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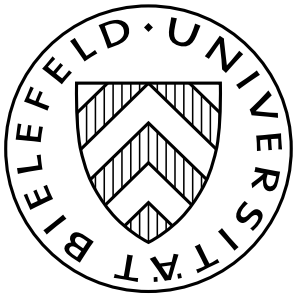

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Existence of Arrow-Debreu Equilibrium with Generalized Stochastic Differential Utility

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Abstract

This paper establishes, in the setting of Brownian information, a general equilibrium existence result under a stochastic differential formulation of intertemporal recursive utility. The present class of utility functionals is generated by a backward stochastic differential equation and incorporates preference for the local risk of the stochastic utility process.

The setting contains models in which Knightian uncertainty is represented in the subjective and objective sense.

Key words and phrases: BSDE, GSDU, super-gradients, properness, general equilibrium, Knightian uncertainty

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1 Introduction

The aim of this work is to enlarge the class of dynamic utility specification which ensure an equilibrium in continuous time and under uncertainty. We are interested in recursive preference structures which allow consideration of multiple priors.
The standard model in economics assumes an additive utility structure. Applications of these models to rational asset pricing reflects in the equality of marginal utilities and the equilibrium state price. In order to tackle the drawbacks of the standard model, the functional dependency between income and prices forces to broaden the preference structure or the topology of the consumption space. Results related to the second possibility may be found in [HH 92], [BaRi 01] and [MaRi 10] or [DMV 07]. This paper concentrates on the expansion of utilities which guarantee Arrow-Debreu equilibria in the space of adapted consumption rate bundles.
In [DuEp 92] a recursive utility specification is introduced, stochastic differential utility (SDU). This concept factors the future utility of the remaining consumption stream into the evaluation of the present consumption. This enables the agent to distinguish the different concepts of risk aversion and preferences for intertemporal substitution. Whereby in the additive case the systemically relation of these concepts is unavoidable. With such a utility specification [DuGeSk 94] prove the existence of equilibria and discuss the dynamics of efficiency via a system of forward backward integral equation.
In the Brownian setting, [LaQu 03] consider a BSDE approach and extend the notion of recursive utilities in continuous time. The initial value of a BSDE

$$dU_t = -f(t, c_t, U_t, Z_t)dt + Z_t' dB_t, \quad U_T = 0,$$

is the utility of an agent with aggregator f and consumption process c which lies in the set of positive consumption rates. In this family of utility functionals source-dependent first or second order risk aversion, a kind of asymmetric risk aversion, can be modeled. Furthermore, the notion of preference for information, introduced and axiomatized by [Ski 98], in contained in the present GSDU-class, see also [Laz 04].
This framework covers models where agents are faced with imprecise knowledge about the probability distribution with regard to the underlying risk in the economy. For instance, in [ChEp 02] Knightian uncertainty is formalized via the drift uncertainty of an appropriate BSDE. A subclass of the utility

functionals, namely κ-ambiguity, lies in the class of the considered functionals. The implications of ambiguity aversion for financial markets and asset pricing are studied in [EpMi 03] for a two-agent equilibrium setting and in [EpWa 94] for the discrete time case.
Another class of economies which lies in the scope of the existence result is related to incomplete preferences motivated by [Bew 02]. Here, the agents foreclose unmotivated gambles by an inertia principle. In [DaRi 10], Bewley economies are considered in discrete time. They related them to variational preference anchored at the initial endowment e, a special case of variational preferences that have been axiomatized by [MMR 06]. The existence result applies here as well, whenever the consumption space is of $\mathrm{L}^2(\mathbb{P})$-type.
In order to apply our results to the above examples, we cannot assume differentiability on the whole domain of the aggregator f. This leads to the non differentiability of the utility functional and forces to consider supergradients, see [EpWa 94, Section 2.5] for a discussion.
We follow the classical approach dealing with infinite-dimensional commodity and price spaces by introducing the concept of properness (or cone conditions). In [Kl 48], the first relationship between the supporting property of a convex set and a cone condition is formulated. [ChKa 80] translated this result into economic theory. For a first overview we refer to [MaZa 91] and [ATY 00]. The existence proof is an application of the abstract existence result of [Pod 96]. The empty interior of the positive cone of the consumption space requests a pointwise (forward-)properness condition which has to hold at each Pareto optimal consumption.

The paper is organized as follows. Section 2 introduces the model, recasts the notion of GSDU utility functionals and discusses the supergradient. Section 3 considers efficient allocations, proves the boundedness away from zero of the components and the existence of general equilibrium. Section 4 contains concluding remarks. Proofs of auxiliary results are collected in the Appendix.

2 The economy

Fix a time set $[0,T]$, for some $T \in]0,\infty[$. The probability space $(\Omega,\mathcal{F},\mathbb{P})$ is equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by an n-dimensional Brownian motion $\{B_t\}_{t\in[0,T]}$ and satisfying the usual conditions, be given.

2.1 Consumption and price space

For simplicity, the case of $l = 1$ commodity is examined.[1] We introduce the Hilbert lattice of optional processes $c : \Omega \times [0,T] \to \mathbb{R}$, which are finite with respect to the following norm $\|c\|_2 := \mathbb{E}[\int_0^T \|c_t\|^2 dt]^{\frac{1}{2}}$, denoted by $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{P} \otimes dt) := \mathrm{L}^2(\Omega \otimes [0,T], \mathcal{O}, \mathbb{P} \otimes dt)$. The consumption set is given by the positve cone L^2_+.[2]
The Hilbert space structure of the consumption space allows to consider the commodity-price duality $\langle \mathrm{L}^2, (\mathrm{L}^2)^* \rangle$ given by the scalar product of L^2.

2.2 Primitives of the economy and main Theorem

The economy consists of $m \in \mathbb{N}$ agents. Fix the initial endowments $e^i \in \mathrm{L}^2_+$ with $\sum_i^m e^i = e$.
The preference of agent i is described by a utility functional $U^i : \mathrm{L}^2_+ \to \mathbb{R}$ which is given by the initial value $U_0 = \mathbb{E}[\int_0^T f(t, c_t, U_t, Z_t)dt]$ of the solution of the following backward stochastic differential equation (BSDE)

$$dU_t = -f(t, c_t, U_t, Z_t)dt + Z_t' dB_t \quad U_T = 0. \tag{1}$$

By *generalized stochastic differential utility*, GSDU for short, the functional $c \mapsto U_0$ is defined, where U_0 is the $\mathbb{P}$-a.s initial value of (1). This is a rigorous formulation of the utility backward recursion principle, considered first by [DuEp 92] in the continuous time case under uncertainty. In [EKPQ 97, Proposition 3.5, p.35] time consistency of GSDU is shown which can be seen as a benchmark for intertemporal preferences. From an economic point of view, when this BSDE is used to define generalized stochastic differential utility, it increases the modeling degree of freedom when there is a Z component in the aggregator. The quadratic variation of this process is given by $\langle U \rangle_t = \int_0^t |Z_s|^2 ds$. This "intensity process" appears in the intertemporal aggregator, a direct effect of Z, as a component in the aggregator, can explicitly express preferences on "local risk".
The following assumption will ensure the standard properties of the utility functional.

[1] The case of finite commodities can be treated by same argumentation, we refer to [DuGeSk 94].

[2] Measures on $\Omega \times [0,T]$ which allow considerations of terminal consumption are possible. In this case the BSDE in (1) has a non trivial terminal condition.

Assumption 1. *The aggregator f is uniform Lipschitz continuous in u and z with constant $k > 0$ and satisfies a linear growth condition in c. Furthermore, f fullfills:*

1. *For all $(t, u, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$, $f(t, \cdot, u, z)$ is strictly increasing.*
2. *For all $(t, u, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$, $f(t, \cdot, u, z)$ is differentiable on $]0, \infty[$.*
3. $\overline{\delta}_f(c) := \sup_{(t,u,z)} |\partial_c f(t, c, u, z)| < \infty \quad \forall c \in \mathbb{R}_{++}$
4. *Define $\underline{\delta}_f(c) := \inf_{(t,u,z)} |\partial_c f(t, c, u, z)|$. For every positive sequence $\{c_n\}_{n\in\mathbb{N}}$ which converges to 0, we have $\underline{\delta}_f(c_n) \overset{n\to\infty}{\longrightarrow} \infty$.*
5. *For all $t \in [0, T]$, $f(t, \cdot, \cdot, \cdot)$ is a concave and continuous function.*

The Lipschitz-growth assumption on the aggregator guarantee unique existence of (1) for all $c \in \mathrm{L}^2$, see [EKPQ 97].

