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Contributions to Overlapping Generations models: Increasing Returns, Durable Goods and Optimality

vorgelegt von:
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To Bebe Maria, Dada sy Neny

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Abstract

This Ph.D dissertation develops general equilibrium issues of overlapping generations economies (OLG hereafter), which play important roles to study many intertemporal phenomena. They are particularly adapted to study issues on distributions between different generations, pensions systems, debts, taxes, money, monetary policy, interest rates and growth. We use a general equilibrium approach and study classical issues such as the existence of equilibrium, characterization of Pareto optimal allocations.

Many works on OLG have some limitations regarding the production sector. Indeed, standard hypothesis supposes that production set is convex, returns to scale are either constant or decreasing. But many sectors involve high technological revolution which implies internal or external economies of scale, highlighting the importance of increasing returns in growth. The first step of our work is thus to consider, a standard OLG model with production beyond the classical hypothesis of constant returns to scale. We are concerned, in the first place by a formalization of increasing returns in OLG models, described in Balasko, Cass and Shell [3], Balasko and Shell [4], [5]. The production possibilities are described by a sequence of production mapping and the main innovation comes from the fact that we allow for increasing returns to scale of more general type of non-convexities. To describe the behavior of the firms, we consider loss-free pricing rules, which cover the case of the average pricing rule, the competitive behavior when the firms have convex production sets, and the competitive behavior with quantity constraints. We prove the existence of an equilibrium under assumptions, which are at the same level of generality than the ones for the existence in an exchange economy.

Beyond the existence result, we are led to study the mechanism of transfer between generations in order to analyze the possible perpetuation of firms. We then incorporate durable goods which may be stored from one period to a successive period through a linear technology. In this model, we establish not only the existence of an equilibrium but also highlight features of durable goods that entitle consumers, the roles of producers, lenders and borrowers, even at the end of their lifetime. Another important result on the relation between prices allows us to make a link with the Pareto efficiency of equilibrium, confirming their role in restoring the market failure in OLG economies. We review the characterization of Pareto optimal allocations, in the line of Balasko and Shell [4], but in addition we allow for multiple agents and multiple goods per period. Our approach is set-theoretic and geometrical. The consumers characteristics are described by their consumption sets, their preference sets and the associated normal cones. We give conditions of Pareto optimality, under very basic assumptions, by providing a simple and geometric version of the proof of Balasko and Shell [4], encompassing the case of non-complete and non-transitive preferences.

Résumé

Cette thèse s'inscrit dans l'étude des modèles à générations imbriquées qui a aujourd'hui un rôle central pour étudier de nombreux sujets en économie. Les modèles générations font l'objet de travaux tant en microéconomie qu'en macroéconomie à côté des modèles de croissance optimale. Une des limitations des nombreux travaux sur les modèles générations provient de la modélisation du secteur productif, à savoir les hypothèses standards de convexité, les rendements d'échelle constants ou décroissants. Mais certaines productions sont concernées par des la révolutions technologiques impliquant des économies d'échelle interne et externe. De plus, si on considère les modèles de croissance endogène, on assiste à des externalités concernant le capital humain, qui sont des sources de rendement croissant au niveau agrégé. Ainsi, les rendements croissants sont essentiels en économie.

Nous commençons alors par considérer, dans le Chapitre 2, un modèle à généra- tions imbriquées avec production, mais au-delà de l'hypothèse classique de rendements constants. Les capacités de production sont modélisées par une suite de fonctions multivoques de production, de plus, nous ne faisons aucune hypothèse sur les rendements d'échelles, ainsi les rendements croissants sont permis. Les comportements des producteurs sont décrits par des règles de tarification sans perte, qui comprennent la tarification au coût moyen, le comportement compétitif lorsque les ensembles de production sont convexes, ainsi que le comportement compétitif sous contrainte à la Dehez–Drèze [23], [24]. Nous établissons l'existence d'équilibre sous des hypothèses aussi générales que celles des modèles d'échange pure.

Dans le Chapitre 3, nous proposons d'étudier la possibilté de transfert entre les générations. Pour celà, nous introduisons des biens durables qui peuvent être stockés entre deux périodes successives, grâce à une technologie linéaire. Nous démontrons l'existence d'équilibre via une ćonomie équivalente sans bien durable, dans laquelle la période de vie de chaque consommateur est étendue sur trois périodes. Un résultat additionnel concernant la relation entre les prix nous permettrait de faire le lien avec les conditions d'optimalité.

Le Chapitre 4 reprend les conditions d'optimalité vues dans Balasko et Shell [4], mais en plus nous considérons le cas où il y a plusieurs agents par période. Notre approche est ensembliste et géométrique. Chaque consommateur est caractérisé par son ensemble de consommation, ainsi que son ensemble d'allocations préférées et le cône normal correspondant. Nous caractérisons les allocations Pareto optimales en donnant une approche simple et géométrique à la preuve de Balasko et Shell [4], tenant aussi compte des préférences non-complètes et non-transitives.

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Chapter 1

General Introduction

1.1 OLG models of general equilibrium

Overlapping generations models (OLG hereafter) were introduced by Allais [1] in 1947 and Samuelson [54] in 1958. Observing that "*we live in a world where new generations are always coming along*", Samuelson built a model in which generations overlap indefinitely, in contrast to the Arrow-Debreu model which assumes all agents as contemporary. OLG models evolve infinitely many dates, and at each date, a "generation" of new agents is born and agents live within two subsequent dates. Thus, at date t, consumers with different lifetimes coexist: the young consumers born at date t and the old ones born at the preceeding date $t-1$. In addition, at each date, there are finitely many goods. One important feature of the Samuelson's OLG model is this double infinity of goods and consumers, which is source of many unobserved phenomena in Arrow-Debreu economies: money has a value and the first welfare theorem may fail. Indeed, in an economy where there are many dates, consumers may wish to transfer their wealth accross different periods of their lifetime, by means of contracts for future deliveries, debts or money. In a finite horizon case, by backward induction, it is impossible to hold money with a positive value since at the last period it is worthless, so it is at one period before the last one and so on. But in an OLG economy, there is no last date, money can have a positive value even though it does not enter as an argument in the preferences of consumers, and competitive equilibria need not be Pareto efficient. Indeed, by taking into account even the simple demographic structure of OLG models, the wealth of agents at equilibrium prices can be finite although the total endowments have infinite value. OLG models highlight the possibility of inefficiency, which is excluded in the Arrow-Debreu model, and which can be recovered by transfers between generations. OLG models are then more specific and more realistic than the standard and static Arrow-Debreu model.

They are particularly approppriate to study redistribution or reallocation issues between different generations (pension system, debt, taxes etc), intertemporal phenomena such as monetary policy and growth, besides optimal growth theory.

Samuelson's OLG model has been since developped into a more comprehensive model called also "*OLG model of general equilibrium*"as pointed by Geanakoplos in [34]. An OLG model can thus be described as a general equilibrium model with infinitely many agents and goods, and a production sector. The idea is to use a mechanism analysis which aims at finding the "good" allocation for all economic agents, that is, an allocation which satisfies, in term of utility the current generations but which does not penalize the future generations. Our work lies in this respect and treats general equilibrium issues such as equilibrium existence, optimality, but in addition we enlarge the framework by relaxing some classical restrictions and allowing for more general but simpler results. In this line, we go beyond the limiting assumption of constant returns to scale and consider the possibility of increasing returns to scale in the production side. In order to study transfer mechanisms accross generations, we allow for goods to be durable, they can be consumed but can also play other roles such as input factors (in a storage technology) or collaterals for loans. Thus any agent who holds a durable good acts as consumer and producer at the same time, and sees his lifetime artificially extended given the remaining wealth that may still persist after they disappear from the economy. Their role in restoring inefficiency in OLG models leads us to consider Pareto optimality characterizations. For this matter, we propose to revisit the characterizations provided by Balasko and Shell [4], as a note where the setting considers non-transitive and non-complete preferencesand the approach is mainly set-theoretic.

In the section that follows, we highlight the relevence of increasing returns to scale in economics, their driving sources, and what has been documented in literature.

1.2 Increasing Returns to Scale (IRS)

1.2.1 On the sources of increasing returns to scale

Increasing returns to scale occur when the firm adds more to output than to costs, that is there is an increase in output as cost decreases. Thus IRS have their sources in economies of scale, internal or external to the firm, resulting in an increase of the average productivity of the firm.

Internal economies come from the expansion of the firm itself, that is its average costs depend on its size. There are many ways to achieve internal economies of scale. They can occur due to technical economies, which can be related to a large machinery and equipments resulting in a production of large scale so that the average cost declines. In addition, large firms have the possibility to increase productivity by using a labour division, splitting tasks according to the specialization of the workforce. Another form of labour division is the managerial economies of scale, showing the importance of trained and qualified employees who are able to take quicker and better decisions. Good managers are able to find new methods and equipments, more profitable to the firm, cutting wastes and allowing for more efficiency in terms of time thus reducing production costs. Bigger firms have more possibilities to enjoy financial economies of scale since they are more favorable to loans at lower rates than the smaller ones, leading to additionnal resources and opportunities to raise their scale. Another source of internal economies is the monopsy power, which occurs when a firm has access to its factor inputs at lower prices in the market. This is the case of some large firms who are able to negotiate lower prices when purchasing raw materials in some poor countries.

External economies of scale occur outside of the firm, in this case its average costs do not depend on its size, however they can be internal to the industry to which it belongs. This corresponds to Alfred Marshall's[1] treatment of increasing returns: external to the firm but internal to the industry. An important source of external economies of scale is knowledge spillover that benefits industrial clusters. Indeed workers from different firms can easily interact, share ideas and knowledge, allowing firms to take advantage of improvements of human capital, inventions and technical successes of other firms. These informal channels and knowledge diffusions are not costly but crucial for success. These interactions are beneficial especially for large industries. It is evident that external economies can also be brought by scientific progresses of local universities, or by creations of transportations and infrastructures that firms gathered in the surrounding area can benefit.

These two types of economies of scale have different implications in the market. An industry concerned with external economies, for instance can consist of small firms under perfect competition. Indeed, it is possible that given productions of all other firms, a single firm j's production set is a convex cone. Thus increasing returns due to external economies can be compatible, at least partially to the standard competitive model. However, internal economies result in cost advantage of large firms over small ones, leading to deviations from perfect competition and to alternative modelling of firms behaviour. This explains the introduction of pricing rules in literature, and we shall see that, under a nonnegativity condition, this notion also allows to give account of models with increasing returns in macroeconomics, where they are usually associated to imperfect competition.

[1]See Marshall [45] (1920)

Before introducing the notion of pricing rules, let us recall an example provided by Laffont (1979) [39] and used by Bonnisseau (1994) [9] to illustrate that due to externalities, increasing returns are internalized so that at the aggregate level, the production set fails to be convex. More specifically, the iso-output set is non-convex, leading to an optimal production that neither maximizes the profit nor minimizes the cost. The example is as follows:

Consider three goods A, B, C, where C is used as a factor input and A and B are consumption goods produced by two firms. The first firm produces at constant returns to scale, and its production set is given by:

$$Y_1 := \{(a,b,c) \in \mathbb{R}^3 \mid b \leq 0, c \leq 0, a \leq -c\}$$

The second firm incurs negative externalities from the first firm. Given a production plan (a,b,c) its production set is:

$$Y_2(a,b,c) := \{(a,b,c) \in \mathbb{R}^3 \mid a \leq 0, c \leq 0, b \leq -c(1+\min\{\frac{1}{2}, -\frac{c_1}{a_1} - 1\})\}$$

There is a unique consumer, endowed initially with $\omega = (0,0,1)$, and whose utility function is defined by: $u : (x_A, x_B, x_C) \in \mathbb{R}^3_+ \to x_A(x_B)^3$. Although the productions sets are convex, the aggregate production set, given by

$$Y := \{y \in \mathbb{R}^3 \mid \exists y_1 \in Y_1, \exists y_2 \in Y_2(y_1), y = y_1 + y_2\}$$

is not convex. Indeed, for a total input factor equals to 1, the intersection between Y and the plane given by $c = 1$ fails to be convex, as shown in Figure 1.1. Thus Y is not convex, which implies that the graph of Y_2 is not convex. There is a unique Pareto optimum, given by: $x^* = (\frac{1}{6}, \frac{9}{8}, 0)$, $y_1^* = (\frac{1}{6}, 0, -\frac{1}{4})$, $y_2^* = (0, \frac{9}{8}, -\frac{3}{4})$, where both firms operate. Note that the production plan y_1^* is not efficient, since firm 1 can still srictly increase his production in good A at the same input level and the minimization of cost would be reached if only firm 1 operated. Moreover, the aggregate production $y^* = y_1^* + y_2^*$ does not correspond to the maximization of profit at any given price. Thus, when externalities are internalized, increasing returns occur so that the production is not convex anymore, and the profit maximization or cost minimization at given price become meaningless.

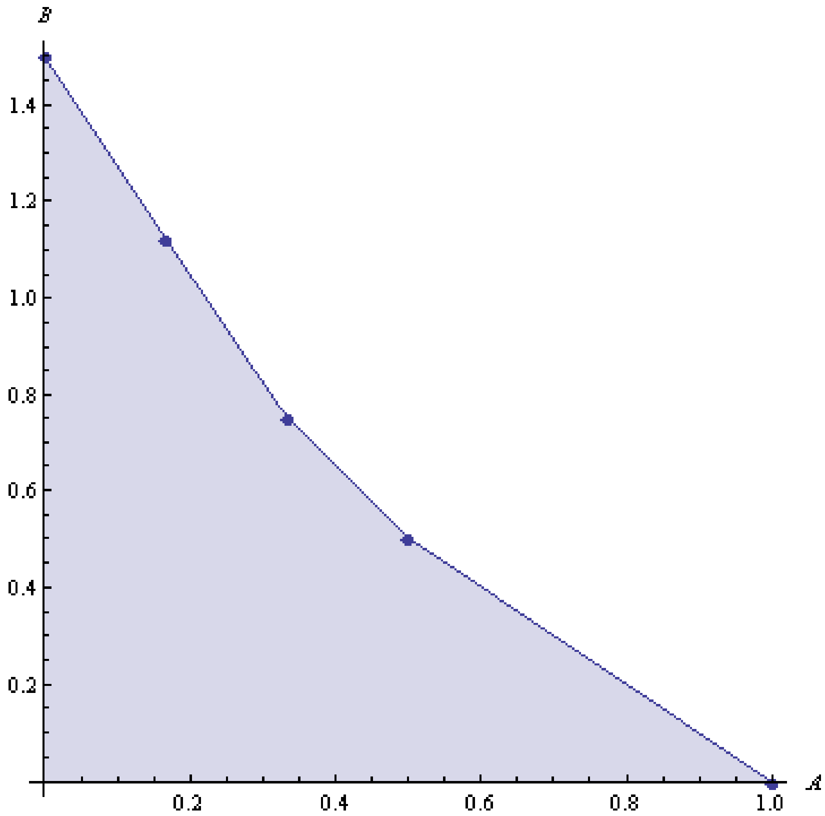

FIGURE 1.1: An example of non-convex production due to externalities ([9], [39])

1.2.2 The concept of Pricing Rules

Increasing returns to scale are sources of non-convexities in production, which makes the profit maximizing behaviour at given prices meaningless. In this line, it is necessary to model the behavior of producers beyond the profit maximizing one.

A pricing rule maps a firm's set of weakly efficient production plans to the set of prices. This concept allows for a more general type of behavior of firms.

Formally, let us consider a finite set of producers $\mathcal{J}$. Let Y^j be the production set of firm j and let F^j denote set of weakly efficient production plans, $F^j := \{y^j \in Y^j \mid y'^j \notin Y^j\}$. A pricing rule is a mapping φ^j which associates to a feasible and weakly efficient production plan $y^j \in F^j$, a set of prices $\varphi^j(y^j)$ compatible with this production. The graph of φ^j describes combinations of prices and production plan that firm j finds "acceptable". This concept allows for different behaviors thus different pricing rules for different firms. When a combination of prices and production plans (p, y) is found acceptable to all of the firms find , it is called a production equilibrium. Such an equilibrium may not be reached if pricing rules fail to be continuous or sensitive. Thus, although it is a general and flexible concept, a pricing rule cannot be completely arbitrary, and some regularity restrictions are needed on the admissible pricing rules. We will then focus on non-empty-, compact- and convex- valued mappings, which graphs are closed. These properties ensure the closedness and connectedness of the graph of the pricing rules. Particular cases are:

- the profit maximization at given prices, in the case of convex production, or external economies of scale,
- the average cost pricing: when firms choose prices which make them just break-even,
- the marginal pricing rule: firms are instructed to sell their outputs at prices that satisfy the first order condition for optimality (in this case, losses may occur under increasing returns to scale),
- the constrained maximization profit: firms maximize profits at given prices, but subject to quantity constraints.

When firms exhibit non-convexities, then losses are possible. It is then natural to require firms to have boundaries on their losses, in order to ensure that a production equilibrium exists. This property ensures sensitivity of the pricing rules to changes in the production plan y^j, and non-emptiness of the set of production equilibria. A pricing rule has the bounded losses property when there exists a scalar $\alpha^j \leq 0$ such that for each $y^j \in F^j$, $q \cdot y^j \geq \alpha^j$, for all $q \in \varphi^j(y^j)$. When pricing rules are regular with bounded losses, the set of admissible prices given a production plan cannot reduce to a singleton, this is a favorable condition to the existence of production equilibrium. A general existence result when firms follow bounded losses pricing rules is established by Bonnisseau and Cornet in [10].

Pricing rules can be endogeneous or exogeneous to the models, they allow for both price-taking and price setting behaviours. In [26], Dierker, Guesnerie and Neuefeind established the existence of equilibrium when some firms are price takers and other firms are price setters: they set the prices of their products given the prices of the inputs. Their result encompasses a wide array of pricing rules, including the marginal cost pricing rule. Their result also takes into account the losses incured by firms, financed through exogeneously given shares.

1.2.3 Some existing results on IRS

There is a wide range of literature in microeconomics and macroeconomics, taking into account increasing returns to scale. Most of such works are centered on endogenous growth.

On one side, many results and analysis on the existence of equilibrium when the behaviors of firms are described in terms of pricing rules have been established in literature, see Cornet [22] (1988), Dehez and Drèze [23], [24], Heal [38], and Villar [59] (2000), with the latter work covering very comprehensive results on convex and non-convex production economies. More recent results also can be found in Bonnisseau and Jamin [11], [12] (2008, 2009), where [12] treats increasing returns in intertemporal economies, which is then directly applicable to OLG models, when extended to the infinite period case.

Brown and Heal [18] (1979) has reached a result standing for the possibility of increasing returns to be an essential ingredient to attempt an economic development, when they treat equity and efficiency in a Walrasian framework with production and increasing returns. In [18], the owner of a firm that my incur losses has no interest to close it down. It is indeed concluded that "*with increasing returns to scale in production, it may be possible to remove some endowment from one person, give it to another, and make both better off*". Translating this result to development economics supposes that in an under-developed country with a highly unequal distribution of income, a redistribution away from the very rich may in the long run make all better off, because the acquisition of purchasing power by the middle and lower income groups may lead to development of a mass market and a substantial increase in industrial profits.

On the other side, many works and results have been developed on models of endogeneous growth based on increasing returns. Young [60] (1928) pointed out, in his analysis increasing returns as source of economic progress, and in the 1980s appears a remarkable revival of interest in economic growth. For instance, Romer [53] (1986) proposed an alternative view on long-run growth driven by the accumulation of knowledge, which has an increasing marginal product on the production of consumption good; thre, externalities, increasing returns in the production of output, and decreasing returns in the production of new knowledge are keys. In List and Zhou [40] (2007), increasing returns to scale arising from fixed costs of production and internal to the firm generate positive growth. Moreover, Lucas [41] (2002), Barro and Sala-i Martin [6] (2003) provide a thoughtful literature synthesis on economic growth.

Now, we consider some models of neoclassical growth with increasing returns introduced by Benhabib and Farmer [7] (1994), Gali [33] (1995), and Barseghyan and DiCecio [8] (2006). [12] focuses on indeterminacy caused by increasing returns, while [33] and [8] conclude on multiple steady states equilibria leading to poverty traps, for some degree of increasing returns to scale, a conclusion that is in opposition to the intuition proposed by Brown and Heal [18]. The common feature of these studies centers on the modelling of the production sector, where increasing returns are exclusively associated to imperfect competitions. The structure of the market consists of considering two main sectors of production: one for intermediate goods and the other one for final goods. Firms in the sector of final goods operate under perfect competition while each intermediate goods producer has a monopoly power over the good it produces [2]. This formalization allows then for the application of competitive behaviour approach on the production sector despite the increasing returns to scale. This imperfect competition structure, takes into account the market power that firms with increasing returns possess, allowing them to make positive profit. If a main reason of incompatibility between competition and increasing returns lies on the supply mappings that may fail to be well defined, this market structure allows to write a closed form making equilibrium possible in the presence of incresing returns. This modelling is actually a variant of pricing rules that generates no losses, but as concluded in Barseghyan and DiCecio [8], "*bad equilibria*"and poverty trap may occur. We then wonder whether this structure is not neutral to this result and if going beyond it by letting firms behave differently according more general pricing policies would not result in good equilibria only, in favour of increasing returns in development economics.

1.3 Approach and Overview of the results

Although the focus and the motivations are not necessarely the same, the results enumerated above bring further issues to be deeply considered. We are concerned, in the first place by a formalization of increasing returns in OLG models, described in Balasko, Cass and Shell [3], Balasko and Shell [4], [5]. This thesis consists of three mains chapters which correspond to three main papers, on increasing returns to scale [13], on durable goods [51] and on optimality characterizations [14].

[2] In [42] is provided the notion of Staggered pricing model: the idea is to embed intermediate goods produced by firms with market power. Intermediate goods are bundled without cost (without labor or capital) into a final good by a competitive firm. This brings their market power to intermediate goods producers

1.3.1 OLG models with increasing returns

We start with the OLG models, described in Balasko, Cass and Shell [3], Balasko and Shell [4], [5], see also Tvede [57] for a more intuitive version of OLG models. Our work relies on existing results on equilibrium studies in economies with increasing returns to scale, which namely consist of modelizing the production sector in a more general way using the concept of pricing rules.

Given the demographic structure of OLG economies, where at each date $t = 1, 2, \ldots$ appears a new cohort of individuals living for two successive dates, we model production capacities sequentially through set-valued mappings (F_t^j) which associate, to a vector of inputs at date t, a vector of outputs at date $t+1$. This implicitely supposes that production takes time, and an investment at a period one cannot give any return before the subsequent period. Since the production is described in a recursive way, one can think of a pricing rule, defined in a similar way. Thus at each period t, there is a set-valued mapping φ_t^j which associates to a weakly efficient production plan y_t^j of firm j a set of admissible prices. This corresponds to the idea that production decisions are taken at each period, and that compatible prices are also determined at each period. A global pricing rule would correspond to a production and a price decides at the firt period $t = 1$ and established for all the future periods. This approach is less suitable with the structure of OLG models, where at each period, new consumers take part to the economical decision and thus to the production decision related to their lifetime.

