\!1_{\Lambda_m}(f_n - f)\| \to 0, n \to \infty,\end{aligned}$$

for some compact $\Lambda_m \subset \mathbb{R}^d$. Next,

$$\begin{aligned}\|1\!\!1_\Lambda(a * (f_n - f))\| &\leq \|1\!\!1_\Lambda(a_m * (f_n - f))\| + \|1\!\!1_\Lambda((a - a_m) * (f_n - f))\| \\ &\leq D\|1\!\!1_{\Lambda_m}(f_n - f)\| + (C + \|f\|)\|a - a_m\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

and the second term may be arbitrary small by a choice of m. □

If $\kappa_2 = 0$ then the non-local competition (a^-) is not presented in (2.1). In this case the comparison principle holds for all nonnegative initial conditions in $L^\infty(\mathbb{R}^d)$ (see Theorem 3.1 below). In particular (2.18) can play a role of bound from above with $u(0) = \|u_0\|_\infty$, which yields that any solution to (2.1) is bounded globally in time.

If $\kappa_2 > 0$ then the comparison principle does not hold in general and another approach is needed to prove global boundedness of the solution, what is shown in the next theorem, which is an adaptation of [59, Theorem 1.2].

Below, $|\cdot| = |\cdot|_{\mathbb{R}^d}$ denotes the Euclidean norm in $\mathbb{R}^d$, $B_r(x)$ is a closed ball in $\mathbb{R}^d$ with the center at $x \in \mathbb{R}^d$ and the radius $r > 0$; and b_r is a volume of this ball. Consider also, for any $z \in \mathbb{Z}^d$, $q > 0$, a hypercube in $\mathbb{R}^d$ with the center at $2qz \in \mathbb{R}^d$ and the side $2q$:

$$H_q(z) := \{y \in \mathbb{R}^d \mid 2z_i q - q \leq y_i \leq 2z_i q + q, i = 1, \ldots, d\}.$$

Theorem 2.8. *Let $\kappa_2 > 0$. Suppose that, for some $q \in \bigl(0, \frac{r_0}{2\sqrt{d}}\bigr]$,*

$$a_q^+ := \sum_{z \in \mathbb{Z}^d} \sup_{x \in H_q(z)} a^+(x) < \infty \tag{2.21}$$

(e.g. let, for some $\varepsilon > 0$, $A > 0$, one have $a^+(x) \leq \frac{A}{1+|x|^{d+\varepsilon}}$, for a.a. $x \in \mathbb{R}^d$). If $\kappa_1 = 0$ we additionally suppose that there exists $r_0 > 0$ such that

$$\alpha := \inf_{|x| \leq r_0} a^-(x) > 0. \tag{2.22}$$

Then, the solution $u \geq 0$ to (2.1), with $0 \leq u_0 \in C_{ub}(\mathbb{R}^d)$, belongs to $\mathcal{X}_\infty$.

Proof. If $m \geq \varkappa^+$ then the statement is trivially followed from (2.20). Suppose that $m < \varkappa^+$ and rewrite (2.1) in the form

$$\frac{\partial}{\partial t} u(x,t) = (L_{a^+} u)(x,t) + u(x,t)\bigl(\varkappa^+ - \bigl(Gu\bigr)(x,t)\bigr), \tag{2.23}$$

where $\theta = \dfrac{\varkappa^+ - m}{\varkappa^-} > 0$ and the operator L_{a^+} acts in x and is given by (1.4).

Suppose first that $\kappa_1 = 0$. It is easily seen that $H_q(z) \subset B_{q\sqrt{d}}(2qz)$, $z \in \mathbb{Z}^d$, $q > 0$. Take any $q \leq \frac{r_0}{2\sqrt{d}}$ such that (2.21) holds, and set $r = q\sqrt{d} \leq \frac{r_0}{2}$. Define

$$v(x,t) := (1\!\!1_{B_r(0)} * u)(x,t) = \int_{B_r(x)} u(y,t)\,dy. \tag{2.24}$$

By Lemma 2.1, Theorem 2.2, (2.20), $0 \le v \in \mathcal{X}_T$, $T > 0$, and

$$\|v(\cdot,t)\| \le b_r e^{(\varkappa^+ - m)t}\|u_0\|, \quad t \ge 0.$$

Note that, by (1.4),

$$L_{a^+}v = \varkappa^+ a^+ * 1\!\!1_{B_r(0)} * u - \varkappa^+ 1\!\!1_{B_r(0)} * u = 1\!\!1_{B_r(0)} * (L_{a^+}u).$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}v(x,t) - (L_{a^+}v)(x,t) =& \Bigl(1\!\!1_{B_r(0)} * \frac{\partial}{\partial t}u\Bigr)(x,t) - \bigl(1\!\!1_{B_r(0)} * (L_{a^+}u)\bigr)(x,t) \\ =& \int_{B_r(x)} u\,(y,t)\,\bigl(\varkappa^+ - \bigl(Gu\bigr)(y,t)\bigr)\,dy. \end{aligned} \tag{2.25}$$

By (2.24), one has $\|v(\cdot,0)\| \le b_r\|u_0\|$. Set

$$M > \max\Bigl\{b_r\,\|u_0\|\,,\,\frac{\varkappa^+ - m}{\alpha\kappa_2}\Bigr\}. \tag{2.26}$$

First, we will prove that

$$\|v(\cdot,t)\| \le M, \qquad t \ge 0. \tag{2.27}$$

On the contrary, suppose that there exists $t' > 0$ such that $\|v(\cdot,t')\| > M$. By (2.24) and Lemma 2.1, $\|v(\cdot,t)\|$ is continuous in t. Next, since $\|v(\cdot,0)\| < M$, there exists $t_0 > 0$ such that $\|v(\cdot,t_0)\| = M$ and $\|v(\cdot,t)\| < M$, for all $t \in [0,t_0)$.

Consider the sequence $\{x_n\} \subset \mathbb{R}^d$ such that $v(x_n,t_0) \to M$, $n \to \infty$. Define the following functions:

$$u_n(x,t) := u(x+x_n,t), \quad v_n(x,t) := v(x+x_n,t) = (1\!\!1_{B_r(0)} * u_n)(x,t),$$

for $x \in \mathbb{R}^d, t \ge 0$. Take any $T > 0$. Evidently, $u \in C_{ub}(\mathbb{R}^d \times [0,T])$, then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x,y \in \mathbb{R}^d$, $t,s \in [0,T]$, with $|x-y|_{\mathbb{R}^d} + |t-s| < \delta$, one has $|u_n(x,t) - u_n(y,s)| = |u(x+x_n,t) - u(y+x_n,s)| < \varepsilon$. And, by (2.20),

$$\|u_n(\cdot,t)\| \le \|u(\cdot,t)\| \le e^{(\varkappa^+ - m)T}\|u_0\|, \quad n \in \mathbb{N}, t \in [0,T]. \tag{2.28}$$

Hence $\{u_n\}$ is a uniformly bounded and uniformly equicontinuous sequence of functions on $\mathbb{R}^d \times [0,T]$. Thus, by a version of the Arzelà–Ascoli Theorem, see e.g. [36, Appendix C.8], there exists a subsequence $\{u_{n_k}\}$ and a continuous function u_∞ on $\mathbb{R}^d \times [0,T]$ such that $u_{n_k} \overset{\mathrm{loc}}{\Longrightarrow} u_\infty$. Moreover, one can easily show that $u_\infty \in C_{ub}(\mathbb{R}^d \times [0,T])$. By (2.28) and Lemma 2.7, $v_{n_k} \overset{\mathrm{loc}}{\Longrightarrow} v_\infty = 1\!\!1_{B_r(0)} * u_\infty$, moreover, $v_\infty \in C_{ub}(\mathbb{R}^d \times [0,T])$.

It is easily seen that both parts of (2.25) belong to $\mathcal{X}_T$. Hence one can integrate (2.25) on $[0,t] \subset [0,T]$, namely,

$$\begin{aligned} v(x,t) = v(x,0) &+ \int_0^t (L_{a^+}v)(x,s)\,ds \\ &+ \int_0^t \int_{B_r(x)} u\,(y,s)\,\bigl(\varkappa^+ - \bigl(Gu\bigr)(y,s)\bigr)\,dy\,ds. \end{aligned} \tag{2.29}$$

Substitute $x + x_{n_k}$ instead of x into (2.29) and use twice the integration by substitution in the second integral, then one gets the same equality (2.29), but for v_{n_k}, u_{n_k} instead of v, u, respectively. Next, by Lemma 2.7 and the dominated convergence arguments, one can pass to the limit in k in the obtained equality. As a result, one get (2.29) for v_∞, u_∞ instead of v and u, respectively. Next, since $C_{ub}(\mathbb{R}^d \times [0,T]) \subset \mathcal{X}_T$, the integrands with respect to s in the left hand side (l.h.s. in the sequel) of the modified equation (2.29) (with $u_\infty, v_\infty \in \mathcal{X}_T$) will belong to $\mathcal{X}_T$ as well. As a result, v_∞ will be differentiable in t in the sense of the norm in $C_{ub}(\mathbb{R}^d)$. Finally, after differentiation, one get (2.25) back, but for v_∞, u_∞, namely,

$$\frac{\partial}{\partial t} v_\infty(x,t) - (L_{a^+} v_\infty)(x,t) = \int_{B_r(x)} u_\infty(y,t) \bigl(\varkappa^+ - \bigl(G u_\infty\bigr)(y,t)\bigr)\, dy. \tag{2.30}$$

Going back to the definition of x_n, one can see that

$$v_\infty(0,t_0) = \lim_{k\to\infty} v_{n_k}(0,t_0) = \lim_{k\to\infty} v(x_{n_k},t_0) = M, \tag{2.31}$$

whereas, for any $x \in \mathbb{R}^d$, $t \in [0,t_0)$, $v_\infty(x,t) = \lim\limits_{k\to\infty} v(x+x_{n_k},t) \le M$. Therefore, $\frac{\partial}{\partial t} v_\infty(0,t_0) \ge 0$ and, by (1.4), $(L_{a^+} v_\infty)(0,t_0) \le 0$. Then, by (2.30),

$$\int_{B_r(0)} u_\infty(y,t_0) \bigl(\varkappa^+ - \bigl(G u_\infty\bigr)(y,t_0)\bigr)\, dy \ge 0. \tag{2.32}$$

Next, the function $u_\infty(\cdot,t_0)$, by the construction above, is nonnegative. It can not be identically equal to 0 on $B_r(0)$, since otherwise, by (2.24), $v_\infty(0,t_0) = 0$ that contradicts (2.31). Hence by (2.32), the function $\varkappa^+ - \bigl(G u_\infty\bigr)(\cdot,t_0)$ cannot be strictly negative on $B_r(0)$. Thus, there exists $y_0 \in B_r(0)$ such that $\varkappa^+ \ge \bigl(G u_\infty\bigr)(y_0,t_0)$. Since $2r \le r_0$, one has that $\inf\limits_{x\in B_{2r}(0)} a^-(x) \ge \alpha$, cf. (2.22). Therefore, one can continue:

$$\begin{aligned}
\frac{\varkappa^+ - m}{\kappa_2} &\ge (a^- * u_\infty)(y_0,t_0) \ge \int_{B_{2r}(0)} a^-(y) u_\infty(y_0 - y, t_0)\, dy \\
&\ge \alpha \int_{B_{2r}(0)} u_\infty(y_0 - y, t_0)\, dy = \alpha \int_{B_{2r}(y_0)} u_\infty(y,t_0)\, dy \\
&\ge \alpha \int_{B_r(0)} u_\infty(y,t_0)\, dy = \alpha v_\infty(0,t_0) = \alpha M,
\end{aligned}$$

that contradicts (2.26). Therefore, our assumption was wrong, and (2.27) holds.

We proceed now to show that $\|u(\cdot,t)\|$ is uniformly bounded in time. By (2.24), (2.27), (2.21), one has, for $r = q\sqrt{d}$,

$$\begin{aligned}
(a^+ * u)(x,t) &= \sum_{z\in\mathbb{Z}^d} \int_{H_q(z)} a^+(y) u(x-y,t)\, dy \\
&\le \sum_{z\in\mathbb{Z}^d} \sup_{y\in H_q(z)} a^+(y) \int_{B_r(2qz)} u(x-y,t)\, dy \\
&= \sum_{z\in\mathbb{Z}^d} \sup_{y\in H_q(z)} a^+(y) \int_{B_r(x-2qz)} u(y,t)\, dy \le M a_q^+.
\end{aligned} \tag{2.33}$$

Therefore, by (2.1), (2.33), using the same arguments as for the proof of (2.20) one gets that

$$\begin{aligned} 0 \le u(x,t) &\le e^{-mt}u_0(x) + \int_0^t e^{-(t-s)m}\varkappa^+ M a_q^+ \, ds \\ &= e^{-mt}u_0(x) + \frac{\varkappa^+ M a_q^+}{m}(1-e^{-mt}) \\ &\le \max\Big\{\frac{\varkappa^+ M a_q^+}{m}, \|u_0\|\Big\}, \qquad x \in \mathbb{R}^d,\ t \ge 0. \end{aligned} \tag{2.34}$$

Suppose that $\kappa_1 > 0$. We can repeat the previous prove for $v(x,t) = u(x,t)$ and $M > \max\{\|u_0\|, \frac{\varkappa^+ - m}{\kappa_1}\}$. In this case (2.32) has the following form

$$u_\infty(0,t_0)\big(\varkappa^+ - \big(Gu_\infty\big)(0,t_0)\big) > 0,$$

where $u_\infty(0,t_0) = M$. Hence $\frac{\varkappa^+ - m}{\kappa_1} > u_\infty(0,t_0)$, that contradicts the choice of M. The proof is fulfilled. □

Remark 2.9. It should be stressed that we essentially used the uniform continuity of the solution to prove Theorem 2.8.

The following proposition shows that if $\kappa_1 = 0$ then an additional assumption on a^- (c.f. (2.22)) might be necessary for the global boundedness of the solution.

Proposition 2.10. *Let $\varkappa^+ > m$ and $\kappa_1 = 0$ ($\kappa_2 = \varkappa^-$). For $x \in \mathbb{R}$, we define*

$$\begin{aligned} u_0(x) &= 1 + k cos(\frac{\pi x}{l}), \qquad a^-(x) = \frac{1}{2}(\delta_l(x) + \delta_{-l}(x)), \\ a^+(x) &= a_\alpha(x) = \frac{2}{\alpha}\mathbb{1}_{[-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}]}(x) cos(\alpha x). \end{aligned}$$

There exist $\alpha, l > 0$ such that, for any $k > 0$, the solution to (2.1) *is globally unbounded in time, namely,*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \to \infty,\ t \to \infty.$$

Proof. Let us first note that, for any $\alpha,\ \beta > 0$, the functions $p_\beta(x) = cos(\beta x)$ and $q_\beta(x) = sin(\beta x)$ are eigenvectors of the convolution operator

$$Af(x) = (a_\alpha * f)(x),$$

with eigenvalues $\lambda_{\alpha,\beta} = \frac{4}{\alpha^2 - \beta^2} cos\frac{\pi\beta}{2\alpha}$ ($\lambda_{\alpha,\alpha} = \frac{\pi}{\alpha^2}$), namely the following equalities hold

$$(a_\alpha * p_\beta)(x) = \lambda_{\alpha,\beta} p_\beta(x), \qquad (a_\alpha * q_\beta)(x) = \lambda_{\alpha,\beta} q_\beta(x), \qquad x \in \mathbb{R}.$$

Since u_0 is $2l$ periodic, then, for all $t > 0$, u is $2l$ periodic and it satisfies

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \varkappa^+(a^+ * u)(x,t) - mu(x,t) - \frac{\varkappa^-}{2}u(x,t)\big(u(x-l,t) + u(x+l,t)\big) \\ &= \varkappa^+(a^+ * u)(x,t) - mu(x,t) - \varkappa^- u(x,t)u(x+l,t). \end{aligned}$$

We set $v(x,t) = u(x+l,t)$, $w(x,t) = u(x,t) - v(x,t)$. Then $u,\ v,\ w$ satisfy

$$\frac{\partial u}{\partial t} = \varkappa^+ a^+ * u - mu - \varkappa^- uv, \qquad \frac{\partial v}{\partial t} = \varkappa^+ a^+ * v - mv - \varkappa^- uv,$$
$$\frac{\partial w}{\partial t} = \varkappa^+ a^+ * w - mw.$$

We are looking for a solution to (2.35) in the following form $w(x,t) = X(x)T(t)$, where $T(0) = 1$,

$$\begin{cases} \dfrac{\partial T}{\partial t}(t)X(x) = \big(\varkappa^+(a_\alpha * X)(x) - mX(x)\big)T(t), \\ \qquad X(x) = 2kp_{\frac{\pi}{L}}(x) = u_0(x) - u_0(x+l). \end{cases} \tag{2.35}$$

The following equation holds

$$\begin{aligned} \varkappa^+(a_\alpha * X - mX) &= 2k\left(\varkappa^+ a_\alpha * p_{\frac{\pi}{l}} - mp_{\frac{\pi}{l}}\right) \\ &= 2k\left(\varkappa^+\lambda_{\alpha,\frac{\pi}{l}} - m\right)p_{\frac{\pi}{l}} = \left(\varkappa^+\lambda_{\alpha,\frac{\pi}{l}} - m\right)X. \end{aligned}$$

Hence

$$T(t) = e^{(\varkappa^+\lambda_{\alpha,\frac{\pi}{l}} - m)t}.$$

Since $\varkappa^+ > m$ there exist α, l such that $\varkappa^+\lambda_{\alpha,\frac{\pi}{L}} > m$. Hence $\|w(\cdot,t)\| \to \infty$, as $t \to \infty$. However

$$\|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} + \|v(\cdot,t)\|_{L^\infty(\mathbb{R})} = 2\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}.$$

Finally, one concludes $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \to \infty$, as $t \to \infty$. The proof is fulfilled. □

Under conditions of Theorem 2.8, the solution u will be uniformly continuous on $\mathbb{R}^d \times \mathbb{R}_+$, namely, the following simple proposition holds true.

Proposition 2.11. *Let u be a solution to* (2.1) *with $u_0 \in C_{ub}(\mathbb{R}^d)$, and suppose that there exists $C > 0$, such that*

$$|u(x,t)| \le C, \quad x \in \mathbb{R}^d, \ t \ge 0.$$

Then $u \in C_{ub}(\mathbb{R}^d \times \mathbb{R}_+)$. Moreover, $\|u(\cdot,t)\| \in C_{ub}(\mathbb{R}_+)$.

Proof. Being solution to (2.1), u satisfies the integral equation

$$u(x,t) = u_0(x) + \int_0^t \big(\varkappa^+(a^+ * u)(x,s) - u(x,s)\big(Gu\big)(x,s)\big)\,ds.$$

Hence for any $x, y \in \mathbb{R}^d$, $0 \le \tau < t$, one has

$$\begin{aligned} |u(x,t) - u(y,\tau)| &\le \int_\tau^t (2\varkappa^+C + 2\varkappa^-C^2 + 2mC)ds \\ &= 2(\varkappa^+ + \varkappa^-C + m)C(t-\tau), \end{aligned}$$

that fulfills the proof of the first statement. Then, the second one follows from the inequality $\big|\|u(\cdot,t)\| - \|u(\cdot,\tau)\|\big| \le \|u(\cdot,t) - u(\cdot,\tau)\|$. □

3 Around the comparison principle

The comparison principle is one of the basic tools for the study of elliptic and parabolic PDE. It is widely use for the nonlocal diffusion equation (1.6) (see e.g. [22]), however, it does not hold, in general if a^- is presented in reaction (see e.g. [3, 59] and the references therein). We will find the sufficient conditions (see (A1) and (A2) below), under which the comparison principle for the equation (2.1) does hold and which will be the basic conditions for all our further settings. Moreover, one can show a necessity of these conditions (Remark 3.7). Subsection 3.2 is devoted to the maximum principle, which is a counterpart of the comparison one for parabolic ODE. In particular, Theorem 3.10 states that graphs of two different solutions to (2.1) never touch. The last Subsection gives further technical tools which will be explored through the paper.

3.1 Comparison principle

Let $T>0$ be fixed. Define the sets $\mathcal{X}_T^1$ and $\tilde{\mathcal{X}}_T^1$ of functions from $\mathcal{X}_T$, respectively, $\tilde{\mathcal{X}}_T$, which are continuously differentiable on $(0,T]$ in the sense of the norm in $C_{ub}(\mathbb{R}^d)$, respectively, in $L^\infty(\mathbb{R}^d)$. Here and below we consider the left derivative at $t=T$ only. For any u from $\mathcal{X}_T^1$ one can define the following function

$$(\mathcal{F}u)(x,t) := \frac{\partial u}{\partial t}(x,t) - \varkappa^+(a^+ * u)(x,t) + u(x,t)\big(Gu\big)(x,t). \tag{3.1}$$

for all $t\in(0,T]$ and all $x\in\mathbb{R}^d$. Moreover, for any $u\in\tilde{\mathcal{X}}_T^1$, one can consider the function $\dfrac{\partial u}{\partial t}(\cdot,t)\in L^\infty(\mathbb{R}^d)$, for all $t\in(0,T]$. Then, one can also define (3.1), which will considered a.e. in $x\in\mathbb{R}^d$ now.

Theorem 3.1. *Let there exist $c>0$, such that*

$$\varkappa^+a^+(x)\geq c\kappa_2 a^-(x), \quad a.a.\ x\in\mathbb{R}^d. \tag{3.2}$$

Let $T\in(0,\infty)$ be fixed and functions $u_1,u_2\in\mathcal{X}_T^1$ be such that, for any $(x,t)\in\mathbb{R}^d\times(0,T]$,

$$(\mathcal{F}u_1)(x,t)\leq(\mathcal{F}u_2)(x,t), \tag{3.3}$$

$$0\leq u_1(x,t), \qquad 0\leq u_2(x,t)\leq c, \qquad u_1(x,0)\leq u_2(x,0). \tag{3.4}$$

Then $u_1(x,t)\leq u_2(x,t)$, for all $(x,t)\in\mathbb{R}^d\times[0,T]$. In particular, $u_1\leq c$.

Proof. Define the following function

$$f(x,t) := (\mathcal{F}u_2)(x,t) - (\mathcal{F}u_1)(x,t)\geq 0, \quad x\in\mathbb{R}^d, t\in(0,T], \tag{3.5}$$

cf. (3.3). We set

$$K = m + \varkappa^-\|u_1\|_T + \kappa_1 c, \tag{3.6}$$

and consider a linear mapping

$$\begin{aligned} F(t,w) := Kw - mw + \varkappa^+(a^+ * w) - \kappa_2 w(a^- * u_1)& \\ - \kappa_2 u_2(a^- * w) - \kappa_1 u_1 w - \kappa_1 u_2 w + e^{Kt} f,& \end{aligned} \tag{3.7}$$

for $w\in\mathcal{X}_T$. By (3.4), (3.5), (3.6), (3.2), (2.4), $w\geq0$ implies

$$F(t,w)\geq(\varkappa^+a^+ - c\kappa_2 a^-)*w\geq 0. \tag{3.8}$$

Define also the function

$$v(x,t) := e^{Kt}(u_2(x,t)-u_1(x,t)), \quad x\in\mathbb{R}^d, t\in[0,T].$$

Clearly, $v\in\mathcal{X}_T^1$, and it is straightforward to check that

$$F(t,v(s,t)) = \frac{\partial}{\partial t}v(x,t), \tag{3.9}$$

for all $x \in \mathbb{R}^d$, $t \in (0,T]$. Therefore, v solves the following integral equation in $C_{ub}(\mathbb{R}^d)$:

$$\begin{cases} v(x,t) = v(x,0) + \displaystyle\int_0^t F(s, v(x,s))ds, & (x,t) \in \mathbb{R}^d \times (0,T], \\ v(x,0) = u_2(x,0) - u_1(x,0), & x \in \mathbb{R}^d, \end{cases} \tag{3.10}$$

where $v(x,0) \geq 0$, by (3.4).

Consider also another integral equation in $C_{ub}(\mathbb{R}^d)$:

$$\tilde{v}(x,t) = (\Psi\tilde{v})(x,t) \tag{3.11}$$

where

$$(\Psi w)(x,t) := v(x,0) + \int_0^t \max\{F(s, w(x,s)), 0\}\, ds, \qquad w \in \mathcal{X}_T. \tag{3.12}$$

It is easily seen that $w \in \mathcal{X}_T^+$ yields $\Phi w \in \mathcal{X}_T^+$. Next, for any $\tilde{T} < T$ and for any $w_1, w_2 \in \mathcal{X}_{\tilde{T}}^+$, one gets from (3.7), (3.12), that

$$\begin{aligned} \|\Psi w_1 - \Psi w_2\|_{\tilde{T}} &\leq \tilde{T}\big(K + m + \varkappa^+ + \varkappa^-(\|u_1\|_T + c)\big)\|w_2 - w_1\|_{\tilde{T}} \\ &:= q_T\tilde{T}\|w_2 - w_1\|_{\tilde{T}}, \end{aligned} \tag{3.13}$$

where we used the elementary inequality $|\max\{a,0\} - \max\{b,0\}| \leq |a-b|$, $a, b \in \mathbb{R}$. Therefore, for $\tilde{T} < (q_T)^{-1}$, Ψ is a contraction on $\mathcal{X}_{\tilde{T}}^+$. Thus, there exists a unique solution to (3.11) on $[0,\tilde{T}]$. In the same way, the solution can be extended on $[\tilde{T}, 2\tilde{T}], [2\tilde{T}, 3\tilde{T}], \ldots$, and therefore, on the whole $[0,T]$. By (3.11), (3.12),

$$\tilde{v}(x,t) \geq v(x,0) \geq 0, \tag{3.14}$$

hence, by (3.8), (3.12),

$$\tilde{v}(x,t) = v(x,0) + \int_0^t F(s, \tilde{v}(x,s))\, ds =: \Xi(\tilde{v})(x,t). \tag{3.15}$$

Since $\tilde{v} \in \mathcal{X}_T$, (3.15) implies that $\tilde{v}$ is a solution to (3.10) as well. The same estimate as in (3.13) shows that Ξ is a contraction on $\mathcal{X}_{\tilde{T}}$, for small enough $\tilde{T}$. Thus $\tilde{v} = v$ on $\mathbb{R}^d \times [0,\tilde{T}]$, and one continue this consideration as before on the whole $[0,T]$. Then, by (3.14), $v(x,t) \geq 0$ on $\mathbb{R}^d \times [0,T]$, that yields the statement. □

The weaker inequality between a^+ and a^- could be assumed in Theorem 3.1. In this case global in time bound on u_1 is a priory required, as one can see in the following theorem.

Theorem 3.2. *Let there exist $d > 0$, such that*

$$\varkappa^+ a^+(x) \geq d\kappa_2 a^-(x), \quad a.a.\ x \in \mathbb{R}^d.$$

Let $T \in (0,\infty)$ be fixed and functions $u_1, u_2 \in \mathcal{X}_T^1$ be such that, for any $(x,t) \in \mathbb{R}^d \times (0,T]$,

$$\begin{gathered} (\mathcal{F}u_1)(x,t) \leq (\mathcal{F}u_2)(x,t), \\ 0 \leq u_1(x,t) \leq d, \qquad 0 \leq u_2(x,t) \leq c, \qquad u_1(x,0) \leq u_2(x,0), \end{gathered}$$

where $c > 0$. Then $u_1(x,t) \leq u_2(x,t)$, for all $(x,t) \in \mathbb{R}^d \times [0,T]$.

Proof. The prove is similar to the prove of Theorem 3.1. The only difference is that one need to define

$$F(t,w) := Kw - mw + \varkappa^+(a^+ * w) - \kappa_2 w(a^- * u_2) \\ - \kappa_2 u_1(a^- * w) - \kappa_1 u_1 w - \kappa_1 u_2 w + e^{Kt} f,$$

where $K = m + \varkappa^- c + \kappa_1 d$. □

Remark 3.3. The previous theorems hold true in $\tilde{\mathcal{X}}_T^1$. Here and below, for the L^∞-case, one can assume that (3.3), (3.4) hold almost everywhere in x only.

From the proof of Theorem 3.1, one can see that we used the fact that u_1, u_2 belong to $\mathcal{X}_T^1$ to ensure that (3.9) implies (3.10) only. For technical reasons we will need to extend the result of Theorem 3.1 for a wider class of functions. Naturally, to get (3.10) from (3.9), it is enough to assume absolute continuity of $v(x,t)$ in t, for a fixed x. Consider the corresponding statement.

For any $T \in (0,\infty]$, define the set $\mathscr{D}_T$ of all functions $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$, such that, for all $t \in [0,T)$, $u(\cdot,t) \in C_{ub}(\mathbb{R}^d)$, and, for all $x \in \mathbb{R}^d$, the function $f(x,t)$ is absolutely continuous in t on $[0,T)$. Then, for any $u \in \mathscr{D}_T$, one can define the function (3.1), for all $x \in \mathbb{R}^d$ and a.a. $t \in [0,T)$.

Proposition 3.4. *The statement of Theorem 3.1 remains true, if we assume that $u_1, u_2 \in \mathscr{D}_T$ and, for any $x \in \mathbb{R}^d$, the inequality* (3.3) *holds for a.a. $t \in (0,T)$ only.*

Proof. One can literally repeat the proof of Theorem 3.1; for any $x \in \mathbb{R}^d$, the function (3.5) and the mapping (3.7) will be defined for a.a. $t \in (0,T)$ now (and it will not be a mapping on $\mathcal{X}_T$, of course). Similarly, (3.8) and (3.9) hold, for all x and a.a. t. However, for any $x \in \mathbb{R}^d$, one gets that (3.10) holds still for all $t \in [0,T]$. Hence, the rest of the proof remains the same, stress that, in general, $F(t)v \notin \mathcal{X}_T$, whereas $\Xi(v) \in \mathcal{X}_T$, cf. (3.15). □

The standard way to use Theorem 3.1 is to take u_1 and u_2 which solve (2.1), thus, $\mathcal{F}u_1 = \mathcal{F}u_2 = 0$, and (3.3) holds. Then Theorem 3.1 gives a comparison between these solutions provided that there exists a comparison between the initial conditions. However, to do this, one needs to know *à priori* that $u_2(x,t) \le c$. For example, one can demand that c is not smaller than the constant in the r.h.s. of (2.34). Another possibility is to compare the solution to (2.1) with the solution to its homogeneous version (2.19) (with $t_0 = 0$).

Namely, let (3.2) hold, $0 < v \le c$, and, cf. (2.18),

$$\psi(t,v) := \frac{v}{v g_\theta(t) + \exp(-\theta\varkappa^- t)} \ge 0,$$
$$g_\theta(t) := \lim_{y\to\theta} \frac{1-\exp(-y\varkappa^- t)}{y} \ge 0.$$

It is easily seen that, for $\theta \le 0$, $\psi(t,v)$ decreases monotonically to 0 on $t \in [0,\infty)$: exponentially fast, for $\theta < 0$, and linearly fast, for $\theta = 0$. In particular, $\psi(t,v) \le v \le c$, $t \ge 0$. As a result,

- if $\varkappa^+ \le m$ and $0 \le u_0 \in C_{ub}(\mathbb{R}^d)$ be such that $\|u_0\| \le c$, then $\|u(\cdot,t)\| \le \psi\bigl(t,\|u_0\|\bigr)$. In particular, u converges to 0 uniformly in space as $t \to \infty$.

Next, for $\theta > 0$, the function $\psi(t,v)$ increases monotonically to θ on $t \in [0,\infty)$, if $v < \theta$; and it decreases monotonically to θ, if $v > \theta$, and, clearly, $\psi(t) \equiv \theta$, if $v = \theta$. Therefore, if (3.2) holds with $c > \theta$ and $0 < \|u_0\| \le c$ then $\psi(t,\|u_0\|) \le \|u_0\| \le c$, and therefore, $\|u(\cdot,t)\| \le \psi\bigl(t,\|u_0\|\bigr) \to \theta$, $t \to \infty$. Set also $\inf_{\mathbb{R}^d} u_0(x) =: \beta \ge 0$, then one can apply the comparison principle to the functions $u_1 = \psi(t,\beta)$ and $u_2 = u$. (Note that $\psi(t,0) = 0$.) As a result,

- if $\varkappa^+ > m$ and $0 \le u_0 \in C_{ub}(\mathbb{R}^d)$ be such that $0 < \|u_0\| \le c$, then $\psi(t,\beta) \le u(x,t) \le \psi\bigl(t,\|u_0\|\bigr)$, $x \in \mathbb{R}^d$, $t \ge 0$, where $\beta = \inf\limits_{\mathbb{R}^d} u_0(x) \ge 0$. In particular, if $\beta > 0$ then u converges to θ exponentially fast as $t \to \infty$ and uniformly in space.

Consider the case in which (3.2) holds with $c \ge \theta$ and $\|u_0\| \le \theta$, in more details. Then, one can set $u_2 \equiv \theta$ (that is a solution to (2.1)), and $\|u(\cdot,t)\| \le \theta = \psi(t,\theta)$. Of course, for this case it is enough to have (3.2) with $c = \theta$ only. The latter constitutes the following basic assumptions for the most part of our further results:

$$\varkappa^+ > m, \tag{A1}$$

$$\varkappa^+ a^+(x) \ge \kappa_2 \theta a^-(x), \quad \text{a.a. } x \in \mathbb{R}^d. \tag{A2}$$

Proposition 3.5. *Suppose that* (A1) *and* (A2) *hold. Let* $0 \le u_0 \in C_{ub}(\mathbb{R}^d)$ *be an initial condition to* (2.1) *and* $u \in \mathcal{X}_T$ *be the corresponding solutions on any* $[0,T]$, $T > 0$. *Suppose that* $0 \le u_0(x) \le \theta$, $x \in \mathbb{R}^d$. *Then* $u \in \mathcal{X}_\infty$, *with* $\|u\|_\infty \le \theta$.

Let $v_0 \in C_{ub}(\mathbb{R}^d)$ *be another initial condition to* (2.1) *such that* $u_0(x) \le v_0(x) \le \theta$, $x \in \mathbb{R}^d$*; and* $v \in \mathcal{X}_\infty$ *be the corresponding solution. Then*

$$u(x,t) \le v(x,t), \quad x \in \mathbb{R}^d, t \ge 0.$$

If, additionally, $\beta := \inf\limits_{x\in\mathbb{R}^d} u_0(x) > 0$*, then*

$$\frac{\beta\theta}{\beta + (\theta - \beta)\exp(-\theta\varkappa^- t)} \le u(x,t) \le \theta, \quad x \in \mathbb{R}^d, t \ge 0. \tag{3.16}$$

In particular,

$$\|u(\cdot,t) - \theta\| \le \frac{\theta(\theta-\beta)}{\beta}\exp(-\theta\varkappa^- t), \quad t \ge 0.$$

Proof. The first two parts were proved above; note that $\theta\varkappa^- = \varkappa^+ - m$. The last one is followed from the definition of the function ψ above and the estimate for the difference between low and upper bounds in (3.16). □

Remark 3.6. The same result may be formulated for $\tilde{\mathcal{X}}_T$ and $\tilde{\mathcal{X}}_\infty$. All inequalities will hold true almost everywhere only.

We did not consider all possible relations between c, $\theta > 0$, and $\|u_0\|$. In particular, the previous-type considerations do not cover the situation in which (3.2) holds with $c < \theta$. In such a case, the solution to (2.19) (with $t_0 = 0$) can not be considered as a function u_2 in Theorem 3.1 since that solution tends to θ as $t \to \infty$, hence, (3.4) will not hold. This situation remains open.

Another case, which is not covered by the comparison method is the following: let $\theta > 0$, i.e. (A1) holds, and $\|u_0\| > c$. However, it may be analyzed using stability arguments provided that $c \ge \theta$, the latter evidently implies (A2).
We set, cf. (3.19) below,

$$J_\theta(x) := \varkappa^+ a^+(x) - \kappa_2 \theta a^-(x), \quad x \in \mathbb{R}^d. \tag{3.17}$$

Next, denote the r.h.s. of (2.1) by $H(u)$. Recall, that $H(\theta) = H(0) = 0$, hence, $u^* \equiv \theta$ and $u_* \equiv 0$ are stationary solutions to (2.1). Consider the stability property of these solution. To do this, find the linear operator $H'(u)$ on $C_{ub}(\mathbb{R}^d)$: for $v \in C_{ub}(\mathbb{R}^d)$,

$$\begin{aligned} H'(u)v &= \frac{d}{ds} H(u+sv)\bigg|_{s=0} \\ &= \varkappa^+(a^+ * v) - mv - \kappa_2 v(a^- * u) - \kappa_2 u(a^- * v) - 2\kappa_1 uv. \end{aligned} \tag{3.18}$$

Therefore, by (3.17),

$$H'(\theta)v = \varkappa^+(a^+ * v) - mv - \kappa_2\theta v - \kappa_2\theta(a^- * v) - 2\kappa_1\theta v = J_\theta * v - (m + 2\kappa_1\theta + \kappa_2\theta)v.$$

By (3.17), $\int_{\mathbb{R}^d} J_\theta(x)\,dx = \varkappa^+ - \kappa_2\theta$, thus, the spectrum $\sigma(A)$ of the operator $Av := J_\theta * v$ on $C_{ub}(\mathbb{R}^d)$ is a subset of $\{z \in \mathbb{C} \mid |z| \le \varkappa^+ - \kappa_2\theta\}$. Therefore,

$$\sigma(H'(\theta)) \subset \{z \in \mathbb{C} \mid |z + m + 2\kappa_1\theta + \kappa_2\theta| \le \varkappa^+ - \kappa_2\theta\}.$$

If (A1) holds then $\sigma(H'(\theta)) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$. Hence, by e.g. [27, Chapter VII], $u^* \equiv \theta$ is uniformly and asymptotically stable solution, in the sense of Lyapunov, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any solution $u \in C_{ub}(\mathbb{R}^d)$ to (2.1) and for all $t_1 \ge 0$, the inequality $\|u(\cdot, t_1) - \theta\| < \delta$ implies that, for any $t \ge t_1$, $\|u(\cdot, t) - \theta\| < \varepsilon$; and, for some $\delta_0 > 0$, the inequality $\|u(\cdot, t_1) - \theta\| < \delta_0$ yields $\lim\limits_{t\to\infty} \|u(\cdot, t) - \theta\| = 0$. In particular, it works if $\theta < \|u_0\| \le \theta + \delta_0$. Moreover, it is possible to show that $u^* \equiv \theta$ is a globally asymptotically (exponentially) stable solution to (2.1), that means, in particular, that $\|u_0\| > \theta$ may be arbitrary.

Note also, that, by (3.18), $H'(0)v = \varkappa^+(a^+ * v) - mv$. If (A1) holds, then the operator $H'(0)$ has an eigenvalue $\varkappa^+ - m > 0$ whose corresponding eigenfunctions will be constants on $\mathbb{R}^d$. Therefore $\sigma(H'(0))$ has points in the right half-plane and since $H''(0)$ exists, one has, again by [27, Chapter VII], that $u_* \equiv 0$ is unstable, i.e. there exists a solution u such that $\inf\limits_{\mathbb{R}^d} |u(x,t)| \ge \varepsilon$, for some $\varepsilon > 0$, for all $x \in \mathbb{R}^d$ and for all $t \ge t_0 = t_0(\varepsilon)$.

Remark 3.7. The condition (3.2) is the necessary one to have a comparison principle for non-negative (essentially) bounded by the constant c solutions to (2.1), provided that $c \ge \theta$. To show this, consider, for simplicity, the case $c = \theta$. Let the condition (A2) fails in a ball $B_r(y_0)$ only, $r > 0$, $y_0 \in \mathbb{R}^d$, i.e. $J_\theta(x) < 0$, for a.a. $x \in B_r(y_0)$, where J_θ is given by (3.17). Take any $y \in B_r(y_0)$ with $\frac{r}{4} < |y - y_0| < \frac{3r}{4}$, then $y_0 \notin B_{\frac{r}{4}}(y)$ whereas $B_{\frac{r}{4}}(y) \subset B_r(y_0)$. Take $u_0 \in C_{ub}(\mathbb{R}^d)$ such that $u_0(x) = \theta$, $x \in \mathbb{R}^d \setminus B_{\frac{r}{4}}(y_0 - y)$, and $u_0(x) < \theta$, $x \in B_{\frac{r}{4}}(y_0 - y)$. Since $\int_{\mathbb{R}^d} J_\theta(x)\,dx = \varkappa^+ - \kappa_2\theta = m + \kappa_1\theta$, one has

$$\begin{aligned} \frac{\partial u}{\partial t}(y_0, 0) &= -(m + \kappa_1\theta)\theta + \varkappa^+(a^+ * u)(y_0, 0) - \kappa_2\theta(a^- * u)(y_0, 0) \\ &= (J_\theta * u)(y_0, 0) - (\varkappa^+ - \kappa_2\theta)\theta = (J_\theta * (u_0 - \theta))(y_0) \\ &= \int_{B_{\frac{r}{4}}(y)} J_\theta(x)(u_0(y_0 - x) - \theta)\,dx > 0, \end{aligned}$$

Therefore, $u(y_0, t) > u(y_0, 0) = \theta$, for small enough $t > 0$, and hence, the statement of Proposition 3.5 does not hold in this case. The similar counterexample may be considered if (3.2) fails, for $c > \theta$. Note that the case $c < \theta$ is again unclear.

3.2 Maximum principle

The maximum principle is a 'standard counterpart' of the comparison principle, see e.g. [20].

We will present sufficient conditions that solutions to (2.1) never reach at positive times the stationary values θ and 0, provided that the corresponding initial conditions were not these constants. Moreover, we will prove the so-called strong maximum principle (Theorem 3.10), cf. e.g. [22].

Through the rest of the paper we will suppose that (A1), (A2) hold and $\theta > 0$ is given by (2.17). Under these assumptions, for any $q \in (0, \theta]$, one can generalize the function (3.17) as follows

$$\begin{aligned} J_q(x) : &= \varkappa^+ a^+(x) - q\kappa_2 a^-(x), \\ &\geq \varkappa^+ a^+(x) - \theta\kappa_2 a^-(x) \geq 0, \qquad x \in \mathbb{R}^d. \end{aligned} \tag{3.19}$$

since (A2) holds.

Definition 3.8. For $\theta > 0$, given by (2.17), consider the following sets

$$U_\theta := \{f \in C_{ub}(\mathbb{R}^d) \mid 0 \leq f(x) \leq \theta,\ x \in \mathbb{R}^d\}, \tag{3.20}$$

$$L_\theta := \{f \in L^\infty(\mathbb{R}^d) \mid 0 \leq f(x) \leq \theta,\ \text{for a.a. } x \in \mathbb{R}^d\}. \tag{3.21}$$

We introduce also the following assumption:

$$\text{there exists } \rho, \delta > 0 \text{ such that } a^+(x) \geq \rho, \ \text{for a.a. } x \in B_\delta(0). \tag{A3}$$

Prove that then the solutions to (2.1) (or, equivalently, (2.23)) are strictly positive; this is quite common feature of linear parabolic equations, however, in general, it may fail for nonlinear ones.

Proposition 3.9. *Let* (A1), (A2), (A3) *hold. Let $u_0 \in U_\theta$, $u_0 \not\equiv 0$, $u_0 \not\equiv \theta$, be the initial condition to* (2.1), *and $u \in \mathcal{X}_\infty$ be the corresponding solution. Then*

$$u(x,t) > \inf_{\substack{y \in \mathbb{R}^d \\ s > 0}} u(y,s) \geq 0, \qquad x \in \mathbb{R}^d, t > 0.$$

Proof. By Theorem 2.2 and Proposition 3.5, $0 \leq u(x,t) \leq \theta$, $x \in \mathbb{R}^d$, $t \geq 0$. Then, by (2.23),

$$\frac{\partial u}{\partial t}(x,t) - (L_{a^+}u)(x,t) \geq 0. \tag{3.22}$$

Prove that, under (3.22), u cannot attain its infimum on $\mathbb{R}^d \times (0,\infty)$ without being a constant. Indeed, suppose that, for some $x_0 \in \mathbb{R}^d$, $t_0 > 0$,

$$u(x_0,t_0) \leq u(x,t), \quad x \in \mathbb{R}^d, t > 0. \tag{3.23}$$

Then, clearly,

$$\frac{\partial u}{\partial t}(x_0,t_0) = 0, \tag{3.24}$$

and (3.22) yields $(L_{a^+}u)(x_0,t_0) \leq 0$. On the other hand, (3.23) and (1.4) imply $(L_{a^+}u)(x_0,t_0) \geq 0$. Therefore,

$$\int_{\mathbb{R}^d} a^+(x_0 - y)(u(y,t_0) - u(x_0,t_0))\, dy = 0. \tag{3.25}$$

Then, by (A3), for all $y \in B_\delta(x_0)$,

$$u(y, t_0) = u(x_0, t_0). \tag{3.26}$$

By the same arguments, for an arbitrary $x_1 \in \partial B_\delta(x_0)$, we obtain (3.26), for all $y \in B_\delta(x_1)$. Hence, (3.26) holds on $B_{2\delta}(x_0)$, and so on. As a result, (3.26) holds, for all $y \in \mathbb{R}^d$, thus $u(\cdot, t_0)$ is a constant. Then, considering (2.1) at (x_0, t_0), and taking into account (3.24), one gets $u(x_0, t_0)\bigl(\varkappa^+ - \bigl(Gu\bigr)(x_0, t_0)\bigr)= 0$ with $u(x, t_0) = u(x_0, t_0)$, $x \in \mathbb{R}^d$; cf. (2.19). By (3.23), $u(x_0, t_0) = \theta \geq \sup_{y\in\mathbb{R}^d, s>0} u(y, s)$ implies $u \equiv \theta$, that contradicts $u_0 \not\equiv \theta$. Hence $u(x, t_0) = u(x_0, t_0) = 0$, $x \in \mathbb{R}^d$. Then, by (2.18), $u(x, t) = 0$, $x \in \mathbb{R}^d$, $t \geq t_0$. And now one can consider the reverse time in (2.1) starting from $t = t_0$. Namely, we set $w(x, t) := u(x, t_0 - t)$, $t \in [0, t_0]$, $x \in \mathbb{R}^d$. Then $w(x, 0) = v(t_0) = 0$, $x \in \mathbb{R}^d$, and

$$\frac{\partial w}{\partial t}(x, t) = w(x, t)\bigl(Gw\bigr)(x, t) - \varkappa^+(a^+ * w)(x, t). \tag{3.27}$$

The equation (3.27) has a unique classical solution in $C_{ub}(\mathbb{R}^d)$ on $[0, t_0]$. Indeed, if $w_1, w_2 \in \mathcal{X}_{t_0}$ both solve (3.27), then the difference $w_2 - w_1$ is a solution to the following linear equation

$$\begin{aligned}\frac{\partial h}{\partial t}(x, t) = mh(x, t) - \varkappa^+(a^+ * h)(x, t) + \kappa_2 h(x, t)(a^- * w_2)(x, t)\\ + \kappa_2 w_1(x, t)(a^- * h)(x, t) + \kappa_1 h(x, t)\bigl(w_2(x, t) + w_1(x, t)\bigr),\end{aligned} \tag{3.28}$$

with $h(x, 0) = 0$, $x \in \mathbb{R}^d$. The r.h.s. of (3.28), for any $w_1, w_2 \in \mathcal{X}_{t_0}$, is a bounded linear operator on $C_{ub}(\mathbb{R}^d)$, therefore, there exists a unique solution to (3.28), hence, $h \equiv 0$. As a result, $w_1 \equiv w_2$. Since $w \equiv 0$ satisfies (3.27) with the initial condition above, one has $u(x, t_0 - t) = 0$, $t \in [0, t_0]$, $x \in \mathbb{R}^d$. Hence, $u(\cdot, t) \equiv 0$, for all $t \geq 0$, that contradicts $u_0 \not\equiv 0$. Thus, the initial assumption was wrong, and (3.23) can not hold. □

In contrast to the case of the infimum, the solution to (2.1) may attain its supremum but not the value θ. One can prove this under a modified version of (A3): suppose that, cf. (3.19),

$$\begin{gathered}\text{there exists } \rho, \delta > 0, \text{ such that}\\ J_\theta(x) = \varkappa^+ a^+(x) - \kappa_2 \theta a^-(x) \geq \rho, \text{ for a.a. } x \in B_\delta(0).\end{gathered} \tag{A4}$$

As a matter of fact, under (A4), a much stronger statement than unattainability of θ does hold.

Theorem 3.10. *Let* (A1), (A2), (A4) *hold. Let $u_1, u_2 \in \mathcal{X}_\infty$ be two solutions to* (2.1), *such that $u_1(x, t) \leq u_2(x, t) \leq \theta$, $x \in \mathbb{R}^d$, $t \geq 0$. Then either $u_1(x, t) = u_2(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$ or $u_1(x, t) < u_2(x, t)$, $x \in \mathbb{R}^d$, $t > 0$.*

Proof. Let $u_1(x, t) \leq u_2(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$, and suppose that there exist $t_0 > 0$, $x_0 \in \mathbb{R}^d$, such that $u_1(x_0, t_0) = u_2(x_0, t_0)$. Define $w := u_2 - u_1 \in \mathcal{X}_\infty$. Then $w(x, t) \geq 0$ and $w(x_0, t_0) = 0$, hence $\frac{\partial}{\partial t} w(x_0, t_0) = 0$. Since both u_1 and u_2 solve (2.1), one easily gets that w satisfies the following linear equation

$$\begin{aligned}\frac{\partial}{\partial t} w(x, t) = (J_\theta * w)(x, t) + \kappa_2(\theta - u_1(x, t))(a^- * w)(x, t)\\ - w(x, t)\bigl(\kappa_1\bigl(u_2(x, t) + u_1(x, t)\bigr) + \kappa_2(a^- * u_2)(x, t) + m\bigr);\end{aligned} \tag{3.29}$$

or, at the point (x_0, t_0), we will have

$$0 = (J_\theta * w)(x_0, t_0) + \kappa_2(\theta - u_1(x_0, t_0))(a^- * w)(x_0, t_0). \tag{3.30}$$

Since the both summands in (3.30) are nonnegative, one has $(J_\theta * w)(x_0, t_0) = 0$. Then, by (A4), we have that $w(x, t_0) = 0$, for all $x \in B_\delta(x_0)$. Using the same arguments as in the proof of Proposition 3.9, one gets that $w(x, t_0) = 0$, $x \in \mathbb{R}^d$. Then, by Corollary 2.4, $w(x, t) = 0$, $x \in \mathbb{R}^d$, $t \geq t_0$. Finally, one can reverse the time in the linear equation (3.29) (cf. the proof of Proposition 3.9), and the uniqueness arguments imply that $w \equiv 0$, i.e. $u_1(x, t) = u_2(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$. The statement is proved. □

By choosing $u_2 \equiv \theta$ in Theorem 3.10, we immediately get the following

Corollary 3.11. *Let* (A1)*,* (A2)*,* (A4) *hold. Let* $u_0 \in U_\theta$*,* $u_0 \not\equiv \theta$*, be the initial condition to* (2.1)*, and* $u \in \mathcal{X}_\infty$ *be the corresponding solution. Then* $u(x, t) < \theta$*,* $x \in \mathbb{R}^d$*,* $t > 0$*.*

3.3 Further toolkits

We start with the proof that any solution to (2.1) is locally stable with respect to the locally uniform convergence of Definition 2.6, provided that (3.2) holds. This stability is very 'weak', for example, $u_* \equiv 0$, being unstable solution (see Subsection 3.1 above), will be still locally stable.

Theorem 3.12. *Let* (A1)*,* (A2) *hold. Let* $T > 0$ *be fixed. Consider a sequence of functions* $u_n \in \mathcal{X}_T$ *which are solutions to 2.1 with uniformly bounded initial conditions:* $u_n(\cdot, 0) \in U_\theta$*,* $n \in \mathbb{N}$*. Let* $u \in \mathcal{X}_T$ *be a solution to* (2.1) *with initial condition* $u(\cdot, 0)$ *such that* $u_n(\cdot, 0) \overset{\text{loc}}{\Longrightarrow} u(\cdot, 0)$*. Then* $u_n(\cdot, t) \overset{\text{loc}}{\Longrightarrow} u(\cdot, t)$*, uniformly in* $t \in [0, T]$*.*

Proof. It is easily seen that $u(\cdot, 0) \in U_\theta$. By Proposition 3.5, $u_n(\cdot, t), u(\cdot, t) \in U_\theta$, $n \in \mathbb{N}$, for any $t \geq 0$. We define, for any $n \in \mathbb{N}$, the following functions on $\mathbb{R}^d$:

$$\overline{u}_n(x, 0) := \max\{u_n(x, 0), u(x, 0)\}, \qquad \underline{u}_n(x, 0) := \min\{u_n(x, 0), u(x, 0)\}.$$

Then, clearly, $0 \leq \underline{u}_n(x, 0) \leq u(x, 0) \leq \overline{u}_n(x, 0) \leq \theta$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$. Hence the corresponding solutions $\overline{u}_n(x, t)$, $\underline{u}_n(x, t)$ to (2.1) belongs to U_θ as well. By Theorem 3.5, one has

$$\underline{u}_n(x, t) \leq u(x, t) \leq \overline{u}_n(x, t), \quad x \in \mathbb{R}^d, t \in [0, T].$$

In the same way, one gets $\underline{u}_n(x, t) \leq u_n(x, t) \leq \overline{u}_n(x, t)$ on $\mathbb{R}^d \times [0, T]$. Therefore, it is enough to prove that $\overline{u}_n$ and $\underline{u}_n$ converge locally uniformly to u.

Prove that $\overline{u}_n \overset{\text{loc}}{\Longrightarrow} u$. For any $n \in \mathbb{N}$, the function $h_n(\cdot, t) = \overline{u}_n(\cdot, t) - u(\cdot, t) \in U_\theta$, $t \geq 0$, satisfies the equation $\frac{\partial}{\partial t} h_n = A_n h_n$ with $h_{n,0}(x) := h_n(x, 0) = \overline{u}_n(x, 0) - u(x, 0) \geq 0$, $x \in \mathbb{R}^d$, where, for any $0 \leq h \in \mathcal{X}_T$,

$$A_n h := -mh + \varkappa^+(a^+ * h) - \kappa_2 h(a^- * \overline{u}_n) - \kappa_2 u(a^- * h) - \kappa_1 h(u + \overline{u}_n).$$

For any u_n and u, A_n is a bounded linear operator on $C_{ub}(\mathbb{R}^d)$, therefore, $h_n(x, t) = (e^{tA_n} h_{n,0})(x)$, $x \in \mathbb{R}^d$, $t \in [0, T]$. Since $u \geq 0$, one has that, for any $0 \leq h \in \mathcal{X}_T$, $(A_n h)(x, t) \leq (Ah)(x, t)$, $x \in \mathbb{R}^d$, $t \in [0, T]$, where a bounded linear operator A is given on $C_{ub}(\mathbb{R}^d)$ by

$$Ah := \varkappa^+(a^+ * h) - \kappa_2 u(a^- * h) - \kappa_1 uh.$$

Next, the series expansions for e^{tA_n} and e^{tA} converge in the topology of norms of operator on the space $C_{ub}(\mathbb{R}^d)$. Then, for any $n \in \mathbb{N}$, and for $x \in \mathbb{R}^d$, $t \in [0, T]$,

$$h_n(x, t) = (e^{tA_n} h_{n,0})(x) \leq (e^{TA} h_{n,0})(x) = \sum_{m=0}^{\infty} \frac{T^m}{m!} A^m h_{n,0}, \tag{3.31}$$

and, moreover, for any $\varepsilon > 0$ one can find $M = M(\varepsilon) \in \mathbb{N}$, such that we get from (3.31) that

$$h_n(x,t) \leq \sum_{m=0}^{M} \frac{T^m}{m!} A^m h_{n,0}(x) + \varepsilon\theta, \quad x \in \mathbb{R}^d, t \in [0,T]. \tag{3.32}$$

as $h_{n,0} \in U_\theta$, $n \in \mathbb{N}$. Finally, the assumptions of the statement yield that $h_{n,0} \overset{\mathrm{loc}}{\Longrightarrow} 0$. Then, by (3.31) and Lemma 2.7, $h_n(x,t) \overset{\mathrm{loc}}{\Longrightarrow} 0$ uniformly in $t \in [0,T]$. Hence, $\overline{u}_n \overset{\mathrm{loc}}{\Longrightarrow} u$ uniformly on $[0,T]$. The convergence $\underline{u}_n \overset{\mathrm{loc}}{\Longrightarrow} u$ may be proved by an analogy. □

Remark 3.13. An analogous statement holds in the space $\tilde{\mathcal{X}}_T$, $T > 0$.

In the case of measurable bounded functions, cf. Remark 3.13, we will need also a weaker form of the local stability above.

Proposition 3.14. *Let* (A1), (A2) *hold. Let $T > 0$ be fixed. Consider a sequence of functions $u_n \in \tilde{\mathcal{X}}_T$ which are solutions to 2.1 with uniformly bounded initial conditions: $u_n(\cdot,0) \in L_\theta$, $n \in \mathbb{N}$. Let $u \in \tilde{\mathcal{X}}_T$ be a solution to 2.1 with initial condition $u(\cdot,0)$ such that $u_n(x,0) \to u(x,0)$, for a.a. $x \in \mathbb{R}^d$. Then $u_n(x,t) \to u(x,t)$, for a.a. $x \in \mathbb{R}^d$, uniformly in $t \in [0,T]$.*

Proof. The proof will be fully analogous to that for Theorem 3.12 until the inequality (3.32), in which $M = M(\varepsilon, x)$ now. The rest of the proof is the same, taking into account that an analogue of Lemma 2.7 with both convergences almost everywhere holds true by the dominated convergence theorem. □

In the sequel, it will be useful to consider the solution to (2.1) as a nonlinear transformation of the initial condition.

Definition 3.15. For a fixed $T > 0$, define the mapping Q_T on $L^\infty_+(\mathbb{R}^d) := \{f \in L^\infty(\mathbb{R}^d) \mid f \geq 0 \text{ a.e.}\}$, as follows

$$(Q_T f)(x) := u(x,T), \quad x \in \mathbb{R}^d, \tag{3.33}$$

where $u(x,t)$ is the solution to (2.1) with the initial condition $u(x,0) = f(x)$.

Let us collect several properties of Q_T needed below.

Proposition 3.16. *Let* (A1), (A2) *hold. The mapping $Q = Q_T : L^\infty_+(\mathbb{R}^d) \to L^\infty_+(\mathbb{R}^d)$ satisfies the following properties*

(Q1) *$Q : L_\theta \to L_\theta$, $Q : U_\theta \to U_\theta$,*

(Q2) *let $T_y : L^\infty_+(\mathbb{R}^d) \to L^\infty_+(\mathbb{R}^d)$, $y \in \mathbb{R}^d$, be a translation operator, given by*

$$(T_y f)(x) = f(x-y), \quad x \in \mathbb{R}^d; \tag{3.34}$$

then

$$(QT_y f)(x) = (T_y Q f)(x), \quad x, y \in \mathbb{R}^d, \tag{3.35}$$

(Q3) *$Q0 = 0$, $Q\theta = \theta$, and $Qr > r$, for any constant $r \in (0,\theta)$,*

(Q4) *if $f(x) \leq g(x)$, for a.a. $x \in \mathbb{R}^d$, then $(Qf)(x) \leq (Qg)(x)$, for a.a. $x \in \mathbb{R}^d$;*

(Q5) *if $f_n \overset{\mathrm{loc}}{\Longrightarrow} f$, then $(Qf_n)(x) \to (Qf)(x)$, for a.a. $x \in \mathbb{R}^d$.*

Proof. The property (Q1) follows from Remark 3.6 and Proposition 3.5. To prove (Q2) we note that, by (2.3), $T_y(a^\pm * u) = a^\pm * (T_y u)$, and then, by (2.7), $B(T_y v) = T_y(Bv)$, therefore, by (2.6), if $\tau = 0$ and $u_\tau = T_y f$, then $\Phi_\tau T_y = T_y \Phi$, where Φ is given by (2.6) with f in place of u_τ only. As a result, $\Phi_\tau^n T_y = T_y \Phi^n$ hence

$$Q_\Upsilon(T_y f) = \lim_{n\to\infty} \Phi_\tau^n T_y f = \lim_{n\to\infty} T_y \Phi^n f = T_y(Q_\Upsilon f);$$

and one can continue the same considerations on the next time-interval. The property (Q3) is a straightforward consequence of Corollary 2.4; indeed, (2.18) implies, for $\alpha_T := \exp(-\theta\varkappa^- T) \in (0,1)$,

$$Q_T r - r = \frac{\theta r}{r(1-\alpha_T) + \theta\alpha_T} - r = \frac{r(\theta - r)(1-\alpha_T)}{r(1-\alpha_T) + \theta\alpha_T} > 0.$$

The property (Q4) holds also by Remark 3.6 and Proposition 3.5 The property (Q5) is a weaker version of Remark 3.13 and Proposition 3.14. □

Let S^{d-1} denotes a unit sphere in $\mathbb{R}^d$ centered at the origin:

$$S^{d-1} = \bigl\{x \in \mathbb{R}^d \bigm| |x| = 1\bigr\}; \tag{3.36}$$

in particular, $S^0 = \{-1, 1\}$.

Definition 3.17. A function $f \in L^\infty(\mathbb{R}^d)$ is said to be increasing (decreasing, constant) along the vector $\xi \in S^{d-1}$ if, for a.a. $x \in \mathbb{R}^d$, the function $f(x + s\xi) = (T_{-s\xi} f)(x)$ is increasing (decreasing, constant) in $s \in \mathbb{R}$, respectively.