The pure exchange economy is given by $\mathtt{E} = \left\{\mathrm{L}^2_+, U^i, e^i\right\}_{1\le i\le m}$.
An element $(\bar{c}^1, \ldots, \bar{c}^m; \Pi) \in (\mathrm{L}^2)^m \times \mathrm{L}^{2\prime}$, consisting of a feasible allocation and a non-zero linear price functional, is called a contingent Arrow-Debreu equilibrium if for each i, $\bar{c}^i$ maximizes agent i's utility over all $c \in \mathrm{L}^2_+$ satisfying $\Pi(c^i - e^i) \le 0$ and the allocation $\bar{c}$ is $\sum \bar{c}^i = e$. The main result is the following.

Theorem 1. *Suppose the endowment $e \in \mathrm{L}^2_+$ of the economy is bounded away from zero. For each agent i Assumption 1 holds and $\|e^i\|_2 > 0$.*
Then there exists a contingent Arrow-Debreu equilibrium $(\bar{c}^1, \ldots, \bar{c}^m; \Pi)$ with the following properties:

1. *For every i, $\bar{c}^i$ is bounded away from zero.*
2. *Π has a Riesz Representation $\pi \in \mathrm{L}^2_+$. For every i there is a $\mu_i > 0$ such that $\pi = \mu_i \pi^i(\bar{c}^i)$ where $\pi^i(\bar{c}^i)$ is a super gradient density given by*

$$\pi^i_t(\bar{c}^i) = \mathcal{E}(D_U f^i, D_Z f^i)_t \partial_c f^i(t, c_t, U_t, Z_t)$$

where $(D_U f^i_t, D_Z f^i_t) \in \partial_{U,Z} f(c_t)$.

The process $\mathcal{E}(\cdot,\cdot)$ and the corespondence $\partial_{U,Z} f(\cdot)$ are defined in Proposition 3 below. The proof of the existence is an application to the abstract existence result of [Pod 96]. The properties of the equilibrium are based on a priori estimates, stochastic Gronwall inequalities and the full characterization of the superdifferential. The main step is to prove that optimal allocations are bounded away from zero.

2.3 Examples

In the following, we present a number of economies such that Theorem 1 applies. We note that, the additive case with a discounting β of future consumption is related to $f(t,c,z,u) = u(c) - \beta u$ and can be seen as the canonical special case. Subjective beliefs are related to $f(t,c,z,u) = u(c) - \beta u - \theta' z$, we refer to [LaQu 03, Section 3.1, p.158].

κ-ambiguity
An agent may not know the real world probability measure and is confronted with a set of prior probability measures. This uncertainty or unmeasurable risk is referred to as ambiguity. In [ChEp 02], a continuous time model is introduced which models the set of priors in terms of the density kernel related to each prior. Let $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n_+$ and define the set of priors

$$\Theta = \left\{ \theta : [0,T] \times \Omega \to \mathbb{R}^n, \mathcal{O}\text{-measurable} : \theta^i_t \in [-\kappa_i, \kappa_i], \quad \forall 1 \leq i \leq n \right\}.$$

We implement Θ by considering, for each prior $\theta \in \Theta$, a single SDU model $U^\theta_t = \mathbb{E}^\theta[\int_t^T g(c_t, U_t)|\mathcal{F}_t]$ such that risk aversion and intertemporal preferences can be encoded in g. One can show, see [ChEp 02, Theorem 2.2, p.1414], that

$$U_t = \min_{\theta \in \Theta} U^\theta_t, \quad t \in [0,T],$$

where the process U solves the BSDE

$$\begin{aligned} dU_t &= -(g(c_t, U_t) + \min_{\theta \in \Theta} \theta_t \cdot Z_t)dt + Z_t dB_t \\ &= -g(c_t, U_t) + \kappa \cdot |Z_t| dt + Z_t dB_t, \quad U_T = 0. \end{aligned}$$

Put $K = [-\kappa_1, \kappa_1] \times \ldots \times [-\kappa_n, \kappa_n]$. Since $z \mapsto \max_{\theta \in K} \theta \cdot z$ is the convex conjugate of the indicator function 1_K, f is concave in z. Lipschitz continuity in z is implied by to the boundedness of each κ_i. The differentiability is not

satisfied.
Ambiguity aversion is referred to the consideration of the worst case utility. Heuristically speaking, the bigger κ_i is, the more ambiguity aversion is assigned to the agent.
For each agent k let $\kappa^k = (\kappa_1^k, \dots, \kappa_n^k) \in \mathbb{R}_+^n$ and Θ^k the corresponding set of priors. In order to apply the thoerem, we briefly discuss the conditions in Assumption 1. We may take an SDU aggregator $g : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ which is conform. The concavity, uniform Lipschitz continuity and hence continuity with respect to the z-component are obvious.
We compute a subset of the super-differential explicitly for a case when the consumption process generates the solution of the associated BSDE: $c^k \mapsto (U^k, Z^k)$. Define the worst case Girsanov kernels:

$$\Theta_c^k = \left\{ \theta \in \Theta^k : \theta^* \in \arg\max_{y \in \Theta_t} y \cdot Z_t^k \ \ \forall t \in [0,T] \right\}$$

According to Theorem 1 the equilibrium allocation components are bounded away from zero. By Proposition 3 below, the super-differential can be written in the form

$$\partial U^k(c^k) \ \ = \left\{ \pi : \exists \theta \in \Theta_c^k, \{\pi_t\}_{t\in[0,T]} = \left\{ \lambda_t^k \cdot \partial_c g^k(c_t^k, U_t^k) \cdot z_t^\theta \right\}_{t\in[0,T]} \right\}$$

where $dz_t^\theta = -\theta_t z_t^\theta dB_t$ and $z_0^\theta = 1$.
In comparison with [ChEp 02, Section 5.1] our Inada condtion on $f(t, \cdot, u, z)$ (instead of a growth condition on $\partial f(t, \cdot, u, z)$) allows a full characterization of the superdifferential $\partial U^k(c^k)$.

Anchored preferences
This example studies an auxilary economy with variational preferences which guarantee the existence of a Bewley equilibrium.

In [DaRi 10] the concept of a discrete time Bewley economy is considered, where preferences are allowed to be incomplete. The existence of a Bewley equilibrium with inertia is established by considering an auxiliary economy with complete static variational preferences. The set of priors of agent k is given Θ^k introduced in Example 1. Let $\mathcal{P}^k$ denotes the corresponding set of probability measures. Fix $c, e \in \mathrm{L}_+^2$ and a strictly increasing, concave utility function $u : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ satisfying the Inada conditions in its second variable.

As in Definition 2.14 of [DaRi 10], we define the variational utility functional anchored at e by

$$V(x) = \min_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}[\int_0^T u(t, x_t) - u(t, e_t)dt] \tag{2}$$

Using similar arguments as in [ChEp 02, Theorem 2.2 (c), p.1414], one obtains that $V(x)$ is the initial value of the solution of the following BSDE

$$dU_t = \big[-(u(t, c_t) - u(t, e_t)) + \max_{\theta\in\Theta} \theta_t \cdot Z_t\big]dt + Z_t dB_t, \quad U_T = 0$$

The existence of the BSDE follows by the same arguments as in Example 1.

2.4 Properties of GSDU

The next propositions deliver the properties of the utility functional.

Proposition 1. *1. If the aggregator f is (strictly) concave in consumption, then the related utility functional is (strictly) concave.*

2. If the aggregator f is (strictly) increasing, then the related utility functional is (strictly) increasing. Moreover, if $c \geq \hat{c}$, then for any $t \in [0, T]$ we have $U_t(c) \geq U_t(\hat{c})$.