We do not make any assumption on the returns to scale, but we posit classical hypothesis such as closedness, free-disposal and possibility of inaction. We consider firms to be active forever once they are set, but this does not exclude the possibility of considering productions starting at each date and active for two subsequent dates as in Tvede [57]. In this case, we have a sequence of set-valued mapping $(F_t^j)_t$ started at each date t, active at dates t and $t+1$, and inactive at dates $t' \neq t, t+1$, that is $(F_{t'}^j)_{t'}$ are identically null.

We suppose in addition that firms are privately owned and we will naturally make use of the properties of regularity and of bounded losses in our model with production. In particular, our approach consists of cases where firms admit only non-negative profits, thus we will focus on loss-free pricing rules, that is the bound α^j defined in section 1.2.2 is reduced to zero. This restriction is natural when firms are not regulated and when $0 \in Y_t^j$, that is firms are privately owned and can always refuse to produce, instead of incuring losses. These pricing rules may be considered when there are increasing returns to scale, fixed costs, when there is a fixed capital that is indivisible input, or the production function is S-shaped.

Particular loss-free pricing rules are the constrained profit maximization, the average cost pricing, or more generally the mark-up pricing, which consists of adding a profit component to the average cost. More details on this class of pricing rules are provided in Villar [59]. As already remarked earlier, increasing returns are usually associated to imperfect competition, especially in macroeconomics, but the pricing that results from such a market is actually a variant of loss-free pricing rule. For instance, the one proposed by Benhabib and Farmer [7] is a mark-up pricing rule where the mark-up is constant.

Given the private ownership and the free losses assumptions, profits possibly strictly positive are redistributed among consumers according to their exogeneous shares on firms. We then prove the existence of an equilibrium under assumptions, which are at the same level of generality than the ones for the existence in an exchange economy. It is important to remark that the free-disposal assumption is key in this existence result. Indeed, as established in Bonnisseau and Cornet (1988) [10] [3], this assumption allows the usual existence to hold when production sets are non-convex. This explains why we introduce positive polar cones in our proof, which at least permits our global production to satisfy a weak form of free disposal.

However, this result relies on an assumption of private ownership with given shares, which is rather restrictive in an economy where firms are perpetuated by successive generations, but whose transfers of firms ownerships are not formalized since no stock markets are introduced. This issue,clearly ignored if firms exhibited constant returns to scale with zero profits highlights the usefulness of considering a general wealth distribution function and a mechanism of wealth transfers between successive generations, which will allow to keep the firms active forever. Letting newly born individuals be endowed with exogeneously given shares can be explained through bequest motives, that is old agents of previous generation, since they are about to die, freely leave to the young their ownership. In this way, agents can prevent the disappearance of their firms from the economy, and their offsprings who become the new owners of the firms will distribute the profits between them according to the shares they have inherited, leave the ownerships to the next cohort and so on. Another possibility, implicit to our model, is the existence of hidden financial markets, where agents either save at risk-free rate or invest in firms by buying shares, but either choice is supposed to bring them the same return, under the absence of arbitrage condition and the equi-profitability among the firms.

Although our model also accounts for a succession of firms active only for two dates, the possibility of perpetual firms drives us to consider durable goods, through which,

[3] See [10] Lemma 5.1

transfer of wealth, and in particular of capital of firms could be meaningful accross generations.

1.3.2 OLG models with durable goods

Throughout chapter 3, we will see that durable goods serve in transferring wealth accross generations, either through markets or bequest motives, but in addition, they can also provide to their owners, possibilities to have access to a loan backed up by collaterals, to hold inputs for additionnal endowments that can be useful to the future, not only for their well-being during their old age but also for the well-being of their offsprings.

We start considering a simple pure exchange economy with durable goods and will make use of the intuition obtained in this framework to understand the transfer mechanism between generations. Our model incorporates then durable goods which may be stored from one period to a successive period through a linear technology. We show that the existence of an equilibrium can be established by considering an equivalent economy "*without*" durable goods, where the agents economic activity is extended over three successive periods. This intermediate step helps to confirm that a consumption of durable goods by young agents has an impact, both on their consumption when old, but also after their lifetime, this explains the extension from a two-period lifetime into three-period one. It is clear that the consumption behaviour has impacts not only on their close future or their own generation but also on the succeeding generations.

However, above the existence result, this paper provides a mechanism within which generations successively transfer their wealth, by means of durable goods, and shows that their consumptions, thanks to their durability, entail their owners the role of producer: they have in their possession a technology that allows them to store or transfer their consumption to the next period. Durable good can be then assimilated to a saving, an asset holding that permits young agents to shift their endowment to a preferred consumption plan. In the same spirit, our model implicitely supposes the existence of a financial markets, where agents take loans and back them up with their durable goods, which will be systematically seized once they disappear from the economy. Basically, our model entitles agents to buy and hold durable goods, seen as assets, even at their old age; this situation can be explained by a lifetime contract called also viager, when a house, a durable good, allows old agents to enjoy not only of that house but also of an additionnal resource corresponding to the financial value of it on the future market. Our model is not incompatible with uncertain lifetimes, where agents may hold assets or be involved in any liability at the end of their lifetime. If indeed, at each period, they have a probability to die in the next period, then defaults are prevented thanks to the durable goods that serve as collaterals to loans. In both certain and uncertain frameworks, we remark that a purchase of durable goods by old can be motivated by bequest motives, where agents are supposed to be altruistic.

While we mainly focus on the mechanism of wealth transfer between generations, it is important to note that many works, theoretical and empirical, on durable goods already exist in the literature, to study different issues, such as savings, borrowing constraints and collaterals. Furtherore, since durable goods are used as component in wealth, they are useful to study wealth distribution, see Diaz and Luenngo-Prado in [25], who in addition relate the two with precautionary savings. Indeed, in [25], the liquidity or illiquuidity nature of durable goods, have impact on the behaviour and wealth composition of agents, reflected to the notion of precautionary saving, especially when there is uncertainty or risk in the economy. Such an issue is not treated in our case, especially since the main feature of the durable goods we consider is their desirability. But this framework can be worth consideration for the continuation of this work, where liquid assets can be represented by shares on firms that may be may make positive profits. Moreover, these studies which involve empirical analysis consider durable goods which are not easily divisible, and entail very high transaction costs, as in [46]. In [25], they are assumed specific to households and cannot be traded or rented without first converting them back to a productive capital. Our model does not have this specificity and allows for a durable good to be divisible when sold on a future market to the young. But in case of production, this work meets that property and goes in the line of our aim to study the perpetuation of firms through transfers of shares and property rights accross generations. For instance, agents partipate in firms by putting together their investment so that each durable capital, possibly specific to each firm is kept in its entirety.

Another important result involves a relation between prices at equilibrium, which is similar to the one provided by Balaso and Shell in [4]. Indeed, equilibria may fail to be efficient in OLG models, and durable goods such as money, or an infinitely lived asset like a land, could restore the market failure. We then propose to revisit the Pareto optimality characterizations provided by Balaso and Shell.

1.3.3 Characterization of optimal allocations in OLG models with multiple goods

We propose to review in chapter 4 the characterization of Pareto optimal allocations, in the line of Balasko and Shell [4], but in addition we allow for multiple agents per period. Our approach is set-theoretic and geometrical. The consumers characteristics are described by their consumption sets, their preference sets and the associated normal cones. We give conditions of Pareto optimality, under very basic assumptions, by providing a simple and geometric version of the proof of Balasko and Shell [4].

Balasko and Shell [4] provide a criterion based on the asymptotic behavior of the norm of the prices to characterize Pareto optimal allocation without durable good or infinitely lived asset. Burke [17] revisits this criterion by focusing in particular on the right definition of the Gaussian curvature of the indifference surface. Actually, these authors provide a proof with a first step considering the special case of a single commodity per period. Then, the generalization to several commodities is only sketched.

Our purpose in this paper is three fold: to provide a simpler, direct proof of the Balasko-Shell Criterion considering in one step several consumers for each generation and several commodities; to encompass the case of non-complete, non-transitive preferences; to compute explicitly a Pareto improving transfer when the allocation does not satisfy the Balasko-Shell Criterion. Nevertheless, note that the structure of the proof is based strongly on Balasko-Shell's one.

It is important to remark that a geometrical approach has already been provided by Borglin and Keiding [16]. [16] considers infinite horizons economies, and treats the particular case of OLG models. They center the notion of Pareto optimality to its weak form, and consider characterizations based on parameters that describe the economy such as supporting prices and curvatures of indifference surfaces, thus an approach that easily meets our model.

We consider this contribution as a first step to be able in future works to tackle the question in presence of durable commodities and with heterogeneous longevities of the agents.

Throughout these three chapters, we have raised aditionnal issues but we have also accumulated further tools and intuitions that will be important and helpful for further studies, especially in production economies where increasing returns are allowed and thus growth can be expected.

Chapter 2

Existence of an equilibrium in an OLG model with increasing returns

Abstract

We consider a standard overlapping generations economy with a simple demographic structure where a new cohort of agents appears at each period and whose economic activity is extended over two successive periods, and finitely many firms are active forever. The production possibilities are described by a sequence of production set-valued mappings and the main innovation comes from the fact that we allow for increasing returns to scale of more general type of non-convexities. To describe the behavior of the firms, we consider loss-free pricing rules, which covers the case of the average pricing rule, the competitive behavior when the firms have convex production sets, and the competitive behavior with quantity constraints à la Dehez-Drèze. We prove the existence of an equilibrium under assumptions, which are at the same level of generality than the ones for the existence in an exchange economy[1].

JEL classification: C62, D50, D62.

Keywords: Overlapping generations model, increasing returns to scale, loss-free pricing rules, equilibrium, existence

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[1]This paper is based on "*Existence of an equilibrium in an OLG model with increasing returns*" [13]

2.1 Introduction

Overlapping generations models are studied both in microeconomics and in macroeconomics to analyze intertemporal phenomena. These models involve infinitely many dates, goods and consumers. This double infinity is source of many unobservable phenomenons in Arrow-Debreu economy even if the space of goods is of infinite dimension.

Regarding the production side, if we consider endogenous growth models, externalities might be introduced for example via the level of human capital, which are source of increasing returns at the aggregate level. But these returns are not taken into account by the agents, who have a myopic behavior in the sense that they do not care about the externalities they create.

We thus plan to study a standard overlapping generations model with production allowing increasing returns to scale and a behavior of the producers, which goes beyond the competitive one.

The basic model is the one introduced [3–5], see also [57] for a very intuitive approach. The production knowledge of a producer is described by generalized production correspondences, which define the possible outputs at one date given the vector of inputs consumed at the previous date. This sequential approach of the production allows to consider innovation along the time and heterogeneity of producers.

The equilibrium concept is the standard one but for the behavior of the producers since we do not assume that the production sets are convex. Hence the standard competitive behavior is meaningless.

In models allowing for non-convex technologies, the firms follow general pricing rules to describe a large range of possible behaviors including the profit maximizing behavior at given prices. The literature considers pricing rules which associate a set of admissible prices to a weakly efficient production. For a comprehensive introduction see [10, 22, 26, 59]. Since the production is defined in a recursive way, we propose to define also the pricing rule recursively, so that the prices for two successive dates depend on the production possibilities for these two dates and not for the other ones.

We consider loss-free pricing rules, meaning that the firms are restricted to get a non negative profit over two successive periods. This covers the case of the average pricing rule, the competitive behavior when the firms have convex production sets, and the competitive behavior with quantity constraints à la Dehez-Drèze, [23, 24].

Contrary to the case of a constant return technology, it is crucial to determine how profits (or losses) of producers are distributed among consumers. Indeed, the optimality of the equilibrium allocation depends on the repartition scheme. In this first paper, we only consider private ownership economies and we assume that the shares are given exogenously. It would be meaningful to introduce a stock market at each date allowing the old generation to sell the shares to the young generation.

In this paper, we provide an existence result under sufficient conditions at the same level of generality than those for an exchange economy. On the production side, we need to assume the free-disposal condition as for the static models. On the pricing rule, we just need a continuity condition.

2.2 Description of the model

We consider an economy with infinitely many dates ($t = 1, 2 \cdots$). For all $t \in \mathbb{N}^*$, there exists a finite set $\mathcal{L}_t$ of commodities available in the world. We denote $\#\mathcal{L}_t = L_t$.

Consumers

At each period $t \in \mathbb{N}$ (including at period 0), a finite and non-empty set of consumers $\mathcal{I}_t$, called generation t, are born. We denote $\#\mathcal{I}_t = I_t$ and $\mathcal{I} = \cup_{t \in \mathbb{N}} \mathcal{I}_t$. Each individual lives two periods (an agent born at period t lives at t and $t+1$ and is assumed to have no economic activity before t and after $t+1$).

The consumption set of each individual $i \in \mathcal{I}_t$, $t \geq 1$ is the subset $X^i = \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$. Thus consumption of each consumer of generation t is limited to his lifetime t and $t+1$. The consumption set of consumers of generation 0 is $\mathbb{R}_+^{L_1}$.

Consumers preferences are represented by a utility function $u^i : X^i \to \mathbb{R}$. This means that preferences are complete and transitive.

The vector $e^i \in \mathbb{R}_{++}^{L_t} \times \mathbb{R}_{++}^{L_{t+1}}$ represents the initial endowment of the agent i of the generation t, which is null outside his lifetime.

Producers

We assume the set of producers $\mathcal{J}$ to be finite. Each firm is supposed to be active for all dates. We denote $\#\mathcal{J} = J$.

The production possibilities are represented by production set-valued production mappings which associate to a given vector of inputs at date t, a set of possible outputs produced at the next period. This supposes that the production process takes time, the consumption of an input at date t has no influence on the output at this date. For each firm j, $(F_t^j)_{t=1}^{\infty}$ is a sequence of mappings from $-\mathbb{R}_+^{L_t}$ to $\mathbb{R}^{L_{t+1}}$. For a given inputs vector z_t^j, $F_t^j(z_t^j)$ is the set of possible vector of outputs the firm can produce. We do not assume the nonnegativity of the output vectors in order to allow for productions with free-disposal, but the hypothesis we posit later in the next section will show that only the nonnegative output vectors will be relevant.

Let us associate to each firm j at each period t an elementary production set Z_t^j defined by:

$$Z_t^j = \{(z_t^j, \zeta_{t+1}^j) \in -\mathbb{R}_+^{L_t} \times \mathbb{R}^{L_{t+1}} \mid \zeta_{t+1}^j \in F_t^j(z_t^j)\}$$

Notice that Z_j^t is the graph of the mapping F_t^j. We define the global inter-temporal production set of firm j by:

$$Y^j = \left\{(y_t^j)_{t=1}^{\infty} \in \prod_{t=1}^{\infty} \mathbb{R}^{L_t} \mid \forall t, \exists (z_t^j, \zeta_{t+1}^j) \in Z_t^j : y_t^j = z_t^j + \zeta_t^j \text{ with } \zeta_1^j = 0\right\}$$

We remark that, although we suppose firms to be active forever, we do not exclude the possibility of firms starting at each date t and operating only for two successive dates t and $t+1$ as in Tvede [57]. In this case, we call for all $t \geq 1$, $\mathcal{J}_t$ the finite set of firms started at each date t. Thus, firm $j \in \mathcal{J}_t$ is active at dates t and $t+1$, and inactive at dates $t' \neq t, t+1$. We can thus consider the sequence of set-valued mapping $(F_t^j)_t$ where $(F_{t'}^j)_{t'}$ are identically null, if $t \neq t, t+1$

Feasibility condition

An allocation $((x^i)_{i\in\mathcal{I}}, (y^j)_{j\in\mathcal{J}}) \in \prod_{t=0}^{\infty} \prod_{i\in\mathcal{I}_t} X^i \times \prod_{j\in\mathcal{J}} Y^j$ is feasible if for all $t \in \mathbb{N}^*$, it satisfies the market-clearing condition:

$$\sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} x_t^i = \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} e_t^i + \sum_{j\in\mathcal{J}} y_t^j \tag{2.1}$$

We denote by $\mathcal{A}(\mathcal{E})$ the set of feasible allocations.

Pricing Rule

The price vector p is an element of $\prod_{t=1}^{\infty} \mathbb{R}_+^{L_t}$, and p_{th} is the market price of the commodity h at date t.

Since the model we consider allows for increasing returns, the producers behavior cannot only be characterized by a competitive and profit maximization behavior. So we describe the behavior of the producers by general pricing rules. See Cornet [22], Dierker, Guesnerie and Neuefeind [26] and Villar [59] for a survey on the representation of economic behavior of producers by pricing rules. All the firms are allowed to follow different pricing rules. Since the production possibilities are defined in a recursive way, we define the pricing rule in a similar way. This approach corresponds to the idea that at each period, newly born individuals appear, taking part to the economic decisions thus to the determination of productions carried over their lifetime. For a producer j at a period t, the pricing rule φ_t^j is a set-valued mapping defined on the set of weakly efficient productions of Z_t^j with values in $\mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$. So, given a weakly efficient production $y^j \in Y^j$ and a price p, the pair (y^j, p) is compatible with the behavior of the jth producer if for all t, $(p_t, p_{t+1}) \in \varphi_t^j(z_t^j, \zeta_{t+1}^j)$ where $(z_t^j, \zeta_{t+1}^j) \in Z_t^j$ and $y_t^j = z_t^j + \zeta_t^j$.

A state $((y^j), p)$ is called a *production equilibrium* if for all t, each firm $j \in \mathcal{J}$ finds acceptable the price (p_t, p_{t+1}) according to his pricing rule for the given weakly production plan (y^j), that is for all $j \in \mathcal{J}$, $(p_t, p_{t+1}) \in \varphi_t^j(z_t^j, \zeta_{t+1}^j)$, where $(z_t^j, \zeta_{t+1}^j) \in Z_t^j$ and $y_t^j = z_t^{j*} + \zeta_t^j$.

Budget Constraint

We assume that we are in a private ownership economy. Each agent $i \in \mathcal{I}_t$ holds a share $\theta^{ij} \geq 0$ of the firm j such that for all j, $\sum_{i \in \mathcal{I}_t} \theta^{ij} = 1$.

The budget constraint, for each agent $i \in \mathcal{I}_t$, $t \in \mathbb{N}^*$ is given by:

$$p_t \cdot x_t^i + p_{t+1} \cdot x_{t+1}^i \leq p_t \cdot e_t^i + p_{t+1} \cdot e_{t+1}^i + \sum_{j \in \mathcal{J}} \theta^{ij}(p_t \cdot z_t^j + p_{t+1} \cdot \zeta_{t+1}^j)$$

and for $i \in \mathcal{I}_0$, $p_1 \cdot x_1^i \leq p_1 \cdot e_1^i$

Equilibrium

We are now able to state the definition of an equilibrium in this overlapping generation economy with production.

Definition 2.2.1. An equilibrium in the OLG economy $\mathcal{E}$ is an element $(p^*, (x^{i*}), (y^{j*})) \in \prod_{t=1}^{\infty} \mathbb{R}_+^{L_t} \times \prod_{i \in \mathcal{I}} X^i \times \prod_{j \in \mathcal{J}} Y^j$ such that:

a) for all $t \in \mathbb{N}^*$, for all $i \in \mathcal{I}_t$, x^{i*} is a maximal element of u^i in the budget set:

$$\{x^i \in X^i \mid p_t^* \cdot x_t^i + p_{t+1}^* \cdot x_{t+1}^i \leq p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i + \sum_{j \in \mathcal{J}} \theta^{ij}(p_t^* \cdot z_t^{j*} + p_{t+1}^* \cdot \zeta_{t+1}^{j*})\},$$

and, for all $i \in \mathcal{I}_0$, x^{i*} is a maximal element of u^i in the budget set $\{x^i \in X^i \mid p_1^* \cdot x_1^i \leq p_1^* \cdot e_1^i\}$;

b) for all $j \in \mathcal{J}$, for all t, $(p_t^*, p_{t+1}^*) \in \varphi_t^j(z_t^{j*}, \zeta_{t+1}^{j*})$, where $(z_t^{j*}, \zeta_{t+1}^{j*}) \in Z_t^j$ and $y_t^{j*} = z_t^{j*} + \zeta_t^{j*}$;

c) for all $t \in \mathbb{N}^*$, $\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^{i*} = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i + \sum_{j \in \mathcal{J}} y_t^{j*}$.

An equilibrium is thus a list of prices and allocations such that: *(a)* every consumer maximizes her utility at given prices within her budget set; *(b)* all the firms are at equilibrium at $((y^{j*}), p^*)$; and *(c)* all markets clear at every date $t \in \mathbb{N}^*$.

2.3 Existence of equilibrium

We consider standard assumptions on the consumption side.

Assumption C.

a) For all $t \in \mathbb{N}^*$, for all individuals $i \in \mathcal{I}_t$, $X^i = \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_t+1}$ and for all $i \in \mathcal{I}_0$, $X^i = \mathbb{R}_+^{L_1}$.

b) For all individuals in $\mathcal{I}$, u^i is continuous, quasi-concave and locally non-satiated;

c) For all $t \in \mathbb{N}^*$, there exists $i_0(t) \in \mathcal{I}_t$ such that for all $x_t \in \mathbb{R}_+^{L_t}$, $u^{i_0(t)}(x_t, \cdot)$ is locally non-satiated and $i_1(t) \in \mathcal{I}_t$ such that for all $x_{t+1} \in \mathbb{R}_+^{L_{t+1}}$, $u^{i_1(t)}(\cdot, x_{t+1})$ is locally non-satiated.

Assumption E. For all $t \in \mathbb{N}^*$, for all $i \in \mathcal{I}_t$, $e^i \in \mathbb{R}_{++}^{L_t} \times \mathbb{R}_{++}^{L_t+1}$ and for all $i \in \mathcal{I}_0$, $e^i \in \mathbb{R}_{++}^{L_1}$.

We posit the following assumption on the production mappings.

Assumption F.

a) For all $(j, t) \in \mathcal{J} \times \mathbb{N}^*$, F_t^j has a closed graph;

b) for all $z_t^j \in -\mathbb{R}_+^{L_t}$, $0 \in F_t^j(z_t^j)$;

c) for all $z_t^j \in -\mathbb{R}_+^{L_t}$, $F_t^j(z_t^j) \cap \mathbb{R}_+^{L_{t+1}}$ is bounded;

d) for all $z_t^j, z_t^{j\prime} \in -\mathbb{R}_+^{L_t}$, if $z_j^t \leq z_t^{j\prime}$ then $F_t^j(z_t^{j\prime}) \subset F_t^j(z_j^t)$;

e) $F_t^j(z_t^j) = (F_t^j(z_t^j) \cap \mathbb{R}_+^{L_{t+1}}) - \mathbb{R}_+^{L_{t+1}}$.

Assumption F implies that Y^j is closed for the product topology and satisfies the posibility of inaction and the free-disposal assumption. The set of weakly efficient production plans coincides then with the frontier of the production set. Furthermore, negative outputs correspond to the disposal of some part of the production. We do not make assumptions on the returns to scale, thus increasing returns are allowed.

We make the following regularity condition on the pricing rule. As pointed in Bonnisseau and Cornet [10], this regularity condition helps to avoid the case where the set of admissible prices is reduced to a singleton, otherwise there is no possibility of equilibria.