Proposition 3.18. *Let* (A1), (A2) *hold. Let $u_0 \in L_\theta$ be the initial condition for the equation* (2.1) *which is increasing (decreasing, constant) along a vector $\xi \in S^{d-1}$; and $u(\cdot,t) \in L_\theta$, $t \geq 0$, be the corresponding solution (cf. Proposition 3.5 and Remark 3.6). Then, for any $t > 0$, $u(\cdot,t)$ is increasing (decreasing, constant, respectively) along the ξ.*

Proof. Let u_0 be decreasing along a $\xi \in S^{d-1}$. Take any $s_1 \leq s_2$ and consider two initial conditions to (2.1): $u_0^i(x) = u_0(x + s_i\xi) = (T_{-s_i\xi} u_0)(x)$, $i = 1, 2$. Since u_0 is decreasing, $u_0^1(x) \geq u_0^2(x)$, $x \in \mathbb{R}^d$. Then, by Proposition 3.16,

$$T_{-s_1\xi} Q_t u_0 = Q_t T_{-s_1\xi} u_0 = Q_t u_0^1 \geq Q_t u_0^2 = Q_t T_{-s_2\xi} u_0 = T_{-s_2\xi} Q_t u_0,$$

that proves the statement. The cases of a decreasing u_0 can be considered in the same way. The constant function along a vector is decreasing and decreasing simultaneously. □

For the sequel, we need also to show that any solution to (2.1) is bounded from below by a solution to the corresponding equation with 'truncated' kernels $a^\pm$. Namely, suppose that the conditions (A1), (A2) hold. Consider a family of Borel sets $\{\Delta_R \mid R > 0\}$, such that $\Delta_R \nearrow \mathbb{R}^d$, $R \to \infty$. Define, for any $R > 0$, the following kernels:

$$a_R^\pm(x) = 1\!\!1_{\Delta_R}(x) a^\pm(x), \quad x \in \mathbb{R}^d, \tag{3.37}$$

and the corresponding 'truncated' equation, cf. (2.1),

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \varkappa^+(a_R^+ * w)(x,t) - m w(x,t) - \kappa_1 w^2(x,t) \\ \qquad\qquad - \kappa_2 w(x,t)(a_R^- * w)(x,t), \qquad x \in \mathbb{R}^d,\ t > 0, \\ w(x,0) = w_0(x), \qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^d. \end{cases} \tag{3.38}$$

We set

$$A_R^{\pm} := \int_{\Delta_R} a^{\pm}(x)\, dx \nearrow 1, \quad R \to \infty, \tag{3.39}$$

by (2.2). Then the non-zero constant solution to (3.38) is equal to

$$\theta_R = \frac{\varkappa^+ A_R^+ - m}{\kappa_2 A_R^- + \kappa_1} \to \theta, \quad R \to \infty, \tag{3.40}$$

however, the convergence θ_R to θ is, in general, not monotonic. Clearly, by (A1), $\theta_R > 0$ if only

$$A_R^+ > \frac{m}{\varkappa^+} \in (0, 1). \tag{3.41}$$

Proposition 3.19. *Let* (A1), (A2) *hold, and* $R > 0$ *be such that* (3.41) *holds, cf.* (3.39). *Let* $w_0 \in C_{ub}(\mathbb{R}^d)$ *be such that* $0 \leq w_0(x) \leq \theta_R$, $x \in \mathbb{R}^d$. *Then there exists the unique solution* $w \in \mathcal{X}_\infty$ *to* (3.38), *such that*

$$0 \leq w(x,t) \leq \theta_R, \quad x \in \mathbb{R}^d, \ t > 0. \tag{3.42}$$

Let $u_0 \in U_\theta$ *and* $u \in \mathcal{X}_\infty$ *be the corresponding solution to* (2.1). *If* $w_0(x) \leq u_0(x), x \in \mathbb{R}^d$, *then*

$$w(x,t) \leq u(x,t), \quad x \in \mathbb{R}^d, \ t > 0. \tag{3.43}$$

Proof. Denote $\Delta_R^c := \mathbb{R}^d \setminus \Delta_R$. We have

$$\begin{aligned} \theta - \theta_R &= \frac{\kappa_2 \theta A_R^- + \kappa_1 \theta - \varkappa^+ A_R^+ + m}{\varkappa^-(\kappa_2 A_R^- + \kappa_1)} = \frac{\varkappa^+(1 - A_R^+) - \kappa_2 \theta (1 - A_R^-)}{\varkappa^-(\kappa_2 A_R^- + \kappa_1)} \\ &= \frac{1}{\varkappa^-(\kappa_2 A_R^- + \kappa_1)} \int_{\Delta_R^c} \bigl(\varkappa^+ a^+(x) - \kappa_2 \theta a^-(x)\bigr)\, dx \geq 0, \end{aligned}$$

by (A2). Therefore,

$$0 < \theta_R \leq \theta. \tag{3.44}$$

Clearly, (A2) and (3.44) yield

$$\varkappa^+ a_R^+(x) \geq \theta_R \varkappa^- a_R^-(x), \quad x \in \mathbb{R}^d. \tag{3.45}$$

Thus one can apply Proposition 3.5 to the equation (3.38) using trivial equalities $a_R^{\pm}(x) = A_R^{\pm} \tilde{a}_R^{\pm}(x)$, where the kernels $\tilde{a}_R^{\pm}(x) = (A_R^{\pm})^{-1} a_R^{\pm}(x)$ are normalized, cf. (2.2); and the inequality (3.45) is the corresponding analog of (A2), according to (3.40). This proves the existence and uniqueness of the solution to (3.38) and the bound (3.42).

Next, for $\mathcal{F}$ given by (3.1), one gets from (3.37) and (3.38), that the solution w to (3.38) satisfies the following equality

$$(\mathcal{F}w)(x,t) = -\varkappa^+ \int_{\Delta_R^c} a^+(y) w(x-y,t)\, dy + \kappa_2 w(x,t) \int_{\Delta_R^c} a^-(y) w(x-y,t)\, dy. \tag{3.46}$$

By (3.42), (3.44), (A2), one gets from (3.46) that

$$\begin{aligned} (\mathcal{F}w)(x,t) &\leq -\varkappa^+ \int_{\Delta_R^c} a^+(y) w(x-y,t)\, dy + \kappa_2 \theta \int_{\Delta_R^c} a^-(y) w(x-y,t)\, dy \\ &\leq 0 = (\mathcal{F}u)(x,t), \end{aligned}$$

where u is the solution to (2.1). Therefore, we may apply Theorem 3.1 to get the statement. □

Remark 3.20. The statements of Proposition 3.19 remains true for the functions from $L^\infty(\mathbb{R}^d)$ (the inequalities will hold a.e. only).

4 Traveling waves

Traveling waves were studied intensively for the original Fisher–KPP equation (1.7), see e.g. [6,13,57]; for locally nonlinear equation with nonlocal diffusion (1.6), see e.g. [22,88,99]; and for nonlocal nonlinear equation with local diffusion see e.g. [3,8,59,73].

Through this section we will mainly work in L^∞-setting, see Remarks 2.3, 3.3, 3.6, 3.13 above. Recall that we will always assume that (A1) and (A2) hold, and $\theta > 0$ is given by (2.17).

Let us give a brief overview for the results of this Section. First, we will show (Proposition 4.4) that the study of a traveling wave solution to the equation (2.1) in a direction $\xi \in S^{d-1}$ (cf. Definition 4.3 below) may be reduced to the study of the corresponding one-dimensional equation (4.4), whose kernels are given by (4.6). The existence and properties of the traveling wave solutions will be considered under the so-called Mollison condition (4.10), cf. e.g. [2,12,22,25,71,72]. Namely, in Theorem 4.9 we will prove that, for any $\xi \in S^{d-1}$, there exists $c_*(\xi) \in \mathbb{R}$, such that, for any $c \geq c_*(\xi)$, there exists a traveling wave with the speed c, and, for any $c < c_*(\xi)$, such a traveling wave does not exist. Moreover, we will find an expression for $c_*(\xi)$, see (4.80). We will that the profile of a traveling wave with a non-zero speed is smooth, whereas the zero-speed traveling wave (provided it exists, i.e. if $c_*(\xi) \leq 0$) has a continuous profile (Proposition 4.11, Corollary 4.12). In Theorem 4.23, we will show a connection between traveling wave speeds and the corresponding profiles. Next, using the Ikehara–Delange-type Tauberian theorem (Proposition 4.28), we will find the exact asymptotic of a decaying traveling wave profile at $+\infty$ (Proposition 4.31). This will allow us to prove the uniqueness (up to shifts) of a traveling wave wave profile with a given speed $c \geq c_*(\xi)$ (Theorem 4.33).

4.1 Existence and properties of traveling waves

Definition 4.1. Let $\mathcal{M}_\theta(\mathbb{R})$ denote the set of all decreasing and right-continuous functions $f : \mathbb{R} \to [0, \theta]$.

Remark 4.2. There is a natural embedding of $\mathcal{M}_\theta(\mathbb{R})$ into $L^\infty(\mathbb{R})$. According to this, for a function $f \in L^\infty(\mathbb{R})$, the inclusion $f \in \mathcal{M}_\theta(\mathbb{R})$ means that there exists $g \in \mathcal{M}_\theta(\mathbb{R})$, such that $f = g$ a.s. on $\mathbb{R}$.

Definition 4.3. Let $\tilde{\mathcal{X}}^1_\infty := \tilde{\mathcal{X}}_\infty \cap C^1((0,\infty) \to L^\infty(\mathbb{R}^d))$. A function $u \in \tilde{\mathcal{X}}^1_\infty$ is said to be a traveling wave solution to the equation (2.1) with a speed $c \in \mathbb{R}$ and in a direction $\xi \in S^{d-1}$ if and only if (iff, in the sequel) there exists a function $\psi \in \mathcal{M}_\theta(\mathbb{R})$, such that

$$\begin{aligned} &\psi(-\infty) = \theta, \qquad \psi(+\infty) = 0, \\ &u(x,t) = \psi(x \cdot \xi - ct), \quad t \geq 0, \text{ a.a. } x \in \mathbb{R}^d. \end{aligned} \tag{4.1}$$

Here and below S^{d-1} is defined by (3.36) and $x \cdot y = (x,y)_{\mathbb{R}^d}$ is the scalar product in $\mathbb{R}^d$. The function ψ is said to be the profile for the traveling wave, whereas c is its speed.

We will use some ideas and results from [99].

To study traveling wave solutions to (2.1), it is natural to consider the corresponding initial conditions of the form

$$u_0(x) = \psi(x \cdot \xi), \tag{4.2}$$

for some $\xi \in S^{d-1}$, $\psi \in \mathcal{M}_\theta(\mathbb{R})$. Then the solutions will have a special form as well, namely, the following proposition holds.

Proposition 4.4. *Let $\xi \in S^{d-1}$, $\psi \in \mathcal{M}_\theta(\mathbb{R})$, and an initial condition to* (2.1) *be given by $u_0(x) = \psi(x \cdot \xi)$, a.a. $x \in \mathbb{R}^d$; let also $u \in \tilde{\mathcal{X}}_\infty^+$ be the corresponding solution. Then there exist a function $\phi : \mathbb{R} \times \mathbb{R}_+ \to [0, \theta]$, such that $\phi(\cdot, t) \in \mathcal{M}_\theta(\mathbb{R})$, for any $t \geq 0$, and*

$$u(x,t) = \phi(x \cdot \xi, t), \quad t \geq 0, \text{ a.a. } x \in \mathbb{R}^d. \tag{4.3}$$

Moreover, there exist functions $\check{a}^\pm$ (depending on ξ) on $\mathbb{R}$ with $0 \leq \check{a}^\pm \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} \check{a}^\pm(s)\, ds = 1$, such that ϕ is a solution to the following one-dimensional version of (2.1)*:*

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(s,t) = \varkappa^+ (\check{a}^+ * \phi)(s,t) - m\phi(s,t) - \kappa_1 \phi^2(s,t) \\ \qquad\qquad - \kappa_2 \phi(s,t)(\check{a}^- * \phi)(s,t), \qquad t > 0, \text{ a.a. } s \in \mathbb{R}, \\ \phi(s,0) = \psi(s), \qquad \text{a.a. } s \in \mathbb{R}. \end{cases} \tag{4.4}$$

Proof. Choose any $\eta \in S^{d-1}$ which is orthogonal to the ξ. Then the initial condition u_0 is constant along η, indeed, for any $s \in \mathbb{R}$,

$$u_0(x + s\eta) = \psi((x+s\eta)\cdot \xi) = \psi(x \cdot \xi) = u_0(x), \quad \text{a.a. } x \in \mathbb{R}^d.$$

Then, by Proposition 3.18, for any fixed $t > 0$, the solution $u(\cdot, t)$ is constant along η as well. Next, for any $\tau \in \mathbb{R}$, there exists $x \in \mathbb{R}^d$ such that $x \cdot \xi = \tau$; and, clearly, if $y \cdot \xi = \tau$ then $y = x + s\eta$, for some $s \in \mathbb{R}$ and some η as above. Therefore, if we just set, for a.a. $x \in \mathbb{R}^d$, $\phi(\tau, t) := u(x,t)$, $t \geq 0$, this definition will be correct a.e. in $\tau \in \mathbb{R}$; and it will give (4.3). Next, for a.a. fixed $x \in \mathbb{R}^d$, $u_0(x + s\xi) = \psi(x \cdot \xi + s)$ is decreasing in s, therefore, u_0 is decreasing along the ξ, and by Proposition 3.18, $u(\cdot, t)$, $t \geq 0$, will be decreasing along the ξ as well. The latter means that, for any $s_1 \leq s_2$, we have, by (4.3),

$$\phi(x \cdot \xi + s_1, t) = u(x + s_1 \xi, t) \geq u(x + s_2 \xi, t) = \phi(x \cdot \xi + s_2, t),$$

and one can choose in the previous any x which is orthogonal to ξ to prove that ϕ is decreasing in the first coordinate.

To prove the second statement, for $d \geq 2$, choose any $\{\eta_1, \eta_2, \ldots, \eta_{d-1}\} \subset S^{d-1}$ which form a complement of $\xi \in S^{d-1}$ to an orthonormal basis in $\mathbb{R}^d$. Then, for a.a. $x \in \mathbb{R}^d$, with $x = \sum_{j=1}^{d-1} \tau_j \eta_j + s\xi$, $\tau_1, \ldots, \tau_{d-1}, s \in \mathbb{R}$, we have (using an analogous expansion of y inside the integral below an taking into account that any linear transformation of orthonormal bases preserves volumes)

$$\begin{aligned} &(a^\pm * u)(x,t) = \int_{\mathbb{R}^d} a^\pm(y) u(x-y,t) dy \\ &= \int_{\mathbb{R}^d} a^\pm\biggl(\sum_{j=1}^{d-1} \tau_j' \eta_j + s'\xi\biggr) u\biggl(\sum_{j=1}^{d-1} (\tau_j - \tau_j')\eta_j + (s-s')\xi, t\biggr)\, d\tau_1' \ldots d\tau_{d-1}' ds' \\ &= \int_{\mathbb{R}} \biggl(\int_{\mathbb{R}^{d-1}} a^\pm\biggl(\sum_{j=1}^{d-1} \tau_j'\eta_j + s'\xi\biggr)\, d\tau_1' \ldots d\tau_{d-1}'\biggr) u\bigl((s-s')\xi, t\bigr)\, ds', \end{aligned} \tag{4.5}$$

where we used again Proposition 3.18 to show that u is constant along the vector $\eta = \sum_{j=1}^{d-1}(\tau_j - \tau_j')\eta_j$ which is orthogonal to the ξ.

Therefore, one can set

$$\check{a}^\pm(s) := \begin{cases} \displaystyle\int_{\mathbb{R}^{d-1}} a^\pm(\tau_1\eta_1 + \ldots + \tau_{d-1}\eta_{d-1} + s\xi)\, d\tau_1 \ldots d\tau_{d-1}, & d \geq 2, \\ a^\pm(s\xi), & d = 1. \end{cases} \tag{4.6}$$

It is easily seen that $\check{a}^\pm = \check{a}^\pm_\xi$ does not depend on the choice of $\eta_1, \ldots, \eta_{d-1}$, which constitute a basis in the space $H_\xi := \{x \in \mathbb{R}^d \mid x \cdot \xi = 0\} = \{\xi\}^\perp$. Note that, clearly,

$$\int_{\mathbb{R}} \check{a}^\pm(s)\, ds = \int_{\mathbb{R}^d} a^\pm(y)\, dy = 1. \tag{4.7}$$

Next, by (4.3), $u\bigl((s-s')\xi, t\bigr) = \phi(s-s',t)$, therefore, (4.5) may be rewritten as

$$(a^\pm * u)(x,t) = \int_{\mathbb{R}} \check{a}^\pm(s')\phi(s-s',t)\, ds' =: (\check{a}^\pm * \phi)(s,t),$$

where $s = x\cdot\xi$. The rest of the proof is obvious now. □

Remark 4.5. Let $\xi \in S^{d-1}$ be fixed and $\check{a}^\pm$ be defined by (4.6). Let ϕ be a traveling wave solution to the equation (4.4) (in the sense of Definition 4.3, for $d=1$) in the direction $1 \in S^0 = \{-1,1\}$, with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and a speed $c \in \mathbb{R}$. Then the function u given by

$$u(x,t) = \psi(x\cdot\xi - ct) = \psi(s-ct) = \phi(s,t), \tag{4.8}$$

for $x \in \mathbb{R}^d$, $t \geq 0$, $s = x\cdot\xi \in \mathbb{R}$, is a traveling wave solution to (2.1) in the direction ξ, with the profile ψ and the speed c.

Remark 4.6. One can realize all previous considerations for increasing traveling wave, increasing solution along a vector ξ etc. Indeed, it is easily seen that the function $\tilde{u}(x,t) = u(-x,t)$ with the initial condition $\tilde{u}_0(x) = u_0(-x)$ is a solution to the equation (2.1) with $a^\pm$ replaced by $\tilde{a}^\pm(x) = a^\pm(-x)$; note that $(a^\pm * u)(-x,t) = (\tilde{a}^\pm * \tilde{u})(x,t)$.

Remark 4.7. It is a straightforward application of (3.35), that if $\psi \in \mathcal{M}_\theta(\mathbb{R})$, $c \in \mathbb{R}$ gets (4.1) then, for any $s \in \mathbb{R}$, $\psi(\cdot + s)$ is a traveling wave to (2.1) with the same c.

We will need also the following simple statement.

Proposition 4.8. *Let* (A1), (A2) *hold and* $\xi \in S^{d-1}$ *be fixed. Define, for an arbitrary* $T > 0$, *the mapping* $\tilde{Q}_T : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ *as follows:* $\tilde{Q}_T\psi(s) = \phi(s,T)$, $s \in \mathbb{R}$, *where* $\phi : \mathbb{R}\times\mathbb{R}_+ \to [0,\theta]$ *solves* (4.4) *with* $0 \leq \psi \in L^\infty_+(\mathbb{R})$. *Then such a* $\tilde{Q}_T$ *is well-defined, satisfies all properties of Proposition 3.16 (with* $d=1$*), and, moreover,* $\tilde{Q}_T(\mathcal{M}_\theta(\mathbb{R})) \subset \mathcal{M}_\theta(\mathbb{R})$.

Proof. Consider one-dimensional equation (4.4), where $\check{a}^\pm$ are given by (4.6). The latter equality together with (A2) imply that

$$\varkappa^+\check{a}^+(s) \geq \kappa_2\theta\check{a}^-(s), \quad \text{a.a. } s \in \mathbb{R}. \tag{4.9}$$

Therefore, all previous results (e.g. Theorem 2.2) hold true for the solution to (4.4) as well. In particular, all statements of Proposition 3.16 hold true, for $Q = \tilde{Q}_T$, $d=1$. Moreover, by the proof of Theorem 2.2 (in the L^∞-case, cf. Remark 2.3), since the mappings B and Φ_τ, cf. (2.7), (2.6), map the set $\mathcal{M}_\theta(\mathbb{R})$ into itself, we have that $\tilde{Q}_T$ has this property as well, cf. Remark 4.2. □

Now we are going to prove the existence of the traveling wave solution to (2.1). Denote, for any $\lambda > 0$, $\xi \in S^{d-1}$,

$$\mathfrak{a}_\xi(\lambda) := \int_{\mathbb{R}^d} a^+(x)e^{\lambda x\cdot\xi}\, dx \in [0,\infty]. \tag{4.10}$$

For a given $\xi \in S^{d-1}$, consider the following assumption on a^+:

$$\text{there exists } \mu = \mu(\xi) > 0 \text{ such that } \mathfrak{a}_\xi(\mu) < \infty. \tag{A5}$$

Theorem 4.9. *Let* (A1) *and* (A2) *hold and* $\xi \in S^{d-1}$ *be fixed. Suppose also that* (A5) *holds. Then there exists* $c_*(\xi) \in \mathbb{R}$ *such that*

1) *for any* $c \geq c_*(\xi)$, *there exists a traveling wave solution, in the sense of Definition 4.3, with a profile* $\psi \in \mathcal{M}_\theta(\mathbb{R})$ *and the speed* c,

2) *for any* $c < c_*(\xi)$, *such a traveling wave does not exist.*

Proof. Let $\mu > 0$ be such that (A5) holds. Then, by (4.6),

$$\begin{aligned}\int_{\mathbb{R}} \check{a}^+(s) e^{\mu s} ds &= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} a^{\pm}(\tau_1 \eta_1 + \ldots + \tau_{d-1}\eta_{d-1} + s\xi) e^{\mu s}\, d\tau_1 \ldots d\tau_{d-1} ds \\ &= \mathfrak{a}_\xi(\mu) < \infty. \end{aligned} \tag{4.11}$$

Clearly, the integral equality in (4.11) holds true for any $\lambda \in \mathbb{R}$ as well, with $\mathfrak{a}_\xi(\lambda) \in [0, \infty]$.

Let $\mu > 0$ be such that (A5) holds. Define a function from $\mathcal{M}_\theta(\mathbb{R})$ by

$$\varphi(s) := \theta \min\{e^{-\mu s}, 1\}. \tag{4.12}$$

Let us prove that there exists $c \in \mathbb{R}$ such that $\bar{\phi}(s,t) := \varphi(s - ct)$ is a super-solution to (4.4), i.e.

$$\mathcal{F}\bar{\phi}(s,t) \geq 0, \quad s \in \mathbb{R}, t \geq 0, \tag{4.13}$$

where $\mathcal{F}$ is given by (3.1) (in the case $d = 1$). We have

$$\begin{aligned}(\mathcal{F}\bar{\phi})(s,t) = &-c\varphi'(s-ct) - \varkappa^+(\check{a}^+ * \varphi)(s-ct) + m\varphi(s-ct) \\ &+ \kappa_2 \varphi(s-ct)(\check{a}^- * \varphi)(s-ct) + \kappa_1 \varphi^2(s-ct),\end{aligned}$$

hence, to prove (4.13), it is enough to show that, for all $s \in \mathbb{R}$,

$$\mathcal{J}_c(s) := c\varphi'(s) + \varkappa^+(\check{a}^+ * \varphi)(s) - m\varphi(s) - \kappa_2\varphi(s)(\check{a}^- * \varphi)(s) - \kappa_1\varphi^2(s) \leq 0. \tag{4.14}$$

By (4.12), (4.9), for $s < 0$, we have

$$\begin{aligned}\mathcal{J}_c(s) &= \varkappa^+(\check{a}^+ * \varphi)(s) - m\theta - \kappa_2\theta(\check{a}^- * \varphi)(s) - \kappa_1\theta^2 \\ &\leq \big((\varkappa^+\check{a} - \kappa_2\theta\check{a}^-) * \theta\big)(s) - m\theta - \kappa_1\theta^2 = 0.\end{aligned}$$

Next, by (4.12),

$$(\check{a}^+ * \varphi)(s) \leq \theta \int_{\mathbb{R}} \check{a}^+(\tau) e^{-\mu(s-\tau)}\, d\tau = \theta e^{-\mu s} \mathfrak{a}_\xi(\mu),$$

therefore, for $s \geq 0$, we have

$$\mathcal{J}_c(s) \leq -\mu c\theta e^{-\mu s} + \varkappa^+ \theta e^{-\mu s}\mathfrak{a}_\xi(\mu) - m\theta e^{-\mu s};$$

and to get (4.14) it is enough to demand that $\varkappa^+\mathfrak{a}_\xi(\mu) - m - \mu c \leq 0$, in particular,

$$c = \frac{\varkappa^+\mathfrak{a}_\xi(\mu) - m}{\mu}. \tag{4.15}$$

As a result, for $\bar{\phi}(s,t) = \varphi(s - ct)$ with c given by (4.15), we have

$$\mathcal{F}\bar{\phi} \geq 0 = \mathcal{F}(\tilde{Q}_t\varphi), \tag{4.16}$$

as $\tilde{Q}_t\varphi$ is a solution to (4.4). Then, by (A2) and the inequality $\bar{\phi} \leq \theta$, one can apply Proposition 3.4 and get that

$$\tilde{Q}_t\varphi(s') \leq \bar{\phi}(t, s') = \varphi(s' - ct), \quad \text{a.a. } s' \in \mathbb{R},$$

where c is given by (4.15); note that, by (4.12), for any $s \in \mathbb{R}$, the function $\bar{\phi}(s,t)$ is absolutely continuous in t. In particular, for $t = 1$, $s' = s + c$, we will have

$$\tilde{Q}_1\varphi(s + c) \leq \varphi(s), \quad \text{a.a. } s \in \mathbb{R}. \tag{4.17}$$

And now one can apply [99, Theorem 5] which states that, if there exists a flow of abstract mappings $\tilde{Q}_t$, each of them maps $\mathcal{M}_\theta(\mathbb{R})$ into itself and has properties (Q1)–(Q5) of Proposition 3.16, and if, for some t (e.g. $t = 1$), for some $c \in \mathbb{R}$, and for some $\varphi \in \mathcal{M}_\theta(\mathbb{R})$, the inequality (4.17) holds, then there exists $\psi \in \mathcal{M}_\theta(\mathbb{R})$ such that, for any $t \geq 0$,

$$(\tilde{Q}_t\psi)(s + ct) = \psi(s), \quad \text{a.a. } s \in \mathbb{R}, \tag{4.18}$$

that yields the solution to (4.4) in the form (4.8), and hence, by Remark 4.5, we will get the existence of a solution to (2.1) in the form (4.1). It is worth noting that, in [99], the results were obtained for increasing functions. By Remark 4.6, the same results do hold for decreasing functions needed for our settings.

Next, by [99, Theorem 6], there exists $c_* = c_*(\xi) \in (-\infty, \infty]$ such that, for any $c \geq c_*$, there exists $\psi = \psi_c \in \mathcal{M}_\theta(\mathbb{R})$ such that (4.18) holds, and for any $c < c_*$ such a ψ does not exist. Since for c given by (4.15) such a ψ exists, we have that $c_* \leq c < \infty$, moreover, one can take any μ in (4.15) for that (A5) holds. Therefore,

$$c_* \leq \inf_{\lambda > 0} \frac{\varkappa^+ \mathfrak{a}_\xi(\lambda) - m}{\lambda}. \tag{4.19}$$

The statement is proved. □

Remark 4.10. It can be seen from the proof above that we didn't use the special form (4.12) of the function φ after the inequality (4.16). Therefore, if a function $\varphi_1 \in \mathcal{M}_\theta(\mathbb{R})$ is such that the function $\bar{\phi}(s,t) := \varphi_1(s - ct)$, $s \in \mathbb{R}$, $t \geq 0$, is a super-solution to (4.4), for some $c \in \mathbb{R}$, i.e. if (4.13) holds, then there exists a traveling wave solution to (4.4), and hence to (2.1), with some profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and the same speed c.

Next two statements describe the properties of a traveling wave solution.

Proposition 4.11. *Let $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and $c \in \mathbb{R}$ be such that there exists a solution $u \in \tilde{\mathcal{X}}^1_\infty$ to the equation* (2.1) *such that* (4.1) *holds, for some $\xi \in S^{d-1}$. Then $\psi \in C^1(\mathbb{R} \to [0, \theta])$, for $c \neq 0$, and $\psi \in C(\mathbb{R} \to [0, \theta])$, otherwise.*

Proof. The condition (4.1) implies (4.2) for the $\xi \in S^{d-1}$. Then, by Proposition 4.4, there exists ϕ given by (4.3) which solves (4.4); moreover, by Remark 4.5, (4.8) holds.

Let $c \neq 0$. It is well-known that any monotone function is differentiable almost everywhere. Prove first that ψ is differentiable everywhere on $\mathbb{R}$. Fix any $s_0 \in \mathbb{R}$. It follows directly from Proposition 4.4, that $\phi \in C^1((0, \infty) \to L^\infty(\mathbb{R}))$. Therefore, for any $t_0 > 0$ and for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that, for all $t \in \mathbb{R}$ with $|ct| < \delta$ and $t_0 + t > 0$, the following inequalities hold, for a.a. $s \in \mathbb{R}$,

$$\frac{\partial \phi}{\partial t}(s, t_0) - \varepsilon < \frac{\phi(s, t_0 + t) - \phi(s, t_0)}{t} < \frac{\partial \phi}{\partial t}(s, t_0) + \varepsilon, \tag{4.20}$$

$$\frac{\partial \phi}{\partial t}(s, t_0) - \varepsilon < \frac{\partial \phi}{\partial t}(s, t_0 + t) < \frac{\partial \phi}{\partial t}(s, t_0) + \varepsilon. \tag{4.21}$$

Set, for the simplicity of notations, $x_0 = s_0 + ct_0$. Take any $0 < h < 1$ with $2h < \min\bigl\{\delta, |c|t_0, |c|\delta\bigr\}$. Since ψ is a decreasing function, one has, for almost all $s \in (x_0, x_0 + h^2)$,

$$\begin{aligned}\frac{\psi(s_0+h)-\psi(s_0)}{h} &\leq \frac{\psi(s-ct_0+h-h^2)-\psi(s-ct_0)}{h}\\ &= \frac{\phi(s,t_0+\frac{h^2-h}{c})-\phi(s,t_0)}{\frac{h^2-h}{c}}\,\frac{h^2-h}{ch} \leq \biggl(\frac{\partial\phi}{\partial t}(s,t_0)\mp\varepsilon\biggr)\,\frac{h-1}{c},\end{aligned} \tag{4.22}$$

by (4.20) with $t = \frac{h^2-h}{c}$; note that then $|ct| = h - h^2 < h < \delta$, and $t_0 + t > 0$ (the latter holds, for $c < 0$, because of $t_0 + t > t_0$ then; and, for $c > 0$, it is equivalent to $ct_0 > -ct = h - h^2$, that follows from $h < ct_0$). Stress, that, in (4.22), one needs to choose $-\varepsilon$, for $c > 0$, and $+\varepsilon$, for $c < 0$, according to the left and right inequalities in (4.20), correspondingly.

Similarly, for almost all $s \in (x_0 - h^2, x_0)$, one has

$$\begin{aligned}\frac{\psi(s_0+h)-\psi(s_0)}{h} &\geq \frac{\psi(s-ct_0+h+h^2)-\psi(s-ct_0)}{h}\\ &= \frac{\phi(s,t_0-\frac{h^2+h}{c})-\phi(s,t_0)}{-\frac{h^2+h}{c}}\,\frac{h^2+h}{-ch} \geq \biggl(\frac{\partial\phi}{\partial t}(s,t_0)\pm\varepsilon\biggr)\,\frac{h+1}{-c},\end{aligned} \tag{4.23}$$

where we take again the upper sign, for $c > 0$, and the lower sign, for $c < 0$; note also that $h + h^2 < 2h < \delta$. Next, one needs to 'shift' values of s in (4.23) to get them the same as in (4.22). To do this note that, by (4.8),

$$\phi\Bigl(s+h^2, t_0+\frac{h^2}{c}\Bigr) = \phi(s,t_0), \quad \text{a.a. } s \in \mathbb{R}^d. \tag{4.24}$$

As a result,

$$\begin{aligned}(\check{a}^{\pm}*\phi)\Bigl(s+h^2,t_0+\frac{h^2}{c}\Bigr) &= \int_{\mathbb{R}} \check{a}^{\pm}(s')\phi\Bigl(s-s'+h^2,t_0+\frac{h^2}{c}\Bigr)\,ds\\ &= (\check{a}^{\pm}*\phi)(s,t_0), \quad \text{a.a. } s\in\mathbb{R}^d.\end{aligned} \tag{4.25}$$

Then, by (4.4), (4.24), (4.25), one gets

$$\frac{\partial}{\partial t}\phi\Bigl(s+h^2,t_0+\frac{h^2}{c}\Bigr) = \frac{\partial}{\partial t}\phi(s,t_0), \quad \text{a.a. } s \in \mathbb{R}^d. \tag{4.26}$$

Therefore, by (4.26), one gets from (4.23) that, for almost all $s \in (x_0, x_0 + h^2)$, cf. (4.22),

$$\frac{\psi(s_0+h)-\psi(s_0)}{h} \geq \biggl(\frac{\partial\phi}{\partial t}\Bigl(s,t_0+\frac{h^2}{c}\Bigr)\pm\varepsilon\biggr)\,\frac{h+1}{-c},$$

and, since $\bigl|\frac{h^2}{c}\bigr| < \delta$, one can apply the right and left inequalities in (4.21), for $c > 0$ and $c < 0$, correspondingly, to continue the estimate

$$\geq \biggl(\frac{\partial\phi}{\partial t}(s,t_0)\pm 2\varepsilon\biggr)\,\frac{h+1}{-c}. \tag{4.27}$$

Combining (4.22) and (4.27), we obtain

$$\begin{aligned}\biggl(\operatorname*{esssup}_{s\in(x_0,x_0+h^2)}\frac{\partial\phi}{\partial t}(s,t_0)\pm 2\varepsilon\biggr)\,\frac{h+1}{-c} &\leq \frac{\psi(s_0+h)-\psi(s_0)}{h}\\ &\leq \biggl(\operatorname*{esssup}_{s\in(x_0,x_0+h^2)}\frac{\partial\phi}{\partial t}(s,t_0)\mp\varepsilon\biggr)\,\frac{h-1}{c}.\end{aligned} \tag{4.28}$$

For fixed $s_0 \in \mathbb{R}$, $t_0 > 0$ and for $x_0 = s_0 + ct_0$, the function

$$f(h) := \operatorname*{esssup}_{s\in(x_0,x_0+h^2)} \frac{\partial \phi}{\partial t}(s,t_0), \quad h \in (0,1),$$

is bounded, as $|f(h)| \leq \bigl\|\frac{\partial\phi}{\partial t}(\cdot,t_0)\bigr\|_\infty < \infty$, and monotone; hence there exists $\bar f = \lim\limits_{h\to 0+} f(h)$. As a result, for small enough h, (4.28) yields

$$(\bar f \pm 2\varepsilon)\frac{1}{-c} - \varepsilon \leq \frac{\psi(s_0+h)-\psi(s_0)}{h} \leq (\bar f \mp \varepsilon)\frac{-1}{c} + \varepsilon,$$

and, therefore, there exists $\dfrac{\partial\psi}{\partial s}(s_0+) = \dfrac{-\bar f}{c}$. In the same way, one can prove that there exists $\dfrac{\partial\psi}{\partial s}(s_0-) = \dfrac{-\bar f}{c}$, and, therefore, ψ is differentiable at s_0. As a result, ψ is differentiable (and hence continuous) on the whole $\mathbb{R}$.

Next, for any $s_1, s_2, h \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\frac{\psi(s_1+h)-\psi(s_1)}{h} - \frac{\psi(s_2+h)-\psi(s_2)}{h}\right| \\
&\qquad = \frac{1}{|c|}\left|\frac{\phi\bigl(s_1+ct_0, t_0-\frac{h}{c}\bigr)-\phi(s_1+ct_0,t_0)}{-\frac{h}{c}}\right. \\
&\qquad\qquad \left. - \frac{\phi\bigl(s_1+ct_0, t_0+\frac{s_1-s_2}{c}-\frac{h}{c}\bigr)-\phi\bigl(s_1+ct_0,t_0+\frac{s_1-s_2}{c}\bigr)}{-\frac{h}{c}}\right|;
\end{aligned}$$

and if we pass h to 0, we get

$$\begin{aligned}
|\psi'(s_1)-\psi'(s_2)| &= \frac{1}{|c|}\left|\frac{\partial}{\partial t}\phi(s_1+ct_0,t_0) - \frac{\partial}{\partial t}\phi\Bigl(s_1+ct_0, t_0+\frac{s_1-s_2}{c}\Bigr)\right| \\
&\leq \frac{1}{|c|}\left\|\frac{\partial}{\partial t}\phi(\cdot,t_0) - \frac{\partial}{\partial t}\phi\Bigl(\cdot, t_0+\frac{s_1-s_2}{c}\Bigr)\right\|.
\end{aligned} \tag{4.29}$$

And now, by the continuity of $\frac{\partial}{\partial t}\phi(\cdot,t)$ in t in the sense of the norm in $L^\infty(\mathbb{R})$, we have that, by (4.21), the inequality $|s_1-s_2| \leq |c|\delta$ implies that, by (4.29),

$$|\psi'(s_1)-\psi'(s_2)| \leq \frac{1}{|c|}\varepsilon.$$

As a result, $\psi'(s)$ is uniformly continuous on $\mathbb{R}$ and hence continuous.

Finally, consider the case $c=0$. Then (4.8) implies that $\phi(s,t)$ must be constant in time, i.e. $\phi(s,t)=\psi(s)$, for a.a. $s\in\mathbb{R}$. Thus one can rewrite (4.4) as follows

$$\begin{aligned}
0 &= -\varkappa^+(\check a^+ * \psi)(s) + m\psi(s) + \kappa_2\psi(s)(\check a^- * \psi)(s) + \kappa_1\psi^2(s) \\
&= \kappa_1\psi^2(s) + A(s)\psi(s) - B(s),
\end{aligned} \tag{4.30}$$

where $A(s) = m + \kappa_2(\check a^- * \psi)(s)$ and $B(s) = \varkappa^+(\check a^+ * \psi)(s)$. Equivalently,

$$\psi(s) = \frac{\sqrt{A^2(s)+4\kappa_1 B(s)} - A(s)}{4\kappa_1}. \tag{4.31}$$

Since $\psi \in L^\infty(\mathbb{R})$, then, by Lemma 2.1, the r.h.s. of (4.31) is a continuous in s function, and hence $\psi \in C(\mathbb{R})$. $\square$

Let $u \in \tilde{\mathcal{X}}^1_\infty$ be a traveling wave solution to (2.1), in the sense of Definition 4.3, with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and a speed $c \in \mathbb{R}$. Then, by Remark 4.5 and Proposition 4.11, for any $c \neq 0$, one can differentiate $\psi(s - ct)$ in $t \geq 0$. Thus (cf. also Lemma 2.1) we get

$$c\psi'(s) + \varkappa^+(\check{a}^+ * \psi)(s) - m\psi(s) - \kappa_2\psi(s)(\check{a}^- * \psi)(s) - \kappa_1\psi^2(s) = 0, \quad s \in \mathbb{R}. \quad (4.32)$$

For $c = 0$, one has (4.30), i.e. (4.32) holds in this case as well.

Let $k \in \mathbb{N} \cup \{\infty\}$ and $C_b^k(\mathbb{R})$ denote the class of all functions on $\mathbb{R}$ which are k times differentiable and whose derivatives (up to the order k) are continuous and bounded on $\mathbb{R}$.

Corollary 4.12. *In conditions and notations of Proposition 4.11, for any speed $c \neq 0$, the profile $\psi \in C_b^\infty(\mathbb{R})$.*

Proof. By Lemma 2.1, $\check{a}^\pm * \psi \in C_b(\mathbb{R})$. Then (4.32) yields $\psi' \in C_b(\mathbb{R})$, i.e. $\psi \in C_b^1(\mathbb{R})$. By e.g. [86, Proposition 5.4.1], $\check{a}^\pm * \psi \in C_b^1(\mathbb{R})$ and $(\check{a}^\pm * \psi)' = \check{a}^\pm * \psi'$, therefore, the equality (4.32) holds with ψ' replaced by ψ'' and ψ replaced by ψ'. Then, by the same arguments $\psi \in C_b^2(\mathbb{R})$, and so on. The statement is proved. □

Proposition 4.13. *In conditions and notations of Proposition 4.11, ψ is a strictly decaying function, for any speed c.*

Proof. Let $c \in \mathbb{R}$ be the speed of a traveling wave with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ in a direction $\xi \in S^{d-1}$. By Proposition 4.11, $\psi \in C(\mathbb{R})$. Suppose that ψ is not strictly decaying, then there exists $\delta_0 > 0$ and $s_0 \in \mathbb{R}$, such that $\psi(s) = \psi(s_0)$, for all $|s - s_0| \leq \delta_0$. Take any $\delta \in \bigl(0, \frac{\delta_0}{2}\bigr)$, and consider the function $\psi^\delta(s) := \psi(s + \delta)$. Clearly, $\psi^\delta(s) \leq \psi(s)$, $s \in \mathbb{R}$. By Remark 4.7, ψ^δ is a profile for a traveling wave with the same speed c. Therefore, one has two solutions to (2.1): $u(x,t) = \psi(x \cdot \xi - ct)$ and $u^\delta(x,t) = \psi^\delta(x \cdot \xi - ct)$ and hence $u(x,t) \leq u^\delta(x,t)$, $x \in \mathbb{R}^d$, $t \geq 0$. By the maximum principle, see Theorem 3.10, either $u \equiv u^\delta$, that contradicts $\delta > 0$ or $u(x,t) < u^\delta(x,t)$, $x \in \mathbb{R}^d$, $t > 0$. The latter, however, contradicts the equality $u(x,t) = u^\delta(x,t)$, which holds e.g. if $x \cdot \xi - ct = s_0$. Hence ψ is a strictly decaying function. □

Under assumptions (A1) and (A2), define the following function, cf. (3.19),

$$\check{J}_\upsilon(s) := \varkappa^+\check{a}^+(s) - \upsilon\kappa_2\check{a}^-(s), \quad s \in \mathbb{R}, \upsilon \in (0, \theta]. \quad (4.33)$$

Then, by (4.9),

$$\check{J}_\upsilon(s) \geq \check{J}_\theta(s) \geq 0, \quad s \in \mathbb{R}, \upsilon \in (0, \theta].$$

Proposition 4.14. *Let* (A1) *and* (A2) *hold. Then, in the conditions and notations of Proposition 4.11, there exists $\mu = \mu(c, a^+, \varkappa^-, \theta) > 0$ such that*

$$\int_{\mathbb{R}} \psi(s) e^{\mu s}\, ds < \infty.$$

Proof. At first, we prove that $\psi \in L^1(\mathbb{R}_+)$. Let $\upsilon \in (0, \theta)$ and $\check{J}_\upsilon(s) > 0$, $s \in \mathbb{R}$ be given by (4.33). Since $\int_{\mathbb{R}} \check{J}_\upsilon(s)\, ds = \varkappa^+ - \upsilon\kappa_2 > m + \kappa_1\upsilon$, one can choose $R_0 > 0$, such that

$$\int_{-R_0}^{R_0} \check{J}_\upsilon(s)\, ds = m + \kappa_1\upsilon. \quad (4.34)$$

We rewrite (4.32) as follows

$$c\psi'(s) + (\check{J}_\upsilon * \psi)(s) + \bigl(\upsilon - \psi(s)\bigr)\bigl(\kappa_1\psi(s) + \kappa_2(\check{a}^- * \psi)(s)\bigr) - (m + \kappa_1\upsilon)\psi(s) = 0, \quad s \in \mathbb{R}. \quad (4.35)$$

Fix arbitrary $r_0 > 0$, such that

$$\psi(r_0) < \upsilon. \tag{4.36}$$

Let $r > r_0 + R_0$. Integrate (4.35) over $[r_0, r]$; one gets

$$c(\psi(r) - \psi(r_0)) + A + B = 0, \tag{4.37}$$

where

$$\begin{aligned} A &:= \int_{r_0}^{r} (\check{J}_\upsilon * \psi)(s)\, ds - (m + \kappa_1 \upsilon) \int_{r_0}^{r} \psi(s) ds, \\ B &:= \int_{r_0}^{r} (\upsilon - \psi(s))\bigl(\kappa_1 \psi(s) + \kappa_2 (\check{a}^- * \psi)(s)\bigr)\, ds. \end{aligned}$$

By (4.33), (4.34), one has

$$\begin{aligned} A &\geq \int_{r_0}^{r} \int_{-R_0}^{R_0} \check{J}_\upsilon(\tau) \psi(s - \tau) d\tau ds - (m + \kappa_1 \upsilon) \int_{r_0}^{r} \psi(s)\, ds \\ &= \int_{-R_0}^{R_0} \check{J}_\upsilon(\tau) \left(\int_{r_0 - \tau}^{r - \tau} \psi(s)\, ds - \int_{r_0}^{r} \psi(s)\, ds \right) d\tau \\ &= \int_{0}^{R_0} \check{J}_\upsilon(\tau) \left(\int_{r_0 - \tau}^{r_0} \psi(s)\, ds - \int_{r - \tau}^{r} \psi(s)\, ds \right) d\tau \\ &\quad + \int_{-R_0}^{0} \check{J}_\upsilon(\tau) \left(\int_{r}^{r - \tau} \psi(s)\, ds - \int_{r_0}^{r_0 - \tau} \psi(s)\, ds \right) d\tau; \end{aligned} \tag{4.38}$$

and since ψ is a decreasing function and $r - R_0 > r_0$, we have from (4.38), that

$$\begin{aligned} A &\geq (\psi(r_0) - \psi(r - R_0)) \int_{0}^{R_0} \tau \check{J}_\upsilon(\tau)\, d\tau + (\psi(r + R_0) - \psi(r_0)) \int_{-R_0}^{0} (-\tau) J_\upsilon(\tau)\, d\tau \\ &\geq -\theta \int_{-R_0}^{0} (-\tau) J_\upsilon(\tau)\, d\tau =: -\theta \bar{J}_{\upsilon, R_0}. \end{aligned} \tag{4.39}$$

Next, (4.36) and monotonicity of ψ imply

$$B \geq (\upsilon - \psi(r_0)) \int_{r_0}^{r} \bigl(\kappa_1 \psi(s) + \kappa_2 (\check{a}^- * \psi)(s)\bigr)\, ds. \tag{4.40}$$

Then, by (4.37), (4.39), (4.40), (4.36), one gets

$$\begin{aligned} 0 &\leq (\upsilon - \psi(r_0)) \int_{r_0}^{r} \bigl(\kappa_1 \psi(s) + \kappa_2 (\check{a}^- * \psi)(s)\bigr)\, ds \\ &\qquad \leq \theta \bar{J}_{\upsilon, R_0} + c(\psi(r_0) - \psi(r)) \to \theta \bar{J}_{\upsilon, R_0} + c\psi(r_0) < \infty, \quad r \to \infty, \end{aligned}$$

therefore, $\kappa_1 \psi + \kappa_2 \check{a}^- * \psi \in L^1(\mathbb{R}_+)$. Finally, (4.7) implies that there exist a measurable bounded set $\Delta \subset \mathbb{R}$, with $m(\Delta) := \int_\Delta ds \in (0, \infty)$, and a constant $\mu > 0$, such that $\check{a}^-(\tau) \geq \mu$, for a.a. $\tau \in \Delta$. Let $\delta = \inf \Delta \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$, one has

$$(\check{a}^- * \psi)(s) \geq \int_\Delta \check{a}^-(\tau) \psi(s - \tau)\, d\tau \geq \mu \psi(s - \delta) m(\Delta).$$

Therefore $\psi \in L^1(\mathbb{R}_+)$.

For any $N \in \mathbb{N}$, we define $\varphi_N(s) := 1\!\!1_{(-\infty,N)}(s) + e^{-\lambda(s-N)} 1\!\!1_{[N,\infty)}(s)$, where $\lambda > 0$. By the proved above, $\psi, \check{a}^\pm * \psi \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R})$ hence, by (4.32), $c\psi' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R})$. Therefore, all terms of (4.32) being multiplied on $e^{\lambda s}\varphi_N(s)$ are integrable over $\mathbb{R}$. After this integration, (4.32) will be read as follows

$$I_1 + I_2 + I_3 = 0, \tag{4.41}$$

where (recall that $\varkappa^-\theta - \varkappa^+ = -m$)

$$\begin{aligned}
I_1 &:= c \int_{\mathbb{R}} \psi'(s) e^{\lambda s} \varphi_N(s)\, ds,\\
I_2 &:= \varkappa^+ \int_{\mathbb{R}} \bigl((\check{a}^+ * \psi)(s) - \psi(s)\bigr) e^{\lambda s}\varphi_N(s)\, ds,\\
I_3 &:= \int_{\mathbb{R}} \psi(s)\bigl(\varkappa^+ - m - \kappa_1\psi(s) - \kappa_2(\check{a}^- * \psi)(s)\bigr) e^{\lambda s}\varphi_N(s)\, ds
\end{aligned}$$

We estimate now I_1, I_2, I_3 from below.

We start with I_2. One can write

$$\begin{aligned}
&\int_{\mathbb{R}} (\check{a}^+ * \psi)(s) e^{\lambda s}\varphi_N(s)\, ds = \int_{\mathbb{R}}\int_{\mathbb{R}} \check{a}^+(s-\tau)\psi(\tau) e^{\lambda s}\varphi_N(s)\, d\tau ds\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \check{a}^+(s) e^{\lambda s}\varphi_N(\tau+s)\, ds\, e^{\lambda\tau}\psi(\tau)\, d\tau\\
&\geq \int_{\mathbb{R}} \Bigl(\int_{-\infty}^{R} \check{a}^+(s) e^{\lambda s}\, ds\Bigr) \varphi_N(\tau+R) e^{\lambda\tau}\psi(\tau)\, d\tau,
\end{aligned} \tag{4.42}$$

for any $R > 0$, as φ is nonincreasing. By (4.7), one can choose $R > 0$ such that

$$\int_{-\infty}^{R} \check{a}^+(\tau)\, d\tau > 1 - \frac{\varkappa^-\theta}{4}.$$

By continuity arguments, there exists $\nu > 0$ such that, for any $0 < \lambda < \nu$,

$$\int_{-\infty}^{R} \check{a}^+(\tau) e^{\lambda\tau}\, d\tau \geq \Bigl(1 - \frac{\varkappa^-\theta}{4}\Bigr) e^{\lambda R}. \tag{4.43}$$

Therefore, combining (4.42) and (4.43), we get

$$\begin{aligned}
I_2 &\geq \int_{\mathbb{R}} \Bigl(1 - \frac{\varkappa^-\theta}{4}\Bigr) e^{\lambda R}\varphi_N(\tau+R) e^{\lambda\tau}\psi(\tau)\, d\tau - \int_{\mathbb{R}} \psi(s) e^{\lambda s}\varphi_N(s)\, ds\\
&= \int_{\mathbb{R}} \Bigl(1 - \frac{\varkappa^-\theta}{4}\Bigr) \varphi_N(\tau) e^{\lambda\tau}\psi(\tau - R)\, d\tau - \int_{\mathbb{R}} \psi(s) e^{\lambda s}\varphi_N(s)\, ds\\
&\geq -\frac{\varkappa^-\theta}{4} \int_{\mathbb{R}} \psi(s) e^{\lambda s}\varphi_N(s)\, ds,
\end{aligned} \tag{4.44}$$

as $\psi(\tau - R) \geq \psi(\tau)$, $\tau \in \mathbb{R}$, $R > 0$.

Now we estimate I_3. By (4.1), it is easily seen that the function $(\check{a}^- * \psi)(s)$ decreases monotonically to 0 as $s \to \infty$. Suppose additionally that $R > 0$ above is such that

$$\kappa_1\psi(s) + \kappa_2(\check{a}^- * \psi)(s) < \frac{\varkappa^-\theta}{2}, \quad s > R.$$

Then, one gets

$$\begin{aligned}I_3 &\geq \frac{\varkappa^-\theta}{2}\int_R^\infty \psi(s)e^{\lambda s}\varphi_N(s)\,ds\\&\quad+\int_{-\infty}^R \psi(s)\bigl(\varkappa^-\theta-\kappa_1\psi(s)-\kappa_2(\check{a}^-*\psi)(s)\bigr)e^{\lambda s}\varphi_N(s)\,ds\\&\geq \frac{\varkappa^-\theta}{2}\int_R^\infty \psi(s)e^{\lambda s}\varphi_N(s)\,ds,\end{aligned}$$

as $0\leq\psi\leq\theta$, $\varphi_N\geq 0$, $(\check{a}^-*\psi)(s)\leq\theta$.

It remains to estimate I_1 (in the case $c\neq 0$). Since $\lim\limits_{s\to\pm\infty}\psi(s)e^{\lambda s}\varphi_N(s)=0$, we have from the integration by parts formula, that

$$I_1=-c\int_{\mathbb{R}}\psi(s)(\lambda\varphi_N(s)+\varphi_N'(s))e^{\lambda s}\,ds.$$

For $c>0$, one can use that $\varphi_N'(s)\leq 0$, $s\in\mathbb{R}$, and hence

$$I_1\geq -c\lambda\int_{\mathbb{R}}\psi(s)\varphi_N(s)e^{\lambda s}\,ds.$$

For $c<0$, we use that, by the definition of φ_N, $\lambda\varphi_N(s)+\varphi_N'(s)=0$, $s\geq N$; therefore,

$$I_1=-c\lambda\int_{-\infty}^N\psi(s)\,ds>0. \tag{4.45}$$

Therefore, combining (4.44)–(4.45), we get from (4.41), that

$$0\geq -\lambda\bar{c}\int_{\mathbb{R}}\psi(s)\varphi_N(s)e^{\lambda s}\,ds-\frac{\varkappa^-\theta}{4}\int_{\mathbb{R}}\psi(s)e^{\lambda s}\varphi_N(s)\,ds+\frac{\varkappa^-\theta}{2}\int_R^\infty\psi(s)e^{\lambda s}\varphi_N(s)\,ds,$$

where $\bar{c}=\max\{c,0\}$.

The latter inequality can be easily rewritten as

$$\begin{aligned}\Bigl(\frac{\varkappa^-\theta}{4}-\lambda\bar{c}\Bigr)\int_R^\infty\psi(s)e^{\lambda s}\varphi_N(s)\,ds&\leq\Bigl(\frac{\varkappa^-\theta}{4}+\lambda\bar{c}\Bigr)\int_{-\infty}^R\psi(s)\varphi_N(s)e^{\lambda s}\,ds\\&\leq\Bigl(\frac{\varkappa^-\theta}{4}+\lambda\bar{c}\Bigr)\theta\int_{-\infty}^R e^{\lambda s}\,ds=:I_{\lambda,R}<\infty,\end{aligned}\tag{4.46}$$

for any $0<\lambda<\nu$.

Take now $\mu<\min\bigl\{\nu,\frac{\varkappa^-\theta}{4c}\bigr\}$, for $c>0$, and $\mu<\nu$, otherwise. Then, by (4.46), for any $N>R$, one get

$$\infty>\Bigl(\frac{\varkappa^-\theta}{4}-\mu\bar{c}\Bigr)^{-1}I_{\mu,R}>\int_R^\infty\psi(s)e^{\mu s}\varphi_N(s)\,ds\geq\int_R^N\psi(s)e^{\mu s}\,ds,$$

thus,

$$\begin{aligned}\int_{\mathbb{R}}\psi(s)e^{\mu s}\,ds&=\int_{-\infty}^R\psi(s)e^{\mu s}\,ds+\int_R^\infty\psi(s)e^{\mu s}\,ds\\&\leq\theta\int_{-\infty}^R e^{\mu s}\,ds+I_{\mu,R}\Bigl(\frac{\varkappa^-\theta}{4}-\mu\bar{c}\Bigr)^{-1}<\infty,\end{aligned}$$

that gets the statement. □

4.2 Speed and profile of a traveling wave

Through this subsection we will suppose, additionally to (A1) and (A2), that

$$a^+ \in L^\infty(\mathbb{R}^d). \tag{A6}$$

Clearly, (A2) and (A6) imply $a^- \in L^\infty(\mathbb{R}^d)$.

Remark 4.15. All further statements remain true if we change (A6) on the condition $\check{a}^+ \in L^\infty(\mathbb{R})$, where $\check{a}^+$ is given by (4.6); evidently, the latter condition is, for $d \geq 2$, weaker than (A6).

Let $\xi \in S^{d-1}$ be fixed and (A5) hold. Assume also that

$$\int_{\mathbb{R}^d} |x \cdot \xi| \, a^+(x) \, dx < \infty. \tag{A7}$$

Under assumption (A7), we define

$$\mathfrak{m}_\xi := \int_{\mathbb{R}^d} x \cdot \xi \; a^+(x) \, dx. \tag{4.47}$$

Suppose also, that the following modification of (A3) holds:

$$\begin{gathered} \text{there exist } r = r(\xi) \geq 0,\ \rho = \rho(\xi) > 0,\ \delta = \delta(\xi) > 0, \text{ such that} \\ a^+(x) \geq \rho, \ \text{ for a.a. } x \in B_\delta(r\xi). \end{gathered} \tag{A8}$$

For an $f \in L^\infty(\mathbb{R})$, let $\mathfrak{L}f$ be a bilateral-type Laplace transform of f, cf. [96, Chapter VI]:

$$(\mathfrak{L}f)(z) = \int_{\mathbb{R}} f(s) e^{zs} \, ds, \quad \operatorname{Re} z > 0. \tag{4.48}$$

We collect several results about $\mathfrak{L}$ in the following lemma.

Lemma 4.16. *Let $f \in L^\infty(\mathbb{R})$.*

(L1) *There exists $\lambda_0(f) \in [0, \infty]$ such that the integral (4.48) converges in the strip $\{0 < \operatorname{Re} z < \lambda_0(f)\}$ (provided that $\lambda_0(f) > 0$) and diverges in the half plane $\{\operatorname{Re} z > \lambda_0(f)\}$ (provided that $\lambda_0(f) < \infty$).*

(L2) *Let $\lambda_0(f) > 0$. Then $(\mathfrak{L}f)(z)$ is analytic in $\{0 < \operatorname{Re} z < \lambda_0(f)\}$, and, for any $n \in \mathbb{N}$,*

$$\frac{d^n}{dz^n}(\mathfrak{L}f)(z) = \int_{\mathbb{R}} e^{zs} s^n f(s) \, ds, \quad 0 < \operatorname{Re} z < \lambda_0(f).$$

(L3) *Let $f \geq 0$ a.e. and $0 < \lambda_0(f) < \infty$. Then $(\mathfrak{L}f)(z)$ has a singularity at $z = \lambda_0(f)$. In particular, $\mathfrak{L}f$ has not an analytic extension to a strip $0 < \operatorname{Re} z < \nu$, with $\nu > \lambda_0(f)$.*

(L4) *Let $f' := \frac{d}{ds} f \in L^\infty(\mathbb{R})$, $f(\infty) = 0$, and $\lambda_0(f') > 0$. Then $\lambda_0(f) \geq \lambda_0(f')$ and, for any $0 < \operatorname{Re} z < \lambda_0(f')$,*

$$(\mathfrak{L}f')(z) = -z(\mathfrak{L}f)(z). \tag{4.49}$$

(L5) *Let $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\lambda_0(f) > 0$, $\lambda_0(g) > 0$. Then $\lambda_0(f * g) \geq \min\{\lambda_0(f), \lambda_0(g)\}$ and, for any $0 < \operatorname{Re} z < \min\{\lambda_0(f), \lambda_0(g)\}$,*

$$\bigl(\mathfrak{L}(f * g)\bigr)(z) = (\mathfrak{L}f)(z)(\mathfrak{L}g)(z). \tag{4.50}$$

(L6) *Let* $0 \le f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $\lambda_0(f) > 0$. *Then* $\lim\limits_{\lambda\to 0+} (\mathfrak{L}f)(\lambda) = \int_{\mathbb{R}} f(s)\, ds$.

(L7) *Let* $f \ge 0$, $\lambda_0(f) \in (0,\infty)$ *and* $A := \int_{\mathbb{R}} f(s) e^{\lambda_0(f)s}\, ds < \infty$. *Then* $\lim\limits_{\lambda\to\lambda_0(f)-} (\mathfrak{L}f)(\lambda) = A$.

(L8) *Let* $f \ge 0$ *be decreasing on* $\mathbb{R}$, *and let* $\lambda_0(f) > 0$. *Then, for any* $0 < \lambda < \lambda_0(f)$,

$$f(s) \le \frac{\lambda e^\lambda}{e^\lambda - 1} (\mathfrak{L}f)(\lambda) e^{-\lambda s}, \quad s \in \mathbb{R}. \tag{4.51}$$

Moreover,

$$\lambda_0\bigl(f^2\bigr) \ge 2\lambda_0(f), \tag{4.52}$$

and for any $0 \le g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $\lambda_0(g) > 0$,

$$\lambda_0\bigl(f(g * f)\bigr) \ge \lambda_0(f) + \min\bigl\{\lambda_0(g), \lambda_0(f)\bigr\}. \tag{4.53}$$

Proof. We can rewrite $\mathfrak{L} = \mathfrak{L}^+ + \mathfrak{L}^-$, where

$$(\mathfrak{L}^\pm f)(z) = \int_{\mathbb{R}_\pm} f(s) e^{zs}\, ds, \quad \operatorname{Re} z > 0,$$

$\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_- = (\infty, 0]$. Let $\mathcal{L}$ denote the classical (unilateral) Laplace transform:

$$(\mathcal{L}f)(z) = \int_{\mathbb{R}_+} f(s) e^{-zs}\, ds,$$

and $\mathfrak{l}_0(f)$ be its abscissa of convergence (see details, e.g. in [96, Chapter II]). Then, clearly, $(\mathfrak{L}^+ f)(z) = (\mathcal{L}f)(-z)$, $(\mathfrak{L}^- f)(z) = (\mathcal{L}f^-)(z)$, where $f^-(s) = f(-s)$, $s \in \mathbb{R}$. As a result, $\lambda_0(f) = -\mathfrak{l}_0(f)$.

It is easily seen that, for $f \in L^\infty(\mathbb{R})$, $\mathfrak{l}_0(f^-) \le 0$, in particular,the function $(\mathfrak{L}^- f)(z)$ is analytic on $\operatorname{Re} z > 0$.

Therefore, the properties (L1)–(L3) are direct consequences of [96, Theorems II.1, II.5a, II.5b], respectively. The property (L4) may be easily derived from [96, Theorem II.2.3a, II.2.3b], taking into account that $f(\infty) = 0$. The property (L5) one gets by a straightforward computation, cf. [96, Theorem VI.16a]; note that $f * g \in L^\infty(\mathbb{R})$.

Next, $\lambda_0(f) > 0$ implies $\mathfrak{l}_0(f) < 0$, therefore, $\mathfrak{L}^+ f$ can be analytically continued to 0. If $\mathfrak{l}(f^-) < 0$, then $\mathfrak{L}^- f$ can be analytically continued to 0 as well, and (L6) will be evident. Otherwise, if $\mathfrak{l}(f^-) = 0$ then (L6) follows from [96, Theorem V.1]. Similar arguments prove (L7).

To prove (L8) for decreasing nonnegative f, note that, for any $0 < \lambda < \lambda_0(f)$,

$$f(s) \int_{s-1}^{s} e^{\lambda\tau}\, d\tau \le \int_{s-1}^{s} f(\tau) e^{\lambda\tau}\, d\tau \le (\mathfrak{L}f)(\lambda), \quad s \in \mathbb{R},$$

that implies (4.51). Next, by (L5), $\lambda_0(g * f) > 0$, and conditions on g yield that $g * f \ge 0$ is decreasing as well. Therefore, by (4.51), for any $0 < \lambda < \lambda_0(g * f)$,

$$\begin{aligned}
\bigl|\bigl(\mathfrak{L}(f(g * f))\bigr)(z)\bigr| &\le \int_{\mathbb{R}} f(s)(g * f)(s) e^{s \operatorname{Re} z}\, ds \\
&\le \frac{\lambda e^\lambda}{e^\lambda - 1} \bigl(\mathfrak{L}(g * f)\bigr)(\lambda) \int_{\mathbb{R}} f(s) e^{-s\lambda} e^{s \operatorname{Re} z}\, ds < \infty,
\end{aligned}$$

provided that $\operatorname{Re} z < \lambda_0(f) + \lambda < \lambda_0(f) + \lambda_0(g * f)$. As a result, $\lambda_0\bigl(f(g * f)\bigr) \ge \lambda_0(f) + \lambda_0(g * f)$ that, by (L5), implies (4.53). Similarly one can prove (4.52). $\square$

Fix any $\xi \in S^{d-1}$. Then, by (4.11), one has that $\lambda_0(\check{a}^\pm) > 0$. Consider, cf. (4.15), (4.19), the following complex-valued function

$$G_\xi(z) := \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(z) - m}{z}, \quad \operatorname{Re} z > 0, \tag{4.54}$$

which is well-defined on $0 < \operatorname{Re} z < \lambda_0(\check{a}^+)$. Note that, by (4.11),

$$(\mathfrak{L}\check{a}^+)(\lambda) = \mathfrak{a}_\xi(\lambda), \qquad G_\xi(\lambda) = \frac{\varkappa^+\mathfrak{a}_\xi(\lambda) - m}{\lambda}, \qquad 0 < \lambda < \lambda_0(\check{a}^+),$$

and hence, by (4.19),

$$c_*(\xi) \le \inf_{\lambda>0} G_\xi(\lambda), \tag{4.55}$$

where $c_*(\xi)$ is the minimal speed of traveling waves, cf. Theorem 4.9. We will show below that in fact there exists equality in (4.55), and hence in (4.19).

We start with the following notations to simplify the further statements.

Definition 4.17. Let $m > 0$, $\varkappa^\pm > 0$, $0 \le a^- \in L^1(\mathbb{R}^d)$ be fixed, and (A1) holds. For an arbitrary $\xi \in S^{d-1}$, denote by $\mathcal{U}_\xi$ the subset of functions $0 \le a^+ \in L^1(\mathbb{R}^d)$ such that (A2) and (A5)–(A8) hold.