3. If the aggregator f is continuous, then the related utility functional is $\|\cdot\|_2$-continuous.

Next, we mention the explicit formula for the differential of the utility functional. We do not assume differentiability on the whole domain of the aggregator since concavity allows us to consider super-differentials. The partial super-gradient of the aggregator with respect to the corresponding component x is denoted by $D_x f(t, \cdot, x, \cdot)$. The partial super-differential in utility and intensity, namely $\partial_{U,Z} f$ at (t, c, u, z), consists of all $(D_U f(t, c, u, z), D_Z f(t, c, u, z)) = (a, b) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$f(t, c + x, u + y_1, z + y_2) \leq f(t, c, u, z) + \partial_c f(t, c, u, z)x + ay_1 + by_2.$$

For $k = U, Z$, the stochastic process $\{D_k f(t, c_t, U_t, Z_t)\}_{t\in[0,T]}$ is denoted by $D_k f$. For any process $(a, b) \in \mathrm{L}^1(\mathbb{P} \otimes dt) \times \mathrm{L}^2(\mathbb{P} \otimes dt; \mathbb{R}^n)$, we introduce the

stochastic exponential process $\mathcal{E}(a,b)$ which is defined as the solution of the SDE

$$\frac{d\mathcal{E}_t(a,b)}{\mathcal{E}_t(a,b)} = a_t dt + b_t dB_t, \quad \mathcal{E}_0(a,b) = 1.$$

Proposition 2. *Let Assumption 1 holds,* $c \in \mathrm{L}^2_+$ *be bounded away from zero and* (U,Z) *be a solution of the BSDE (1). Then, for every process* $(D_U f, D_Z f) \in \partial_{U,Z} f(c,U,Z)$*,* $\pi = \{\pi_t\}_{t\in[0,T]}$ *is a super-gradient density of* U_0 *at* c*, where*

$$\pi_t = \mathcal{E}_t(D_U f, D_Z f) \cdot \partial_c f(t, c_t, U_t, Z_t). \tag{3}$$

Now, we characterize the super differential in terms of super gradient densities. According to [Gi 82, Theorem 3, p.122] the set of supergradients of a concave functional U at c is

$$\partial U(c) = \left\{ f \in \mathrm{L}^2 : \nabla^+ U(c)(y) \le f(y) \le \nabla^- U(c)(y) \quad \forall y \in \mathrm{L}^2 \right\},$$

where $\nabla^- U$ $(\nabla^+ U)$ is the left(right)-hand Gateaux derivative. Comparing these procceses with (3), we observe a supergradient $g \in \partial U(c)$ may change for every (ω,t) the supergradient of the aggregator f.

Proposition 3. *Suppose the conditions of Proposition 2 hold. Define the correspondences* $(\partial_{U,Z} f(c)_t)_{t\in[0,T]}$*, for each* t*, where* $\partial_{U,Z} f(c)_t : \Omega \to \mathcal{P}(\mathbb{R}^{1+n})$ *be given by*

$$\partial_{U,Z} f(c)_t(\omega) = \partial_{U,Z} f(t, c_t(\omega), U_t(\omega), Z_t(\omega)).$$

Then we have

$$\partial U(c) = \left\{ \mathcal{E}(a,b)\partial_c f(\cdot, c_\cdot, U_\cdot, Z_\cdot) : \mathcal{E}_t(a,b) \in \partial_{U,Z} f(c)_t \ \forall t \in [0,T] \right\}.$$

3 Existence of equilibria

The objective of this section is to identify efficient and equilibrium allocations for GSDU preferences.
We begin to characterize Pareto optimal allocations with the solution of a social planning problem and prove the existence of a solution. Afterwards we introduce the first order conditions. Moreover we show that Assumption 1 is sufficient to guarantee that the components of the efficient allocation are bounded away from zero.

3.1 Preliminaries

Let $e \in \mathrm{L}^2_+$ denote the endowment process which has to be shared among the $m \in \mathbb{N}$ agents.
We define the usual norm on the underlying space for allocations $\mathrm{L}^{2,m} = (\mathrm{L}^2)^m$, $\|c\|^{2,m} := (\sum_{i=1}^m \mathbb{E}[\int_0^T |c_t^i|^2 dt])^{\frac{1}{2}}$, where $c = (c^1, \ldots, c^m)$. By $\mathrm{L}^{2,m}_+$, we denote the positive cone of $\mathrm{L}^{2,m}$ and by $\mathrm{L}^{2,m}_{++}$ the quasi interior. The set of *feasible allocations* is defined by $\Lambda(e) := \left\{(c^1, ..., c^m) \in \mathrm{L}^{2,m}_+ : e \geq \sum c^i\right\}$.
A weighting $\alpha \in \mathbb{R}^m_+$ induces a *representative utility* $U^\alpha : \mathrm{L}^{2,m}_+ \to \mathbb{R}$, $U^\alpha(c^1, \ldots, c^m) := \sum_{i=1}^m \alpha_i U^i(c^i)$. U^α is related to the social planner of the economy.
An allocation $(\hat{c}^1, \ldots, \hat{c}^m) \in \mathrm{L}^{2,m}_+$ is *α-efficient* if it achieves the maximum over $\Lambda(e)$, i.e. $U^\alpha(\hat{c}) = \max_{c \in \Lambda(e)} U^\alpha(c)$.
A feasible allocation $c = (c^1, \ldots, c^m) \in \Lambda(e)$ is *Pareto optimal* if there is no feasible allocation $\bar{c} = (\bar{c}^1, ..., \bar{c}^m) \in \Lambda(e)$ such that

$$\forall i \in \{1, \ldots, m\}\ U^i(\bar{c}^i) \geq U^i(c^i) \text{ and } \exists i \in \{1, \ldots, m\}\ U^i(\bar{c}^i) > U^i(c^i).$$

In the following, we state the relation to α-efficiency when utility functionals are concave.

Proposition 4. *Suppose the utility functionals are of GSDU type. Each aggregator $f^i : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous in utility and intensity and satisfies a linear growth condition in consumption. On $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n$ the f^i's are concave and continuous. Then there is an α-efficient allocation. Moreover, Pareto optimal allocations exists.*

Proposition 2 derived the super-differential of the utility functional generated by an aggregator satisfying Assumption 1. We begin with the first order conditions of optimality for concave and not necessarily Gateaux differentiable functionals. Define the set of *feasible directions* at c^i is given by $F(c^i) = \left\{h \in \mathrm{L}^2 : \exists \mu > 0 \quad c^i + \mu h \in \mathrm{L}^2_+\right\}$ and the set of *feasible transfers* $H(c) = \left\{h \in \mathrm{L}^{2,m} : \sum h^i = 0, h^i \in F(c^i), 1 \leq i \leq m\right\}$.

Proposition 5. *Assume that for each i, the utility functional U^i is upper semi-continuous, strictly increasing, concave. Moreover, U^i takes nowhere the value ∞ and is not equal $-\infty$. The aggregate endowment e is bounded away from zero.*
Then the α-efficiency of $\hat{c} \in \Lambda(e)$ is equivalent to the existence of a $DU^i(\hat{c}^i) \in$

$\partial U^i(\hat{c}^i)$, for each i, such that

$$0 \geq \sum \langle D\alpha_i U^i(\hat{c}^i), h^i \rangle \quad h \in H(\hat{c}).$$

Corollary 1. *If, for each i, the utility functional U^i is upper semi-continuous, strictly increasing, concave and it takes no where the value ∞ and is not equal $-\infty$, we have the following:*

1. *If $\bigcap_{i=1}^m \partial\alpha_i U^i(\hat{c}^i) \neq \emptyset$, then $\hat{c} = (\hat{c}^1, \ldots, \hat{c}^m)$ is α-efficient.*
2. *If $(\hat{c}_1, \ldots, \hat{c}_m)$ is α-efficient and ,for all i, $\hat{c}^i$ is bounded away from zero, then $\bigcap_{i=1}^m \partial\alpha_i U^i(\hat{c}^i) \neq \emptyset$ holds.*

3.2 GSDU and efficient allocations

In order to apply the previous result to GSDU we have to establish a criterion which ensures that the components of the efficient allocation are bounded away from zero. We use the assumption of Section 2, related to the present aggregator $f^i : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

Lemma 1. *Suppose Assumption 1 holds. e is bounded away from zero. Fix an α-efficient allocation $c = (c^1, \ldots, c^m) \in \mathrm{L}^{2,m}_{++}$. Then, for each i, c^i is bounded away from zero.*