Assumption PR. For all $(j,t) \in \mathcal{J} \times \mathbb{N}^*$,

a) φ_t^j has a closed graph and for all $(z_t^j, \zeta_{t+1}^j) \in \partial Z_t^j$, $\varphi_t^j(z_t^j, \zeta_{t+1}^j)$ is a closed convex cone in $\mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$ different from $\{(0,0)\}$;

b) for all $(z_t^j, \zeta_{t+1}^j) \in \partial Z_t^j$, for all $(p_t, p_{t+1}) \in \varphi_t^j(z_t^j, \zeta_{t+1}^j)$, if $\zeta_{t+1,k} < 0$ then $p_{t+1,k} = 0$.

Although losses are possible in the presence of increasing returns, we will particularly focus on loss-free pricing rules: only combinations of prices and productions yielding nonnegative profits will be found acceptable by the producers.

Assumption LF. (Loss-free assumption) For all $(j,t) \in \mathcal{J} \times \mathbb{N}$, for all $(z_t^j, \zeta_{t+1}^j) \in Z_t^j$, for all $(p_t, p_{t+1}) \in \varphi_t^j(z_t^j, \zeta_{t+1}^j)$,

$$p_t \cdot z_t^j + p_{t+1} \cdot \zeta_{t+1}^j \geq 0$$

Assumption LF is naturally associated to firms which operate in unregulated markets and where inaction is possible: in this way, whenever the markets conditions are not advantageous, the owners of the firms can always decide to close them down without incuring any cost, thus profits are guaranteed to stay nonnegative.

Pricing rules satisfying Assumptions PR and LF always exist. Villar in [59] has pointed out two main examples: the constrained profit maximization and the mark-up pricing rule.

In our framework, the constrained profit maximization is given by:

$\varphi_t^{jCPM}(z_t^j, \zeta_{t+1}^j) = \{(p_t, p_{t+1}) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}} \setminus (0,0) \mid p_t \cdot z_t^j + p_{t+1} \cdot \zeta_{t+1}^j \geq p_t \cdot z_t^{j\prime} + p_{t+1} \cdot \zeta_{t+1}^{j\prime}, \forall (z_t^{j\prime}, \zeta_{t+1}^{j\prime}) \in Z_t^j,$ with $z_{t+1}^{j\prime} \geq z_{t+1}^j\}$, for $(z_t^j, \zeta_{t+1}^j) \neq (0,0)$,

$\varphi_t^{jCPM}(0,0)$ is the closed convex hull of $(p_t, p_{t+1}) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}} \setminus (0,0)$ for which there exist sequences $(z_t^{j\nu}, \zeta_{t+1}^{j\nu}) \in \partial Z_t^j \setminus \{(0,0)\}$ and $(p_t^\nu, p_{t+1}^\nu) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$, such that $(z_t^{j\nu}, \zeta_{t+1}^{j\nu}) \to (0,0)$ and $(p_t^\nu, p_{t+1}^\nu) \to (p_t, p_{t+1})$ with $(p_t^\nu, p_{t+1}^\nu) \in \varphi_t^{jCPM}(z_t^{j\nu}, \zeta_{t+1}^{j\nu})$.

This pricing rule results in combinations of prices and productions such that no other combinations using fewer inputs but yielding to higher profits are possible. It can be used to modelize the behaviour of firms when the increasing returns are due to fixed costs or the use of a fixed capital like land or machinery.

In order to introduce the mark-up pricing rule, we define the average cost pricing rules as combinations of prices and productions plans which make firms to just break even:

$\varphi_t^{jAC}(z_t^j, \zeta_{t+1}^j) = \{(p_t, p_{t+1}) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}} \mid p_t \cdot z_t^j + p_{t+1} \cdot \zeta_{t+1}^j = 0\}$, if $(z_t^j, \zeta_{t+1}^j) \neq (0,0)$,

$\varphi_t^{jAC}(0,0)$ is the closed convex hull of $(p_t, p_{t+1}) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}} \setminus (0,0)$ for which there exist sequences $(z_t^{j\nu}, \zeta_{t+1}^{j\nu}) \in \partial Z_t^j \setminus \{(0,0)\}$ and $(p_t^\nu, p_{t+1}^\nu) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$, such that $(z_t^{j\nu}, \zeta_{t+1}^{j\nu}) \to (0,0)$ and $(p_t^\nu, p_{t+1}^\nu) \to (p_t, p_{t+1})$ with $p_t^\nu \cdot z_t^{j\nu} + p_{t+1}^\nu \cdot \zeta_{t+1}^{j\nu} = 0\}$.

Fom the average-cost pricing rule, we define the “mark-up pricing”as $\varphi_t^{jAC}(z_t^j(1 + \rho_t^j), \zeta_{t+1}^j)$. This pricing rule thus consists of prices that cover the costs of the firms to which is added a profit component ρ_t^j called mark-up. A mark-up can be related conditions on the firms, such as the existence of fixed cots, fixed capital or the presence of entry barriers. By definition of the average-cost pricing rule, $\rho_t^j = -\frac{p_t \cdot z_t^j + p_{t+1} \cdot \zeta_{t+1}^j}{p_t \cdot z_t^j}$; this ratio expresses the measure of the profitability of firm j when it advances some capital or investment at date t.

Clearly, average cost pricing is a particular mark-up pricing where the mark-up is equal to zero.

The main result of this paper is the following:

Theorem 2.3.1. Under Assumptions C, E, F, PR and LF, the OLG economy $\mathcal{E}$ has an equilibrium.

Remark 2.3.1. This result encompasses the known existence results for exchange economies. Indeed, it suffices to consider that there is only one producer with a constant production correspondence F_t defined by $F_t(z_t) = -\mathbb{R}_+^{L_{t+1}}$ and the pricing rule corresponding to the competitive behavior, that is,

$$\varphi_t(z_t, \zeta_{t+1}) = \{(p_t, p_{t+1}) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}} \mid p_t \cdot z_t + p_{t+1} \cdot \zeta_{t+1} = 0\}.$$

Remark 2.3.2. If we further assume that F_t^j has a convex graph for all (j,t) and that the pricing rule φ_t^j describes the competitive behavior, that is,

$\varphi_t^j(z_t^j, \zeta_{t+1}^j) = \{(p_t, p_{t+1}) \in \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}} \mid p_t \cdot z_t^j + p_{t+1} \cdot \zeta_{t+1}^j \geq p_t \cdot z_t^{j\prime} + p_{t+1} \cdot \zeta_{t+1}^{j\prime}, \forall (z_t^{j\prime}, \zeta_{t+1}^{j\prime}) \in Z_t^j\}$, then Assumptions PR and LF are satisfied and Theorem 2.3.1 gives the existence of a competitive equilibrium in the OLG economy.

Remark 2.3.3. Note that Assumption PR (b) implies that for all $t \in \mathbb{N}^*$, for all $k \in \mathcal{L}_{t+1}$, then $\zeta^{j*}_{t+1,k} \geq 0$ if commodity k is desirable by at least one consumer of generation t or $t+1$. So, even if we do not a priori exclude negative quantities of output when we define the production mappings, at equilibrium, the production of an output is always non negative for desirable commodities.

2.4 Equilibrium in truncated economies

We will proceed as in exchange economies (see Balasko et al. [3]) to establish the existence of equilibrium in $\mathcal{E}$: first we show the existence of pseudo-equilibrium in the truncated economies with a finite horizon

$$\mathcal{E}_\tau = \left((u^{\tau i}, X^{\tau i}, e^{\tau i}, \theta^i)_{i \in \mathcal{I}_0^{\tau-1}}, (Y^{tj}, \tilde{\varphi}^{tj})_{\substack{t=1,\ldots,\tau-1 \\ j \in \mathcal{J}}} \right)$$

then we prove that prices and allocations remains in a compact space of a suitable linear space and we finally show that a cluster point is an equilibrium of the OLG economy.

Notations.

$\mathcal{I}_0^{\tau-1} = \cup_{t=0}^{\tau-1} \mathcal{I}_t$ is the set of all the individuals born up to period $\tau - 1$.

For each $i \in \mathcal{I}_1$,

$X^{\tau i} = \{x \in \prod_{t=1}^{\tau} \mathbb{R}_+^{L_t} \mid x_{t'} = 0, \forall t' > 1\}$

$u^{\tau i}(x) = u^i(x_1)$

$e^{\tau i} = (e^{\tau i}_{t'})_{t'=1}^{\tau}$ such that $e^{\tau i}_1 = e^i_1$, and $e^{\tau i}_{t'} = 0$ if $t' > 1$.

For each $t = 1, \ldots, \tau - 1$, for each $i \in \mathcal{I}_t$,

$X^{\tau i} = \{x \in \prod_{t=1}^{\tau} \mathbb{R}_+^{L_t} \mid x_{t'} = 0, \forall t' \neq t, t+1\}$

$u^{\tau i}(x) = u^i(x_t, x_{t+1})$

$e^{\tau i} = (e^{\tau i}_{t'})_{t'=1}^{\tau}$ such that $e^{\tau i}_t = e^i_t$, $e^{\tau i}_{t+1} = e^i_{t+1}$ and $e^{\tau i}_{t'} = 0$ if $t' \neq t, t+1$.

For each t, we choose an arbitrary closed convex cone C_t, called free-disposal cone, included in $\mathbb{R}^{L_t}_{++} \cup \{0\}$ containing $\mathbf{1}^t = (1, \ldots, 1) \in \mathbb{R}^{L_t}_{++} \cup \{0\}$ in its interior. We denote by C_t^+ the positive polar cone of C_t[2].

[2] $C_t^+ = \{v \in \mathbb{R}^{L_t} \mid v \cdot u \geq 0, \forall u \in C_t\}$

We define the extended production set Y^{tj} for $t = 1, \dots, \tau - 1$ as follows:

$$Y^{tj} = \{(y^{tj}_{t'})^{\tau}_{t'=1} \in \prod_{t'=1}^{\tau} \mathbb{R}^{L_{t'}} \mid (y^{tj}_t, y^{tj}_{t+1}) \in Z^j_t, \forall t', y^{tj}_{t'} \in -C_{t'}\}$$

This extension is necessary since the existence result for economies with non-convex production sets require that the production sets satisfies the free-disposal assumption or at least a weak form of it, namely, with our notations the fact that $Y^{tj} - \prod_{t'=1}^{\tau} C_{t'} = Y^{tj}$. We also extend the pricing rules as follows: for all $y^{tj} \in \partial Y_{tj}$,

$$\tilde{\varphi}^{tj}(y^{tj}) = \{p \in \prod_{t'=1}^{\tau} C^+_t \mid (p_t, p_{t+1}) \in \varphi^j_t(y^{tj}_t, y^{tj}_{t+1}), p_{t'} \cdot y^{tj}_{t'} = 0, \forall t' \neq t, t+1\}$$

We thus extend the production sets into $\prod_{t'=1}^{\tau} \mathbb{R}^{L_{t'}}$, but we focus only on the production at date t and restrict the activity at dates $t' \neq t$ to the free disposal at zero cost. We also remark that if $p \in \tilde{\varphi}^{tj}(y^{tj})$ and $p_{t'} \in \mathbb{R}^{L_{t'}}_+ \setminus \{0\}$ for some $t' \neq t, t+1$, then $y^{tj}_{t'} = 0$.

The truncation of the economy leads us to a weak notion of equilibrium termed as pseudo-equilibrium.

Definition 2.4.1. A pseudo-equilibrium in the truncated economy $\mathcal{E}_\tau$ is an element $(p^*, (x^{i*}), (y^{tj*})) \in \prod_{t=1}^{\tau} C^+_t \times \prod_{i \in \mathcal{I}_0^{\tau-1}} X^{\tau i} \times \prod_{j \in \mathcal{J}} \prod_{t=1}^{\tau-1} Y^{tj}$ such that:

a) for all $t = 1, 2 \dots \tau - 1$, for all $i \in \mathcal{I}_t$, x^{i*} is a maximal element of $u^{\tau i}$ in the budget set

$$\{x^i \in X^{\tau i} \mid p^* \cdot x_i \leq p^* \cdot e^{\tau i} + \sum_{j \in \mathcal{J}} \theta^{ij}_t p^* \cdot y^{tj*}\};$$

for all $i \in \mathcal{I}_0$, x^{i*} is a maximal element of $u^{\tau i}$ in the budget set $\{x^i \in X^{\tau i} \mid p^* \cdot x^i \leq p^* \cdot e^{\tau i}\}$;

b) for all $j \in \mathcal{J}$, for all $t = 1, \dots, \tau - 1$, $p^* \in \tilde{\varphi}^{tj}(y^{tj*})$;

c) For all $t = 1, \dots, \tau-1$, $\sum_{i \in \mathcal{I}_0^{\tau-1}} x^{i*}_t = \sum_{i \in \mathcal{I}_0^{\tau-1}} e^{\tau i}_t + \sum_{j \in \mathcal{J}} \sum_{t'=1}^{\tau-1} y^{t'j*}_t$ and $\sum_{i \in \mathcal{I}_0^{\tau-1}} x^{i*}_\tau \leq \sum_{i \in \mathcal{I}_0^{\tau-1}} e^{\tau i}_\tau + \sum_{i \in \mathcal{I}_\tau} e^i_\tau + \sum_{j \in \mathcal{J}} \sum_{t'=1}^{\tau-1} y^{t'j*}_\tau$

Remark 2.4.1. The difference between a pseudo-equilibrium and an equilibrium is that we do not require the market clearing condition at the last period τ and we artificially increase the initial endowments by adding those of the consumers of the generation τ. This particular feature is useful to show below that if $\tau' > \tau$, then the restriction of a pseudo-equilibrium of $\mathcal{E}_{\tau'}$ to the $\tau - 1$ first generations is a pseudo-equilibrium of $\mathcal{E}_\tau$.

Remark 2.4.2. Since Condition (c) of the above definition is weaker on the last period τ than the standard market clearing condition, an equilibrium of $\mathcal{E}_\tau$ is clearly a pseudo-equilibrium.

Remark 2.4.3. In the definition of a pseudo-equilibrium, the price p^* is supposed to be in $\prod_{t=1}^{\tau} C_t^+$. Actually, we remark that it belongs to the smaller set $\prod_{t=1}^{\tau} \mathbb{R}_+^{L_t}$. This is a consequence of Condition (b) and the fact that φ_t^j takes its values in $\mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$. Consequently, we deduce from the definition of $\tilde{\varphi}^{tj}$ that at equilibrium $y_{t'}^{tj*} = 0$ for all $t' \neq t, t+1$.

Remark 2.4.4. From the definition of the truncated economy and the definition of a pseudo-equilibrium, we remark that if $\bar{\tau} > \tau$ and $(p^*, (x^{i*}), (y^{tj*}))$ is a pseudo-equilibrium in the economy $\mathcal{E}_{\bar{\tau}}$, then the price and the allocations restricted to the τ first periods $\left(q^*, (\chi^{i*})_{i \in \mathcal{I}_0^{\tau-1}}, (\xi^{tj*})_{\substack{j \in \mathcal{J} \\ t=1,\ldots,\tau-1}}\right)$ defined by

$q^* = (p_t^*)_{t=1}^{\tau}$,

for all $i \in \mathcal{I}_0^{\tau-1}$, $\chi^{i*} = (x_t^{i*})_{t=1}^{\tau}$,

for all $j \in \mathcal{J}$, for all $t = 1, \ldots, \tau - 1$, $\xi^{tj*} = (y_{t'}^{tj*})_{t'=1}^{\tau}$,

is a pseudo-equilibrium in the economy $\mathcal{E}_\tau$.

Indeed, from the definition of a pseudo-equilibrium, we just have to look at Condition (c) for the period τ. Since $(p^*, (x^{i*}), (y^{tj*}))$ is a pseudo-equilibrium in the economy $\mathcal{E}_{\bar{\tau}}$ and $\bar{\tau} > \tau$, one has:

$$\sum_{i \in \mathcal{I}_0^{\bar{\tau}-1}} x_\tau^{i*} = \sum_{i \in \mathcal{I}_0^{\bar{\tau}-1}} e_\tau^{\bar{\tau}i} + \sum_{j \in \mathcal{J}} \sum_{t'=1}^{\bar{\tau}-1} y_\tau^{t'j*}$$

From the definition of $X^{\bar{\tau}i}$, for all $i \in \cup_{t=\tau+1}^{\bar{\tau}-1} \mathcal{I}_t$, $x_\tau^{i*} = 0$. From the definition of $e^{\bar{\tau}i}$, for all $i \in \cup_{t=\tau+1}^{\bar{\tau}-1} \mathcal{I}_t$, $e_\tau^{\bar{\tau}i} = 0$. From the previous remark, for all $t' = \tau + 1, \ldots, \bar{\tau} - 1$, for all j, $y_\tau^{t'j*} = 0$. Furthermore, for all j, $y_\tau^{\tau j*} \leq 0$ and for all $i \in \mathcal{I}_\tau$, $x_\tau^{i*} \geq 0$. So, one deduces that

$$\sum_{i \in \mathcal{I}_0^{\bar{\tau}-1}} x_\tau^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-1}} x_\tau^{i*} + \sum_{i \in \mathcal{I}_\tau} x_\tau^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-1}} e_\tau^{\bar{\tau}i} + \sum_{i \in \mathcal{I}_\tau} e_\tau^{\bar{\tau}i} + \sum_{j \in \mathcal{J}} \sum_{t'=1}^{\tau-1} y_\tau^{t'j*} + \sum_{j \in \mathcal{J}} y_\tau^{\tau j*}$$

which implies that

$$\sum_{i \in \mathcal{I}_0^{\tau-1}} x_\tau^{i*} \leq \sum_{i \in \mathcal{I}_0^{\tau-1}} e_\tau^{\bar{\tau}i} + \sum_{i \in \mathcal{I}_\tau} e_\tau^{\bar{\tau}i} + \sum_{j \in \mathcal{J}} \sum_{t'=1}^{\tau-1} y_\tau^{t'j*}$$

So we get Condition (c) for the period τ since $x_\tau^{i*} = \chi_\tau^{i*}$ and $e_\tau^{\bar{\tau}i} = e_\tau^{\tau i}$ for all $i \in \mathcal{I}_0^{\tau-1}$ and $e_\tau^{\bar{\tau}i} = e_\tau^i$ for all $i \in \mathcal{I}_\tau$.

Since the truncated economy $\mathcal{E}_\tau$ does not satisfy the strong survival assumption but its weak form, we are going to deduce the existence of pseudo-equilibrium from a quasi-equilibrium. One has:

Definition 2.4.2. A quasi-equilibrium in the truncated economy $\mathcal{E}_\tau$ is a list $(p^*, (x^{i*}), (y^{tj*}))$ in $\prod_{t=1}^{\tau} C_t^+ \times \prod_{i\in\mathcal{I}_0^{\tau-1}} X^{\tau i} \times \prod_{j\in\mathcal{J}} \prod_{t=1}^{\tau-1} Y^{tj}$ satisfying:

a') for all $t = 1, 2 \ldots \tau - 1$, x^{i*} is an element of the budget set:

$$\{x^i \in X^{\tau i} \mid p^* \cdot x_i^* \leq p^* \cdot e^{\tau i} + \sum_{j\in\mathcal{J}} \theta_t^{ij} p^* \cdot y^{tj*}\}$$

and for all $x^i \in X^{\tau i}$ such that: $p^* \cdot x_i < p^* \cdot e^{\tau i} + \sum_{j\in\mathcal{J}} \theta_t^{ij} p^* \cdot y^{tj},\ u^{\tau i}(x^i) \leq u^{\tau i}(x^{i*})$,

for all $i \in \mathcal{I}_0$, $x^{i*} \in \{x^i \in X^i \mid p^* \cdot x^i \leq p^* \cdot e^{\tau i}\}$ and for all $x^i \in X^{\tau i}$ such that $p^* \cdot x^i < p^* \cdot e^{\tau i},\ u^{\tau i}(x^i) \leq u^{\tau i}(x^{i*})$,

b) for all $j \in \mathcal{J}$, for all $t = 1, \ldots, \tau - 1$, $p^* \in \tilde{\varphi}^{\tau j}(y^{tj*})$;

c) $\sum_{i\in\mathcal{I}_0^{\tau-1}} x^{i*} = \sum_{i\in\mathcal{I}_0^{\tau-1}} e^i + \sum_{j\in\mathcal{J}} \sum_{t=1}^{\tau-1} y^{tj*}$;

d) $p^* \neq 0$.

Proposition 2.4.1. Under the assumptions of Theorem 2.3.1, for all $\tau \geq 2$, there exists a quasi-equilibrium of the economy $\mathcal{E}_\tau$.

Proof. The proof is based on the fact that $\mathcal{E}_\tau$ satisfies the necessary assumption of the existence of a (quasi)-equilibrium. See Bonnisseau-Cornet [10] for the existence of equilibrium with bounded-losses pricing rules and in particular of losses-free pricing rules, Gourdel [37] for the existence of quasi-equilibrium and the way to go from quasi-equilibrium to equilibrium, and Bonnisseau-Jamin [11] for the existence of equilibrium with a weaker version of the free-disposal assumption.

Indeed, the existence of quasi-equilibrium is ensured by Assumptions (C) and (E), and the facts that :

- $\tilde{\varphi}^{tj}$ satisfies Assumption (PR)(a) since φ_t^j satisfies this assumption and C_t is a closed convex cone.
- for all $(y^{tj}) \in \prod \partial Y^{tj}$, if $p \in \cap_{j,t} \tilde{\varphi}^{tj}(y^{tj}), p \cdot e^{\tau i} + \sum_{j\in\mathcal{J}} \theta_t^{ij} p \cdot y^{tj} \geq 0$, thanks to Assumptions (LF) and (E), and Remark 2.4.3, that is $p^* \in \prod_{t=1}^{\tau} \mathbb{R}_+^{L_t}$.
- $Y^{tj} - \prod_{t'=1}^{\tau} C_{t'} = Y^{tj}$ (free-disposal)

and the boundedness assumption stated by the following lemma. □

Let $e \in \prod_{t\in\mathbb{N}^*} \mathbb{R}_+^{L_t}$ defined by $e_t = \sum_{i\in\mathcal{I}_t\cup\mathcal{I}_{t-1}} e_t^i$. Let $e' \in \prod_{t\in\mathbb{N}^*} \mathbb{R}_+^{L_t}$ such that $e' \geq e$. We denote by $\tilde{\mathcal{A}}(\mathcal{E}_\tau(e'))$, the set of allocations satisfying the market clearing condition for a pseudo-equilibrium (Condition (c) of Definition 3.4.1) for the economy $\mathcal{E}_\tau$. We establish that feasible allocations are bounded for all greater initial endowments.

Lemma 2.1. *For all $e' \geq e$, for all $j \in \mathcal{J}$, there exists a sequence of non negative real numbers (m^{tj}) such that for all τ, for all $((x^i),(y^{tj})) \in \tilde{\mathcal{A}}(\mathcal{E}_\tau(e'))$,*

for all $i \in \mathcal{I}_0^{\tau-1}$, for all $t = 1,\dots,\tau-1$, $0 \leq x_t^i \leq e'_t + \left(\sum_{j\in\mathcal{J}} m^{tj}\right)\mathbf{1}^t$;

for all $j \in \mathcal{J}$, for all $t = 1,\dots,t-1$, for all $t' \neq t+1$,

$0 \geq y_{t'}^{tj} \geq -e'_{t'} - \sum_{j\in\mathcal{J}} m^{t'j}\mathbf{1}^{t'}$,

$m^{(t+1)j}\mathbf{1}^{t+1} \geq y_{t+1}^{tj} \geq -e'_{t+1} - \sum_{j\in\mathcal{J}} m^{(t+1)j}\mathbf{1}^{t+1}$.