For $a^+ \in \mathcal{U}_\xi$, denote also the interval $I_\xi \subset (0,\infty)$ by

$$I_\xi := \begin{cases} (0,\infty), & \text{if } \ \lambda_0(\check{a}^+) = \infty, \\ \bigl(0, \lambda_0(\check{a}^+)\bigr), & \text{if } \lambda_0(\check{a}^+) < \infty \ \text{ and } \ (\mathfrak{L}\check{a}^+)\bigl(\lambda_0(\check{a}^+)\bigr) = \infty \\ \bigl(0, \lambda_0(\check{a}^+)\bigr], & \text{if } \lambda_0(\check{a}^+) < \infty \ \text{ and } \ (\mathfrak{L}\check{a}^+)\bigl(\lambda_0(\check{a}^+)\bigr) < \infty. \end{cases}$$

Proposition 4.18. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Then there exists a unique $\lambda_* = \lambda_*(\xi) \in I_\xi$ such that*

$$\inf_{\lambda>0} G_\xi(\lambda) = \min_{\lambda \in I_\xi} G_\xi(\lambda) = G_\xi(\lambda_*) > \varkappa^+\mathfrak{m}_\xi. \tag{4.56}$$

Moreover, G_ξ is strictly decreasing on $(0,\lambda_*]$ and G_ξ is strictly increasing on $I_\xi \setminus (0,\lambda_*]$ (the latter interval may be empty).

Proof. First of all, by (4.6), the condition (A7) implies, cf. (4.11),

$$\mathfrak{m}_\xi = \int_{\mathbb{R}} s\check{a}^+(s)\,ds \in \mathbb{R}. \tag{4.57}$$

Next, to simplify notations, we set $\lambda_0 := \lambda_0(\check{a}^+) \in (0,\infty]$. Denote also

$$F_\xi(\lambda) := \varkappa^+\mathfrak{a}_\xi(\lambda) - m = \lambda G_\xi(\lambda), \qquad \lambda \in I_\xi. \tag{4.58}$$

By (L2), for any $\lambda \in (0,\lambda_0)$,

$$\mathfrak{a}_\xi''(\lambda) = \int_{\mathbb{R}} s^2\check{a}^+(s)e^{\lambda s}\,ds > 0, \tag{4.59}$$

therefore, $\mathfrak{a}_\xi'(\lambda)$ is increasing on $(0,\lambda_0)$; in particular, by (4.57), we have, for any $\lambda\in(0,\lambda_0)$,

$$\int_{\mathbb{R}} s\breve{a}^+(s)e^{\lambda s}\,ds = \mathfrak{a}_\xi'(\lambda) > \mathfrak{a}_\xi'(0) = \int_{\mathbb{R}} s\breve{a}^+(s)\,ds = \mathfrak{m}_\xi. \tag{4.60}$$

Next, by (L6), $F_\xi(0+) = \varkappa^+ - m > 0$, hence,

$$G_\xi(0+) = \infty. \tag{4.61}$$

Finally, for $\lambda\in(0,\lambda_0)$, we have

$$G_\xi'(\lambda) = \lambda^{-2}\bigl(\lambda F_\xi'(\lambda) - F_\xi(\lambda)\bigr) = \lambda^{-1}\bigl(F_\xi'(\lambda) - G_\xi(\lambda)\bigr), \tag{4.62}$$
$$G_\xi''(\lambda) = \lambda^{-1}(F_\xi''(\lambda) - 2G_\xi'(\lambda)). \tag{4.63}$$

We will distinguish two cases.

Case 1. There exists $\mu\in(0,\lambda_0)$ with $G_\xi'(\mu)=0$. Then, by (4.63), (4.59),

$$G_\xi''(\mu) = \mu^{-1}F_\xi''(\mu) = \mu^{-1}\varkappa^+\mathfrak{a}_\xi''(\mu) > 0.$$

Hence any stationary point of G_ξ is with necessity a point of local minimum, therefore, G_ξ has at most one such a point, thus it will be a global minimum. Moreover, by (4.62), (4.60), $G'(\mu)=0$ implies

$$G_\xi(\mu) = F_\xi'(\mu) = \varkappa^+\mathfrak{a}_\xi'(\mu) > \varkappa^+\mathfrak{m}_\xi. \tag{4.64}$$

Therefore, in the Case 1, one can choose $\lambda_*=\mu$ (which is unique then) to fulfill the statement.

List the conditions under which the Case 1 is possible.

1. Let $\lambda_0=\infty$. Note that (A8) implies that there exist $\delta'>0$, $\rho'>0$, such that $\breve{a}^+(s)\geq\rho'$, for a.a. $s\in[r-\delta',r+\delta']$. Indeed, fix, for the case $d\geq 2$, a basis $\eta_1,\ldots,\eta_{d-1}$ of $H_\xi=\{\xi\}^\perp$, cf. definition of (4.6), then

   $$B_\delta(r\xi)\supset\Bigl\{(r+\sigma)\xi+\tau_1\eta_1+\ldots+\tau_{d-1}\eta_{d-1}\Bigm| |\sigma|\leq\frac{\delta}{\sqrt{d}}, |\tau_i|\leq\frac{\delta}{\sqrt{d}}\Bigr\}.$$

   Therefore, by (4.6) and (A8),

   $$\breve{a}^+(s)\geq\rho\Bigl(\frac{2\delta}{\sqrt{d}}\Bigr)^{d-1} =: \rho', \quad s\in[r-\delta',r+\delta'], \quad \delta' := \frac{\delta}{\sqrt{d}}. \tag{4.65}$$

   Hence if $\lambda_0=\infty$, then

   $$\frac{1}{\lambda}\mathfrak{a}_\xi(\lambda)\geq\frac{1}{\lambda}\int_r^{r+\delta'}\breve{a}^+(s)e^{\lambda s}\,ds\geq\rho'\frac{1}{\lambda^2}\bigl(e^{\lambda(r+\delta')}-e^{\lambda r}\bigr)\to\infty, \tag{4.66}$$

   as $\lambda\to\infty$. Then, in such a case, $G_\xi(\infty)=\infty$. Therefore, by (4.61), there exists a zero of G_ξ'.

2. Let $\lambda_0<\infty$ and $\mathfrak{a}_\xi(\lambda_0)=\infty$. Then, again, (4.61) implies the existence of a zero of G_ξ' on $(0,\lambda_0)$.

3. Let $\lambda_0 < \infty$ and $\mathfrak{a}_\xi(\lambda_0) < \infty$. By (4.58), (4.62),

$$\lim_{\lambda\to 0+} \lambda^2 G'_\xi(\lambda) = -F_\xi(0+) = -(\varkappa^+ - m) < 0.$$

Therefore, the function G'_ξ has a zero on $(0, \lambda_0)$ if and only if takes a positive value at some point from $(0, \lambda_0)$.

Now, one can formulate and consider the opposite to the Case 1.

Case 2. Let $\lambda_0 < \infty$, $\mathfrak{a}_\xi(\lambda_0) < \infty$, and

$$G'_\xi(\lambda) < 0, \quad \lambda \in (0, \lambda_0). \tag{4.67}$$

Therefore,

$$\inf_{\lambda>0} G_\xi(\lambda) = \inf_{\lambda\in(0,\lambda_0]} G_\xi(\lambda) = \lim_{\lambda\to\lambda_0-} G_\xi(\lambda) = G_\xi(\lambda_0), \tag{4.68}$$

by (L7). Hence we have the first equality in (4.56), by setting $\lambda_* := \lambda_0$. To prove the second inequality in (4.56), note that, by (4.62), the inequality (4.67) is equivalent to $F'_\xi(\lambda) < G_\xi(\lambda)$, $\lambda \in (0, \lambda_0)$. Therefore, by (4.68), (4.58), (4.60),

$$G_\xi(\lambda_0) = \inf_{\lambda\in\left(\frac{\lambda_0}{2},\lambda_0\right)} G_\xi(\lambda) \geq \inf_{\lambda\in\left(\frac{\lambda_0}{2},\lambda_0\right)} F'_\xi(\lambda) \geq \varkappa^+ \mathfrak{a}'_\xi\Bigl(\frac{\lambda_0}{2}\Bigr) > \varkappa^+ \mathfrak{m}_\xi,$$

where we used again that, by (4.59), $\mathfrak{a}'_\xi$ and hence F'_ξ are increasing on $(0, \lambda_0)$. The statement is fully proved now. □

The second case in the proof of Proposition 4.18 requires additional analysis. Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$, $\lambda_0 := \lambda_0(\check{a}^+)$. By (L2), one can define the following function

$$\mathfrak{t}_\xi(\lambda) := \varkappa^+ \int_{\mathbb{R}} (1 - \lambda s)\check{a}^+(s) e^{\lambda s}\, ds \in \mathbb{R}, \qquad \lambda \in [0, \lambda_0). \tag{4.69}$$

Note that

$$\int_{\mathbb{R}_-} |s| \check{a}^+(s) e^{\lambda_0 s}\, ds < \infty, \tag{4.70}$$

and $\int_{\mathbb{R}_+} s\check{a}^+(s)e^{\lambda_0 s}\, ds \in (0, \infty]$ is well-defined. Then, in the case $\lambda_0 < \infty$ and $\mathfrak{a}_\xi(\lambda_0) < \infty$, one can continue $\mathfrak{t}_\xi$ at λ_0, namely,

$$\mathfrak{t}_\xi(\lambda_0) := \varkappa^+ \int_{\mathbb{R}} (1 - \lambda_0 s)\check{a}^+(s) e^{\lambda_0 s}\, ds \in [-\infty, \varkappa^+). \tag{4.71}$$

To prove the latter inclusion, i.e. that $\mathfrak{t}_\xi(\lambda_0) < \varkappa^+$, consider the function $f_0(s) := (1 - \lambda_0 s)e^{\lambda_0 s}$, $s \in \mathbb{R}$. Then, $f'_0(s) = -\lambda_0^2 s e^{\lambda_0 s}$, and thus $f_0(s) < f_0(0) = 1$, $s \neq 0$. Moreover, the function $g_0(s) = f_0(-s) - f_0(s)$, $s \geq 0$ is such that $g'_0(s) = \lambda_0^2 s(e^{\lambda_0 s} - e^{-\lambda_0 s}) > 0$, $s > 0$. As a result, for any $\delta > 0$, $f_0(-\delta) > f_0(\delta)$, and

$$\int_{\mathbb{R}} f_0(s)\check{a}^+(s)\, ds \leq f_0(-\delta) \int_{\mathbb{R}\setminus[-\delta,\delta]} \check{a}^+(s)\, ds + \int_{[-\delta,\delta]} \check{a}^+(s)\, ds < \int_{\mathbb{R}} \check{a}^+(s)\, ds = 1.$$

Proposition 4.19. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Suppose also that $\lambda_0 := \lambda_0(\check{a}^+) < \infty$ and $\mathfrak{a}_\xi(\lambda_0) < \infty$. Then (4.67) holds iff*

$$\mathfrak{t}_\xi(\lambda_0) \in (0, \varkappa^+), \tag{4.72}$$

$$m \leq \mathfrak{t}_\xi(\lambda_0). \tag{4.73}$$

Proof. Define the function, cf. (4.58),

$$H_\xi(\lambda) := \lambda F'_\xi(\lambda) - F_\xi(\lambda), \quad \lambda \in (0, \lambda_0). \tag{4.74}$$

By (4.62), the condition (4.67) holds iff H_ξ is negative on $(0, \lambda_0)$. By (4.74), (4.59), one has $H'_\xi(\lambda) = \lambda F''_\xi(\lambda) > 0$, $\lambda \in (0, \lambda_0)$ and, therefore, H_ξ is (strictly) increasing on $(0, \lambda_0)$. By Proposition 4.18, G'_ξ, and hence H_ξ, are negative on a right-neighborhood of 0. As a result, $H_\xi(\lambda) < 0$ on $(0, \lambda_0)$ iff

$$\lim_{\lambda \to \lambda_0 -} H_\xi(\lambda) \leq 0. \tag{4.75}$$

On the other hand, by (4.58), (4.69), one can rewrite $H_\xi(\lambda)$ as follows:

$$H_\xi(\lambda) = -\mathfrak{t}_\xi(\lambda) + m, \quad \lambda \in (0, \lambda_0). \tag{4.76}$$

By the monotone convergence theorem,

$$\lim_{\lambda \to \lambda_0 -} \int_{\mathbb{R}_+} (\lambda s - 1)\check{a}^+(s) e^{\lambda s}\, ds = \int_{\mathbb{R}_+} (\lambda_0 s - 1)\check{a}^+(s) e^{\lambda_0 s}\, ds \in (-1, \infty].$$

Therefore, by (4.70), (4.76), $\mathfrak{t}_\xi(\lambda_0) \in \mathbb{R}$ iff $H_\xi(\lambda_0) = \lim\limits_{\lambda \to \lambda_0 -} H_\xi(\lambda) \in \mathbb{R}$. Next, clearly, $H_\xi(\lambda_0) \in (m - \varkappa^+, 0]$ holds true iff both (4.73) and (4.72) hold.

As a result, (4.67) is equivalent to (4.75) and the latter, by (4.70), implies that $\mathfrak{t}_\xi(\lambda_0) \in \mathbb{R}$ and hence $H_\xi(\lambda_0) \in (m - \varkappa^+, 0]$. Vice versa, (4.72) yields $\mathfrak{t}_\xi(\lambda_0) \in \mathbb{R}$ that together with (4.73) give that $H_\xi(\lambda_0) \leq 0$, i.e. that (4.67) holds. □

According to the above, it is natural to consider two subclasses of functions from $\mathcal{U}_\xi$, cf. Definition 4.17.

Definition 4.20. Let $\xi \in S^{d-1}$ be fixed. We denote by $\mathcal{V}_\xi$ the class of all kernels $a^+ \in \mathcal{U}_\xi$ such that one of the following assumptions does hold:

1. $\lambda_0 := \lambda_0(\check{a}^+) = \infty$;
2. $\lambda_0 < \infty$ and $\mathfrak{a}_\xi(\lambda_0) = \infty$;
3. $\lambda_0 < \infty$, $\mathfrak{a}_\xi(\lambda_0) < \infty$ and $\mathfrak{t}_\xi(\lambda_0) \in [-\infty, m)$, where $\mathfrak{t}_\xi(\lambda_0)$ is given by (4.71).

Correspondingly, we denote by $\mathcal{W}_\xi$ the class of all kernels $a^+ \in \mathcal{U}_\xi$ such that $\lambda_0 < \infty$, $\mathfrak{a}_\xi(\lambda_0) < \infty$, and $\mathfrak{t}_\xi(\lambda_0) \in [m, \varkappa^+)$. Clearly, $\mathcal{U}_\xi = \mathcal{V}_\xi \cup \mathcal{W}_\xi$.

As a result, combining the proofs and statements of Propositions 4.18 and 4.19, one immediately gets the following corollary.

Corollary 4.21. *Let $\xi \in S^{d-1}$ be fixed, $a^+ \in \mathcal{U}_\xi$, and λ_* be the same as in Proposition 4.18. Then $\lambda_* < \lambda_0 := \lambda_0(\check{a}^+)$ iff $a^+ \in \mathcal{V}_\xi$; moreover, then $G'(\lambda_*) = 0$. Correspondingly, $\lambda_* = \lambda_0$ iff $a^+ \in \mathcal{W}_\xi$; in this case,*

$$\lim_{\lambda \to \lambda_0 -} G'_\xi(\lambda) = \frac{m - \mathfrak{t}_\xi(\lambda_0)}{\lambda_0^2} \leq 0. \tag{4.77}$$

Example 4.22. To demonstrate the cases of Definition 4.20 on an example, consider the following family of functions

$$\check{a}^+(s) := \frac{\alpha e^{-\mu|s|^p}}{1+|s|^q}, \quad s\in\mathbb{R}, p\geq 0, q\geq 0, \mu>0, \tag{4.78}$$

where $\alpha>0$ is a normalising constant to get (4.7). Clearly, the case $p\in[0,1)$ implies $\lambda_0(\check{a}^+)=0$, that is impossible under assumption (A5). Next, $p>1$ leads to $\lambda_0(\check{a}^+)=\infty$, in particular, the corresponding $a^+\in\mathcal{V}_\xi$. Let now $p=1$, then $\lambda_0(\check{a}^+)=\mu$. The case $q\in[0,1]$ gives $\mathfrak{a}_\xi(\lambda_0)=\infty$, i.e. $a^+\in\mathcal{V}_\xi$ as well. In the case $q\in(1,2]$, we will have that $\mathfrak{a}_\xi(\lambda_0)<\infty$, however, $\int_{\mathbb{R}} s\check{a}^+(s)e^{\mu s}\,ds=\infty$, i.e. $\mathfrak{t}_\xi(\mu)=-\infty$, and again $a^+\in\mathcal{V}_\xi$. Let $q>2$; then, by (4.69),

$$\begin{aligned}\mathfrak{t}_\xi(\mu) &= \varkappa^+\alpha\int_{\mathbb{R}_-}\frac{1-\mu s}{1+|s|^q}e^{2\mu s}\,ds+\varkappa^+\alpha\int_{\mathbb{R}_+}\frac{1-\mu s}{1+s^q}\,ds\\ &\geq \varkappa^+\alpha\int_{\mathbb{R}_+}\frac{1-\mu s}{1+s^q}\,ds=\frac{\pi\varkappa^+\alpha}{q}\left(\frac{1}{\sin\frac{\pi}{q}}-\frac{\mu}{\sin\frac{2\pi}{q}}\right)\geq m,\end{aligned}$$

if only $\mu\leq 2\cos\frac{\pi}{q}-\frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q}$ (note that $q>2$ implies $\sin\frac{2\pi}{q}>0$); then we have $a^+\in\mathcal{W}_\xi$. On the other hand, using the inequality $te^{-t}\leq e^{-1}$, $t\geq 0$, one gets

$$\begin{aligned}\mathfrak{t}_\xi(\mu) &= \varkappa^+\alpha\int_{\mathbb{R}_+}\frac{(1+\mu s)e^{-2\mu s}+1-\mu s}{1+s^q}\,ds \\ &\leq \varkappa^+\alpha\int_{\mathbb{R}_+}\frac{1+\frac{1}{2e}+1-\mu s}{1+s^q}\,ds=\frac{\pi\varkappa^+\alpha}{q}\left(\frac{1+4e}{2e\sin\frac{\pi}{q}}-\frac{\mu}{\sin\frac{2\pi}{q}}\right)<m,\end{aligned}\tag{4.79}$$

if only $\mu>\frac{1+4e}{e}\cos\frac{\pi}{q}-\frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q}$; then we have $a^+\in\mathcal{V}_\xi$. Since

$$\frac{d}{d\mu}\bigl((1+\mu s)e^{-2\mu s}+1-\mu s\bigr)=-se^{-2\mu s}(1+2s\mu)-s<0,\quad s>0,\mu>0,$$

we have from (4.79), that $\mathfrak{t}_\xi(\mu)$ is strictly decreasing and continuous in μ, therefore, there exist a critical value

$$\mu_*\in\Bigl(2\cos\frac{\pi}{q}-\frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q},(4+e^{-1})\cos\frac{\pi}{q}-\frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q}\Bigr),$$

such that, for all $\mu>\mu_*$, $a^+\in\mathcal{V}_\xi$, whereas, for $\mu\in(0,\mu_*]$, $a^+\in\mathcal{W}_\xi$.

Now we are ready to prove the main statement of this subsection.

Theorem 4.23. *Let $\xi\in S^{d-1}$ be fixed and $a^+\in\mathcal{U}_\xi$. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 4.9, and let, for any $c\geq c_*(\xi)$, the function $\psi=\psi_c\in\mathcal{M}_\theta(\mathbb{R})$ be a traveling wave profile corresponding to the speed c. Let $\lambda_*\in I_\xi$ be the same as in Proposition 4.18. Denote, as usual, $\lambda_0:=\lambda_0(\check{a}^+)$.*

1. *The following relations hold*

$$c_*(\xi)=\min_{\lambda>0}\frac{\varkappa^+\mathfrak{a}_\xi(\lambda)-m}{\lambda}=\frac{\varkappa^+\mathfrak{a}_\xi(\lambda_*)-m}{\lambda_*}>\varkappa^+\mathfrak{m}_\xi, \tag{4.80}$$

$$\lambda_0(\psi)\in(0,\lambda_*], \tag{4.81}$$

$$(\mathfrak{L}\psi)\bigl(\lambda_0(\psi)\bigr)=\infty; \tag{4.82}$$

and the mapping $(0,\lambda_*] \ni \lambda_0(\psi) \mapsto c \in [c_*(\xi),\infty)$ *is a (strictly) monotonically decreasing bijection, given by*

$$c = \frac{\varkappa^+ \mathfrak{a}_\xi\bigl(\lambda_0(\psi)\bigr) - m}{\lambda_0(\psi)}. \tag{4.83}$$

In particular, $\lambda_0(\psi_{c_*(\xi)}) = \lambda_*$.

2. *For* $a^+ \in \mathcal{V}_\xi$*, one has* $\lambda_* < \lambda_0$ *and there exists another representation for the minimal speed than* (4.83)*, namely,*

$$\begin{aligned} c_*(\xi) &= \varkappa^+ \int_{\mathbb{R}^d} x\cdot\xi\, a^+(x) e^{\lambda_* x\cdot\xi}\, dx \\ &= \varkappa^+ \int_{\mathbb{R}} s\check{a}^+(s) e^{\lambda_* s}\, ds > \varkappa^+ \mathfrak{m}_\xi. \end{aligned} \tag{4.84}$$

Moreover, for all $\lambda \in (0,\lambda_*]$,

$$\mathfrak{t}_\xi(\lambda) \geq m, \tag{4.85}$$

and the equality holds for $\lambda = \lambda_*$ *only.*

3. *For* $a^+ \in \mathcal{W}_\xi$*, one has* $\lambda_* = \lambda_0$*. Moreover, the inequality* (4.85) *also holds as well as, for all* $\lambda \in (0,\lambda_*]$,

$$c \geq \varkappa^+ \int_{\mathbb{R}} s\check{a}^+(s) e^{\lambda s}\, ds, \tag{4.86}$$

whereas the equalities in (4.85) *and* (4.86) *hold true now for* $m = \mathfrak{t}_\xi(\lambda_0)$*,* $\lambda = \lambda_*$*,* $c = c_*(\xi)$ *only.*

Proof. By Theorem 4.9, for any $c \geq c_*(\xi)$, there exists a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$, cf. Remark 4.7, which define a traveling wave solution (4.1) to (2.1) in the direction ξ. Then, by (4.32), we get

$$-c\psi'(s) = \varkappa^+(\check{a}^+ * \psi)(s) - m\psi(s) - \kappa_1\psi^2(s) - \kappa_2\psi(s)(\check{a}^- * \psi)(s), \quad s \in \mathbb{R}. \tag{4.87}$$

Step 1. By Proposition 4.14, we have that $\lambda_0(\psi) > 0$. Note also that the condition (A2) implies (4.9), therefore, $\lambda_0(\check{a}^-) \geq \lambda_0(\check{a}^+) > 0$, if $\kappa_2 > 0$. Take any $z \in \mathbb{C}$ with

$$0 < \operatorname{Re} z < \min\bigl\{\lambda_0(\check{a}^+), \lambda_0(\psi)\bigr\} \leq \lambda_0(\psi) < \min\{\lambda_0(\psi^2), \lambda_0\bigl(\psi(\check{a}^- * \psi)\bigr)\}, \tag{4.88}$$

where the later inequality holds by (4.52) and (4.53). As a result, by (L5), (L8), being multiplied on e^{zs} the l.h.s. of (4.87) will be integrable (in s) over $\mathbb{R}$. Hence, for any z which satisfies (4.88), $(\mathfrak{L}\psi')(z)$ converges. By (L4), it yields $\lambda_0(\psi) \geq \lambda_0(\psi') \geq \min\bigl\{\lambda_0(\check{a}^+), \lambda_0(\psi)\bigr\}$.

Therefore, by (4.49), (4.50), we get from (4.87)

$$\begin{aligned} cz(\mathfrak{L}\psi)(z) = \varkappa^+(\mathfrak{L}\check{a}^+)(z)(\mathfrak{L}\psi)(z) - m(\mathfrak{L}\psi)(z) \\ - \kappa_1\bigl(\mathfrak{L}(\psi^2)\bigr)(z) - \kappa_2\bigl(\mathfrak{L}(\psi(\check{a}^- * \psi))\bigr)(z), \end{aligned} \tag{4.89}$$

if only

$$0 < \operatorname{Re} z < \min\bigl\{\lambda_0(\check{a}^+), \lambda_0(\psi)\bigr\}. \tag{4.90}$$

Since $\psi \not\equiv 0$, we have that $(\mathfrak{L}\psi)(z) \neq 0$, therefore, one can rewrite (4.89) as follows

$$G_\xi(z) - c = \frac{\kappa_1\bigl(\mathfrak{L}(\psi^2)\bigr)(z) + \kappa_2\bigl(\mathfrak{L}(\psi(\check{a}^- * \psi))\bigr)(z)}{z(\mathfrak{L}\psi)(z)}, \tag{4.91}$$

if (4.90) holds. By (4.88), both nominator and denominator in the r.h.s. of (4.91) are analytic on $0 < \operatorname{Re} z < \lambda_0(\psi)$, therefore. Suppose that $\lambda_0(\psi) > \lambda_0(\check{a}^+)$, then (4.91) holds on $0 < \operatorname{Re} z < \lambda_0(\check{a}^+)$, however, the r.h.s. of (4.91) would be analytic at $z = \lambda_0(\check{a}^+)$, whereas, by (L3), the l.h.s. of (4.91) has a singularity at this point. As a result,

$$\lambda_0(\check{a}^+) \geq \lambda_0(\psi), \tag{4.92}$$

for any traveling wave profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$. Thus one gets that (4.91) holds true on $0 < \operatorname{Re} z < \lambda_0(\psi)$.

Prove that

$$\lambda_0(\psi) < \infty. \tag{4.93}$$

Since $0 \leq \psi \leq \theta$ yields $0 \leq a^- * \psi \leq \theta$, one gets from (4.91) that, for any $0 < \lambda < \lambda_0(\psi)$,

$$c \geq G_\xi(\lambda) - \varkappa^- \frac{\theta}{\lambda} = \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(\lambda) - \varkappa^+}{\lambda}. \tag{4.94}$$

If $\lambda_0(\check{a}^+) < \infty$ then (4.93) holds by (4.92). Suppose that $\lambda_0(\check{a}^+) = \infty$. By (4.66), the r.h.s. of (4.94) tends to ∞ as $\lambda \to \infty$, thus the latter inequality cannot hold for all $\lambda > 0$; and, as a result, (4.93) does hold.

Step 2. Recall that (4.55) holds. Suppose that $c \geq c_*(\xi)$ is such that, cf. (4.56),

$$c \geq G_\xi(\lambda_*) = \inf_{\lambda_0 \in (0,\lambda_*]} G_\xi(\lambda) = \inf_{\lambda_0 \in I_\xi} G_\xi(\lambda). \tag{4.95}$$

Then, by Proposition 4.18, the equation $G_\xi(\lambda) = c$, $\lambda \in I_\xi$, has one or two solutions. Let λ_c be the unique solution in the first case or the smaller of the solutions in the second one. Since G_ξ is decreasing on $(0, \lambda_*]$, we have $\lambda_c \leq \lambda_*$. Since the nominator in the r.h.s. of (4.91) is positive, we immediately get from (4.91) that

$$(\mathfrak{L}\psi)(\lambda_c) = \infty, \tag{4.96}$$

therefore, $\lambda_c \geq \lambda_0(\psi)$. On the other hand, one can rewrite (4.91) as follows

$$(\mathfrak{L}\psi)(z) = \frac{\kappa_1\bigl(\mathfrak{L}(\psi^2)\bigr)(z) + \kappa_2\bigl(\mathfrak{L}(\psi(\check{a}^- * \psi))\bigr)(z)}{z(G_\xi(z) - c)}. \tag{4.97}$$

By (4.91), $G_\xi(z) \neq c$, for all $0 < \operatorname{Re} z < \lambda_0(\psi) \leq \lambda_c \leq \lambda_* \leq \lambda_0(\check{a}^+)$. As a result, by (4.88), (L1), and (L3), $\lambda_c = \lambda_0(\psi)$, that together with (4.96) proves (4.81) and (4.82), for waves whose speeds satisfy (4.95). By (4.11), we immediately get, for such speeds, (4.83) as well. Moreover, (4.83) defines a strictly monotone function $(0, \lambda_*] \ni \lambda_0(\psi) \mapsto c \in [G_\xi(\lambda_*), \infty)$.

Next, by (4.69), (L2), (4.58), (4.62), we have that, for any $\lambda \in I_\xi$,

$$\mathfrak{t}_\xi(\lambda) = \varkappa^+ \mathfrak{a}_\xi(\lambda) - \varkappa^+ \lambda \mathfrak{a}'_\xi(\lambda) = m + F_\xi(\lambda) - \lambda F'_\xi(\lambda) = m - \lambda^2 G'_\xi(\lambda). \tag{4.98}$$

Recall that, by Proposition 4.18, the function G_ξ is strictly decreasing on $(0, \lambda_*)$. Then (4.98) implies that $\mathfrak{t}_\xi(\lambda) > m$, $\lambda \in (0, \lambda_*)$. On the other hand, by the second equality in (4.62), the inequality $G_\xi'(\lambda) < 0$, $\lambda \in (0, \lambda_*)$, yields $G_\xi(\lambda) > F_\xi'(\lambda)$, for such a λ. Let $c > G_\xi(\lambda_*)$. By (4.83), (4.58), we have then $c > \varkappa^+ \mathfrak{a}_\xi'(\lambda)$, for all $\lambda \in [\lambda_0(\psi), \lambda_*)$. By (4.59), F_ξ' is increasing, hence, by (L2), the strict inequality in (4.86) does hold, for $\lambda \in (0, \lambda_*)$.

Let again $c \geq G_\xi(\lambda_*)$, and let $a^+ \in \mathcal{V}_\xi$. Then, by Corollary 4.21, $\lambda_* < \lambda_0(\check{a}^+)$ and $G'(\lambda_*) = 0$. By (4.62), the latter equality and (4.98) give $\mathfrak{t}_\xi(\lambda_*) = m$, that fulfills the proof of (4.85), for such a^+ and m. Moreover, by (4.64),

$$G_\xi(\lambda_*) = \varkappa^+ \mathfrak{a}_\xi'(\lambda_*) = \varkappa^+ \int_{\mathbb{R}} s\check{a}^+(s) e^{\lambda_* s}\, ds. \tag{4.99}$$

Let $a^+ \in \mathcal{W}_\xi$, then $\lambda_* = \lambda_0(\check{a}^+)$. It means that $\mathfrak{t}_\xi(\lambda_*) = m$ if $m = \mathfrak{t}_\xi(\lambda_0)$ only, otherwise, $\mathfrak{t}_\xi(\lambda_*) > m$. Next, we get from (4.95), (4.62) (4.77),

$$c \geq G_\xi(\lambda_*) \geq \lim_{\lambda \to \lambda_* -} F_\xi'(\lambda) = \varkappa^+ \int_{\mathbb{R}} s\check{a}^+(s) e^{\lambda_* s}\, ds, \tag{4.100}$$

where the latter equality may be easily verified if we rewrite, for $\lambda \in (0, \lambda_*)$,

$$F_\xi'(\lambda) = \varkappa^+ \int_{\mathbb{R}_-} s\check{a}^+(s) e^{\lambda s}\, ds + \varkappa^+ \int_{\mathbb{R}_+} s\check{a}^+(s) e^{\lambda s}\, ds, \tag{4.101}$$

and apply the dominated convergence theorem to the first integral and the monotone convergence theorem for the second one. On the other hand, (4.77) implies that the second inequality in (4.100) will be strict iff $m < \mathfrak{t}_\xi(\lambda_0)$, whereas, for $c = G_\xi(\lambda_*) = \inf\limits_{\lambda>0} G_\xi(\lambda)$ and $m = \mathfrak{t}_\xi(\lambda_0)$, we will get all equalities in (4.100).

Step 3. Let now $c \geq c_*(\xi)$ and suppose that $\lambda_0(\check{a}^+) > \lambda_0(\psi)$. Prove that (4.95) does hold. On the contrary, suppose that the c is such that

$$c_*(\xi) \leq c < \inf_{\lambda \in (0, \lambda_*]} G_\xi(\lambda) = \inf_{\lambda>0} G_\xi(\lambda). \tag{4.102}$$

Again, by (4.91), $G_\xi(z) \neq c$, for all $0 < \operatorname{Re} z < \lambda_0(\psi)$, and (4.97) holds, for such a z. Since we supposed that $\lambda_0(\check{a}^+) > \lambda_0(\psi)$, one gets from (4.88), that both nominator and denominator of the r.h.s. of (4.97) are analytic on

$$\{0 < \operatorname{Re} z < \nu\} \supsetneq \{0 < \operatorname{Re} z < \lambda_0(\psi)\},$$

where $\nu = \min\bigl\{\lambda_0(\check{a}^+), \lambda_0\bigl(\psi(\check{a}^- * \psi)\bigr), \lambda_0(\psi^2)\bigr\}$. On the other hand, (L3) implies that $\mathfrak{L}\psi$ has a singularity at $z = \lambda_0(\psi)$. Since

$$\min\{\bigl(\mathfrak{L}(\psi^2)\bigr)(\lambda_0(\psi)), \bigl(\mathfrak{L}(\psi(\check{a}^- * \psi))\bigr)(\lambda_0(\psi))\} > 0,$$

the equality (4.97) would be possible if only $G_\xi(\lambda_0(\psi)) = c$, that contradicts (4.102).

Step 4. By (4.92), it remains to prove that, for $c \geq c_*(\xi)$, (4.95) does holds, provided that we have $\lambda_0(\check{a}^+) = \lambda_0(\psi)$. Again on the contrary, suppose that (4.102) holds. For $0 < \operatorname{Re} z < \lambda_0(\psi)$, we can rewrite (4.89) as follows

$$z(\mathfrak{L}\psi)(z)(G_\xi(z) - c) = \kappa_1\bigl(\mathfrak{L}(\psi^2)\bigr)(z) + \kappa_2\bigl(\mathfrak{L}(\psi(\check{a}^- * \psi))\bigr)(z). \tag{4.103}$$

In the notations of the proof of Lemma 4.16, the functions $\mathfrak{L}^-\psi$ and $\mathfrak{L}^-\check{a}^+$ are analytic on $\operatorname{Re} z > 0$. Moreover, $(\mathfrak{L}^+\psi)(\lambda)$ and $(\mathfrak{L}^+\check{a}^+)(\lambda)$ are increasing on $0 < \lambda < \lambda_0(\check{a}^+) = \lambda_0(\psi)$. Then, cf. (4.101), by the monotone convergence theorem, we will get from (4.103) and (4.88), that

$$\int_{\mathbb{R}} \psi(s) e^{\lambda_0(\psi)s}\, ds < \infty, \qquad \int_{\mathbb{R}} \check{a}^+(s) e^{\lambda_0(\check{a}^+)s}\, ds < \infty. \tag{4.104}$$

We are going to apply now Proposition 3.19, in the case $d = 1$, to the equation (4.4), where the initial condition ψ is a wave profile with the speed c which satisfies (4.102). Namely, we set $\Delta_R := (-\infty, R) \nearrow \mathbb{R}$, $R \to \infty$, and let $\check{a}^\pm_R$, $\check{A}^\pm_R$ be given by (3.37), (3.39) respectively with $d = 1$ and $a^\pm$ replaced by $\check{a}^\pm$. Consider a strictly monotonic sequence $\{R_n \mid n \in \mathbb{N}\}$, such that $0 < R_n \to \infty$, $n \to \infty$ and

$$\check{A}^+_{R_n} > \frac{m}{\varkappa^+}, \tag{4.105}$$

cf. (3.41). Let $\theta_n := \theta_{R_n}$ be given by (3.40) with $A^\pm_R$ replaced by $\check{A}^\pm_{R_n}$. Then, by (3.44), $\theta_n \leq \theta$, $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. Consider the 'truncated' equation (3.38) with $d = 1$, $a^\pm_R$ replaced by $a^\pm_{R_n}$, and the initial condition $w_0(s) := \min\{\psi(s), \theta_n\} \in C_{ub}(\mathbb{R})$. By Proposition 3.19, there exists the unique solution $w^{(n)}(s,t)$ of the latter equation. Moreover, if we denote the corresponding nonlinear mapping, cf. Definition 3.15 and Proposition 4.8, by $\tilde{Q}^{(n)}_t$, we will have from (3.42) and (3.43) that

$$(\tilde{Q}^{(n)}_t w_0)(s) \leq \theta_n, \quad s \in \mathbb{R}, t \geq 0, \tag{4.106}$$

and

$$(\tilde{Q}^{(n)}_t w_0)(s) \leq \phi(s,t), \tag{4.107}$$

where ϕ solves (4.4). By (4.8), we get from (4.107) that $(\tilde{Q}^{(n)}_1 w_0)(s + c) \leq \psi(s)$, $s \in \mathbb{R}$. The latter inequality together with (4.106) imply

$$(\tilde{Q}^{(n)}_1 w_0)(s + c) \leq w_0(s). \tag{4.108}$$

Then, by the same arguments as in the proof of Theorem 4.9, cf. (4.17), we obtain from [99, Theorem 5] that there exists a traveling wave ψ_n for the equation (3.38) (with $d = 1$ and $a^\pm_R$ replaced by $a^\pm_{R_n}$), whose speed will be exactly c (and c satisfies (4.102)).

Now we are going to get a contradiction, by proving that

$$\inf_{\lambda>0} G_\xi(\lambda) = \lim_{n\to\infty} \inf_{\lambda>0} G^{(n)}_\xi(\lambda), \tag{4.109}$$

where $G^{(n)}_\xi$ is given by (4.54) with $\check{a}^\pm$ replaced by $\check{a}^\pm_n := \check{a}^\pm_{R_n}$. The sequence of functions $G^{(n)}_\xi$ is point-wise monotone in n and it converges to G_ξ point-wise, for $0 < \lambda \leq \lambda_0(\check{a}^+)$; note we may include $\lambda_0(\check{a}^+)$ here, according to (4.104). Moreover, $G^{(n)}_\xi(\lambda) \leq G_\xi(\lambda)$, $0 < \lambda \leq \lambda_0(\check{a}^+)$. As a result, for any $n \in \mathbb{N}$,

$$G^{(n)}_\xi(\lambda^{(n)}_*) = \inf_{\lambda>0} G^{(n)}_\xi(\lambda) \leq \inf_{\lambda>0} G_\xi(\lambda) = G_\xi(\lambda_*). \tag{4.110}$$

Hence if we suppose that (4.109) does not hold, then

$$\inf_{\lambda>0} G_\xi(\lambda) - \lim_{n\to\infty} \inf_{\lambda>0} G^{(n)}_\xi(\lambda) > 0.$$

Therefore, there exist $\delta > 0$ and $N \in \mathbb{N}$, such that

$$G^{(n)}_\xi(\lambda^{(n)}_*) = \inf_{\lambda>0} G^{(n)}_\xi(\lambda) \leq \inf_{\lambda>0} G_\xi(\lambda) - \delta = G_\xi(\lambda_*) - \delta, \quad n \geq N. \tag{4.111}$$

Clearly, (3.37) with $\Delta_{R_n} = (-\infty, R_n)$ implies that $\lambda_0(\check{a}_n^+) = \infty$, hence $G_\xi^{(n)}$ is analytic on $\operatorname{Re} z > 0$. One can repeat all considerations of the first three steps of this proof for the equation (3.38). Let $c_*^{(n)}(\xi)$ be the corresponding minimal traveling wave speed, according to Theorem 4.9. Then the corresponding inequality (4.93) will show that the abscissa of an arbitrary traveling wave to (3.38) is less than $\lambda_0(\check{a}_n^+) = \infty$. As a result, the inequality $c_*^{(n)}(\xi) < \inf\limits_{\lambda>0} G_\xi^{(n)}(\lambda)$, cf. (4.102), is impossible, and hence, by the Step 3,

$$c \geq c_*^{(n)}(\xi) = \inf_{\lambda>0} G_\xi^{(n)}(\lambda) = G_\xi^{(n)}(\lambda_*^{(n)}), \tag{4.112}$$

where $\lambda_*^{(n)}$ is the unique zero of the function $\frac{d}{d\lambda} G_\xi^{(n)}(\lambda)$. Let $\mathfrak{t}_\xi^{(n)}$ be given on $(0,\infty)$ by (4.69) with $\check{a}^+$ replaced by $\check{a}_n^+$. Then

$$\frac{d}{d\lambda}\mathfrak{t}_\xi^{(n)}(\lambda) = -\lambda\varkappa^+ \int_{-\infty}^{R_n} \check{a}^+(s) s^2 e^{\lambda s}\, ds < 0, \quad \lambda > 0. \tag{4.113}$$

By (4.85), the unique point of intersection of the strictly decreasing function $y = \mathfrak{t}_\xi^{(n)}(\lambda)$ and the horizontal line $y = m$ is exactly the point $(\lambda_*^{(n)}, 0)$.

Prove that there exist $\lambda_1 > 0$, such that $\lambda_*^{(n)} > \lambda_1$, $n \geq N$, and there exists $N_1 \geq N$, such that $\mathfrak{t}_\xi^{(n)}(\lambda) \leq \mathfrak{t}_\xi^{(m)}(\lambda)$, $n > m \geq N_1$, $\lambda \geq \lambda_1$. Recall that (4.105) holds; we have

$$\begin{aligned}
\lambda G_\xi^{(n)}(\lambda) &= \varkappa^+ \int_{\mathbb{R}} \check{a}_n^+(s)(e^{\lambda s} - 1)\, ds + \varkappa^+ \check{A}_{R_n}^+ - m \\
&\geq \varkappa^+ \int_{-\infty}^0 \check{a}_n^+(s)(e^{\lambda s} - 1)\, ds + \varkappa^+ \check{A}_{R_1}^+ - m,
\end{aligned}$$

and the inequality $1 - e^{-s} \leq s$, $s \geq 0$ implies that

$$\biggl| \int_{-\infty}^0 \check{a}_n^+(s)(e^{\lambda s} - 1)\, ds \biggr| \leq \lambda \int_{-\infty}^0 \check{a}_n^+(s)|s|\, ds \leq \lambda \int_{\mathbb{R}} \check{a}^+(s)|s|\, ds < \infty,$$

by (A7). As a result, if we set

$$\lambda_1 := (\varkappa^+ \check{A}_{R_1}^+ - m)\biggl(\varkappa^+ \int_{\mathbb{R}} \check{a}^+(s)|s|\, ds + |G_\xi(\lambda_*)|\biggr)^{-1} > 0,$$

then, for any $\lambda \in (0, \lambda_1)$, we have

$$\lambda G_\xi^{(n)}(\lambda) \geq \varkappa^+ \check{A}_{R_1}^+ - m - \lambda_1 \varkappa^+ \int_{\mathbb{R}} \check{a}_n^+(s)|s|\, ds = \lambda_1 |G_\xi(\lambda_*)| \geq \lambda G_\xi(\lambda_*),$$

i.e. $G_\xi^{(n)}(\lambda) \geq G_\xi(\lambda_*) = \inf\limits_{\lambda>0} G_\xi(\lambda)$. Then, for any $n \geq N$, (4.111) implies that $\lambda_*^{(n)}$, being the minimum point for $G_\xi^{(n)}$, does not belong to the interval $(0, \lambda_1)$. Next, let $N_1 \geq N$ be such that $R_n \geq \frac{1}{\lambda_1}$, for all $n \geq N_1$. Then, for any $\lambda \geq \lambda_1$, and for any $n > m \geq N_1$, we have $R_n > R_m$ and

$$\begin{aligned}
\mathfrak{t}_\xi^{(n)}(\lambda) - \mathfrak{t}_\xi^{(m)}(\lambda) &= \varkappa^+ \int_{R_m}^{R_n} (1 - \lambda s)\check{a}^+(s) e^{\lambda s}\, ds \\
&\leq \varkappa^+ \int_{R_m}^{R_n} (1 - \lambda_1 s)\check{a}^+(s) e^{\lambda s}\, ds \leq 0.
\end{aligned}$$

As a result, the sequence $\{\lambda_*^{(n)} \mid n \geq N_1\} \subset [\lambda_1, \infty)$ is monotonically decreasing (cf. (4.113)). We set

$$\vartheta := \lim_{n\to\infty} \lambda_*^{(n)} \geq \lambda_1. \tag{4.114}$$

Next, for any $n, m \in \mathbb{N}$, $n > m \geq N_1$,

$$G_\xi^{(n)}(\lambda_*^{(n)}) \geq G_\xi^{(m)}(\lambda_*^{(n)}) \geq G_\xi^{(m)}(\lambda_*^{(m)}), \tag{4.115}$$

where we used that $G_\xi^{(n)}$ is increasing in n and $\lambda_*^{(m)}$ is the minimum point of $G_\xi^{(m)}$. Therefore, the sequence $\{G_\xi^{(n)}(\lambda_*^{(n)})\}$ is increasing and, by (4.111), is bounded. Then, there exists

$$\lim_{n\to\infty} G_\xi^{(n)}(\lambda_*^{(n)}) =: g \leq G_\xi(\lambda_*) - \delta. \tag{4.116}$$

Fix $m \geq N_1$ in (4.115) and pass n to infinity; then, by the continuity of $G_\xi^{(n)}$,

$$g \geq \lim_{\lambda\to\vartheta+} G_\xi^{(m)}(\lambda) = G_\xi^{(m)}(\vartheta) \geq G_\xi^{(m)}(\lambda^{(m)}), \tag{4.117}$$

in particular, $\vartheta > 0$, as $G_\xi^{(m)}(0+) = \infty$. Next, if we pass m to ∞ in (4.117), we will get from (4.116)

$$\lim_{m\to\infty} G_\xi^{(m)}(\vartheta) = g \leq G_\xi(\lambda_*) - \delta < G_\xi(\lambda_*). \tag{4.118}$$

If $0 < \vartheta \leq \lambda_0(\check{a}^+)$ then

$$\lim_{m\to\infty} G_\xi^{(m)}(\vartheta) = G_\xi(\vartheta) \geq G_\xi(\lambda_*),$$

that contradicts (4.118). If $\vartheta > \lambda_0(\check{a}^+)$, then $\lim\limits_{m\to\infty} G_\xi^{(m)}(\vartheta) = \infty$ (recall again that $\mathfrak{L}^-(\check{a}^+)(\lambda)$ is analytic and $\mathfrak{L}^-(\check{a}^+)(\lambda)$ is monotone in λ), that contradicts (4.118) as well.

The contradiction we obtained shows that (4.109) does hold. Then, for the chosen $c \geq c_*(\xi)$ which satisfies (4.102), one can find n big enough to ensure that, cf. (4.112),

$$c < \inf_{\lambda>0} G_\xi^{(n)}(\lambda) = c_*^{(n)}(\xi).$$

However, as it was shown above, for this n there exists a profile ψ_n of a traveling wave to the 'truncated' equation (3.38) (with, recall, $d = 1$ and $a_R^\pm$ replaced by $a_{R_n}^\pm$). The latter contradicts the statement of Theorem 4.9 applied to this equation, as $c_*^{(n)}(\xi)$ has to be a minimal possible speed for such waves.

Therefore, the strict inequality in (4.102) is impossible, hence, we have equality in (4.55). As a result, (4.11) implies (4.80), and (4.99) may be read as (4.84). The rest of the statement is evident now. □

Remark 4.24. Clearly, the assumption $a^+(-x) = a^+(x)$, $x \in \mathbb{R}^d$, implies $\mathfrak{m}_\xi = 0$, for any $\xi \in S^{d-1}$. As a result, all speeds of traveling waves in any directions are positive, by (4.80).

4.3 Uniqueness of traveling waves

In this subsection we will prove the uniqueness (up to shifts) of a profile ψ for a traveling wave with given speed $c \geq c_*(\xi)$, $c \neq 0$. We will use the almost traditional now approach, namely, we find an *a priori* asymptotic for $\psi(t)$, $t \to \infty$, cf. e.g. [2, 15] and the references therein.

We start with the so-called characteristic function of the equation (2.1). Namely, for a given $\xi \in S^{d-1}$ and for any $c \in [c_*(\xi), \infty)$, we set

$$\mathfrak{h}_{\xi,c}(z) := \varkappa^+(\mathfrak{L}\check{a}^+)(z) - m - zc = zG_\xi(z) - zc, \qquad \operatorname{Re} z \in I_\xi. \tag{4.119}$$

Proposition 4.25. *Let $\xi \in S^{d-1}$ be fixed, $a^+ \in \mathcal{U}_\xi$, $\lambda_0 := \lambda_0(\check{a}^+)$, $c_*(\xi)$ be the minimal traveling wave speed in the direction ξ. Let, for any $c \geq c_*(\xi)$, the function $\psi \in \mathcal{M}_\theta(\mathbb{R})$ be a traveling wave profile corresponding to the speed c. For the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\lambda_0)$, we will assume, additionally, that*

$$\int_{\mathbb{R}} s^2 \check{a}^+(s) e^{\lambda_0 s}\, ds < \infty. \tag{4.120}$$

Then the function $\mathfrak{h}_{\xi,c}$ is analytic on $\{0 < \operatorname{Re} z < \lambda_0(\psi)\}$. Moreover, for any $\beta \in (0, \lambda_0(\psi))$, the function $\mathfrak{h}_{\xi,c}$ is continuous and does not equal to 0 on the closed strip $\{\beta \leq \operatorname{Re} z \leq \lambda_0(\psi)\}$, except the root at $z = \lambda_0(\psi)$, whose multiplicity j may be 1 or 2 only.

Proof. By (4.91) and the arguments around, $\mathfrak{h}_{\xi,c}(z) = z(G_\xi(z) - c)$ is analytic on $\{0 < \operatorname{Re} z < \lambda_0(\psi)\} \subset I_\xi$ and does not equal to 0 there. Then, by (4.83) and Proposition 4.18, the smallest positive root of the function $\mathfrak{h}_{\xi,c}(\lambda)$ on $\mathbb{R}$ is exactly $\lambda_0(\psi)$. Prove that if $z_0 := \lambda_0(\psi) + i\beta$ is a root of $\mathfrak{h}_{\xi,c}$, then $\beta = 0$. Indeed, $\mathfrak{h}_{\xi,c}(z_0) = 0$ yields

$$\varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\lambda_0(\psi)s} \cos\beta s\, ds = m + c\lambda_0(\psi),$$

that together with (4.83) leads to

$$\varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\lambda_0(\psi)s} (\cos\beta s - 1)\, ds = 0,$$

and thus $\beta = 0$.

Regarding multiplicity of the root $z = \lambda_0(\psi)$, we note that, by Proposition 4.18 and Corollary 4.21, there exist two possibilities. If $a^+ \in \mathcal{V}_\xi$, then $\lambda_0(\psi) \leq \lambda_* < \lambda_0(\check{a}^+)$ and, therefore, G_ξ is analytic at $z = \lambda_0(\psi)$. By the second equality in (4.119), the multiplicity j of this root for $\mathfrak{h}_{\xi,c}$ is the same as for the function $G_\xi(z) - c$. By Proposition 4.18, G_ξ is strictly decreasing on $(0, \lambda_*)$ and, therefore, $j = 1$ for $c > c_*(\xi)$. By Corollary 4.21, for $c = c_*(\xi)$, we have $G'_\xi(\lambda_0(\psi)) = G'_\xi(\lambda_*) = 0$ and, since $\mathfrak{h}''_{\xi,c}(\lambda_0) > 0$, one gets $j = 2$.

Let now $a^+ \in \mathcal{W}_\xi$. Then, we recall, $\lambda_* = \lambda_0 := \lambda_0(\check{a}^+) < \infty$, $G_\xi(\lambda_0) < \infty$ and (4.77) hold. For $c > c_*(\xi)$, the arguments are the same as before, and they yield $j = 1$. Let $c = c_*(\xi)$. Then $\mathfrak{h}_{\xi,c}(\lambda_0) = 0$, and, for all $z \in \mathbb{C}$, $\operatorname{Re} z \in (0, \lambda_0)$, one has

$$\begin{aligned} \mathfrak{h}_{\xi,c}(\lambda_0 - z) = \mathfrak{h}_{\xi,c}(\lambda_0 - z) - \mathfrak{h}_{\xi,c}(\lambda_0) &= \varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau)(e^{(\lambda_0 - z)\tau} - e^{\lambda_0 \tau}) d\tau + cz \\ &= z\Bigl(-\varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau) e^{\lambda_0 \tau} \int_0^\tau e^{-zs}\, ds d\tau + c\Bigr). \end{aligned} \tag{4.121}$$

Let $z = \alpha + \beta i$, $\alpha \in (0, \lambda_0)$. Then $\left|e^{\lambda_0 \tau} e^{-zs}\right| = e^{\lambda_0 \tau - \alpha s}$. Next, for $\tau \geq 0$, $s \in [0, \tau]$, we have $e^{\lambda_0 \tau - \alpha s} \leq e^{\lambda_0 \tau}$; whereas, for $\tau < 0$, $s \in [\tau, 0]$, one has $e^{\lambda_0 \tau - \alpha s} = e^{\lambda_0(\tau - s)} e^{(\lambda_0 - \alpha)s} \leq 1$. As a result, $\left|e^{\lambda_0\tau}e^{-zs}\right| \le e^{\lambda_0\max\{\tau,0\}}$. Then, using that $a^+\in\mathcal{W}_\xi$ implies $\int_{\mathbb{R}}\check{a}^+(\tau)e^{\lambda_0\max\{\tau,0\}}\,ds<\infty$, one can apply the dominated convergence theorem to the double integral in (4.121); we get then

$$\lim_{\substack{\operatorname{Re} z\to 0+\\ \operatorname{Im} z\to 0}}\frac{\mathfrak{h}_{\xi,c}(\lambda_0-z)}{z} = -\varkappa^+\int_{\mathbb{R}}\check{a}^+(\tau)e^{\lambda_0\tau}\tau d\tau + c. \tag{4.122}$$

According to the statement 3 of Theorem 4.23, for $m<\mathfrak{t}_\xi(\lambda_0)$, the r.h.s. of (4.122) is positive, i.e. $j=1$ in such a case. Let now $m=\mathfrak{t}_\xi(\lambda_0)$, then the r.h.s. of (4.122) is equal to 0. It is easily seen that one can rewrite then (4.121) as follows

$$\begin{aligned}\frac{\mathfrak{h}_{\xi,c}(\lambda_0-z)}{z} &= \varkappa^+\int_{\mathbb{R}}\check{a}^+(\tau)e^{\lambda_0\tau}\int_0^\tau (1-e^{-zs})\,dsd\tau\\ &= z\varkappa^+\int_{\mathbb{R}}\check{a}^+(\tau)e^{\lambda_0\tau}\int_0^\tau\int_0^s e^{-zt}\,dt\,ds\,d\tau.\end{aligned} \tag{4.123}$$

Similarly to the above, for $\operatorname{Re} z\in(0,\lambda_0)$, one has that $|e^{\lambda_0\tau-zt}|\le e^{\lambda_0\max\{\tau,0\}}$. Then, by (4.120) and the dominated convergence theorem, we get from (4.123) that

$$\lim_{\substack{\operatorname{Re} z\to 0+\\ \operatorname{Im} z\to 0}}\frac{\mathfrak{h}_{\xi,c}(\lambda_0-z)}{z^2} = \frac{\varkappa^+}{2}\int_{\mathbb{R}}\check{a}^+(\tau)e^{\lambda_0\tau}\tau^2 d\tau\in(0,\infty).$$

Thus $j=2$ in such a case. The statement is fully proved now. □

Remark 4.26. Combining results of Theorem 4.23 and Proposition 4.25, we immediately get that, for the case $j=2$, the minimal traveling wave speed $c_*(\xi)$ always satisfies (4.84).

Remark 4.27. If $\check{a}^+$ is given by (4.78), then, cf. Example 4.22, the case $a^+\in\mathcal{W}_\xi$, $m=\mathfrak{t}_\xi(\lambda_0)$ together with (4.120) requires $p=1$, $\mu<\mu_*$, $q>3$.

We consider now the following Ikehara–Delange type Tauberian theorem, cf. [28, 61, 89].

For any $\mu>\beta>0$, $T>0$, we set

$$K_{\beta,\mu,T} := \bigl\{z\in\mathbb{C}\bigm|\beta\le\operatorname{Re} z\le\mu,\ |\operatorname{Im} z|\le T\bigr\}.$$

Let, for any $D\subset\mathbb{C}$, $\mathcal{A}(D)$ be the class of all analytic functions on D.

Proposition 4.28. *Let $\mu>\beta>0$ be fixed. Let $\varphi\in C^1(\mathbb{R}_+\to\mathbb{R}_+)$ be a non-increasing function such that, for some $a>0$, the function $\varphi(t)e^{(\mu+a)t}$ is non-decreasing, and the integral*

$$\int_0^\infty e^{zt}\varphi'(t)dt,\quad 0<\operatorname{Re} z<\mu, \tag{4.124}$$

converges. Suppose also that there exist a constant $j\in\{1,2\}$ and complex-valued functions $H,F:\{0<\operatorname{Re} z\le\mu\}\to\mathbb{C}$, such that $H\in\mathcal{A}(0<\operatorname{Re} z\le\mu)$, $F\in\mathcal{A}(0<\operatorname{Re} z<\mu)\cap C(0<\operatorname{Re} z\le\mu)$, and, for any $T>0$,

$$\lim_{\sigma\to0+}(\log\sigma)^{2-j}\sup_{\tau\in[-T,T]}\bigl|F(\mu-2\sigma-i\tau)-F(\mu-\sigma-i\tau)\bigr| = 0, \tag{4.125}$$

and also that the following representation holds

$$\int_0^\infty e^{zt}\varphi(t)dt = \frac{F(z)}{(z-\mu)^j}+H(z),\quad 0<\operatorname{Re} z<\mu. \tag{4.126}$$

Then φ has the following asymptotic

$$\varphi(t)\sim F(\mu)e^{-\mu t}t^{j-1},\quad t\to+\infty. \tag{4.127}$$

The proof of Proposition 4.28 is based on the following Tenenbaum's result.

Lemma 4.29 ("Effective" Ikehara–Ingham Theorem, cf. [89, Theorem 7.5.11])**.** *Let $\alpha(t)$ be a non-decreasing function such that, for some fixed $a > 0$, the following integral converges:*

$$\int_0^\infty e^{-zt} d\alpha(t), \quad \operatorname{Re} z > a. \tag{4.128}$$

Let also there exist constants $D \geq 0$ and $j > 0$, such that for the functions

$$G(z) := \frac{1}{a+z} \int_0^\infty e^{-(a+z)t} d\alpha(t) - \frac{D}{z^j}, \quad \operatorname{Re} z > 0, \tag{4.129}$$

$$\eta(\sigma, T) := \sigma^{j-1} \int_{-T}^{T} \bigl| G(2\sigma + i\tau) - G(\sigma + i\tau) \bigr| d\tau, \quad T > 0,$$

one has that

$$\lim_{\sigma \to 0+} \eta(\sigma, T) = 0, \quad T > 0. \tag{4.130}$$

Then

$$\alpha(t) = \left\{ \frac{D}{\Gamma(j)} + O\bigl(\rho(t)\bigr) \right\} e^{at} t^{j-1}, \quad t \geq 1, \tag{4.131}$$

where

$$\rho(t) := \inf_{T \geq 32(a+1)} \Bigl\{ T^{-1} + \eta\bigl(t^{-1}, T\bigr) + (Tt)^{-j} \Bigr\}. \tag{4.132}$$

Proof of Proposition 4.28. We first express $\int_0^\infty e^{\lambda t} \varphi(t) dt$ in the form (4.128). By the assumption, the function $\alpha(t) := e^{(\mu+a)t} \varphi(t)$ is non-decreasing. For any $0 < \operatorname{Re} z < \mu$, one has

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = (\mu + a) \int_0^\infty e^{(\mu - z)t} \varphi(t) dt + \int_0^\infty e^{(\mu - z)t} \varphi'(t) dt, \tag{4.133}$$

and the r.h.s. of (4.133) converges, by (4.124) and (L4). Then, by [96, Corollary II.1.1a], the l.h.s. of (4.133) converges, for $\operatorname{Re} z > 0$, and hence, by [96, Theorem II.2.3a], one gets

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = -\varphi(0) + (a + z) \int_0^\infty e^{(\mu - z)t} \varphi(t) dt. \tag{4.134}$$

Therefore, by (4.126) and (4.134), we have

$$\frac{1}{a+z} \int_0^\infty e^{-(a+z)t} d\alpha(t) = \frac{F(\mu - z)}{z^j} + K(z), \quad 0 < \operatorname{Re} z < \mu,$$

where $K(z) := H(\mu - z) - \dfrac{\varphi(0)}{a+z}$, $0 < \operatorname{Re} z \leq \mu$.

Let now G be given by (4.129) with $\alpha(t)$ as above and $D := F(\mu)$. Check the condition (4.130); one can assume, clearly, that $0 < \sigma < 2\sigma < \mu$. Since $K \in \mathcal{A}(0 < \operatorname{Re} z \le \mu)$, one easily gets that

$$\begin{aligned}
&\lim_{\sigma\to 0+} \sigma^{j-1} \int_{-T}^{T} \bigl|G(2\sigma+i\tau) - G(\sigma+i\tau)\bigr| d\tau \\
&\le \lim_{\sigma\to 0+} \sigma^{j-1} \int_{-T}^{T} \Bigl|\frac{F(\mu-2\sigma-i\tau)-F(\mu)}{(2\sigma+i\tau)^j} - \frac{F(\mu-\sigma-i\tau)-F(\mu)}{(\sigma+i\tau)^j}\Bigr| d\tau \\
&\le \lim_{\sigma\to 0+} \sigma^{j-1} \int_{-T}^{T} \Bigl|\frac{F(\mu-2\sigma-i\tau)-F(\mu-\sigma-i\tau)}{(\sigma+i\tau)^j}\Bigr| d\tau \\
&\quad + \lim_{\sigma\to 0+} \sigma^{j-1} \int_{-T}^{T} \bigl|F(\mu-2\sigma-i\tau)-F(\mu)\bigr| \Bigl|\frac{1}{(2\sigma+i\tau)^j} - \frac{1}{(\sigma+i\tau)^j}\Bigr| d\tau, \\
&=: \lim_{\sigma\to 0+} A_j(\sigma) + \lim_{\sigma\to 0+} B_j(\sigma).
\end{aligned} \tag{4.135}$$

One has

$$A_j(\sigma) \le \sup_{\tau\in[-T,T]} \bigl|F(\mu-2\sigma-i\tau) - F(\mu-\sigma-i\tau)\bigr| \sigma^{j-1} \int_{-T}^{T} \frac{1}{|\sigma+i\tau|^j} d\tau,$$

and since

$$\sigma^{j-1} \int_{-T}^{T} \frac{1}{|\sigma+i\tau|^j} d\tau = \sigma^{j-1} \int_{-T}^{T} \frac{1}{(\sigma^2+\tau^2)^{\frac{j}{2}}} d\tau = \begin{cases} 2\log \dfrac{\sqrt{T^2+\sigma^2}+T}{\sigma}, & j=1, \\ 2\arctan \dfrac{T}{\sigma}, & j=2, \end{cases}$$

we get, by (4.125), that $\lim\limits_{\sigma\to 0+} A_j(\sigma) = 0$.