Proof of lemma 1. Let $\nu = \mathbb{P} \otimes dt$ and take a $c \in \mathrm{L}^{2,m}_{++}$. For every i we have $U^i(c^i) > U^i(0)$ since each U^i strictly increasing.
Suppose some c^j is not bounded away from zero. Then for every $h > 0$ there is an $\hat{H} = \hat{H}(h) \in \mathcal{O}$ such that $\nu(\hat{H}) > 0$ and $c^j \leq h$ on $\hat{H}$. Since e is bounded away from zero, we have $e > C$ ν-a.e. for some constant $C > 0$. This gives us, if C is taken small enough, that there is an agent k such that $c^k \geq \frac{C}{m}$ on $H' \subset \hat{H}$. We choose $H = \{c^j < h\} \cap \{\frac{C}{m} \leq c^k \leq C^k\}$ which has positive measure.
On the other hand, since c is in the quasi interior of $\mathrm{L}^{2,m}_{+}$, for every i, there is a set $A^i \in \mathcal{O}$ with $\nu(A^i) > 0$ and a number $a^i > 0$ such that $c^i \geq a^i$ on A^i. We show a Pareto improvement when multiples of H and A^j are traded between agent j and k .
Let $\lambda^k \in]0, h[$ and $\lambda^j \in]0, \frac{a_j}{2}[$. Define the following BSDEs:

$$c^j \mapsto (U, Z),\ c^j - \lambda^j 1_{A^j} \mapsto (U^A, Z^A) \text{ and } c^j - \lambda^j 1_{A^j} + \lambda^k 1_H \mapsto (U^{AH}, Z^{AH}),$$

where $U_0 = U^j(c^j)$, $U_0^A = U^j(c^j - \lambda^j 1_{A^j})$ and $U_0^{AH} = U^j(c^j - \lambda^j 1_{A^j} + \lambda^k 1_H)$ are the corresponding evaluated utility functionals. We derive

$$\begin{aligned} & U^j(c^j - \lambda^j 1_{A^j} + \lambda^k 1_H) - U^j(c^j) \\ \geq \; & \mathbb{E}[\int_0^T \underline{\delta}_{f^j}(2h)\lambda^k 1_H(t) - k^j | Z_t^{AH} - Z_t^A | dt] \\ - \; & \mathbb{E}[\int_0^T \overline{\delta}_{f^j}(\frac{a^j}{2})\lambda^j 1_{A^j}(t) + k^j | Z_t - Z_t^A | dt] \end{aligned}$$

The inequality applied the estimates in Lemma 4 and Lemma 5, given in Appendix A.2. Next, we compute appropriate estimates for the Z parts. By the Cauchy-Schwartz inequality and the a priori estimates in [EKPQ 97], with $\lambda^2 = 2k$, $\mu = 1$ and $\beta \geq 2k(1+k)+1$, we derive:

$$\begin{aligned} & \mathbb{E}[\int_0^T |Z_s - Z_s^A| ds] \\ \leq \; & \Big(\frac{T\lambda^2}{\mu^2(\lambda^2 - k)} \mathbb{E}[\int_0^T e^{\beta s} |f^j(s, c_s^j, U_s, Z_s) - f^j(s, c_s^j - \lambda^j 1_{A^j}, U_s, Z_s)|^2 ds]\Big)^{\frac{1}{2}} \\ \leq \; & T^{1/2} \cdot \Big(2e^{\beta T} \mathbb{E}[\int_0^T |f^j(s, c_s^j, U_s, Z_s) - f^j(s, c_s^j - \lambda^j 1_{A^j}, U_s, Z_s)|^2 ds]\Big)^{\frac{1}{2}} \\ = \; & (2Te^{\beta T})^{1/2} \mathbb{E}[\int_0^T |\partial_c f^j(s, c_s^j - \lambda^j 1_{A^j} + \xi_s^c, U_s, Z_s)\lambda^j 1_{A^j}(s)|^2 ds]^{\frac{1}{2}} \\ \leq \; & (2Te^{\beta T})^{1/2} \mathbb{E}[\int_0^T \overline{\delta}_{f^j}(\frac{a^j}{2})^2 {\lambda^j}^2 1_{A^j}(s) ds]^{\frac{1}{2}} \\ \leq \; & (2Te^{\beta T})^{1/2} \mathbb{E}[\int_0^T \overline{\delta}_{f^j}(\frac{a^j}{2})\lambda^j 1_{A^j}(s) ds] \end{aligned}$$

The last equality is a pointwise application of the mean value theorem with some positive process ξ^c. The last but one inequality is true since we assumed $\lambda^j < \frac{a^j}{2}$ and $c^j \geq a_j$ on A^j and because $\partial_c f^j$ is decreasing. Analogous arguments yields,

$$\mathbb{E}[\int_0^T |Z_s^{AH} - Z_s^A| ds] \leq (2Te^{\beta T})^{1/2} \mathbb{E}[\int_0^T \overline{\delta}_{f^j}(\frac{a^j}{2})\lambda^l 1_H(s) ds].$$

Since h can be taken arbitrarily small, $\underline{\delta}_f(2h)$ becomes arbitrarily large and

by the last two derivations:

$$\begin{aligned}
& U^j(c^j - \lambda^j 1_{A^j} + \lambda^l 1_H) - U^j(c^j) \\
\geq\ & e^{-k_j T}\underline{\delta}_{f^j}(2h)\mathbb{E}[\int_0^T \lambda^l 1_H(t)dt] - e^{-k_j T}\overline{\delta}_{f^j}(\frac{a^j}{2})\mathbb{E}[\int_0^T \lambda^j 1_{A^j}(t)dt] \\
& -\mathbb{E}[\int_0^T k_j|Z_t^{AH} - Z_t^A|dt\ ] - \mathbb{E}[\int_0^T k_j|Z_t - Z_t^A|dt] \\
=\ & \lambda^k\Big(\underbrace{e^{-k_j T}}_{=b_1^j}\underline{\delta}_{f^j}(2h)\nu(H) - \underbrace{k_j(2T)^{1/2}e^{\beta T}}_{=b_2^j}\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j)\Big) \\
& -\lambda^j\Big(b_1^j\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j) + b_2^j\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(H)\Big)
\end{aligned}$$

A utility improvement of agent j is related to strict positivity of the last term. An analogous derivation and a modification of Lemma 4 and Lemma 5 yield the corresponding inequality for agent k.
Hence, in order to achieve a Pareto improvement

$$1 \ > \ \frac{b_1^j\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j) + b_2^j\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(H)}{b_1^j\underline{\delta}_{f^j}(2h)\nu(H) - b_2^j\overline{\delta}_{f^j}(\frac{a^j}{2})\nu(A^j)} \cdot \frac{b_1^k\overline{\delta}_{f^k}(\frac{C}{2m})\nu(A^j) + b_2^k\overline{\delta}_{f^k}(\frac{a^k}{2})\nu(H)}{b_1^k\underline{\delta}_{f^k}(2C^k)\nu(H) - b_2^k\overline{\delta}_{f^k}(\frac{a^j}{2})\nu(A^j)}$$

must hold. If we take a sufficiently small h, then, by the Inada style condition, $\underline{\delta}_f(2h)$ becomes arbitrary large. Consequently $\nu(\hat{H})$ and hence $\nu(H)$ becomes small. We may choose A^j such that $\nu(H) = \nu(A^j) > 0$, this gives us

$$1 \ > \ \frac{b_1^j\overline{\delta}_{f^j}(\frac{a^j}{2}) + b_2^j\overline{\delta}_{f^j}(\frac{a^j}{2})}{b_1^j\underline{\delta}_{f^j}(2h) - b_2^j\overline{\delta}_{f^j}(\frac{a^j}{2})} \cdot \frac{b_1^k\overline{\delta}_{f^k}(\frac{C}{2m}) + b_2^k\overline{\delta}_{f^k}(\frac{a^k}{2})}{b_1^k\underline{\delta}_{f^k}(2C^k) - b_2^k\overline{\delta}_{f^k}(\frac{a^j}{2})},$$

by choosing appropriate multiples $\lambda^k \in]0, h[$ and $\lambda^j \in]0, \frac{a_j}{2}[$:

$$U^j(c^j - \lambda^j 1_{A^j} + \lambda^k 1_H) > U^j(c^j)\ \textit{and}\ U^k(c^k + \lambda^j 1_{A^j} - \lambda^k 1_H) > U^k(c^k)$$

This yields a Pareto improvement, contradicting that $(c^1, \ldots, c^m)$ is a Pareto optimal allocation. By Corollary 1, this is equivalent to α-efficiency. Therefore, each c^j of the efficient allocation is bounded away from zero. ■

3.3 Properness and the proof of the main proof

In this section we deal with the existence of an equilibrium in the sense of Arrow Debreu. In Appendix A.3, we consider an economy defined on an

abstract lattice and state the existence of a quasi equilibrium.
The notion of F-properness at x, see Definition 1 in Appendix A.3, can be written as:

> There is a $v \in \mathrm{L}^2_+$ and an $\varepsilon > 0$ such that for all z with $\|z\|_{\mathrm{L}^2} < \varepsilon$, $U(x + \lambda(v - z)) > U(x)$, for small $\lambda > 0$ and $x + \lambda(v - z) \in \mathrm{L}^2_+$.