Proof. Let $((x^i),(y^{tj}))$ be an element of $\tilde{\mathcal{A}}(\mathcal{E}_\tau(e'))$. Then, for all $t = 1,\dots,\tau$,

$$\sum_{i\in\mathcal{I}_0^{\tau-1}} x_t^i \leq e'_t + \sum_{j\in\mathcal{J}}\sum_{t'=1}^{\tau-1} y_t^{t'j}$$

For all $j \in \mathcal{J}$, we define the sequence (m^{tj}) as follows: $m^{1j} = 0$ and m^{t+1j} is a positive real number so that:

$F_t^j(-e'_t - \sum_{j\in\mathcal{J}} m^{tj}\mathbf{1}^t) \subset m^{t+1j}\mathbf{1}^{t+1} - \mathbb{R}_+^{L_{t+1}}$.

Such real number exists from the boundedness Assumption F(c). Since $0 \leq \sum_{i\in\mathcal{I}_0^{\tau-1}} x_1^i$, we get $\sum_{j\in\mathcal{J}} y_1^{1j} + \sum_{j\in\mathcal{J}}\sum_{t=2}^{\tau-1} y_1^{tj} \geq -e'_1$. Since for all $j \in \mathcal{J}$, $y_1^{1j} \leq 0$ and for all $t = 1,\dots,\tau-1$, $y_1^{tj} \leq 0$, we obtain $0 \geq y_1^{1j} \geq -e'_1$, for all j and $0 \geq y_1^{tj} \geq -e'_1$ for all t.

For the second period, we have

$$\sum_{j\in\mathcal{J}} y_2^{1j} + \sum_{j\in\mathcal{J}} y_2^{2j} + \sum_{j\in\mathcal{J}}\sum_{t=3}^{\tau-1} y_2^{tj} \geq -e'_2$$

For all $j \in \mathcal{J}$, $y_2^{2j} \leq 0$ and for all $t = 3,\dots,\tau-1$, $y_2^{tj} \leq 0$. From the above inequalities and Assumption F(d), $y_2^{1j} \in F_1^j(y_1^{1j}) \subset F_1^j(-e'_1) \subset m^{2j}\mathbf{1}^2 - \mathbb{R}_+^{L_2}$. Thus, for all $j \in \mathcal{J}$,

$0 \geq y_2^{tj} \geq -e'_2 - \sum_{j\in\mathcal{J}} m^{2j}\mathbf{1}^2$, for all $t = 2\cdots\tau-1$,

$m^{2j}\mathbf{1}^2 \geq y_2^{1j} \geq -e'_2 - \sum_{j\neq j'} y_2^{1j'} \geq -e'_2 - \sum_{j\neq j'} m^{2j'}\mathbf{1}^2 \geq -e'_2 - \sum_{j'\in\mathcal{J}} m^{2j'}\mathbf{1}^2$

By an induction argument taking into account the definition of the sequences (m^{tj}) we prove the result for all period.

For the consumptions, since they are all non-negative, $0 \leq x^i \leq \sum_{i'\in\mathcal{I}_0^{\tau-1}} x^{i'}$. So, for all t,

$0 \leq x_t^i \leq e_t' + \sum_{j \in \mathcal{J}} \sum_{t'=1}^{\tau-1} y_t^{t'j} \leq e_t' + \sum_{j \in \mathcal{J}} y_t^{t-1j} \leq e_t' + \sum_{j \in \mathcal{J}} m^{tj} \mathbf{1}^t$

□

The following lemma ensures that a quasi-equilibrium in $\mathcal{E}_\tau$ is an equilibrium.

Lemma 2.2. *If $(p^*, (x^{i*}), (y^{tj*}))$ is a quasi-equilibrium, then $p_t^* \neq 0$ for all t and $(p^*, (x^{i*}), (y^{tj*}))$ is an equilibrium.*

Proof. Since the utility functions are continuous, the condition for a quasi-equilibrium $(p^*, (x^{i*}), (y^{tj*}))$ to be an equilibrium is that the indidvidual wealth is strictly above the subsistence level, that is: $w^{i*} = p^* \cdot e^{\tau i} + \sum_{j \in \mathcal{J}} \theta_t^{ij} p^* \cdot y^{tj*} > \inf p^*.X^{\tau i}$, for all $i \in \mathcal{I}_0^{\tau-1}$. As already remarked (See Remark 2.4.3), $p^* \in \prod_{t=1}^{\tau} \mathbb{R}_+^{L_t}$, so $\inf p^*.X^{\tau i} = 0$. Hence, from Assumptions E and LF, it suffices to show that $p_t^* \neq 0$ for all $t = 1, \ldots, \tau$.

Suppose that there exists t such that $p_t^* = 0$. Knowing that p^* is not equal to 0, there exists $\bar{t}$ such that $p_{\bar{t}}^* \neq 0$ and $p_{\bar{t}+1}^* = 0$ or $p_{\bar{t}}^* = 0$ and $p_{\bar{t}+1}^* \neq 0$. We deal with the first case, the proof being the same for the second case.

Since $p_{\bar{t}}^* \in \mathbb{R}_+^{L_{\bar{t}}} \setminus \{0\}$, the consumer i_1 in $\mathcal{I}_{\bar{t}}$ given by Assumption C(c) has a strictly positive wealth $w^{i_1*} > 0$. Then $(x_{\bar{t}}^{i_1*}, x_{\bar{t}+1}^{i_1*})$ is a demand of consumer i_1. But, then, the local non-satiation of the partial utility function $u^{i_1}(x_{\bar{t}}^{i_1*}, \cdot)$ is incompatible with $p_{\bar{t}+1}^* = 0$.

Thus, necessarily $p_t^* \neq 0$ for all t, and $w^{i*} > \inf p^*.X^{\tau i} = 0$.

□

From Remark 2, an equilibrium is a pseudo-equilibrium, thus we have proved the following result.

Proposition 2.4.2. Under the assumptions of Theorem 2.3.1, for all $\tau \geq 2$, there exists a pseudo-equilibrium of the economy $\mathcal{E}_\tau$.

In the following lemma, we provide two properties of the pseudo-equilibrium, which will be useful for the limit argument in the next section. In the following, a non zero equilibrium price p^* is normalized so that $\sum_{t=1}^{\tau} \sum_{\ell \in L_t} p_{t\ell}^* = 1$.

Lemma 2.3. *If $(p^*, (x^{i*}), (y^{tj*})$ is a pseudo-equilibrium, then $p_t^* \neq 0$ for all t.*

The set of pseudo-equilibria of the economy $\mathcal{E}_\tau$ with a normalized price is closed.

Proof. The first part uses the same argument as for Lemma 2.2.

We now consider a sequence of pseudo-equilibria $(p^\nu, (x^{i\nu}), (y^{tj\nu}))$ that converges to $(\bar{p}, (\bar{x}^i), (\bar{y}^{tj})$. We prove that $(\bar{p}, (\bar{x}^i), (\bar{y}^{tj})$ is also a pseudo-equilibrium.

It is easy to establish that $(\bar{p}, (\bar{x}^i), (\bar{y}^{tj})$ satisfies the condition (b) in Definition 2, since $\tilde{\varphi}^{\tau j}$ has closed graph, and also the condition (c). So it remains to show that the condition (a) is also satisfied.

Denote by $(w^{i\nu})$ the associated wealth sequence and by $\bar{w}^i$ its limit. One easily shows that the budget constraint is satisfied by $\bar{x}^i$. If $\bar{w}^i > 0$, then $\bar{x}^i$ maximizes the utility function under the budget constraint. Indeed, if $\bar{p} \cdot x^i < \bar{w}^i$, then for ν large enough, $p^\nu \cdot x^i \leq w^{i\nu}$. But this implies that $u^i(x^i) \leq u^i(x^{i\nu})$, and by the continuity of u^i, $u^i(x^i) \leq u^i(\bar{x}^i)$. If $\bar{p} \cdot x^i = \bar{w}^i > 0$. Let $\lambda < 1$. Then $\bar{p} \cdot (\lambda x^i) < \bar{w}^i$. So, from above, $u^i(\lambda x^i) \leq u^i(\bar{x}^i)$. Using again the continuity of u^i, $u^i(x^i) = \lim_{\lambda \to 1} u^i(\lambda x^i) \leq u^i(\bar{x}^i)$.

Let us now prove that $\bar{p}_t \neq 0$, for all t. Since $\bar{p} \neq 0$ by normalization, there exists t such that $\bar{p}_t \neq 0$. Hence, for the consumer $i_0(t) \in \mathcal{I}_t$ and $i_1(t-1) \in \mathcal{I}_{t-1}$, $\bar{w}^{i_0(t)} > 0$ and $\bar{w}^{i_1(t-1)} > 0$. So the agents $i_0(t)$ and $i_1(t-1)$ are utility maximizer hence, from Assumption C(c), $\bar{p}_{t+1} \neq 0$ and $\bar{p}_{t-1} \neq 0$. Doing again the same argument, we conclude that the prices at each period is different from 0.

Since $\bar{p}_t \neq 0$, for all t, $\bar{w}^i > 0$ for all consumers, hence all of them are maximizing utility at the price $\bar{p}$. □

2.5 From truncated equilibria to equilibrium

The proof of Theorem 2.3.1 consists of considering a sequence of pseudo-equilibrium in the truncated economy with an horizon increasing to infinity. First, we establish that the sequence of equilibrium prices in the truncated economies remains in a compact set for the product topology on $\prod_{t=1}^{\infty} \mathbb{R}^{L_t}$. Then we show that the sequence of T-equilibrium remains in a compact set and we prove that a cluster point is an equilibrium of the OLG economy $\mathcal{E}$.

From the previous section, for all $T \geq 2$, there exists a T-equilibrium $(p^T, (x^{iT}), (y^{tjT}))$ of the economy $\mathcal{E}_T$. Since we have proved in the previous section (see Lemma 2.2) that $p_1^T \neq 0$, we normalize p^T so that $\sum_{\ell \in L_1} p_{1\ell}^T = 1$.

We extend the price and the allocations to the whole space $\prod_{t=1}^{\infty} \mathbb{R}^{L_t}$ by adding zeros for the missing components without modifying the notations. So, now the sequences (p^T), (x^{iT}) and (y^{tjT}) are in $\prod_{t=1}^{\infty} \mathbb{R}^{L_t}$.

We now prove that the sequence of prices (p^T) remains in a compact subset of $\prod_{t=1}^{\infty} \mathbb{R}^{L_t}$.

Lemma 2.4. *For all t, there exists $\tilde{k}_t \in \mathbb{R}_+$ such that for all T, $0 \leq p_t^T \leq \tilde{k}_t \mathbf{1}^t$.*

Proof. If it is not true, there exist $\bar{t}$ and an increasing sequence (T^ν) such that $p^{T^\nu}_{\bar{t}} \geq \nu \mathbf{1}^{\bar{t}}$. Let $\tau > \bar{t} + 2$. We assume without any loss of generality that $T^\nu > \tau$ for all ν.

Now we consider the restriction to the τ first period of the T^ν-equilibrium $(p^{T^\nu}, (x^{iT^\nu}), (y^{tjT^\nu}))$:

- for all $i \in \mathcal{I}_0^{\tau-1}$, $x^{i\nu}$ is the restriction of x^{iT^ν} to $\prod_{t=1}^{\tau} \mathbb{R}^{L_t}$;
- for all $j \in \mathcal{J}$, for all $t = 1, \ldots, \tau - 1$, $y^{tj\nu}$ is the restriction of y^{tjT^ν} to $\prod_{t=1}^{\tau} \mathbb{R}^{L_t}$;
- p^ν is the restriction of p^{T^ν} to $\prod_{t=1}^{\tau} \mathbb{R}^{L_t}$.

From Remark 2.4.4 in the previous section, $(p^\nu, (x^{i\nu}), (y^{tj\nu}))$ is a pseudo-equilibrium of the truncated economy $\mathcal{E}_\tau$. We now renormalize the price p^ν as follows:

$$\pi^\nu = \frac{1}{\sum_{t=1}^{\tau} \sum_{\ell \in L_t} p^\nu_{t\ell}} p^\nu$$

Since π^ν is non negative, the sequence π^ν remains in the simplex of $\prod_{t=1}^{\tau} \mathbb{R}^{L_t}$, which is compact. From Lemma 2.1 in the previous section, the sequence $(x^{i\nu}), (y^{tj\nu}))$ remains in the compact subset $\tilde{\mathcal{A}}(\mathcal{E}_\tau(e))$. So the sequence $(\pi^\nu, (x^{i\nu}), (y^{tj\nu}))$ has a cluster point $(\bar{\pi}, (\bar{x}^i), (\bar{y}^{tj}))$. From Lemma 3.2, $(\bar{\pi}, (\bar{x}^i), (\bar{y}^{tj}))$ is also a pseudo-equilibrium of the truncated economy $\mathcal{E}_\tau$. But $\bar{\pi}_1 = 0$ since $(\sum_{t=1}^{\tau} \sum_{\ell \in L_t} p^\nu_{t\ell})$ converges to $+\infty$ and $0 \leq p^\nu_{1\ell} \leq 1$ for all $\ell \in L_1$. Hence we get a contradiction since Lemma 3.2 shows that for all $t = 1, \ldots, \tau$, $\bar{\pi}_t \neq 0$. □

Proof of Theorem 2.3.1.

From Lemma 2.1 and the above lemma, the sequence of T-equilibrium of the economy $\mathcal{E}_T$, $(p^T, (x^{iT}), (y^{tjT}))$, remains in a compact set for the product topology of $\prod_{t=1}^{\infty} \mathbb{R}^{L_t} \times \prod_{t'=1}^{\infty} \prod_{i \in \mathcal{I}_{t'}} \prod_{t=1}^{\infty} \mathbb{R}^{L_t} \times \prod_{j \in \mathcal{J}} \prod_{t'=1}^{\infty} \prod_{t=1}^{\infty} \mathbb{R}^{L_t}$. Since this is a countable product of finite dimensional spaces, the product topology is metrizable on the compact sets and there exists a sub-sequence $(p^{T^\nu}, (x^{iT^\nu}), (y^{tjT^\nu}))$ of $(p^T, (x^{iT}), (y^{tjT}))$, which converges to $(p^*, (x^{i*}), (y^{tj*}))$. We recall that the convergence for the product topology implies the usual convergence when we consider only a finite number of components.

For each $\tau \geq 2$, for ν large enough, the restriction of $(p^{T^\nu}, (x^{iT^\nu}), (y^{tjT^\nu}))$ to the τ first periods is a pseudo-equilibrium of $\mathcal{E}_\tau$ (see Remark 2.4.4 and it converges to the restriction of $(p^*, (x^{i*}), (y^{tj*}))$ to the τ first periods. From Lemma 3.2, this restriction is a pseudo-equilibrium of $\mathcal{E}_\tau$. From Definition 3.4.1 and the notations above, one deduces that $(p^*, (\xi^{i*}), (y^{j*}))$ defined as follows in an equilibrium for the OLG economy $\mathcal{E}$:

- for all $t \geq 1$, for all $i \in \mathcal{I}_t$, $\xi^{i*} = (x^{i*}_t, x^{i*}_{t+1})$ and for all $i \in \mathcal{I}_0$, $\xi^{i*} = x^{i*}_1$;

- for all $j \in \mathcal{J}$ for all $t \geq 1$, $z_t^{j*} = y_t^{tj*}$, $\zeta_{t+1}^{j*} = y_{t+1}^{tj*}$ and $y_t^{j*} = z_t^{j*} + \zeta_t^{j*}$ with $\zeta_0^{j*} = 0$. □

2.6 A Numerical Example

This subsection intends to illustrate a case of a simple stationary OLG economy with a single firm. At every date $t \geq 1$, there is only one good $|\mathcal{L}_t| = 1$, and one individual per generation $|\mathcal{I}_t| = 1$.

Consumers

All consumers are identical except the old consumer at date $t = 0$. They live two periods, and are described by their consumption set $X = \mathbb{R}_+^2$, their initial endowment vector (e^y, e^o) and their utility function $u : X \to \mathbb{R}$. The consumer at date $t = 0$ who is old at date $t = 1$ has consumption set $\mathbb{R}_+$.

We consider a standard time-separable utility function, namely the Cobb-Doublas utility function: $u(x_t^y, x_t^o) = a \ln x_t^y + (1-a) \ln x_t^o$, $0 \leq a \leq 1$.

The firm

The firm is described by a production function $f : -\mathbb{R}_+ \to \mathbb{R}$ which transforms the good at date t into the good at date $t+1$.

We assume that f is of the form: $f(z_t) = \begin{cases} -\gamma(z_t - \hat{z}), & \text{if } z_t \leq \hat{z} \leq 0 \\ 0, & \text{if } \hat{z} \leq z_t \leq 0 \end{cases}$, where $\gamma > 1$, $\hat{z} < 0$ and $z_t \leq 0$ the input used at date t. The output at time $t+1$ is then given by: $\zeta_{t+1} = f(z_t)$.

This technology exhibits strict increasing returns due to the fixed cost $\hat{z}$.

Pricing rule

The price vector p is an element of $\mathbb{R}_+^2$. Define the relative price: $\pi_t = \frac{p_t}{p_{t+1}}$. We describe the behavior of the producer by the average cost pricing: $p_t z_t + p_{t+1}\zeta_{t+1} = 0$, or in term of relative prices: $\pi_t z_t + \zeta_{t+1} = 0$.

The consumer's demand

Each consumer t is maximizing her utility function u^t under her budget constraint. Given a price (p_t, p_{t+1}), the consumer t's demand is:

$$\begin{cases} x_t^y = \frac{aw^t}{p_t} = a(e^y + \frac{e^o}{\pi_t}) \\ x_t^o = \frac{(1-a)w^t}{p_{t+1}} = (1-a)(\pi_t e^y + e^o) \end{cases}$$

Clearly, if the relative price π_t increases, the agent will decrease his consumption when young and increase it again when old. His lifetime utility is an increasing function of $\pi_t \geq 0$. However, we note that at each period t, the relative price π_t cannot be too small. Indeed, if we let π_t tend to 0^+ at date t, the consumption of young x_t^y will be infinitely high while the total resource at each date is finite.

Supply function

Whenever $z_t \leq \hat{z}$, the firm can decide to produce $\zeta_{t+1} = -\gamma(z_t - \hat{z})$ following an average cost pricing. Thus we can write: $\zeta_{t+1} = -\pi_t z_t$ and:

- $\zeta_{t+1} = \frac{\gamma \hat{z} \pi_t}{\pi_t - \gamma}$
- $z_t = -\frac{\gamma \hat{z}}{\pi_t - \gamma}$

Remarks:
i) z_t and ζ_{t+1} are well defined whenever $\pi_t < \gamma$;
ii) for all $t \geq 0$, for $\pi_t \in (0, \gamma)$, $\zeta_{t+1} > 0$ and $z_t < 0$, in addition, ζ_{t+1} is an increasing function of π_t;
iii)$\lim_{\pi_t \to \gamma^-} \zeta_{t+1} = \lim_{\pi_t \to \gamma^-} \frac{\gamma \hat{y} \pi_t}{\pi_t - \gamma} = +\infty$
iv) In case of constant returns to scale, that is no fixed cost, it is known that the firm, while maximizing its profit would exhibit a discontinuous supply function. The fixed cost associated to the average cost pricing result in a smooth supply function on the range $(0, \gamma)$.

Equilibrium

An equilibrium is an element $(p_t^*, (x_0^{*o}, (x_t^{*y}, x_t^{*o})_{t=1}^{\infty}), (z_t,^* \zeta_{t+1}^*))$ such that:
a) for all t, (x_t^{*y}, x_t^{*o}) is a solution to

$$\begin{array}{ll} max & u(x_t^t, x_{t+1}^t) \\ s.t & \pi_t^* x_t^{*y} + x_t^{*o} \leq \hat{w}^{*t} \end{array}$$

b) For all t, $\pi_t^* z_t^* + \zeta_{t+1}^* = 0$;
c) $x_t^{*y} + x_{t-1}^{*o} = e^y + e^o + z_t^* + \zeta_{t-1}^*$;
where $\pi_t^* = \frac{p_t^*}{p_{t+1}^*}$

Characterization of equilibria

The sequence of prices (p_t^*) is an equilibrium price system if the sequence of relative prices (π_t^*) is a solution to:

$$(1-a)e^y(\pi_{t-1} - 1) - \frac{\gamma \hat{y}}{\pi_{t-1} - \gamma}\pi_{t-1} = ae^o(1 - \frac{1}{\pi_t}) - \frac{\gamma \hat{y}}{\pi_t - \gamma}$$

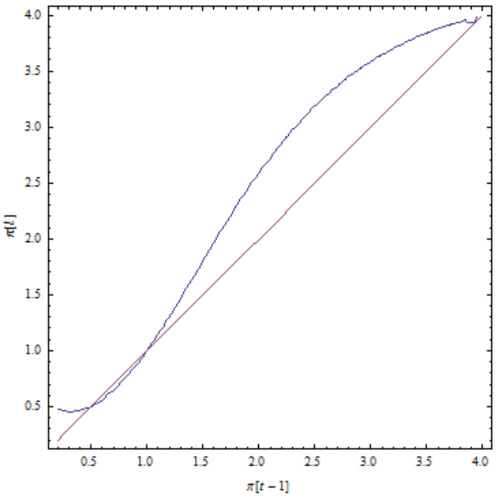

FIGURE 2.1: Asymptotic efficiency: when the average-cost prices tend to the marginal cost with no fixed cost

with $\pi_t \in (0, \gamma)$ for all $t \geq 1$, $\gamma > 1$, $0 \geq \hat{y} \geq -e^y$ and $1 \geq a \geq 0$.

The following figure illustrates this equation in terms of (π_{t-1}, π_t) for the particular case where $\gamma = 4$ the initial endowment at each period $\omega = 2$, the initial endowment when old $e = 0.8$, the propension to consume when young $a = 0.02$, the fixed cost $\hat{z} = -1$. Here, the consumer is ready to invest in the production since his endowment is large enough when young and his preferences put a higher weight when old.

We have multiple steady states equilibria: a low and stable steady state $\pi^* < 1$ and $\pi^{**} = 1$ which is not stable. Suppose that at each date t_0, $\pi_{t_0} < 1$, then the economy will display a succession of inflationary equilibria since all the following prices will be below 1 and will be decreasing to the low steady state π^*. If we instead have $\pi_{t_0} > 1$, then the successive price $\pi_{t_0+1} > 1$ so are all the following prices. This case will lead to a non stationary equilibria where the relative prices will increase and tend to the marginal price γ without reaching it, that is: $1 < \pi_{t_0} < \pi_{t_0+1} < \ldots < \gamma$. Thus from period t_0, the production will be increasing, so will be the welfare of all the successive generations since they will benefit from a deflation: $p_{t_0} > p_{t_0+1} > \ldots$. Thus we have an economy which is asymptotically efficient: it converges toward an equilibrium which is associated to an economy without fixed costs, at which the utility is at its maximum. For instance, if u^*, u^{**} and $\bar{u}$ are respectively the utility levels at π^*, π^{**} and γ, then given the same parameters above, we have: $u^* = 0.255 < u^{**} = 0.595 < \bar{u} = 1.597$.