Next, since $F \in C(K_{\beta,\mu,T})$, there exists $C_1 > 0$ such that $|F(z)| \le C_1$, $z \in K_{\beta,\mu,T}$. Therefore,

$$\begin{aligned}
B_j(\sigma) &\le \sigma^{j-1} \sup_{|\tau|\le\sqrt{\sigma}} \bigl|F(\mu-2\sigma-i\tau)-F(\mu)\bigr| \int\limits_{|\tau|\le\sqrt{\sigma}} \Bigl|\frac{1}{(2\sigma+i\tau)^j} - \frac{1}{(\sigma+i\tau)^j}\Bigr| d\tau \\
&\quad + 2C_1\sigma^{j-1} \int\limits_{\sqrt{\sigma}\le|\tau|\le T} \Bigl|\frac{1}{(2\sigma+i\tau)^j} - \frac{1}{(\sigma+i\tau)^j}\Bigr| d\tau.
\end{aligned} \tag{4.136}$$

Note that, for any $a < b$,

$$\int_a^b \Bigl|\frac{1}{2\sigma+i\tau} - \frac{1}{\sigma+i\tau}\Bigr| d\tau = \int_a^b \frac{\sigma}{\sqrt{(2\sigma^2-\tau^2)^2+9\sigma^2\tau^2}} d\tau = \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} h_1(x)dx;$$

$$\int_a^b \Bigl|\frac{1}{(2\sigma+i\tau)^2} - \frac{1}{(\sigma+i\tau)^2}\Bigr| d\tau = \sigma \int_a^b \frac{\sqrt{9\sigma^2+4\tau^2}}{(2\sigma^2-\tau^2)^2+9\sigma^2\tau^2} d\tau = \frac{1}{\sigma} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} h_2(x)dx,$$

where

$$h_1(x) := \frac{1}{\sqrt{4+5x^2+x^4}}, \qquad h_2(x) := \frac{\sqrt{9+4x^2}}{4+5x^2+x^4}.$$

Now, one can estimate terms in (4.136) separately. We have

$$\sigma^{j-1} \int\limits_{|\tau|\leq\sqrt{\sigma}} \Bigl|\frac{1}{(2\sigma+i\tau)^j} - \frac{1}{(\sigma+i\tau)^j}\Bigr| d\tau = \int\limits_{|x|\leq\frac{\sqrt{\sigma}}{\sigma}} h_j(x)dx < \int_{\mathbb{R}} h_j(x)dx < \infty.$$

Next, since F is uniformly continuous on $K_{\beta,\mu,T}$, we have that, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(\mu,\sigma,\tau) := \bigl|F(\mu-2\sigma-i\tau) - F(\mu)\bigr| < \varepsilon$, if only $4\sigma^2+\tau^2 < \delta$. Therefore, if $\sigma > 0$ is such that $4\sigma^2+\sigma < \delta$ then $\sup_{|\tau|\leq\sqrt{\sigma}} f(\mu,\sigma,\tau) < \varepsilon$ hence

$$\sup_{|\tau|\leq\sqrt{\sigma}} \bigl|F(\mu-2\sigma-i\tau) - F(\mu)\bigr| \to 0, \quad \sigma \to 0+.$$

Finally,

$$\sigma^{j-1} \int\limits_{\sqrt{\sigma}\leq|\tau|\leq T} \Bigl|\frac{1}{(2\sigma+i\tau)^j} - \frac{1}{(\sigma+i\tau)^j}\Bigr| = 2\int\limits_{\frac{\sqrt{\sigma}}{\sigma}}^{\frac{T}{\sigma}} h_j(x)dx \to 0, \quad \sigma \to 0+,$$

as $\int_{\mathbb{R}} h_j(x)dx < \infty$. As a result, (4.136) gives $B_j(\sigma) \to 0$, as $\sigma \to 0+$. Combining this with $A_j(\sigma) \to 0$, one gets (4.130) from (4.135); and we can apply Lemma 4.29. Namely, by (4.131), there exist $C > 0$ and $t_0 \geq 1$, such that

$$De^{at}t^{j-1} \leq \varphi(t)e^{(\mu+a)t} \leq \{D + C\rho(t)\}\, e^{at}t^{j-1}, \quad t \geq t_0.$$

as $\Gamma(j) = 1$, for $j \in \{1,2\}$. By (4.130), (4.132) $\rho(t) \to 0$ as $t \to \infty$. Therefore,

$$\varphi(t)e^{(\mu+a)t} \sim De^{at}t^{j-1}, \quad t \to \infty,$$

that is equivalent to (4.127) and finishes the proof. ☐

To apply Proposition 4.28 to our settings, we will need the following statement, which is an adaptation of [101, Lemma 3.2, Proposition 3.7].

Proposition 4.30. *Let $\xi \in S^{d-1}$ be fixed, $a^+ \in \mathcal{U}_\xi$, $c_*(\xi)$ be the minimal traveling wave speed in the direction ξ. Let a traveling wave profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ correspond to a speed $c \geq c_*(\xi)$, $c \neq 0$. Then there exists $\nu > 0$, such that $\psi(t)e^{\nu t}$ is a monotonically increasing function.*

Proof. We start from the case $c > 0$. Since $\psi(t) > 0,\ t \in \mathbb{R}$, it is sufficient to prove that

$$\frac{\psi'(t)}{\psi(t)} > -\nu, \quad t \in \mathbb{R}. \tag{4.137}$$

Fix any $\mu \geq \dfrac{\varkappa^+}{c} > 0$. Then, clearly,

$$\kappa_1\psi^2(t) + \kappa_2(\check{a}^- * \psi)(t) + m \leq \varkappa^-\theta + m = \varkappa^+ \leq c\mu,$$

and we will get from (4.87), that

$$0 \geq c\psi'(s) + \varkappa^+(\check{a}^+ * \psi)(s) - c\mu\psi(s), \quad s \in \mathbb{R}. \tag{4.138}$$

Multiply both parts of (4.138) on $e^{-\mu s} > 0$ and set

$$w(s) := \psi(s)e^{-\mu s} > 0, \quad s \in \mathbb{R}.$$

Then $w'(s) = \psi'(s)e^{-\mu s} - \mu w(s)$ and one can rewrite (4.138) as follows

$$\begin{aligned} 0 &\geq cw'(s) + \varkappa^+(\check{a}^+ * \psi)(s)e^{-\mu s} \\ &= cw'(s) + \varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau}d\tau, \quad s \in \mathbb{R}. \end{aligned} \tag{4.139}$$

As it was shown in the proof of Proposition 4.18, (A8) implies that there exists $\varrho > 0$, such that

$$\int_{2\varrho}^{\infty} \check{a}^+(s)e^{-\mu s}ds > 0; \tag{4.140}$$

indeed, it is enough to set $2\varrho := r + \frac{\delta'}{2}$ in (4.65).

Integrate (4.139) over $s \in [t, t+\varrho]$; one gets

$$0 \geq c(w(t+\varrho) - w(t)) + \varkappa^+ \int_t^{t+\varrho} \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau}d\tau ds. \tag{4.141}$$

Since $w(t)$ is a monotonically decreasing function, we have

$$\begin{aligned} &\int_t^{t+\varrho} \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau}d\tau ds \geq \varrho \int_{\mathbb{R}} \check{a}^+(\tau)w(t+\varrho-\tau)e^{-\mu\tau}d\tau \\ \geq \varrho &\int_{2\varrho}^{\infty} \check{a}^+(\tau)w(t+\varrho-\tau)e^{-\mu\tau}d\tau \geq \varrho w(t-\varrho) \int_{2\varrho}^{\infty} \check{a}^+(\tau)e^{-\mu\tau}d\tau. \end{aligned} \tag{4.142}$$

We set, cf. (4.140),

$$C(\mu, \rho) := \frac{\varkappa^+}{c} \int_{2\varrho}^{\infty} \check{a}^+(s)e^{-\mu s}ds > 0.$$

Then (4.141) and (4.142) yield

$$w(t) - \varrho C(\mu, \rho)w(t-\varrho) \geq w(t+\varrho) > 0, \quad t \in \mathbb{R}. \tag{4.143}$$

Now we integrate (4.139) over $s \in [t-\varrho, t]$. Similarly to above, one gets

$$\begin{aligned} 0 &\geq c(w(t) - w(t-\varrho)) + \varkappa^+ \int_{t-\varrho}^{t} \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau}d\tau ds \\ &\geq c(w(t) - w(t-\varrho)) + \varrho\varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau)w(t-\tau)e^{-\mu\tau}d\tau. \end{aligned} \tag{4.144}$$

By (4.143) and (4.144), we have

$$\frac{1}{\varrho C(\mu, \rho)} \geq \frac{w(t-\varrho)}{w(t)} \geq 1 + \frac{\varrho\varkappa^+}{c} \int_{\mathbb{R}} \check{a}^+(\tau)\frac{w(t-\tau)}{w(t)}e^{-\mu\tau}d\tau. \tag{4.145}$$

On the other hand, (4.87) implies that

$$-\frac{\psi'(t)}{\psi(t)} \leq \frac{\varkappa^+}{c}\frac{(\check{a}^+ * \psi)(t)}{\psi(t)} = \frac{\varkappa^+}{c} \int_{\mathbb{R}} \check{a}^+(\tau)\frac{w(t-\tau)}{w(t)}e^{-\mu\tau}d\tau, \quad t \in \mathbb{R}. \tag{4.146}$$

Finally, (4.145) and (4.146) yield (4.137) with $\nu = \dfrac{1}{\rho^2 C(\mu,\rho)} > 0$.

Let now $c < 0$. For any $\nu \in \mathbb{R}$, one has

$$\psi'(s) = e^{-\nu s}(\psi(s)e^{\nu s})' - \nu\psi(s), \quad s \in \mathbb{R}.$$

Hence, by (4.32), (A2),

$$\begin{aligned}
0 =& ce^{-\nu s}(\psi(s)e^{\nu s})' - c\nu\psi(s) + \chi^+(\check{a}^+ * \psi)(s) \\
&\qquad\qquad - \kappa_1\psi^2(s) - \kappa_2\psi(s)(\check{a}^- * \psi)(s) - m\psi(s) \\
\geq& ce^{-\nu s}(\psi(s)e^{\nu s})' - c\nu\psi(s) + \chi^+(\check{a}^+ * \psi)(s) \\
&\qquad\qquad - \kappa_1\theta\psi(s) - \kappa_2\theta(\check{a}^- * \psi)(s) - m\psi(s) \\
\geq& ce^{-\nu s}(\psi(s)e^{\nu s})' - c\nu\psi(s) - \kappa_1\theta\psi(s) - m\psi(s), \quad s \in \mathbb{R}.
\end{aligned}$$

As a result, choosing $\nu > \dfrac{m + \kappa_1\theta}{-c}$, one gets

$$-ce^{-\nu s}(\psi(s)e^{\nu s})' \geq (-c\nu - \kappa_1\theta - m)\psi(s) > 0, \quad s \in \mathbb{R},$$

i.e. $\psi(s)e^{\nu s}$ is an increasing function. □

Now, we can apply Proposition 4.28 to find the asymptotic of the profile of a traveling wave.

Proposition 4.31. *In conditions and notations of Proposition 4.25, for $c \neq 0$, there exists $D = D_j > 0$, such that*

$$\psi(t) \sim De^{-\lambda_0(\psi)t}t^{j-1}, \quad t \to \infty. \tag{4.147}$$

Proof. We set $\mu := \lambda_0(\psi)$ and

$$\begin{aligned}
f(z) &:= \kappa_1\big(\mathfrak{L}(\psi^2)\big)(z) + \kappa_2\big(\mathfrak{L}(\psi(\check{a}^- * \psi))\big)(z), & g_j(z) &:= \frac{\mathfrak{h}_{\xi,c}(z)}{(z-\mu)^j}, \\
H(z) &:= -\int_{-\infty}^{0} \psi(t)e^{zt}\,dt, & F(z) &:= \frac{f(z)}{g_j(z)}.
\end{aligned} \tag{4.148}$$

By (4.88) and Lemma 4.16, we have that $f, H \in \mathcal{A}(0 < \operatorname{Re} z \leq \mu)$; in particular, for any $T > 0$, $\beta > 0$,

$$\bar{f} := \sup_{z \in K_{\beta,\mu,T}} |f(z)| < \infty. \tag{4.149}$$

By Proposition 4.25, the function g_j is continuous and does not equal to 0 on the strip $\{0 < \operatorname{Re} z \leq \mu\}$, in particular, for any $T > 0$, $\beta > 0$,

$$\bar{g}_j := \inf_{z \in K_{\beta,\mu,T}} |g(z)| > 0. \tag{4.150}$$

Therefore, $F \in \mathcal{A}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \leq \mu)$. As a result, one can rewrite (4.97) in the form (4.126), with $\varphi = \psi$ and with F, H as in (4.148).

Taking into account Proposition 4.30, to apply Proposition 4.28 it is enough to prove that (4.125) holds. Assume that $0 < 2\sigma < \mu$.

Let $j=2$. Clearly, $F\in C(0<\operatorname{Re} z\leq \mu)$ implies that F is uniformly continuous on $K_{\beta,\mu,T}$. Then, for any $\varepsilon>0$ there exists $\delta>0$ such that, for any $\tau\in[-T,T]$, the inequality

$$|\sigma| = |(\mu-2\sigma-i\tau)-(\mu-\sigma-i\tau)|<\delta,$$

implies

$$|F(\mu-2\sigma-i\tau)-F(\mu-\sigma-i\tau)|<\varepsilon,$$

and hence (4.125) holds (with $j=2$).

Let now $j=1$. If $F\in\mathcal{A}(K_{\beta,\mu,T})$, we have, evidently, that F' is bounded on $K_{\beta,\mu,T}$, and one can apply a mean-value-type theorem for complex-valued functions, see e.g. [37], to get that F is a Lipschitz function on $K_{\beta,\mu,T}$. Therefore, for some $K>0$,

$$|F(\mu-2\sigma-i\tau)-F(\mu-\sigma-i\tau)|<K|\sigma|,$$

for all $\tau\in[-T,T]$, that yields (4.125) (with $j=1$). By Proposition 4.18 and Corollary 4.21, the inclusion $F\in\mathcal{A}(K_{\beta,\mu,T})$ always holds for $c>c_*$; whereas, for $c=c_*$ it does hold iff $a^+\in\mathcal{V}_\xi$. Moreover, the case $a^+\in\mathcal{W}_\xi$ with $m=\mathfrak{t}_\xi(\lambda_0)$ and $c=c_*$ implies, by Proposition 4.25, $j=2$ and hence it was considered above.

Therefore, it remains to prove (4.125) for the case $a^+\in\mathcal{W}_\xi$ with $m<\mathfrak{t}_\xi(\lambda_0)$, $c=c_*$ (then $j=1$). Denote, for simplicity,

$$z_1:=\mu-\sigma-i\tau, \qquad z_2:=\mu-2\sigma-i\tau. \tag{4.151}$$

Then, by (4.148), (4.149), (4.150), one has

$$\begin{aligned}\bigl|F(z_2)-F(z_1)\bigr| &\leq \Bigl|\frac{f(z_2)}{g_1(z_2)}-\frac{f(z_1)}{g_1(z_2)}\Bigr|+\Bigl|\frac{f(z_1)}{g_1(z_2)}-\frac{f(z_1)}{g_1(z_1)}\Bigr|\\ &\leq \frac{1}{\bar{g}_1}\bigl|f(z_2)-f(z_1)\bigr|+\frac{\bar{f}}{\bar{g}_1^2}|g_1(z_1)-g_1(z_2)|.\end{aligned} \tag{4.152}$$

Note that, if $0<\phi\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ be such that $\lambda_0(\phi)>\mu$ then

$$\begin{aligned}\bigl|(\mathfrak{L}\phi)(z_2)-(\mathfrak{L}\phi)(z_1)\bigr| &\leq \int_{\mathbb{R}}\phi(s)e^{\mu s}|e^{-2\sigma s}-e^{-\sigma s}|ds\\ &\leq \sigma\int_0^\infty \phi(s)e^{(\mu-\sigma)s}sds+\sigma\int_{-\infty}^0\phi(s)e^{(\mu-2\sigma)s}|s|\,ds=O(\sigma),\end{aligned} \tag{4.153}$$

as $\sigma\to 0+$, where we used that $\sup_{s<0}e^{(\mu-2\sigma)s}|s|<\infty$, $0<2\sigma<\mu$, and that (L2) holds. Applying (4.153) to $\phi=\psi(\check{a}^-*\psi)\leq\theta^2\check{a}^-\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, one gets

$$\sup_{\tau\in[-T,T]}\bigl|f(z_2)-f(z_1)\bigr|=O(\sigma), \quad \sigma\to 0+.$$

Therefore, by (4.152), it remains to show that

$$\lim_{\sigma\to 0+}\log\sigma\sup_{\tau\in[-T,T]}|g_1(z_1)-g_1(z_2)|=0. \tag{4.154}$$

Recall that, in the considered case $c = c_*$, one has $\mathfrak{h}_{\xi,c}(\mu) = 0$. Therefore, by (4.119), (4.148), (4.151), we have

$$
\begin{aligned}
|g_1(z_1) - g_1(z_2)| &= \left| \frac{\mathfrak{h}_{\xi,c}(z_1) - \mathfrak{h}_{\xi,c}(\mu)}{z_1 - \mu} - \frac{\mathfrak{h}_{\xi,c}(z_2) - \mathfrak{h}_{\xi,c}(\mu)}{z_2 - \mu} \right| \\
&= \left| \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(z_1) - \varkappa^+(\mathfrak{L}\check{a}^+)(\mu)}{z_1 - \mu} - \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(z_2) - \varkappa^+(\mathfrak{L}\check{a}^+)(\mu)}{z_2 - \mu} \right| \\
&\leq \varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\mu s} \left| \frac{1 - e^{(-\sigma - i\tau)s}}{\sigma + i\tau} - \frac{1 - e^{(-2\sigma - i\tau)s}}{2\sigma + i\tau} \right| ds \\
&= \varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\mu s} \left| \int_0^s \bigl(e^{(-\sigma - i\tau)t} - e^{(-2\sigma - i\tau)t}\bigr)\, dt \right| ds \\
&\leq \varkappa^+ \int_0^\infty \check{a}^+(s) e^{\mu s} \int_0^s \left| e^{-\sigma t} - e^{-2\sigma t} \right| dt\, ds \\
&\quad + \varkappa^+ \int_{-\infty}^0 \check{a}^+(s) e^{\mu s} \int_s^0 \left| e^{-\sigma t} - e^{-2\sigma t} \right| dt\, ds
\end{aligned} \tag{4.155}
$$

and since, for $t \geq 0$, $\left|e^{-\sigma t} - e^{-2\sigma t}\right| \leq \sigma t$; and, for $s \leq t \leq 0$,

$$
\left|e^{-\sigma t} - e^{-2\sigma t}\right| = e^{-2\sigma t}\left|e^{\sigma t} - 1\right| \leq e^{-2\sigma s}\sigma|t|,
$$

one can continue (4.155)

$$
\leq \frac{1}{2}\sigma\varkappa^+ \int_0^\infty \check{a}^+(s) e^{\mu s} s^2\, ds + \frac{1}{2}\sigma\varkappa^+ \int_{-\infty}^0 \check{a}^+(s) e^{(\mu - 2\sigma)s} s^2\, ds.
$$

Since $\mu > 2\sigma$, one has $\sup_{s \leq 0} e^{(\mu - 2\sigma)s} s^2 < \infty$, therefore, by (4.120), one gets

$$
\sup_{\tau \in [-T,T]} |g_1(z_1) - g_1(z_2)| \leq \mathrm{const} \cdot \sigma,
$$

that proves (4.154). The statement is fully proved now. □

Remark 4.32. By (4.127) and (4.148), one has that the constant $D = D_j$ in (4.147) is given by

$$
D = D(\psi) = \bigl(\kappa_1\bigl(\mathfrak{L}(\psi^2)\bigr)(\mu) + \kappa_2\bigl(\mathfrak{L}(\psi(\check{a}^- * \psi))\bigr)(\mu)\bigr) \lim_{z \to \mu} \frac{(z - \mu)^j}{\mathfrak{h}_{\xi,c}(z)},
$$

where $\mu = \lambda_0(\psi)$. Note that, by Proposition 4.25, the limit above is finite and does not depend on ψ. Next, by Remark 4.7, for any $q \in \mathbb{R}$, $\psi_q(s) := \psi(s + q)$, $s \in \mathbb{R}$ is a traveling wave with the same speed, and hence, by Theorem 4.23, $\lambda_0(\psi_q) = \lambda_0(\psi)$. Moreover,

$$
\begin{aligned}
\bigl(\mathfrak{L}(\psi_q(\check{a}^- * \psi_q))\bigr)(\mu) &= \int_{\mathbb{R}} \psi(s + q) \int_{\mathbb{R}} \check{a}^-(t)\psi(s - t + q)\, dt\, e^{\mu s}\, ds \\
&= e^{-\mu q}\bigl(\mathfrak{L}(\psi(\check{a}^- * \psi))\bigr)(\mu), \\
\bigl(\mathfrak{L}(\psi_q^2)\bigr)(\mu) &= \int_{\mathbb{R}} \psi^2(s + q) e^{\mu s} ds = e^{-\mu q}\bigl(\mathfrak{L}(\psi^2)\bigr)(\mu).
\end{aligned}
$$

Thus, for a traveling wave profile ψ one can always choose a $q \in \mathbb{R}$ such that, for the shifted profile ψ_q, the corresponding $D = D(\psi_q)$ will be equal to 1.

Finally, we are ready to prove the uniqueness result.

Theorem 4.33. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Suppose, additionally, that* (A4) *holds. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 4.9. For the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\lambda_0)$, we will assume, additionally, that* (4.120) *holds. Then, for any $c \geq c_*$, such that $c \neq 0$, there exists a unique, up to a shift, traveling wave profile ψ for* (2.1).

Proof. We will follow the sliding technique from [22]. Let $\psi_1, \psi_2 \in C^1(\mathbb{R}) \cap \mathcal{M}_\theta(\mathbb{R})$ are traveling wave profiles with a speed $c \geq c_*$, $c \neq 0$, cf. Proposition 4.11. By Proposition 4.31 and Remark 4.32, we may assume, without lost of generality, that (4.147) holds for both ψ_1 and ψ_2 with $D = 1$. By the proof of Proposition 4.25, the corresponding $j \in \{1, 2\}$ depends on $a^\pm$, $\varkappa^\pm$, m only, and does not depend on the choice of ψ_1, ψ_2. By Theorem 4.23, $\lambda_0(\psi_1) = \lambda_0(\psi_2) =: \lambda_c \in (0, \infty)$.

Step 1. Prove that, for any $\tau > 0$, there exists $T = T(\tau) > 0$, such that

$$\psi_1^\tau(s) := \psi_1(s - \tau) > \psi_2(s), \quad s \geq T. \tag{4.156}$$

Indeed, take an arbitrary $\tau > 0$. Then (4.147) with $D = 1$ yields

$$\lim_{s\to\infty} \frac{\psi_1^\tau(s)}{(s-\tau)^{j-1} e^{-\lambda_c(s-\tau)}} = 1 = \lim_{s\to\infty} \frac{\psi_2(s)}{s^{j-1} e^{-\lambda_c s}}.$$

Then, for any $\varepsilon > 0$, there exists $T_1 = T_1(\varepsilon) > \tau$, such that, for any $s > T_1$,

$$\frac{\psi_1^\tau(s)}{(s-\tau)^{j-1} e^{-\lambda_c(s-\tau)}} - 1 > -\varepsilon, \qquad \frac{\psi_2(s)}{s^{j-1} e^{-\lambda_c s}} - 1 < \varepsilon.$$

As a result, for $s > T_1 > \tau$,

$$\begin{aligned} \psi_1^\tau(s) - \psi_2(s) &> (1-\varepsilon)(s-\tau)^{j-1} e^{-\lambda_c(s-\tau)} - (1+\varepsilon) s^{j-1} e^{-\lambda_c s} \\ &= s^{j-1} e^{-\lambda_c s} \biggl(\Bigl(1 - \frac{\tau}{s}\Bigr)^{j-1} e^{\lambda_c \tau} - 1 - \varepsilon \Bigl(\Bigl(1 - \frac{\tau}{s}\Bigr)^{j-1} e^{\lambda_c \tau} + 1 \Bigr) \biggr) \\ &\geq s^{j-1} e^{-\lambda_c s} \biggl(\Bigl(1 - \frac{\tau}{T_1}\Bigr)^{j-1} e^{\lambda_c \tau} - 1 - \varepsilon \bigl(e^{\lambda_c \tau} + 1 \bigr) \biggr) > 0, \end{aligned} \tag{4.157}$$

if only

$$0 < \varepsilon < \frac{\Bigl(1 - \dfrac{\tau}{T_1}\Bigr)^{j-1} e^{\lambda_c \tau} - 1}{e^{\lambda_c \tau} + 1} =: g(\tau, T_1). \tag{4.158}$$

For $j = 1$, the nominator in the r.h.s. of (5.64) is positive. For $j = 2$, consider $f(t) := \bigl(1 - \frac{t}{T_1}\bigr) e^{\lambda_c t} - 1$, $t \geq 0$. Then $f'(t) = \frac{1}{T_1} e^{\lambda_c t} (\lambda_c T_1 - \lambda_c t - 1) > 0$, if only $T_1 > t + \frac{1}{\lambda_c}$, that implies $f(t) > f(0) = 0$, $t \in \bigl(0, T_1 - \frac{1}{\lambda_c}\bigr)$.

As a result, choose $\varepsilon = \varepsilon(\tau) > 0$ with $\varepsilon < g\bigl(\tau, \tau + \frac{1}{\lambda_c}\bigr)$, then, without loss of generality, suppose that $T_1 = T_1(\varepsilon) = T_1(\tau) > \tau + \frac{1}{\lambda_c} > \tau$. Therefore, $0 < \varepsilon < g\bigl(\tau, \tau + \frac{1}{\lambda_c}\bigr) \leq g(\tau, T_1)$, that fulfills (4.158), and hence (4.157) yields (4.156), with any $T > T_1$.

Step 2. Prove that there exists $\nu > 0$, such that, cf. (4.156),

$$\psi_1^\nu(s) \geq \psi_2(s), \quad s \in \mathbb{R}. \tag{4.159}$$

Let $\tau > 0$ be arbitrary and $T = T(\tau)$ be as above. Choose any $\delta \in \bigl(0, \frac{\theta}{4}\bigr)$. By (4.7), (4.1), and the dominated convergence theorem,

$$\lim_{s\to-\infty}(\check{a}^- * \psi_2)(s) = \lim_{s\to-\infty}\int_{\mathbb{R}} \check{a}^-(\tau)\psi_2(s-\tau)\,d\tau = \theta > \delta. \tag{4.160}$$

Then, one can choose $T_2 = T_2(\delta) > T$, such that, for all $s < -T_2$,

$$\psi_1^\tau(s) > \theta - \delta, \tag{4.161}$$

$$\kappa_1\psi_2(s) + \kappa_2(\check{a}^- * \psi_2)(s) > \delta. \tag{4.162}$$

Note also that (4.156) holds, for all $s \geq T_2 > T$, as well. Clearly, for any $\nu \geq \tau$,

$$\psi_1^\nu(s) = \psi_1(s-\nu) \geq \psi_1(s-\tau) > \psi_2(s), \quad s > T_2.$$

Next, $\lim\limits_{\nu\to\infty}\psi_1^\nu(T_2) = \theta > \psi_2(-T_2)$ implies that there exists $\nu_1 = \nu_1(T_2) = \nu_1(\delta) > \tau$, such that, for all $\nu > \nu_1$,

$$\psi_1^\nu(s) \geq \psi_1^\nu(T_2) > \psi_2(-T_2) \geq \psi_2(s), \quad s \in [-T_2, T_2].$$

Let such a $\nu > \nu_1$ be chosen and fixed. As a result,

$$\psi_1^\nu(s) \geq \psi_2(s), \quad s \geq -T_2, \tag{4.163}$$

and, by (4.161),

$$\psi_1^\nu(s) + \delta > \theta > \psi_2(s), \quad s < -T_2. \tag{4.164}$$

For the $\nu > \nu_1$ chosen above, define

$$\varphi_\nu(s) := \psi_1^\nu(s) - \psi_2(s), \quad s \in \mathbb{R}. \tag{4.165}$$

To prove (4.159), it is enough to show that $\varphi_\nu(s) \geq 0$, $s \in \mathbb{R}$.

On the contrary, suppose that φ_ν takes negative values. By (4.163), (4.164),

$$\varphi_\nu(s) \geq -\delta, \quad s < -T_2; \qquad \varphi_\nu(s) \geq 0, \quad s \geq -T_2. \tag{4.166}$$

Since $\lim\limits_{s\to-\infty}\varphi_\nu(s) = 0$ and $\varphi_\nu \in C^1(\mathbb{R})$, our assumption implies that there exists $s_0 < -T_2$, such that

$$\varphi_\nu(s_0) = \min_{s\in\mathbb{R}}\varphi_\nu(s) \in [-\delta, 0). \tag{4.167}$$

We set also

$$\delta_* := -\varphi_\nu(s_0) = \psi_2(s_0) - \psi_1^\nu(s_0) \in (0, \delta]. \tag{4.168}$$

Next, both ψ_1^ν and ψ_2 solve (4.32). Let $\check{J}_\theta$ be given by (4.33). Then, recall, $\int_{\mathbb{R}}\check{J}_\theta(s)\,ds = \varkappa^+ - \kappa_2\theta$. Denote, cf. (1.4), $\check{L}_\theta\varphi := \check{J}_\theta * \varphi - (\varkappa^+ - \kappa_2\theta)\varphi$. Then one can rewrite (4.32), cf. (4.35),

$$c\psi'(s) + (\check{L}_\theta\psi)(s) + (\theta - \psi(s))\bigl(\kappa_1\psi(s) + \kappa_2(\check{a}^- * \psi)(s)\bigr) = 0.$$

Writing the latter equation for ψ_1^ν and ψ_2 and subtracting the results, one gets

$$\begin{gathered} c\varphi_\nu'(s) + (\check{L}_\theta\varphi_\nu)(s) + A(s) = 0, \\ A(s) := (\theta - \psi_1^\nu(s))\bigl(\kappa_1\psi_1^\nu(s) + \kappa_2(\check{a}^- * \psi_1^\nu)(s)\bigr) \\ -(\theta - \psi_2(s))\bigl(\kappa_1\psi_2(s) + \kappa_2(\check{a}^- * \psi_2)(s)\bigr). \end{gathered} \tag{4.169}$$

Consider (4.169) at the point s_0. By (4.167),

$$\varphi_\nu'(s_0) = 0, \qquad (\check{L}_\theta \varphi_\nu)(s_0) \geq 0. \tag{4.170}$$

Next, (4.168) yields

$$\begin{aligned} A(s_0) =&(\theta - \psi_1^\nu(s_0))\bigl(\kappa_1\psi_1^\nu(s_0) + \kappa_2(\check{a}^- * \psi_1^\nu)(s_0)\bigr) \\ &+ (\delta_* - (\theta - \psi_1^\nu(s_0))\bigl(\kappa_1\psi_2(s_0) + \kappa_2(\check{a}^- * \psi_2)(s_0)\bigr) \\ =&(\theta - \psi_1^\nu(s_0))\bigl(\kappa_1\varphi_\nu(s_0) + \kappa_2(\check{a}^- * \varphi_\nu)(s_0)\bigr) \\ &+ \delta_*\bigl(\kappa_1\psi_2(s_0) + \kappa_2(\check{a}^- * \psi_2)(s_0)\bigr) \\ =&(\theta - \psi_1^\nu(s_0))\bigl(\kappa_1\varphi_\nu(s_0) + \kappa_2(\check{a}^- * (\varphi_\nu + \delta_*))(s_0)\bigr) \\ &+ \delta_*\bigl(\kappa_1\psi_2(s_0) + \kappa_2(\check{a}^- * \psi_2)(s_0) - (\theta - \psi_1^\nu(s_0))\bigr) \\ >&0, \end{aligned} \tag{4.171}$$

because of (4.167), (4.161), and (4.162). The strict inequality in (4.171) together with (4.170) contradict to (4.169). Therefore, (4.159) holds, for any $\nu > \nu_1$.

Step 3. Prove that, cf. (4.159),

$$\vartheta_* := \inf\{\vartheta > 0 \mid \psi_1^\vartheta(s) \geq \psi_2(s), s \in \mathbb{R}\} = 0. \tag{4.172}$$

On the contrary, suppose that $\vartheta_* > 0$. Let $\varphi_* := \varphi_{\vartheta_*}$ be given by (4.165). By the continuity of the profiles, $\varphi_* \geq 0$.

First, assume that $\varphi_*(s_0) = 0$, for some $s_0 \in \mathbb{R}$, i.e. φ_* attains its minimum at s_0. Then (4.170) holds with ϑ replaced by ϑ_*, and, moreover, cf. (4.169),

$$A(s_0) = \kappa_2(\theta - \psi_1^\vartheta(s_0))(\check{a}^- * \varphi_*)(s_0) \geq 0.$$

Therefore, (4.169) implies

$$(\check{L}_\theta \varphi_*)(s_0) = 0. \tag{4.173}$$

By the same arguments as in the proof of Proposition 4.18, one can show that (A4) implies that the function $\check{J}_\theta$ also satisfies (A4), for $d = 1$, with some another constants. Then, arguing in the same way as in the proof of Proposition 3.9 (with $d = 1$ and a^+ replaced by $\check{J}_\theta$), one gets that (4.173) implies that φ_* is a constant, and thus $\varphi_* \equiv 0$, i.e. $\psi_1^{\vartheta_*} \equiv \psi_2$. The latter contradicts (4.156).

Therefore, $\varphi_*(s) > 0$, i.e. $\psi_1^{\vartheta_*}(s) > \psi_2(s)$, $s \in \mathbb{R}$. By (4.156) and (4.160), there exists $T_3 = T_3(\vartheta_*) > 0$, such that $\psi_1^{\frac{\vartheta_*}{2}}(s) > \psi_2(s)$, $s > T_3$, and also, for any $s < -T_3$, (4.162) holds and (4.164) holds with ϑ replaced by $\frac{\vartheta_*}{2}$ (for some fixed $\delta \in \bigl(0, \frac{\theta}{4}\bigr)$). For any $\varepsilon \in \bigl(0, \frac{\vartheta_*}{2}\bigr)$, $\psi_1^{\vartheta_* - \varepsilon} \geq \psi_1^{\frac{\vartheta_*}{2}}$, therefore,

$$\psi_1^{\vartheta_* - \varepsilon}(s) > \psi_2(s), \quad s > T_3,$$

and also (4.164) holds with ϑ replaced by $\vartheta_* - \varepsilon$, for $s < -T_3$. We set

$$\alpha := \inf_{t \in [-T_3, T_3]} (\psi_1^{\vartheta_*}(s) - \psi_2(s)) > 0.$$

Since the family $\bigl\{\psi_1^{\vartheta_* - \varepsilon} \mid \varepsilon \in \bigl(0, \frac{\vartheta_*}{2}\bigr)\bigr\}$ is monotone in ε, and $\lim\limits_{\varepsilon \to 0} \psi_1^{\vartheta_* - \varepsilon}(t) = \psi_1^{\vartheta_*}(t)$, $t \in \mathbb{R}$, we have, by Dini's theorem, that the latter convergence is uniform on $[-T_3, T_3]$. As a result, there exists $\varepsilon = \varepsilon(\alpha) \in \bigl(0, \frac{\vartheta_*}{2}\bigr)$, such that

$$\psi_1^{\vartheta_*}(s) \geq \psi_1^{\vartheta_* - \varepsilon}(s) \geq \psi_2(s), \quad s \in [-T_3, T_3].$$

Then, the same arguments as in the Step 2 prove that $\psi_1^{\vartheta_*-\varepsilon}(s) \geq \psi_2(s)$, for all $s \in \mathbb{R}$, that contradicts the definition (4.172) of ϑ_*.

As a result, $\vartheta_* = 0$, and by the continuity of profiles, $\psi_1 \geq \psi_2$. By the same arguments, $\psi_2 \geq \psi_1$, that fulfills the statement. □

5 Front propagation with a constant speed

We will study here the behavior of $u(tx, t)$, where u solves (2.1), for big $t \geq 0$. The results of Section 4 together with the comparison principle imply that if an initial condition $u_0(x)$ to (2.1) has a minorant/majorant which has a form $\psi(x \cdot \xi)$, $\xi \in S^{d-1}$, where $\psi \in \mathcal{M}_\theta(\mathbb{R})$ is a traveling wave profile in the direction ξ with a speed $c \geq c_*(\xi)$, then for the corresponding solution u to (2.1), the function $u(tx, t)$ will have the minorant/majorant $\psi(t(x \cdot \xi - c))$, correspondingly. In particular, if the initial condition is "below" of any traveling wave in a given direction, then one can estimate the corresponding value of $u(tx, t)$ (Theorem 5.4). Considering such a behavior in different directions, one can obtain a (bounded, cf. Proposition 5.7) set, out of which the solution exponentially decays to 0 (Theorem 5.9). Inside of this set the solution will uniformly converge to θ (Theorem 5.10). We will study stationary solutions (Proposition 5.12) and consider the case of slow decaying kernels $a^\pm$ (Subsection 5.4) as well.

5.1 Long-time behavior along a direction

We will follow the abstract scheme proposed in [93]. Note that all statements there were considered in the space $C_b(\mathbb{R}^d)$, however, it can be checked straightforward that they remain true in the space $C_{ub}(\mathbb{R}^d)$. We will assume that (A1) and (A2) hold. Recall that θ, U_θ, L_θ are given by (2.17), (3.20), and (3.21), respectively.

Consider the set N_θ of all nonincreasing functions $\varphi \in C(\mathbb{R})$, such that $\varphi(s) = 0$, $s \geq 0$, and

$$\varphi(-\infty) := \lim_{s\to-\infty} \varphi(s) \in (0, \theta).$$

It is easily seen that $N_\theta \subset U_\theta$.

For arbitrary $s \in \mathbb{R}$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, define the following mapping $V_{s,c,\xi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}^d)$

$$(V_{s,c,\xi} f)(x) = f(x \cdot \xi + s + c), \quad x \in \mathbb{R}^d. \tag{5.1}$$

Fix an arbitrary $\varphi \in N_\theta$. For $T > 0$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, consider the mapping $R_{T,c,\xi} :\ L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, given by

$$(R_{T,c,\xi} f)(s) = \max\bigl\{\varphi(s), (Q_T(V_{s,c,\xi} f))(0)\bigr\}, \quad s \in \mathbb{R}, \tag{5.2}$$

where Q_T is given by (3.33), cf. Proposition 3.16. Consider now the following sequence of functions

$$f_{n+1}(s) = (R_{T,c,\xi} f_n)(s), \quad f_0(s) = \varphi(s), \qquad s \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}. \tag{5.3}$$

By Proposition 3.16 and [93, Lemma 5.1], $\varphi \in U_\theta$ implies $f_n \in U_\theta$ and $f_{n+1}(s) \geq f_n(s)$, $s \in \mathbb{R}$, $n \in \mathbb{N}$; hence one can define the following limit

$$f_{T,c,\xi}(s) := \lim_{n\to\infty} f_n(s), \quad s \in \mathbb{R}. \tag{5.4}$$

Also, by [93, Lemma 5.1], for fixed $\xi \in S^{d-1}$, $T > 0$, $n \in \mathbb{N}$, the functions $f_n(s)$ and $f_{T,c,\xi}(s)$ are nonincreasing in s and in c; moreover, $f_{T,c,\xi}(s)$ is a lower semicontinuous function of s, c, ξ, as a result, this function is continuous from the right in s and in c. Note also, that $0 \le f_{T,c,\xi} \le \theta$. Then, for any c, ξ, one can define the limiting value

$$f_{T,c,\xi}(\infty) := \lim_{s\to\infty} f_{T,c,\xi}(s).$$

Next, for any $T > 0$, $\xi \in S^{d-1}$, we define

$$c_T^*(\xi) = \sup\{c \mid f_{T,c,\xi}(\infty) = \theta\} \in \mathbb{R} \cup \{-\infty, \infty\},$$

where, as usual, $\sup \emptyset := -\infty$. By [93, Propositions 5.1, 5.2], one has

$$f_{T,c,\xi}(\infty) = \begin{cases} \theta, & c < c_T^*(\xi), \\ 0, & c \ge c_T^*(\xi), \end{cases} \tag{5.5}$$

cf. also [93, Lemma 5.5]; moreover, $c_T^*(\xi)$ is a lower semicontinuous function of ξ. It is crucial that, by [93, Lemma 5.4], neither $f_{T,c,\xi}(\infty)$ nor $c_T^*(\xi)$ depends on the choice of $\varphi \in N_\theta$. Note that the monotonicity of $f_{T,c,\xi}(s)$ in s and (5.5) imply that, for $c < c_T^*(\xi)$, $f_{T,c,\xi}(s) = \theta$, $s \in \mathbb{R}$.

Proposition 5.1. *Let $\xi \in S^{d-1}$ and suppose that* (A1)*,* (A2)*, and* (A5) *hold. Let $c_*(\xi)$ be as in Theorem 4.9. Then*

$$c_T^*(\xi) = Tc_*(\xi), \quad T > 0. \tag{5.6}$$

Proof. Take any $c \in \mathbb{R}$ with $cT \ge c_T^*(\xi)$. Then, by (5.5), $f_{T,cT,\xi} \not\equiv \theta$. By (5.2), (5.3), one has

$$f_{n+1}(s) \ge (Q_T(V_{s,cT,\xi} f_n))(0), \quad s \in \mathbb{R}. \tag{5.7}$$

Since $f_n(s)$ is nonincreasing in s, one gets, by (5.1), that, for a fixed $x \in \mathbb{R}^d$, the function $(V_{s,cT,\xi} f_n)(x)$ is also nonincreasing in s. Next, by (5.1), (5.4) and Propositions 3.14,

$$(Q_T(V_{s,cT,\xi} f_n))(x) \to (Q_T(V_{s,cT,\xi} f_{T,cT,\xi}))(x), \text{ a.a. } x \in \mathbb{R}^d. \tag{5.8}$$

Note that, by (5.1) and Proposition 4.4,

$$(Q_T(V_{s,cT,\xi} f_{T,cT,\xi}))(x) = \phi(x\cdot\xi, T), \tag{5.9}$$

where $\phi(\tau, t)$, $\tau \in \mathbb{R}$, $t \in \mathbb{R}_+$ solves (4.4) with $\psi(\tau) = f_{T,cT,\xi}(\tau + s + cT)$ (note that s is a parameter now, cf. (4.4)). On the other hand, the evident equality $(V_{s,cT,\xi} f_{T,cT,\xi})(x + \tau\xi) = f_{T,cT,\xi}(x\cdot\xi + \tau + s + cT)$, $\tau \in \mathbb{R}$ shows that the function $V_{s,cT,\xi} f_{T,cT,\xi}$ is a decreasing function on $\mathbb{R}^d$ along the $\xi \in S^{d-1}$, cf. Definition 3.17, as $f_{T,cT,\xi}$ is a decreasing function on $\mathbb{R}$. Then, by Proposition 3.18 and (5.9), the function $\mathbb{R}^d \ni x \mapsto \phi(x\cdot\xi, T) \in [0, \theta]$ is decreasing along the ξ as well, i.e. $\phi(x\cdot\xi + \tau, T) = \phi((x + \tau\xi)\cdot\xi, T) \le \phi(x\cdot\xi, T)$, $\tau \ge 0$. As a result, the function $\phi(s, T)$ is monotone (almost everywhere) in s. Since $f_{T,cT,\xi}(s)$ was continuous from the right in s, one gets from (5.7), (5.8), that

$$f_{T,cT,\xi}(s) \ge (\tilde{Q}_T f_{T,cT,\xi})(s + cT),$$

where $\tilde{Q}_T$ is given as in Proposition 4.8. Since $f_{T,cT,\xi} \not\equiv \theta$, one has that, by [99, Theorem 5] (cf. the proof of Theorem 4.9), there exists a traveling wave profile with speed c. By Theorem 4.9, we have that $c \ge c_*(\xi)$, and hence $c_T^*(\xi) \ge Tc_*(\xi)$.

Take now any $c \ge c_*(\xi)$ and consider, by Theorem 4.9, a traveling wave in a direction $\xi \in S^{d-1}$, with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and a speed c. Then, by (5.1) and (4.1),

$$(Q_T(V_{s,cT,\xi}\psi))(x) = \psi((x\cdot\xi - cT) + s + cT) = \psi(x\cdot\xi + s).$$

Choose $\varphi \in N_\theta$ such that $\varphi(s) \le \psi(s)$, $s \in \mathbb{R}$ (recall that all constructions are independent on the choice of φ). Then, one gets from (5.2) and (Q4) of Proposition 3.16, that

$$(R_{T,cT,\xi}\varphi)(s) \le (R_{T,cT,\xi}\psi)(s) = \psi(s), \quad s \in \mathbb{R}.$$

Then, by (5.3) and (5.4), $f_{T,cT,\xi}(s) \le \psi(s)$, $s \in \mathbb{R}$, and thus (5.5) implies $cT \ge c_T^*(\xi)$; as a result, $Tc_*(\xi) \ge c_T^*(\xi)$, that fulfills the statement. □

We describe now how the solution to (2.1) behaves, for big times, along a direction $\xi \in S^{d-1}$. We start with a result about an exponential decaying along such a direction. It is worth noting that we do not need to assume either (A1) or (A2) to prove Proposition 5.2 below.

For any $\xi \in S^{d-1}$ and $\lambda > 0$, consider the following set of bounded functions on $\mathbb{R}^d$:

$$E_{\lambda,\xi}(\mathbb{R}^d) := \bigl\{ f \in L^\infty(\mathbb{R}^d) \bigm| \|f\|_{\lambda,\xi} := \operatorname*{esssup}_{x\in\mathbb{R}^d} |f(x)| e^{\lambda x\cdot\xi} < \infty \bigr\}. \tag{5.10}$$

Evidently, for $f \in L^\infty(\mathbb{R}^d)$,

$$\operatorname*{esssup}_{x\in\mathbb{R}^d} |f(x)| e^{\lambda x\cdot\xi} < \infty \quad \text{if and only if} \quad \operatorname*{esssup}_{x\cdot\xi\ge 0} |f(x)| e^{\lambda x\cdot\xi} < \infty,$$

therefore,

$$E_{\lambda,\xi}(\mathbb{R}^d) \subset E_{\lambda',\xi}(\mathbb{R}^d), \quad \lambda > \lambda' > 0, \xi \in S^{d-1}.$$

Proposition 5.2. *Let $\xi \in S^{d-1}$ and $\lambda > 0$ be fixed and suppose that* (A5) *holds with $\mu = \lambda$. Let $0 \le u_0 \in E_{\lambda,\xi}(\mathbb{R}^d)$ and let $u = u(x,t)$ be a solution to* (2.1)*. Then*

$$\|u(\cdot,t)\|_{\lambda,\xi} \le \|u_0\|_{\lambda,\xi} e^{pt}, \quad t \ge 0, \tag{5.11}$$

where

$$p = p(\xi,\lambda) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x) e^{\lambda x\cdot\xi}\, dx - m \in \mathbb{R}. \tag{5.12}$$

Proof. First, we note that, for any $a \in L^1(\mathbb{R}^d)$,

$$\begin{aligned} \bigl|(a * f)(x) e^{\lambda x\cdot\xi}\bigr| &\le \int_{\mathbb{R}^d} |a(x-y)| e^{\lambda(x-y)\cdot\xi} |f(y)| e^{\lambda y\cdot\xi}\, dy \\ &\le \|f\|_{\lambda,\xi} \int_{\mathbb{R}^d} |a(y)| e^{\lambda y\cdot\xi}\, dy. \end{aligned} \tag{5.13}$$

We will follow the notations from the proof of Theorem 2.2, cf. Remark 2.3. Let p is given by (5.12) and suppose that, for some $\tau \in [0,T)$, $\|u_\tau\|_{\lambda,\xi} \le \|u_0\|_{\lambda,\xi}\, e^{p\tau}$. Take any $v \in \mathcal{X}^+_{\tau,\Upsilon}(r)$ with Υ, r given by (2.13), (2.15), such that

$$\|v(\cdot,t)\|_{\lambda,\xi} \le \|u_0\|_{\lambda,\xi}\, e^{pt}, \quad t \in [\tau,\Upsilon]. \tag{5.14}$$

Then, by (2.6), (2.7), one gets, for any $t \in [\tau,\Upsilon]$,

$$\begin{aligned} 0 &\le (\Phi_\tau v)(x,t) e^{\lambda x\cdot\xi} \\ &\le e^{-(t-\tau)m} u_\tau(x) e^{\lambda x\cdot\xi} + \int_\tau^t e^{-m(t-s)} \varkappa^+ (a^+ * v)(x,s) e^{\lambda x\cdot\xi}\, ds \\ &\le \|u_0\|_{\lambda,\xi}\, e^{-m(t-\tau)} e^{p\tau} + \|u_0\|_{\lambda,\xi}\, \varkappa^+ \int_{\mathbb{R}^d} a^+(y) e^{\lambda y\cdot\xi}\, dy \int_\tau^t e^{-m(t-s)} e^{ps}\, ds, \\ &= \|u_0\|_{\lambda,\xi}\, e^{-mt} e^{(p+m)\tau} + \|u_0\|_{\lambda,\xi}\, (p+m) e^{-mt} \int_\tau^t e^{(p+m)s}\, ds \\ &= \|u_0\|_{\lambda,\xi}\, e^{pt}, \end{aligned}$$

where we used (5.13) and (5.14). Therefore, $\|(\Phi_\tau v)(\cdot,t)\|_{\lambda,\xi} \le \|u_0\|_{\lambda,\xi}\, e^{pt}$, $t\in[\tau,\Upsilon]$. As a result,

$$\|(\Phi_\tau^n v)(\cdot,t)\|_{\lambda,\xi} \le \|u_0\|_{\lambda,\xi}\, e^{pt}, \quad n\in\mathbb{N},\ t\in[\tau,\Upsilon].$$

Then $\|u(\cdot,t)\|_{\lambda,\xi}$ satisfies the same inequality on $[\tau,\Upsilon]$; and, by the proof of Theorem 2.2, we have the statement. □

Remark 5.3. It follows from (L1) of Lemma 4.16 and the considerations thereafter, that the statement of Proposition 5.2 remains true if (A5) holds for some $\mu > \lambda$, provided that we assume, additionally, (A6).

Define now the following set

$$\Upsilon_{T,\xi} = \bigl\{x\in\mathbb{R}^d \mid x\cdot\xi \le c_T^*(\xi)\bigr\}, \quad \xi\in S^{d-1}, T>0. \tag{5.15}$$

Clearly, the set $\Upsilon_{T,\xi}$ is convex and closed. Moreover, by (5.6),

$$\Upsilon_{T,\xi} = T\Upsilon_{1,\xi}. \tag{5.16}$$

Here and below, for any measurable $A\subset\mathbb{R}^d$, we define $tA := \{tx \mid x\in A\}\subset\mathbb{R}^d$. We are going to explain now how a solution $u(x,t)$ to (2.1) behaves outside of the set $t\Upsilon_{1,\xi} = \Upsilon_{t,\xi}$, $t>0$.

Theorem 5.4. *Let $\xi\in S^{d-1}$ and $a^+\in\mathcal{U}_\xi$; i.e. all conditions of Definition 4.17 hold. Let $\lambda_* = \lambda_*(\xi)\in I_\xi$ be the same as in Proposition 4.18. Suppose that $u_0\in E_{\lambda_*,\xi}(\mathbb{R}^d)\cap L_\theta$ and let $u\in\tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). Let $\mathscr{O}_\xi\subset\mathbb{R}^d$ be an open set, such that $\Upsilon_{1,\xi}\subset\mathscr{O}_\xi$ and $\delta := \operatorname{dist}(\Upsilon_{1,\xi},\mathbb{R}^d\setminus\mathscr{O}_\xi) > 0$. Then the following estimate holds*

$$\operatorname*{esssup}_{x\notin t\mathscr{O}_\xi} u(x,t) \le \|u_0\|_{\lambda_*,\xi} e^{-\lambda_*\delta t}, \quad t>0. \tag{5.17}$$

Proof. Let $p_* := p(\xi,\lambda_*)$ be given by (5.12). By (5.11), (5.10), one has

$$0\le u(x,t)\le \|u_0\|_{\lambda_*,\xi}\exp\bigl\{p_*t - \lambda_* x\cdot\xi\bigr\}, \quad \text{a.a. } x\in\mathbb{R}^d. \tag{5.18}$$

Next, by (5.15) and Proposition 5.1, for any $t>0$ and for all $x\in\mathbb{R}^d\setminus t\mathscr{O}_\xi$, one has $x\cdot\xi \ge tc_1^*(\xi) + t\delta = tc_*(\xi) + t\delta$. Then, by (4.83),

$$\begin{aligned}\inf_{x\notin t\mathscr{O}_\xi}(\lambda_* x\cdot\xi) &\ge t\lambda_* c_*(\xi) + t\lambda_*\delta \\ &= t\Bigl(\varkappa^+\int_{\mathbb{R}^d} a^+(x)e^{\lambda_* x\cdot\xi}\,dx - m\Bigr) + t\lambda_*\delta = tp_* + t\lambda_*\delta.\end{aligned}$$

Therefore, (5.18) implies the statement. □

Remark 5.5. The assumption $u_0\in E_{\lambda_*,\xi}(\mathbb{R}^d)$ is close, in some sense, to the weakest possible assumption on an initial condition $u_0\in L_\theta$ for the equation (2.1) to have

$$\lim_{t\to\infty}\operatorname*{esssup}_{x\notin t\mathscr{O}_\xi} u(x,t) = 0, \tag{5.19}$$

for an arbitrary open set $\mathscr{O}_\xi\supset\Upsilon_{1,\xi}$, where $\Upsilon_{1,\xi}$ is defined by (5.15). Indeed, take any λ_1,λ with $0<\lambda_1<\lambda<\lambda_* = \lambda_*(\xi)$. By Theorem 4.23, there exists a traveling wave solution to (2.1) with a profile $\psi_1\in\mathcal{M}_\theta(\mathbb{R})$ such that $\lambda_0(\psi_1) = \lambda_1$. By Proposition 4.31 (with $j=1$ as $\lambda_1<\lambda_*$) we have that $\psi_1(t)\sim De^{-\lambda_1 t}$, $t\to\infty$. It is easily seen that one can choose a function $\varphi\in\mathcal{M}_\theta(\mathbb{R})\cap C(\mathbb{R})$

such that there exist $p>0$, $T>0$, such that $\varphi(t)\geq\psi_1(t)$, $t\in\mathbb{R}$ and $\varphi(t)=pe^{-\lambda t}$, $t>T$. Take now $u_0(x)=\varphi(x\cdot\xi)$, $x\in\mathbb{R}^d$. We have $u_0\in E_{\lambda,\xi}(\mathbb{R}^d)\setminus E_{\lambda_*,\xi}(\mathbb{R}^d)$. Then, by Proposition 4.4, the corresponding solution has the form $u(x,t)=\phi(x\cdot\xi,t)$. By Proposition 3.5 applied to the equation (4.4), $\phi(s,t)\geq\psi_1(s-c_1t)$, $s\in\mathbb{R}$, $t\geq0$, where $c_1=G_\xi(\lambda_1)>c_*(\xi)$, cf. (4.54) and (4.83). Take $c\in(c_*(\xi),c_1)$ and consider an open set $\mathscr{O}_\xi:=\{x\in\mathbb{R}^d\mid x\cdot\xi<c\}$, then $\Upsilon_{1,\xi}\subset\mathscr{O}_\xi\subset\{x\in\mathbb{R}^d\mid x\cdot\xi\leq c_1\}=:A_1$. One has

$$\begin{aligned}\sup_{x\notin t\mathscr{O}_\xi}u(x,t)&\geq\sup_{x\in tA_1\setminus t\mathscr{O}_\xi}\phi(x\cdot\xi,t)\\&\geq\sup_{ct<s\leq c_1t}\psi_1(s-c_1t)=\psi_1(ct-c_1t)>\psi_1(0),\end{aligned}$$

as $c<c_1$ and ψ_1 is decreasing. As a result, (5.19) does not hold.

On the other hand, if $\psi_*\in\mathcal{M}_\theta(\mathbb{R})$ is a profile with the minimal speed $c_*(\xi)\neq0$ and if $j=2$, cf. Proposition 4.25, then $u_0(x):=\psi_*(x\cdot\xi)$ does not belong to the space $E_{\lambda_*,\xi}(\mathbb{R}^d)$, and the arguments above do not contradict (5.19) anymore. In the next remark, we consider this case in more details.

Remark 5.6. In connection with the previous remark, it is worth noting also that one can easily generalize Theorem 5.4 in the following way. Let $u_0\in E_{\lambda,\xi}(\mathbb{R}^d)\cap L_\theta$, for some $\lambda\in(0,\lambda_*]$, and let $u\in\tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). Consider the set $A_{c,\xi}:=\bigl\{x\in\mathbb{R}^d\mid x\cdot\xi\leq c\bigr\}$, where $c=\lambda^{-1}(\varkappa^+\mathfrak{a}_\xi(\lambda)-m)$ cf. (4.83). Then, for any open set $B_{c,\xi}\supset A_{c,\xi}$ with $\delta_c:=\operatorname{dist}(A_{c,\xi},\mathbb{R}^d\setminus B_{c,\xi})>0$, one gets

$$\operatorname*{esssup}_{x\notin tB_{c,\xi}}u(x,t)\leq\|u_0\|_{\lambda,\xi}e^{-\lambda\delta_ct}. \tag{5.20}$$

Therefore, if $u_0(x)=\psi_*(x\cdot\xi)$, where ψ_* is as in Remark 5.5 above, then, evidently, $u_0\in E_{\lambda,\xi}(\mathbb{R}^d)$, for any $\lambda\in(0,\lambda_*)$. Then, for any open $\mathscr{O}_\xi\supset\Upsilon_{1,\xi}$ with $\delta:=\operatorname{dist}(\Upsilon_{1,\xi},\mathbb{R}^d\setminus\mathscr{O}_\xi)>0$ one can choose, for any $\varepsilon\in(0,1)$, $c_1=c_*(\xi)+\delta\varepsilon$. By Theorem 4.23, there exists a unique $\lambda_1=\lambda_1(\varepsilon)\in(0,\lambda_*)$ such that $c_1=\lambda_1^{-1}(\varkappa^+\mathfrak{a}_\xi(\lambda_1)-m)$. Then $u_0\in E_{\lambda_1,\xi}(\mathbb{R}^d)$ and $A_{c_1,\xi}\subset\mathscr{O}_\xi$, i.e. $\mathscr{O}_\xi$ may be considered as a set $B_{c_1,\xi}$, cf. above. As a result, (5.20) gives (5.17), with the constant $\|u_0\|_{\lambda_1,\xi}<\|u_0\|_{\lambda_*,\xi}$, and with $\lambda_*\delta$ replaced by $\lambda_1\delta(1-\varepsilon)$. Note that, clearly, $\|u_0\|_{\lambda_1,\xi}\nearrow\|u_0\|_{\lambda_*,\xi}$, $\lambda_1\nearrow\lambda_*$, $\varepsilon\to0$.

5.2 Global long-time behavior

We are going to consider now the global long-time behavior along all possible directions $\xi\in S^{d-1}$ simultaneously. Define, cf. (5.15),

$$\Upsilon_T=\bigl\{x\in\mathbb{R}^d|x\cdot\xi\leq c_T^*(\xi),\ \xi\in S^{d-1}\bigr\},\quad T>0. \tag{5.21}$$

By (5.15), (5.6), (5.16),

$$\Upsilon_T=\bigcap_{\xi\in S^{d-1}}\Upsilon_{T,\xi}=\bigcap_{\xi\in S^{d-1}}T\Upsilon_{1,\xi}=T\Upsilon_1,\quad T>0. \tag{5.22}$$

Clearly, the set Υ_T, $T>0$ is convex and closed. To have an analog of Theorem 5.4 for the set Υ_T, one needs to have $a^+\in\mathcal{U}_\xi$, for all $\xi\in S^{d-1}$, cf. Definition 4.17.

Since $\int_{x\cdot\xi\leq0}a^+(x)e^{\lambda x\cdot\xi}\,dx\in[0,1]$, $\xi\in S^{d-1}$, $\lambda>0$, we have the following observation. If, for some $\xi\in S^{d-1}$, there exist $\mu^\pm>0$, such that, cf. (4.10), $\mathfrak{a}_{\pm\xi}(\mu^\pm)<\infty$, i.e. if (A5) holds for both ξ and $-\xi$, then, for $\mu = \min\{\mu^+, \mu^-\}$,

$$\begin{aligned}\int_{\mathbb{R}^d} a^+(x)e^{\mu|x\cdot\xi|}\,dx &= \int_{x\cdot\xi\geq 0} a^+(x)e^{\mu x\cdot\xi}\,dx + \int_{x\cdot\xi<0} a^+(x)e^{-\mu x\cdot\xi}\,dx \\ &\leq \int_{x\cdot\xi\geq 0} a^+(x)e^{\mu^+ x\cdot\xi}\,dx + \int_{x\cdot(-\xi)>0} a^+(x)e^{\mu^- x\cdot(-\xi)}\,dx < \infty. \end{aligned} \tag{5.23}$$

Let now $\{e_i \mid 1 \leq i \leq d\}$ be an orthonormal basis in $\mathbb{R}^d$. Let (A5) holds for $2d$ directions $\{\pm e_i \mid 1 \leq i \leq d\} \subset S^{d-1}$ and let $\mu_i = \min\{\mu(e_i), \mu(-e_i)\}$, $1 \leq i \leq d$, cf. (5.23). Set $\mu = \frac{1}{d}\min\{\mu_i \mid 1 \leq i \leq d\}$. Then, by the triangle and Jensen's inequalities and (5.23), one has

$$\begin{aligned}\int_{\mathbb{R}^d} a^+(x)e^{\mu|x|}\,dx &\leq \int_{\mathbb{R}^d} a^+(x)\exp\biggl(\sum_{i=1}^d \frac{1}{d}\mu_i|x\cdot e_i|\biggr)\,dx \\ &\leq \sum_{i=1}^d \frac{1}{d}\int_{\mathbb{R}^d} a^+(x)e^{\mu_i|x\cdot e_i|}\,dx < \infty.\end{aligned}$$

As a result, the assumption that (A5) holds, for all $\xi \in S^{d-1}$, is equivalent to the following one

$$\text{there exists } \mu_d > 0, \text{ such that } \int_{\mathbb{R}^d} a^+(x)e^{\mu_d|x|}\,dx < \infty. \tag{A9}$$

Clearly, (A9) implies

$$\int_{\mathbb{R}^d} |x|a^+(x)\,dx < \infty, \tag{5.24}$$

and thus (A7) holds, for any $\xi \in S^{d-1}$. Then, one can define the (global) first moment vector of a^+, cf. (4.47),

$$\mathfrak{m} := \int_{\mathbb{R}^d} xa^+(x)\,dx \in \mathbb{R}^d. \tag{5.25}$$

The most 'anisotropic' assumption is (A8). We will assume, for simplicity, that (A3) holds; then (A8) holds with $r(\xi) = 0$, for all $\xi \in S^{d-1}$.

Proposition 5.7. *Let* (A1), (A2), (A3), (A6), (A9) *hold. Then, for any* $T > 0$, $T\varkappa^+\mathfrak{m} \in \mathbb{R}^d$ *is an interior point of* Υ_T*, and* Υ_T *is a bounded set.*

Proof. By (5.22), it is enough to prove the statement, for $T = 1$. By (4.47), for any orthonormal basis $\{e_i \mid 1 \leq i \leq d\} \subset S^{d-1}$, $\mathfrak{m} = \sum_{i=1}^d \mathfrak{m}_{e_i}$. As it was shown above, the assumptions of the statement imply that Theorem 4.23 holds, for any $\xi \in S^{d-1}$. Therefore, by (4.80) and Proposition 5.1,

$$(\varkappa^+\mathfrak{m})\cdot\xi = \varkappa^+\int_{\mathbb{R}^d} x\cdot\xi a^+(x)\,dx = \varkappa^+\mathfrak{m}_\xi < c_*(\xi) = c_1^*(\xi), \tag{5.26}$$

for all $\xi \in S^{d-1}$; thus $\varkappa^+\mathfrak{m} \in \Upsilon_1$. Since the inequality in (5.26) is strict, the point $\varkappa^+\mathfrak{m}$ is an interior point of Υ_1.