Now, we establish the existence of an equilibrium when the utility functional $U^i : \mathrm{L}^2_+ \to \mathbb{R}$ is given by a GSDU. Assumption 1 ensures that every component of the efficient allocation is bounded away from zero.
We prove the existence of the equilibrium by an application of Theorem 2. Therefore, we need the F-properness to hold at certain points. This is proven in the following lemma. The principle goes back to [LeV 96] where the case of separable utilities was treated. Apart from Lemma 1, the main work was already done in Proposition 2, where the square integrability of the super gradient density $\pi(c) = \mathcal{E}\partial_c f^i(\cdot, c., U., Z.)$ is proved. In this case, $\langle \pi(c), \cdot \rangle$ is the supporting linear functional at c.

Lemma 2. *Suppose that $c = (c^1, \ldots, c^m)$ is a Pareto optimal allocation with $U^i(c^i) \geq U^i(e^i)$. Under the assumptions of Theorem 1, the F-properness at each c^i holds.*

Proof of lemma 2. By a modification of Lemma 1, each c^i is bounded away from zero. The assumption of a quasi interior allocation may be substituted by individual rationality.
Fix $v \equiv 1$ as the properness vector. According to Proposition 2, a super gradient density $\pi(D_{U,Z} f^i) \in \mathrm{L}^2_{++}$ at c^i is given by

$$\pi(D_{U,Z} f^i)_t = \mathcal{E}(D_U f^i, D_Z f^i)_t \cdot \partial_c f^i(t, c_t, U_t, Z_t).$$

The parametrization is related to the super-differential $\partial_{U,Z} f^i$ of the aggregator. Define, in accordance to the argumentation below,

$$V(D_{U,Z} f^i) = \left\{ z \in \mathrm{L}^2 : \langle \pi(D_{U,Z} f^i), (1 - z) \rangle_{\mathrm{L}^2} > 0 \right\}$$

and

$$V := \bigcap_{D_{U,Z} f^i \in \partial_{U,Z} f^i} V(D_{U,Z} f^i).$$

V is a neighborhood of 0 in L^2:
We show that for each $D_{U,Z}f^i$ there exists an open ball around zero which is contained in $V(D_{U,Z}f^i)$. Choose some

$$z \in \left\{ y \in \mathrm{L}^2 : \|y\|_{\mathrm{L}^2} < \frac{\|\pi(D_{U,Z}f^i)\|_{\mathrm{L}^1}}{\|\pi(D_{U,Z}f^i)\|_{\mathrm{L}^2}} \right\} \textit{ arbitrary.}$$

The positivity of π implies

$$\langle \pi(D_{U,Z}f^i), z\rangle_{\mathrm{L}^2} < \|\pi\|_{\mathrm{L}^1}\langle \pi(D_{U,Z}f^i), 1\rangle_{\mathrm{L}^2},$$

where the last equality holds by the positivity of $\pi(D_{U,Z}f^i)$. We have $z \in V(D_{U,Z}f^i)$. Hence, there is an open ball which is contained in V.
Let $c^i + \lambda(1-z) \in \mathrm{L}^2_+$, where $z \in V$ is arbitrary and $\lambda > 0$ is sufficiently small. When $\lambda > 0$ tends to zero, the term $\lambda^{-1}U^i(c^i + \lambda(1-z)) - U^i(c^i)$ increases, due to the concavity of U^i. Fix some $z \in V$. Whenever $\lambda \searrow 0$, the limit of the quotient exists by [Gi 82, Theorem 1, p.117] and we have

$$\lim_{\lambda \searrow 0} \frac{U^i(c^i + \lambda(1-z)) - U^i(c^i)}{\lambda} \geq \langle \pi(D_{U,Z}f^i)_t(1-z)\rangle_{\mathrm{L}^2} > 0.$$

The first inequality holds by [Gi 82, Theorem 3, p.122]. The second inequality is valid since $z \in V \supset B_\varepsilon(0)$. Now, consider a sufficiently small λ with $U^i(c^i + \lambda(1-z)) > U^i(c^i)$. In other words, U^i is F-proper at c^i . ■

Proof of theorem 1. Assumption 1 implies strict monotonicity, concavity and norm continuity for each utility functional U^i. The F-properness at each c^i is the content of Lemma 2, where $(c^1, \ldots, c^m)$ is an α-efficient allocation with $U^i(c^i) \geq U^i(e^i)$ for all i. The existence of the equilibrium follows from Theorem 2. We prove the properties of the equilibrium.

1. Pareto optimality follows from Theorem 2 in Appendix A.3. This implies α-efficiency for some $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$. By Lemma 1, each $\bar{c}^i$ is bounded away from zero.

2. The linear functional Π is L^2-continuous. Since $\mathbb{P}\otimes dt$ is a finite measure and therefore σ-finite, we conclude that there is a $\pi \in \mathrm{L}^2$ such that $\Pi(\cdot) = \langle \cdot, \pi\rangle_{\mathrm{L}^2}$. Each $\bar{c}^i$ is bounded away from zero. Therefore the set of feasible directions $F(\bar{c}^i)$ is norm-dense in L^2. The equilibrium allocation maximizes the utility of each agent:

$$U^i(\bar{c}^i) = \max_{c \in \mathrm{L}^2_+ : \Pi(c^i - e^i) \leq 0} U^i(c^i) = \max_{c \in \mathrm{L}^2_+ : g(c^i) \leq 0} U^i(c^i)$$

Each $\bar{c}^i$ is bounded away from zero and hence the Slater condition is satisfied with $g(\frac{\bar{c}^i}{2}) < 0$. By the Kuhn-Tucker Theorem, [BaPr 85, Theorem 3.1.4, p.177], for concave functionals, it is necessary and sufficient for the optimality of $\bar{c}^i$ that there is a $\mu_i \geq 0$ such that

$$0 \in \partial - U^i(\bar{c}^i) + \mu_i \Pi(\bar{c}^i), \quad \mu_i g(\bar{c}^i) = 0$$

on the set of feasible directions $F(\bar{c}^i)$.
U^i is strictly monotone, consequently $g(\bar{c}^i) = 0$. $\mu_i = 0$ would imply $0 \in \partial U^i(\bar{c}^i)$ and this contradicts the strict monotonicity of U^i. This proves the strict positivity of μ_i and we have, for all $h \in F(\bar{c}^i)$, $DU^i(\bar{c}^i)(h) = \mu_i \Pi(h)$, for some $DU^i(\bar{c}^i) \in \partial U^i(\bar{c}^i)$. Each $\bar{c}^i$ is bounded away from zero. By Proposition 3 each supergradient has the stated form. ■

4 Concluding Remarks

We discuss the equilibrium in terms of generic existence and a possible Radner embedding.