The existence of fixed cost and the average cost pricing make the firm decide to produce at each date. So the fixed cost is not always necessarely a barrier to production. It is also important to remark that a production at the marginal price γ is still possible even though it will generate losses equal to $\gamma p_{t+1}\hat{z} = p_t\hat{z} \leq 0$. Indeed, the demand would be:

$$\begin{cases} x_t^y = a(e^y + \hat{z} + \frac{e^o}{\gamma}) \\ x_t^o = (1-a)(\gamma(e^y + \hat{z}) + e^o) \end{cases}$$

The corresponding market clearing equation gives a positive production equals to 0.768, given the same parameters above, and the associated utility at γ is 0.344 which is clearly lower than u^{**} thus lower than $\bar{u}$. This confirms the "*superiority* of average cost over over marginal cost when $\pi_{t_0} > 1$.

Note that the choice of a is primordial and that this result relies on the particular fact that consumers have higher initial endowment but lower incentive to consume when young, they are more focused on future consumption.

2.7 Discussions and Concluding Remarks

As already pointed out earlier, loss-free pricing rules are relevant in unregulated markets and can be applied to firms whose increasing returns are due to fixed cost, or associated to a S-shape production function, which means that the technology displays first increasing returns to scale until some level of production then decreasing returns to scale. As provided in Villar [59], important examples are the constrained profit maximization and the mark-up pricing rule which also include the average-cost pricing as a particular case. These two groups of loss-free pricing rules are known to be the closest extensions of competitive behavior when there are increasing returns. In particular, they follow the basic profit that a firm will produce only when the profits are non-negative, otherwise it closes down.

Increasing returns and imperfect competition:

It is important to remark that increasing returns are usually associated to imperfect competition. In particular, firms benefiting from increasing returns may become not negligible since their productivity will be as high as their size, so that they have some market power and their behavior goes beyond the price-taker's one. These firms act then as monopolies but because of the social inefficiencies they cause in the market, they may be considered publicly owned thus regulated. Regulation supposes that the firms are assigned to follow marginal pricing, which implies losses when there are increasing returns to scale, leading us beyond the loss-free framework. This calls for the need to model the behavior of firms with monopoly power in an unregulated approach, which face serious difficulties when these firms are not identical and interact between them.

Some models that treat growth theory have provided structures that permit to reconcile competitive behavior of firms and increasing returns.

In this line, Benhabib and Farmer [7] have provided such a model in which the firms behaviors and the associated pricing are described as follows. There is a continuum of intermediate goods $y(i)$, $i \in [0,1]$, they are used to produce a final consumption good according to the production function: $Y = \left[\int_0^1 (y(i))^\lambda di\right]^{1/\lambda}$ where $\lambda \in (0,1)$. Clearly, this production function exhibits constant returns to scale. The final good sector is assumed to be competitive. If $p(i)$ is the price of intermediate good i with respect to the final good, then the problem of the final good sector is given by:

$$\max_{y(i)} \left\{ \left[\int_0^1 (y(i))^\lambda di\right]^{1/\lambda} - \int_0^1 p(i)y(i)di \right\}$$

The first order condition implies that the demand for the intermediate goods i is given by: $y(i) = p_t(i)^{1/\lambda} Y$, or equivalently, $p(i) = \left(\frac{y(i)}{Y}\right)^{\lambda-1}$, which is well defined whenever $Y > 0$. The intermediate goods are, in turns produced by firms that are involved in a monopolistic competition as in Dixit and Stiglitz [27]. Their production, à la Cobb-Douglas makes use of capital and labor, $y(i) = K(i)^\alpha N(i)^\beta$, where $\alpha, \beta > 0$, $\alpha + \beta > 1$, thus the increasing returns to scale are introduced in the intermediate goods sector. It is assumed that the technology used to produce intermediate goods is identical to each firm and that the factor market is competitive so that the capital and the labor are paid their marginal products. Let r and ω be respectively the capital rent and the wage relative to the final consumption good. The decision of each firm consists of maximizing for $p(i)$ its profit: $\Pi(i) = p(i)y(i) - \omega N(i) - rK(i) = Y^{1-\lambda} N(i)^{\beta\lambda} K(i)^{\alpha\lambda} - \omega N(i) - rK(i)$. Here, the parameter λ represents the degree of monoppoly power of each firm, and for $\lambda(\alpha + \beta) \leq 1$, this profit function is concave in $N(i)$ and $K(i)$, which confirms the consistency between increasing returns in the technology and monopolistic competition. The first order condition gives: $r = \alpha\lambda \frac{p(i)y(i)}{K(i)}$ and $\omega = \beta\lambda \frac{p(i)y(i)}{N(i)}$. The assumption of symmetry leads to a solution where $N(i) = N$, $K(i) = K$ and $p(i) = \bar{p}$. However, the competitivity of the final good sector leads to zero profit, that is $Y - \int_0^1 \bar{p}y(i)di = 0$, and using the demand function in intermediate goods, we obtain that $p(i) = \bar{p} = 1$. But thanks to the monopoly power λ, the intermediate goods sector makes positive profits. Indeed, by substituting this price $\bar{p}$ to the expressions of ω and r and then to $\pi(i)$, we have that: $\Pi := \int_0^1 \pi(i)di = Y(1 - \lambda(\alpha + \beta)) > 0$, whenever $Y > 0$ and $\lambda(\alpha + \beta) < 1$.

Structuring the production sector in this way allows firms to exploit their market power by charging prices higher than their marginal costs, making then excess profits contrary to the regulated approach, thus in this approach, one basically makes use a variant of loss-free pricing rules. Indeed, in this case, the production set consists of the set of

(K, N, Y) such that $Y \leq K^{\alpha} N^{\beta}$ and a (weakly) efficient production (K, N, Y) satisfies $Y = K^{\alpha} N^{\beta}$, with $\alpha + \beta \geq 1$. According to the expressions of r, ω above and the fact that the final good Y is the reference good, the application φ defined by $\varphi(K, N, Y) := (\alpha\lambda K^{\alpha-1} N^{\beta}, \beta\lambda K^{\alpha} N^{\beta-1}, 1)$, which confirms that this notion of loss-free pricing rule gives account of models with increasing returns in macroeconomics. Here, we have an example of mark-up pricing à la Villar, where the mark-up, for $Y > 0$ is given by: $\frac{\Pi}{Y} = 1 - \lambda(\alpha + \beta)$ that is positive and constant. Another way of associating increasing returns and imperfect competition in macroeconomic models can be seen in Seegmuller [55], where increasing returns are due to a fixed cost, firms are identical, have perfect information on the final good's demand, can enter or exit the market for free, behave competitively in the factor market, and compete à la Cournot in the final good market. His model then shows for all the firms, a same positive mark-up that depends on the number of active firms in the market . This case actually follows the equi-profitability principle by Villar in [59].

Ownership transfer of firms:

Another important remark concerns the possible perpetuation of firms accross generations. Indeed, although we obtain an existence result which conditions are rather as general as the ones used for an exchange economy, an important feature was the assumption of exogenously given shares on firms, that all individuals received systematically at their birth, this settles the distribution of profits among consumers. However, according to the budget constraint, these profits in addition to the initial endowments are completely spent to finance lifetime consumptions of each agent, and it is not explicitly shown how firms can be perpetuated forever. The literature provides many ways to treat intergenerational transfers. One of these ways is the notion of bequest: agents, supposed altruistic, choose to devolve their properties and assets to other agents through a will at the end of their lifetime. For instance, we can see the shares θ^{ij}'s at date t as the bequest left to each child $i \in \mathcal{I}_t$ by cohort $t-1$. In this way, agents can prevent the disappearance of their firms from the economy, and their offsprings who become the new owners of the firms will distribute the profits between them according to the shares they have inherited, leave the ownerships to the next cohort and so on.

There is also a non altruistic way to transfer firms ownership among generations by allowing agents to buy shares of firms through savings. Indeed, the lifetime budget constraint implicitely supposes the existence of a hidden financial market, we can for instance assume that each individual of cohort t decides to save s^i when young at rate r_{t+1}. The budget balance of agent $i \in \mathcal{I}_t$ can be spread into: $p_t \cdot x_t^i = p_t \cdot e_t^i - s^i$ and $p_{t+1} \cdot x_{t+1}^i = p_{t+1} \cdot e_{t+1}^i + s^i(1 + r_{t+1})$. This savings can be used to invest in the firms so that agent i holds a portfolio (θ^{ij}), and $s^i := -\sum_{j \in \mathcal{J}} \theta^{ij} p_t \cdot z_t^j$. By assuming the no arbitrage conditions: $-p_t \cdot z_t^j = \frac{1}{1+r_{t+1}} p_{t+1} \cdot \zeta_{t+1}^j$ and that all firms j that are active at date t have the same return or profitability, agent i will be indifferent between holding a bond or investing in any firm j, so that the only thing that matters is his gain at date $t+1$, that is $s^i r_{t+1} = \sum_j \theta^{ij} \pi_t^j$, where π_t^j is the profit made by firm j at date t. We thus end up with the same lifetime budget constraint as in our model, that is: $p_t \cdot x_t^i + p_{t+1} \cdot x_{t+1}^i = p_t \cdot e_t^i + p_{t+1} \cdot e_{t+1}^i - s^i + (1 + r_{t+1}) s^i = p_t \cdot e_t^i + p_{t+1} \cdot e_{t+1}^i + \sum_j \theta^{ij} (p_t \cdot z_t^j + p_{t+1} \cdot \zeta_{t+1}^j)$, for $i \in \mathcal{I}_t$, $t \geq 1$.

Note that assuming uniform profitability among firms is an interesting feature. Indeed, This case is already presented above, and it captures the situation where firms operate in the same economical environment and may face the same conditions that generate increasing returns to scale.

However, the shares on firms, θ^{ij} remains undeterminated, it is not explicitely provided how the shares are attributed to each agent, this leavies unexplained a key element of the model. This issue will appear again later in the next chapter, in which we also aim at defining a plausible intergenerational transfer. The paper is partially inspired by Magill and Quinzii in [43], and consists of a modest approach, intending to analyze the mechanism of transfer between generations, by means of durable goods.

Chapter 3

OLG models with durable goods

Abstract

We consider a standard pure exchange overlapping generations economy. The demographic structure consists of a new cohort of agents at each period with an economic activity extended over two successive periods. Our model incorporates durable goods that may be stored from one period to a successive period through a linear technology. In this model, we intend to study the mechanism of transfer between generations, and we show that the existence of an equilibrium can be established by considering an equivalent economy "*without*" durable goods, where the agents economic activity is extended over three successive periods[1].

JEL classification: C62, D50, D62.

Keywords: Overlapping generations model, durable goods, irreducibility, equilibrium, existence

[1]This paper is based on "*Existence of equilibrium in OLG economies with durable goods*", [51]

3.1 Introduction

Many works in OLG models involve the standard hypothesis of constant returns to scale. This hypothesis constitutes a limitation to the production sector. Indeed, by bahaving competitively, producers will reach their maximal profit at zero. Thus, the issue on the repartition of profits among consumers who own the firms is avoided, as well as the one on the existence of market which allows the transfert of property right from one generation to another one. Moving beyond this restriction constitutes one of our motivations in a previous work [13]. In this paper, we indeed proposed to go beyond the standard hypothesis of constant returns and considered a model of overlapping generations with production where increasing returns are allowed. The equilibrium existence rests on the following facts. There is a finite number of firms that are active forever but owned successively by 2-period lived individuals who receive exogeneous shares on firms at their birth. The producers are instructed to make non-negative profits while they choose a combination of prices and production plans. The consumers wealth consists of the value of their initial endowments and the profits they obtain from the firms according to their respective shares. They use all their wealth to pay for their lifetime consumption, leaving nothing to the next generations. Thus the transfers of property rights between generations are excluded.

We propose then to improve this work by studying the possibility of transfer of firms property across generations. This paper constitutes a first step to this improvement. We are working in the line of [43] who considers a model of overlapping generations with production. There are durable goods which depreciate over time, and they perpetuate the firms by allowing the transmission of the ownership shares accross generations through a stock market. However, we will first focus our attention to a pure exchange overlapping generations model as in [3–5], but in addition, commodities can be durable. We intend to study, in such a model, the mechanism of wealth transfer between generations, and we consider that the existence of durable goods makes the transfer of shares between generations meaningful.

We are in a model with infinitely many dates, and for simplicity, the set of commodities, perishable or durable, is the same at each date. Furthermore, there is no uncertainty, and every individual is supposed to correctly anticipate the future prices. There is a technology that permits to store and transfer the durable goods from one period to the next one. This storing technology is linear. The transfered good will implicitely act like an additionnal endowment at the date it is available. This technology is considered as a production function in the sense that a consumer who purchases a durable good can consume it and also use it as input to produce a good at the succeeding period. The relation between the spot prices in Proposition 3.2.1, which is like an arbitrage-free condition illustrates this, where price at each date can consist of two components: the input investment and the consumption cost. Malinvaud already considers this kind of phenomen in [44]. There, he introduces a forward market at each date, where agents can trade goods available only in the future. We can assimilate our work to a model à la Malinvaud with production.

The existence of durable goods in the model implies that agents in the end of their lifetime, will still own some goods which they will not need anymore in the next period. Thus we introduce, at each date a futures market that allows trades of goods available in a future date. This market helps the old generations to sell, at the end of their lifetime, their remaining durable goods after consumption. As mentionned above, this futures market is similar to a market of shares in a firm: at each date, old agents sell to the young the right to dispose of the remaining durable goods. This creates an additional resource to the old agents and a possibility for the young agents to increase their endowment when they become old. Purchase by the young on this market can be seen as savings that will finance the retirement of the old. The old generations, in return, will leave, at the end of their lifetime some commodities, to which the young generations, one period after, can have access. We can see those contracts as lifetime sale contracts called also "viager", where the old people can sell, for instance their house for an annuity, while they still occupy it until the end of their life, the buyers will then own the house right after.

In our case, there is no uncertainty, but durable goods can also act as collaterals in mortgage loans, in which case the reimbursement takes place one period after the agreement, during which the borrower seizes the collateral itself. This collateral role of durable goods is not new, it has been treated in intertemporal general equilibrium theory, as in [36] in the case of two-period model, or also in [2] in which agents have infinte lifetime and trade long-lived assets secured by durable goods. Thus, even though default may imply disequilibrium, by incorporating durable goods to secure assets, default becomes possible and equilibrium exists under standard hypothesis. In our case, in order to establish the existence result, we make an important assumption on the desirability of durable goods, which ensures that prices are positive, and in addition, there are no wastes at equilibrium: all the durable goods owned by the old will be bought by the young generations, that is all lendings will be payed back.

Our model can easily be extended to treat the case where lifetimes are uncertain and agents may hold assets or be involved in any liability at the end of their lifetime. If indeed, at each period, they have a probability to die in the next period, then defaults are prevented thanks to the durable goods that serve as collaterals to loans. In both frameworks of certain and uncertain lifetimes, we remark that a purchase of durable goods by old can be also the result of bequest motives, where agents are supposed to be altruistic.

While we mainly focus on the mechanism of wealth transfer between generations, it is important to note that many works, theoretical and empirical, on durable goods already exist in literature, to study different issues, such as savings, borrowing constraints and collaterals. Furthermore, since durable goods are used as components in wealth, they are useful to study wealth distribution, see Diaz and Luenngo-Prado in [25], who in addition relate the two with precautionary savings. In [25], for instance the liquidity or illiquuidity nature of durable goods, have impact on the behaviour of consumers, reflect to the notion of precautionary saving, especially when there is uncertainty or risk in the economy. Such an issue is not treated in our case, especially since the main feature of the durable goods we consider is their desirability, thus without taking into account how liquid they are. Moreover, these studies which involve empirical analysis consider durable goods which are not easily divisible, and entail very high transaction costs, as in Martin [46]. In [25], they are assumed specific to households and cannot be traded or rented without first converting them back to a productive capital. Our model does not have this specificity and allows for a durable good to be divisible when sold on a future market to the young. But in case of production, this work meets that property and goes in the line of our aim to study the perpetuation of firms through transfers of shares and property rights accross generations. For instance, agents partipate in firms by putting together their investment so that each durable capital, possibly specific to each firm is kept in its entirety.

The paper is organized as follows. The model is descibed in Section 2. We make classical assumptions on the consumers, at the same level as for a pure exchange economy with perishable goods, but in addition we assume that goods are desirable. A first result of the paper establishes a relation between the spot prices and the futures prices at equilibrium.

Our main result is the existence of an equilibrium in this economy. The arbitrage-free conditions on equilibrium prices and the condition of no wastes allow us to consider a so-called "reduced equilibrium". This arbitrage-free condition at equilibrium also implies an indeterminacy concerning the purchase of young agents on the futures market. Indeed, thanks to the relations between spot and futures prices like no-arbitrage conditions, they will be indifferent between buying today on the futures market or buying tomorrow on the spot market.

We then establish the existence of the reduced equilibrium, for that we reformulate the model into an equivalent economy "without" durable goods as defined in Section 3. In this associated economy, all individuals will artificially live over three periods, and the consumption sets are transformed so that they will not consist of the positive orthant anymore. Furthermore the strong survival assumption is not satisfied since the initial endowments are no longer interior points.

The existence result is concluded in Section 4, where the proof is similar to Balasko et al in [3–5] but we also use the notion of irreducibility, used by Florenzano as seen in [28, 29]. Irreducibility ensures that no matter how we allocate the individuals into two groups, each of the groups has some good for which the other group is willing to exchange with some goods of its own. This condition is easily obtained in our model thanks to the presence of durable and desirable goods and the connections between all the generations: they are indeed involved in a trade, either directly when they have common periods of life, or indirectly, in which case, individuals of each generation will successively act as intermediaries between them. The existence of equilibrium in the original model with durable goods follows the existence of equilibrium in the equivalent economy without durable goods. This passage through an equivalent economy gives account to the involvement of durable goods in the utility of the agents.

3.2 The Model

We consider an overlapping generations economy with discrete and infinitely many dates $(t = 1, 2 \dots)$.

Commodities

There exists a finite set $\mathcal{L}$ of commodities available for consumption and trade in the world. We denote $\#\mathcal{L} = L$. Goods can be perishable or durable, and may suffer transformations from one period to an immediate successive period.

We represent these transformations by linear mappings $\Gamma^t : \mathbb{R}^L \to \mathbb{R}^L$ which transform each consumption $x_t \in \mathbb{R}^L_+$ at date t into a bundle of goods
$\Gamma^t(x_t) \in \mathbb{R}^L_+$ at date $t+1$. The commodity $\ell \in \mathcal{L}_t$ is perishable if $\Gamma^t(\delta_\ell) = 0$, where $\delta_\ell \in \mathbb{R}^L$ consists of one unit of commodity ℓ and nothing else.

So each good can be seen as a consumption good and an input if we think of Γ^t as a production function.

Consumers

A generation 0, $\mathcal{I}_0$, lives only one period. At each period $t \in \mathbb{N}^*$, there is a finite and non-empty set of consumers $\mathcal{I}_t$ called generation t, who are born and live for two periods. We denote $\#\mathcal{I}_t = I_t$ and $\mathcal{I} = \cup_{t\in\mathbb{N}}\mathcal{I}_t$.

The consumption set of each individual $i \in \mathcal{I}_t$ is a subset $X^i = \mathbb{R}^L_+ \times \mathbb{R}^L_+$. The consumption set of consumers of generation 0 is $\mathbb{R}^L_+$.

Consumers preferences are represented by a utility function $u^i : X^i \to \mathbb{R}$.

The vector $e^i \in \mathbb{R}^L \times \mathbb{R}^L$ represents the initial endowment of the agent i of the generation t.

Assumption C.

a) For all individuals in $\mathcal{I}$, u^i is continuous, quasi-concave and locally non-satiated.

b) For all $t \in \mathbb{N}^*$, there exists $i_0(t) \in \mathcal{I}_t$ such that $u^{i_0(t)}$ is strictly monotonic.

Assumption C is a classical assumption in a standard finite economy.

Assumption E. For all $t \in \mathbb{N}^*$, for all $i \in \mathcal{I}_t$, $e^i \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{++}$, for all $i \in \mathcal{I}_0$, $e^i \in \mathbb{R}^L_{++}$.

Markets and Prices

At each date t, there is a spot market for consumption. The price vector p is an element of $\prod_{t=1}^{\infty} \mathbb{R}^L_+$ and $p_{t\ell}$ is the spot price of commodity ℓ at date t.

Furthermore, we allow for trade between generations. To make clear how this trade takes place within one period, consider an individual born at date t who purchases x^i_{t+1} when old. This consumption gives right to x^i_{t+1} at date $t+1$ and to $\Gamma^{t+1}(x^i_{t+1})$ available only at date $t+2$ that is, after his lifetime. So at the end of date $t+1$ he may wish to sell $\Gamma^{t+1}(x^i_{t+1})$ to young. We write f^i_{t+1} the purchase of a young i of generation $t+1$ from old agents of generation t at date t+1, and Π_{t+1} the vector price at which the trade is agreed, the future price vector Π is an element of $\prod_{t=1}^{\infty} \mathbb{R}^L_+$. We remark that f^i_{t+1} is not available before date $t+2$. This market can be seen as a futures market, where agents trade goods available only in the next period.

Each individual $i \in \mathcal{I}_t$ purchases f_t^i at date t from generation $t-1$ at price Π_t, which is available only at date $t+1$ and gives right to $\Gamma^t(f_t^i)$ at price p_{t+1}. At date $t+1$, the same individual earns $\Gamma^t(x_t^i)$ from his previous consumption and sells x_{t+1}^i on the futures market, when he is old, at the end of his lifetime. These operations are summarized by the following budget constraint at prices (p, Π): for agent $i \in \mathcal{I}_t$,

$$p_t \cdot x_t^i + p_{t+1} \cdot x_{t+1}^i + \Pi_t \cdot f_t^i \leq p_t \cdot e_t^i + p_{t+1} \cdot e_{t+1}^i + p_{t+1} \cdot \Gamma^t(x_t^i) + p_{t+1} \cdot \Gamma^t(f_t^i) + \Pi_{t+1} \cdot x_{t+1}^i$$

for agent $i \in \mathcal{I}_0$, $p_1 \cdot x_1^i \leq p_1 \cdot e_1^i + \Pi_1 \cdot x_1^i$.

Note that the consumption when old x_{t+1}^i appears at both sides as expenditure and additionnal income.

We denoted by $B^i(p, \Pi)$ the budget constraint associated to (p, Π).

Feasibility conditions

An allocation $((x^i), (f^i))$ in $\prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} \mathbb{R}_+^L$ is feasible if:

$$\sum_{i \in \mathcal{I}_{t-1}} x_{t,h}^i = \sum_{i \in \mathcal{I}_t} f_{t,h}^i, \text{ for } t \geq 1, \text{ for } h \text{ durable },$$

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^i = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(x_{t-1}^i) + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(f_{t-1}^i), \text{ for } t > 1,$$

and

$$\sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x_1^i = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} e_1^i$$

The first equation indicates that there are no wastes, all the durable goods owned by old agents at the end of their lifetime will be bought by the young agents at that time.