Next, by Proposition 5.1, $x \in \Upsilon_1$ implies that, for any fixed $\xi \in S^{d-1}$, $x \cdot \xi \leq c_*(\xi)$ and $x \cdot (-\xi) \leq c_*(-\xi)$, i.e.

$$-c_*(-\xi) \leq x \cdot \xi \leq c_*(\xi), \quad x \in \Upsilon_1, \xi \in S^{d-1}. \tag{5.27}$$

Then (5.27) implies

$$|x \cdot \xi| \leq \max\bigl\{|c_*(\xi)|, |c_*(-\xi)|\bigr\}, \quad x \in \Upsilon_1, \xi \in S^{d-1};$$

in particular, for an orthonormal basis $\{e_i \mid 1 \leq i \leq d\}$ of $\mathbb{R}^d$, one gets

$$|x| \leq \sum_{i=1}^{d} |x \cdot e_i| \leq \sum_{i=1}^{d} \max\bigl\{|c_*(e_i)|, |c_*(-e_i)|\bigr\} =: R < \infty, \quad x \in \Upsilon_1,$$

that fulfills the statement. □

Remark 5.8. It is worth noting that, by (4.80), (4.47), the following inequality holds, cf. (5.27),

$$c_*(\xi) + c_*(-\xi) > \varkappa^+(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi}) = 0.$$

For any $T > 0$, consider the set $\mathcal{M}(T)$ of all subsets from $\mathbb{R}^d$ of the following form:

$$M_T = M_{T,\varepsilon,K,\xi_1,\ldots,\xi_K} = \bigl\{x \in \mathbb{R}^d \mid x \cdot \xi_i \leq c_T^*(\xi_i) + \varepsilon, \ i = 1, \ldots, K\bigr\}, \tag{5.28}$$

for some $\varepsilon > 0$, $K \in \mathbb{N}$, $\xi_1, \ldots, \xi_K \in S^{d-1}$.

We are ready now to prove a result about the long-time behavior at infinity in space.

Theorem 5.9. *Let the conditions* (A1), (A2), (A3), (A6), (A9) *hold. Let $u_0 \in L_\theta$ be such that for all $\lambda > 0$,*

$$|||u_0||| := \operatorname*{esssup}_{x \in \mathbb{R}^d} u_0(x) e^{\lambda |x|} < \infty, \tag{5.29}$$

and let $u \in \tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). *Then, for any open set $\mathscr{O} \supset \Upsilon_1$, there exists $\nu = \nu(\mathscr{O}) > 0$, such that*

$$\operatorname*{esssup}_{x \notin t\mathscr{O}} u(x,t) \leq |||u_0||| e^{-\nu t}, \quad t > 0.$$

Proof. By Proposition 5.7, the set Υ_1 is bounded and nonempty. Then, by [93, Lemma 7.2], there exists $\varepsilon > 0$, $K \in \mathbb{N}$, $\xi_1, \ldots, \xi_K \in S^{d-1}$ and a set $M \in \mathcal{M}(1)$ of the form (5.28), with $T = 1$, such that

$$\Upsilon_1 \subset M \subset \mathscr{O}. \tag{5.30}$$

Choose now

$$\mathscr{O}_{\xi_i} = \Bigl\{x \in \mathbb{R}^d \Bigm| x \cdot \xi_i < c_1^*(\xi_i) + \frac{\varepsilon}{2}\Bigr\} \supset \Upsilon_{1,\xi_i}, \quad 1 \leq i \leq K.$$

Then, by (5.30),

$$\Upsilon_1 = \bigcap_{\xi \in S^{d-1}} \Upsilon_{1,\xi} \subset \bigcap_{i=1}^{K} \Upsilon_{1,\xi_i} \subset \bigcap_{i=1}^{K} \mathscr{O}_{\xi_i} \subset M \subset \mathscr{O},$$

and, therefore,

$$\mathbb{R}^d \setminus \mathscr{O} \subset \bigcup_{i=1}^{K} (\mathbb{R}^d \setminus \mathscr{O}_{\xi_i}). \tag{5.31}$$

By (5.10), the assumption (5.29) implies,

$$\begin{aligned}\|u_0\|_{\lambda,\xi} &\leq \max\Bigl\{\operatorname*{esssup}_{x\cdot\xi\geq 0}|u_0(x)|e^{\lambda x\cdot\xi}, \operatorname*{esssup}_{x\cdot\xi<0}|u_0(x)|\Bigr\}\\ &\leq \max\Bigl\{\operatorname*{esssup}_{x\cdot\xi\geq 0}|u_0(x)|e^{\lambda|x|}, \operatorname*{esssup}_{x\cdot\xi<0}|u_0(x)|\Bigr\} \leq \operatorname*{esssup}_{x\in\mathbb{R}^d}|u_0(x)|e^{\lambda|x|} \leq |||u_0|||,\end{aligned}$$

for any $\lambda > 0$, $\xi \in S^{d-1}$. Denote

$$\nu_i := \lambda_*(\xi_i)\mathrm{dist}\,(\Upsilon_{1,\xi_i}, \mathbb{R}^d \setminus \mathscr{O}_{\xi_i}) = \lambda_*(\xi_i)\frac{\varepsilon}{2}, \quad 1 \leq i \leq K.$$

Then, by Theorem 5.4 and (5.31), one gets, for any $t > 0$,

$$\operatorname*{esssup}_{x\notin t\mathscr{O}} u(x,t) \leq \max_{1\leq i\leq K} \operatorname*{esssup}_{x\notin t\mathscr{O}_{\xi_i}} u(x,t) \leq \|u_0\|_{\lambda_*(\xi_i),\xi_i} e^{-\nu_i t} \leq |||u_0|||e^{-\nu t},$$

with $\nu := \min\{\nu_i \mid 1 \leq i \leq K\}$. □

Our second main result about the long-time behavior states that the solution $u \in \mathcal{X}_\infty$ uniformly converges to θ inside the set $t\Upsilon_1 = \Upsilon_t$. The proof of this result is quite technical. For the convenience of the reader, we present here the statement of Theorem 5.10 only, and explain the proof in the next subsection.

For a closed set $A \subset \mathbb{R}^d$, we denote by $\mathrm{int}(A)$ the interior of A.

Theorem 5.10. *Let the conditions* (A1), (A2), (A4), (A6), (A9) *hold. Let* $u_0 \in U_\theta$, $u_0 \not\equiv 0$, *and* $u \in \mathcal{X}_\infty$ *be the corresponding solution to* (2.1)*. Then, for any compact set* $\mathscr{C} \subset \mathrm{int}(\Upsilon_1)$,

$$\lim_{t\to\infty} \min_{x\in t\mathscr{C}} u(x,t) = \theta. \tag{5.32}$$

Corollary 5.11. *Let the conditions* (A1), (A2), (A4), (A6), (A9) *hold. Let* $u_0 \in L_\theta$ *be such that there exist* $x_0 \in \mathbb{R}^d$, $\eta > 0$, $r > 0$, *with* $u_0 \geq \eta$, *for a.a.* $x \in B_r(x_0)$*. Let* $u \in \tilde{\mathcal{X}}_\infty$ *be the corresponding solution to* (2.1)*. Then, for any compact set* $\mathscr{C} \subset \mathrm{int}(\Upsilon_1)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x\in t\mathscr{C}} u(x,t) = \theta.$$

Proof. The assumption on u_0 implies that there exists a function $v_0 \in U_\theta \subset L_\theta$, $v_0 \not\equiv 0$, such that $u_0(x) \geq v_0(x)$, for a.a. $x \in \mathbb{R}^d$. Then, by Remark 3.6, $u(x,t) \geq v(x,t)$, for a.a. $x \in \mathbb{R}^d$, and for all $t \geq 0$, where $v \in \tilde{\mathcal{X}}_\infty$ is the corresponding to v_0 solution to (2.1). By Proposition 3.5, $v \in \mathcal{X}_\infty$, and one has (5.32) for v, with the same Υ_1, cf. (Q1) of Proposition 3.16. The statement follows then from the evident inequality

$$\min_{x\in t\mathscr{C}} v(x,t) = \operatorname*{essinf}_{x\in t\mathscr{C}} v(x,t) \leq \operatorname*{essinf}_{x\in t\mathscr{C}} u(x,t) \leq \theta.$$ □

As an important application of Theorem 5.10, we will prove that there are not stationary solutions $u \geq 0$ to (2.1) (i.e. solutions with $\frac{\partial}{\partial t}u = 0$), except $u \equiv 0$ and $u \equiv \theta$, provided that the origin belongs to $\mathrm{int}(\Upsilon_1)$.

Proposition 5.12. *Let the conditions* (A1), (A2), (A4), (A6), (A9), *and* (2.22) *hold. Let the origin belongs to* $\mathrm{int}(\Upsilon_1)$. *Then there exist only two non-negative stationary solutions to* (2.1) *in* $L^\infty(\mathbb{R}^d)$*, namely,* $u = 0$ *and* $u = \theta$.

Proof. Since $\frac{\partial}{\partial t}u = 0$, one gets from (2.1) that

$$u(x) = \frac{\pm\sqrt{D(x)} - \bigl(m + B(x)\bigr)}{\kappa_1}, \quad x \in \mathbb{R}^d, \tag{5.33}$$

where

$$A(x) = \varkappa^+(a^+ * u)(x), \ B(x) = \kappa_2(a^- * u)(x),$$
$$D(x) = \bigl(m + B(x)\bigr)^2 + 4\kappa_1 A(x) \ge m > 0.$$

Then, by Lemma 2.1, one easily gets that $u \in C_{ub}(\mathbb{R}^d)$.

Denote $M := \|u\| = \sup_{x\in\mathbb{R}^d} u(x)$. We are going to prove now that $M \le \theta$. On the contrary, suppose that $M > \theta$. One can rewrite (5.33) as follows:

$$mu(x) + \kappa_1 u^2(x) + \kappa_2(a^- * u)(x)(u(x) - \theta)$$
$$= (J_\theta * u)(x) \le M(\varkappa^+ - \kappa_2\theta), \tag{5.34}$$

where $J_\theta \ge 0$ is given by (3.19) and hence $\int_{\mathbb{R}^d} J_\theta(x)\,dx = \varkappa^+ - \kappa_2\theta$.

Choose a sequence $x_n \in \mathbb{R}^d$, $n \in \mathbb{N}$, such that $u(x_n) \to M$, $n \to \infty$. Substitute x_n to the inequality (5.34) and pass $n \to \infty$. Since $M > \theta$ and $u \ge 0$, one gets then that $(a^- * u)(x_n) \to 0$, $n \to \infty$. Passing to a subsequence of $\{x_n\}$ and keeping the same notation, for simplicity, one gets that

$$(a^- * u)(x_n) \le \frac{1}{n}, \ n \ge 1.$$

For all $n \ge r_0^{-2d}$, set $r_n := n^{-\frac{1}{2d}} \le r_0$; then the inequality (2.22) holds, for any $x \in B_{r_n}(0)$, and hence

$$\frac{1}{n} \ge (a^- * u)(x_n) \ge \alpha(1\!\!1_{B_{r_n}(0)} * u)(x_n) \ge \alpha V_d(r_n) \min_{x\in B_{r_n}(x_n)} u(x), \tag{5.35}$$

where $V_d(R)$ is a volume of a sphere with the radius $R > 0$ in $\mathbb{R}^d$. Since $V(r_n) = r_n^d V_d(1) = n^{-\frac{1}{2}} V_d(1)$, we have from (5.35), that, for any $n \ge r_0^{-2d}$, there exists $y_n \in B_{r_n}(x_n)$, such that

$$u(y_n) \le \frac{1}{\alpha\sqrt{n}V_d(1)}.$$

Thus $u(y_n) \to 0$, $n \to \infty$. Recall that $u(x_n) \to M > 0$, $n \to \infty$, however, $|x_n - y_n| \le r_n = n^{-\frac{1}{2d}}$, that may be arbitrary small. This contradicts the fact that $u \in C_{ub}(\mathbb{R}^d)$.

As a result, $0 \le u(x) \le \theta = M$, $x \in \mathbb{R}^d$. Let $u \not\equiv 0$. By Theorem 5.10, for any compact set $\mathscr{C} \subset \mathrm{int}(\Upsilon_1)$, $\min_{x\in t\mathscr{C}} u(x) \to \theta$, $t \to \infty$, as $u(x,t) = u(x)$ now. Since $0 \in \mathrm{int}(\Upsilon_1)$, the latter convergence is obviously possible for $u \equiv \theta$ only. □

Remark 5.13. It is worth noting that, by (5.15), (5.16), and (5.6), the assumption $0 \in \mathrm{int}(\Upsilon_1)$ implies that $c_*(\xi) \ge 0$, for all $\xi \in S^{d-1}$. It means that all traveling waves in all directions have nonnegative speeds only.

5.3 Proof of Theorem 5.10

We will do as follows. At first, in Proposition 5.18, we apply results of [93] for discrete time, to prove (5.32) for continuous time, provided that u_0 is separated from 0 on a big enough set. Next, in Proposition 5.19, we show that there exists a proper subsolution to (2.1), which will reach (as we explain thereafter) any needed level after a finite time. Finally, we properly use in Proposition 5.20 the results of [14], to prove that the solution to (2.1) will dominate the subsolution after a finite time.

We start with the following Weinberger's result (rephrased in our settings). Note that (A4) implies (A3), hence, under conditions of Theorem 5.10, we have by Proposition 5.7, that $\Upsilon_T \neq \emptyset$, $T > 0$.

Lemma 5.14 (cf. [93, Theorem 6.2])**.** *Let* (A1)*,* (A2)*,* (A4)*,* (A6)*,* (A9) *hold. Let* $u_0 \in U_\theta$ *and* $T > 0$ *be arbitrary, and* Q_T *be given by* (3.33)*. Define*

$$u_{n+1}(x) := (Q_T u_n)(x), \quad n \geq 0. \tag{5.36}$$

Then, for any compact set $\mathscr{C}_T \subset \operatorname{int}(\Upsilon_T)$ *and for any* $\sigma \in (0, \theta)$*, one can choose a radius* $r_\sigma = r_\sigma(Q_T, \mathscr{C}_T)$*, such that*

$$u_0(x) \geq \sigma, \quad x \in B_{r_\sigma}(0), \tag{5.37}$$

implies

$$\lim_{n\to\infty} \min_{x \in n\mathscr{C}_T} u_n(x) = \theta. \tag{5.38}$$

Remark 5.15. By the proof of [93, Theorem 6.2], the radius $r_\sigma(Q_T, \mathscr{C}_T)$ is not defined uniquely. In the sequel, $r_\sigma(Q_T, \mathscr{C}_T)$ means just a radius which fulfills the assertion of Lemma 5.14 for the chosen Q_T and $\mathscr{C}_T$, rather than a function of Q_T and $\mathscr{C}_T$.

Remark 5.16. It is worth noting, that, by (3.33) and the uniqueness of the solution to (2.1), the iteration (5.36) is just given by

$$u_n(x) = u(x, nT), \quad x \in \mathbb{R}^d, n \in \mathbb{N} \cup \{0\}. \tag{5.39}$$

Therefore, (5.38) with $T = 1$ yields (5.32), for $\mathbb{N} \ni t \to \infty$, namely,

$$\lim_{n\to\infty} \min_{x \in n\mathscr{C}} u(x, n) = \theta, \tag{5.40}$$

provided that (5.37) holds with $r_\sigma = r_\sigma(Q_1, \mathscr{C})$, $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$.

Lemma 5.17. *Let the conditions of Theorem 5.10 hold. Fix a* $\sigma \in (0, \theta)$ *and a compact set* $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$*. Let* $u_0 \in U_\theta$ *be such that* $u_0(x) \geq \sigma$*,* $x \in B_{r_\sigma(Q_1, \mathscr{C})}(0)$*. Then, for any* $k \in \mathbb{N}$*,*

$$\lim_{n\to\infty} \min_{x \in \frac{n}{k}\mathscr{C}} u\Bigl(x, \frac{n}{k}\Bigr) = \theta. \tag{5.41}$$

Proof. Since $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$, one can choose a compact set $\tilde{\mathscr{C}} \subset \operatorname{int}(\Upsilon_1)$ such that

$$\mathscr{C} \subset \operatorname{int}(\tilde{\mathscr{C}}). \tag{5.42}$$

By (5.39) and Lemma 5.14 (with $T = 1$), the assumption $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1, \mathscr{C})}(0)$ implies (5.40). Fix $k \in \mathbb{N}$, take $p = \frac{1}{k}$; then choose and fix the radius $r_\sigma\bigl(Q_p, p\tilde{\mathscr{C}}\bigr)$. By (5.40), there exists an $N = N(k) \in \mathbb{N}$, such that

$$u(x, N) \geq \sigma, \quad x \in N\mathscr{C},$$
$$B_{r_\sigma(Q_p, p\tilde{\mathscr{C}})}(0) \subset N\mathscr{C}.$$

Apply now Lemma 5.14, with $u_0(x) = u(x, N)$, $x \in \mathbb{R}^d$, $T = p$, and

$$\mathscr{C}_T = \mathscr{C}_p := p\tilde{\mathscr{C}} \subset p\,\mathrm{int}(\Upsilon_1) = \mathrm{int}(\Upsilon_p),$$

as, by (5.22), $p\Upsilon_1 = \Upsilon_p$. We will get then

$$\lim_{n\to\infty} \min_{x\in np\tilde{\mathscr{C}}} u(x, N + np) = \theta. \tag{5.43}$$

By (5.42), there exists $M \in \mathbb{N}$ such that one has

$$\Bigl(\frac{N}{n} + p\Bigr)\mathscr{C} \subset p\tilde{\mathscr{C}}, \quad n \geq M. \tag{5.44}$$

Therefore, by (5.44), one gets, for $n \geq M$,

$$\begin{aligned}\min_{x\in np\tilde{\mathscr{C}}} u(x, N + np) &\leq \min_{x\in n(\frac{N}{n}+p)\mathscr{C}} u(x, N + np)\\ &= \min_{x\in (Nk+n)\frac{1}{k}\mathscr{C}} u\Bigl(x, (Nk + n)\frac{1}{k}\Bigr) \leq \theta.\end{aligned} \tag{5.45}$$

By (5.43) and (5.45), one gets the statement. □

Now, one can prove Theorem 5.10, under assumption on the initial condition.

Proposition 5.18. *Let the conditions of Theorem 5.10 hold. Fix a $\sigma \in (0, \theta)$ and a compact set $\mathscr{C} \subset \mathrm{int}(\Upsilon_1)$. Let $u_0 \in U_\theta$ be such that $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1,\mathscr{C})}(0)$, and $u \in \mathcal{X}_\infty$ be the corresponding solution to* (2.1)*. Then*

$$\lim_{t\to\infty} \min_{x\in t\mathscr{C}} u(x, t) = \theta. \tag{5.46}$$

Proof. Suppose (5.46) were false. Then, there exist $\varepsilon > 0$ and a sequence $t_N \to \infty$, such that $\min\limits_{x\in t_N\mathscr{C}} u(x, t_N) < \theta - \varepsilon$, $n \in \mathbb{N}$. Since $t_N\mathscr{C}$ is a compact set and $u(\cdot, t) \in U_\theta$, $t \geq 0$, there exists $x_N \in t_N\mathscr{C}$, such that

$$u(x_N, t_N) < \theta - \varepsilon, \quad n \in \mathbb{N}. \tag{5.47}$$

Next, by Proposition 2.11, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $x', x'' \in \mathbb{R}^d$ and for all $t', t'' > 0$, with $|x' - x''| + |t' - t''| < \delta$, one has

$$|u(x', t') - u(x'', t'')| < \frac{\varepsilon}{2}. \tag{5.48}$$

Since $\mathscr{C}$ is a compact, $p(\mathscr{C}) := \sup\limits_{x\in\mathscr{C}} \|x\| < \infty$. Choose $k \in \mathbb{N}$, such that $\frac{1}{k} < \frac{\delta}{1+p(\mathscr{C})}$. By (5.41), there exists $M(k) \in \mathbb{N}$, such that, for all $n \geq M(k)$,

$$u\Bigl(x, \frac{n}{k}\Bigr) > \theta - \frac{\varepsilon}{2}, \quad x \in \frac{n}{k}\mathscr{C}. \tag{5.49}$$

Choose $N > N_0$ big enough to ensure $t_N > \frac{M(k)}{k}$. Then, there exists $n \geq M(k)$, such that $t_N \in \bigl[\frac{n}{k}, \frac{n+1}{k}\bigr)$. Hence

$$\Bigl|t_N - \frac{n}{k}\Bigr| < \frac{1}{k} < \frac{\delta}{1 + p(\mathscr{C})}. \tag{5.50}$$

Next, for the chosen N, there exists $y_N \in \mathscr{C}$, such that $x_N = t_N y_N$. Set $t' = t_N$, $t'' = \frac{n}{k}$, $x' = x_N = t_N y_N$, and $x'' = \frac{n}{k} y_N$. Then, by (5.50),

$$|t' - t''| + |x' - x''| = \Big| t_N - \frac{n}{k} \Big| \big(1 + |y_N|\big) < \delta.$$

Therefore, one can apply (5.48). Combining this with (5.47), one gets

$$u\Big(\frac{n}{k} y_N, \frac{n}{k}\Big) = u\Big(\frac{n}{k} y_N, \frac{n}{k}\Big) - u(t_N y_N, t_N) + u(x_N, t_N) < \frac{\varepsilon}{2} + \theta - \varepsilon = \theta - \frac{\varepsilon}{2},$$

that contradicts (5.49), as $\frac{n}{k} y_N \in \frac{n}{k}\mathscr{C}$. Hence the statement is proved. □

Next two statements will allow us to get rid the restriction on u_0 in Proposition 5.18.

Proposition 5.19. *Let* (A1), (A2), *and* (A4) *hold; assume also that* (5.24) *holds, and* $\mathfrak{m}$ *is given by* (5.25). *Then there exists* $\alpha_0 > 0$, *such that, for all* $\alpha \in (0, \alpha_0)$, *there exists* $q_0 = q_0(\alpha) \in (0, \theta)$, *such that there exists* $T = T(\alpha, q_0) > 0$, *such that, for all* $q \in (0, q_0)$, *the function*

$$w(x,t) = q \exp\Big(-\frac{|x - t\mathfrak{m}|^2}{\alpha t}\Big), \quad x \in \mathbb{R}^d, t > T, \tag{5.51}$$

is a subsolution to (2.1) *on* $t > T$; *i.e.* $\mathcal{F}w(x,t) \le 0$, $x \in \mathbb{R}^d$, $t > T$, *where* $\mathcal{F}$ *is given by* (3.1).

Proof. Let J_q, $q \in (0, \theta)$ be given by (3.19), and consider the function (5.51). Since $w(x,t) \le q$, we have from (3.1), that

$$\begin{aligned}(\mathcal{F}w)(x,t) &= w(x,t)\Big(\frac{|x|^2}{\alpha t^2} - \frac{|\mathfrak{m}|^2}{\alpha}\Big) - \varkappa^+ (a^+ * w)(x,t) \\ &\quad + \kappa_1 w^2(x,t) + \kappa_2 w(x,t)(a^- * w)(x,t) + m w(x,t) \\ &\le w(x,t)\Big(\frac{|x|^2}{\alpha t^2} - \frac{|\mathfrak{m}|^2}{\alpha}\Big) - (J_q * w)(x,t) + (\kappa_1 q + m) w(x,t). \end{aligned} \tag{5.52}$$

Since, for any $q_0 \in (0, \theta)$ and for any $q \in (0, q_0)$, $J_q(x) \ge J_{q_0}(x)$, $x \in \mathbb{R}^d$, one gets from (5.52), that, to have $\mathcal{F}w \le 0$, it is enough to claim that, for all $x \in \mathbb{R}^d$,

$$\kappa_1 q_0 + m + \frac{|x|^2}{\alpha t^2} - \frac{|\mathfrak{m}|^2}{\alpha} \le \exp\Big(\frac{|x - t\mathfrak{m}|^2}{\alpha t}\Big) \int_{\mathbb{R}^d} J_{q_0}(y) \exp\Big(-\frac{|x - y - t\mathfrak{m}|^2}{\alpha t}\Big)\, dy.$$

By changing x onto $x + t\mathfrak{m}$ and a simplification, one gets an equivalent inequality

$$\kappa_1 q_0 + m + \frac{|x|^2}{\alpha t^2} + \frac{2\, x \cdot \mathfrak{m}}{\alpha t} \le \int_{\mathbb{R}^d} J_{q_0}(y) \exp\Big(\frac{2x \cdot y}{\alpha t}\Big) \exp\Big(-\frac{|y|^2}{\alpha t}\Big)\, dy =: I(t). \tag{5.53}$$

One can rewrite $I(t) = I_0(t) + I^+(t) + I^-(t)$, where

$$I_0(t) := \int_{\mathbb{R}^d} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}}\, dy; \qquad I^+(t) := \int_{x \cdot y \ge 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} \Big(e^{\frac{2x \cdot y}{\alpha t}} - 1\Big) dy;$$

$$I^-(t) := \int_{x \cdot y < 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} \Big(e^{\frac{2x \cdot y}{\alpha t}} - 1\Big) dy.$$

Using that $e^s - 1 \ge s$, for all $s \in \mathbb{R}$, and $e^s - 1 \ge s + \frac{s^2}{2}$, for all $s \ge 0$, one gets the following estimates

$$I^+(t) \ge \frac{2}{\alpha t} \int_{x \cdot y \ge 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} (x \cdot y) dy + \frac{2}{\alpha^2 t^2} \int_{x \cdot y \ge 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} (x \cdot y)^2 dy,$$

and

$$I^-(t) \geq \frac{2}{\alpha t}\int_{x\cdot y<0} J_{q_0}(y)e^{-\frac{|y|^2}{\alpha t}}(x\cdot y)dy.$$

Therefore,

$$I(t) \geq I_0(t) + \frac{2}{\alpha t}\Bigl(x\cdot\int_{\mathbb{R}^d} J_{q_0}(y)e^{-\frac{|y|^2}{\alpha t}}ydy\Bigr) + \frac{2}{\alpha^2t^2}\int_{x\cdot y\geq 0} J_{q_0}(y)e^{-\frac{|y|^2}{\alpha t}}(x\cdot y)^2dy. \tag{5.54}$$

By the dominated convergence theorem,

$$I_0(t) \nearrow \int_{\mathbb{R}^d} J_{q_0}(x)\,dx = \varkappa^+ - q_0\kappa_2 > m + \kappa_1 q_0, \quad t\to\infty, \tag{5.55}$$

for any $q_0 \in (0,\theta)$. Set also

$$I_1(t) := \int_{\mathbb{R}^d} J_{q_0}(y)e^{-\frac{|y|^2}{\alpha t}}y\,dy.$$

By (5.24) and (A2), one has $\int_{\mathbb{R}^d} a^-(x)|x|\,dx<\infty$ and hence $\int_{\mathbb{R}^d} J_{q_0}(x)|x|\,dx<\infty$. Then, by the dominated convergence theorem,

$$I_1(t) \to \int_{\mathbb{R}^d} J_{q_0}(y)ydy =: \mu(q_0) \in \mathbb{R}^d, \quad t\to\infty. \tag{5.56}$$

Since $0 \leq J_{q_0}(x) \leq \varkappa^+a^+(x)$, $x\in\mathbb{R}^d$, we have, by (5.25) and the dominated convergence theorem, that $m(q_0)\to\mathfrak{m}$, $q_0\to 0$.

For any $\varepsilon>0$ with $m+2\varepsilon<\varkappa^+$, one can choose $q_0=q_0(\varepsilon)\in(0,\theta)$, such that

$$\varkappa^+ > \varkappa^+ - \kappa_2 q_0 > \kappa_1 q_0 + m + 2\varepsilon, \qquad |\mathfrak{m}-\mu(q_0)| < \frac{\varepsilon}{2}. \tag{5.57}$$

By (5.55), (5.56), there exists $T_1 = T_1(\varepsilon, q_0)>0$, such that, for all $\alpha>0$ and $t>0$ with $\alpha t > T_1$, one has, cf. (5.57),

$$\varkappa^+ \geq I_0(t) > \kappa_1 q_0 + m + \varepsilon, \qquad |I_1(t)-\mu(q_0)| < \frac{\varepsilon}{2}. \tag{5.58}$$

Let $T>\frac{T_1}{\alpha}$ be chosen later. The function

$$I_2(t) := \int_{x\cdot y\geq 0} J_{q_0}(y)e^{-\frac{|y|^2}{\alpha t}}(x\cdot y)^2dy$$

is also increasing in $t>0$. Clearly, from (5.57) and (5.58), one has $|I_1(t)-\mathfrak{m}|<\varepsilon$. Therefore, by (5.54) and (5.58), one gets, for $t>T>\frac{T_1}{\alpha}$,

$$\begin{aligned} I(t) &> \kappa_1 q_0 + m + \varepsilon + \frac{2}{\alpha t}x\cdot(I_1(t)-\mathfrak{m}) + \frac{2}{\alpha t}x\cdot\mathfrak{m} + \frac{2}{\alpha^2t^2}I_2(t) \\ &\geq \kappa_1 q_0 + m + \varepsilon - \frac{2\varepsilon}{\alpha t}|x| + \frac{2}{\alpha t}x\cdot\mathfrak{m} + \frac{2}{\alpha^2t^2}I_2(T). \end{aligned} \tag{5.59}$$

Next, by (A2), (A4), and (3.19), $J_{q_0}(y)\geq\rho$, for a.a. $y\in B_\delta(0)$. For an arbitrary $x\in\mathbb{R}^d$, consider the set

$$B_x = \Bigl\{y\in\mathbb{R}^d \Bigm| |y|\leq\delta, \frac{1}{2}\leq\frac{x\cdot y}{|x||y|}\leq 1\Bigr\}.$$

Then

$$I_2(T) \geq \frac{\rho}{4}|x|^2 \int_{B_x} |y|^2 e^{-\frac{|y|^2}{\alpha T}}\, dy. \tag{5.60}$$

The set B_x is a cone inside the ball $B_\delta(0)$, with the apex at the origin, the height which lies along x, and the apex angle $2\pi/3$. Since the function inside the integral in the r.h.s. of (5.60) is radially symmetric, the integral does not depend on x. Fix an arbitrary $\bar{x} \in \mathbb{R}^d$ and denote

$$A(\tau) = A(\tau, \delta) = \int_{B_{\bar{x}}} |y|^2 e^{-\frac{|y|^2}{\tau}}\, dy \nearrow \int_{B_{\bar{x}}} |y|^2 dy =: \bar{B}_\delta, \quad \tau \to \infty. \tag{5.61}$$

Then, by (5.59) and (5.60), one has, for $t > T$,

$$I(t) > \kappa_1 q_0 + m + \varepsilon - \frac{2\varepsilon}{\alpha t}|x| + \frac{2}{\alpha t} x \cdot \mathfrak{m} + \frac{\rho A(\alpha T)}{2\alpha^2 t^2}|x|^2. \tag{5.62}$$

By (5.62), to prove (5.53), it is enough to show that

$$\varepsilon - \frac{2\varepsilon}{\alpha t}|x| + \frac{\rho A(\alpha T)}{2\alpha^2 t^2}|x|^2 \geq \frac{|x|^2}{\alpha t^2}, \qquad t > T,\ x \in \mathbb{R}^d,$$

or, equivalently, for $2\alpha < \rho A(\alpha T)$,

$$\left(\sqrt{\frac{\rho A(\alpha T) - 2\alpha}{2}} \frac{|x|}{\alpha t} - \varepsilon \sqrt{\frac{2}{\rho A(\alpha T) - 2\alpha}} \right)^2 + \varepsilon - \varepsilon^2 \frac{2}{\rho A(\alpha T) - 2\alpha} \geq 0. \tag{5.63}$$

To get (5.63), we proceed as follows. For a given $\rho > 0$, $\delta > 0$ which provide (A4), we set $\alpha_0 := \frac{1}{2}\rho \bar{B}_\delta$, cf. (5.61). Then, for any $\alpha \in (0, \alpha_0)$, there exists $T_2 = T_2(\alpha) > 0$, such that

$$2\alpha < \rho A(\alpha T_2) < \rho \bar{B}_\delta.$$

Choose now $\varepsilon = \varepsilon(\alpha) > 0$, such that $m + 2\varepsilon < \varkappa^+$ and

$$\varepsilon < \frac{1}{2}(\rho A(\alpha T_2) - 2\alpha) < \frac{1}{2}(\rho A(\alpha T) - 2\alpha), \quad T > T_2. \tag{5.64}$$

For the chosen ε, find $q_0 = q_0(\alpha) \in (0, \theta)$ which ensures (5.57). Then, find $T_1 = T_1(\alpha, q_0) > 0$ which gives (5.58); and, finally, take $T = T(\alpha, q) > T_2$ such that $\alpha T > T_1$. As a result, for $t > T$, one has $\alpha t > \alpha T > T_1$, thus (5.58) holds, whereas (5.64) yields (5.63). The latter inequality gives (5.53), and hence, for all $q \in (0, q_0)$, $\mathcal{F} w \leq 0$, for w given by (5.51). The statement is proved. □

Proposition 5.20. *Let* (A1), (A2), *and* (A4) *hold. Then, there exists $t_1 > 0$, such that, for any $t > t_1$ and for any $\tau > 0$, there exists $q_1 = q_1(t, \tau) > 0$, such that the following holds. If $u_0 \in L_\theta$ is such that there exist $\eta > 0$, $r > 0$, $x_0 \in \mathbb{R}^d$ with $u_0(x) \geq \eta$, $x \in B_r(x_0)$ and $u \in \tilde{\mathcal{X}}_\infty$ is the corresponding solution to* (2.1), *then*

$$u(x, t) \geq q_1 e^{-\frac{|x - x_0|^2}{\tau}}, \quad x \in \mathbb{R}^d. \tag{5.65}$$

Proof. At first, we note that (5.65) may be rewritten as follows:

$$q_1 e^{-\frac{|x|^2}{\tau}} \leq u(x + x_0, t_0) = T_{-x_0} Q_{t_0} u_0(x) = Q_{t_0} T_{-x_0} u_0(x),$$

cf. (3.33), (3.34), (3.35), and one has

$$T_{-x_0}u_0(x) = u_0(x+x_0) \geq \eta, \quad |(x+x_0)-x_0| = |x| \leq r.$$

Therefore, it is enough to prove the statement for $x_0 = 0$.

Consider now arbitrary functions $b, v_0 \in C^\infty(\mathbb{R}^d)$, such that

$$\begin{aligned}
\operatorname{supp} b = B_\delta(0), \qquad & 0 < b(x) = b(|x|) \leq \rho, && x \in \operatorname{int}(B_\delta(0));\\
\operatorname{supp} v_0 = B_r(0), \qquad & 0 < v_0(x) \leq \eta, && x \in \operatorname{int}(B_r(0));\\
\exists\, 0 < p < \min\{r,1\},\ 0 < \nu < \eta, \qquad & \text{such that } v_0(x) \geq \nu, && x \in B_p(0),
\end{aligned}$$

where ρ and δ are the same as in (A4). Set $\langle b \rangle := \int_{\mathbb{R}^d} b(x)\,dx > 0$. Define two bounded operators in the space $L^\infty(\mathbb{R}^d)$, cf. (1.4): $Bu = b * u$, $L_b u = Bu - \langle b \rangle u$. One can rewrite (2.1) as follows

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= (J_\theta * u)(x,t) - mu(x,t) + (\theta - u(x,t))\big(\kappa_1 u(x,t) + \kappa_2 (a^- * u)(x,t)\big)\\
&= (b*u)(x,t) - mu(x,t) + f(x,t),
\end{aligned}$$

where, for any $x \in \mathbb{R}^d$, $t \geq 0$,

$$f(x,t) := ((J_\theta - b) * u)(x,t) + (\theta - u(x,t))\big(\kappa_1 u(x,t) + \kappa_2 (a^- * u)(x,t)\big).$$

By (A4) and the choice of b, $J_\theta(x) \geq b(x)$, $x \in \mathbb{R}^d$. In particular, $m = \int_{\mathbb{R}^d} J_\theta(x)\,dx \geq \langle b \rangle$, and $f(x,t) \geq 0$, $x \in \mathbb{R}^d$, $t \geq 0$. Next, for any $t \geq 0$, $\|f(\cdot,t)\|_\infty \leq \theta(m - \langle b \rangle) + \varkappa^- \theta^2 < \infty$. Since $b \geq 0$ and $Bu = b * u$ defines a bounded operator on $L^\infty(\mathbb{R}^d)$, one has that $e^{tB} f(x,s) \geq 0$, for all $t, s \geq 0$, $x \in \mathbb{R}^d$. By the same argument, $u_0(x) \geq \eta 1\!\!1_{B_r(0)}(x) \geq v_0(x) \geq 0$ implies $(e^{tB}u_0)(x) \geq (e^{tB}v_0)(x)$. Therefore,

$$\begin{aligned}
u(x,t) &= e^{-tm}(e^{tB}u_0)(x) + \int_0^t e^{-(t-s)m}(e^{(t-s)B}f)(x,s)ds\\
&\geq e^{-tm}(e^{tB}u_0)(x) \geq e^{-(m-\langle b \rangle)t}(e^{tL_b}v_0)(x), \quad x \in \mathbb{R}^d. \qquad (5.66)
\end{aligned}$$

We are going to apply now the results of [14]. To do this, set $\beta := \langle b \rangle^{-1}$. Then

$$(e^{tL_b}v_0)(x) = (e^{\langle b \rangle t(\beta L_b)}v_0)(x) = v(x, \langle b \rangle t), \qquad (5.67)$$

where v solves the differential equation $\frac{d}{dt}v = \beta L_b$. Since $\int_{\mathbb{R}^d} \beta b(x)\,dx = 1$, then, by [16, Theorem 2.1, Lemma 2.2],

$$v(x,t) = e^{-t}v_0(x) + (w * v_0)(x,t), \qquad (5.68)$$

where $w(x,t)$ is a smooth function. Moreover, by [14, Proposition 5.1], for any $\omega \in (0,\delta)$ there exist $c_1 = c_1(\omega) > 0$ and $c_2 = c_2(\omega) \in \mathbb{R}$, such that

$$\begin{aligned}
&w(x,t) \geq h(x,t), \quad x \in \mathbb{R}^d, t \geq 0,\\
&h(x,t) := c_1 t \exp\Big(-t - \frac{1}{\omega}|x|\log|x| + (\log t - c_2)\Big[\frac{|x|}{\omega}\Big]\Big).
\end{aligned} \qquad (5.69)$$

Here $[\alpha]$ means the entire part of an $\alpha \in \mathbb{R}$, and $0 \log 0 := 1$, $\log 0 := -\infty$.

Set $t_1 = e^{c_2} > 0$. Since $[\alpha] > \alpha - 1$, $\alpha \in \mathbb{R}$, one has, for $t > t_1$,

$$h(x,t) \geq c_1 e^{c_2} \exp\Big(-t - \frac{1}{\omega}|x|\log|x| + (\log t - c_2)\frac{|x|}{\omega}\Big) \geq c_3 g(x,t),$$

where $c_3 = c_1 e^{c_2} > 0$ and

$$g(x,t) := \exp\Bigl(-t - \frac{1}{\omega}|x| \log|x|\Bigr), \quad x \in \mathbb{R}^d, t > t_1.$$

Since $v_0 \geq \nu 1\!\!1_{B_p(0)}$, one gets from (5.68) and (5.69), that

$$v(x,t) \geq \nu e^{-t} 1\!\!1_{B_p(0)}(x) + \nu c_3 \int_{B_p(x)} g(y,t)\, dy \tag{5.70}$$

Set $V_p := \int_{B_p(0)} dx$. For any fixed $t > t_1$, since $g(\cdot, t) \in C(B_p(x))$, there exists $y_0, y_1 \in B_p(x)$, such that $g(y,t)$ attains its minimal and maximal values on $B_p(x)$ at these points, respectively. Since $B_p(x)$ is a convex set, one gets that, for any $\gamma \in (0,1)$, $y_\gamma := \gamma y_1 + (1-\gamma) y_0 \in B_p(x)$. Then

$$V_p g(y_0, t) \leq \int_{B_p(x)} g(y_\gamma, t)\, dy \leq V_p g(y_1, t).$$

Therefore, by the intermediate value theorem there exists, $\tilde{y}_t = \tilde{y}(x,t) \in B_p(x)$, $t > t_1$, $x \in \mathbb{R}^d$, such that $\int_{B_p(x)} g(y,t)\, dy = V_p g(\tilde{y}_t, t)$. Hence one gets from (5.66), (5.67), (5.70), that

$$u(x,t) \geq c_4 e^{-(m - \langle b \rangle)t} g\bigl(\tilde{y}_t, \langle b \rangle t\bigr) = c_4 \exp\Bigl(-mt - \frac{1}{\omega}|\tilde{y}_t| \log|\tilde{y}_t|\Bigr), \tag{5.71}$$

for $\tilde{y}_t = \tilde{y}(x,t) \in B_p(x)$, $t > t_1$; here $c_4 = c_3 \nu V_p > 0$.

As a result, to get the statement, it is enough to show that, for any $t > t_1$ and for any $\tau > 0$, there exists $q_1 = q_1(t,\tau) > 0$, such that the r.h.s. of (5.71) is estimated from below by $q_1 e^{-\frac{|x|^2}{\tau}}$, i.e. that

$$mt + \frac{1}{\omega}|\tilde{y}_t| \log|\tilde{y}_t| - \log c_4 \leq \frac{|x|^2}{\tau} - \log q_1, \quad x \in \mathbb{R}^d, \tag{5.72}$$

Note that $\tilde{y}_t \in B_p(x)$ implies $|\tilde{y}_t| \leq p + |x|$, $x \in \mathbb{R}^d$.

Let $p + |x| \leq 1$. Then $\log|\tilde{y}_t| \leq 0$, and the l.h.s. of (5.72) is majorized by $mt - \log c_4$. Therefore, to get (5.72), it is enough to have $q_1 < c_4 e^{-mt}$, regardless of τ.

Let now $|x| + p > 1$. Recall that we chose $p < 1$. The function $s \log s$ is increasing on $s > 1$. Hence to get (5.72), we claim

$$(|x| + 1)\log(|x| + 1) \leq \frac{\omega}{\tau}|x|^2 - \omega mt + \omega \log c_4 - \omega \log q_1. \tag{5.73}$$

Consider now the function $f(s) = as^2 - (s+1)\log(s+1)$, $s \geq 0$, $a = \frac{\omega}{\tau} > 0$. Then $f(0) = 0$, $f'(s) = 2as - \log(s+1) - 1$, $f'(0) = -1$, $f''(s) = 2a - \frac{1}{s+1}$. Since $f''(s) \nearrow 2a > 0$, $s \to \infty$, there exists $s_0 > 0$, such that $f''(s) > 0$, for all $s > s_0$, i.e. $f'(s)$ increases on $s > s_0$. Since $f'(s) \to \infty$, $s \to \infty$, there exists $s_1 > s_0$, such that $f'(s) > 0$, for all $s > s_1$, i.e. f is increasing on $s > s_1$. Finally, for any $t > t_1$, one can choose $q_1 = q_1(t,\tau) > 0$ small enough, to get

$$\min_{s \in [0, s_1]} f(s) - \omega mt + \omega \log c_4 - \omega \log q_1 > 0$$

and to fulfill (5.73), for all $x \in \mathbb{R}^d$. The statement is proved. □

Now, we are ready to prove the main Theorem 5.10.

Proof of Theorem 5.10. For $u_0 \equiv \theta$, the statement is trivial. Hence let $u_0 \not\equiv \theta$, $u_0 \not\equiv 0$. Next, recall that, (A4) implies (A3) and (A9) implies (5.24). Therefore, one may use the statements of Propositions 5.7, 5.19, 5.20.

According to Proposition 5.19, choose any $\alpha \in (0, \alpha_0)$ and take the corresponding $q_0 = q_0(\alpha) \in (0, \theta)$ and $T = T(\alpha, q_0) > 0$. Choose then arbitrary $t_2 > T$. Let $\mathfrak{m}$ be given by (5.25). Set $x_0 = t_2\mathfrak{m} \in \mathbb{R}^d$. By Proposition 3.9, there exist $\eta = \eta(t_2) > 0$ and $r = r(t_2) > 0$, such that $u(x, t_2) \geq \eta$, $|x - x_0| = |x - t_2\mathfrak{m}| \leq r$. Apply now Proposition 5.20, with $u_0(x) = u(x, t_2)$; let t_1 be the moment of time stated there. Take, for the α chosen above, $\tau = \alpha t_2 > 0$. Take any $t_3 > \max\{t_1, t_2\}$ and the corresponding $q_1 = q_1(t_3, \tau) > 0$. We will get then, by (5.65), that

$$u(x, t_3 + t_2) \geq q_1 \exp\Bigl(-\frac{|x - t_2\mathfrak{m}|^2}{\alpha t_2}\Bigr), \quad x \in \mathbb{R}^d. \tag{5.74}$$

Of course, one can assume that $q_1 < q_0$ (otherwise, we just pass to a weaker inequality in (5.74)).

We are going to apply now Theorem 3.1, with $c = \theta$ and, for $t \geq 0$,

$$\begin{aligned} u_1(x, t) &= q_1 \exp\Bigl(-\frac{|x - (t + t_2)\mathfrak{m}|^2}{\alpha(t + t_2)}\Bigr) \geq 0, \\ u_2(x, t) &= u(x, t + t_3 + t_2) \in [0, \theta]. \end{aligned}$$

By (5.74), $u_1(x, 0) \leq u_2(x, 0)$, $x \in \mathbb{R}^d$. Since u solve (2.1), $\mathcal{F}u_2 \equiv 0$. Next, by Proposition 5.19, if we set $q = q_1$, we will have $\mathcal{F}u_1 \leq 0$, as $t + t_2 \geq t_2 > T$. Therefore, by Theorem 3.1,

$$u(x, t + t_3 + t_2) \geq q_1 \exp\Bigl(-\frac{|x - (t + t_2)\mathfrak{m}|^2}{\alpha(t + t_2)}\Bigr), \quad t \geq 0, x \in \mathbb{R}^d,$$

or, equivalently,

$$u(x + (t + t_2)\mathfrak{m}, t + t_3 + t_2) \geq q_1 \exp\Bigl(-\frac{|x|^2}{\alpha(t + t_2)}\Bigr), \quad t \geq 0,\ x \in \mathbb{R}^d,$$

Let now $\mathscr{K} \subset \operatorname{int}(\Upsilon_1)$ be a compact set. Choose any $\sigma \in (0, q_1)$ and consider a radius $r_\sigma = r_\sigma(Q_1, \mathscr{K})$ which fulfills Proposition 5.18, cf. Remark 5.15. Then $|x| \leq r_\sigma$ implies that there exists $t_4 = t_4(\sigma, \mathscr{K}) > 0$, such that, for all $t \geq t_4$,

$$q_1 \exp\Bigl(-\frac{|x|^2}{\alpha(t + t_2)}\Bigr) \geq q_1 \exp\Bigl(-\frac{r_\sigma^2}{\alpha(t + t_2)}\Bigr) > \sigma.$$

Then, one can apply Proposition 5.18 with $u_0(x) = u(x + (t_4 + t_2)\mathfrak{m}, t_4 + t_3 + t_2)$, $x \in \mathbb{R}^d$; by (5.46), we have

$$\lim_{t\to\infty} \min_{x \in t\mathscr{K}} u(x + (t_2 + t_4)\mathfrak{m}, t + t_2 + t_3 + t_4) = \theta. \tag{5.75}$$

Let, finally, $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$ be an arbitrary compact set from the statement of Theorem 5.10. It is well-known, that the distance between disjoint compact and closed sets is positive; in particular, one can consider the compact $\mathscr{C}$ and the closure of $\mathbb{R}^d \setminus \Upsilon_1$. Therefore, there exists a compact set $\mathscr{K} \subset \operatorname{int}(\Upsilon_1)$, such that $\mathscr{C} \subset \operatorname{int}(\mathscr{K})$. Let $\delta_0 > 0$ be the distance between $\mathscr{C}$ and the closure of $\mathbb{R}^d \setminus \mathscr{K}$. One has then that (5.75) does hold with $t_4 = t_4(\sigma, \mathscr{K}) > 0$.

By (5.75), for any $\varepsilon > 0$, there exists $t_5 > 0$ such that, for all $t > t_2 + t_3 + t_4 + t_5 =: t_6 > 0$ and for all $y \in \mathscr{K}$,

$$u\bigl((t - t_2 - t_3 - t_4)y + (t_2 + t_4)\mathfrak{m}, t\bigr) > \theta - \varepsilon \tag{5.76}$$

Without loss of generality we can assume that t_5 is big enough to ensure

$$(t_2 + t_3 + t_4) \max_{x\in\mathscr{C}} |x| + (t_2 + t_4)|\mathfrak{m}| < \delta_0 t_5. \tag{5.77}$$

Then, for any $x \in \mathscr{C}$ and for any $t > t_6$, the vector

$$y(x,t) := \frac{tx - (t_2 + t_4)\mathfrak{m}}{t - t_2 - t_3 - t_4}$$

is such that

$$|y(x,t) - x| = \frac{\bigl|(t_2 + t_3 + t_4)x - (t_2 + t_4)\mathfrak{m}\bigr|}{t - t_2 - t_3 - t_4} < \delta_0,$$

where we used (5.77). Therefore, $y(x,t) \in \mathscr{K}$, for all $x \in \mathscr{C}$ and $t > t_6$, and hence (5.76), being applied for any such $y(x,t)$, yields $u(tx,t) > \theta - \varepsilon$, $x \in \mathscr{C}$, $t > t_6$, that fulfils the proof. □

5.4 Fast propagation for slow decaying dispersal kernels

All result above about traveling waves and long-time behavior of the solutions were obtained under exponential integrability assumptions, cf. (A5) or (A9). In [52], it was proved, for the equation (1.6) on $\mathbb{R}$ with local nonlinear term, that the case with a^+ which does not satisfy such conditions leads to 'accelerating' solutions, i.e. in this case the equality like (5.32) holds for arbitrary big compact $\mathscr{C} \subset \mathbb{R}^d$. The aim of this Subsection is to show an analogous result for the equation (2.1). The detailed analysis of the propagation for the slow decaying a^+ will be done in a forthcoming paper.

We will prove an analog of the first statement in [52, Theorem 1].

Theorem 5.21. *Let the conditions* (A1), (A2), (A4), (A6), *and* (5.24) *hold. Suppose also there exists a function* $0 \le b \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, *such that* $a^+(x) \ge b(|x|)$, *for a.a.* $x \in \mathbb{R}^d$, *and that, cf.* (A9), *for any* $\lambda > 0$ *and for any* $\xi \in S^{d-1}$,

$$\int_{\mathbb{R}^d} b(|x|) e^{\lambda\, x\cdot\xi} dx = \infty. \tag{5.78}$$

Let $u_0 \in L_\theta$ *be such that there exist* $x_0 \in \mathbb{R}^d$, $\eta > 0$, $r > 0$, *with* $u_0 \ge \eta$, *for a.a.* $x \in B_r(x_0)$. *Let* $u \in \tilde{\mathcal{X}}_\infty$ *be the corresponding solution to* (2.1). *Then, for any compact set* $\mathscr{K} \subset \mathbb{R}^d$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x\in t\mathscr{K}} u(x,t) = \theta. \tag{5.79}$$

Proof. By the same arguments as in the proof of Corollary 5.11, there exists $v_0 \in U_\theta$, $v_0 \not\equiv 0$, such that $u_0(x) \ge v_0(x)$, for a.a. $x \in \mathbb{R}^d$, and $u(x,t) \ge v(x,t)$, for a.a. $x \in \mathbb{R}^d$ and for all $t \ge 0$, where $v \in \tilde{\mathcal{X}}_\infty$ is the corresponding to v_0 solution to (2.1), moreover, $v \in \mathcal{X}_\infty$.

Let $\bar\theta \in (0,\theta)$ be chosen and fixed. We are going to apply now Proposition 3.19 to (3.37)–(3.39) with $\Delta_R := B_R(0) \nearrow \mathbb{R}^d$, $R \to \infty$. Consider an increasing sequence $\{R_n \mid n \in \mathbb{N}\}$, such that

(i) $\delta < R_n \to \infty$, $n \to \infty$, where δ is the same as in (A4);
(ii) $A^+_{R_n} > \frac{m}{\varkappa^+}$, $n \in \mathbb{N}$, cf. (3.41);
(iii) $\bar\theta < \theta_{R_n} \le \theta$, cf. (3.40), (3.44).

Let $w_0 \in C_{ub}(\mathbb{R}^d)$, $w_0 \not\equiv 0$ be such that $0 \le w_0(x) \le v_0(x)$, $x \in \mathbb{R}^d$ and $\|w_0\| \le \bar\theta$. Let, for any $n \in \mathbb{N}$, $w^{(n)} \in \mathcal{X}_\infty$ be the corresponding solution to the equation (3.38) with R replaced by R_n. Then, by (3.43), $w^{(n)}(x,t) \le v(x,t)$, for all $x \in \mathbb{R}^d$, $t \ge 0$, $n \in \mathbb{N}$. As a result,

$$w^{(n)}(x,t) \le v(x,t) \le \theta, \quad \text{a.a. } x \in \mathbb{R}^d,\ t \ge 0,\ n \in \mathbb{N}. \tag{5.80}$$

For an arbitrary $\xi \in S^{d-1}$, consider the corresponding $\check{a}^+_{R_n}$, cf. (4.6). Clearly, $\lambda_0(\check{a}^+_{R_n}) = \infty$, i.e. $a^+_{R_n} \in \mathcal{V}_\xi$, $n \in \mathbb{N}$, cf. Definition 4.20. Let $\mathfrak{a}^{(n)}_\xi(\lambda)$, $n \in \mathbb{N}$, $\lambda > 0$ be defined by (4.10), with a^+ replaced by $a^+_{R_n}$. Finally, let $c^{(n)}_*(\xi)$ be the corresponding minimal traveling wave's speed for the equation (3.38) (with R replaced by R_n). Prove that

$$\lim_{n\to\infty} \inf_{\xi \in S^{d-1}} c^{(n)}_*(\xi) = \infty. \tag{5.81}$$

By (4.80), it is enough to show that, cf. (5.24), for any

$$C > \varkappa^+ \int_{\mathbb{R}^d} a^+(x)|x|dx, \tag{5.82}$$

there exists $N = N(C) \in \mathbb{N}$, such that, for all $\lambda > 0$,

$$\frac{1}{\lambda}\bigl(\varkappa^+ \mathfrak{a}^{(n)}_\xi(\lambda) - m\bigr) \geq C, \qquad \xi \in S^{d-1},\ n \geq N. \tag{5.83}$$

Denote $\Xi^\pm_\xi := \{x \in \mathbb{R}^d \mid \pm x\cdot\xi \geq 0\}$; i.e. $\Xi^+_\xi \cup \Xi^-_\xi = \mathbb{R}^d$. Then, by (ii) above,

$$\begin{aligned}
\varkappa^+ \mathfrak{a}^{(n)}_\xi(\lambda) - m &= \varkappa^+ \int_{\mathbb{R}^d} a^+_{R_n}(x)(e^{\lambda x\cdot\xi} - 1)dx + \varkappa^+ A^+_{R_n} - m\\
&\geq \varkappa^+ \int_{\Xi^-_\xi} a^+_{R_n}(x)(e^{\lambda x\cdot\xi} - 1)dx + \varkappa^+ A^+_{R_1} - m,
\end{aligned}$$

as $\int_{\Xi^+_\xi} a^+_{R_n}(x)(e^{\lambda x\cdot\xi} - 1)dx \geq 0$. By the inequality $1 - e^{-s} \leq s$, $s \geq 0$, one has that

$$\left|\int_{\Xi^-_\xi} a^+_{R_n}(x)(e^{\lambda x\cdot\xi} - 1)dx\right| \leq \lambda \int_{\Xi^-_\xi} a^+_{R_n}(x)|x\cdot\xi|dx \leq \lambda \int_{\mathbb{R}^d} a^+(x)|x|dx.$$

Hence, cf. (ii), (5.24), and (5.82), if we set

$$\lambda_1 := \frac{\varkappa^+ A^+_{R_1} - m}{2C} > 0,$$

then, for any $\lambda \in (0, \lambda_1)$ and for any $\xi \in S^{d-1}$,

$$\varkappa^+ \mathfrak{a}^{(n)}_\xi(\lambda) - m \geq \varkappa^+ A^+_{R_1} - m - \lambda_1 \varkappa^+ \int_{\mathbb{R}^d} a^+(x)|x|dx \geq \frac{\varkappa^+ A^+_{R_1} - m}{2} > C\lambda,$$

i.e. (5.83) holds.

On the other hand, (A4) and the condition (i) imply that, for any $n \in \mathbb{N}$, the assumption (A8) holds with a^+ replaced by $a^+_{R_n}$, where $r = 0$ and ρ, δ are the same as in (A4), and thus are independent on n. Hence, by (4.66),

$$\frac{1}{\lambda}\mathfrak{a}^{(n)}_\xi(\lambda) \geq \rho' \frac{1}{\lambda^2}(e^{\lambda\delta'} - 1) \to \infty,$$

for all $n \in \mathbb{N}$, and here ρ', δ' are independent on n and on ξ. Therefore, there exists $\lambda_2 > 0$, such that, for all $\lambda > \lambda_2$, $\xi \in S^{d-1}$, $n \in \mathbb{N}$, (5.83) holds.

Let, finally, $\lambda \in [\lambda_1, \lambda_2]$. Since $a^+_{R_n}$ are compactly supported, one has

$$\frac{d}{d\lambda}\mathfrak{a}^{(n)}_\xi(\lambda) = \int_{\Xi^+_\xi} a^+_{R_n}(x)(x\cdot\xi)e^{\lambda x\cdot\xi}dx + \int_{\Xi^-_\xi} a^+_{R_n}(x)(x\cdot\xi)e^{\lambda x\cdot\xi}dx. \tag{5.84}$$

The inequality $se^{-s} \leq \frac{1}{e}$, $s \geq 0$ implies

$$\left|\int_{\Xi^-_\xi} a^+_{R_n}(x)(x\cdot\xi)e^{\lambda x\cdot\xi}dx\right| \leq \frac{1}{e}\int_{\Xi^-_\xi} a^+_{R_n}(x)dx \leq \frac{1}{e}. \tag{5.85}$$

Since

$$\int_{x\cdot\xi\leq 1} b(|x|)e^{\lambda\, x\cdot\xi}dx \leq e^\lambda < \infty, \quad \lambda > 0,$$

one has, by (5.78), that

$$\int_{x\cdot\xi\geq 1} b(|x|)e^{\lambda\, x\cdot\xi}dx = \infty, \quad \lambda > 0. \tag{5.86}$$

Then, by (5.84), (5.85), (5.86), for all $\lambda \geq \lambda_1$,

$$\begin{aligned}\frac{d}{d\lambda}\int_{\mathbb{R}^d} a^+_{R_n}(x)e^{\lambda x\cdot\xi}dx &\geq \int_{\Xi^+_\xi} a^+_{R_n}(x)(x\cdot\xi)e^{\lambda x\cdot\xi}dx - \frac{1}{e}\\ &\geq \int_{x\cdot\xi\geq 1} a^+_{R_n}(x)e^{\lambda_1\, x\cdot\xi}dx - \frac{1}{e}\\ &\geq \int_{x\cdot\xi\geq 1} b(|x|)1\!\!1_{B_{R_n}(0)}(x)e^{\lambda_1\, x\cdot\xi}dx - \frac{1}{e} \to \infty, \quad n\to\infty,\end{aligned}$$

and the latter integral, evidently, does not depend on $\xi \in S^{d-1}$. Therefore, there exists $N_1 = N_1(\lambda_1) \in \mathbb{N}$, such that, for all $n \geq N_1$ and for all $\xi \in S^{d-1}$, the function $\mathfrak{a}^{(n)}_\xi(\lambda)$ is increasing on $[\lambda_1, \lambda_2]$. As a result, for $\lambda \in [\lambda_1, \lambda_2]$, $n \geq N_1$, $\xi \in S^{d-1}$,

$$\begin{aligned}\frac{1}{\lambda}\bigl(\varkappa^+\mathfrak{a}^{(n)}_\xi(\lambda) - m\bigr) &\geq \frac{1}{\lambda_2}\bigl(\varkappa^+\mathfrak{a}^{(n)}_\xi(\lambda_1) - m\bigr)\\ &\geq \frac{1}{\lambda_2}\left(\varkappa^+\int_{\mathbb{R}^d} b(|x|)1\!\!1_{B_{R_n}(0)}(x)e^{\lambda_1\, x\cdot\xi}dx - m\right) \to \infty, \quad n\to\infty,\end{aligned}$$

and, again, the latter expression does not depend on $\xi \in S^{d-1}$, thus the convergence is uniform in ξ. Therefore, one gets (5.83), for a big enough $N > N_1$ and all $\lambda \in [\lambda_1, \lambda_2]$, $\xi \in S^{d-1}$.

As a result, we have (5.81). Take an arbitrary compact $\mathscr{K} \in \mathbb{R}^d$. Choose $n \in \mathbb{N}$ big enough to ensure that

$$\max_{x\in\mathscr{K},\xi\in S^{d-1}} x\cdot\xi < \min_{\xi\in S^{d-1}} c^{(n)}_*(\xi).$$

As a result, $\mathscr{K} \in \operatorname{int}(\Upsilon^{(n)}_1)$, where $\Upsilon^{(n)}_1$ is defined according to (5.21), but for the kernels $a^\pm_{R_n}$. Then (5.32), with $\mathscr{C} = \mathscr{K}$, yields $\min\limits_{x\in t\mathscr{K}} w^{(n)}(x,t) = \theta$, $t\to\infty$. Hence the inequality (5.80) fulfills the statement. □

Corollary 5.22. *Let conditions of Theorem 5.21 hold. Then there does not exist a traveling wave solution, in the sense of Definition 4.3, to the equation* (2.1).

Proof. Suppose that, for some $\xi \in S^{d-1}$, $c \in \mathbb{R}$, and $\psi \in \mathcal{M}_\theta(\mathbb{R})$, (4.1) holds. Then $u_0(x) = \psi(x \cdot \xi)$ satisfies the assumptions of Theorem 5.21. Take a compact set $\mathscr{K} \subset \mathbb{R}^d$, such that $c_1 := \max\limits_{y \in \mathscr{K}} y \cdot \xi > c$. Then (5.79) implies

$$\begin{aligned}\theta &= \lim_{t\to\infty} \operatorname*{essinf}_{x \in t\mathscr{K}} \psi(x \cdot \xi - ct) = \lim_{t\to\infty} \operatorname*{essinf}_{y \in \mathscr{K}} \psi\bigl(t(y \cdot \xi - c)\bigr) \\ &= \lim_{t\to\infty} \psi\bigl(t(c_1 - c)\bigr) = 0,\end{aligned}$$

where we used that ψ is decreasing. One gets a contradiction which proves the statement. □

6 Accelerating front propagation

As it was shown in Theorem 5.21 if a^+ decays slowly than it is impossible to estimate from above a solution to (2.1) by a function which propagates with a constant speed (linearly). In this section it will be shown that if either a^+ or u_0 decay slowly than the front propagation of the solution is faster than linear.

The important point to note here is that estimates from above and form below on the solution in this section will be close to each other only for radially symmetric initial condition and a^+. Therefore without loss of generality we can assume

$$\begin{gathered}\text{there exists } R_0 > 0, \text{ such that, for all } R \geq R_0, \\ \int_{B_R(0)} x a^+(x)\, dx = 0.\end{gathered} \tag{A10}$$

An evident sufficient condition, to get (A10), is $a^+(-x) = a^+(x)$, $x \in \mathbb{R}^d$. The assumption (A10) is sufficient to have $\{0\} \in \Upsilon_1^n$, for all $R_n \geq R$, where R is sufficiently large and $\Upsilon_1^{(n)}$ is defined according to (5.21), but for the kernels $a^\pm_{R_n}$. The following proposition follows from Corollary 5.11.