Generic existence
The framework of the present economy opens the question on generic existence of equilibria. In finite-dimensional commodity spaces, the usual notion of genericity corresponds to full Lebesgue measure. In an infinite-dimensional framework, one way out is to consider sets of first category, meaning that the set is contained in a countable union of closed sets with empty interior. In [ArMo 91] and [Mo 94], it is proven that the set of equilibria is of first category. Strong assumptions of the form $\partial u(0) = \infty$ and $\{\partial u(c_t)\}_{t\in[0,T]} \in \mathrm{L}^2$ ensure the existence of equilibria. In our recursive GSDU setting, the second assumption can be written as

$$\{\mathcal{E}_t(D_U f, D_Z f) \cdot \partial_c f(t, c_t, U_t, Z_t)\}_{t\in[0,T]} \in \mathrm{L}^2.$$

Such a condition is in some sense "in" the model since an integrability condition with respect to the gradient of the utility functional is related to new factors which influence the gradient.
From a topological point of view, Baire's Category theorem establishes an empty interior for first category sets. This notion has little measure theoretic connection. As mentioned in [Ma 85, p.318], topological genericity "has to

be thought of much less sharp than measure-theoretic concept available in the finite-dimensional case".
The parameter space of the economy is the space of endowments $\mathrm{L}_+^{2,m}$. We deal with strong assumption on the (aggregate) endowment process. A possible rationale for this is the alternative and more satisfactory genericity concept, namely prevalence. The principle is introduced in [AnZa 01].

$$\left\{(e^1,\ldots,e^m)\in \mathrm{L}_+^{2,m} : \sum e^k \text{ is bounded away from zero } \right\}$$

is a set of first category.
But as shown in [AnZa 01, Theorem 3.2, p.17], this set is finite prevalent in $\mathrm{L}_+^{2,m}$. Examples related to (finite) shyness and (finite) prevalence can be found in [AnZa 01, Appendix B].

The state price density and financial markets
As a first step to establish this result as an equilibrium foundation for mathematical finance one has to discuss the structure of the state price density. Under an additional assumption, the appearance of the intensity process Z in the supergradients still ensures the semimartingale property of the equilibrium state price density $\pi^i(c^i) = Df_c^i(\cdot, c_\cdot^i, U_\cdot^i, Z_\cdot^i)\mathcal{E}^i$. This can be seen as follows.
Let the aggregate endowment e be a special semimartingale. $\mathcal{E}$ is a special semimartingale by an application of integration by parts. Now, assume that the partial superdifferential $\partial_c f$ does not depend on z and is three times continuously differentiable, then, following the lines of [DuGeSk 94] there is a twice continuously differentiable function K_i, depending on $(t, e, \mathcal{E}, U)$, such that the α-efficient allocation can be written as $\{K_i(\cdot, e_\cdot, \mathcal{E}_\cdot, U_\cdot)\}_{i=1}^m$.
Since the process U is a semimartingale, we may apply Ito's formula and observe the special semimartingale property of π. The absolute continuity of the bounded variation component allow an interpretation of an money market captured by an interest rate process.
Such a "Radner embedding"-procedure can be used to observe a consumption based capital asset pricing model, see [DuZa 89]. For the κ-ambiguity case these can be found in [ChEp 02, Section 5.4, p.1430]. An ambiguity premium can be observed. This can be used to tackle the so called equity premium puzzle.

A Appendix

A.1 Proofs of section 2

PROOF OF PROPOSITION 1. The first two assertions can be found in [EKPQ 97], the third is a modification of [DuEp 92, Proposition 1, p.391]. ■

PROOF OF PROPOSITION 2. This follows from Lemma 3, with $t = 0$. ■

Lemma 3. *Fix $t \in [0,T]$ and suppose the conditions of Proposition 2 hold, then for any direction $h \in \mathrm{L}^2(\mathbb{P} \otimes dt)$ such that $c + h \in \mathrm{L}^2_{++}(\mathbb{P} \otimes dt)$ we have*

$$U_t(c+h) - U_t(c) \leq \mathbb{E}[\int_t^T \frac{\mathcal{E}_s}{\mathcal{E}_t} \partial_c f(s, c_s, U_s, Z_s) h_s ds | \mathcal{F}_t].$$

PROOF OF LEMMA 3. Take a process $c \in \mathrm{L}^2_{++}$ that is bounded away from zero and fix a process $h \in \mathrm{L}^2$ such that $c + h \in \mathrm{L}^2_{++}$. The related utility processes U and U^h are given by

$$dU_t = -f(t, c_t, U_t, Z_t)dt + Z_t dB_t \quad U_T = 0$$

$$\textit{and} \qquad dU^h_t = -f(t, c_t + h_t, U^h_t, Z^h_t)dt + Z^h_t dB_t \quad U^h_T = 0.$$

We define $\mathcal{E}_t := \mathcal{E}_t(D_U f, D_Z f)$ and prove the following

claim : We have $\mathbb{E}[\sup_{t\in[0,T]} \mathcal{E}_t^2] < \infty$.
proof: The process $\mathcal{E}$ admits a decomposition $\mathcal{E}_t = \lambda_t \cdot \Gamma_t$ and hence by the boundedness of the super-gradient w.r.t. the aggregator in utility

$$\lambda_t = \exp(\int_0^t D_U f(s, c_s, U_s, Z_s) ds) \leq \exp(kt). \tag{4}$$

Boundedness of the super-gradient w.r.t. aggregator in the intensity component z implies

$$\mathbb{E}[\exp(\frac{1}{2}\int_0^t |D_Z f(s, c_s, U_s, Z_s)|^2 ds)] \leq \mathbb{E}[\exp(\frac{1}{2}\int_0^t k^2 ds)] < \infty,$$

the Novikov criterion is satisfied, hence the process Γ, given by

$$\Gamma_t = \exp(-\frac{1}{2}\int_0^t |D_Z f(s, c_s, U_s, Z_s)|^2 ds + \int_0^t D_Z f(s, c_s, U_s, Z_s)' dB_s), \tag{5}$$

is indeed a martingale. With regard to the local martingale $\int_0^{\cdot} \Gamma_s d\Gamma_s$, we take a localizing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ such that $\tau_n \overset{n\to\infty}{\longrightarrow} T$ $\mathbb{P}$-a.s., and we see that for each n, $\left\{\int_0^{t\wedge\tau_n} \Gamma_s d\Gamma_s\right\}_{t\in[0,T]}$ is a martingale.
By Itô's formula, the quadratic variation of Γ, the boundedness of the supergradient in the intensity component z and Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}[\Gamma^2_{t\wedge\tau_n}] &= \mathbb{E}[1 + 2\int_0^{t\wedge\tau_n} \Gamma_s d\Gamma_s + \frac{1}{2}\int_0^{t\wedge\tau_n} 2d\langle\Gamma\rangle_s] \\
&= \mathbb{E}[1 + \int_0^{t\wedge\tau_n} \Gamma_s^2 D_Z f(s, c_s, U_s, Z_s)^2 ds] \\
&\leq 1 + \int_0^t \mathbb{E}[\Gamma^2_{s\wedge\tau_n}] k^2 ds.
\end{aligned}$$

Applying the Gronwall lemma with $g(s) = \mathbb{E}[\Gamma^2_{s\wedge\tau_n}]$, we conclude that

$$\mathbb{E}[\Gamma^2_{T\wedge\tau_n}] \leq \exp(Tk^2) < \infty$$

and by the dominated convergence, $\mathbb{E}[\Gamma^2_T] \leq \exp(Tk^2)$. Since Γ is a martingale, it follows that Γ^2 is a submartingale. By virtue of Doob's maximal inequality and (5), we deduce

$$\mathbb{E}[\sup_{t\in[0,T]} \mathcal{E}_t^2] = \mathbb{E}[\sup_{t\in[0,T]} \lambda_t^2 \sup_{t\in[0,T]} \Gamma_t^2] \leq \exp(kT)^2 4\mathbb{E}[\Gamma_T^2] < \infty.$$

□

claim : We have $\mathcal{E}\partial_c f \in \mathrm{L}^2$.
proof: There is a constant $C > 0$ with $c > C$ $\mathbb{P}\otimes dt$-a.e. and, since f is a regular aggregator, the process $t \mapsto \partial_c f(t, c_t, U_t, Z_t)$ takes values in $[0, K]$ $\mathbb{P}\otimes dt$-a.e., where

$$K = \sup_{(t,u,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^n} \partial_c f(t, C, u, z).$$

Since c is bounded away from zero, we have $\partial_c f \in \mathrm{L}^\infty(\mathbb{P}\otimes dt)$ and the claim follows by the previous claim. □

The rest of the prove follows from [SchSk 03, Lemma A.5, p.197]. ■

Proof of proposition 3. Following the proof of [Al 97, Theorem 4.3, p.425] and applying the concave alternative of [FGH 57], we can show that

the right- and left-hand derivatives represent the superdifferential in terms of an order interval:

$$\partial U(c) = \{g \in \mathrm{L}^2 : \nabla^+ U(c)(\omega)_t \leq g_t(\omega) \leq \nabla^- U(c)(\omega)_t\}$$