We denote by $\mathcal{A}(\mathcal{E})$ the set of all feasible allocations.

Equilibrium

Definition 3.2.1. An equilibrium of the economy $(\mathcal{E})$ is a list $(p^*, \Pi^*, (x^{i*}), (f^{i*}))$ in $\prod_{t=1}^{\infty} \mathbb{R}_+^L \times \prod_{t=1}^{\infty} \mathbb{R}_+^L \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} \mathbb{R}_+^L$ such that:

a) for all $t \geq 1$, for all $i \in \mathcal{I}_t$, (x^{i*}, f^{i*}) is a maximal element of u^i satisfying the budget constraint:

$$p_t^* \cdot x_t^i + p_{t+1}^* \cdot x_{t+1}^i + \Pi_t^* \cdot f_t^i \leq p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i + p_{t+1}^* \cdot \Gamma^t(x_t^i) + p_{t+1}^* \cdot \Gamma^t(f_t^i) + \Pi_{t+1}^* \cdot x_{t+1}^i,$$

for all $i \in \mathcal{I}_0$, x^{i*} is a maximal element of u^i satisfying: $p_1^* \cdot x_1^i \leq p_1^* \cdot e_1^i + \Pi_1^* \cdot x_1^i$.
b) the allocation $((x^{i*}), (f^{i*}))$ is feasible:

$$\sum_{i \in \mathcal{I}_{t-1}} x_t^{i*} = \sum_{i \in \mathcal{I}_t} f_t^{i*}, \text{ for } t \geq 1,$$

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^{i*} = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(x_{t-1}^{i*}) + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(f_{t-1}^{i*}), \text{ for } t > 1,$$

$$\sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x_1^{i*} = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} e_1^i.$$

Remark 3.2.1. The first equation in Condition *b) implies:*

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^{i*} = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i + \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_{t-2}} \Gamma^{t-1}(x_{t-1}^{i*}), \text{ for } t > 1$$

This equation states that the consumptions at date t involve consumptions of the preceeding generations.

In the following, we denote by γ^t the transpose of Γ^t.

Proposition 3.2.1. *i)* If (p^*, Π^*) is an equilibrium price, then $(p^*, \hat{\Pi}^*)$ where $\hat{\Pi}_t^* := \gamma^t(p_{t+1}^*)$ for all t, is also an equilibrium.

ii) For all $t \in \mathbb{N}^*$, $p_t^* \gg \gamma^t(p_{t+1}^*)$.

Remark 3.2.2. *i)* The equilibrium price $(p^*, \hat{\Pi}^*)$ coincides with the equilibrium defined by Malinvaud in [44], where he considered an intertemporal economy with perishable commodities. These commodities do not cross time but may be available to agents only in future dates; in this case, agents are allowed to trade on forward markets at forward prices. Our model is then similar to the model à la Malinvaud with a production with constant returns.

ii) Furthermore, if we think x_t^{i*} as a bundle that gives right to consumption at date t as well as "input" for date $t+1$, we may write $p_t^* = \gamma^t(p_{t+1}^*) + p_t'$, where $\gamma^t(p_{t+1}^*)$ can be seen as the "input" cost while p_t' is the consumption cost at date t.

Proof. At equilibrium, $\Pi^*_{t+1} \geq \gamma^{t+1}(p^*_{t+2})$, for all t. Otherwise, all the young agents at date $t+1$ will have an arbitrage opportunity by buying a commodity h on the futures market and reselling it on the spot market at date $t+2$. Then, there is no solution to the utility maximization problem under the budget constraint since the utility functions are locally nonsatiated.

Furthermore, if $\Pi^*_{t+1,h} > (\gamma^{t+1}(p_{t+2}))_h$, the young agents have no incentive to buy on the futures market because it is better to wait until the next period to make the purchase on the spot market, so $f^{i*}_{t+1,h} = 0$ for all i. But this implies that $x^{i*}_{t+1,h} = 0$ for all i. If we decrease the future price from $\Pi^*_{t+1,h}$ to $\hat{\Pi}^*_{t+1,h}$, then the budget set $B^i(p^*, \hat{\Pi}^*)$ associated to $(p^*, \hat{\Pi}^*)$ is smaller and included in the budget set $B^i(p^*, \Pi^*)$ associated to (p^*, Π^*). Moreover, $((x^{i*}), (f^{i*}))$ belongs to $B^i(p^*, \hat{\Pi}^*)$. Hence as it is optimal in $B(p^*, \Pi^*)$, it is still optimal in the smaller set $B^i(p^*, \Pi^*)$. So $(p^*, \hat{\Pi}^*)$ is an equilibrium price with the same consumption allocations.

Now, let us suppose that there exists a durable commodity h in $\mathcal{L}$ such that $p^*_{th} \leq (\gamma^t(p^*_{t+1}))_h$. Thus, either $p^*_{th} < (\gamma^t(p^*_{t+1}))_h$ or $p^*_{th} = (\gamma^t(p^*_{t+1}))_h$ If the first case holds, then young agents at date t, will have an arbitrage opportunity by buying the commodity h at price p^*_{th} on the spot market, and reselling it at the price $(\gamma^t(p^*_{t+1}))_h$ at date $t+1$. In the second case, the agent would be willing to buy as much as she wants of good h when she is young, since her utility is locally nonsatiated, thus there would be no solution to the utility maximization problem. So necessarely, arbitrage-free condition implies $p^*_t \gg \gamma^t(p^*_{t+1})$, for all $t \in \mathbb{N}^*$.
□

Thus the list $(p^*, (x^{i*}), (f^{i*}))$ in $\prod_{t=1}^{\infty} \mathbb{R}^L_+ \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} \mathbb{R}^L_+$ such that:

a) for all $t \geq 1$, for all $i \in \mathcal{I}_t$, x^{i*} is a maximal element of u^i in the budget set:

$$\{x^i \in X^i \mid (p^*_t - \gamma^t(p^*_{t+1})) \cdot x^i_t + (p^*_{t+1} - \gamma^{t+1}(p^*_{t+2})) \cdot x^i_{t+1} \leq p^*_t \cdot e^i_t + p^*_{t+1} \cdot e^i_{t+1}\},$$

for all $i \in \mathcal{I}_0$, x^{i*} is a maximal element of u^i in the budget set:
$\{x^i \in X^i \mid (p^*_1 - \gamma^1(p^*_2)) \cdot x^i_1 \leq p^*_1 \cdot e^i_1\}$.

b) the allocations $((x^{i*}), (f^{i*}))$ are feasible:

$$\sum_{i \in \mathcal{I}_{t-1}} x^{i*}_t = \sum_{i \in \mathcal{I}_t} f^{i*}_t, \text{ for } t \geq 1,$$

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x^{*i}_t = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e^i_t + \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_{t-2}} \Gamma^{t-1}(x^{i*}_{t-1}), \text{ for } t > 1,$$

$$\sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} x_1^{i*} = \sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} e_1^i.$$

is an equilibrium with durable goods.

The budget constraint in *a)* indicates that the agents anticipate the future prices. Indeed the price vector p^*_{t+2} appears in the budget constraint of the agents of generation t because of the trade they make when old with the young of generation $t+1$ whose budget constraint involves prices at date $t+2$: at equilibrium these agents anticipate Π_{t+1} to be equal to $\gamma^{t+1}p_{t+2}$.

Moreover, we note an indetermination for the (f^{i*}) given in *b)*. As a matter of fact, the individuals maximize their utility under a budget constraint that does not depend on the f^{i*} anymore. The f^{i*}'s are only given by $\sum_{i\in\mathcal{I}_{t-1}} x_t^{i*} = \sum_{i\in\mathcal{I}_t} f_t^{i*}$, for $t \geq 1$, which means that the agents are indifferent between buying today on the futures market or buying tomorrow on the spot market. This is an usual direct consequence of the no-arbitrage condition on spot and futures prices.

To prove the existence of an equilibrium with durable goods, we will focus on the so-called "*reduced equilibrium*" defined as follows:

Definition 3.2.2. A "*reduced equilibrium*" is a list $(p^*, (x^{i*}))$ of $\prod_{t=1}^{\infty}\mathbb{R}^L_+ \times \prod_{t=0}^{\infty}\prod_{i\in\mathcal{I}_t} X^i$ such that:

a) for all $t \geq 1$, for all $i \in \mathcal{I}_t$, x^{i*} is a maximal element of u^i in the budget set:

$$\{x^i \in X^i \mid (p_t^* - \gamma^t(p^*_{t+1}))\cdot x_t^i + (p^*_{t+1} - \gamma^{t+1}(p^*_{t+2}))\cdot x^i_{t+1} \leq p_t^*\cdot e_t^i + p^*_{t+1}\cdot e^i_{t+1}\},$$

for all $i \in \mathcal{I}_0$, x^{i*} is a maximal element of u^i in the budget set:
$\{x^i \in X^i \mid (p_1^* - \gamma^1(p_2^*))\cdot x_1^i \leq p_1^*\cdot e_1^i\}$.

b) the allocations $((x^{i*}))$ are feasible:

$$\sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} x_t^{*i} = \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} e_t^i + \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_{t-2}} \Gamma^{t-1}(x^{i*}_{t-1}),\ \text{for } t > 1,$$

$$\sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} x_1^{i*} = \sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} e_1^i.$$

Indeed, if $(p^*, (x^{i*}))$ is a reduced equilibrium, then the list $(p^*, \Pi^*, (x^{i*}), (f^{i*}))$ where $\Pi^*_t := \gamma^t(p^*_{t+1})$ for all t, and the f^{i*}'s are such that $\sum_{i\in\mathcal{I}_{t-1}} x^{i*}_t = \sum_{i\in\mathcal{I}_t} f^{i*}_t$, for $t \geq 1$, is an equilibrium.

Our main result is the following existence theorem.

Theorem 3.2.1. Under Assumptions C and E, the economy $\mathcal{E}$ has an equilibrium.

3.3 An equivalent economy without durable goods

In the following, since the budget constraint of each individual involves prices over three periods of time, we build an *equivalent* economy $\tilde{\mathcal{E}}$ "without" durable goods, where each individual's lifetime is extended over three periods. This equivalent economy is similar to the standard pure exchange OLG model with perishable goods, and we will establish the existence of an equilibrium in $\tilde{\mathcal{E}}$ to prove the existence of an equilibrium in an economy with durable goods.

3.3.1 Description of the equivalent economy $\tilde{\mathcal{E}}$

We consider an overlapping generations model with discrete and infinitely many dates $t = 1, 2\ldots$, and the same commodity space $\mathcal{L}$ at each date.
At each date t, the set of consumers, called generation t is the same, and denoted by $\mathcal{I}_t$.

To describe the characteristics of the consumers we define the following linear mappings:
$\phi^t : (\mathbb{R}^L)^2 \to (\mathbb{R}^L)^3$, for $t \geq 1$, by $\phi^t(x^i_t, x^i_{t+1}) = (x^i_t, \xi^i_{t+1}, \zeta^i_{t+2})$, with
$\xi^i_{t+1} = x^i_{t+1} - \Gamma^t(x^i_t)$, and $\zeta^i_{t+2} = -\Gamma^{t+1}(\xi^i_{t+1} + \Gamma^t(x^i_t))$,
$\phi^0 : \mathbb{R}^L \to (\mathbb{R}^L)^2$, by $\phi^0(x^i_1) = (\xi^i_1, \zeta^i_2)$, with $\xi^i_1 = x^i_1$, and $\zeta^i_2 = -\Gamma^1(\xi^i_1)$.

The consumption sets are now defined as follows:

For each $i \in \mathcal{I}_0$, $\tilde{X}^i := \phi^0(\mathbb{R}^L_+)$, and for each $i \in \mathcal{I}_t,\ t \geq 1$, $\tilde{X}^i := \phi^t((\mathbb{R}^L_+)^2)$.

Thus the consumption set of an agent i of generation t, $\tilde{X}^i$ is defined over three periods: t, t+1 and t+2, and that of generation 0 is now defined over two periods: $t = 1$ and $t = 2$.

The initial endowment is defined as follows:
$\tilde{e}^i = (e^i_1, 0) \in \mathbb{R}^L_{++} \times \mathbb{R}^L$, for $i \in \mathcal{I}_1$, and $\tilde{e}^i = (e^i_t, e^i_{t+1}, 0) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{++} \times \mathbb{R}^L$, for $i \in \mathcal{I}_t$.

For each t, the function ϕ^t is injective, the function $\phi^t_{|(\mathbb{R}^L_+)^2} : (\mathbb{R}^L_+)^2 \to \tilde{X}^i$ is bijective. Its inverse $\psi^t : \tilde{X}^i \to (\mathbb{R}^L_+)^2$ is thus well defined. In the same way, the function $\phi^0_{|\mathbb{R}^L_+} : \mathbb{R}^L_+ \to \tilde{X}^i$ is bijective, and its inverse $\psi^0 : \tilde{X}^i \to \mathbb{R}^L_+$ is thus also well defined.

We can now define the new utility functions $\tilde{u}^i : \tilde{X}^i \to \mathbb{R}$, by $\tilde{u}^i = u^i \circ \psi^t$.

Note that for each $i \in \mathcal{I}_0$, $\tilde{u}^i(\xi^i_1, \zeta^i_2) = u^i(\xi^i_1)$, and for each $i \in \mathcal{I}_t,\ t \geq 1$, $\tilde{u}^i(x^i_t, \xi^i_{t+1}, \zeta^i_{t+2}) = u^i(x^i_t, \xi^i_{t+1} + \Gamma^t(x^i_t))$.

This new utility function clearly shows the role of any purchase of durable goods by young agents on their consumption when old. Re-defining consumption bundles in this way supposes an "*internalisation*" of the durability technology in the utility.

The definition below coincides with the standard definition of a competitive equilibrium in an OLG economy without durable goods.

Definition 3.3.1. An equilibrium in $\tilde{\mathcal{E}}$ is a list $(p^*, (\chi^{i*}))$ in $\prod_{t=1}^{\infty} \mathbb{R}^L_+ \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} \tilde{X}^i$, such that:
a) for all $t \in \mathbb{N}$, for all $i \in \mathcal{I}_t$, $\chi^{i*} = (x^{i*}_t, \xi^{i*}_{t+1}, \zeta^{i*}_{t+2})$ is a maximal element of $\tilde{u}^i$ satisfying the equivalent budget constraint:
$p^*_t \cdot x^{i*}_t + p^*_{t+1} \cdot \xi^{i*}_{t+1} + p^*_{t+2} \cdot \zeta^{i*}_{t+2} \leq p^*_t \cdot \tilde{e}^i_t + p^*_{t+1} \cdot \tilde{e}^i_{t+1}$
b) the consumption plan (χ^{i*}) is feasible:
$\sum_{i \in \mathcal{I}_0} \xi^{i*}_1 + \sum_{i \in \mathcal{I}_1} x^{i^*}_1 = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} \tilde{e}^i_1$
$\sum_{i \in \mathcal{I}_{t-2}} \zeta^{i*}_t + \sum_{i \in \mathcal{I}_{t-1}} \xi^{i*}_t + \sum_{i \in \mathcal{I}_t} x^{i*}_t = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} \tilde{e}^i_t,\ t > 1$

Proposition 3.3.1. If $(p^*, (\chi^{i*}))$, where $\chi^{i*} = (x^{i*}_t, \xi^{i*}_{t+1}, \zeta^{i*}_{t+2})$ for $i \in \mathcal{I}_t$ is an equilibrium of the equivalent economy, then $(p^*, (x^{i*}))$ is a reduced equilibrium, where $x^{i*} = (x^{i*}_t, \xi^{i*}_{t+1} + \Gamma(x^{i*}_t))$ for $i \in \mathcal{I}_t$, $t \geq 1$.

Proof. Indeed, by construction, if (χ^{i*}) is feasible in $\tilde{\mathcal{E}}$, then, $((x^{i*}))$ defined by $x^{i*} = (x^{i*}_t, \xi^{i*}_{t+1} + \Gamma(x^{i*}_t))$ is feasible, that is:

$$\sum_{i \in \mathcal{I}_0} \xi^{i*}_1 + \sum_{i \in \mathcal{I}_1} x^{i^*}_1 = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x^{i^*}_1 = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} \tilde{e}^i_1$$

$$\sum_{i \in \mathcal{I}_{t-2}} \zeta^{i*}_t + \sum_{i \in \mathcal{I}_{t-1}} \xi^{i*}_t + \sum_{i \in \mathcal{I}_t} x^{i*}_t = \sum_{i \in \mathcal{I}_{t-2}} -\Gamma^{t-1}(x^{i*}_{t-1}) + \sum_{i \in \mathcal{I}_{t-1}} (x^{i*}_t - \Gamma^{t-1}(x^{i*}_{t-1})) + \sum_{i \in \mathcal{I}_t} x^{i*}_t$$

Thus:

$$\sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} x_t^{*i} = \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} \tilde{e}_t^i + \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_{t-2}} \Gamma^{t-1}(x_{t-1}^{i*}),\ \text{for } t>1$$

Furthermore the optimality of χ^{i*} for the utility function $\tilde{u}^i$ under the equivalent budget constraint above implies, by construction, the optimality of $x^{i*} = (x_t^{i*}, \xi_{t+1}^{i*} + \Gamma(x_t^{i*}))$ for the utility function u^i and under the budget constraint:

$$p_t^* \cdot x_t^i + p_{t+1}^* \cdot \underbrace{(x_{t+1}^i - \Gamma^t(x_t^i))}_{\xi_{t+1}^i} + p_{t+2}^* \cdot \underbrace{(-\Gamma^{t+1}(x_{t+1}^i))}_{\zeta_{t+2}^i} \le p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i$$

$$p_1^* \cdot \underbrace{x_1^i}_{\xi_1^i} + p_2^* \cdot \underbrace{(-\Gamma^1(x_1^i))}_{\zeta_2^i} \le p_1^* \cdot x_1^i,\ i\in\mathcal{I}_0.$$

□

3.3.2 Some properties of the equivalent economy

- For all $t = 1, 2, \ldots$, for all $i \in \mathcal{I}_t$, the consumption sets $\tilde{X}^i$ are non-empty, closed, and convex.
  We note that the $\tilde{X}^i$'s are not the positive orthants, but the consumptions at date t for $i \in \mathcal{I}_t$ are bounded from below. Indeed, for each individual i of generation t, we allow for negative consumptions at dates $t+1$ and $t+2$, but the consumptions at date $t+1$ and $t+2$ are constrained by the consumptions at the preceeding dates. These kind of consumption sets are considered by Florenzano in [30], in which nonnegative components are called consumptions and nonpositive ones deliveries: the activity of an agent of generation t at date $t+1$ artificially consists of delivering the remaining goods she holds at this period to agents of the next genaration.

- For all $t = 1, 2, \ldots$, for all $i \in \mathcal{I}_t$, the utility functions $\tilde{u}^i$ defined above inherit the conditions on u^i in Assumption C, thanks to the linearity of ϕ^t and ψ^t. In particular, there exists $i_0(t)$ in $\mathcal{I}_t$ such that $\tilde{u}^{i_0}$ is strictly monotonic with respect to the two first variables $(x_t^{i_0}, \xi_{t+1}^{i_0})$.

- The set of feasible allocations $\mathcal{A}(\tilde{\mathcal{E}})$, that is the set of allocations satisfying the market clearing condition (Condition (b) of Definition 3.3.1) for the economy $\tilde{\mathcal{E}}$ is a subset of a compact set for the product topology. Indeed, let $e \in \prod_{t\in\mathbb{N}^*} \mathbb{R}_+^L$ be defined by $e_t = \sum_{i\in\mathcal{I}_t\cup\mathcal{I}_{t-1}} e_t^i$. Let $e' \in \prod_{t\in\mathbb{N}^*} \mathbb{R}_+^L$ such that $e' \ge e$. Then, there exists a sequence of nonnegative vectors $(M_t)_{t\ge1}$ such that for all $(\chi^i) \in \mathcal{A}(\tilde{\mathcal{E}})$,

with $\chi^i = (x_t^i, \xi_{t+1}^i, \zeta_{t+2}^i)$, $i \in \mathcal{I}_t$, for all $t = 1, 2 \ldots$, we have:
$0 \leq x_t^i \leq M_t$, $-\Gamma^t(M_t) \leq \xi_{t+1}^i \leq M_{t+1}$, and $-\Gamma^{t+1}(M_{t+1}) \leq \zeta_{t+2}^i \leq 0$.
M_t is recursively defined by: $M_t = e'_t + \Gamma^{t-1}(M_{t-1} + \Gamma^{t-2}(M_{t-2}) \cdots + \Gamma^1(M_1))$, where $M_1 = e'_1$. (See Appendix A)

3.4 Existence of equilibrium in the equivalent economy

To establish the existence of equilibrium in $\tilde{\mathcal{E}}$, we first truncate the economy $\tilde{\mathcal{E}}$ at a finite horizon τ and consider the set of all individuals born up to period $\tau - 2$, $\mathcal{I}_0^{\tau-2} = \cup_{t=0}^{\tau-2} \mathcal{I}_t$.

For each $i \in \mathcal{I}_0$,

$\tilde{X}^{\tau i} = \{a \in (\mathbb{R}_+^L)^\tau \mid (a_1, a_2) \in \tilde{X}^i, a_{t'} = 0, \forall t' > 2\}$

$\tilde{u}^{\tau i}(a) = \tilde{u}^i(a_1, a_2)$

$\tilde{e}^{\tau i} = (\tilde{e}_{t'}^{\tau i})_{t'=1}^\tau$ such that $\tilde{e}_1^{\tau i} = \tilde{e}_1^i$, and $\tilde{e}_{t'}^{\tau i} = 0$ if $t' > 1$.

For each $i \in \mathcal{I}_t, t = 1, 2 \ldots \tau - 3$,

$\tilde{X}^{\tau i} = \{a \in (\mathbb{R}_+^L)^\tau \mid (a_t, a_{t+1}, a_{t+2}) \in \tilde{X}^i,\ a_{t'} = 0, \forall t' \neq t, t+1, t+2\}$

For each $i \in \mathcal{I}_{\tau-2}$,

$\tilde{X}^{\tau i} = \{a \in (\mathbb{R}_+^L)^\tau \mid (a_{\tau-2}, a_{\tau-1}, -\Gamma^{\tau-1}(a_{\tau-1} + \Gamma^{\tau-2}(a_{\tau-2})) \in \tilde{X}^i,$
$a_{t'} = 0, \forall t' \neq \tau - 1, \tau - 2\}$

$\tilde{u}^{\tau i}(a) = \tilde{u}^i(a_t, a_{t+1}, a_{t+2})$

$\tilde{e}^{\tau i} = (\tilde{e}_{t'}^{\tau i})_{t'=1}^\tau$ such that $\tilde{e}_t^{\tau i} = \tilde{e}_t^i$, $\tilde{e}_{t+1}^{\tau i} = \tilde{e}_{t+1}^i$ and $\tilde{e}_{t'}^{\tau i} = 0$ if $t' \neq t, t+1$.