Proposition 6.1. *Let assumptions* (A1), (A2), (A4), (A6), (A10) *hold. Let $u_0 \in L_\theta$ be such that there exist $x_0 \in \mathbb{R}^d$, $\mu_0 \in (0, \theta)$, $\delta_0 > 0$, such that $u_0(x) \geq \mu_0$, for a.a. $x \in B_{\delta_0}(x_0)$. Let $u \in \tilde{\mathcal{X}}_\infty$ be the corresponding solution to* (2.1)*. Then, for any $\mu \in (0, \theta)$ and for any $r > 0$, there exists $t_\mu(r) > 0$, such that, we have that $u(x, t) \geq \mu$, for a.a. $x \in B_r(0)$ and for all $t \geq t_\mu(r)$.*

Remark 6.2. It is easy to see that the result and proof of Proposition 6.1 remains unchanged if we would treat $B_R(x_0)$ as the ball with the centre at $x_0 \in \mathbb{R}^d$ and the radius $R > 0$ with respect to any other (non Euclidean) norm on $\mathbb{R}^d$.

6.1 Technical tools

6.1.1 Functions with heavy tails on $\mathbb{R}$

Definition 6.3. A function $b : \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) long-tailed* if there exists $\rho = \rho_b \geq 0$, such that $b(s) > 0$, for all $s \geq \rho$; and, for any $\tau \geq 0$,

$$\lim_{s\to\infty} \frac{b(s+\tau)}{b(s)} = 1. \tag{6.1}$$

Remark 6.4. By [33, formula (2.18)], the convergence in (6.1) is equivalent to the locally uniform in τ convergence, namely, (6.1) can be replaced by the assumption that, for all $h > 0$,

$$\lim_{s\to\infty} \sup_{|\tau|\leq h} \left| \frac{b(s+\tau)}{b(s)} - 1 \right| = 0. \tag{6.2}$$

A long-tailed function has to have a 'heavier' tail than any exponential function; namely, the following statement holds.

Lemma 6.5 ([33, Lemma 2.17])**.** *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function. Then, for any $k > 0$,*

$$\lim_{s\to\infty} e^{ks} b(s) = \infty. \tag{6.3}$$

The constant h in (6.2) may be arbitrary big. It is quite natural to ask what will be if h increases to ∞ consistently with s.

Lemma 6.6 (cf. [33, Lemma 2.19, Proposition 2.20])**.** *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function. Then there exists a function $h : (0, \infty) \to (0, \infty)$, with $h(s) < \frac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$, such that, cf.* (6.2)*,*

$$\lim_{s\to\infty} \sup_{|\tau|\leq h(s)} \left| \frac{b(s+\tau)}{b(s)} - 1 \right| = 0. \tag{6.4}$$

We will say then that b is h-insensitive. Of course, for a given long-tailed function b the function h that fulfills (6.4) is not unique, see also [33, Proposition 2.20].

The convergence in (6.1) may be, in general, very 'non-regular' in s. Evidently, if $b(s)$ is decreasing for big values of s, then the l.h.s. of (6.1) converges to 1 from below (for $\tau \geq 0$). Let us specify the corresponding class of functions.

Definition 6.7. A function $b : \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) tail-decreasing* if there exists a number $\rho = \rho_b \geq 0$ such that $b = b(s)$ is strictly decreasing on $[\rho, \infty)$ to 0. In particular, $b(s) > 0$, $s \geq \rho$.

Proposition 6.8. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a tail-decreasing function. Let $h : (0, \infty) \to (0, \infty)$, with $h(s) < \frac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$. Then* (6.4) *holds, if and only if*

$$\lim_{s\to\infty} \frac{b(s \pm h(s))}{b(s)} = 1. \tag{6.5}$$

Proof. Let $\rho = \rho_b > 0$ be as in the Definition 6.7. Then, for the given h and for any $s > 2\rho$, one has that $s - h(s) > \frac{s}{2} > \rho$. Hence, for a fixed $s > 2\rho$, the function $b(s + \tau)$ is decreasing in $\tau \in [-h(s), h(s)]$. Therefore, considering separately positive and negative τ, one gets that, for all $s > 2\rho$,

$$\sup_{|\tau|\leq h(s)} \left| \frac{b(s+\tau)}{b(s)} - 1 \right| = \max\left\{ 1 - \frac{b(s+h(s))}{b(s)}, \frac{b(s-h(s))}{b(s)} - 1 \right\},$$

that yields the statement. □

However, even for a long-tailed tail-decreasing function b, the convergence in (6.1) will not be, in general, monotone in s. To get this monotonicity, we consider the following class of functions.

Definition 6.9. A function $b : \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) tail-log-convex*, if there exists $\rho = \rho_b > 0$ such that $b(s) > 0$, $s \geq \rho$, and the function $\log b$ is convex on $[\rho, \infty)$.

Remark 6.10. It is well-known that any function which is convex on an open interval is continuous there. Therefore, a tail-log-convex function $b = \exp(\log b)$ is continuous on (ρ_b, ∞) as well.

Lemma 6.11. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be tail-log-convex, with $\rho = \rho_b$. Then, for any $\tau > 0$, the function $\dfrac{b(s+\tau)}{b(s)}$ is non-decreasing in $s \in [\rho, \infty)$.*

Proof. Take any $s_1 > s_2 \geq \rho$. Set $B(s) := \log b(s) \leq 0$, $s \in [\rho, \infty)$. Then the desired inequality

$$\frac{b(s_1+\tau)}{b(s_1)} \geq \frac{b(s_2+\tau)}{b(s_2)},$$

is equivalent to

$$B(s_1+\tau) + B(s_2) \geq B(s_2+\tau) + B(s_1).$$

Since B is convex, we have, for $\lambda = \dfrac{\tau}{s_1 - s_2 + \tau} \in (0,1)$,

$$\begin{aligned} B(s_1) &= B\bigl(\lambda s_2 + (1-\lambda)(s_1+\tau)\bigr) \leq \lambda B(s_2) + (1-\lambda)B(s_1+\tau), \\ B(s_2+\tau) &= B\bigl((1-\lambda)s_2 + \lambda(s_1+\tau)\bigr) \leq (1-\lambda)B(s_2) + \lambda B(s_1+\tau), \end{aligned}$$

that implies the needed inequality. □

The next statement describes a crucial property of a long-tailed tail-log-convex function which decays at ∞ fast enough.

Lemma 6.12 (cf. [33, Theorem 4.15])**.** *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed tail-log-convex function such that $b \in L^1(\mathbb{R}_+)$. Suppose that, for a function $h : (0,\infty) \to (0,\infty)$, with $h(s) < \frac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$, the asymptotic (6.4) holds, and that*

$$\lim_{s\to\infty} s\, b\bigl(h(s)\bigr) = 0. \tag{6.6}$$

Set

$$b_+(s) := 1\!\!1_{\mathbb{R}_+}(s) \biggl(\int_{\mathbb{R}_+} b(\tau) d\tau\biggr)^{-1} b(s), \qquad s \in \mathbb{R}, \tag{6.7}$$

Then

$$\begin{aligned} (b_+ * b_+)(s) &= \int_{\mathbb{R}} b_+(s-\tau) b_+(\tau)\, d\tau \\ &= \int_0^s b_+(s-\tau) b_+(\tau)\, d\tau \sim 2b_+(s), \qquad s \to \infty. \end{aligned} \tag{6.8}$$

In the literature, see e.g. [33], a long-tailed probability density b_+ on $\mathbb{R}_+$ that satisfies (6.8) is called a *sub-exponential density* on $\mathbb{R}_+$. This gives a reason for the following definition.

Definition 6.13. We will say that a function $b : \mathbb{R} \to \mathbb{R}_+$ is *weakly (right-side) sub-exponential on* $\mathbb{R}$ if b is long-tailed, $b \in L^1(\mathbb{R}_+)$, and the function b_+, being given by (6.7), satisfies (6.8).

Remark 6.14. Let $b : \mathbb{R} \to \mathbb{R}_+$ be a weakly sub-exponential function on $\mathbb{R}$. Then, by (6.7), (6.8), we have

$$\int_0^s b(s-\tau)b(\tau)\,d\tau \sim 2\biggl(\int_{\mathbb{R}_+} b(\tau)d\tau\biggr)b(s), \quad s \to \infty. \tag{6.9}$$

Suppose, additionally, that $b \in L^1(\mathbb{R})$. Then, in general, the asymptotic, cf. (6.8),

$$(b * b)(s) = \int_{\mathbb{R}} b(s-\tau)b(\tau)\,d\tau \sim 2\biggl(\int_{\mathbb{R}} b(\tau)d\tau\biggr)b(s), \quad s \to \infty, \tag{6.10}$$

may not hold; one needs an additional condition on b, see (6.11) below.

Definition 6.15. We will say that a function $b : \mathbb{R} \to \mathbb{R}_+$ is *strongly (right-side) sub-exponential on* $\mathbb{R}$ if b is long-tailed, $b \in L^1(\mathbb{R})$, and the asymptotic (6.10) holds.

Remark 6.16. By [33, Lemma 4.12], a strongly sub-exponential function on $\mathbb{R}$ is weakly sub-exponential there.

Lemma 6.17 (cf. [33, Lemma 4.13])**.** *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be a weakly sub-exponential function on $\mathbb{R}$. Suppose that there exists $\rho = \rho_b > 0$ and $K = K_b > 0$ such that*

$$b(s+\tau) \leq Kb(s), \quad s > \rho, \ \tau > 0. \tag{6.11}$$

Then (6.10) *holds, i.e. b is strongly sub-exponential on* $\mathbb{R}$.

Remark 6.18. Evidently, a tail-decreasing function defined by Definition 6.7 satisfies (6.11), with the same ρ and $K = 1$.

It is naturally to expect that asymptotically small changes in the behaviour at infinity preserves the sub-exponential property of a function. Namely, consider the following definition.

Definition 6.19. Two functions $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ are said to be *weakly tail-equivalent* if

$$0 < \liminf_{s\to\infty} \frac{b_1(s)}{b_2(s)} \leq \limsup_{s\to\infty} \frac{b_1(s)}{b_2(s)} < \infty, \tag{6.12}$$

or, in other words, if there exist $\rho > 0$ and $C_2 \geq C_1 > 0$, such that,

$$C_1 b_1(s) \leq b_2(s) \leq C_2 b_1(s), \quad s \geq \rho. \tag{6.13}$$

Proposition 6.20. *Let $b_1 : \mathbb{R} \to \mathbb{R}_+$ be a weakly sub-exponential on $\mathbb{R}$ function. Let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function which is weakly tail-equivalent to b_1. Then b_2 is weakly sub-exponential on $\mathbb{R}$ as well. If, additionally,* (6.11) *holds, for $b = b_1$, then b_2 is strongly sub-exponential on* $\mathbb{R}$.

Proof. Let b_1 be weakly sub-exponential on $\mathbb{R}$, cf. Definition 6.13, and the functions $b_{1,+}$ and $b_{2,+}$ be defined according to (6.7). Then, evidently, $b_{1,+}$ and $b_{2,+}$ will be also weakly tail-equivalent, and, moreover, $b_{2,+}$ will be long-tailed. Then, by [33, Theorem 4.8], $b_{2,+}$ is a also sub-exponential density on $\mathbb{R}_+$, i.e. (6.8) holds, for $b_+ = b_{2,+}$. As a result, by Definition 6.13, b_2 is weakly sub-exponential on $\mathbb{R}$. Next, let (6.11) holds, for $b = b_1$. Then, by (6.13), we have, for all $s \geq \rho$,

$$b_2(s+\tau) \leq C_2 b_1(s+\tau) \leq C_2 K b_1(s) \leq \frac{C_2}{C_1} K b_2(s),$$

i.e. (6.11) holds, for $b = b_2$ as well. As a result, by Lemma 6.17, both b_1 and b_2 are strongly sub-exponential on $\mathbb{R}$. □

We consider a useful for the sequel class of functions.

Definition 6.21. We will say a function $b : \mathbb{R} \to \mathbb{R}_+$ belongs to the class $\mathcal{S}(\mathbb{R})$ iff

1. $b \in L^1(\mathbb{R}_+)$ and b is bounded on $\mathbb{R}$;
2. there exists $\rho = \rho_b > 1$, such that b is log-convex and strictly decreasing to 0 on $[\rho, \infty)$ (i.e. b is simultaneously tail-decreasing and tail-log-convex), and (without loss of generality) $b(\rho) \leq 1$;
3. there exist $\delta = \delta_b \in (0,1)$ and an increasing function $h = h_b : (0,\infty) \to (0,\infty)$, with $h(s) < \frac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$, such that the asymptotic (6.5) holds, and, cf. (6.6),

$$\lim_{s\to\infty} b\bigl(h(s)\bigr) s^{1+\delta} = 0. \tag{6.14}$$

For any $n \in \mathbb{N}$, we denote by $\mathcal{S}_n(\mathbb{R})$ the subclass of functions b from $\mathcal{S}(\mathbb{R})$ such that

$$\int_{-\infty}^{\rho} b(s)\, ds + \int_{\rho}^{\infty} b(s) s^{n-1}\, ds < \infty. \tag{6.15}$$

Remark 6.22. It is worth noting again that, for a tail-decreasing function, (6.5) implies that b is long-tailed.

Remark 6.23. By Lemma 6.12 and Remark 6.14, any function $b \in \mathcal{S}(\mathbb{R})$ is weakly sub-exponential on $\mathbb{R}$. Moreover, by Lemma 6.17 and Remark 6.18, any function $b \in \mathcal{S}_1(\mathbb{R})$ is strongly sub-exponential on $\mathbb{R}$.

Remark 6.24. Let $b \in \mathcal{S}(\mathbb{R})$, and $s_0 > 0$ be such that $h(2s_0) > \rho$. Then the monotonicity of b and h implies $b(s) \leq b(h(2s))$, $s > s_0$; and hence, because of (6.14), for $B := 2^{-1-\delta}$, there exists $s_1 \geq s_0$, such that

$$b(s) \leq \frac{B}{s^{1+\delta}}, \quad s \geq s_1. \tag{6.16}$$

In particular, this implies that if $b \in \mathcal{S}(\mathbb{R}) \cap L^1((-\infty,0))$, then $b \in \mathcal{S}_1(\mathbb{R})$.

Below we will show that $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}_n(\mathbb{R})$, $n \in \mathbb{N}$ are closed under some simple transformations of functions. For an arbitrary function $b \in \mathcal{S}(\mathbb{R})$, we consider the following transformed functions:

1. for fixed $p > 0$, $q > 0$, $r \in \mathbb{R}$, we set
$$\widetilde{b}(s) := pb(qs + r), \quad s \in \mathbb{R}; \tag{6.17}$$
2. for a fixed $s_0 > 0$ and a fixed bounded function $c : \mathbb{R} \to \mathbb{R}_+$, we set
$$\breve{b}(s) := 1\!\!1_{(-\infty,s_0)}(s)c(s) + 1\!\!1_{[s_0,\infty)}(s)b(s), \quad s \in \mathbb{R}; \tag{6.18}$$
3. for any $\alpha \in (0,1]$, we denote
$$b_\alpha(s) := \bigl(b(s)\bigr)^\alpha, \quad s \in \mathbb{R}. \tag{6.19}$$

Theorem 6.25. *1. Let $b \in \mathcal{S}(\mathbb{R})$. Then the functions $\widetilde{b}$ and $\breve{b}$ defined in* (6.17) *and* (6.18), *correspondingly, also belong to $\mathcal{S}(\mathbb{R})$, for all admissible values of their parameters. If, additionally, there exists $\alpha' \in (0,1)$ such that $b_{\alpha'} \in L^1(\mathbb{R}_+)$, then there exists $\alpha_0 \in (\alpha',1)$, such that $b_\alpha \in \mathcal{S}(\mathbb{R})$, for all $\alpha \in [\alpha_0, 1]$.*

2. *Let $b \in \mathcal{S}_n(\mathbb{R})$, for some $n \in \mathbb{N}$. Then $\widetilde{b} \in \mathcal{S}_n(\mathbb{R})$. If, additionally, the function c in (6.18) is integrable on $(-\infty, s_0)$, then $\breve{b} \in \mathcal{S}_n(\mathbb{R})$. Finally, if there exists $\alpha' \in (0,1)$ such that (6.15) holds, for $b = b_{\alpha'}$, then there exists $\alpha_0 \in (\alpha',1)$, such that $b_\alpha \in \mathcal{S}_n(\mathbb{R})$, for all $\alpha \in [\alpha_0, 1]$. Moreover, in the latter case, there exist $B_0 > 0$ and $\rho_0 > 0$, such that, for all $\alpha \in (\alpha_0, 1]$,*

$$\int_{\mathbb{R}} \bigl(b(s-\tau)\bigr)^\alpha \bigl(b(\tau)\bigr)^\alpha \, d\tau \leq B_0 \bigl(b(s)\bigr)^\alpha, \quad s \geq \rho_0, \tag{6.20}$$

Proof. It is very straightforward to check that if b is long-tailed, tail-decreasing and tail-log-convex, then $\widetilde{b}, \breve{b}, b_\alpha$ also have these properties, for all admissible values of their parameters. Let $h : (0,\infty) \to (0,\infty)$ be such that $h(s) < \dfrac{s}{2}$, $\lim\limits_{s\to\infty} h(s) = \infty$, and (6.5) hold. Let also (6.14) hold, for some $\delta > 0$.

(i) Evidently, both (6.5) and (6.14) hold, with b replaced by $\breve{b}$. Next, $\breve{b} \in L^1(\mathbb{R}_+)$ and $\breve{b}$ is bounded. Hence $\breve{b} \in \mathcal{S}(\mathbb{R})$. If $b \in \mathcal{S}_n(\mathbb{R})$ and c is integrable on $(-\infty, s_0)$, then (6.15) holds, for b replaced by $\breve{b}$.

(ii) Set, for the given $q > 0$, $r \in \mathbb{R}$,

$$\widetilde{h}(s) := \frac{1}{q} h(qs+r) - \frac{r}{2q} 1\!\!1_{\mathbb{R}_+}(r), \quad s \in [s_1, \infty),$$

where $s_1 > 0$ is such that $qs_1 + r > 0$ and $h(qs+r) > \frac{r}{2q}$, for all $s \geq s_1$. Clearly, $\widetilde{h}$ is increasing on $[s_1, \infty)$, $\lim\limits_{s\to\infty} \widetilde{h}(s) = \infty$, and $\widetilde{h}(s) \leq \frac{1}{2q}(qs+r) - \frac{r}{2q} 1\!\!1_{\mathbb{R}_+}(r) \leq \frac{s}{2}$, for all $s \in [s_1, \infty)$. The interval $(0, s_1)$ is not so 'important', one can choose any increasing $\widetilde{h}$ there, such that $\widetilde{h}(s) < \min\bigl\{\frac{s}{2}, \widetilde{h}(s_1)\bigr\}$, $s \in (0, s_1)$. By Proposition 6.8, (6.5) is equivalent to (6.4). Then, by (6.17), we have

$$\begin{aligned}
\sup_{|\tau| \leq \widetilde{h}(s)} \left| \frac{\widetilde{b}(s+\tau)}{\widetilde{b}(s)} - 1 \right| &= \sup_{q|\tau| \leq h(qs+r) - \frac{r}{2} 1\!\!1_{\mathbb{R}_+}(r)} \left| \frac{b(qs+r+q\tau)}{b(qs+r)} - 1 \right| \\
&\leq \sup_{q|\tau| \leq h(qs+r)} \left| \frac{b(qs+r+q\tau)}{b(qs+r)} - 1 \right| \to 0,
\end{aligned}$$

as $s \to \infty$. Therefore, again by Proposition 6.8, (6.5) holds, for b replaced by $\widetilde{b}$. Next, set $\nu(r) := \frac{r}{2}$, $r \geq 0$, and $\nu(r) := r$, $r < 0$, then

$$\begin{aligned}
\widetilde{b}\bigl(\widetilde{h}(s)\bigr) s^{1+\delta} &= p b\bigl(h(qs+r) + \nu(r)\bigr) s^{1+\delta} \\
&= p \frac{b\bigl(h(qs+r) + \nu(r)\bigr)}{b\bigl(h(qs+r)\bigr)} b\bigl(h(qs+r)\bigr)(qs+r)^{1+\delta} \Bigl(\frac{s}{qs+r}\Bigr)^{1+\delta} \to 0,
\end{aligned}$$

as $s \to \infty$, because of (6.1), (6.14). Therefore, $\widetilde{b} \in \mathcal{S}(\mathbb{R})$. Finally, $b \in \mathcal{S}_n(\mathbb{R})$, for some $n \in \mathbb{N}$, trivially implies $\widetilde{b} \in \mathcal{S}_n(\mathbb{R})$.

(iii) Evidently, the convergence (6.5) implies the same one with b replaced by b_α, with the same h and for any $\alpha \in (0,1)$. Next, let $\alpha' \in (0,1)$ be such that $b_{\alpha'} \in L^1(\mathbb{R}_+)$. By the well-known log-convexity of L^p-norms (for $p > 0$), for any $\alpha \in (\alpha', 1)$ and for $\beta := \frac{\alpha - \alpha'}{\alpha(1-\alpha')} \in (0,1)$, we have $\frac{1}{\alpha} = \frac{1-\beta}{\alpha'} + \beta$ and

$$\|b\|_{L^\alpha(\mathbb{R}_+)} \leq \|b\|^{1-\beta}_{L^{\alpha'}(\mathbb{R}_+)} \|b\|^{\beta}_{L^1(\mathbb{R}_+)} < \infty, \tag{6.21}$$

i.e. $b_\alpha \in L^1(\mathbb{R}_+)$, for all $\alpha \in (\alpha', 1)$. Take and fix now, an arbitrary $\alpha_0 \in \Bigl(\max\Bigl\{\alpha', \dfrac{1}{1+\delta}\Bigr\}, 1\Bigr)$. Then, for any $\alpha \in [\alpha_0, 1)$, we have that $\delta' := \alpha(1+\delta) - 1 \in (0, \delta]$, and hence, by (6.14),

$$\lim_{s\to\infty} b_\alpha\bigl(h(s)\bigr)s^{1+\delta'} = \lim_{s\to\infty}\Bigl(b\bigl(h(s)\bigr)s^{1+\delta}\Bigr)^\alpha = 0.$$

Therefore, $b_\alpha \in \mathcal{S}(\mathbb{R})$, $\alpha \in [\alpha_0, 1]$.

Let, additionally, (6.15) hold, for both b and $b_{\alpha'}$ (i.e., in particular, $b \in \mathcal{S}_n(\mathbb{R})$) and for some $n \in \mathbb{N}$. Then one can use again the log-convexity of L^p-norms, now for $L^p\bigl((\rho,\infty), s^n\,ds\bigr)$ spaces, to deduce that $b_\alpha \in \mathcal{S}_n(\mathbb{R})$, $\alpha \in [\alpha_0, 1]$.

Finally, $b, b_{\alpha_0} \in \mathcal{S}_n(\mathbb{R})$, $n \in \mathbb{N}$, implies $b, b_{\alpha_0} \in \mathcal{S}_1(\mathbb{R})$, and hence, cf. Remark 6.23, b and b_{α_0} are strongly sub-exponential on $\mathbb{R}$, i.e. (6.10) holds, for both b and b_{α_0}. Therefore, for an arbitrary $\varepsilon \in (0,1)$, there exists $\rho_0 = \rho_0(\varepsilon, b, b_{\alpha_0}) > \rho$ (where ρ is from Definition 6.21) and

$$B_0 = 2(1+\varepsilon)\max\Bigl\{\int_{\mathbb{R}} b(s)\,ds, \int_{\mathbb{R}} b_{\alpha_0}(s)\,ds\Bigr\} > 0,$$

such that, for all $s \geq \rho_0$,

$$\begin{aligned} &\int_{\mathbb{R}} b(s-\tau)b(\tau)\,d\tau \leq B_0 b(s), \\ &\int_{\mathbb{R}} b_{\alpha_0}(s-\tau)b_{\alpha_0}(\tau)\,d\tau \leq B_0 b_{\alpha_0}(s). \end{aligned} \tag{6.22}$$

Then, applying again the norm log-convexity arguments, cf. (6.21), one gets, for any fixed $s \geq \rho_0$ and for all $\alpha \in (\alpha_0, 1)$

$$\int_{\mathbb{R}}\bigl(b(s-\tau)b(\tau)\bigr)^\alpha d\tau \leq \biggl(\int_{\mathbb{R}}\bigl(b(s-\tau)b(\tau)\bigr)^{\alpha_0} d\tau\biggr)^{\frac{1}{\alpha_0}(1-\beta)\alpha}\biggl(\int_{\mathbb{R}} b(s-\tau)b(\tau)d\tau\biggr)^{\beta\alpha},$$

where $\beta = \frac{\alpha-\alpha_0}{\alpha(1-\alpha_0)} \in (0,1)$. Combining the latter inequality with (6.22), one gets

$$\int_{\mathbb{R}}\bigl(b(s-\tau)b(\tau)\bigr)^\alpha d\tau \leq \bigl(B_0(b(s))^{\alpha_0}\bigr)^{\frac{1}{\alpha_0}(1-\beta)\alpha}\bigl(B_0 b(s)\bigr)^{\beta\alpha} = B_0\bigl(b(s)\bigr)^\alpha.$$

The theorem is fully proved now. ☐

By Definition 6.21, to check whether a function b belongs to $\mathcal{S}(\mathbb{R})$, one naturally needs a precise information about an appropriate function h, such that (6.5) holds, cf. the proof of Theorem 6.25. However, if $b_1 \in \mathcal{S}(\mathbb{R})$ and b_2 is weakly tail-equivalent to b_1, then, besides of Proposition 6.20, one can not find, in general, an appropriate function h_2 for b_2 such that the analogue of (6.5) and (6.14) would simultaneously hold, having the corresponding function h_1 for b_1 only. Fortunately, we will need results of such type for the functions which are asymptotically tail-proportional only; the latter means that the limits in (6.12) will coincide (and, as a result, the constants C_1 and C_2 in (6.13) will be 'almost equal'). Consider the corresponding statement.

Proposition 6.26. *Let $b_1 \in \mathcal{S}(\mathbb{R})$ and $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function, such that*

$$\lim_{s\to\infty}\frac{b_2(s)}{b_1(s)} = C \in (0,\infty). \tag{6.23}$$

Then $b_2 \in \mathcal{S}(\mathbb{R})$.

Proof. First, we note that (6.23), (6.1) yield that b_2 is long-tailed as b_1 is such. Let $\delta \in (0,1)$ and $h : (0,\infty) \to (0,\infty)$ be an increasing function, such that $h(s) < \frac{s}{2}$, $\lim\limits_{s\to\infty} h(s) = \infty$, and (6.5) and (6.14) hold, for $b = b_1$. Next, take an arbitrary $\varepsilon \in \bigl(0, \min\{1, C\}\bigr)$. Choose $\rho > 1$ such that b_2 is decreasing and log-convex on $[\rho,\infty)$, and $b_2(\rho) \le 1$. By (6.23) and (6.5) (for $b = b_1$), there exists $\rho_1 \ge \rho$, such that, for all $s \ge \rho_1$,

$$0 < (C-\varepsilon)b_1(s) \le b_2(s) \le (C+\varepsilon)b_1(s), \tag{6.24}$$

$$\left|\frac{b_1(s \pm h(s))}{b_1(s)} - 1\right| < \varepsilon. \tag{6.25}$$

Since b_2 is bounded and $b_1 \in L^1(\mathbb{R}_+)$, we have from (6.24) that $b_2 \in L^1(\mathbb{R}_+)$. Moreover, by (6.24), for any $s \ge \rho_1$,

$$\begin{aligned}
\frac{C-\varepsilon}{C+\varepsilon}\biggl(\frac{b_1(s\pm h(s))}{b_1(s)} - 1\biggr) + \frac{C-\varepsilon}{C+\varepsilon} &= \frac{(C-\varepsilon)b_1(s\pm h(s))}{(C+\varepsilon)b_1(s)} \\
&\le \frac{b_2(s\pm h(s))}{b_2(s)} \\
&\le \frac{(C+\varepsilon)b_1(s\pm h(s))}{(C-\varepsilon)b_1(s)} = \frac{C+\varepsilon}{C-\varepsilon}\biggl(\frac{b_1(s\pm h(s))}{b_1(s)} - 1\biggr) + \frac{C+\varepsilon}{C-\varepsilon},
\end{aligned}$$

and, therefore, by (6.25),

$$\left|\frac{b_2(s\pm h(s))}{b_2(s)} - 1\right| < \max\biggl\{\varepsilon\frac{C+\varepsilon}{C-\varepsilon} + \frac{C+\varepsilon}{C-\varepsilon} - 1,\ \varepsilon\frac{C-\varepsilon}{C+\varepsilon} + 1 - \frac{C-\varepsilon}{C+\varepsilon}\biggr\}.$$

Since the latter expression may be arbitrary small, by an appropriate choice of ε, one gets that (6.5) holds, for $b = b_2$. Finally, (6.14), for $b = b_1$, and (6.23) imply that (6.14) holds, for $b = b_2$ and the same δ and h. ☐

Remark 6.27. In the assumptions of the previous theorem, if, additionally, $b_1 \in \mathcal{S}_n(\mathbb{R})$, for some $n \in \mathbb{N}$, and b_2 is integrable on $(-\infty, -\rho_2)$, for some $\rho_2 > 0$, then $b_2 \in \mathcal{S}_n(\mathbb{R})$ (because of (6.24) and the boundedness of b_2).

On the other hand, if one can check that both functions b_1 and b_2 satisfy (6.5) with the same function $h(s)$, then the sufficient condition to verify (6.14) for $b = b_2$, provided that it holds for $b = b_1$, is much weaker than (6.23). To present the corresponding statement, consider the following definition.

Definition 6.28. Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ and, for some $\rho \ge 0$, $b_i(s) > 0$ for all $s \in [\rho,\infty)$, $i = 1,2$. The functions b_1 and b_2 are said to be *(asymptotically) log-equivalent*, if

$$\log b_1(s) \sim \log b_2(s), \quad s \to \infty. \tag{6.26}$$

Proposition 6.29. *Let $b_1 \in \mathcal{S}(\mathbb{R})$ and let h be the function corresponding to Definition 6.21 with $b = b_1$. Let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function, such that* (6.5) *holds with $b = b_2$ and the same h. Suppose that b_1 and b_2 are log-equivalent. Then $b_2 \in \mathcal{S}(\mathbb{R})$. If, additionally, there exists $\alpha' \in (0,1)$, such that* (6.15) *holds with $b = (b_1)^{\alpha'}$ and b_2 is integrable on $(-\infty,\rho)$, then $b_2 \in \mathcal{S}_n(\mathbb{R})$.*

Proof. Let $\delta \in (0,1)$ be such that (6.14) holds for b replaced by b_1. Take an arbitrary $\varepsilon \in \bigl(0, \frac{\delta}{1+\delta}\bigr)$. By (6.26), there exists $\rho_\varepsilon > 0$, such that $b_i(s) < 1$, $s > \rho_\varepsilon$, $i = 1,2$, and

$$\begin{gathered}
-(1-\varepsilon)\log b_1(s) \le -\log b_2(s) \le -(1+\varepsilon)\log b_1(s), \quad s > \rho_\varepsilon, \\
b_1(s)^{1+\varepsilon} \le b_2(s) \le b_1(s)^{1-\varepsilon}, \quad s > \rho_\varepsilon.
\end{gathered} \tag{6.27}$$

Since $h(s) \to \infty$, $s \to \infty$, there exists $\rho_0 > \rho_\varepsilon$, such that $h(s) > \rho_\varepsilon$ for any $s > \rho_0$. Then, by (6.27), we have, for all $s > \rho_0$,

$$b_2(h(s))s^{(1+\delta)(1-\varepsilon)} < b_1(h(s))^{1-\varepsilon}s^{(1+\delta)(1-\varepsilon)} = \bigl(b_1(h(s))s^{1+\delta}\bigr)^{1-\varepsilon},$$

and therefore, (6.14) holds with $b = b_2$ and δ replaced by

$$(1+\delta)(1-\varepsilon) - 1 = \delta - \varepsilon(1+\delta) \in (0,1),$$

that proves the first statement. To prove the second one, assume, additionally, that $\varepsilon < 1 - \alpha'$. Then, by (6.27), we have, for all $s > \rho_\varepsilon$,

$$b_2(s)s^{n-1} \le b_1(s)^{1-\varepsilon}s^{n-1} < b_1(s)^{\alpha'}s^{n-1},$$

as $b_1(s) < 1$ here. □

6.2 Level sets for heavy tailed functions on $\mathbb{R}$

Let $b : \mathbb{R} \to \mathbb{R}_+$ be a tail-decreasing function, cf. Definition 6.7. Choose and fix the corresponding $\rho > 1$, such that $b(\rho) \le 1$. We will consider time dependent level sets of the function b, namely, we are interested in the sets $\{s \in \mathbb{R} : b(s) \le e^{-\beta t}\}$, for different $\beta > 0$ and $t > 0$.

For arbitrary $\varepsilon \in (0,1)$ and $\beta > 0$ one can define the following constants

$$\beta_\varepsilon^- := (1-\varepsilon)\beta > 0, \qquad \beta_\varepsilon^+ := (1+\varepsilon)\beta > 0. \tag{6.28}$$

For any tail-decreasing function b as the above, we can consider the inverse function $b^{-1} = b^{-1}(s)$ for $s \in \bigl(0, b(\rho)\bigr]$, which is decreasing there. For an arbitrary $\varepsilon \in (0,1)$, we set

$$t_{\rho,\varepsilon}^\pm = t_{\rho,\varepsilon}^\pm(b) := -\frac{1}{\beta_\varepsilon^\pm}\log b(\rho) \ge 0. \tag{6.29}$$

Since $\bigl(0, b(\rho)\bigr] \subset (0,1]$, one can define, for $t \ge t_{\rho,\varepsilon}^- > t_{\rho,\varepsilon}^+$, all the following functions

$$\eta(t) = \eta(t,b) := b^{-1}\bigl(e^{-\beta t}\bigr), \tag{6.30}$$

$$\eta_\varepsilon^+(t) = \eta_\varepsilon^+(t,b) := \eta\bigl((1+\varepsilon)t\bigr) = b^{-1}\bigl(e^{-\beta_\varepsilon^+ t}\bigr), \tag{6.31}$$

$$\eta_\varepsilon^-(t) = \eta_\varepsilon^-(t,b) := \eta\bigl((1-\varepsilon)t\bigr) = b^{-1}\bigl(e^{-\beta_\varepsilon^- t}\bigr). \tag{6.32}$$

Clearly, all these functions are increasing to ∞, and

$$\eta_\varepsilon^+(t) \ge \eta(t) \ge \eta_\varepsilon^-(t) \ge \rho, \qquad t \ge t_{\rho,\varepsilon}^-. \tag{6.33}$$

Lemma 6.30. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be tail-decreasing and long-tailed. Then, for any $\varepsilon \in (0,1)$ and for any $c > 0$,*

$$\eta_\varepsilon^-(t) - ct \to \infty, \quad t \to \infty.$$

Proof. Since b is long-tailed, (6.3) holds, for any $k > 0$. Therefore, one has

$$\begin{aligned}\exp\Bigl(\frac{\beta_\varepsilon^-}{c}\bigl(\eta_\varepsilon^-(t) - ct\bigr)\Bigr) &= \exp\Bigl(\frac{\beta_\varepsilon^-}{c}\eta_\varepsilon^-(t)\Bigr)e^{-\beta_\varepsilon^- t}\\ &= \exp\Bigl(\frac{\beta_\varepsilon^-}{c}\eta_\varepsilon^-(t)\Bigr)b(\eta_\varepsilon^-(t)) \to \infty, \quad t \to \infty.\end{aligned}$$

□

Since $\eta(t)$ is an increasing function, one gets that, for any $0 < \varepsilon_1 < \varepsilon_2 < 1$, one has (cf. 6.29)

$$\eta^-_{\varepsilon_2}(t) \le \eta^-_{\varepsilon_1}(t) \le \eta^+_{\varepsilon_1}(t) \le \eta^+_{\varepsilon_2}(t), \qquad t \ge t^-_{\rho,\varepsilon_2} > t^-_{\rho,\varepsilon_1}. \tag{6.34}$$

The following simple lemma shows that the latter inequalities hold for different big enough times as well.

Lemma 6.31. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a tail-decreasing function. For any $0 < \varepsilon_1 < \varepsilon_2 < 1$ and for any $t_1, t_2 \ge t^-_{\rho,\varepsilon_2} > t^-_{\rho,\varepsilon_1}$, there exists $\tau = \tau(t_1, t_2, \varepsilon_1, \varepsilon_2) \ge 0$, such that, for all $t \ge \tau$,*

$$\eta^-_{\varepsilon_2}(t_2 + t) \le \eta^-_{\varepsilon_1}(t_1 + t) \le \eta^+_{\varepsilon_1}(t_1 + t) \le \eta^+_{\varepsilon_2}(t_2 + t). \tag{6.35}$$

Proof. By (6.33), all expressions in (6.35) are not smaller than ρ. Since b is decreasing on $[\rho, \infty)$, we have from (6.31), (6.32), that (6.35) is equivalent to

$$e^{-\beta^-_{\varepsilon_2}(t+t_2)} \ge e^{-\beta^-_{\varepsilon_1}(t+t_1)} \ge e^{-\beta^+_{\varepsilon_1}(t+t_1)} \ge e^{-\beta^+_{\varepsilon_2}(t+t_2)},$$

that always holds if only, cf. (6.28),

$$t \ge \frac{1}{\varepsilon_2 - \varepsilon_1} \max\bigl\{0, t_2(1 - \varepsilon_2) - t_1(1 - \varepsilon_1), t_1(1 + \varepsilon_1) - t_2(1 + \varepsilon_2)\bigr\} \ge 0.$$

The statement is proved. □

Moreover, $\eta(t, b)$ is 'increasing' in function b as well. Namely, one has the following result.

Lemma 6.32. *Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ be two tail-decreasing functions, such that, for some $\rho \ge \max\{\rho_{b_1}, \rho_{b_2}\}$ (cf. Definition 6.7),*

$$0 < b_1(s) \le b_2(s), \quad s \ge \rho. \tag{6.36}$$

Then, for any $\varepsilon \in (0, 1)$ and $t \ge -\frac{1}{\beta^-_\varepsilon} \log b_1(\rho)$,

$$\eta^\pm_\varepsilon(t, b_1) \le \eta^\pm_\varepsilon(t, b_2). \tag{6.37}$$

Proof. First of all, note that, by (6.36) and the tail-decreasing property of b_1, b_2, we have

$$\begin{aligned} -\frac{1}{\beta^-_\varepsilon} \log b_1(\rho) &\ge \max\bigl\{-\frac{1}{\beta^-_\varepsilon} \log b_2(\rho), -\frac{1}{\beta^-_\varepsilon} \log b_1(\rho_1)\bigr\} \\ &\ge \max\bigl\{-\frac{1}{\beta^-_\varepsilon} \log b_2(\rho_2), -\frac{1}{\beta^-_\varepsilon} \log b_1(\rho_1)\bigr\} = \max\bigl\{t^-_{\varepsilon,\rho_1}(b_1), t^-_{\varepsilon,\rho_2}(b_2)\bigr\}. \end{aligned}$$

Next,

$$\begin{aligned} \eta^\pm_\varepsilon(t, b_2) &= b_2^{-1}(e^{-\beta^\pm_\varepsilon t}) = b_2^{-1}\bigl(b_1(\eta^\pm_\varepsilon(t, b_1))\bigr) \\ &\ge b_2^{-1}\bigl(b_2(\eta^\pm_\varepsilon(t, b_1))\bigr) = \eta^\pm_\varepsilon(t, b_1), \end{aligned}$$

where we used that b_2^{-1} decreases and (6.36) holds, for $s = \eta^\pm_\varepsilon(t, b_1) \ge \rho$. □

The following statement shows that for 'logarithmically equivalent' functions the corresponding η_ε's are quite close.

Proposition 6.33. *Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ be two tail-decreasing functions which are log-equivalent, i.e.* (6.26) *holds. Then, for any $0 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 1$, there exists $\tau = \tau(\varepsilon, \varepsilon_1, \varepsilon_2) > 0$, such that, for all $t \geq \tau$,*

$$\eta^-_{\varepsilon_2}(t, b_2) \leq \eta^-_{\varepsilon}(t, b_1) \leq \eta^-_{\varepsilon_1}(t, b_2) \leq \eta^+_{\varepsilon_1}(t, b_2) \leq \eta^+_{\varepsilon}(t, b_1) \leq \eta^+_{\varepsilon_2}(t, b_2). \tag{6.38}$$

Proof. Let $\rho_0 > 0$ be such that b_1 and b_2 are both positive and decreasing to 0 on $[\rho_0, \infty)$ and $b_i(\rho_0) < 1$, $i = 1, 2$. Let $0 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 1$ be fixed.

Consider functions $g_i(s) := -\log b_i(s)$, $s \in \mathbb{R}$, $i = 1, 2$. By (6.26), for a $\delta = \delta(\varepsilon, \varepsilon_1, \varepsilon_2) \in (0, 1)$, which will be specify later, there exists $\rho_\delta > \rho_0$ such that

$$(1 - \delta) g_2(s) \leq g_1(s) \leq (1 + \delta) g_2(s), \quad s > \rho_\delta. \tag{6.39}$$

By (6.34), (6.29), all expressions in (6.38) are bigger than the number $\min\bigl\{\eta^-_{\varepsilon_2}(t, b_1), \eta^-_{\varepsilon_2}(t, b_2)\bigr\}$, provided that $t > \frac{1}{\beta^-_{\varepsilon_2}} \max\bigl\{-\log b_1(\rho_0), -\log b_2(\rho_0)\bigr\}$. Then, since $\eta^\pm_\varepsilon(t)$ are increasing to ∞, there exists $\rho = \rho(\varepsilon_2) > \rho_\delta > \rho_0$ and $\tau = \tau(\rho, \rho_\delta) = \tau(\varepsilon, \varepsilon_1, \varepsilon_2) > 0$, such that all expressions in (6.38) are bigger than ρ, if only $t > \tau$.

Since the functions g_i, $i = 1, 2$ are increasing to ∞ on $[\rho, \infty)$, we have, by (6.31), (6.39),

$$\begin{aligned} &\exp\bigl\{-(1+\delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr)\bigr\} \\ &\qquad \leq \exp\bigl\{-g_1\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr)\bigr\} = b_1\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr) = \exp(-\beta^\pm_\varepsilon t) \\ &\qquad\qquad \leq \exp\bigl\{-(1-\delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr)\bigr\}, \end{aligned} \tag{6.40}$$

for all $t > \tau$. Then, by (6.28), we have

$$(1 - \delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr) < (1 \pm \varepsilon)\beta t < (1 + \delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr), \quad t > \tau.$$

Hence, for $t > \tau$,

$$\begin{aligned} \frac{1+\varepsilon}{1+\delta}\beta t < g_2\bigl(\eta^+_\varepsilon(t, b_1)\bigr) < \frac{1+\varepsilon}{1-\delta}\beta t, \\ \frac{1-\varepsilon}{1+\delta}\beta t < g_2\bigl(\eta^-_\varepsilon(t, b_1)\bigr) < \frac{1-\varepsilon}{1-\delta}\beta t. \end{aligned} \tag{6.41}$$

It is straightforward to verify that the inequality $\varepsilon_1 < \varepsilon < \varepsilon_2$ implies

$$\begin{aligned} 1 + \varepsilon_1 < \frac{1+\varepsilon}{1+\delta} < \frac{1+\varepsilon}{1-\delta} < 1 + \varepsilon_2, \\ 1 - \varepsilon_2 < \frac{1-\varepsilon}{1+\delta} < \frac{1-\varepsilon}{1-\delta} < 1 - \varepsilon_1, \end{aligned}$$

if only we choose δ such that

$$0 < \delta < \min\biggl\{\frac{\varepsilon_2 - \varepsilon}{1 + \varepsilon_2}, \frac{\varepsilon - \varepsilon_1}{1 + \varepsilon_1}\biggr\}. \tag{6.42}$$

Then, we get from (6.41)

$$\begin{aligned} g_2\bigl(\eta^+_{\varepsilon_1}(t, b_2)\bigr) < g_2\bigl(\eta^+_\varepsilon(t, b_1)\bigr) < g_2\bigl(\eta^+_{\varepsilon_2}(t, b_2)\bigr), \\ g_2\bigl(\eta^-_{\varepsilon_2}(t, b_2)\bigr) < g_2\bigl(\eta^-_\varepsilon(t, b_1)\bigr) < g_2\bigl(\eta^-_{\varepsilon_1}(t, b_2)\bigr), \end{aligned}$$

for $t > \tau$. Since g_2 is increasing, we obtain the statement. □

Remark 6.34. Note that, by (6.42) and because of the choice of $\tau = \tau(\rho_\delta)$ in the proof above, we have, in general, that $\tau \to \infty$ when either $\varepsilon_1 \nearrow \varepsilon$ or $\varepsilon_2 \searrow \varepsilon$.

Corollary 6.35. *Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ be two tail-decreasing functions which are weakly tail-equivalent. Then* (6.38) *holds.*

Proof. The proof follows directly from Proposition 6.33, since the inequalities (6.13) imply (6.26). □

Remark 6.36. In view of Proposition 6.33 and Remark 6.35, it is natural to consider the case $b_1(s) \sim b_2(s)$, $s \to \infty$. It is evident that then (using the notations of the proof of Proposition 6.33) $\lim\limits_{s\to\infty}(g_1(s) - g_2(s)) = 0$. Then, for any $\delta \in (0,1)$, there exists $\rho_\delta > 0$, such that $|g_1(s) - g_2(s)| < \delta$, for $s > \rho_\delta$. Fix an arbitrary $\varepsilon_0 \in (0,1)$. Then, for any $\delta \in (0,1)$, there exists $\tau = \tau(\delta, \varepsilon_0)$, such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $t > \tau$, one gets, by (6.34),

$$\eta_\varepsilon^\pm(t, b_i) \geq \eta_{\varepsilon_0}^-(t, b_i) > \rho_\delta, \quad i = 1, 2.$$

As a result, instead of (6.41), one gets

$$\beta_\varepsilon^\pm t - \delta < g_2\bigl(\eta_\varepsilon^\pm(t, b_1)\bigr) < \beta_\varepsilon^\pm t + \delta, \quad t > \tau,$$

and the latter inequalities yield

$$\bigl|g_2\bigl(\eta_\varepsilon^\pm(t, b_1)\bigr) - g_2\bigl(\eta_\varepsilon^\pm(t, b_2)\bigr)\bigr| < \delta, \quad t > \tau. \tag{6.43}$$

Stress that τ does not depend on ε, cf. Remark 6.34; in particular, one can put $\varepsilon = 0$. To get from this that $\eta(t, b_1) - \eta(t, b_2) \to 0$ or, better, that $\eta(t, b_1) \sim \eta(t, b_2)$, $t \to \infty$, one needs some additional assumptions on the function g_2. The simplest one is considered in the Subsubsection 6.3.1 below.

6.3 Examples

We consider now main examples of functions $b \in \mathcal{S}(\mathbb{R})$ and describe the corresponding $\eta_\varepsilon^\pm(t, b)$. Because of Propositions 6.29 and 6.33, we will classify these functions 'up to log-equivalence', i.e. by the asymptotic behaviour of

$$l(s) := -\log b(s).$$

For all functions b of the same class of log-equivalence the corresponding $\eta_\varepsilon^\pm(t, b)$ will be same 'up to ε'.

Next, taking into account the result of Theorem 6.25 concerning the function $\breve{b}$, it will be enough to define b on some (s_0, ∞), $s_0 > 0$ only.

Note also that, by Lemma 6.12, the function b_+ defined by (6.7) is a sub-exponential density on $\mathbb{R}_+$. Therefore, one can use the classical examples of such densities, see e.g. [33]. However, using the result of Theorem 6.25 concerning the function $\widetilde{b}$, one can consider that examples in their 'simplest' forms (ignoring any shifts of the argument or scales of the argument or the function itself). To describe the corresponding $\eta_\varepsilon^\pm(t, \widetilde{b})$, first of all, note that, by (6.17), $\log \widetilde{b}(s) \sim \log b(qs + r)$, $s \to \infty$, therefore, again, the corresponding $\eta_\varepsilon^\pm(t)$ will be the same up to ε. Next, by (6.30) applied for $b_1(s) := b(qs + r)$, one gets

$$\eta_\varepsilon^\pm(t, b_1) = \frac{1}{q}\eta_\varepsilon^\pm(t, b) - \frac{r}{q} \sim \frac{1}{q}\eta_\varepsilon^\pm(t, b), \quad t \to \infty. \tag{6.44}$$

In other words, a scaling of the function b changes $\eta_\varepsilon^\pm(t,b)$ 'up to ε' only, whereas a scaling of its argument will be 'more essential'.

Now we consider different asymptotic of the function $l(s) = -\log b(s)$. In all particular examples below, it is straightforward to check that each particular bounded functions b is such that $b'(s) < 0$ and $(\log b(s))'' > 0$ for all big enough values of s, i.e. b is tail-decreasing and tail-log-convex.

6.3.1 Class 1: $l(s) \sim D\log s$, $s\to\infty$, $D>0$

Polynomial decay For a polynomially decreasing function b one can always describe $\eta(t,b)$ explicitly. Namely, let $b:\mathbb{R}\to\mathbb{R}_+$ be a tail-decreasing function, such that

$$b(s) \sim q s^{-D}, \quad s\to\infty,\ D>0,\ q>0. \tag{6.45}$$

Apply arguments of Remark 6.36 to the function $b_1 = b$ and

$$b_2(s) = 1\!\!1_{(-\infty,1)}(s) + q 1\!\!1_{[1,\infty)}(s) s^{-D}, \quad s\in\mathbb{R}.$$

Then $b_1(s)\sim b_2(s)$, $s\to\infty$, and assuming $\rho_\delta > 1$ in the above, we will get (6.43) with $g_2(s) = D\log s$, $s>\rho_\delta$. Stress again, that one can put $\varepsilon = 0$. Then, evidently $\eta(t,b_1)\sim\eta(t,b_2)$, $t\to\infty$; and we can find $\eta(t,b_2)$, by solving the equation $b_2\bigl(\eta(t,b_2)\bigr) = e^{-\beta t}$. As a result,

$$\eta(t,b) \sim q^{\frac{1}{D}} \exp\Bigl(\frac{\beta t}{D}\Bigr), \quad t\to\infty. \tag{6.46}$$

To show when b which satisfies (6.45) belongs to $\mathcal{S}(\mathbb{R})$, consider the following example of such b. Namely, let, for an arbitrary $D>1$,

$$b(s) = 1\!\!1_{\mathbb{R}_+}(s)\frac{1}{(1+s)^D}, \quad s\in\mathbb{R}.$$

For an arbitrary $\gamma\in(0,1)$, consider $h(s) = s^\gamma$, $s>0$. Then

$$\frac{b(s\pm h(s))}{b(s)} = \left(\frac{1+s}{1+s\pm s^\gamma}\right)^D \to 1, \quad s\to\infty.$$

Finally,

$$b\bigl(h(s)\bigr)s^{1+\delta} = \frac{s^{1+\delta}}{(1+s^\gamma)^D} \to 0, \quad s\to\infty,$$

provided that we will choose h above with $\gamma\in\bigl(\frac{1}{D},1\bigr)$ and take $\delta\in(0,\gamma D-1)\subset(0,1)$. As a result, $b\in\mathcal{S}(\mathbb{R})$. Clearly, $b\in\mathcal{S}_n(\mathbb{R})$ for $D>n$.

Let now $b:\mathbb{R}\to\mathbb{R}_+$ be a bounded tail-decreasing tail-log-convex function, such that (6.45) holds, with $D>1$. Then, clearly, $b(s)\sim q(s+1)^{-D}$, $s\to\infty$, and using the previous result and Proposition 6.26, one has that $b\in\mathcal{S}(\mathbb{R})$. Again, $D>n$ for some $n\in\mathbb{N}$, together with the integrability of b on $-\infty$ would lead to $b\in\mathcal{S}_n(\mathbb{R})$. The function $\eta(t,b)$ is described by (6.46).

Consider now several classical examples.

Example 6.37. 1. *Student's t-function.* Let, for $p>\frac{1}{2}$,

$$\mathscr{T}(s) = \frac{1}{(1+s^2)^p}, \quad s>0.$$

The probability density of Student's t-distribution is given, for $p = \frac{\nu+1}{2}$, $\nu > 0$, by $\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\mathscr{T}\bigl(\frac{s}{\sqrt{\nu}}\bigr)$, and extended symmetrically on the whole $\mathbb{R}$. Then $\mathscr{T} \in \mathcal{S}_n(\mathbb{R})$, $n \in \mathbb{N}$, if only $p > \frac{n}{2}$. The case $p = 1$ is referred to the Cauchy distribution, the corresponding function belongs to $\mathcal{S}_n(\mathbb{R})$ for $n = 1$ only.

2. *The Lévy function.* Let, for $c > 0$,

$$\mathscr{L}(s) = s^{-\frac{3}{2}} \exp\Bigl(-\frac{c}{s}\Bigr), \quad s > 0.$$

The probability density of the Lévy distribution is $\sqrt{\frac{c}{\pi}}\mathscr{L}(s-\mu)$, $\mu \in \mathbb{R}$, $s > \mu$.

3. *The Burr function.* Let, for $c > 0$, $k > 0$,

$$\mathscr{B}(s) = \frac{s^{c-1}}{(1+s^c)^{k+1}}, \quad s > 0.$$

The probability density of the so-called Burr IV distribution is just $ck\mathscr{B}(s)$. Note that the case $c = 1$ is related to the Pareto distribution; the latter has the density $kp^k\mathscr{B}(s - 1)1\!\!1_{[p,\infty)}(s)$ for any $p > 0$.

Logarithmic perturbation of the polynomial decay Let $D > 1$, $\nu \in \mathbb{R}$, and

$$b(s) = 1\!\!1_{(1,\infty)}(s)\frac{(\log s)^\nu}{s^D}, \quad s \in \mathbb{R}.$$

We are going to apply Proposition 6.29 now, with $b_1(s) = s^{-D}$ and $b_2(s) = (\log s)^\nu s^{-D}$. Indeed, then (6.26) evidently holds. It remains to check that (6.5) holds for both b_1 and b_2 with the same $h(s) = s^\gamma$, $\gamma \in (0,1)$. One has

$$\frac{\log(s \pm s^\gamma)}{\log s} = \frac{\log s + \log\bigl(1 \pm s^{\gamma-1}\bigr)}{\log s} \to 1, \quad s \to \infty,$$

that yields the needed. The corresponding $\eta_\varepsilon^\pm(t,b)$ can be estimated by (6.46) using (6.38).

6.3.2 Class 2: $l(s) \sim D(\log s)^q$, $s \to \infty$, $q > 1$, $D > 0$

Consider the function

$$N(s) := 1\!\!1_{\mathbb{R}_+}(s)\exp\bigl(-D(\log s)^q\bigr), \quad s \in \mathbb{R}.$$

Take $h(s) = 1\!\!1_{[\rho,\infty)}(s)s^{\frac{1}{q}}$, where $\rho > 1$ is chosen such that $h(s) < \frac{s}{2}$ for $s \geq \rho$. Prove that (6.5) holds. We have

$$\frac{N(s \pm h(s))}{N(s)} = \exp\biggl\{D(\log s)^q\biggl(1 - \Bigl(1 + \frac{\log(1 \pm s^{\frac{1}{q}-1})}{\log s}\Bigr)^q\biggr)\biggr\}.$$

Since $q > 1$, we have that $t(s) := \dfrac{\log(1 \pm s^{\frac{1}{q}-1})}{\log s} \to 0$, $s \to \infty$. Redefine then ρ to have that $|t(s)| < 1$, if only $s > \rho$. Use the binomial series

$$(1+t)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}t^k, \quad \alpha > 0, \tag{6.47}$$

which converges for $|t|<1$. One gets then (for $\alpha=q$)

$$\frac{N(s\pm h(s))}{N(s)}=\exp\biggl(-D(\log s)^q\sum_{k=1}^\infty\binom{q}{k}t(s)^k\biggr)\to 1,\quad s\to\infty.$$

Indeed, using the well-known inequality

$$\left|\binom{\alpha}{k}\right|\leq\frac{M}{k^{1+\alpha}},\quad k\in\mathbb{N},\ M=M(\alpha), \tag{6.48}$$

one gets, for any $s>\rho$,

$$\left|-D(\log s)^q\sum_{k=1}^\infty\binom{q}{k}t(s)^k\right|\leq D(\log s)^q|t(s)|\sum_{k=1}^\infty\frac{M}{k^{1+q}}\to 0,\quad s\to\infty,$$

since the latter series converges, and, for $q>1$,

$$(\log s)^q|t(s)|\sim(\log s)^{q-1}s^{\frac{1}{q}-1}\to 0,\quad s\to\infty.$$

Finally, for any $\delta\in\mathbb{R}$,

$$N\bigl(s^{\frac{1}{q}}\bigr)s^{1+\delta}=\exp\bigl(-Dq^{-q}(\log s)^q+(1+\delta)\log s\bigr)\to 0,\quad s\to\infty,$$

since $q>1$.

As a result, $N\in\mathcal{S}(\mathbb{R})$. Moreover, evidently, $N\in\mathcal{S}_n(\mathbb{R})$, for any $n\in\mathbb{N}$. To find $\eta(t,N)$, one has to solve the equation $b(s)=e^{-\beta t}$. One has then

$$\eta(t,N)=\exp\biggl(\Bigl(\frac{\beta}{D}t\Bigr)^{\frac{1}{q}}\biggr).$$

We may also consider Proposition 6.29 for $b_1=b$ and $b_2=pb$, where b_2 is tail-decreasing and tail-log-convex function, such that $\log p=o(\log b)$ (that is equivalent to $\log b_1\sim\log b_2$) and p satisfies (6.5) with $h(s)=s^{\frac{1}{q}}$. According to Subsubsection 6.3.1, a natural example of such $p(s)$ might be s^D, $D\in\mathbb{R}$. As a result, then $b_2\in\mathcal{S}_n(\mathbb{R})$, $n\in\mathbb{N}$. The corresponding $\eta_\varepsilon^\pm(t,b_2)$ can be estimated then by $\eta_\varepsilon^\pm(t,N)$ using (6.38). Consider now a classical example of such function b_2.

Example 6.38. *The log-normal function.* Let, for $\gamma>0$,

$$\mathscr{N}(s)=\frac{1}{s}\exp\Bigl(-\frac{(\log s)^2}{2\gamma^2}\Bigr),\quad s>0.$$

By the above, $\mathscr{N}\in\mathcal{S}_n(\mathbb{R})$, $n\in\mathbb{N}$. The log-normal distribution has the density $\frac{1}{\gamma\sqrt{2\pi}}\mathscr{N}(se^{-\mu})$ for an arbitrary $\mu\in\mathbb{R}$.

6.3.3 Class 3: $l(s)\sim s^\alpha$, $\alpha\in(0,1)$

Consider, for any $\alpha\in(0,1)$, the so-called *fractional exponent*

$$w(s)=1\!\!1_{\mathbb{R}_+}(s)e^{-s^\alpha},\quad s\in\mathbb{R}. \tag{6.49}$$

Set $h(s) = 1\!\!1_{[\rho,\infty)}(s)(\log s)^{\frac{2}{\alpha}}$, where $\rho > 0$ is chosen such that $h(s) < \frac{s}{2}$ for $s \geq \rho$. Then, in particular, $t(s) := \dfrac{h(s)}{s} < 1$, $s \geq \rho$. Prove that (6.5) holds. Using (6.47) for $t = \pm t(s)$, one gets

$$\frac{w\bigl(s \pm h(s)\bigr)}{w(s)} = \exp\biggl(-s^\alpha \sum_{k=1}^{\infty} \binom{\alpha}{k} (\pm t(s))^k\biggr) \to 1, \quad s \to \infty,$$

similarly to the arguments in Subsubsection 6.3.2, by using (6.48) and the evident convergence $s^\alpha t(s) \to 0$, $s \to \infty$, $\alpha \in (0,1)$.

Finally, for any $\delta \in \mathbb{R}$,

$$w\bigl(h(s)\bigr)s^{1+\delta} = \exp\bigl(-(\log s)^2 + (1+\delta)\log s\bigr) \to 0, \quad s \to \infty. \tag{6.50}$$

As a result, $w \in \mathcal{S}(\mathbb{R})$. It is clear also that $w \in \mathcal{S}_n(\mathbb{R})$ for all $n \in \mathbb{N}$. To find $\eta(t,w)$, one has to solve the equation $e^{-s^\alpha} = e^{-\beta t}$; therefore,

$$\eta(t,w) = (\beta t)^{\frac{1}{\alpha}}. \tag{6.51}$$

Similarly to the above, one can show that $pw \in \mathcal{S}(\mathbb{R})$, provided that, in particular, $\log p = o(\log w)$ and (6.5) holds for $b = p$ and $h(s) = (\log s)^{\frac{2}{\alpha}}$. Again, one can consider $p(s) = s^D$, $D \in \mathbb{R}$, since it satisfies (6.5) with $h(s) = s^\gamma > (\log s)^{\frac{2}{\alpha}}$, $\alpha, \gamma \in (0,1)$, and big enough s. Consider the corresponding classical example.

Example 6.39. *The Weibull function.* Let, for $\alpha \in (0,1)$,

$$\mathscr{W}(s) = \frac{\exp(-s^\alpha)}{s^{1-\alpha}}, \quad s \geq \rho > 0.$$

Note that $\int_s^\infty \mathscr{W}(\tau)\,d\tau = \frac{1}{\alpha} w(s)$, where w is given by (6.49). By the above, $\mathscr{W} \in \mathcal{S}_n(\mathbb{R})$, $n \in \mathbb{N}$. The probability density of the Weibull distribution is $\frac{\alpha}{\beta}\mathscr{W}\bigl(\frac{s}{\beta}\bigr)$, $s > 0$ for any $\beta > 0$. Note that the density itself is unbounded near 0.