The proof that

$$\lim_{\alpha \searrow 0} \frac{U(c) - U(c + \alpha h)}{\alpha} = \langle \nabla^- U(c), h \rangle = \langle \mathcal{E}(D_U^r f, D_Z^r f) \cdot \partial_c f, h \rangle$$

is an application of results on BSDE depending on parameters, see [EKPQ 97, Proposition 2.4, p.26]. In this case the closed formula of the adjoint process is given by $\mathcal{E}$. We have

$$\partial U(c) = [\mathcal{E}(D_U^r f, D_Z^r f) \cdot \partial_c f, \mathcal{E}(D_U^l f, D_Z^l f) \cdot \partial_c f]$$

and the assertion follows. ■

A.2 Proofs of section 3

Proof of proposition 4. The properties of the aggregator imply the norm continuity and concavity of the utility functionals. Alaouglu's theorem implies the weak compactness of $\Lambda(e)$. Under concavity and upper semi continuity of the utility functionals α-efficient allocation exists by an abstract Weierstrass argument. The equivalence between α-efficiency an Pareto optimality is standard in economic theory. ■

By $\partial_{\mathrm{L}^{2,m}} U$ we denote the super-differential of a functional U on $\mathrm{L}^{2,m}$. We write $\langle DU(c), h \rangle$ for $DU(c)(h)$, where $DU(c)$ is a super-gradient and an element of the super-differential at c.

Proof of proposition 5. Let $g(c^1, \ldots, c^m) = \sum c^i - e$ and $g_i(c^1, \ldots, c^m) = -c^i$. Then α-efficiency of $\hat{c} = (\hat{c}^1, \ldots, \hat{c}^m)$ can be written as

$$U^\alpha(\hat{c}) = \max_{c' \in \Lambda(e)} U^\alpha(c') = \min_{c' \in \mathrm{L}^{2,m} : g_i(c'), g(c') \leq 0} -U^\alpha(c').$$

Since e is bounded away from zero, the Slater condition holds. We apply the Kuhn-Tucker theorem, [BaPr 85, Theorem 3.1.4, p.177], to $-U^\alpha$. Hence, $\hat{c}$ is α-efficient if and only if there are constants $\mu_i, \mu \geq 0$ such that

$$0 \in (\partial_{\mathrm{L}^{2,m}} - U^\alpha)(\hat{c}) + \mu \nabla_{\mathrm{L}^{2,m}} g(\hat{c}) + \sum \mu_i \nabla_{\mathrm{L}^{2,m}} g_i(\hat{c})$$

and $\mu g(\hat{c}) = 0, \quad \mu_i g_i(\hat{c}) = 0 \quad i = 1, \ldots, m$. Taking the non-negativity constraints into account and the existence of a $-D_{\mathrm{L}^{2,m}} U^\alpha(\hat{c}) \in (\partial_{\mathrm{L}^{2,m}} - U^\alpha)(\hat{c})$, this is equivalent to

$$0 \leq -D_{\mathrm{L}^{2,m}} U^\alpha(\hat{c}) + \mu \nabla_{\mathrm{L}^{2,m}} g(\hat{c}) \text{ and } \quad \mu g(\hat{c}) = 0.$$

Taking the feasible transfers $h \in H(\hat{c})$ into account, we have

$$\begin{aligned} 0 &\leq \langle -D_{\mathrm{L}^{2,m}} U^\alpha(\hat{c}), h \rangle_{\mathrm{L}^{2,m}} + \langle \mu \cdot \nabla_{\mathrm{L}^{2,m}} g(\hat{c}), h \rangle_{\mathrm{L}^{2,m}} \\ &= -\sum \alpha_i DU^i(\hat{c}^i) h_i + \mu \sum h_i. \end{aligned}$$

Since the U^i's are strictly increasing, $g(\hat{c}) = 0$ follows. ■

Proof of corollary 1. 1. Let $(h^1, \ldots, h^m) = h \in H(\hat{c})$. By assumption there is a $DU \in \bigcap_{i=1}^m \partial \alpha_i U^i(\hat{c}^i)$, with Riesz representation π. This means for each i, there is a $D\alpha_i U^i(\hat{c}^i) \in \partial \alpha_i U^i(\hat{c}^i)$ such that $D\alpha_i U^i(\hat{c}^i) = \langle \pi, \cdot \rangle$ and therefore

$$\sum \langle D\alpha_i U^i(\hat{c}^i), h^i \rangle = \sum \langle \pi, h^i \rangle = \langle \pi, \sum h^i \rangle = \langle \pi, 0 \rangle = 0.$$

Since the U^i's satisfy the conditions of Proposition 5, $(\hat{c}_1, \ldots, \hat{c}_m)$ is an α-efficient allocation .

2. For each i, the consumption process $\hat{c}^i$ is bounded away from zero. This implies $\mathrm{L}^\infty(\mathbb{P} \otimes dt) \subseteq F(\hat{c}^i)$. Suppose the converse, there are two agents i and j such that $\partial \alpha_i U^i(\hat{c}^i) \cap \partial \alpha_j U^j(\hat{c}^j) = \emptyset$. Then there is an $h_i \in F(\hat{c}^i) \setminus \{0\}$, an $h_j \in F(\hat{c}^j) \setminus \{0\}$ and an $h \in H(\hat{c})$ with $h^k = 0$ if $k \notin \{i, j\}$ such that, for all $D\alpha_i U^i(c^i) \in \partial \alpha_i U^i(c^i)$ and $D\alpha_j U^j(\hat{c}^j) \in \partial \alpha_j U^j(\hat{c}^j)$, we have

$$\begin{aligned} 0 &< \mathbb{E}[\int_0^T h_t^i \pi^i(\hat{c}^i)_t - h_t^i \pi^j(\hat{c}^j)_t dt] \\ &= \mathbb{E}[\int_0^T h_t^i \pi^i(\hat{c}^i)_t + h_t^j \pi^j(\hat{c}^j)_t dt] = \sum \langle D\alpha_i U^i(\hat{c}^i), h^i \rangle_{\mathrm{L}^2}, \end{aligned}$$

where $\pi^j(\hat{c}^j)$ is the Riesz representation of $D\alpha_j U^j(\hat{c}^j)$. But this contradicts Proposition 5. ■

The following two results are used in Lemma 1 and in the proof of the F-properness in Section 3.3. The approach goes back to [DuZa 89]. The aggregator is not differentiable in u and z (but concave) and hence we need the following mean value theorem for convex functions.

Proposition 6. *Let $g : \mathbb{R}^r \to \mathbb{R}$ be a lower semi-continuous and convex function and it takes no where the value $-\infty$ and is not equal $+\infty$. Let X be a convex set in $dom(g)$, then*

$$x^1, x^2 \in ri(X) := \{x \in X : \forall y \in X \exists z \in X, \exists t \in]0,1[: x = ty + (1-t)z\}$$

implies that there exist a $t \in]0,1[$ and a vector $x^* \in \partial g(tx^1 + (1-t)x^2) \subset \mathbb{R}^n$ such that $g(x^2) = g(x^1) + \langle (x^2 - x^1), x^* \rangle_{\mathbb{R}^n}$.

Proof of proposition 6. [Weg 74,Theorem, p.207]

In the present case Lemma 4 and Lemma 5 are formulated such that application to the contradiction argument in Lemma 1 "fits" the agent j.