We note that the standard survival assumption is not satisfied because the initial endowment $\tilde{e}^{\tau i}$ may not belong to the consumption set $\tilde{X}^{\tau i}$. Indeed, for $i \in \mathcal{I}_t$, $(\tilde{e}_t^i, \tilde{e}_{t+1}^i, \tilde{e}_{t+2}^i)$ may not be in $\tilde{X}^i$ since $\tilde{e}_{t+2}^i = 0 \neq -\Gamma^{t+1}(\tilde{e}_{t+1}^i + \Gamma^t(\tilde{e}_t^i))$ if $\Gamma^{t+1} \neq 0$. So in order to overcome this difficulty, we work with a free-disposal equilibrium by introducing the free-disposal cone $Y := -(\mathbb{R}_+^L)^\tau$. Then $\tilde{e}^{\tau i} \in \tilde{X}^{\tau i} - Y$. See [28] and [29] for the existence of free-disposal equilibrium in a pure exchange economy.

Now, we introduce a weak notion of equilibrium, called pseudo-equilibrium, in which we do not require the market clearing condition at periods $\tau - 1$ and τ. Indeed, the truncation of an equilibrium is not an equilibrium but a pseudo-equilibrium. (See Lemma 3.1 below).

Definition 3.4.1. A pseudo-equilibrium in the truncated economy $\tilde{\mathcal{E}}_\tau$ is an element $(p^*, (a^{i*}) \in (\mathbb{R}_+^L)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$ such that:

a) for all $t = 1, 2 \ldots \tau - 2$, for all $i \in \mathcal{I}_t$, a^{i*} is a maximal element of $\tilde{u}^{\tau i}$ in the budget set

$$\{a^i \in \tilde{X}^{\tau i} \mid p^* \cdot a_i \leq p^* \cdot \tilde{e}^{\tau i}\};$$

for all $i \in \mathcal{I}_0$, a^{i*} is a maximal element of $\tilde{u}^{\tau i}$ in the budget set $\{a^i \in \tilde{X}^{\tau i} \mid p^* \cdot a^i \leq p^* \cdot \tilde{e}^{\tau i}\}$;

b) For all $t = 1, \ldots, \tau - 2$,

$$\sum_{i \in \mathcal{I}_0^{\tau-2}} a_t^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}_t^{\tau i},$$

$$\sum_{i \in \mathcal{I}_0^{\tau-2}} a_{\tau-1}^{i*} \leq \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}_{\tau-1}^{\tau i} + \sum_{i \in \mathcal{I}_{\tau-1}} \tilde{e}_{\tau-1}^{i}.$$

According to this definition, at period $\tau - 1$, we artificially increase the initial endowments by adding those of the consumers of the generation $\tau - 1$.

Lemma 3.1. *If $\bar{\tau} > \tau$ and $(\bar{p}^*, (\bar{a}^{i*}))$ is a pseudo-equilibrium in the economy $\mathcal{E}_{\bar{\tau}}$, then the price and the allocations restricted to the $\tau - 1$ first periods $\left(\hat{p}^*, (\hat{a}^{i*})_{i \in \mathcal{I}_0^{\tau-2}}\right)$ defined by*

$\hat{p}^* = (\bar{p}_t^*)_{t=1}^{\tau-1}$,

$\hat{a}^{i*} = (\bar{a}_t^{i*})_{t=1}^{\tau-1}$, *for all $i \in \mathcal{I}_0^{\tau-2}$, is a pseudo-equilibrium in the economy $\mathcal{E}_\tau$.*

In the following, we will establish the existence of a pseudo-equilibrium in $\tilde{\mathcal{E}}_\tau$. For that, we use the fact that an equilibrium with free-disposal is a pseudo-equilibrium. But since $\tilde{e}^{\tau i} \in \tilde{X}^{\tau i} - Y$, the problem of non-interiority of the initial endowments leads us to first make use of the notion of quasi-equilibrium with free-disposal as an intermediate step.

Definition 3.4.2. A quasi-equilibrium with free-disposal in $\tilde{\mathcal{E}}_\tau$ is a list $(p^*, (a^{i*}), y^*)$ in $(\mathbb{R}_+^L)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i} \times Y$ such that:
a') for all $t = 1, 2 \ldots \tau - 1$, a^{i*} is an element of the budget set:

$$\{a^i \in \tilde{X}^{\tau i} \mid p^* \cdot a_i \leq p^* \cdot \tilde{e}^{\tau i}\}$$

and for all $a^i \in \tilde{X}^{\tau i}$ such that: $p^* \cdot a_i < p^* \cdot \tilde{e}^{\tau i}$, $\tilde{u}^{\tau i}(a^i) \leq \tilde{u}^{\tau i}(a^{i*})$, for all $i \in \mathcal{I}_0$, $a^{i*} \in \{a^i \in \tilde{X}^i \mid p^* \cdot a^i \leq p^* \cdot \tilde{e}^{\tau i}\}$ and for all $a^i \in \tilde{X}^{\tau i}$ such that $p^* \cdot a^i < p^* \cdot \tilde{e}^{\tau i}$, $\tilde{u}^{\tau i}(x^i) \leq \tilde{u}^{\tau i}(x^{i*})$,

b) $p^* \cdot y \leq p^* \cdot y^* = 0$ for all $y \in Y$

c) $\sum_{i \in \mathcal{I}_0^{\tau-2}} a^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}^{\tau i} + y^*$

d) $p^* \neq 0$.

Proposition 3.4.1. For all $\tau \geq 3$, $\tilde{\mathcal{E}}_\tau$ has a quasi-equilibrium with free-disposal $(p^*, (a^{i*}), (y*))$ in $(\mathbb{R}_+^L)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i} \times Y$.

Proof

Indeed, $\tilde{\mathcal{E}}_\tau$ satisfies all the necessary conditions of existence of quasi-equilibrium in an exchange economy where free-disposal activities are possible. [See Florenzano in [29], Proposition 2.2.2]

- $\tilde{e}^{\tau i} \in \tilde{X}^{\tau i} - Y$ and $\sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}^{\tau i} \in int\left(\sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{X}^{\tau i} - Y\right)$
- for all i, $\tilde{u}^{\tau i}$ satisfies the classical conditions of continuity, quasi-concavity and local non-satiation,
- the set of feasible allocations $\mathcal{A}(\tilde{\mathcal{E}}_\tau)$ is a subset of a compact set of $(\mathbb{R}^L)^\tau$. □

Actually, one way to go from a quasi-equilibrium to an equilibrium is the notion of McKenzie-Debreu irreducibility. But following Assumption C and Assumption D made on the original economy, we establish that the truncated economy $\tilde{\mathcal{E}}_\tau$ is McKenzie-Debreu irreducible.

Proposition 3.4.2. The truncated economy $\tilde{\mathcal{E}}_\tau$, equiped with the disposal activity Y is McKenzie-Debreu irreducible, that is for all non-empty disjoint subsets J_1, J_2 of $\mathcal{I}_0^{\tau-2}$, $J_1, J_2 \neq \mathcal{I}_0^{\tau-2}$, $\mathcal{I}_0^{\tau-2} = J_1 \sqcup J_2$,and for all feasible allocation $(a^i) \in \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$, there exists an allocation $(a'^i) \in \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$ such that:

1- $\tilde{u}^{\tau i}(a'^i) \geq \tilde{u}^{\tau i}(a^i)$ for all $i \in J_1$ and $\exists j \in J_1, \tilde{u}^{\tau j}(a'^j) > \tilde{u}^{\tau j}(a^j)$,

2- $\sum_{i \in \mathcal{I}_0^{\tau-2}}(a'^i - \tilde{e}^{\tau i}) - \sum_{i \in J_2}(\tilde{e}^{\tau i} - a^i) \in Y$

Taking into account the feasibility of the allocation (a^i), condition *2* can also be written as: $\sum_{i \in \mathcal{I}_0^{\tau-2}} a'^i - \sum_{i \in J_1} a^i + \sum_{i \in J_2} \tilde{e}^{\tau i} \in Y$.

The irreducibility condition says that whenever the individuals are allocated into two nonempty and disjoint groups J_1 and J_2, then for any feasible allocation (a^i) and after disposing of any eventual surplus, $\sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}^{\tau i} + \sum_{i \in J_2}(\tilde{e}^{\tau i} - x^i)$ can be allocated to group J_1 improving the situation of its members as given by Relation 1.

Proof.

First case, there exists t such that $\mathcal{I}_t \cap J_1 \neq \emptyset$, and $\mathcal{I}_t \cap J_2 \neq \emptyset$. So let i_1 and i_2 be in $\mathcal{I}_t$ such that $i_1 \neq i_2$ and $i_1 \in J_1$, $i_2 \in J_2$. Since $\tilde{e}_t^{\tau i}$ and $\tilde{e}_{t+1}^{\tau i}$ are positive for $i = i_1, i_2$, each one is able to provide some good for which the other one is willing to exchange with some good of its own thanks to Assumption C. For instance, take $a'^{i_1} = a^{i_1} + \epsilon$ where $\epsilon > 0$ is arbitrarely small, $a'^{i_2} = \tilde{e}^{\tau i_2} - \epsilon \gg 0$ for ϵ small enough, $a'^i = a^i$, for $i \in J_1, i \neq i_1$, and $a'^i = \tilde{e}^{\tau i}$, for $i \in J_2, i \neq i_2$. Clearly, a'^i satisfies Relations 1 and 2: the situation of one group will be moved to a preferred position, by adding a feasible trade from the other group.

Suppose now that there does not exist t such that $\mathcal{I}_t \cap J_1 \neq \emptyset$ and $\mathcal{I}_t \cap J_2 \neq \emptyset$; let us define: $\bar{t}_1 := max\{t \mid \mathcal{I}_t \subset J_1\}$, $\bar{t}_2 := max\{t \mid \mathcal{I}_t \subset J_2\}$.
Note that the sets $\{t \mid \mathcal{I}_t \subset J_1\}$ and $\{t \mid \mathcal{I}_t \subset J_2\}$ are disjoint and thei union is $\{1, 2, \dots \tau - 2\}$.

If $\bar{t}_1 \neq \tau - 2$, then $\mathcal{I}_{\bar{t}_1} \subset J_1$ and $\mathcal{I}_{\bar{t}_1+1} \subset J_2$. If $\bar{t}_1 = \tau - 2$, then since $\bar{t}_2 \neq \tau - 2$, $\mathcal{I}_{\bar{t}_2} \subset J_2$ and $\mathcal{I}_{\bar{t}_2+1} \subset J_1$. Since the two sub-cases can be treated similarly, we deal only with the first one, in which $\mathcal{I}_{\bar{t}_1} \subset J_1$ and $\mathcal{I}_{\bar{t}_1+1} \subset J_2$. In this sub-case, since $\tilde{e}_{\bar{t}_1}^{\tau i}, \tilde{e}_{\bar{t}_1+1}^{\tau i}$ are positive for $i \in \mathcal{I}_{\bar{t}_1}$, as well as $\tilde{e}_{\bar{t}_1+1}^{\tau i}$ for $i \in \mathcal{I}_{\bar{t}_1+1}$, both generations are able to provide some commodity during their common period of life. In particular, consider the individual $i_0(\bar{t}_1)$ mentionned in Assumption C, then a young individual i of generation $\bar{t}_1 + 1$ can provide some goods to $i_0(\bar{t}_1)$, improving the utility of $i_0(\bar{t}_1)$ when he is old at date $\bar{t}_1 + 1$. If we keep the allocations of the other individuals of J_1 unchanged, and let the other members of J_2 consume their initial endowments, Relations 1 and 2 are satisfied. □

Thanks to Assumptions C and D on the original economy, the McKenzie-Debreu irreducibility of the truncated economy $\tilde{\mathcal{E}}_\tau$ and the interiority of the total initial endowment, we get that a quasi-equilibrium with free-disposal $(p^*, (a^{i*})) \in \prod_{t=1}^{\tau} \mathbb{R}_+^L \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$ is an equlibrium with free-disposal. (See Florenzano [29], Proposition 2.3.2 and Corollary 2.3.2)

Remark 3.4.1. The strict monotonicity of the utility function u^{i_0} of an individual i_0 in $\mathcal{I}_t$ in Assumption C implies that $p_t^* \gg 0$ for all $t = 1, 2, \dots \tau - 1$, thus $y_t^* = 0$ for all $t = 1, 2, \dots \tau - 1$.

Thus, since the equilibrium is realized without disposal of surplus, we get that the equilibrium with free-disposal is actually an equilibrium so a pseudo-equilibrium. Hence, one finally obtains:

Proposition 3.4.3. For all $\tau \geq 3$, there exists a pseudo-equilibrium of the economy $\tilde{\mathcal{E}}_\tau$.

The following lemma gives properties of the pseudo-equilibrium. We normalize a non zero equilibrium price p^* so that $\sum_{t=1}^{\tau} \sum_{\ell \in \mathcal{L}} p^*_{t\ell} = 1$.

Lemma 3.2. *If $(p^*, (a^{i*})) \in (\mathbb{R}^L_+)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau -}} X^{\tau i}$ is a pseudo-equilibrium, then $p^*_t \gg 0$, for all t.*
Furthermore, the set of pseudo-equilibria of the economy $\tilde{\mathcal{E}}_\tau$ with a normalized price is closed.

Proof: See Appendix A.

The last step of the existence of equilibrium in the reduced economy consists of considering a sequence of pseudo-equilibria in the truncated economy with an horizon increasing to infinity. We follow [13], and establish that the sequence of equilibrium prices in the truncated economies remains in a compact set for the product topology on $\prod_{t=1}^{\infty} \mathbb{R}^L$.

From the previous section, for all $T \geq 2$, there exists a pseudo-equilibrium $(p^T, (a^{iT}))$ of the economy $\tilde{\mathcal{E}}_T$. Since we have previously proved that $p^T_1 \neq 0$, we normalize p^T so that $\sum_{\ell \in \mathcal{L}} p^T_{1\ell} = 1$.

We extend the price and the allocations to the whole space $\prod_{t=1}^{\infty} \mathbb{R}^L$ by adding zeros for the missing components without modifying the notations. So, now the sequences (p^T), (a^{iT}) are in $\prod_{t=1}^{\infty} \mathbb{R}^L$.

Lemma 3.3. *For all t, there exists $\tilde{k}_t \in \mathbb{R}_+$ such that for all T, $0 \leq p^T_t \leq \tilde{k}_t \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^L$.*

Proof. See Appendix A.

Now we show that the sequence of pseudo-equilibria remains in a compact set and we prove that a cluster point is an equilibrium of the OLG economy $\mathcal{E}$.

From the compactness of $\mathcal{A}(\tilde{\mathcal{E}})$ and the above lemma, the sequence of T-equilibrium of the economy $\mathcal{E}_T$, $(p^T, (a^{iT}))$, remains in a compact set for the product topology of $\prod_{t=1}^{\infty} \mathbb{R}^L \times \prod_{t'=1}^{\infty} \prod_{i \in \mathcal{I}_{t'}} \prod_{t=1}^{\infty} \mathbb{R}^L$. Since this is a countable product of finite dimensional spaces, the product topology is metrizable on the compact sets and there exists a subsequence $(p^{T^\nu}, (a^{iT^\nu}))$ of $(p^T, (a^{iT}))$, which converges to $(p^*, (a^{i*}))$. We recall that the convergence for the product topology implies the usual convergence when we consider only a finite number of components.

For each $\tau \geq 3$, for ν large enough, the restriction of $(p^{T^\nu}, (a^{iT^\nu}))$ to the τ first periods is a pseudo-equilibrium of $\tilde{\mathcal{E}}_\tau$ (see Lemma 3.1) and it converges to the restriction of $(p^*, (a^{i*}))$

to the τ first periods. From Lemma 3.2, this restriction is a pseudo-equilibrium of $\tilde{\mathcal{E}}_\tau$. From Definition 3.4.1 and the notations above, one deduces that $(p^*, (\alpha^{i*}))$ defined as follows is an equilibrium for the OLG economy $\tilde{\mathcal{E}}$:

$\alpha^{i*} = (x_t^{i*}, \xi_{t+1}^{i*}, \zeta_{t+1}^{i*})$, for all $t \geq 1$ and for all $i \in \mathcal{I}_t$,

$\alpha^{i*} = (\xi_1^{i*}, \zeta_2^{i*})$, for all $i \in \mathcal{I}_0$.

□

3.5 Concluding Remarks, Discussions and Possible extensions

To summarize, we have established an existence result that relies on another important result such as the fact that a pure exchange economy with durable goods is equivalent to a standard pure exchange economy "without" durable goods. This equivalent economy helps to confirm that a consumption of durable goods by young agents has an impact, both on their consumption when old, but also after their lifetime, this explains the extension from a two-period lifetime into three-period one. This equivalent economy satisfies classical assumptions of a standard pure exchange economy, except the survival assumption. In order to recover this difficulty, we introduce an artificial free-disposal cone, a classical technical solution, provided by Florenzano in [28–30]. The ultimate step from an equilibrium with free disposal into an equilibrium is ensured by the desirablility of durable goods, which confirms why this assumption is primordial in our model.

However, above the existence results of equilibria, this paper provides a mechanism within which generations successively transfer their wealth, by means of durable goods, and shows that their consumptions, thanks to their durability, entail their owners a role of producer. It is clear that the consumption behaviour has impacts not only on their close future or their own generation but also on the succeeding generations. The irreducibility property that links generations to each other appears then natural to our economy, and this feature was important to our existence result. In the following, we will see more in detail the role of the durable goods in our model.

Durable goods act like consumption goods and inputs for future dates, so they not only enter the budget constraints but are involved in the utility function of each agent as well, contrary to many existing literatures. Many other models consider durables not as consumption goods but as goods which bring services, and it is theses services that are involved in the utility of agents. In our model, according to the relations on prices, durable goods can be "consumed" and differ from perishable goods in the sense that their prices contain an additionnal component which measures their ability to be used as inputs or any other investment goods for the future. Therefore, any agent who holds a durable good acts like a consumer and a producer at the same time, in this line, we can think of the linear mapping Γ^t as a production technology to which each consumer can have access in order to store or transfer the good accross successive periods. This production takes time: from a consumption at period t is produced a good, available at date $t+1$ that will add to the existing allocations.

Financial markets and Arbitrage-free condition

It is important that durable goods must be desired, by at least one individual of each generation, and no wastes are allowed. Since the future is assumed perfectly antipated, old agents will never purchase any durable goods unless they are sure that they can sell them again on the futures market. Whenever a durable good still remain at the end of the life of its owner, it will be traded on the futures market. At equilibrium, under the no-wastes condition and given the perfect foresight, consumptions of old agents may include durable goods, which will bring an additionnal financial resource thanks to the futures market, this relates the notion of viager if we associate the durable good to housing: agents can enjoy of their house during their old ages while receiving the future market value of that house. The budget initial constraint of agent $i \in \mathcal{I}_t$ illustrates it by showing the consumption x^i_{t+1} at both the expenditure and the income sides.

As mentionned before, the trade of deliverable goods in the future can be associated to borrowings and lendings, that are balanced at each date according to the market clearing condition. More precisely, at each date t, consider the individual $i \in \mathcal{I}_t$ and denote by $l^i_t := \Pi_t \cdot f^i_t$ the amount that he spends on the futures market when young. Note that l^i_t can be seen as saving when young that may be borrowed by old of previous generation, and will bring $(1+r_t)l^i_t$ at date $t+1$, where r_t is the rate of return. Note also that agent i is allowed to borrow b^i_{t+1} when old from the young of next generation but up to $\bar{b}^i_{t+1} := \Pi_{t+1} \cdot x^i_{t+1}$, that is, the durable good he holds when old, x^i_{t+1} will serve as collateral to his lending. Then, the budget constraint of agent i writes as:

$$p_t \cdot x^i_t + p_{t+1} \cdot x^i_{t+1} + l^i_t \leq p_t \cdot e^i_t + p_{t+1} \cdot e^i_{t+1} + p_{t+1} \cdot \Gamma^t(x^i_t) + (1+r_t)l^i_t + b^i_{t+1}$$

At equilibrium, thanks to the arbitrage-free condition $(1+r_t)l^i = (1+r_t)\Pi_t \cdot f^i_t = \gamma^t(p_{t+1}) \cdot f^i_t$, this budget constraint is the same as the initial one introduced previously in our model. In addition, the no-wastes condition: $\sum_{i\in\mathcal{I}_t} f^i_t = \sum_{i\in\mathcal{I}t-1} x^i_t$ implies that $\sum_{i\in\mathcal{I}_t} b^i_t = \sum_{i\in\mathcal{I}_t} \bar{b}^i_t = \sum_{i\in\mathcal{I}_{t-1}} l^i_t$, this confirms that all borrowings are paid back, all collaterals are seized.

Moreover, from the arbitrage-free condition $((1+r_t)\Pi_t - \gamma^t(p_{t+1}))\cdot f_t^i = 0$, we know that if $r_t \leq 0$, then $\Pi_t \geq \gamma^t(p_{t+1})$, a necessary condition at equilibrium as remarked in the proof of Proposition 3.2.1. But the case where $r_t < 0$ would lead to a lack of incentive to save or lend to the old, leaving the remaining durable goods valueless. By considering $r_t = 0$, we have the so called reduced equilibrium where $\Pi_t = \gamma^t(p_{t+1})$, which confirms that at equilibrium the (f^i)'s will be completely indeterminate. As a remark on the shares on firms θ^{ij}'s in the previous chapter, this indetermination is again an immediate consequence of the non-arbitrage condition. Here in our framework, it stipulates that young agents are indifferent between buying today on the futures market or waiting the next period to buy on the spot market. The only condition on the f^i's is the market clearing condition that prevents wastes and defaults in case of lending.

One important fact from our model is its structure which allows the agents to buy or hold an asset at the end of their lifetime, for instance, many agents buy a house at their old age. But the condition of no-arbitrage resulting in an indetermination of the f^i's limits our result, a priori, the indifference of agents between trading with the old in futures market or waiting to buy in the spot market cannot ensure that the transfer actually happens, thus the f^i can eventually be zero. Another way to recover this limitation is the altruism, which will ensure that agents have incentive to hold durable goods and transfer them to the next generation, even if it does not bring them any financial compensation.

Bequest motives:

It might happen that old agents leave freely their remaining durable goods to the next generation as a bequest. In such a situation, supposing altruism implies that their utility may be affected by the utility of their heir who benefits from the bequests. Indeed, we then have a particular case of Seghir and Martinez [56], where we first suppose that lifetimes are not uncertain. Our model can be adapted in the following way. For each t and each $i \in \mathcal{I}_t$, the bequest he intendes to leave to the next generation is denoted by $B^i = \sum_{k\in\mathcal{I}_{t+1}} B^i_k$. Then we introduce a function $V^i : \mathbb{R}^L_+ \to \mathbb{R}_+$ as the utility that agent $i \in \mathcal{I}_t$ gets from his bequest motives. This bequest consists of some part of the remaining durable goods x^i_{t+1} that the old agents decide to put on their will at date $t+1$ so that some individuals of generation $t+1$ will have the rights over them. The other part, ξ^i_{t+1} will be traded on the future market at price Π_{t+1} as described in our original model. Note that whether it concerns trade on futures market or bequest, the delivery will take place at date $t+2$ when the agent disappears from the economy. We do not require that the agents know from whom they benefit of the transfer, so they receive it in an anonymous way, and we call T^i the total nominal transfer that agent i receives. This transfer will consist of the bequest they inherit from the previous generation $t-1$, registered at date t, delivered at date $t+1$ thus valued at the price p_{t+1}. Thus, on one side, old agents leave bequests to the next generation and on the other side, they also receive some transfer from the old of the previous generation.