6.3.4 Class 4: $l(s) \sim \dfrac{s}{(\log s)^\alpha}$, $\alpha > 1$

Consider also a function which decays 'slightly' slowly than an exponential function. Namely, let, for an arbitrary fixed $\alpha > 1$,

$$g(s) = 1\!\!1_{\mathbb{R}_+}(s) \exp\Bigl(-\frac{s}{(\log s)^\alpha}\Bigr), \quad s \in \mathbb{R}. \tag{6.52}$$

Take, for an arbitrary $\gamma \in (1,\alpha)$, $h(s) = (\log s)^\gamma$, $s > 0$; and denote, for a brevity, $p(s) := \dfrac{h(s)}{s} \to 0$, $s \to \infty$. Then, $\log(s + h(s)) = \log s + \log(1 + p(s))$. Set also

$$q(s) = \frac{\log(1+p(s))}{\log s} \to 0, \quad s \to \infty.$$

Then, for any $s > e^{\alpha+1}$, we have

$$\begin{aligned}
&\log \frac{g(s+h(s))}{g(s)} = \frac{s}{(\log s)^\alpha}\biggl(1 - \frac{1+p(s)}{\bigl(1+q(s)\bigr)^\alpha}\biggr) \\
&= \frac{1}{\bigl(1+q(s)\bigr)^\alpha}\biggl(\alpha \frac{\bigl(1+q(s)\bigr)^\alpha - 1}{\alpha q(s)} \frac{\log(1+p(s))}{p(s)} (\log s)^{\gamma-\alpha-1} - (\log s)^{\gamma-\alpha}\biggr) \\
&\to 0, \quad s \to \infty,
\end{aligned}$$

as $\gamma < \alpha$; and similarly $\log \dfrac{g(s-h(s))}{g(s)} \to 0$, $s \to \infty$. Therefore, (6.5) holds for $b = g$. Next,

$$\log\bigl(g(h(s))s^{1+\delta}\bigr) = -(\log s)\Bigl(\frac{(\log s)^{\gamma-1}}{\gamma^\alpha(\log\log s)^\alpha} - (1+\delta)\Bigr) \to -\infty, \quad s \to \infty,$$

that yields (6.14) for $b = g$. As a result, $g \in \mathcal{S}(\mathbb{R})$. Again, evidently, $g \in \mathcal{S}_n(\mathbb{R})$, $n \in \mathbb{N}$. To find $\eta(t, g)$, one has to solve the equation $g(s) = e^{-\beta t}$, i.e. $s(\log s)^{-\alpha} = \beta t$. Making substitution $s = e^\tau$, one easily gets

$$-\frac{\tau}{\alpha}e^{-\frac{\tau}{\alpha}} = -\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}}.$$

Since $s > e^\alpha$ implies $-\frac{\tau}{\alpha} < -1$ and assuming t big enough, to ensure that $-\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}} > -\frac{1}{e}$, one has that the solution to the latter equation can be given in terms of the negative real branch W_{-1} of Lambert W-function, that is the function such that $W_{-1}(\nu)\exp(W_{-1}(\nu)) = \nu$, $W_{-1}(\nu) < -1$, $\nu \in (-e^{-1}, 0)$. Namely, one gets $-\frac{\tau}{\alpha} = W_{-1}\bigl(-\alpha^{-1}(\beta t)^{-\frac{1}{\alpha}}\bigr)$, and, therefore

$$\eta(t, g) = \exp\biggl(-\alpha W_{-1}\Bigl(-\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}}\Bigr)\biggr).$$

However, $\exp(-W_{-1}(\nu)) = \nu^{-1}W_{-1}(\nu)$, therefore,

$$\exp(-\alpha W_{-1}(\nu)) = (-\nu)^{-\alpha}(-W_{-1}(\nu))^\alpha,$$

i.e.

$$\eta(t, g) = \alpha^\alpha \beta t\biggl(-W_{-1}\Bigl(-\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}}\Bigr)\biggr)^\alpha, \quad t > \frac{1}{\beta}\Bigl(\frac{e}{\alpha}\Bigr)^\alpha.$$

To get a feeling about the behaviour of $\eta(t, g)$ for large t, note that $W_{-1}(\nu) \sim \log(-\nu)$, $\nu \to 0-$. As a result,

$$\eta(t, g) \sim \beta t(\log t)^\alpha, \quad t \to \infty. \tag{6.53}$$

Remark 6.40. The analysis of $\eta(t, g)$ above does not require, of course, that $\alpha > 1$. Naturally, $\alpha \in (0, 1]$ gives behaviour of $g(s)$ more 'close' to the exponential function and then $\eta(t, g)$ in (6.53) would be 'almost linear', cf. Lemma 6.30. Unfortunately, our approach does not cover this case: the analysis above shows that $h(s)$, to fulfill even (6.6), must grow faster than $\log s$, whereas so 'big' $h(s)$ would not fulfill (6.5). In general, Lemma 6.12 gives a sufficient condition only, to get a sub-exponential density on $\mathbb{R}_+$. It can be shown, see e.g. [38, Example 1.4.3], that a probability distribution, whose density b on $\mathbb{R}_+$ is such that $\int_s^\infty b(\tau)\,d\tau \sim g(s)$, $s \to \infty$, with $\alpha > 0$, is a sub-exponential distribution (for the latter definition, see e.g. [33, Definition 3.1]). Then we expect that $b(s) \sim -g'(s)$, $s \to \infty$, and it is easy to see that $\log(-g'(s)) \sim \log g(s)$, $s \to \infty$. Therefore, one can apply Proposition 6.33, to estimate $\eta^\pm_\varepsilon(t, b)$ in terms of $\eta^\pm_\varepsilon(t, g)$, whose asymptotic in t may be obtained from (6.53). It should be stressed though that, in general, sub-exponential property of a distribution does not imply the corresponding property of its density, cf. [33, Section 4.2]. Therefore, we can not state that the function b above is a sub-exponential one for $\alpha \in (0, 1]$.

6.4 Technical tools on $\mathbb{R}^d$

Let us fix an orthonormal basis $e_1, \ldots, e_d$ in $\mathbb{R}^d$. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $|x|$ denote the Euclidean norm in $\mathbb{R}^d$, and set

$$\langle x \rangle := \max_{1 \le j \le d} x_j \in \mathbb{R}, \tag{6.54}$$

$$\Delta(x) := \bigl\{ y \in \mathbb{R}^d : y_j \ge x_j,\ 1 \le j \le d \bigr\}. \tag{6.55}$$

We introduce the following classes of functions.

Definition 6.41. 1. Let $\mathcal{D}_d(\mathbb{R})$ be the set of all bounded functions $b : \mathbb{R} \to (0, \infty)$, such that b is (strictly) decreasing to 0 on $\mathbb{R}_+$ and

$$\int_0^\infty b(s) s^{d-1}\, ds < \infty. \tag{6.56}$$

2. Let $\mathcal{R}$ be the set of all bounded radially symmetric functions $c : \mathbb{R}^d \to (0, \infty)$, such that $c(x) = b(|x|)$, $x \in \mathbb{R}^d$ for some $b = b_c \in \mathcal{D}_d(\mathbb{R})$. Note that, because of (6.56), $\mathcal{R} \subset L^1(\mathbb{R}^d)$.

3. Let $\mathcal{M}$ be the set of all bounded functions $c : \mathbb{R}^d \to (0, \infty)$ which satisfy the following monotonicity property: for an arbitrary $x \in \mathbb{R}^d$ and for any $1 \le j \le d$, the function

$$\mathbb{R} \ni s \mapsto c(x + s e_j) \in \mathbb{R}_+ \tag{6.57}$$

is strictly decreasing on $\mathbb{R}$, converges to 0 as $s \to \infty$, and there exists $c_- \in (0, \infty)$, such that, for any $x \in \mathbb{R}^d$,

$$\lim_{s \to -\infty} c\bigl(x + (s, \ldots, s)\bigr) = c_-. \tag{6.58}$$

4. Let $\mathcal{I} \subset \mathcal{M}$ be the set of all functions from $\mathcal{M}$ of the form

$$c(x) = \int_{\Delta(x)} p(y) dy, \qquad p \in \mathcal{R}, \tag{6.59}$$

where $\Delta(x)$ is given by (6.55). Then, clearly, $c_- = \int_{\mathbb{R}^d} p(y) dy$.

Definition 6.42. 1. Let $b \in \mathcal{D}_d(\mathbb{R})$. A function $c \in \mathcal{R} \cup \mathcal{I}$ is said to be *constructed by* b, if $c(x) = b(|x|)$, $x \in \mathbb{R}^d$ (if $c \in \mathcal{R}$) or $c(x) = \int_{\Delta(x)} b(|y|) dy$, $x \in \mathbb{R}^d$ (if $c \in \mathcal{I}$).

2. Let $b \in \mathcal{D}_d(\mathbb{R})$ and $\alpha \in (0, 1)$ be such that $b^\alpha \in \mathcal{D}_d(\mathbb{R})$. Let $c \in \mathcal{R} \cup \mathcal{I}$ be constructed by b. Then we denote by $c_\alpha \in \mathcal{R} \cup \mathcal{I}$ the function constructed by b^α; in other words, for all $x \in \mathbb{R}^d$,

$$c_\alpha(x) := \begin{cases} c(x)^\alpha = b(|x|)^\alpha, & \text{if } c \in \mathcal{R}, \\ \displaystyle\int_{\Delta(x)} b(|y|)^\alpha dy, & \text{if } c \in \mathcal{I}. \end{cases} \tag{6.60}$$

In particular, $c_1 = c$.

Remark 6.43. It is easy to see that, if $b^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$ for some $\alpha_0 \in (0, 1)$, then $b^\alpha \in \mathcal{D}_d(\mathbb{R})$ for all $\alpha \in [\alpha_0, 1]$.

Remark 6.44. Clearly, $c \in \mathcal{R}$ implies $c_\alpha \in \mathcal{R}$, whereas $c \in \mathcal{I}$ implies $c_\alpha \in \mathcal{I}$.

We will always suppose that (A1) hold, and we denote

$$\beta := \varkappa^+ - m > 0. \tag{6.61}$$

Definition 6.45. For any $c \in \mathcal{R} \cup \mathcal{M}$, $t \geq 0$, and $\varepsilon \in (0,1)$, we define the sets

$$\Lambda_\varepsilon^\pm(t,c) := \bigl\{x \in \mathbb{R}^d : c(x)e^{\beta(1\pm\varepsilon)t} \geq 1\bigr\}. \tag{6.62}$$

Clearly, $\Lambda_\varepsilon^-(t,c) \subset \Lambda_\varepsilon^+(t,c)$.

Remark 6.46. Note that, for any $c \in \mathcal{R} \cup \mathcal{M}$ and $\varepsilon \in (0,1)$, there exists $t_{c,\varepsilon} \geq 0$, such that

$$\sup_{x\in\mathbb{R}^d} c(x) \geq e^{-\beta(1-\varepsilon)t_{c,\varepsilon}} \geq e^{-\beta(1+\varepsilon)t_{c,\varepsilon}}.$$

Therefore, for any $t \geq t_{c,\varepsilon}$ the sets $\Lambda_\varepsilon^\pm(t,c)$ are non-empty.

Remark 6.47. For any $c \in \mathcal{R}$ constructed by some $b \in \mathcal{D}_d(\mathbb{R})$ and for any $\rho \geq 0$ such that $b(\rho) \leq 1$, we have that, for any $t \geq t^-_{\rho,\varepsilon}$, cf. (6.29),

$$\Lambda_\varepsilon^\pm(t,c) = \bigl\{x \in \mathbb{R}^d : |x| \leq \eta_\varepsilon^\pm(t,b)\bigr\}. \tag{6.63}$$

Remark 6.48. Note that $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^\pm(t,c)$ is equivalent to $c(x) < e^{-\beta(1\pm\varepsilon)t}$. In particular,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c)} c(x) = 0.$$

On the other hand, for $c \in \mathcal{R}$, we have that $\lim\limits_{|x|\to\infty} c(x) = 0$. Moreover, for $c \in \mathcal{I}$, we have, by (6.55) and (6.59),

$$\lim_{\langle x\rangle\to\infty} c(x) = \lim_{\langle x\rangle\to\infty} \int_{\Delta(x)} b(|y|)dy = 0. \tag{6.64}$$

Proposition 6.49. *Let $c^{(i)} \in \mathcal{I}$ be constructed by $b_i \in \mathcal{D}_d(\mathbb{R})$, $i = 1,2$. Suppose that there exists $\rho > 0$, such that $b_1(s) \leq b_2(s)$ for all $s \geq \rho$. Then, for any $\varepsilon > 0$ there exists $\tau = \tau(\varepsilon, b_1, b_2) > 0$, such that $\Lambda_\varepsilon^\pm(t,c^{(1)}) \subset \Lambda_\varepsilon^\pm(t,c^{(2)})$ for all $t \geq \tau$.*

Proof. First, we note that, for any $c \in \mathcal{M}$, the inequality $c(x) < e^{-\beta(1\pm\varepsilon)t}$ for some big enough t, is equivalent to the existence of some $\rho_t > 0$ such that $\langle x\rangle \geq \rho_t$. Then the inequlity

$$|y| \geq \langle y\rangle \geq \langle x\rangle, \qquad y \in \Delta(x),\ x \in \mathbb{R}^d, \tag{6.65}$$

shows that $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^\pm(t,c)$ implies $|y| \geq \rho_t$, $y \in \Delta(x)$.

Next, we have to prove that $\mathbb{R}^d \setminus \Lambda_\varepsilon^\pm(t,c^{(2)}) \subset \mathbb{R}^d \setminus \Lambda_\varepsilon^\pm(t,c^{(1)})$ for big enough t. Let $\rho_0 \geq \rho$ be such that $b_1(\rho_0) \leq b_2(\rho_0) \leq 1$. Choose $\tau > 0$ such that $t \geq \tau$ implies that $\langle x\rangle \geq \rho_0$ for all $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^\pm(t,c^{(2)})$. By the above, for all $y \in \Delta(x)$, we will have $|y| \geq \rho_0$, and hence $b_1(y) \leq b_2(y)$, $y \in \Delta(x)$. Thus $c^{(1)}(x) \leq c^{(2)}(x)$ and hence $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^\pm(t,c^{(1)})$. □

Remark 6.50. By (6.63), Proposition 6.49 remains evidently true for $c^{(i)} \in \mathcal{R}$, $i = 1,2$ as well.

Remark 6.51. Let $c^{(i)} \in \mathcal{R}$ be constructed by $b_i \in \mathcal{D}_d(\mathbb{R})$, $i = 1, 2$, such that (6.26) holds. Then, by (6.63) and Proposition 6.33, one gets

$$\Lambda^-_{\varepsilon_2}(t, c^{(2)}) \subset \Lambda^-_\varepsilon(t, c^{(1)}) \subset \Lambda^-_{\varepsilon_1}(t, c^{(2)}) \subset \Lambda^+_{\varepsilon_1}(t, c^{(2)}) \subset \Lambda^+_\varepsilon(t, c^{(1)}) \subset \Lambda^+_{\varepsilon_2}(t, c^{(2)}),$$

if only $0 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 1$ and $t \geq \tau(\varepsilon, \varepsilon_1, \varepsilon_2) > 0$. In the sequel, we will need to extend (partially) this result for the case when $b_1 = (b_2)^\alpha$ for an $\alpha < 1$ which is 'close to 1'. Moreover, we will need the corresponding results for functions $c^{(i)} \in \mathcal{I}$, $i = 1, 2$ as well.

Theorem 6.52. *For any $\alpha_0 \in \left(\frac{3}{4}, 1\right)$ there exists $\varepsilon_0 = \varepsilon_0(\alpha_0) \in (0, 1)$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\alpha = \alpha(\varepsilon) \in (\alpha_0, 1)$ such that the following holds. For any $b \in \mathcal{D}_d(\mathbb{R})$ such that $b^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$, let $c, c_\alpha \in \mathcal{R} \cup \mathcal{I}$ be constructed by b and b^α, correspondingly. Then there exists $\tau = \tau(\varepsilon, b) > 0$, such that, for any $t \geq \tau$,*

$$\Lambda^-_\varepsilon(t, c_\alpha) \subset \Lambda^-_{\frac{\varepsilon}{2}}(t, c), \tag{6.66}$$

$$\Lambda^+_{\frac{\varepsilon}{2}}(t, c_\alpha) \subset \Lambda^+_\varepsilon(t, c). \tag{6.67}$$

Remark 6.53. Here and below, we will mean that if $f, g \in \mathcal{R} \cup \mathcal{I}$, then either $f, g \in \mathcal{R}$ or $f, g \in \mathcal{I}$.

Remark 6.54. For any $b \in \mathcal{D}_d(\mathbb{R})$, there exists $\rho > 0$, such that $b(\rho) \leq 1$. Then, for any $s \geq \rho$, one gets that $b(s)^\alpha \geq b(s)$. Therefore, for big enough $t > 0$, the inclusions $\Lambda^\pm_{\frac{\varepsilon}{2}}(t, c) \subset \Lambda^\pm_{\frac{\varepsilon}{2}}(t, c_\alpha)$ follow from Proposition 6.49.

Proof of Theorem 6.52. We will prove (6.67). The proof of (6.66) is fully analogous. Consider two cases separately.

1) For a $c \in \mathcal{R}$. Since $\alpha_0 \in \left(\frac{3}{4}, 1\right)$, one can define

$$\varepsilon_0 := \frac{1 - \alpha_0}{\alpha_0 - \frac{1}{2}} \in (0, 1).$$

Take an arbitrary $\varepsilon \in (0, \varepsilon_0)$, then one easily has that

$$\alpha := \frac{1 + \frac{\varepsilon}{2}}{1 + \varepsilon} \in (\alpha_0, 1). \tag{6.68}$$

Take an arbitrary $b \in \mathcal{D}_d(\mathbb{R})$ such that $b^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$, and let $c \in \mathcal{R}$ be constructed by b. Prove that then there is an equality in (6.67). Indeed, by (6.63), the equality in (6.67) is just equivalent to

$$\eta^+_{\frac{\varepsilon}{2}}(t, b^\alpha) = \eta^+_\varepsilon(t, b), \quad t \geq \tau := t^-_{\rho,\varepsilon}.$$

To prove the latter equality, apply $\log b^\alpha = \alpha \log b$ to both its parts:

$$-\Bigl(1 + \frac{\varepsilon}{2}\Bigr)\beta t = -\alpha(1 + \varepsilon)\beta t,$$

that is equivalent to (6.68).

2) For a $c \in \mathcal{I}$. Prove the following inequality, which is equivalent to (6.68),

$$\mathbb{R}^d \setminus \Lambda^+_\varepsilon(t, c) \subset \mathbb{R}^d \setminus \Lambda^+_{\frac{\varepsilon}{2}}(t, c_\alpha), \quad t \geq \tau. \tag{6.69}$$

Recall that the inclusion $x \in \mathbb{R}^d \setminus \Lambda^+_\varepsilon(t, c)$ is equivalent to

$$c(x) = \int_{\Delta(x)} b(|y|)dy < e^{-\beta(1+\varepsilon)t}. \tag{6.70}$$

We will use Hölder's inequality to estimate $c_\alpha(x)$. It is easy to see that the function

$$f(\alpha) := \alpha - \sqrt{\alpha(1-\alpha)} : \bigl(\tfrac12, 1\bigr) \to (0,1),$$

is increasing. We set $p := p(\alpha) := \frac{1}{f(\alpha)} > 1$ and $q := q(\alpha) := \frac{1}{1-f(\alpha)} > 1$. Then $\frac1p + \frac1q = 1$ and, by (6.70), we have

$$\begin{aligned} c_\alpha(x) &= \int_{\Delta(x)} b(|y|)^{f(\alpha)+(\alpha-f(\alpha))}dy \\ &\leq \biggl(\int_{\Delta(x)} b(|y|)^{f(\alpha)p}dy\biggr)^{\frac1p}\biggl(\int_{\Delta(x)} b(|y|)^{(\alpha-f(\alpha))q}dy\biggr)^{\frac1q} \\ &< e^{-\beta(1+\varepsilon)f(\alpha)t}\biggl(\int_{\Delta(x)} b(|y|)^{(\alpha-f(\alpha))q}dy\biggr)^{\frac1q}. \end{aligned} \tag{6.71}$$

The inclusion $b^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$ means that (6.56) holds with b replaced by b^{α_0}. Therefore, to get the finiteness of the latter integral in (6.71), it is enough to have there α such that $\alpha_0 < g(\alpha) < 1$, where

$$g(\alpha) := (\alpha - f(\alpha))q(\alpha) = \frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{1-\alpha}}, \quad \alpha \in \bigl(\tfrac12, 1\bigr).$$

It is easy to see that $g : \bigl(\frac12,1\bigr) \to \bigl(\frac12,1\bigr)$ is increasing and $g(\alpha) < \alpha$, $\alpha \in \bigl(\frac12,1\bigr)$. Note also that $g\bigl(\frac{9}{10}\bigr) = \frac34$. As a result, for the given $\alpha_0 \in \bigl(\frac34,1\bigr)$, there exists a unique $\alpha_1 \in \bigl(\frac{9}{10},1\bigr)$, such that $\alpha_0 = g(\alpha_1) < \alpha_1$. Hence, for any $\alpha \in (\alpha_1,1) \subset (\alpha_0,1)$, one gets $g(\alpha) > g(\alpha_1) = \alpha_0$, and then $\int_{\mathbb{R}^d} b(|y|)^{g(\alpha)}dy < \infty$; in particular, the latter integral in (6.71) is finite.

Next, the function $h(\varepsilon) = \dfrac{1+\frac{\varepsilon}{2}}{1+\varepsilon} : (0,1) \to \bigl(\frac34,1\bigr)$ is decreasing; cf. (6.68). Therefore, there exists a unique $\varepsilon_0 \in (0,1)$, such that $h(\varepsilon_0) = \alpha_1$; then we have $h : (0,\varepsilon_0) \to (\alpha_1,1)$. Take and fix now an arbitrary $\varepsilon \in (0,\varepsilon_0)$. Since,

$$f : (\alpha_1,1) \to \bigl(f(\alpha_1),1\bigr) \subset (\alpha_1,1) = (h(\varepsilon_0),1),$$

is increasing (we used here that $f(\alpha) < \alpha$), there exists a unique $\alpha = \alpha(\varepsilon) \in (\alpha_1,1)$, such that

$$f(\alpha) = h(\varepsilon) = \frac{1+\frac{\varepsilon}{2}}{1+\varepsilon}. \tag{6.72}$$

Therefore, after $\varepsilon_0, \varepsilon, \alpha$ are chosen, we take an arbitrary $b \in \mathcal{D}_d(\mathbb{R})$ such that $b^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$, and let $c \in \mathcal{I}$ be constructed by b. For this α, by the above, $\int_{\mathbb{R}^d} b(|y|)^{g(\alpha)}dy < \infty$; therefore, there exists $r > 0$, such that, for all $x \in \mathbb{R}^d$ with $\langle x\rangle > r$,

$$\int_{\Delta(x)} b(|y|)^{g(\alpha)}dy \leq 1.$$

The latter inequality together with (6.72) and (6.71) implies that

$$c_\alpha(x) \leq e^{-\beta(1+\frac{\varepsilon}{2})t}, \tag{6.73}$$

provided that $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c)$ (i.e. (6.70) holds) and $\langle x\rangle > r$. In (6.70), $\langle x\rangle \to \infty$ if and only if $t \to \infty$; cf. Remark 6.48. Therefore, there exists $\tau = \tau(r) = \tau(\varepsilon,b) > 0$, such that $t \geq \tau$ in (6.70) implies $\langle x\rangle \geq r$. As a result, for any $t \geq \tau$ and any $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c)$, one gets (6.73), that means that $x \in \mathbb{R}^d \setminus \Lambda_{\frac{\varepsilon}{2}}^+(t,c_\alpha)$; i.e. (6.69) holds. □

Corollary 6.55. *Let $b_1, b_2 \in \mathcal{D}_d(\mathbb{R})$ and $\alpha_0 \in \left(\frac{3}{4}, 1\right)$ be such that $b_i^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$, $i = 1, 2$ and (6.26) holds. Let $c^{(i)} \in \mathcal{R} \cup \mathcal{I}$ be constructed by b_i, $i = 1, 2$. Then there exists $\varepsilon_0 = \varepsilon_0(\alpha_0) \in (0, 1)$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\tau = \tau(\varepsilon) > 0$, such that, for any $t \geq \tau$,*

$$\Lambda_{\varepsilon}^{-}(t, c^{(1)}) \subset \Lambda_{\frac{\varepsilon}{2}}^{-}(t, c^{(2)}), \tag{6.74}$$

$$\Lambda_{\frac{\varepsilon}{2}}^{+}(t, c^{(1)}) \subset \Lambda_{\varepsilon}^{+}(t, c^{(2)}). \tag{6.75}$$

Proof. Let ε_0 by given by Theorem 6.52. Take an arbitrary $\varepsilon \in (0, \varepsilon_0)$ and consider $\alpha = \alpha(\varepsilon) \in (\alpha_0, 1)$ also given by Theorem 6.52. Let $\rho_0 > 0$ be such that $b_i(\rho_0) \leq 1$, $i = 1, 2$. Set $\delta := 1 - \alpha \in (0, 1 - \alpha_0)$. By (6.26), there exists $\rho_\alpha \geq \rho_0$, such that

$$1 - \delta < \frac{-\log b_1(s)}{-\log b_2(s)} < 1 + \delta, \quad s > \rho_\alpha,$$

in particular,

$$b_1(s) < b_2(s)^\alpha, \quad s > \rho_\alpha. \tag{6.76}$$

By Remark 6.43, $b_2^\alpha \in \mathcal{D}_d(\mathbb{R})$, and hence, by (6.76) and Proposition 6.49, applying to b_1 and b_2^α, one gets

$$\Lambda_{\frac{\varepsilon}{2}}^{+}(t, c^{(1)}) \subset \Lambda_{\frac{\varepsilon}{2}}^{+}(t, c_\alpha^{(2)}).$$

The latter inequality together with (6.67) for $c = c^{(2)}$ imply (6.75).

Next, by (6.76), $b_3(s) := b_1(s)^{\frac{1}{\alpha}} < b_2(s)$, if only $s > \rho_\alpha$. From here we have that $b_3 \in \mathcal{D}_d(\mathbb{R})$ and, moreover, by Proposition 6.49, applying to b_3 and b_2,

$$\Lambda_{\frac{\varepsilon}{2}}^{-}(t, c^{(3)}) \subset \Lambda_{\frac{\varepsilon}{2}}^{-}(t, c^{(2)}),$$

where $c^{(3)} \in \mathcal{R} \cup \mathcal{I}$ in constructed by b_3, cf. Remark 6.53. The latter inequality together with (6.66) for $c = c^{(3)}$ imply (6.74). ☐

Remark 6.56. Using the same arguments as in the proof of Proposition 6.49, one can get that the statement of Corollary 6.55 remains true if both functions b_1, b_2 are tail-decreasing only, and $\int_0^\infty b_i(s) s^{d-1}\, ds < \infty$, $i = 1, 2$, cf. Definitions 6.7 and 6.41.

We will need the following analogues of long-tailed functions in $\mathbb{R}^d$.

Definition 6.57.

1. Let $\mathcal{L} \subset \mathcal{R}$ be the set of all functions $c \in \mathcal{R}$, such that $c(x) = b(|x|)$, $x \in \mathbb{R}^d$, where $b \in \mathcal{D}_d(\mathbb{R})$ is tail-log-convex and long-tailed.
2. Let $\mathcal{N} \subset \mathcal{I}$ be the set of all functions $c \in \mathcal{I}$ of the form (6.59) with $p \in \mathcal{L}$.

The reasons fot these definitions are explained by the following lemmas.

Lemma 6.58. *Let $c \in \mathcal{L}$. Then, for any $r > 0$,*

$$\lim_{|x| \to \infty} \sup_{|y| \leq r} \left| \frac{c(x+y)}{c(x)} - 1 \right| = 0. \tag{6.77}$$

Proof. Let c be constructed by a long-tailed $b \in \mathcal{D}_d(\mathbb{R})$. Take arbitrary $r > 0$ and $|x| \geq r$. Then, for any $|y| \leq r$, the monotonicity of b on $\mathbb{R}_+$ implies

$$b(|x| - r) \geq b(|x| - |y|) \geq b(|x + y|) \geq b(|x| + |y|) \geq b(|x| + r).$$

Therefore, for such values of x and y,

$$\left|\frac{c(x+y)}{c(x)} - 1\right| = \left|\frac{b(|x+y|)}{b(|x|)} - 1\right| \leq \max\left\{1 - \frac{b(|x|+r)}{b(|x|)}, \frac{b(|x|-r)}{b(|x|)} - 1\right\},$$

and hence (6.1) implies (6.77). □

Remark 6.59. Note that we have not used here (as well as in the following Lemma) that b is tail-log-convex.

Lemma 6.60. *Let $c \in \mathcal{N}$. Then*

$$\lim_{\langle x\rangle\to\infty} \frac{c(x+h)}{c(x)} = 1, \quad h \in \mathbb{R}^d_+. \tag{6.78}$$

Proof. Let c be constructed by a long-tailed $b \in \mathcal{D}_d(\mathbb{R})$. Take an arbitrary $h \in \mathbb{R}^d_+$, $h \neq 0$. Use Remark 6.4; namely, fix any $R > \langle h\rangle$, and note that, for any $y \in \mathbb{R}^d$, with $y_j \in [x_j, x_j + R]$, $1 \leq j \leq d$, one has

$$\big||y| - |x|\big| \leq |y - x| \leq R\sqrt{d}.$$

Then, by (6.2), for any such y,

$$\left|\frac{b(|y|)}{b(|x|)} - 1\right| \leq \sup_{|\tau|\leq R\sqrt{d}|}\left|\frac{b(|x|+\tau)}{b(|x|)} - 1\right| \to 0, \quad |x| \to \infty.$$

Therefore, for any $\varepsilon \in (0,1)$, there exists $r = r(\varepsilon, R)$, such that, for all $x \in \mathbb{R}^d$ with $\langle x\rangle \geq r$ (that implies $|x| \geq r$, by (6.65)), one has

$$1 - \varepsilon \leq \frac{b(|y|)}{b(|x|)} \leq 1 + \varepsilon, \quad y_j \in [x_j, x_j + R],\ 1 \leq j \leq d.$$

As a result,

$$\begin{aligned} 1 - \frac{c(x+h)}{c(x)} &= \frac{\displaystyle\int_{x_1}^{x_1+h_1} \cdots \int_{x_d}^{x_d+h_d} b(|y|)\,dy}{\displaystyle\int_{x_1}^{\infty} \cdots \int_{x_d}^{\infty} b(|y|)\,dy} \leq \frac{\displaystyle\int_{x_1}^{x_1+\langle h\rangle} \cdots \int_{x_d}^{x_d+\langle h\rangle} \frac{b(|y|)}{b(|x|)}\,dy}{\displaystyle\int_{x_1}^{x_1+R} \cdots \int_{x_d}^{x_d+R} \frac{b(|y|)}{b(|x|)}\,dy} \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\frac{\langle h\rangle^d}{R^d} < \varepsilon, \end{aligned}$$

provided that $R = R(\langle h\rangle, \varepsilon) > \langle h\rangle$ is chosen big enough. The statement is proved. □

Remark 6.61. Note that all previous results remain true if $c \in \mathcal{M}$ is defined by (6.59) with $\Delta(x)$ replaced by $\Delta(x + x_0)$ for a fixed $x_0 \in \mathbb{R}^d$.

6.5 Domain of uniform convergence to positive constant solution

In this Section, we will present sufficient conditions on the kernel a^+ and the initial condition u_0 to the equation (2.1) which imply that there exist a set Λ_t (not necessary bounded) such that $\Lambda_t \nearrow \mathbb{R}^d$, $t \to \infty$ and $\operatorname*{essinf}_{x\in\Lambda_t} u(x,t) \to \theta$, $t \to \infty$, where u is the corresponding solution to (2.1).

Let $c \in \mathcal{R} \cup \mathcal{M}$, $\varepsilon \in (0,1)$, and $\Lambda_\varepsilon^-(t,c)$ be given by (6.62). For any $\lambda > 0$, we define the function

$$g(x,t) = g_{c,\varepsilon,\lambda}(x,t) = \lambda \min\bigl\{1, c(x)e^{\beta_\varepsilon^- t}\bigr\} \tag{6.79}$$

$$= \lambda 1\!\!1_{\Lambda_\varepsilon^-(t,c)}(x) + \lambda c(x) e^{\beta_\varepsilon^- t} 1\!\!1_{\mathbb{R}^d \setminus \Lambda_\varepsilon^-(t,c)}(x), \quad x \in \mathbb{R}^d, t \geq 0. \tag{6.80}$$

Lemma 6.62. *Let $c \in \mathcal{L}$ be given by a (long-tailed and tail-log-convex) function $b \in \mathcal{D}_d(\mathbb{R})$, and $\rho > 0$ be such that $b(\rho) \leq 1$. Define, for any $\lambda > 0$, $\varepsilon \in (0,1)$,*

$$f(s,t) := \lambda 1\!\!1_{s \leq \eta_\varepsilon^-(t)} + \lambda e^{\beta_\varepsilon^- t} b(s)\, 1\!\!1_{s > \eta_\varepsilon^-(t)} \in [0,\lambda], \quad s \in \mathbb{R}_+,\ t \geq t_{\rho,\varepsilon}^-, \tag{6.81}$$

i.e. $g(x,t) = f(|x|,t)$, where g is given by (6.80). Then, for any $\tau > 0$,

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}_+} \left| \frac{f(s+\tau,t)}{f(s,t)} - 1 \right| = 0. \tag{6.82}$$

Proof. Take an arbitrary $\varepsilon \in (0,1)$. For an arbitrary fixed $\tau \in \mathbb{R}_+$, choose $t_0 \geq t_{\rho,\varepsilon}^-$, such that $\eta_\varepsilon^-(t_0) \geq \tau$. Then, for any $t \geq t_0$, the function $F_{\tau,t}(s) := \frac{f(s+\tau,t)}{f(s,t)}$ takes the following values. For $0 \leq s \leq \eta_\varepsilon^-(t) - \tau$, one has $F_{\tau,t}(s) = 1$. For $\eta_\varepsilon^-(t) - \tau < s \leq \eta_\varepsilon^-(t)$, we have $F_{\tau,t}(s) = e^{\beta_\varepsilon^- t} b(s+\tau)$ and, since b is decreasing on $[\eta_\varepsilon^-(t), \infty)$, one gets

$$\frac{b(\eta_\varepsilon^-(t)+\tau)}{b(\eta_\varepsilon^-(t))} = e^{\beta_\varepsilon^- t} b(\eta_\varepsilon^-(t)+\tau) \leq e^{\beta_\varepsilon^- t} b(s+\tau) \leq e^{\beta_\varepsilon^- t} b(\eta_\varepsilon^-(t)) = 1.$$

Finally, for $s > \eta_\varepsilon^-(t)$, we have, $F_{\tau,t}(s) = \frac{b(s+\tau)}{b(s)} \leq 1$ (since b is decreasing) and, by Lemma 6.11,

$$\frac{b(s+\tau)}{b(s)} \geq \frac{b(\eta_\varepsilon^-(t)+\tau)}{b(\eta_\varepsilon^-(t))}.$$

As a result, for all $s \in \mathbb{R}_+$,

$$0 \leq 1 - F_{\tau,t}(s) \leq 1\!\!1_{\{s > \eta_\varepsilon^-(t) - \tau\}}(s) \biggl(1 - \frac{b(\eta_\varepsilon^-(t)+\tau)}{b(\eta_\varepsilon^-(t))}\biggr), \tag{6.83}$$

that implies the statement because of (6.1). □

Lemma 6.63. *Let $c \in \mathcal{N}$ and g be given by (6.80). Then, for any $h \in \mathbb{R}_+^d$,*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \left| \frac{g(x+h,t)}{g(x,t)} - 1 \right| = 0. \tag{6.84}$$

Proof. Take an arbitrary $x \in \mathbb{R}^d$ and $h \in \mathbb{R}_+^d$. By the monotonicity of functions (6.57), we have $c(x+h) \leq c(x)$. Next, it is easy to see that $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^-(t,c)$ implies $x+h \in \mathbb{R}^d \setminus \Lambda_\varepsilon^-(t,c)$, and hence

$$\frac{g(x+h,t)}{g(x,t)} = \frac{c(x+h)}{c(x)} \leq 1.$$

Let $x \in \Lambda_\varepsilon^-(t,c)$. If $x+h \in \Lambda_\varepsilon^-(t,c)$, then $\frac{g(x+h,t)}{g(x,t)} = 1$. Let now h be such that $x+h \in \mathbb{R}^d \setminus \Lambda_\varepsilon^-(t,c)$. Then

$$\frac{g(x+h,t)}{g(x,t)} = e^{\beta_\varepsilon^- t} c(x+h) \leq 1.$$

Moreover, since $x \in \Lambda_\varepsilon^-(t,c)$ implies $c(x)e^{\beta_\varepsilon^- t} \geq 1$, one has for such x, h the following estimate

$$0 \leq 1 - \frac{g(x+h,t)}{g(x,t)} \leq 1 - \frac{c(x+h)}{c(x)}. \tag{6.85}$$

As a result,

$$\left| \frac{g(x+h,t)}{g(x,t)} - 1 \right| = 1 - \frac{g(x+h,t)}{g(x,t)} \leq \sup_{y : c(y+h) < e^{-\beta_\varepsilon^- t}} \left(1 - \frac{c(y+h)}{c(y)} \right).$$

Because of (6.78), for the chosen $h \in \mathbb{R}_+^d$ and for an arbitrary $\delta > 0$, there exists $\rho = \rho(\delta, h) > 0$, such that $\sup\limits_{1 \leq j \leq d} y_j > \rho$ implies

$$0 \leq 1 - \frac{c(y+h)}{c(y)} \leq \delta.$$

Choose now $t_0 = t_0(\rho, \varepsilon, h) = t_0(\delta, \varepsilon, h)$, such that $c\bigl((\rho, \ldots, \rho) + h\bigr) > e^{-\beta_\varepsilon^- t_0}$. Prove that then, for any $t \geq t_0$, the inequality $c(y+h) \leq e^{-\beta_\varepsilon^- t}$ implies $\sup\limits_{1 \leq j \leq d} y_j > \rho$. Indeed, on the contrary, suppose that, for some $t \geq t_0$, the inequality $c(y+h) \leq e^{-\beta_\varepsilon^- t}$ holds, however, $\sup\limits_{1 \leq j \leq d} y_j \leq \rho$. The latter yields

$$e^{-\beta_\varepsilon^- t} \geq c(y+h) \geq c\bigl((\rho, \ldots, \rho) + h\bigr) > e^{-\beta_\varepsilon^- t_0},$$

that contradicts to that $t \geq t_0$. As a result, for all $x \in \mathbb{R}^d$ and $t > t_0$,

$$\left| \frac{g(x+h,t)}{g(x,t)} - 1 \right| \leq \sup_{y : \sup\limits_{1 \leq j \leq d} y_j > \rho} \left(1 - \frac{c(y+h)}{c(y)} \right) < \delta,$$

that implies the statement. □

Definition 6.64. A function $w : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *sub-solution* to (2.1) on $[\tau, \infty)$ for some $\tau \geq 0$, if

$$\frac{\partial w}{\partial t}(x,t) - F(w)(x,t) \leq 0,$$

for a.a. $x \in \mathbb{R}^d$ and for all $t \in [\tau, \infty)$, where

$$F(w)(x,t) := \varkappa^+ (a^+ * w)(x,t) - m w(x,t) - \varkappa^- w(x,t)(a^- * w)(x,t).$$

From Theorem 3.1, we immediately get the following result.

Proposition 6.65. *Let* (A1), (A2) *hold. Let* $0 \leq u \leq \theta$ *be a solution to* (2.1), *and* $w : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a sub-solution to* (2.1) *on* $[\tau, \infty)$, *for some* $\tau \geq 0$. *Suppose that, for some* $t_0, t_1 \geq \tau$, *we have* $u(x, t_0) \geq w(x, t_1)$, *for a.a.* $x \in \mathbb{R}^d$. *Then, for all* $t \geq 0$,

$$u(x, t+t_0) \geq w(x, t+t_1), \quad \textit{for a.a. } x \in \mathbb{R}^d.$$

Proposition 6.66. *Let* (A1), (A2) *hold and* $c \in \mathcal{L} \cup \mathcal{N}$. *Then, for any* $\varepsilon \in (0,1)$ *and* $\lambda_0 \in (0, \varepsilon\theta)$, *there exists* $\tau_0 = \tau_0(\varepsilon, \lambda_0) > 0$, *such that, for any* $\lambda \in [0, \lambda_0]$, *the function* $g = g(x,t)$, *given by* (6.80), *is a sub-solution to* (2.1) *on* $[\tau_0, \infty)$.

Proof. Let, under assumptions (A1), (A2), J_λ be defined by (3.19), $\lambda \in (0, \theta]$. We have then

$$\frac{\partial}{\partial t} g\left(x,t\right) = \lambda \beta_\varepsilon^- e^{\beta_\varepsilon^- t} b\left(|x|\right) 1\!\!1_{|x| > \eta_\varepsilon^-(t)} = \beta_\varepsilon^- g(x,t) 1\!\!1_{|x|>\eta_\varepsilon^-(t)}; \tag{6.86}$$

and

$$\begin{aligned} F\left(g\right) &= \varkappa^+ a^+ * g - mg - \kappa_2 g \left(a^- * g\right) - \kappa_1 g^2 \\ &= \left(\varkappa^+ a^+ - \lambda\kappa_2 a^-\right) * g - (m + \kappa_1\lambda) g + \left(\kappa_2 \left(a^- * g\right) + \kappa_1 g\right)\left(\lambda - g\right) \\ &\geq J_\lambda * g - (m + \kappa_1 \lambda) g. \end{aligned} \tag{6.87}$$

We need to prove that

$$\frac{\partial w}{\partial t}(x,t) - F(w)(x,t) \leq 0.$$

1. Let $c \in \mathcal{L}$, $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Let $\rho \geq 0$ be such that $b(\rho) \leq 1$, and $t_{\rho,\varepsilon}^-$ and $\eta_\varepsilon^-(t) = \eta_\varepsilon^-(t,b)$ be given by (6.29) and (6.32), correspondingly. Since f given by (6.81) is decreasing in its first coordinate, we have

$$\begin{aligned} (J_\lambda * g)(x,t) &= \int_{\mathbb{R}^d} J_\lambda(-y) g(x+y,t)dy = \int_{\mathbb{R}^d} J_\lambda(-y) f(|x+y|,t)dy \\ &\geq \int_{\mathbb{R}^d} J_\lambda(-y) f(|x|+|y|,t)dy = \int_{\mathbb{R}^d} J_\lambda(y) f(|x|+|y|,t)dy \\ &= g(x,t)\int_{\mathbb{R}^d} J_\lambda(y)\frac{f(|x|+|y|,t)}{f(|x|,t)}dy, \end{aligned} \tag{6.88}$$

for a.a. $x \in \mathbb{R}^d$. Note that, by (6.83),

$$0 < \frac{f(|x|+|y|,t)}{f(|x|,t)} \leq 1, \quad x,y \in \mathbb{R}^d, \ t \in \mathbb{R}_+.$$

Then, by (6.82) and the dominated convergence theorem, one gets

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} J_\lambda(y) \sup_{x\in\mathbb{R}^d}\left|\frac{f(|x|+|y|,t)}{f(|x|,t)} - 1\right| dy = 0. \tag{6.89}$$

Since

$$\int_{\mathbb{R}^d} J_\lambda(y)dy = \varkappa^+ - \lambda\kappa_2 > m + \kappa_1\lambda > 0,$$

one can get from (6.89), that, for any (small enough later) $\varepsilon_1 \in (0,1)$, there exists a $\tau_0 \geq t_{\rho,\varepsilon}^-$, such that, for all $t \geq \tau_0$ and for all $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} J_\lambda(y)\frac{f(|x|+|y|,t)}{f(|x|,t)}dy > (1-\varepsilon_1)(\varkappa^+ - \lambda\kappa_2). \tag{6.90}$$

Then, combining (6.86)–(6.90), one gets, for all $t \geq \tau_0$, $x \in \mathbb{R}^d$,

$$\begin{aligned} -\frac{\partial}{\partial t} g\left(x,t\right) + F\left(g\right)\left(x,t\right) &\geq g(x,t)\Big((1-\varepsilon_1)(\varkappa^+ - \lambda\kappa_2) - m - \kappa_1\lambda - \beta_\varepsilon^- 1_{|x|>\eta_\varepsilon^-(t)}\Big) \\ &\geq g(x,t)\Big((1-\varepsilon_1)(\varkappa^+ - \lambda\kappa_2) - m - \kappa_1\lambda - \beta_\varepsilon^-\Big) > 0, \end{aligned}$$

if only

$$\varepsilon_1 \le 1 - \frac{m + \beta_\varepsilon^+ + \lambda\kappa_1}{\varkappa^+ - \lambda\kappa_2} = \frac{\varkappa^-(\varepsilon\theta - \lambda)}{\varkappa^+ - \lambda\varkappa^-}.$$

Evidently, to ensure the latter inequality, it is sufficient to take, for any fixed $\lambda_0 \in (0, \varepsilon\theta)$,

$$\varepsilon_1 < \frac{\varkappa^-(\varepsilon\theta - \lambda_0)}{\varkappa^+}, \tag{6.91}$$

and $\lambda \in [0, \lambda_0]$. The proof, for $c \in \mathcal{R}$, is fulfilled.

2. Let $c \in \mathcal{N}$. Denote, for any $y \in \mathbb{R}^d$,

$$y^+ := \bigl(|y_1|, \ldots, |y_d|\bigr) \in \mathbb{R}^d_+. \tag{6.92}$$

Since the function c is decreasing along all basis directions (i.e. the functions (6.57) are all decreasing, $j = 1, \ldots, d$), we easily get that the function g given by (6.80) has the same property (in x). Therefore, since $y_j \le y_j^+$, $j = 1, \ldots, d$, one gets

$$g(x + y, t) \ge g(x + y^+, t).$$

Therefore, we will have, instead of (6.88),

$$\begin{aligned}(J_\lambda * g)(x,t) &= \int_{\mathbb{R}^d} J_\lambda(-y) g(x+y,t)\, dy \ge \int_{\mathbb{R}^d} J_\lambda(-y) g(x+y^+,t)\, dy \\ &= g(x,t) \int_{\mathbb{R}^d} J_\lambda(y) \biggl(\frac{g(x+y^+,t)}{g(x,t)} - 1 \biggr) dy + g(x,t) \int_{\mathbb{R}^d} J_\lambda(y)\, dy.\end{aligned}$$

Taking into account (6.84), for $h = y^+$, the rest of the proof is fully analogous to the first part. □

Theorem 6.67. *Let the assumptions* (A1), (A2) *and* (A10) *hold. Let $0 \le u_0 \le \theta$ be such that there exist $x_0 \in \mathbb{R}^d$, $\mu_0 \in (0, \theta)$, $\delta_0 > 0$, such that $u_0(x) \ge \mu_0$, for a.a. $x \in B_{\delta_0}(0)$. Suppose also that there exists $c \in \mathcal{L} \cup \mathcal{N}$, such that*

$$(a^+ * u_0)(x) \ge c(x), \quad x \in \mathbb{R}^d. \tag{6.93}$$

Let $\varepsilon \in (0,1)$ and let $\Lambda_\varepsilon^-(t,c)$ be given by (6.62). Then

$$\lim_{t\to\infty} \operatorname*{essinf}_{x \in \Lambda_\varepsilon^-(t,c)} u(x,t) = \theta. \tag{6.94}$$

Proof. We can rewrite (2.1) in the following form:

$$\frac{\partial}{\partial t} u = \varkappa^+(a^+ * u) - \varkappa^+ u + \kappa_2 u(\theta - a^- * u) + \kappa_1 u(\theta - u).$$

Since the solution does exist and $0 \le u \le \theta$, we have, for all $t > 0$ and a.a. $x \in \mathbb{R}^d$,

$$\begin{aligned} u(x,t) &= e^{-\varkappa^+ t} u_0(x) + \varkappa^+ \int_0^t e^{-\varkappa^+(t-s)} (a^+ * u)(x,s) ds \\ &\quad + \int_0^t e^{-\varkappa^+(t-s)} u(x,s) \Bigl(\kappa_2\bigl(\theta - (a^- * u)(x,s)\bigr) + \kappa_1\bigl(\theta - u(x,s)\bigr)\Bigr) ds \\ &\ge e^{-\varkappa^+ t} u_0(x) + \varkappa^+ \int_0^t e^{-\varkappa^+(t-s)} (a^+ * u)(x,s) ds. \end{aligned} \tag{6.95}$$

The same inequality for $u(x,s)$ implies

$$\begin{aligned} u(x,t) &\geq \varkappa^+ \int_0^t e^{-\varkappa^+(t-s)} (a^+ * u)(x,s) ds \\ &\geq \varkappa^+ \int_0^t e^{-\varkappa^+(t-s)} e^{-\varkappa^+ s} (a^+ * u_0)(x) ds \\ &= \varkappa^+ t e^{-\varkappa^+ t} (a^+ * u_0)(x) \geq \varkappa^+ t e^{-\varkappa^+ t} c(x), \end{aligned} \tag{6.96}$$

for all $t \geq 0$ and a.a. $x \in \mathbb{R}^d$, because of (6.93).

Fix an arbitrary $\varepsilon \in (0,1)$. Take any $\delta \in (0,\varepsilon)$ and $\lambda_0 \in (0, \delta\theta)$ and consider $\tau_0 = \tau_0(\delta, \lambda_0)$ given by Proposition 6.66. Set now

$$\lambda := \min\bigl\{\lambda_0, \varkappa^+ \tau_0 e^{-(\varkappa^+ + \beta_\delta^-)\tau_0}\bigr\}. \tag{6.97}$$

Then, by (6.96) and (6.79), we have, for a.a. $x \in \mathbb{R}^d$,

$$u(x,\tau_0) \geq \lambda e^{\beta_\delta^- \tau_0} c(x) \geq \lambda \min\bigl\{e^{\beta_\delta^- \tau_0} c(x), 1\bigr\} = g_{c,\delta,\lambda}(x,\tau_0). \tag{6.98}$$

Therefore, by Propositions 6.66 and 6.65, one gets, for any $\tau \geq 0$,

$$u(x, \tau_0 + \tau) \geq g_{c,\delta,\lambda}(x, \tau_0 + \tau), \quad \text{for a.a. } x \in \mathbb{R}^d.$$

As a result,

$$u(x, \tau_0 + \tau) \geq \lambda, \quad \text{for a.a. } x \in \Lambda_\delta^-(\tau_0 + \tau, c), \tau \geq 0. \tag{6.99}$$

From now we will distinguish two cases.

1. Let $c \in \mathcal{L}$, $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Fix $\tau \geq 0$. Since (6.63) holds, we have that the set

$$\begin{aligned} \widetilde{\Lambda} : &= \{y \in \mathbb{R}^d : B_1(y) \subset \Lambda_\delta^-(\tau_0 + \tau, c)\} \\ &= \bigl\{y \in \mathbb{R}^d : B_1(y) \subset B_{\eta_\delta^-(\tau_0+\tau,b)}(0)\bigr\}. \end{aligned}$$

is nothing but $B_{\eta_\delta^-(\tau_0+\tau,b)-1}(0)$ and, moreover,

$$\Lambda_\delta^-(\tau_0 + \tau, c) = \bigcup_{y \in \widetilde{\Lambda}} B_1(y). \tag{6.100}$$

Take and fix now an arbitrary $y \in \widetilde{\Lambda}$, i.e. $|y| \leq \eta_\delta^-(\tau_0 + \tau) - 1$. Then, by (6.99),

$$u(x, \tau_0 + \tau) \geq \lambda 1\!\!1_{B_1(y)}(x), \quad \text{for a.a. } x \in \mathbb{R}^d.$$

Consider now equation (2.1) with the initial condition

$$v_0(x) := u(x + y, \tau_0 + \tau) \geq \lambda 1\!\!1_{B_1(0)}(x), \quad \text{for a.a. } x \in \mathbb{R}^d.$$

Let $v(x,t)$ be the corresponding solution to (2.1). Let, for an arbitrary $z \in \mathbb{R}^d$, T_z denote the translation operator on functions on $\mathbb{R}^d$, i.e. $(T_z f)(x) = f(x - z)$, $x \in \mathbb{R}^d$. Then, by Proposition 3.16, $v_0(x) = T_{-y} u(x, \tau_0 + \tau)$ implies $v(x,t) = T_{-y} u(x, \tau_0 + \tau + t) = u(x + y, \tau_0 + \tau + t)$, $x \in \mathbb{R}^d$, for all $t \geq 0$.

Take an arbitrary $\mu \in (0,\theta)$. Apply Proposition 6.1 to the solution v with $x_0 = 0$, $\delta_0 = 1$, $\mu_0 = \lambda$; then there exists $t_\mu \geq 1$, such that

$$u(x+y, \tau_0 + \tau + t) = v(x,t) \geq \mu, \quad \text{for a.a. } x \in B_1(0),$$

for all $t \geq t_\mu$. As a result, for a.a. $x \in B_1(y)$,

$$u(x, \tau_0 + s + t_\mu) \geq \mu. \tag{6.101}$$

Stress that t_μ does depends neither on y with $|y| \leq \eta_\delta^-(\tau_0+\tau) - 1$ nor on $\tau \geq 0$. As a result, by (6.100), for any $\delta \in (0,1)$, $\lambda_0 \in (0,\delta\theta)$, $\mu \in (0,\theta)$, there exist $\tau_0 = \tau_0(\delta,\lambda_0)$ and $t_\mu \geq 1$, such that, for all $\tau \geq 0$ and for a.a. x with $|x| \leq \eta_\delta^-(\tau_0+\tau)$, the inequality (6.101) holds.

Apply now Lemma 6.31, for $\varepsilon_2 := \varepsilon > \delta =: \varepsilon_1$, $t_1 = \tau_0$, $t_2 = \tau_0 + t_\mu$. One gets that there exists $\tau_1 \geq 0$, such that, for all $\tau \geq \tau_1$,

$$\eta_\varepsilon^-(\tau + \tau_0 + t_\mu) \leq \eta_\delta^-(\tau + \tau_0),$$

i.e. (6.101) holds, for all $\tau \geq \tau_1$ and a.a. x with $|x| \leq \eta_\varepsilon^-(\tau + \tau_0 + t_\mu)$. Since $\mu \in (0,\theta)$ was arbitrary, the latter fact yields (6.94).

2. Let now $c \in \mathcal{N}$. Use Remark 6.2. Namely, we consider the norm

$$|x|_\infty := |(x_1,\ldots,x_d)|_\infty := \max_{1\leq j\leq d} |x_j|,$$

in $\mathbb{R}^d$. Let $\widetilde{B}_{\frac{1}{2}}(x)$ denote the ball with the centre at an $x \in \mathbb{R}^d$ and the radius $\frac{1}{2}$ w.r.t. the $|\cdot|_\infty$-norm. Then, clearly,

$$\widetilde{B}_{\frac{1}{2}}(x) = \bigtimes_{j=1}^{d} \Bigl[x_j - \frac{1}{2}, x_j + \frac{1}{2}\Bigr] = \bigtimes_{j=1}^{d} \bigl[y_j - 1, y_j\bigr] =: C_1(y),$$

where $y_j = x_j + \frac{1}{2}$, $1 \leq j \leq d$. Stress that, by (6.62), if $c \in \mathcal{N} \subset \mathcal{M}$, i.e. the functions (6.57) are decreasing on $\mathbb{R}$, then $y \in \Lambda_\delta^-(\tau_0+\tau, c)$ implies that

$$C_1(y) \subset \Lambda_\delta^-(\tau_0 + \tau, c).$$

Therefore, cf. (6.100),

$$\Lambda_\delta^-(\tau_0+\tau, c) = \bigcup_{y \in \Lambda_\delta^-(\tau_0+\tau,c)} C_1(y). \tag{6.102}$$

Hence, one can just repeat the previous proof by changing $B_1(y)$ on $C_1(y)$, $y \in \Lambda_\delta^-(\tau_0+\tau,c)$ and using Proposition 6.1 and Remark 6.2. □

The following proposition gives a sufficient condition for (6.93); the result is a generalisation of [33, Theorem 4.2].

Proposition 6.68. *Let $f \in L^1(\mathbb{R}^d \to \mathbb{R}_+)$ and $c : \mathbb{R}^d \to \mathbb{R}_+$ be a bounded function, such that (6.77) holds (e.g. $c \in \mathcal{L}$). Then*

$$\liminf_{|x|\to\infty} \frac{(c*f)(x)}{c(x)} \geq \int_{\mathbb{R}^d} f(y)\,dy. \tag{6.103}$$

In particular, if, additionally, $c(x) > 0$, $x \in \mathbb{R}^d$, then there exists $D > 0$ such that

$$(c*f)(x) \geq Dc(x), \quad x \in \mathbb{R}^d. \tag{6.104}$$

Proof. For any $r > 0$, we have

$$\frac{(c * f)(x)}{c(x)} \geq \int_{|y| \leq r} \frac{c(x-y)}{c(x)} f(y)\, dy$$
$$\geq \biggl(1 - \sup_{|y|\leq r} \Bigl| \frac{c(x-y)}{c(x)} - 1 \Bigr| \biggr) \int_{|y|\leq r} f(y)\, dy.$$

Take an arbitrary $\delta \in (0,1)$ and choose $r = r(\delta) > 0$ such that $\int_{|y|\leq r} f(y)\, dy > (1-\delta) \int_{\mathbb{R}^d} f(y)\, dy$. Next, by (6.77), there exists $\rho = \rho(r) = \rho(\delta) \geq r$, such that $\sup\limits_{|y|\leq r} \bigl| \frac{c(x-y)}{c(x)} - 1 \bigr| < \delta$, for all $|x| \geq \rho$. As a result, for any $\delta \in (0,1)$, there exists $\rho = \rho(\delta) > 0$, such that

$$\frac{(c * f)(x)}{c(x)} > (1-\delta)^2 \int_{\mathbb{R}^d} f(y)\, dy, \quad |x| \geq \rho,$$

that yields (6.103). Finally, by e.g. Lemma 2.1, $c * f$ is a continuous function on $B_\rho(0)$; then, it is easy to see that $c(x) > 0$, $x \in \mathbb{R}^d$ implies that $(c * f)(x) > 0$, $x \in \mathbb{R}^d$. Hence the boundedness of c yields $\inf\limits_{|x|\leq\rho} \frac{(c*f)(x)}{c(x)} > 0$, that fulfilled the statement. □

The use of Theorem 6.67 and Proposition 6.68 under different relations between a^+ and u_0 is presented in Subsection 6.7.

6.6 Domain of uniform convergence to zero solution

Our aim for this Section is to get a counterpart to Theorem 6.67, namely, we are going to find sufficient conditions on a function $c \in \mathcal{L} \cup \mathcal{N}$ to get

$$\lim_{t\to\infty} \operatorname*{esssup}_{x \notin \Lambda_\varepsilon^+(t,c)} u(x,t) = 0, \tag{6.105}$$

where $\varepsilon \in (0,1)$ and $\Lambda_\varepsilon^+(t,c)$ is given by (6.62). Note that, by (6.62), if $c_1, c_2 \in \mathcal{R} \cup \mathcal{M}$ are such that $c_1(x) \leq c_2(x)$, $x \in \mathbb{R}^d$, then $\mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c_2) \subset \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c_1)$, therefore, we are interested to get (6.105), for the 'smallest possible' c, the best is to get it for the same c as in Theorem 6.67.

For a bounded function $\varpi : \mathbb{R}^d \to (0,+\infty)$, we define, for any $f : \mathbb{R}^d \to \mathbb{R}$,

$$\|f\|_\varpi := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{\varpi(x)} \in [0,\infty]. \tag{6.106}$$

If $\varpi(x) = b(|x|)$, $x \in \mathbb{R}^d$, for a bounded $b : \mathbb{R}_+ \to (0,\infty)$, we will use the notation $\|f\|_b := \|f\|_\varpi$.

Proposition 6.69 (cf. Propostion 5.2)**.** *Let a bounded function $\varpi : \mathbb{R}^d \to (0,+\infty)$ be such that, for some $\gamma \in (0,\infty)$,*

$$\frac{(a^+ * \varpi)(x)}{\varpi(x)} \leq \gamma, \quad x \in \mathbb{R}^d. \tag{6.107}$$

Let $0 \leq u_0 \in L^\infty(\mathbb{R}^d)$ and $\|u_0\|_\varpi < \infty$; let u be the corresponding solution to (2.1)*. Then*

$$\|u(\cdot,t)\|_\varpi \leq \|u_0\|_\varpi e^{(\varkappa^+\gamma - m)t}, \quad t \geq 0. \tag{6.108}$$

Proof. First we note that, for any $a \in L^1(\mathbb{R}^d)$ and a bounded ϖ, the convolution $(a^+ * \varpi)(x)$ is a bounded function on $\mathbb{R}^d$ (and even uniformly continuous, see e.g. Lemma 2.1). Next, for any $f : \mathbb{R}^d \to \mathbb{R}$, with $\|f\|_\varpi < \infty$, we have

$$\left| \frac{(a * f)(x)}{\varpi(x)} \right| \leq \int_{\mathbb{R}^d} \frac{|a(y)|\varpi(x-y)}{\varpi(x)} \frac{|f(x-y)|}{\varpi(x-y)} dy \leq \frac{|a| * \varpi(x)}{\varpi(x)} \|f\|_\varpi. \tag{6.109}$$

We will follow the notations from the proof of Theorem 2.2. Suppose that, for some $\tau \in [0, T)$, $\|u_\tau\|_\varpi \leq \|u_0\|_\varpi e^{pt}$, for

$$p := \varkappa^+ \gamma - m. \tag{6.110}$$

Take any $v \in \mathcal{X}^+_{\tau,\Upsilon}(r)$ with Υ, r given by (2.13), (2.15), such that

$$\|v(\cdot, t)\|_\varpi \leq \|u_0\|_\varpi e^{pt}, \ t \in [\tau, \Upsilon]. \tag{6.111}$$

We will check the following inequality

$$\|(\Phi_\tau v)(\cdot, t)\|_\varpi \leq \|u_0\|_\varpi e^{pt}, t \in [\tau, \Upsilon]. \tag{6.112}$$

By (2.6), (2.7), (6.109) and (6.111), one gets, for $t \in [\tau, \Upsilon]$

$$\begin{aligned} 0 \leq \frac{(\Phi_\tau v)(x,t)}{\varpi(x)} &\leq e^{-(t-\tau)m} \frac{u_\tau(x)}{\varpi(x)} + \varkappa^+ \int_\tau^t e^{-m(t-s)} \frac{(a^+ * v)(x,s)}{\varpi(x)} ds \\ &\leq \|u_0\|_\varpi e^{-m(t-\tau)} e^{p\tau} + \varkappa^+ \|u_0\|_\varpi \frac{(a^+ * \varpi)(x)}{\varpi(x)} \int_\tau^t e^{-m(t-s)} e^{ps} ds \\ &= \|u_0\|_\varpi e^{-mt} e^{(p+m)\tau} + \frac{\varkappa^+}{p+m} \|u_0\|_\varpi e^{-mt} \gamma \bigl(e^{(p+m)t} - e^{(p+m)\tau}\bigr) \\ &= \|u_0\|_\varpi e^{pt}, \end{aligned}$$

by (6.110). Since u is the limiting function for the sequence $\Phi^n_\tau v$, $n \in \mathbb{N}$ (see the proof of Theorem 2.2), one gets the statement. □

Remark 6.70. In Proposition 5.2, we consider, for an arbitrary $\lambda > 0$, $\xi \in S^{d-1}$, the function $\varpi(x) = e^{-\lambda x \cdot \xi}$ (which is not bounded though; here and below $x \cdot y = (x, y)_{\mathbb{R}^d}$ stands for the scalar product in $\mathbb{R}^d$). Then, clearly, $\dfrac{(a^+ * \varpi)(x)}{\varpi(x)} \equiv \int_{\mathbb{R}^d} a^+(y) e^{\lambda y \cdot \xi} \, dy =: \gamma$, provided that the latter integral is finite (that is the crucial assumption to get the constant speed of the front).

Proposition 6.71. *Let a bounded function $\omega : \mathbb{R}^d \to (0, +\infty)$ be such that, for any $\lambda > 0$, the set*

$$\Omega_\lambda := \Omega_\lambda(\omega) := \bigl\{x \in \mathbb{R}^d : \omega(x) < \lambda\bigr\} \neq \emptyset, \tag{6.113}$$

whereas

$$\Omega_\lambda \searrow \emptyset, \quad \lambda \searrow 0, \tag{6.114}$$

(i.e., in particular, $\Omega_\lambda \subset \Omega_{\lambda'}$, for $\lambda < \lambda'$). Suppose further that there exists $\eta > 0$, such that

$$\lim_{\lambda \to 0+} \sup_{x \in \Omega_\lambda} \frac{(a^+ * \omega)(x)}{\omega(x)} = \eta. \tag{6.115}$$

Then, for any $\delta \in (0,1)$, there exists $\lambda = \lambda(\delta, \omega) \in (0,1)$, such that (6.107) *holds, with*

$$\varpi(x) := \omega_\lambda(x) := \min\bigl\{\lambda, \omega(x)\bigr\}, \quad x \in \mathbb{R}^d, \tag{6.116}$$

and $\gamma := \max\{1, (1+\delta)\eta\}$.