Lemma 4. *Assume that U is a generalized stochastic differential utility generated by an aggregator f that satisfies Assumption 1. Let $A \in \mathcal{O}$ and $a > 0$ be arbitrary. If $y, x \in \mathrm{L}^2_+$ with $y \geq a$ on A, $x = 0$ on A^c and $x \leq \frac{a}{2}$, then*

$$U(y) - U(y-x) \leq e^{kT} \mathbb{E}[\int_0^T \overline{\delta}_f(\frac{a}{2}) x_t + k|Z_s - \bar{Z}_s| dt].$$

Proof of lemma 4. Let $(U_t, Z_t)_{t \in [0,T]} = (U, Z)$ be the solution of the utility process related to y and $(\bar{U}, \bar{Z})$ the solution of the utility process related to $y - x$ where x is chosen as above. f is, by assumption, differentiable in c. We apply the classical mean value theorem to the consumption component. Since f is uniformly Lipschitz continuous in u and z, upper semi-continuity follows and we apply Proposition 6 to $-f(t, c, \cdot, \cdot)$. We conclude that there is an $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ valued process (ξ^c, ξ^U, ξ^Z) such that

$$\begin{aligned} U_t - \bar{U}_t &= \mathbb{E}[\int_t^T f(s, y_s, U_s, Z_s) - f(s, y_s - x_s, \bar{U}_s, \bar{Z}_s) ds | \mathcal{F}_t] \\ &= \mathbb{E}[\int_t^T \partial_c f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s) x_s \\ &\quad + D_U f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s)(U_s - \bar{U}_s) \\ &\quad + \langle D_Z f(s, y_s + \xi^c_s, U_s + \xi^U_s, Z_s + \xi^Z_s), (Z_s - \bar{Z}_s) \rangle ds | \mathcal{F}_t]. \end{aligned}$$

Observe $U_s - \bar{U}_s \geq 0$, for all $t \in [0,T]$, by Proposition 1. since $x \geq 0$ and f is increasing in consumption. Combined with the boundedness of the

super-gradients, we derive:

$$\begin{aligned} U_t - \bar{U}_t &\leq \mathbb{E}[\int_t^T \partial_c f(s, y_s + \xi_s^c, U_s + \xi_s^U, Z_s + \xi_s^Z) x_s + k(U_s - \bar{U}_s) \\ &\quad + \langle D_Z f(s, y_s + \xi_s^c, U_s + \xi_s^U, Z_s + \xi_s^Z), (Z_s - \bar{Z}_s)\rangle ds | \mathcal{F}_t] \\ &\leq \mathbb{E}[\int_t^T \overline{\delta}_f(\frac{a}{2}) x_s + k(U_s - \bar{U}_s) + k|Z_s - \bar{Z}_s| ds | \mathcal{F}_t] \end{aligned}$$

The last equality holds because $x \mapsto \partial_c f(x, v)$ is decreasing and using the estimate $\overline{\delta}_f(\frac{a}{2})$, since $y_s(\omega) + \xi_s^c(\omega) \geq \frac{a}{2}$ on A.
Finally, the first Stochastic Gronwall inequality, see [DuEp 92, Corollary B, p. 386] evaluated at time zero yields:

$$U(y) - U(y - x) = U_0 - \bar{U}_0 \leq e^{kT} \mathbb{E}[\int_0^T \overline{\delta}_f(\frac{a}{2}) x_s + k|Z_s - \bar{Z}_s| ds] \qquad \blacksquare$$

Lemma 5. *Assume that U is a generalized stochastic differential utility generated by an aggregator f that satisfies Assumption 1. Let $H \in \mathcal{O}$, $h > 0$ and $y \in \mathrm{L}_+^2$ with $y \leq h$ on H. Then $\forall \lambda \in [0, h]$*

$$U(y + \lambda 1_H) - U(y) \geq e^{-kT} \mathbb{E}[\int_0^T \underline{\delta}_f(2h) \lambda 1_H(t) - k|\bar{Z}_s - Z_s| dt].$$

Proof of lemma 5. Let $(U_t, Z_t)_{t \in [0,T]} = (U, Z)$ be the solution of the utility process of the process y and $(\bar{U}, \bar{Z})$ the solution of the utility process of $y + \lambda 1_H$, where x is chosen as above. f is differentiable in consumption and concave in the other components. Applying the mean value theorem for c and Proposition 3.5.2 for U and Z, there is a $\mathbb{R}^{2+n}$ valued process (ξ^c, ξ^V, ξ^Z) and we have

$$\bar{U}_t - U_t \geq \mathbb{E}[\int_t^T \underline{\delta}_f(2h) \lambda 1_H(t) - k(\bar{U}_s - U_s) - k|\bar{Z}_s - Z_s| ds | \mathcal{F}_t].$$

The last inequality follows from the application of the estimates $\underline{\delta}_f(2h)$ (since $y_s(\omega) + \xi_s^c(\omega) \leq 2h$ on H) and arguments similar to Lemma 4.
We have $U_s - \bar{U}_s \geq 0$ since $z \geq 0$ and f is increasing and therefore, by Proposition 1. also the utility functional U.
Finally, the second Stochastic Gronwall inequality, see [DuEp 92, Corollary B, p. 387], evaluated at time zero now gives us:

$$U(y) - U(y - x) = U_0 - \bar{U}_0 \geq e^{-kT} \mathbb{E}[\int_0^T \underline{\delta}_f(2h) \lambda 1_H(t) - k|\bar{Z}_s - Z_s| ds] \qquad \blacksquare$$

A.3 Quasi-equilibrium in normed lattices

This section presents an existence result for economies where the commodity space is a certain vector lattice.
Let (L, τ) be the commodity space, a vector lattice with a Hausdorff, locally convex topology τ. We assume that the positive cone L_+ is τ-closed and the analytic dual space L^* is a sublattice of $L^\star$. The space $L^\star$ consists of all order bounded linear forms. A linear form is *order bounded* if the set $f([x,y]) = \{f(z) : z \in [x,y]\} \subset \mathbb{R}$ is contained in a bounded interval for each order interval $\{z \in X : x \leq z \leq y\} = [x,y] \subset L$.

We fix a pure exchange economy with $m \in \mathbb{N}$ agents $\mathtt{E} = \{L_+, P_i, e^i\}_{1 \leq i \leq m}$ in L such that $P_i : L_+ \to 2^{L_+}$ are the preference relations on the consumption set L_+ and $e^i \in L_+$ is the initial endowment of each agent.
An allocation $(x^1, \ldots, x^m)$ is individually rational if $e^i \notin P_i(x^i)$ for every i.
A *quasi-equilibrium* in the economy $\mathtt{E}$ consists of a feasible allocation $(x^1, \ldots, x^m) \in L_+^m$, i.e. $\sum x^i = e$, and a linear functional $\pi : L \to \mathbb{R}$ with $\pi \neq 0$ such that, for all i $\pi(x^i) \leq \pi(e^i)$ and for any i, $y \in K_+$ with $y \in P(x^i)$ implies $\pi(y) \geq \pi(x^i)$. The quasi-equilibrium is an *equilibrium* if $y \in P(x^i)$ implies $\pi(y) > \pi(x^i)$.

We introduce the notion of forward properness whose principle is a modification of this cone characterization and was introduced in [YaZa 86].

Definition 1. *A preference relation $P : L_+ \to 2^{L_+}$ is* F-proper at $x \in L_+$ *if: There is a $v \in L_+$, some constant $\rho > 0$ and a τ-neighborhood U satisfying, with $\lambda \in]0, \rho[$:*

If $z \in U$, then $x + \lambda v - z \in L_+$ implies $x + \lambda v - \lambda z \in P(x)$

The following standard assumptions are needed to establish the existence of a quasi-equilibrium.

Assumption 2. *The economy satisfies the following conditions:*

1. *$y \notin P_i(y)$ for all $y \in L_+$ and every i*
2. *$P_i(y)$ is a convex set for all $y \in L_+$ and every i.*
3. *There is a Hausdorff vector space topology η on L such that $[0, e]$ is η-compact and such that, for every $i = 1, \ldots, m$, P_i is η-τ continuous, i.e. the graph.*

$$gr(P_i) = \{(x,y) \in L \times L : x \in L_+, y \in P_i(x)\}$$

is a relatively open subset of $L_+ \times L_+$ in the product topology η-τ.

4. *$P_i(y) \cap L(e) \neq \emptyset$ for all $y \in [0, e]$ and every i.*

5. *$L(e)$ is τ-dense in L and if $(x_1, \ldots, x_m) \in L_+^m$ is an individually rational and Pareto-optimal allocation, then, for every i, P_i is F-proper at x_i.*

Theorem 2. *Suppose the economy* E *satisfies Assumption 4.1. Then there is an $x \in L_+^m$ and a $p \in L^*$ such that (x, p) is a non-trivial quasi-equilibrium.*

Proof of theorem 2. [Pod 96, Theorem 2, p.471]

If preferences are strictly monotone and continuous and the total endowment is strictly positive, the notions of equilibrium and quasi-equilibrium coincide. This is can be found in [AB 03, Corollary 8.37, p.233], where it is requested that L^* is a sublattice of $L^\star$.

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