Proof. By (6.116), for an arbitrary $\lambda > 0$, we have $\omega_\lambda(x) \le \lambda$, $x \in \mathbb{R}^d$; then, by (2.2), $(a^+ * \omega_\lambda)(x) \le \lambda$, $x \in \mathbb{R}^d$, as well. In particular, cf. (6.116),

$$(a^+ * \omega_\lambda)(x) \le \omega_\lambda(x), \quad x \in \mathbb{R}^d \setminus \Omega_\lambda. \tag{6.117}$$

Next, by (6.115), for any $\delta > 0$ there exists $\lambda = \lambda(\delta) \in (0,1)$ such that

$$\left| \sup_{x \in \Omega_\lambda} \frac{(a^+ * \omega)(x)}{\omega(x)} - \eta \right| \le \delta\eta,$$

in particular,

$$(a^+ * \omega)(x) \le (1+\delta)\eta\omega(x) = (1+\delta)\eta\omega_\lambda(x), \quad x \in \Omega_\lambda. \tag{6.118}$$

Therefore, for all $x \in \Omega_\lambda$,

$$(a^+ * \omega_\lambda)(x) = (a * \omega)(x) - \bigl(a^+ * (\omega - \omega_\lambda)\bigr)(x) \le (1+\delta)\eta\omega_\lambda(x), \tag{6.119}$$

where we used the obvious inequality: $\omega \ge \omega_\lambda$. By (6.117) and (6.119), one gets the statement. □

Remark 6.72. It is easy to check that any function $\omega \in \mathcal{R} \cup \mathcal{M}$ (where $\rho = 0$ and $\rho = -\infty$, respectively) satisfies (6.113)–(6.114).

Theorem 6.73. *Let the assumption* (A1) *hold. Suppose that* $\omega \in \mathcal{R} \cup \mathcal{M}$ *be such that, cf.* (6.115),

$$\lim_{\lambda \to 0+} \sup_{x \in \Omega_\lambda} \frac{(a^+ * \omega)(x)}{\omega(x)} \le 1. \tag{6.120}$$

Let $0 \le u_0 \le \theta$ *be such that* $\|u_0\|_\omega < \infty$*, and let* $u = u(x,t)$ *be the corresponding solution to* (2.1)*. Then, for any* $\varepsilon \in (0,1)$*, there exist* $A_\varepsilon > 0$ *and* $t_0 = t_0(\varepsilon) > 0$*, such that, cf.* (6.62),

$$\operatorname*{esssup}_{x \notin \Lambda_\varepsilon^+(t,\omega)} u(x,t) \le \bigl(A_\varepsilon + \|u_0\|_\omega\bigr) e^{-\frac{\varepsilon\beta}{2}t}, \quad t \ge t_0. \tag{6.121}$$

Proof. Take an arbitrary $\varepsilon \in (0,1)$ and let $\delta = \delta(\varepsilon) \in (0,1)$ be chosen later. By Remark 6.72 and Proposition 6.71, there exists $\lambda = \lambda(\delta,\omega) = \lambda(\varepsilon,\omega) \in (0,1)$, such that (6.107) holds, with ϖ given by (6.116) and $\gamma = 1 + \delta$. Note that

$$\frac{u_0(x)}{\omega_\lambda(x)} \le \frac{\theta}{\lambda} 1\!\!1_{\mathbb{R}^d \setminus \Omega_\lambda}(x) + \frac{u_0(x)}{\omega(x)} 1\!\!1_{\Omega_\lambda}(x) \le \frac{\theta}{\lambda} + \|u_0\|_\omega < \infty, \tag{6.122}$$

and one can apply Proposition 6.69. Namely, setting $A_\varepsilon := \frac{\theta}{\lambda} > 0$, one gets from (6.122), (6.108) that, for a.a. $x \in \Omega_\lambda$ and for all $t \ge 0$,

$$\begin{aligned} u(x,t) &\le \|u_0\|_{\omega_\lambda} e^{(\varkappa^+(1+\delta)-m)t}\omega_\lambda(x) \\ &\le \bigl(A_\varepsilon + \|u_0\|_\omega\bigr) e^{(\varkappa^+(1+\delta)-m)t}\omega(x). \end{aligned} \tag{6.123}$$

By (6.62) and (6.113),

$$\mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,\omega) = \Omega_{e^{-\beta_\varepsilon^+ t}}, \quad t > 0. \tag{6.124}$$

Set $t_0 = t_0(\varepsilon) := -\frac{1}{\beta_\varepsilon^+}\log\lambda > 0$. By (6.114), cf. Remark 6.72, for any $t \geq t_0$, one gets from (6.124) that

$$\mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,\omega) \subset \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t_0,\omega) = \Omega_\lambda.$$

Hence, by (6.123), (6.124), for a.a. $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,\omega)$, one gets

$$\begin{aligned} u(x,t) &\leq \bigl(A_\varepsilon + \|u_0\|_\omega\bigr) e^{(\varkappa^+(1+\delta)-m)t}\omega(x) \\ &\leq \bigl(A_\varepsilon + \|u_0\|_\omega\bigr) e^{(\varkappa^+(1+\delta)-m)t} e^{-\beta_\varepsilon^+ t}, \end{aligned}$$

and

$$\varkappa^+(1+\delta) - m - \beta_\varepsilon^+ = \beta + \delta\varkappa^+ - \beta(1+\varepsilon) = \delta\varkappa^+ - \beta\varepsilon = -\frac{\varepsilon\beta}{2},$$

if only we set from the very beginning $\delta := \frac{\varepsilon\beta}{2\varkappa^+}$. The statement is proved. □

Remark 6.74. It is easy to see from the proof above, that the denominator 2 in the right-hand side of (6.121) can be changed on $1+\nu$, for an arbitrary $\nu \in (0,1)$; then $t_0 = t_0(\varepsilon,\nu)$.

We are going to find now, for a given a^+, a proper ω to validate (6.120). We will always assume that a^+ is bounded by a radially symmetric function, namely:

$$\begin{gathered} \text{There exists } b^+ \in \mathcal{R}, \text{ such that} \\ a^+(x) \leq b^+(|x|), \ \text{ for a.a. } x \in \mathbb{R}^d. \end{gathered} \tag{A11}$$

We start with the following sufficient condition.

Proposition 6.75. *Let* (A11) *hold with $b^+ \in \mathcal{D}_d(\mathbb{R})$ which is log-equivalent, cf. Definition 6.28, to the function b, given by*

$$b(s) := 1\!\!1_{\mathbb{R}_+}(s)\frac{M}{(1+s)^{d+\mu}}, \quad s \in \mathbb{R}, \tag{6.125}$$

for some $\mu, M > 0$. Then there exists $\alpha_0 \in (0,1)$, such that, for all $\alpha \in (\alpha_0,1)$, the function $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, satisfies (6.120).

Proof. Set $\alpha_0 := \dfrac{d+\frac{\mu}{2}}{d+\mu} \in (0,1)$. Take arbitrary $\alpha \in (\alpha_0,1)$ and $\epsilon \in (0,1-\alpha)$. Take also an arbitrary $\delta \in (0,1)$, and define $h(s) = s^\delta$, $s > 0$. By (6.27), applied to $b_1 = b$ and $b_2 = b^+$, there exists $s_\delta > 2r$ such that, for all $s > s_\delta$,

$$h(s) < \frac{s}{2}, \qquad b^+(s) \leq \bigl(b(s)\bigr)^{1-\epsilon}. \tag{6.126}$$

For an arbitrary $x \in \mathbb{R}^d$ with $|x| > s_\delta$, we have a disjoint expansion $\mathbb{R}^d = D_1(x) \sqcup D_2(x) \sqcup D_3(x)$, where

$$\begin{gathered} D_1(x) := \bigl\{|y| \leq h(|x|)\bigr\}, \quad D_2(x) := \Bigl\{h(|x|) < |y| \leq \frac{|x|}{2}\Bigr\}, \\ D_3(x) = \Bigl\{|y| \geq \frac{|x|}{2}\Bigr\}. \end{gathered}$$

Then, $\frac{(a^+ * \omega)(x)}{\omega(x)} = I_1(x) + I_2(x) + I_3(x)$, where

$$I_j(x) := \int_{D_j(x)} a^+(y) \frac{(1+|x|)^{(d+\mu)\alpha}}{(1+|x-y|)^{(d+\mu)\alpha}} dy, \quad j = 1,2,3.$$

Using the inequality $|x-y| \geq \big||x|-|y|\big|$, $x,y\in\mathbb{R}^d$, one has that $|x-y|\geq |x|-|y|\geq |x|-|x|^\delta$ for $y\in D_1(x)$, $|x|>s_\delta$. Then

$$I_1(x) \leq \biggl(\frac{1+|x|}{1+|x|-|x|^\delta}\biggr)^{(d+\mu)\alpha}\int_{D_1(x)} a^+(y)dy \to 1, \quad |x|\to\infty.$$

Next, we evidently have, for any $|y|<\frac{|x|}{2}$, that $1+|x-y|\geq 1+|x|-|y|\geq \frac{1}{2}(1+|x|)$; therefore,

$$I_2(x) \leq 2^{(d+\mu)\alpha}\int_{\{|y|>|x|^\delta\}} a^+(y)dy \to 0, \quad |x|\to\infty.$$

Finally, by (A11) and (6.126), the inclusions $y\in D_3(x)$ and $|x|>s_\delta$ imply

$$a^+(y)\leq b^+(|y|)\leq b(|y|)^{1-\epsilon}\leq b\Bigl(\frac{|x|}{2}\Bigr)^{1-\epsilon},$$

and, therefore,

$$\begin{aligned}
I_3(x) &\leq M\frac{(1+|x|)^{(d+\mu)\alpha}}{\Bigl(1+\frac{|x|}{2}\Bigr)^{(d+\mu)(1-\epsilon)}}\int_{D_3(x)}\frac{1}{(1+|x-y|)^{(d+\mu)\alpha_0}}dy\\
&\leq M\frac{(1+|x|)^{(d+\mu)\alpha}}{\Bigl(1+\frac{|x|}{2}\Bigr)^{(d+\mu)(1-\epsilon)}}\int_{\mathbb{R}^d}\frac{1}{(1+|y|)^{d+\frac{\mu}{2}}}dy\to 0, \quad |x|\to\infty,
\end{aligned}$$

as $1-\epsilon>\alpha$. Since b is decreasing on $\mathbb{R}_+$, we have, by (6.113), that, for any $\lambda>0$, there exists $\rho_\lambda>0$, such that $\Omega_\lambda=\{x\in\mathbb{R}^d : |x|>\rho_\lambda\}$. As a result, one gets (6.120) from the above. □

Therefore, under quite weak assumption (A11) on $a^+\in L^1(\mathbb{R}^d)$, with b^+ given by (6.125), one gets (6.121), for $\omega(x)=b(|x|)^\alpha$, $x\in\mathbb{R}^d$ (i.e. $\omega\in\mathcal{R}$). We are going to enhance this result in three directions. First, we will get the corresponding result for the case when a^+ is decaying quicker than any inverse polynomial. Next, we will show how to get (6.121) with an $\omega\in\mathcal{I}\subset\mathcal{M}$ (namely, in the form (6.59)). Finally, we will show how to enhance the results by dropping the α appeared.

We start with the following two lemmas.

Lemma 6.76. *Let $b\in L^1(\mathbb{R})$ be even, positive, decreasing to 0 on the whole $\mathbb{R}_+$, and right-side long-tailed function. Suppose that there exist $B, r_b, \rho_b>0$, such that*

$$\int_{r_b}^{\infty} b(s-\tau)b(\tau)\,d\tau \leq Bb(s), \quad s>\rho_b. \tag{6.127}$$

Suppose also that

$$\lim_{|x|\to\infty}\frac{a^+(x)|x|^{d-1}}{b(|x|)}=0. \tag{6.128}$$

Then the inequality (6.120) *holds, for $\omega(x):=b(|x|)$, $x\in\mathbb{R}^d$.*

Proof. The assumption (6.128) implies that

$$g(r):=\sup_{|x|\geq r}\frac{a^+(x)|x|^{d-1}}{\omega(x)}\to 0, \quad r\to\infty. \tag{6.129}$$

Take an arbitrary $\delta \in (0,1)$. By (6.129), one can take then $r = r(\delta) > r_b$ such that $g(r) < \delta$.

Next, by Lemma 6.58, the inequality (6.77) holds, for $c = \omega$. Therefore, there exists $\rho = \rho(\delta, r) = \rho(\delta) > \max\{r, \rho_b\}$, such that

$$\sup_{|y|\leq r} \frac{\omega(x-y)}{\omega(x)} < 1+\delta, \quad |x| \geq \rho. \tag{6.130}$$

Then, by (6.129) and (6.130), we have

$$\begin{aligned}
(a^+ * \omega)(x) &= \omega(x) \int_{\{|y|\leq r\}} a^+(y) \frac{\omega(x-y)}{\omega(x)} dy \\
&\quad + \omega(x) \int_{\{|y|\geq r\}} \frac{a^+(y)|y|^{d-1}}{\omega(y)} \frac{\omega(x-y)\omega(y)}{\omega(x)|y|^{d-1}} dy \\
&\leq \omega(x)(1+\delta) \int_{|y|\leq r} a^+(y) dy \\
&\quad + g(r)\omega(x) \int_{\{|y|\geq r\}} \frac{b(|x-y|)b(|y|)}{b(|x|)|y|^{d-1}} dy,
\end{aligned}$$

and using that b is decreasing on $\mathbb{R}_+$ and the inequality $|x-y| \geq \bigl||x|-|y|\bigr|$, one gets, cf. (2.2) and recall that $g(r) < \delta$,

$$\leq \omega(x)(1+\delta) + \delta\omega(x) \int_{\{|y|\geq r\}} \frac{b\bigl(\bigl||x|-|y|\bigr|\bigr)b(|y|)}{b(|x|)|y|^{d-1}} dy,$$

and using the spherical coordinates, one gets

$$\leq \omega(x)(1+\delta) + \delta\omega(x)\gamma_d \int_r^\infty \frac{b\bigl(|x|-p\bigr)b(p)}{b(|x|)} dp, \tag{6.131}$$

where γ_d is the hyper-surface area of an d-dimensional unit sphere (note that we have omitted an absolute value, as b is even). Finally, using that $r > r_b$ and $\rho > \rho_b$, we obtain from (6.127) and (6.131) that, for any $\delta \in (0,1)$,

$$(a^+ * \omega)(x) \leq \omega(x)\bigl(1 + \delta(1+\gamma_d B)\bigr), \quad |x| > \rho(\delta),$$

that implies the statement. □

Lemma 6.77. *Let $b \in \mathcal{S}_1(\mathbb{R})$ be an even function. Suppose that there exists $\alpha' \in (0,1)$ such that*

$$\int_0^\infty b(s)^{\alpha'} s^{d-1}\, ds < \infty, \tag{6.132}$$

and, for any $\alpha \in (\alpha', 1)$,

$$\lim_{|x|\to\infty} \frac{a^+(x)}{b(|x|)^\alpha} |x|^{d-1} = 0. \tag{6.133}$$

Then there exists $\alpha_0 \in (\alpha', 1)$ such that the inequality (6.120) holds for $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, for all $\alpha \in (\alpha_0, 1)$.

Proof. We apply the second part of Theorem 6.25, for $n = 1$; note that then (6.132) implies (6.15). As a result, for any $\alpha \in (\alpha_0, 1)$, the inequality (6.20) holds; in particular, then (6.127) holds with b replaced by b^α. The latter together with (6.133) allows to apply Lemma 6.76 for b replaced by b^α, that fulfils the statement. □

Remark 6.78. Note that, by Remark 6.43, (6.132) implies that $b \in \mathcal{D}_d(\mathbb{R})$ and hence, cf. Definition 6.21, $b \in \mathcal{S}_d(\mathbb{R})$.

As a result, one gets a counterpart of Proposition 6.75, for the case when the function b^+ in (A11) decays faster than polynomial and $d > 1$.

Proposition 6.79. *Let* (A11) *hold for a function* $b^+ \in \mathcal{D}_d(\mathbb{R})$ *which is loq-equivalent to a function* $b \in \mathcal{S}_1(\mathbb{R})$. *For* $d > 1$, *we suppose, additionally, that*

$$\lim_{s\to\infty} b(s)s^\nu = 0, \qquad \text{for all } \nu \geq 1. \tag{6.134}$$

Then there exists $\alpha_0 \in (0,1)$, *such that, for all* $\alpha \in (\alpha_0, 1)$, *the function* $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$ *satisfies* (6.120).

Proof. We will use Lemma 6.77. For $d > 1$, one gets from (6.134) that, for any $\nu > 0$, there exists $\rho_\nu \geq 1$, such that $b(s) \leq s^{-\nu}$, $s > \rho_\nu$. In particular, for any $\alpha' \in (0,1)$, one has (6.132). For $d = 1$, $\sigma = 0$, we use instead that $b \in \mathcal{S}_1(\mathbb{R})$ implies (6.16), and hence we get (6.132), if only $\alpha' \in \bigl(\frac{1}{1+\delta}, 1\bigr)$.

Next, for any $d \in \mathbb{N}$, choose an arbitrary $\alpha \in (\alpha', 1)$. Then, by (A11) and (6.27) applied for $b_1 = b$ and $b_2 = b^+$, we have that, for any $\epsilon \in (0, 1-\alpha)$, there exists $\rho_\epsilon > 0$, such that, for all $|x| > \rho_\varepsilon$,

$$\frac{a^+(x)}{b(|x|)^\alpha}|x|^{d-1} \leq b(|x|)^{1-\epsilon-\alpha}|x|^{d-1} = \Bigl(b(|x|)|x|^\nu\Bigr)^{1-\epsilon-\alpha}, \tag{6.135}$$

where $\nu = \dfrac{d-1}{1-\epsilon-\alpha} \geq 0$, as $\alpha < 1-\epsilon$. Clearly, (6.135) together with (6.134), in the case $d > 1$, imply (6.133), that fulfills the statement. □

Remark 6.80. Note that, in Proposition 6.75, for the function b given by (6.125), one can choose $\alpha' \in (0,1)$ such that (6.132) holds. The same property we have checked above for the function b which satisfies assumptions of Proposition 6.79. As a result, by Remark 6.43, the functions $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$ in these Propositions are integrable for all $\alpha \in (\alpha_0, 1)$.

Now we are going to find examples of $\omega \in \mathcal{I}$ such that (6.120) holds. We start with the following definition.

Definition 6.81. Let $p \in \mathcal{R}$ be constructed by a function $b \in \mathcal{D}_d(\mathbb{R})$, i.e. $p(x) = b(|x|)$, $x \in \mathbb{R}^d$. For any $\lambda \in \bigl(0, b(0)\bigr)$, we set

$$\Theta_\lambda(p) := \bigl\{x \in \mathbb{R}^d : \Delta(x) \subset \Omega_\lambda(p)\bigr\}, \tag{6.136}$$

where $\Delta(x)$ is given by (6.55).

Remark 6.82. Let $\rho_{\lambda,b} > 0$ be the unique number such that $b\bigl(\rho_{\lambda,b}\bigr) = \lambda$. Then, evidently, $\Omega_\lambda(p) = \bigl\{x \in \mathbb{R}^d : |x| > \rho_{\lambda,b}\bigr\}$. Therefore, by (6.65), one gets, for any $x \in \mathbb{R}^d$ with $\langle x\rangle > \rho_{\lambda,b}$ and for any $y \in \Delta(x)$, that $|y| > \rho_{\lambda,b}$, and hence $y \in \Omega_\lambda(p)$. As a result,

$$\bigl\{x \in \mathbb{R}^d : \langle x\rangle > \rho_{\lambda,b}\bigr\} \subset \Theta_\lambda(p);$$

in particular, the latter set is not empty.

Proposition 6.83. *Let $p \in \mathcal{R}$ be constructed by a function $b \in \mathcal{D}_d(\mathbb{R})$. Suppose that* (6.120) *holds with $\omega = p$ and $\Omega_\lambda = \Omega_\lambda(p)$. Let $c \in \mathcal{I}$ be given by* (6.59)*. Then the following analogue to* (6.120) *holds:*

$$\lim_{\lambda\to 0+} \sup_{x\in\Theta_\lambda(p)} \frac{(a^+ * c)(x)}{c(x)} \leq 1. \tag{6.137}$$

Proof. Take an arbitrary $\delta \in (0,1)$. By (6.120) with $\omega = p$, there exists $\lambda_0 = \lambda_0(\delta)$, such that, for all $\lambda \in (0, \lambda_0)$, we have, cf. (6.118),

$$\frac{(a^+ * p)(x)}{p(x)} \leq 1 + \delta, \quad x \in \Omega_\lambda(p). \tag{6.138}$$

Next, for any $x \in \mathbb{R}^d$, one gets

$$\begin{aligned}(a^+ * c)(x) &= \int_{\mathbb{R}^d} a^+(x-y) \int_{\Delta(y)} p(z)\, dz\, dy \\ &= \int_{\mathbb{R}^d} a^+(x-y) \int_{\Delta(x)} p(z-(x-y))\, dz\, dy \\ &= \int_{\Delta(x)} (a^+ * p)(z)\, dz.\end{aligned}$$

As a result, by (6.138) and (6.136), we have that, for any $x \in \Theta_\lambda(p)$,

$$\frac{(a^+ * c)(x)}{c(x)} = \frac{\displaystyle\int_{\Delta(x)} \frac{(a^+ * p)(z)}{p(z)} p(z)\, dz}{c(x)} \leq 1 + \delta.$$

Since the latter holds for any $\lambda \in (0, \lambda_0)$, one gets the statement. □

To get from (6.137) the inequality (6.120) with $\omega = c$ and $\Omega_\lambda = \Omega_\lambda(c)$, consider the following lemma.

Lemma 6.84. *Let $p \in \mathcal{R}$ be constructed by a long-tailed function $b \in \mathcal{D}_d(\mathbb{R})$ (in particular, let $p \in \mathcal{L}$). Let $c \in \mathcal{I}$ be given by* (6.59)*. Then there exists $\lambda_1 > 0$, such that, for all $\lambda \in (0, \lambda_1)$,*

$$\Omega_\lambda(c) \subset \Theta_\lambda(p). \tag{6.139}$$

Proof. By Lemma 6.58 and Remark 6.59, we have that (6.77) holds with c replaced by p. As a result, for any $\varepsilon > 0$ and $r > 0$, there exists $R = R(\varepsilon, r) > 0$, such that

$$p(x+y) \geq (1-\varepsilon)p(x), \quad |y| \leq r, \ |x| \geq R.$$

Therefore, $x \in \Omega_\lambda(c)$ with $|x| \geq R$ implies that

$$\begin{aligned}\lambda &\geq \int_{x_1}^{x_1+\frac{r}{\sqrt{d}}} \cdots \int_{x_d}^{x_d+\frac{r}{\sqrt{d}}} b\Bigl(\sqrt{y_1^2+\ldots+y_d^2}\Bigr)\, dy_1 \ldots dy_d \\ &\geq \frac{r^d}{d^{\frac{d}{2}}} p\Bigl(x + \Bigl(\frac{r}{\sqrt{d}}, \ldots, \frac{r}{\sqrt{d}}\Bigr)\Bigr) \geq \frac{r^d}{d^{\frac{d}{2}}}(1-\varepsilon)p(x).\end{aligned}$$

Choose now $\varepsilon = \frac{1}{2}$ and $r = 2^{\frac{1}{d}}\sqrt{d} > 0$, and consider the corresponding R. Since $\lambda \downarrow 0$ if and only if $\langle x \rangle \to \infty$, there exists $\lambda_1 > 0$ such that, for all $\lambda \in (0, \lambda_1)$, the inclusion $x \in \Omega_\lambda(c)$

implies $\langle x\rangle > R$ and hence $|x| > R$. Moreover, for any $y \in \Delta(x)$, we have that $y \in \Omega_\lambda(c)$, by the monotonicity of c in each of variables; and, by (6.65), $\langle x\rangle > R$ implies $|y| > R$. As a result, for any $y \in \Delta(x)$ (including $y = x$), we have that $p(y) \le \lambda$, i.e $\Delta(x) \subset \Omega_\lambda(p)$. Then, by (6.136), $x \in \Theta_\lambda(p)$, that proves the statement. □

Theorem 6.85. *Let the assumption* (A1) *hold. Let* $b : \mathbb{R} \to (0,\infty)$ *be an even long-tailed function decreasing on* $\mathbb{R}_+$ *to* 0*, such that for some* $\alpha_0 \in \bigl(\frac{3}{4},1\bigr)$, $b^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$*; and, for any* $\alpha \in (\alpha_0,1)$*, the inequality* (6.120) *holds with* $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$*. (In particular, let the conditions of either Proposition 6.75 or Proposition 6.79 hold.) Let* $c \in \mathcal{R}\cup\mathcal{I}$ *be constructed by the function* b *in the sense of Definition 6.42. Suppose that* $0 \le u_0 \le \theta$ *and* $\|u_0\|_c < \infty$*; and let* $u = u(x,t)$ *be the corresponding solution to* (2.1)*. Then there exists* $\varepsilon_0 \in (0,1)$*, such that, for any* $\varepsilon \in (0,\varepsilon_0)$*, there exist* $A_\varepsilon > 0$ *and* $\tau = \tau(\varepsilon) > 0$*, such that*

$$\operatorname*{esssup}_{x\notin\Lambda_\varepsilon^+(t,c)} u(x,t) \le \bigl(A_\varepsilon + B\|u_0\|_c\bigr)e^{-\frac{\varepsilon\beta}{4}t}, \quad t \ge \tau, \tag{6.140}$$

where $B := \max\bigl\{1, b(0)^{1-\alpha_0}\bigr\}$.

Proof. First, we note that by Remark 6.43, $b^{\alpha_0} \in \mathcal{D}_d(\mathbb{R})$ implies $b \in \mathcal{D}_d(\mathbb{R})$. Next, let $\alpha_0 \in \bigl(\frac{3}{4},1\bigr)$ be given. For any b decreasing on $\mathbb{R}_+$ and for any $\alpha \in (\alpha_0,1)$,

$$b(|x|)^\alpha = \left(\frac{b(|x|)}{b(0)}\right)^\alpha b(0)^\alpha \ge \frac{b(|x|)}{b(0)^{1-\alpha}},$$

since $b(|x|) \le b(0)$, $x \in \mathbb{R}^d$. Let $c_\alpha \in \mathcal{R}\cup\mathcal{I}$ be given by (6.60). Then, $\|u_0\|_c < \infty$ implies

$$\frac{u_0(x)}{c_\alpha(x)} \le \frac{u_0(x)}{c(x)} b(0)^{1-\alpha} \le \|u_0\|_c \max\bigl\{1, b(0)^{1-\alpha_0}\bigr\}, \quad x \in \mathbb{R}^d,$$

i.e. $\|u_0\|_{c_\alpha} \le B\|u_0\|_c$, where $B = \max\bigl\{1, b(0)^{1-\alpha_0}\bigr\}$.

Let $\varepsilon_0 = \varepsilon_0(\alpha_0)$ by given by Theorem 6.52. Take an arbitrary $\varepsilon \in (0,\varepsilon_0)$ and consider $\alpha = \alpha(\varepsilon) \in (\alpha_0,1)$ also given by Theorem 6.52. By the assumed, (6.120) holds for $\omega = p^\alpha$, where $p(x) = b(|x|)$, $x \in \mathbb{R}^d$. Therefore, for $c \in \mathcal{R}$, one gets that (6.120) holds for $\omega = c_\alpha \in \mathcal{R}$, cf. Remark 6.44. Let now $c \in \mathcal{I}$. Since b is long-tailed, the function b^α is long-tailed as well. Then, one can use Lemma 6.84 with p replaced by b^α; one gets then, for some $\lambda_1 > 0$,

$$\Omega_\lambda(c_\alpha) \subset \Theta_\lambda(p^\alpha), \quad \lambda \in (0,\lambda_1).$$

Therefore, Proposition 6.83 implies that (6.120) holds for $\omega = c_\alpha \in \mathcal{I}$.

As a result, one can use now Theorem 6.73 with $\omega = c_\alpha \in \mathcal{R}\cup\mathcal{I}$ and ε replaced by $\frac{\varepsilon}{2}$. Namely, there exist $A_\varepsilon > 0$ and $t_0 = t_0(\varepsilon) > 0$, such that

$$\operatorname*{esssup}_{x\notin\Lambda_{\frac{\varepsilon}{2}}^+(t,c_\alpha)} u(x,t) \le \bigl(A_\varepsilon + B\|u_0\|_c\bigr)e^{-\frac{\varepsilon\beta}{4}t}, \quad t \ge t_0. \tag{6.141}$$

On the other hand, by Theorem 6.52, there exists $\tau = \tau(\varepsilon) > 0$, such that (6.67) holds, i.e.

$$\mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c) \subset \mathbb{R}^d \setminus \Lambda_{\frac{\varepsilon}{2}}^+(t,c_\alpha), \quad t \ge \tau. \tag{6.142}$$

Combining (6.141) and (6.142), one gets (6.140). □

Remark 6.86. By Corollary 6.55 and Remark 6.56, one can get (6.140) for any c constructed in the sense of Definition 6.42 by a function $b_1 : \mathbb{R} \to \mathbb{R}_+$ which is tail-decreasing only and such that $\log b_1(s) \sim \log b(s)$, $s \to \infty$.

Remark 6.87. Using a bit more cumbersome expressions for ε_0 and $\alpha = \alpha(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$ in the proof of Theorem 6.52, one can obtain (6.67) with $\frac{\varepsilon}{2}$ replaced by any $\varepsilon' \in (0, \varepsilon)$. As a result, we may apply Theorem 6.73 inside the proof of Theorem 6.85 with ε replaced by $\frac{\varepsilon}{1+\nu'}$ for any $\nu' \in (0,1)$. Combining this observation with Remark 6.74, one can get, as a result, (6.140), where, in the denominator of the right-hand side, the number 4 will be replaced by $1 + \nu''$ for an arbitrary $\nu'' \in (0,1)$, by a redefining of $\tau = \tau(\varepsilon, \nu'')$.

Remark 6.88. For $c \in \mathcal{R} \cup \mathcal{I}$, the condition $\|u_0\|_c < \infty$ separates, in some sense, the cases of 'decreasing' and 'symmetric' initial conditions. Namely, if, for example, $u_0 \in \mathcal{M}$, then the inequality $\|u_0\|_c < \infty$ is impossible for any $c \in \mathcal{R}$, cf. (6.106); and hence c must be from $\mathcal{I}$.

6.7 Corollaries and examples

The aim of this Section is to provide useful sufficient conditions on functions a^+ and u_0 to get simultaneously (6.94) and (6.140), i.e., in particular, to get that

$$\lim_{t\to\infty} \operatorname*{essinf}_{x\in\Lambda_\varepsilon^-(t,c)} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{esssup}_{x\notin\Lambda_\varepsilon^+(t,c)} u(x,t) = 0, \tag{6.143}$$

if only $c \in \mathcal{L} \cup \mathcal{N}$ and ε is small enough.

Through this section we will always assume that the assumptions (A1), (A2), (A4), (A6), (A10) hold true. As an reinforcement of (A11), consider the assumptions below.

Let $\delta, \rho > 0$, and $R_0 > 0$ be the same as in the assumptions (A4) and (A10), correspondingly. Suppose that there exist constants $\mu, M > 0$, $r \geq R_0$, a point $x_0 \in \mathbb{R}^d$, and functions $b^+ \in \mathcal{D}_d(\mathbb{R})$, $b_+ : \mathbb{R} \to \mathbb{R}_+$, $v^\circ \in \mathcal{R} \cup \mathcal{M}$, $v_\circ : \mathbb{R}^d \to [0, \theta]$, such that

$$b_+(|x|) \leq a^+(x) \leq b^+(|x|), \qquad \text{for a.a. } x \in \mathbb{R}^d; \tag{B.1}$$

$$b^+(s) \leq \frac{M}{(1+s)^{d+\mu}}, \qquad \text{for a.a. } s \geq r; \tag{B.2}$$

$$b_+(s) \geq \delta, \qquad \text{for a.a. } s \in [0, \rho]; \tag{B.3}$$

$$\theta \geq v^\circ(x) \geq u_0(x) \geq v_\circ(x), \qquad \text{for a.a. } x \in \mathbb{R}^d; \tag{B.4}$$

$$v_\circ(x) \geq \delta, \qquad \text{for a.a. } x \in B_\rho(x_0). \tag{B.5}$$

To simplify the formulations below, let us introduce the set $\widetilde{\mathcal{S}}(\mathbb{R}) \subset \mathcal{S}_d(\mathbb{R})$, $d \in \mathbb{N}$ as follows. Let $\widetilde{\mathcal{S}}_1(\mathbb{R}) := \mathcal{S}_1(\mathbb{R})$, whereas, for $d > 1$, let $\widetilde{\mathcal{S}}_d(\mathbb{R})$ be the set of all functions $b \in \mathcal{S}_d(\mathbb{R})$, such that b is either given by (6.125) for some $M, \mu > 0$ or b satisfies (6.134).

We will distinguish several cases. They are described below a bit informally, leaving the exact formulations to the corresponding Propositions (note that if $b_+ = b^+$, $v_\circ = v^\circ$ in the above, then those descriptions become exact).

Case 1. $\lim\limits_{|x|\to\infty} u_0(x) = 0$.

Subcase 1.1. $\sup\limits_{x\in\mathbb{R}^d} \dfrac{u_0(x)}{a^+(x)} < \infty$.

Proposition 6.89. *Let assumptions* (A1), (A2), (A4), (A6), (A10) *and* (B.1)–(B.5) *hold. Suppose that $b_+ \in \mathcal{D}_d(\mathbb{R})$ is a long-tailed and tail-log-convex function, and let both b_+ and b^+ be log-equivalent, cf. Definition 6.28, to a function $b \in \widetilde{\mathcal{S}}_d(\mathbb{R})$. Suppose also that*

$$\sup_{x\in\mathbb{R}^d} \frac{u_0(x)}{b(|x|)} < \infty. \tag{6.144}$$

Then there exist $\varepsilon_0 \in (0,1)$ *and* $B > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$*, there exists* $A = A(\varepsilon) > 0$ *and* $t_1 = t_1(\varepsilon) > 0$*, such that* (6.94) *and* (6.140) *both hold for* $c(x) = b(|x|)$*,* $x \in \mathbb{R}^d$*.*

Proof. Let $\varepsilon_0 \in (0,1)$ be chosen later. Take an arbitrary $\varepsilon \in (0, \varepsilon_0)$.

Let $c_+ \in \mathcal{L} \subset L^1(\mathbb{R}^d)$ be constructed by $b_+ \in \mathcal{D}_d(\mathbb{R})$, cf. Definition 6.42. Then (6.144) implies that $u_0 \in L_1(\mathbb{R}^d)$. Therefore, one can apply Proposition 6.68 with $c = c_+ > 0$ and $f = u_0$; namely, there exists $D > 0$, such that $a^+ * u_0 \geq c_+ * u_0 \geq Dc_+ \in \mathcal{L}$. Then, by Theorem 6.67, for any $\varepsilon_1 \in (0, \varepsilon)$, we have that (6.94) holds, with ε replaced by ε_1 and c replaced by Dc_+. By (6.63),

$$\Lambda^-_{\varepsilon_1}(t, Dc_+) = \bigl\{x \in \mathbb{R}^d : |x| \leq \eta^-_{\varepsilon_1}(t, Db_+)\bigr\}.$$

By the assumed, $\log(Db_+)(s) \sim \log b(s)$, $s \to \infty$. Therefore, one can apply Proposition 6.33; namely, by (6.38) with $b_1 = b$, $b_2 = Db_+$, we have that $\Lambda^-_\varepsilon(t, b) \subset \Lambda^-_{\varepsilon_1}(t, Dc_+)$, and hence (6.94) holds, with $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Note that we had not any restrictions on ε_0 here.

If $d = 1$ or if $d > 1$ and, additionally, (6.134) holds, then one can apply Proposition 6.79. If $d > 1$ and (6.134) does not hold, then, by the assumed, b is given by (6.125), and one can apply Proposition 6.75. In both cases, there exists $\alpha_1 \in (0,1)$, such that, for all $\alpha \in (\alpha_1, 1)$, the function $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, satisfies (6.120). Moreover, we have shown in the proof of Proposition 6.79, that there exists $\alpha' \in (0,1)$, such that (6.132) holds. As a result, taking any $\alpha_0 \in \bigl(\max\{\frac{3}{4}, \alpha_1, \alpha'\}, 1\bigr)$, we will fulfill all conditions of Theorem 6.85, i.e. one can choose above $\varepsilon_0 \in (0,1)$, such that (6.140) holds, with $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. □

Subcase 1.2. $\displaystyle\lim_{|x|\to\infty} \frac{a^+(x)}{u_0(x)} = 0.$

Proposition 6.90. *Let assumptions* (A1), (A2), (A4), (A6), (A10) *and* (B.1)–(B.5) *hold. Let* $v^\circ, v_\circ \in \mathcal{R}$ *be constructed by* $b_\circ, b^\circ \in \mathcal{D}_d(\mathbb{R})$*. Suppose that* $b_\circ$ *is long-tailed and tail-log-convex,* $b^\circ \in \widetilde{\mathcal{S}}_d(\mathbb{R})$*, and let both* $b_\circ$ *and* b° *be log-equivalent to a function* $b \in \widetilde{\mathcal{S}}_d(\mathbb{R})$*. Assume also that*

$$\lim_{|x|\to\infty} \frac{a^+(x)}{b^\circ(|x|)} |x|^{d-1} = 0. \tag{6.145}$$

Then there exist $\varepsilon_0 \in (0,1)$ *and* $B > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$*, there exists* $A = A(\varepsilon) > 0$ *and* $t_1 = t_1(\varepsilon) > 0$*, such that* (6.94) *and* (6.140) *both hold, for* $c(x) = b(|x|)$*,* $x \in \mathbb{R}^d$*.*

Proof. Let $\varepsilon_0 \in (0,1)$ be chosen later. Take an arbitrary $\varepsilon \in (0, \varepsilon_0)$.

The proof of (6.94) is essentially the same as that for Proposition 6.89, with only the difference that we will apply now Proposition 6.68 for $c = c_\circ > 0$ and $f = a^+ \in L^1(\mathbb{R}^d)$, where $c_\circ(x) := b_\circ(|x|)$, $x \in \mathbb{R}^d$.

To prove (6.140), we are going to apply Theorem 6.73 to $\omega(x) = b^\circ(|x|)$, $x \in \mathbb{R}^d$. Clearly, by (B.4), $\|u_0\|_\omega < \infty$. It remains to check (6.120). By the proof of Theorem 6.25, the inclusion $b^\circ \in \mathcal{S}_d(\mathbb{R})$ implies the first inequality in (6.22) with b replaced by b°. This evidently yields (6.127), also with b replaced by b°. Therefore, because of (6.145), one can use Lemma 6.76, and we get (6.120). As a result, we may apply Theorem 6.73: there exists $\varepsilon_0 \in (0,1)$ such that (6.121) holds with ε replaced by $\frac{\varepsilon}{2} < \varepsilon_0$. The rest of the proof will be similar to that in the proof of Proposition 6.89: by using the log-equivalence of b° and b, (6.63), Proposition 6.33, and the evident inequality $\|u_0\|_{b^\circ} \leq \|u_0\|_b$, one gets (6.140). □

Remark 6.91. Because of the assumption $\|u_0\|_c < \infty$ in Theorem 6.85, one has to have $b^\circ \in \widetilde{\mathcal{S}}_d(\mathbb{R})$ instead of just $b \in \widetilde{\mathcal{S}}_d(\mathbb{R})$. A sufficient condition to get the latter inclusion from the former one is given by Proposition 6.29.

Case 2. $\lim\limits_{\rho\to\infty} u_0((\rho,\ldots,\rho)) = 0,\ \lim\limits_{\rho\to-\infty} u_0((\rho,\ldots,\rho)) \in (0,\theta]$.

Subcase 2.1. $\sup\limits_{x\in\mathbb{R}^d} \dfrac{u_0(x)}{\displaystyle\int_{\Delta(x)} a^+(y)dy} < \infty.$

Proposition 6.92. *Let assumptions* (A1), (A2), (A4), (A6), (A10) *and* (B.1)–(B.5) *hold. Let all assumptions of Proposition 6.89 but* (6.144) *hold. Instead of* (6.144), *we suppose that*

1. *there exists* $\xi \in (0,\theta)$, *such that*

$$v_\circ(x) \geq \xi\, 1\!\!1_{\mathbb{R}^d_-}(x), \quad x \in \mathbb{R}^d,\ \mathbb{R}_- := (-\infty, 0], \tag{6.146}$$

2. *for the function* $c \in \mathcal{I}$ *constructed by* b, *cf. Definition 6.42, one has that*

$$\sup_{x\in\mathbb{R}^d} \frac{u_0(x)}{c(x)} < \infty. \tag{6.147}$$

Then there exist $\varepsilon_0 \in (0,1)$ *and* $B > 0$ *such that, for any* $\varepsilon \in (0,\varepsilon_0)$, *there exists* $A = A(\varepsilon) > 0$ *and* $t_1 = t_1(\varepsilon) > 0$, *such that* (6.94) *and* (6.140) *both hold for the function* c.

Proof. Let $\varepsilon_0 \in (0,1)$ be chosen later. Take an arbitrary $\varepsilon \in (0,\varepsilon_0)$. As we have mentioned above, the assumptions about q_+ and b_+ imply that $c_+ \in \mathcal{L}$, where $c_+(x) := b_+(|x|)$, $x \in \mathbb{R}^d$. Then, by (B.1),

$$\begin{aligned}(a^+ * u_0)(x) &\geq \xi \int_{\mathbb{R}^d} c_+(y) 1\!\!1_{\mathbb{R}^d_-}(x-y)\,dy \\ &= \xi \int_{\Delta(x)} c_+(y)\,dy =: \tilde{c}(x), \quad x \in \mathbb{R}^d;\end{aligned}$$

and hence $\tilde{c} \in \mathcal{N}$. Thus, one can apply Theorem 6.67 to get (6.94) with c replaced by $\tilde{c}$ and ε replaced by $\frac{\varepsilon}{2}$. Next, by the log-equivalence between b and b^+, we have that (6.26) holds for $b_1 = b$, $b_2 = \xi b_+$. Therefore, we can apply Corollary 6.55 with $c^{(1)}(x) = c(x) := \int_{\mathbb{R}^d} b(|y|)dy$, $x \in \mathbb{R}^d$, $c^{(2)} = \tilde{c}$, see also Remark 6.56; and then (6.74) leads to (6.94) for this c.

To get (6.140) we will need just to repeat all corresponding arguments from Proposition 6.89 with only the difference that Theorem 6.85 will be applied now for the $c \in \mathcal{I}$. □

Remark 6.93. Using [**?**, Proposition 3.15] in the same way as in the proof of Theorem 6.67, one can replace $\mathbb{R}^d_-$ in (6.146) by $\bigtimes_{j=1}^{d} (-\infty, \overline{y}_j]$, for an arbitrary fixed $\overline{y} \in \mathbb{R}^d$.

Remark 6.94. If, additionally, $u_0(x) = \int_{\Delta(x)} p(y)dy$, $x \in \mathbb{R}^d$ for some $p \in L^1(\mathbb{R}^d)$, then, evidently,

$$\sup_{x\in\mathbb{R}^d} \frac{p(x)}{a^+(x)} < \infty \quad \Longrightarrow \quad \sup_{x\in\mathbb{R}^d} \frac{u_0(x)}{\displaystyle\int_{\Delta(x)} a^+(y)dy} < \infty.$$

Subcase 2.2. $\lim\limits_{\langle x\rangle\to\infty} \dfrac{\int_{\Delta(x)} a^+(y)|y|^{d-1}dy}{u_0(x)} = 0.$

Proposition 6.95. *Let assumptions* (A1), (A2), (A4), (A6), (A10) *and* (B.1)–(B.5) *hold. Let* $v^\circ, v_\circ \in \mathcal{I}$ *be constructed by* $b_\circ, b^\circ \in \mathcal{D}_d(\mathbb{R})$*, cf. Definition 6.42. Suppose that all assumptions of Proposition 6.90 hold; and let* $c \in \mathcal{I}$ *be constructed by the function* b*. Then there exist* $\varepsilon_0 \in (0,1)$ *and* $B > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$*, there exists* $A = A(\varepsilon) > 0$ *and* $t_1 = t_1(\varepsilon) > 0$*, such that* (6.94) *and* (6.140) *both hold.*

Proof. First, we apply Proposition 6.68 with $f = a^+$ and c replaced by $v_\circ$. Then, similarly to the proof of Proposition 6.92, we may apply Theorem 6.67 to get (6.94) with c replaced by $v_\circ$ and ε replaced by $\frac{\varepsilon}{2}$, and, by using the log-equivalence between b and $b_\circ$, Corollary 6.55, and Remark 6.56, we will get (6.94) for the needed c.

Next, in the same way as in the proof of Proposition 6.90, we get (6.120) for $\omega(x) = b^\circ(|x|)$, $x \in \mathbb{R}^d$. Then we apply Proposition 6.83 with b replaced by b°. Hence one gets (6.137) with $c_\alpha = v^\circ$, and one can use Lemma 6.84, that implies (6.120) for $\omega = v^\circ$. Therefore, by Theorem 6.73, one has (6.140) with $\omega = v^\circ$ and ε replaced by $\frac{\varepsilon}{2}$. Again, now by the log-equivalence between b and b°, one can use Corollary 6.55 and Remark 6.56, cf. Remark 6.86, and then (6.75) yields the needed. ☐

Remark 6.96. It is easy to check that the convergence (6.145) indeed implies that

$$\lim_{\langle x\rangle\to\infty} \frac{\displaystyle\int_{\Delta(x)} a^+(y)|y|^{d-1}dy}{\displaystyle\int_{\Delta(x)} b^\circ(y)dy} = 0.$$

The following Corollary summarizes the statements above in the simplest case when $b_+ = b^+$ and $v_\circ = v^\circ$.

Corollary 6.97. *Let* $b, q : \mathbb{R} \to \mathbb{R}_+$ *be bounded functions such that* (B.2) *holds for both* $b^+ = b$ *and* $b^+ = q$*, and* $q(s) \geq \delta$*,* $s \in [0, \rho]$*. Let* (A1), (A2), (A4), (A6), (A10) *hold, and* $a^+(x) = b(|x|)$*,* $x \in \mathbb{R}^d$*. Let one of the following conditions holds*

$$\sup_{s\in\mathbb{R}_+} \frac{q(s)}{b(s)} < \infty, \tag{6.148}$$

$$\lim_{s\to\infty} \frac{b(s)}{q(s)} s^{d-1} = 0. \tag{6.149}$$

1. *Let* $u_0(x) = q(|x|)$*,* $x \in \mathbb{R}^d$ *and* $q : \mathbb{R} \to [0, \theta]$*. Then*
   (a) *if* $b \in \widetilde{\mathcal{S}}_d(\mathbb{R})$ *and* (6.148) *holds, then* (6.143) *holds with* $c = a^+$*;*
   (b) *if* $q \in \widetilde{\mathcal{S}}_d(\mathbb{R})$ *and* (6.149) *holds, then* (6.143) *holds with* $c = u_0$*.*
2. *Let* $u_0(x) = \displaystyle\int_{\Delta(x)} q(|y|)dy$*,* $x \in \mathbb{R}^d$ *with* $\displaystyle\int_0^\infty q(s)s^{d-1}ds \in (0, \theta]$*. Then*
   (a) *if* $b \in \widetilde{\mathcal{S}}_d(\mathbb{R})$ *and* (6.148) *holds, then* (6.143) *holds with*
   $$c(x) := \int_{\Delta(x)} a^+(y)dy, \quad x \in \mathbb{R}^d;$$
   (b) *if* $q \in \widetilde{\mathcal{S}}_d(\mathbb{R})$ *and* (6.149) *holds, then* (6.143) *holds with* $c = u_0$*.*

Proof. Note that $q(s) \geq \delta$, $s \in [0, \rho]$ implies

$$\int_{\Delta(x)} q(|y|)dy \geq \text{const} \cdot 1\!\!1_{\mathbb{R}^d_-}(x), \quad x \in \mathbb{R}^d.$$

All other requirements of Propositions 6.89, 6.90, 6.92, 6.95 evidently hold true. □

Thus, informally speaking, in the Case 1, one gets (6.143) with $c = a^+$ or $c = u_0$, whichever decays slowly, whereas, in the Case 2, one gets (6.143) with $c = \int_\Delta a^+$ or $c = u_0$ whichever decays slowly (and provided that, in both cases, c has ‘’heavy tails’).

Consider now several examples. In all of them we will suppose that the conditions (A1), (A2), (A4), (A6), (A10) hold, and that u_0 is separated from 0 in a neighborhood of the origin.

Example 6.98. Let, for some $\mu > 0$, $\nu \geq 0$, $r, M > 1$, and $\delta \geq 0$, one of the following two pairs of conditions hold, for a.a. $|x| \geq r$,

$$\begin{aligned} (\log|x|)^{-\nu}(1+|x|)^{-d-\mu} \leq a^+(x) \leq (\log|x|)^{\nu}(1+|x|)^{-d-\mu}, \\ u_0(x) \leq M(1+|x|)^{-d-\mu}, \end{aligned} \tag{6.150}$$

or if, additionally, $\delta + \nu > 0$,

$$\begin{aligned} (\log|x|)^{-\nu}(1+|x|)^{-d-\mu} \leq u_0(x) \leq (\log|x|)^{\nu}(1+|x|)^{-d-\mu}, \\ a^+(x) \leq M(1+|x|)^{-2d-\mu-1-\delta}, \end{aligned} \tag{6.151}$$

Then, for (6.150), we just apply Proposition 6.89 with $b(s) = (1+s)^{-(d+\mu)}$.

For (6.151), we will use Proposition 6.90 with $b(s) = (1+s)^{-(d+\mu)}$ and $b^\circ(s) = (\log s)^\nu b(s)$; then $b^\circ \in \mathcal{S}_d(\mathbb{R})$, see Subsubsection 6.3.1. Note that then

$$\frac{a^+(x)}{b^\circ(|x|)}|x|^{d-1} \leq \frac{M(1+|x|)^{d+\mu}}{(\log|x|)^\nu(1+|x|)^{2d+\mu-1+\delta}}|x|^{d-1} = \frac{M}{(\log|x|)^\nu|x|^\delta} \to 0,$$

as $|x| \to \infty$, if only $\delta + \nu > 0$.

In both cases, we will get, see again Subsubsection 6.3.1, that there exists $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{|x| \leq \exp\left(\frac{\beta(1-\varepsilon)t}{d+\mu}\right)} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{esssup}_{|x| \geq \exp\left(\frac{\beta(1+\varepsilon)t}{d+\mu}\right)} u(x,t) = 0, \tag{6.152}$$

Example 6.99. Let now $d = 1$ and, for some $r, \mu, M > 0$, $\xi \in (0, \theta)$,

$$\begin{aligned} a^+(x) = M(1+|x|)^{-1-\mu}, \quad |x| > r, \\ \xi 1\!\!1_{\mathbb{R}_+}(x) \leq u_0(x) \leq \xi \int_x^\infty a^+(y)dy \leq \theta, \quad x \in \mathbb{R}, \xi \in (0,\theta), \end{aligned}$$

and u_0 is decreasing on $\mathbb{R}$. Then the front is described via the function

$$\int_x^\infty a^+(y)dy = \frac{M}{\mu+1}x^{-\mu},$$

if x is big enough. Therefore, by (6.35) and Proposition 6.92,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x \leq \exp\left(\frac{\beta(1-\varepsilon)t}{\mu}\right)} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{esssup}_{x \geq \exp\left(\frac{\beta(1+\varepsilon)t}{\mu}\right)} u(x,t) = 0, \tag{6.153}$$

i.e. the motion of the front goes a bit faster than in (6.152) with $d = 1$.

Example 6.100. Let, for some $\nu, \mu \geq 0$, $r, M > 1$, and $\alpha \in (0,1)$, one of the following two pairs of conditions hold, for a.a. $|x| \geq r$,

$$\begin{aligned} (1+|x|)^{-\nu} \exp(-|x|^\alpha) \leq a^+(x) \leq (1+|x|)^\nu \exp(-|x|^\alpha), \\ u_0(x) \leq M \exp(-|x|^\alpha), \end{aligned} \tag{6.154}$$

or if, additionally, $\nu + \mu > d - 1$,

$$\begin{aligned} (1+|x|)^{-\nu} \exp(-|x|^\alpha) \leq u_0(x) \leq (1+|x|)^\nu \exp(-|x|^\alpha), \\ a^+(x) \leq M(1+|x|)^{-\mu} \exp(-|x|^\alpha), \end{aligned} \tag{6.155}$$

Then, the same arguments as in Example 6.98 related to the function $b(s) = \exp(-s^\alpha)$ will imply that, cf. Subsubsection 6.3.3, there exists $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{|x|\leq \left(\beta(1-\varepsilon)t\right)^{\frac{1}{\alpha}}} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{esssup}_{|x|\geq \left(\beta(1+\varepsilon)t\right)^{\frac{1}{\alpha}}} u(x,t) = 0, \tag{6.156}$$

Example 6.101. Let now $d = 1$ and, for some $r, M > 0$, $\xi \in (0, \theta)$, $\alpha \in (0, 1)$, one of the following two pairs of conditions hold

$$\begin{aligned} a^+(x) = M \exp(-|x|^\alpha), \quad |x| > r, \\ \xi 1\!\!1_{\mathbb{R}_+}(x) \leq u_0(x) \leq \xi \int_x^\infty a^+(y)dy \leq \theta, \quad x \in \mathbb{R}, \xi \in (0,\theta), \end{aligned} \tag{6.157}$$

or if, additionally, $\mu > d - 1$,

$$\begin{aligned} u_0(x) = M \exp(-x^\alpha), \quad x > r, \\ a^+(x) \leq M|x|^{\mu+\alpha-1} \exp(-|x|^\alpha), \quad |x| > r, \end{aligned} \tag{6.158}$$

and, in both cases, u_0 is decreasing on $\mathbb{R}$. Then, for (6.157), by Proposition 6.92, the front is described via the function $M \int_x^\infty \exp(-y^\alpha)dy$ for big enough x. Moreover, by Remark 6.86, the same representation will hold, up to the choice of ε, via the function

$$M \int_x^\infty y^{\alpha-1} \exp(-y^\alpha)dy = \frac{M}{\alpha} \exp(-x^\alpha), \tag{6.159}$$

since the integrands are logarithmically equivalent. Therefore,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x\leq \left(\beta(1-\varepsilon)t\right)^{\frac{1}{\alpha}}} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{esssup}_{x\geq \left(\beta(1+\varepsilon)t\right)^{\frac{1}{\alpha}}} u(x,t) = 0, \tag{6.160}$$

i.e., in contrast to Example 6.99 the motion of the front is the same as in (6.156) with $d = 1$.

For (6.158), we have from (6.159) that $u_0(x) = \int_x^\infty b(y)dy$, where b is also logarithmically equivalent to $\exp(-x^\alpha)$. Therefore, by Proposition 6.95, one gets (6.160) as well.

In the last example, one shows that an 'intermediate' front propagation is possible as well.

Example 6.102. Let, for some $M, P, r, \alpha > 0$ and for all $|x| > r$

$$\begin{aligned} a^+(x) = M \exp\bigl(-(\alpha \log |x|)^2\bigr), \\ u_0(x) \leq P \exp\bigl(-(\alpha \log |x|)^2\bigr). \end{aligned}$$

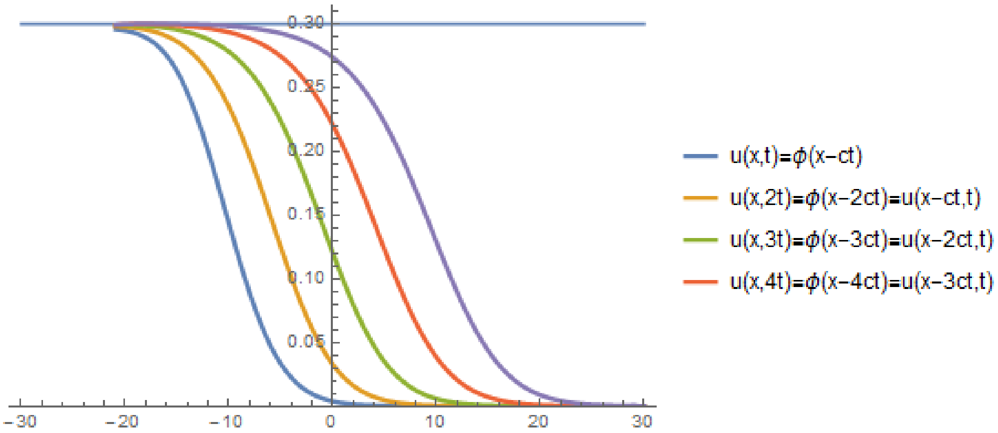
Then, by Proposition 6.89 and Subsubsection 6.3.2, one gets

$$\begin{aligned}
&\lim_{t\to\infty} \operatorname*{essinf}_{|x|\le\exp\left(\sqrt{\alpha\beta(1-\varepsilon)t}\right)} u(x,t) = \theta,\\
&\lim_{t\to\infty} \operatorname*{esssup}_{|x|\ge\exp\left(\sqrt{\alpha\beta(1+\varepsilon)t}\right)} u(x,t) = 0,
\end{aligned} \tag{6.161}$$

Similarly, using Subsubsection 6.3.4, one can construct a^+ and u_0, such that the front will be described approximately by $\beta(1\pm\varepsilon)t(\log t)^\gamma$ for any $\gamma > 1$, that demonstrates slower motion than that in (6.156).

A Pictures

1. A traveling wave. See Definition 4.3.



2. Anisotropic front propagation. For related definitions see (5.15) and (5.21).

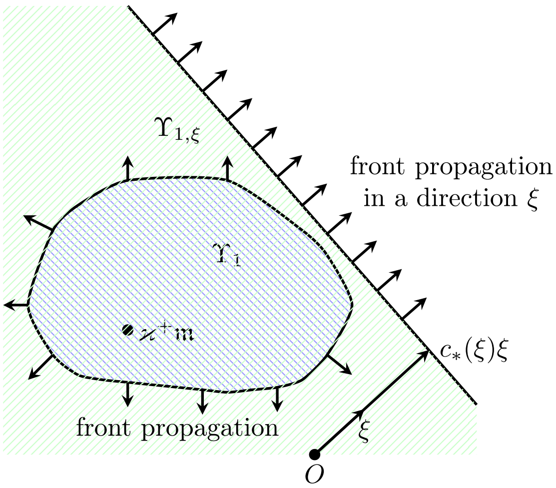


Figure 1: Relationship between the sets $\Upsilon_{1,\xi}$ and Υ_1

3. Front propagation with a constant speed. See Theorem 5.4, 5.9, 5.10.

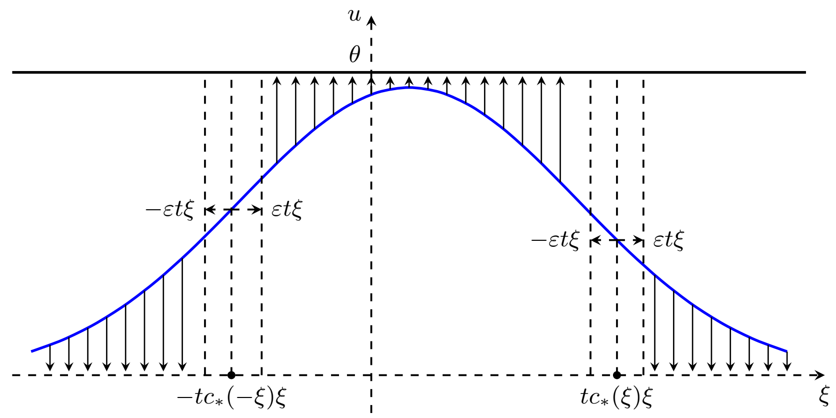


Figure 2: Space-value diagram

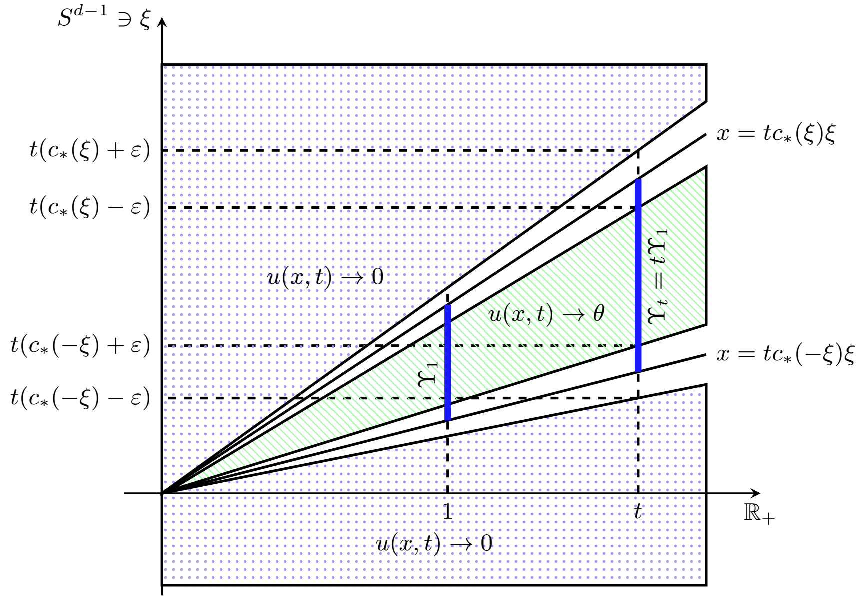


Figure 3: Space-time diagram

4. Accelerating front propagation. See Theorem 6.67, 6.73.

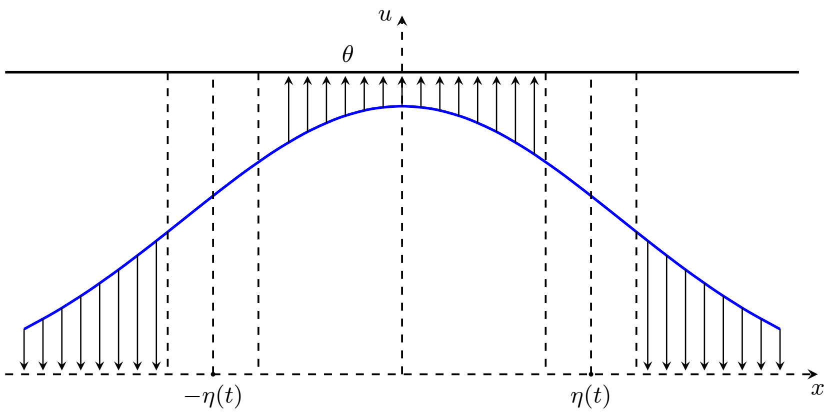


Figure 4: Space-value diagram

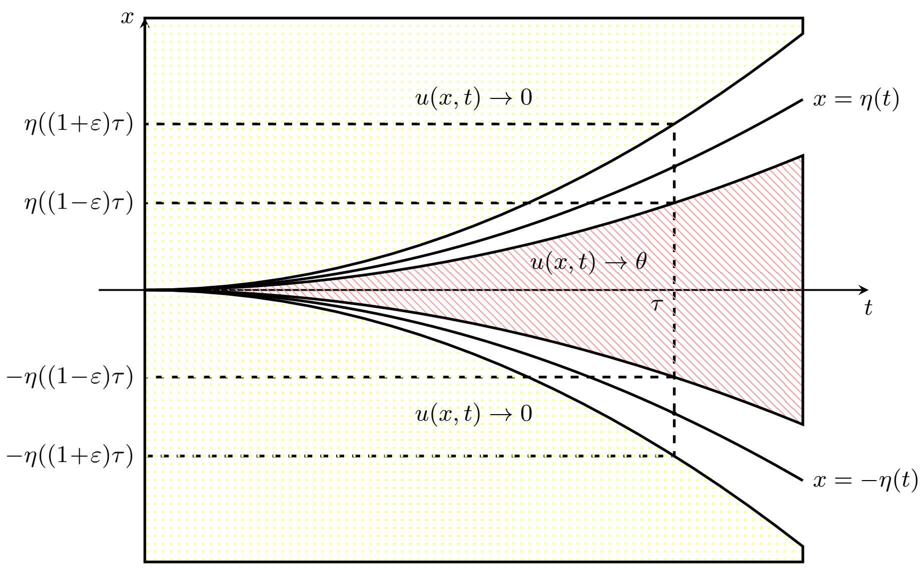


Figure 5: Space-value diagram

Acknowledgements

Financial support of DFG through CRC 701, Research Group “Stochastic Dynamics: Mathematical Theory and Applications” is gratefully acknowledged.

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