The budget constraint of agent $i \in \mathcal{I}_t$ becomes:

$$p_t \cdot x_t^i + p_{t+1} \cdot x_{t+1}^i + \Pi_t f_t^i \leq p_t \cdot e_t^i + p_{t+1} \cdot e_{t+1}^i + p_{t+1} \cdot \Gamma^t(x_t^i) + \Pi_{t+1} \cdot \xi_{t+1}^i + T^i$$

Thus, if we consider again the same model with the borrowing and lending system, the collateral to back up his borrowing will consist of the remaining durable goods net of the bequest, or equivalently, the bequest cannnot exceed the wealth of agent i net of his debt from borrowing. More precisely, we have that $x_{t+1}^i = \xi_{t+1}^i + B^i$, and $\sum_{i \in \mathcal{I}_t} \xi_{t+1}^i = \sum_{i \in \mathcal{I}_{t+1}} f_{t+1}^i$.

At each period t, we can write the no-wastes condition as:

$$\sum_{i \in \mathcal{I}_{t-1}} x_t^i = \sum_{i \in \mathcal{I}_t} f_t^i + \sum_{i \in \mathcal{I}_{t-1}} B^i,$$

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^i = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(x_{t-1}^i) + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(f_{t-1}^i) + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(B^i), \text{ for } t > 1,$$

and $\sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x_1^i = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} e_1^i$.

Finally, an equilibrium would then consist of $(p, (T^i), (x^i), (f^i), (B^i))$ such that:

a) (x^i, f^i, B^i) maximizes the utility function $U^i(x^i, B^i) := u^i(x^i) + V^i(B^i)$ satisfying the budget constraint above;

b) There are no wastes and the market clears at each date t.

c) $T^i = \sum_{k \in \mathcal{I}_{t-1}} p_{t+1} \cdot \Gamma^t(B_i^k)$.

Here, agents are supposed to receive their bequest only after they reimburse their debt. The difference with our initial model is that, old agents now have a lower level of collateral to access to a lending, but he can enjoy of an additionnal wealth through the anonymous transfer from the previous generation, which is not necessarely as high as the value of his whole remaining durable if he would have traded it on the future market. But here, introducing the bequest motive enhances the possibility of transfer accross generations, where the incentive is not necessarely related to the financial issue but to the utility of each agent that is increasing with the bequest he concedes. The concerns of the agents on the next generations well-being can explain then why they still hold assets or buy durable goods at the end of their lifetime.

In our framework, conditions of no-arbitrage on prices at equilibrium are required inducing the indetermination of the f^is and possibility of a null rate of return. Although this transfer mechanism appears to be financially neutral or non-motivating, this is not the case for he economy since it helps to improve the situation of old agents and of all the successive generations. This can explain the role of institutions that make sure that some allocations will be always left to the old and to the future generations to promote their welfare.

Durable goods and uncertainty

Introducing uncertainty to our model is relevant extension. For instance, associating durable goods to collateral loans in the literature supposes that agents are exposed to default risk by lending, since the borrowers, for some reason may fail to pay them back. The existence of durable goods eases the extension of our model to a framework where agents finitely live at least for two consecutive periods, and have a robability to die in the next period, at each period. Indeed, whenever an agent disappears from the economy, then either he has paid his debt back or the lender seizes his durable good that serves as collateral at its current price after incuring the transformation Γ^t. Such a model already exists, we can for instance follow again Seghir and Martinez [56] and adapt it to our OLG model.

We consider the same discrete and infinitely many dates $t = 0, 1, \ldots$. The demographic structure consists of a finite group of agents born at each date t, $\mathcal{I}_t$, and lives for at least two dates, ensuring individuals of different lifetimes to overlap. Consider $i \in \mathcal{I}_t$. He is initially endowed of some allocations $(e^i_{t+k})_{k\geq 0}$ and consumes $(x^i_{t+k})_{k\geq 0}$ for k finite. $e^i_{t+k} = 0$ if agent i lives from t to $t+k-1$ where $k \geq 2$. His utility function is accordingly extended. Each consumption x^i_{t+k} will bring $\Gamma^{t+k}(x^i_{t+k})$ at date $t+k+1$ and can be proposed to be sold on the future market or proposed as collateral for a loan contracted at any $t+k$. All agents alive at this period can take part to this trade, and agent i can also take part to the future contracts issued by his contemporanous.

Call $\mathcal{G}_s$ the set of all agents alive at date s, then, the market clearing condtition at date s is given by:

$$\sum_{i\in\mathcal{G}_s} x^i_s = \sum_{i\in\mathcal{G}_s} f^i_s, \text{ for } s \geq 1,$$

$$\sum_{i\in\mathcal{G}_s} x^i_s = \sum_{i\in\mathcal{G}_s} e^i_s + \sum_{i\in\mathcal{G}_{s-1}} \Gamma^{s-1}(x^i_{s-1}) + \sum_{i\in\mathcal{G}_{s-1}} \Gamma^{s-1}(f^i_{s-1}), \text{ for } s > 1,$$

Consequently, the uncertainty on the duration of lifetime enables agents to trade with all their contemporanous, at any period they are alive, but at each date, all of the borrowing and lending activities will be balanced so that there is no room for default even when a borrower accidently disappears from the economy.

Durable goods, liquid and illiquid wealth

Although our focus is different, it is worth pointing out that the absence of risk or uncertainty limits the roles of durable goods and ignores some issues related to them. This can be seen in the literature that considers durable goods as consumptions and assets, and treats for instance wealth distribution, illiquidity, income risks and precautionary savings. It is not specified in our model, but in the literature, the illiquidity level of durable goods is of significant importance in determining the saving, and can explain also the consumption or expenditure of agents. The illiquidity supposes that, if agents are facing financial issues, in order to sell their durable goods or borrow againt them, they may incur losses. This may then interfere in the demand of durable goods and limit their purchase, as confirmed by Mishkin [47]. This issue is clearly ignored in our model, even if we also link the existence of durable good to savings and to collateral for loans. Indeed, we consider durable goods desirable, thus agents are willing to buy or hold them, their transactions are easy and costless and their future prices are perfectly forseen.

There are additionnal reasons why households are willing to possess an illiquid store of value such as a durable good, when there is uncertainty and the economical environment is risky. If we then go beyond these restrictions of desirability and certainty, an answer is provided by Diaz and Prado [25] who try to clarify the composition of wealth in the USA, where poorer households hold more illiquid wealth than the richer ones. For that, they consider two means of savings: financial assets considered liquid and durable goods, considered illiquid and which main roles are then services providers and collateral for loans. They establish that "*durable goods represent an important fraction of household wealth*", and confirm the fact that "*portfolio becomes more liquid as wealth increases*".

Our model, does not explicit these two types of assets, liquid and illiquid, and does not allow to determine the composition of the wealth of agents. As a crucial hypothesis to our model, the desirability of durable goods ensures the purchase of durable goods both in spot and furures markets, without raising their level of liquidity. But such a feature may be important and correspond to the notion of equity while introducing financial markets in a productive economies. In the case where positive profits are possible and dividends are paid, agents have incentive to buy or hold shares, which liquidity eases the trade. Considering this framework meets one of the mains perspectives to our thesis. Whenever increasing returns are allowed, positive profits can be expected, which needs to raise the notions of distribution among consumers, fundamental values of a firm that now consist not only of its remaining capital but take also into account the future profits it may generate.

The next chapter treats optimality issues, which was motivated by the role that plays durable goods to restore Pareto optimality in OLG models. Indeed, the relations on prices at equilibrium partly reflect the characterization provided in Balasko and Shell [4]. However, our existence result relies on the equivalent economy where consumption sets are not the positive orthant. This calls for the need to establish a proof compatible with such a non conventionnal framework.

Chapter 4

Optimal allocations in OLG models with multiple goods

Abstract

We consider a pure exchange overlapping generations economy with finitely many commodities and consumers per period having possibly non-complete non transitive preferences. We provide a geometric and direct proof of the Balasko-Shell characterization of Pareto optimal allocation (See, [4]). As a by-product, we compute an explicit Pareto improving transfer when the criterion is not satisfied, which is minimal for some suitable distance[1].

JEL classification: C62, D50, D62.

Keywords: Overlapping generations model, preference set, normal cone, equilibrium, Pareto optimality.

[1]This chapter is based on the paper "*Notes on the Characterization of optimal allocations in OLG models with multiple goods*" [14]

4.1 Introduction

As already well known, OLG equilibrium allocations may lack in being Pareto optimal. This market failure was established by Samuelson [54], who attributes this phenomenon as a lack of double coincidence of wants, and suggests to solve this by using the role of money. Geanakoplos [34] clarifies this phenomenon as a result of two facts: generations overlap, and infinite horizon. He points some links between the Samuelson model and the Arrow-Debreu model, and established how a durable good such as money, or an infinitely lived asset like a land, could restore the market failure.

Balasko and Shell [4] provide a criterion based on the asymptotic behavior of the norm of the prices to characterize Pareto optimal allocation without durable good or infinitely lived asset. Burke [17] revisits this criterion by focusing in particular on the right definition of the Gaussian curvature of the indifference surface. Actually, these authors provide a proof with a first step considering the special case of a single commodity per period. Then, the generalization to several commodities is only sketched.

Our purpose in this paper is three fold: to provide a simpler, direct proof of the Balasko-Shell Criterion considering in one step several consumers for each generation and several commodities; to encompass the case of non-complete, non-transitive preferences; to compute explicitly a Pareto improving transfer when the allocation does not satisfy the Balasko-Shell Criterion. Nevertheless, note that the structure of the proof is based strongly on Balasko-Shell's one.

It is important to remark that a geometrical approach has already been provided by Borglin and Keiding [16]. They actually consider infinite horizons economies, and treats the particular case of OLG models. They center the notion of Pareto optimality to its weak form, and consider characterizations based on parameters that describe the economy such as supporting prices and curvatures of indifference surfaces, thus an approach that easily meets our model.

We consider this contribution as a first step to be able in future works to tackle the question in presence of durable commodities and with heterogeneous longevities of the agents.

Since, we have no more a representation of the preferences by utility function, we cannot use the standard differentiability assumption and the link between the curvature of the indifference surfaces and the second derivative of utility function. So we adopt a geometric approach to state the assumptions directly on the preferred sets. The smoothness assumption is obtained by assuming that the normal cone is an half line. For the upper bound of the curvature, we use the notion of prox-regularity of the complementary of the preferred set, introduced in variational analysis by Rockafellar and Poliquin in [49], which is extensively studied by Thibault and Colombo in [21]. For the lower bound, which is related to the strict convexity of the preferred sets, we assume that the truncated preferred sets are included in a suitable ball with a large enough radius.

The simplification of the proof comes from the fact that we show that the parameter α^t used in Balasko-Shell work to describe the Pareto improving transfer has a nice geometric interpretation in terms of a radius of a sphere tangent to the boundary of the preferred set at the allocation. This remark allows us to use this radius as a particular distance function and to compute explicitly the minimal Pareto improving transfer when the criterion is not satisfied.

Details about the model and assumptions are described in Section 2. Some preliminary results are provided in Section 3. In particular, we characterize the weak Pareto optimal allocation in term of the existence of a supporting price using the normal cone to the preferred sets. The end of this Section explains how the multi-consumer case can be simplified by considering aggregate feasible Pareto improving transfer. In Section 4, we provide the proof and we show which assumptions are used for the if part and for the only if part of the criterion. We also provide two examples showing that the result is no more true if one of the conditions on the curvature does not hold true.

4.2 The model

We[2] consider an OLG economy $\mathcal{E}$ with infinitely many dates $t = 1, 2 \ldots$ At each date there is a finite set of commodities $\mathcal{L}_t$, and we denote by L_t its cardinal.

At each date $t \in \mathbb{N}$, a finite set of individuals $\mathcal{I}_t$ is born, living for two periods, young at date t and old at date $t+1$. We start with the first generation 0 which lives for only one period and consists of the old agents at date 1. $\mathcal{I} = \cup_{t=0}^{\infty} \mathcal{I}_t$ denotes the set of all individuals and $\mathcal{I}_{-0} = \cup_{t=1}^{\infty} \mathcal{I}_t$.

At each date $t \geq 1$, we denote by $x^i = (x^i_t, x^i_{t+1})$ the consumption by an individual i in $\mathcal{I}_t$, which is an element of the consumption set $X^i = \mathbb{R}^{L_t}_+ \times \mathbb{R}^{L_{t+1}}_+$. The consumption set of consumers of generation 0 is $X^i = \mathbb{R}^{L_1}_+$.

[2]**Notations.** We consider several finite dimensional Euclidean space $\mathbb{R}^L$. In each of them, $\mathbb{R}^L_+$ is the standard positive cone and $\mathbb{R}^L_{++}$ its interior. $x \leq y$ means that $y - x \in \mathbb{R}^L_+$, $x \cdot y$ denotes the standard inner product of the vectors x and y, $\|x\| = \sqrt{x \cdot x}$ denotes the standard Euclidean norm. If P is a subset of $\mathbb{R}^L$, $\bar{P}$ denotes its closure and int$P$, its interior. $B(x, r)$ (resp. $\bar{B}(x, r)$) denotes the open (resp. closed) ball of center x and radius r.

Consumers preferences are represented by a (strict) preference relation $P^i : X^i \to X^i$: for all $i \in \mathcal{I}$:

$\xi^i \in P^i(x^i)$ means that ξ^i is strictly preferred to x^i.

$\xi^i \in \bar{P}^i(x^i)$, the closure of $P^i(x^i)$, if and only if ξ^i is preferred or indifferent to x^i.

For all $t \geq 0$, $i \in \mathcal{I}_t$, we denote by $N_{\bar{P}^i(x^i)}(x^i)$ the normal cone[3] of $\bar{P}^i(x^i)$ at x^i.

We now posit the main assumption, which is maintained throughout the paper.

Assumption A.

a) For all individual i in $\mathcal{I}$, $P^i(x^i)$ is open in X^i, convex, $x^i \in \bar{P}^i(x^i) \setminus P^i(x^i)$. For all $i \in \mathcal{I}_{-0}$ $P^i(x^i) + (\mathbb{R}^{L_t}_+ \times \mathbb{R}^{L_{t+1}}_+) \subset P^i(x^i)$ and for $i \in \mathcal{I}_0$, $P^i(x^i) + \mathbb{R}^{L_1}_+ \subset P^i(x^i)$.

b) Each consumer i is endowed with some endowments e^i of the goods during his lifetime: for all $i \in \mathcal{I}$, $e^i \in X^i$ and for all $i \in \mathcal{I}_0$, $e^i \in \mathbb{R}^{L_1}_+$. For each period $t \geq 1$, e_t denotes the total endowments at this date, that is, $e_t = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e^i$.

c) For all $i \in \mathcal{I}$, for all x^i in the interior of X^i, $-N_{\bar{P}^i(x^i)}(x^i)$ is a half line $\{t\gamma^i(x^i) \mid t \geq 0\}$, defined by $\gamma^i(x^i)$ which is a continuous mapping on the interior of X^i satisfying $\|\gamma^i(x^i)\| = 1$. For all x_i in the interior of X^i, for all $i \in \mathcal{I}_{-0}$, , $\gamma^i(x^i) \in \mathbb{R}^{L_t}_{++} \times \mathbb{R}^{L_{t+1}}_{++}$ and for all $i \in \mathcal{I}_0$, $\gamma^i(x^i) \in \mathbb{R}^{L_1}_{++}$.

Assumption A is a classical assumption in a standard finite economy. If the preferences are represented by a utility function, it means that it is continuous, quasi-concave, strictly increasing and smooth on the interior of X^i.

At each date t, there is a spot market for the L commodities. The spot price vector p is an element of $\prod_{t=1}^{\infty} \mathbb{R}^{L_t}_{++}$ and $p_{t\ell}$ is the spot price of commodity ℓ at date t. We consider the set of normalized prices $\Delta := \{p \in \prod_{t=1}^{\infty} \mathbb{R}^{L_t}_{++} \mid \|p_1\| = 1\}$. We denote by $\Pi_t = (p_t, p_{t+1})$, for $t \geq 1$, and $\Pi_0 = p_1$.

Budget Constraints

The budget constraint at a given price p, for each agent $i \in \mathcal{I}_{-0}$ is given by:

$$\Pi_t \cdot x^i = p_t \cdot x^i_t + p_{t+1} \cdot x^i_{t+1} \leq \Pi_t \cdot e^i = p_t \cdot e^i_t + p_{t+1} \cdot e^i_{t+1},$$

[3] $N_{\bar{P}^i(x^i)}(x^i) = \{q \in \mathbb{R}^L \times \mathbb{R}^L \mid q \cdot (z - x^i) \leq 0, \forall z \in \bar{P}^i(x^i)\}$

and for each agent $i \in \mathcal{I}_0$,

$$\Pi_0 \cdot x^i = p_1 \cdot x_1^i \leq \Pi_0 \cdot e^i = p_1 \cdot e_1^i.$$

4.3 Preliminary results

Let us recall some basic definitions of equilibrium and optimal allocations in a standard pure exchange OLG economy with multiple commodities.

Definition 4.3.1. An equilibrium of the economy $\mathcal{E}$ is a list $(p^*, (x^{i*}))$ in $\Delta \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i$ such that:

a) for all $i \in \mathcal{I}$, (x^{i*}) is a maximal element for P^i in the budget set associated to the equilibrium price p^*, that is, for all $i \in \mathcal{I}$, $\Pi_t \cdot x^{i*} \leq \Pi_t \cdot e^i$, and for all $x^i \in P^i(x^{i*})$, $\Pi_t \cdot x^i > \Pi_t \cdot e^i$.

b) the allocation (x^{i*}) is feasible:

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^{i*} = e_t = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i, \text{ for } t \geq 1$$

Proposition 4.3.1. Under Assumption A, if the initial endowments are strictly positive, the OLG economy $\mathcal{E}$ has an equilibrium.

Proof. The existence of an equilibrium can be proved following the same procedure as in [3], using the existence result in finite dimension of Gale and Mas-Colell [31], [32], see also Florenzano [28]. [4]

Definition 4.3.2. The feasible allocation x in $\prod_{i \in \mathcal{I}} X^i$ is Pareto optimal (PO) (resp. weakly Pareto optimal (WPO)) if there is no (y^i) in $\prod_{i \in \mathcal{I}} X^i$ such that:

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} y_t^i = e_t, \text{ for } t \geq 1$$

and for all $i \in \mathcal{I}$, $y^i \in \bar{P}^i(x^i)$, with $y^i \in P^i(x^i)$ for at least one individual i (resp. and there exists $\underline{t} \geq 1$ such that for all $t \geq \underline{t}$, for all $i \in \mathcal{I}_t$, $y^i = x^i$).

We remark that a PO allocation is WPO.

Definition 4.3.3. Let $x = (x^i)$ be an allocation in $\prod_{i \in \mathcal{I}} X^i$. The price $p \in \Delta$ is said to support x if for each $t \in \mathbb{N}$, for all $i \in \mathcal{I}_t$ and for all $\xi^i \in P^i(x^i)$, $\Pi_t \cdot \xi^i > \Pi_t \cdot x^i$.

[4] Main steps of the proof are provided in Appendix B

Remark 4.3.1. Every competitive allocation $x^* = (x^{i*})$ associated with the equilibrium price $p^* \in \Delta$ is supported by p^*.

If $x = (x^i)$, an interior allocation, is supported by the price p, then for all $i \in \mathcal{I}$, there exists $\lambda^i > 0$ such that $\gamma^i(x^i) = \lambda^i \Pi_t$.

Lemma 4.1. *If $x = (x^i)$, an interior allocation, is WPO, then, for all $t \geq 0$, $\gamma^i(x^i) = \gamma^j(x^j)$ for all $i, j \in \mathcal{I}_t$.*

Proof. Let $t \geq 0$. Since (x^i) is WPO, $\sum_{i \in \mathcal{I}_t} x^i \notin \sum_{i \in \mathcal{I}_t} P^i(x^i)$. Indeed, if it would not hold, then there exists $(\xi^i)_{i \in \mathcal{I}_t} \in \prod_{i \in \mathcal{I}_t} P^i(x^i)$, such that $\sum_{i \in \mathcal{I}_t} x^i = \sum_{i \in \mathcal{I}_t} \xi^i$. Then, one easily checks that the allocation (y^i) defined by $y^i = x^i$ for $i \notin \mathcal{I}_t$ and $y^i = \xi^i$ for $i \in \mathcal{I}_t$ is feasible and Pareto dominates (x^i), which is in contradiction with the weak Pareto optimality of (x^i).

By Assumption A, $(P^i(x^i))_{i \in \mathcal{I}_t}$ are convex and nonempty, and for all i, $x^i \notin P^i(x^i)$ and $x^i \in \bar{P}^i(x^i)$. So, for each $i \in \mathcal{I}_t$, there exists a sequence $(\xi^{i\nu})$ of $P^i(x^i)$, which converges to x^i. The set $\sum_{i \in \mathcal{I}_t} P^i(x^i)$ being convex, so by using the standard separation theorem for convex sets in finite-dimensional space for $\sum_{i \in \mathcal{I}_t} x^i$ and $\sum_{i \in \mathcal{I}_t} P^i(x^i)$, there exists $q \neq 0$ such that $q \cdot \sum_{i \in \mathcal{I}_t} x^i \leq q \cdot \sum_{i \in \mathcal{I}_t} \xi^i$ for all $(\xi^i) \in \prod_{i \in \mathcal{I}_t} P^i(x^i)$. Consider an individual i_0 in $\mathcal{I}_t$. Then for all $\xi^{i_0} \in P^{i_0}(x^{i_0})$, $q \cdot x^{i_0} + q \cdot \sum_{i \in \mathcal{I}_t, i \neq i_0} x^i \leq q \cdot \xi^{i_0} + q \cdot \sum_{i \in \mathcal{I}_t, i \neq i_0} \xi^{i\nu}$. By taking the limit, we obtain: $q \cdot x^{i_0} \leq q \cdot \xi^{i_0}$, which means that q belongs to the cone $-N_{\bar{P}^{i_0}(x^{i_0})}(x^{i_0})$, thus $q = \|q\| \gamma^{i_0}(x^{i_0})$. By repeating the same reasoning for all individuals of $\mathcal{I}_t$, we establish that $\gamma^i(x^i) = \frac{1}{\|q\|} q$ for all $i \in \mathcal{I}_t$. □

Lemma 4.2. *The interior allocation $x = (x^i)$ is WPO if and only if there exists a price sequence $p \in \Delta$ which supports $x = (x^i)$.*

Proof. Let x be supported by a price sequence p. Assume that x is not WPO, then there exists a feasible allocation (y^i) and some $\underline{t} \geq 1$ such that $x^i = y^i$ for all $i \in \mathcal{I}_t$, $t \geq \underline{t}$, where $y^i \in \bar{P}^i(x^i)$ for all $i \in \mathcal{I}$ and $y^i \in P^i(x^i)$ for at least an individual. Therefore, for all t and for all $i \in \mathcal{I}_t$, $\Pi_t \cdot y^i \geq \Pi_t \cdot x^i$ with at least one strict inequality for $i_0 \in \mathcal{I}_{t_0}$ where $t_0 < \underline{t}$. Thus, we have:

$$
\begin{aligned}
&p_1 \cdot \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} y_1^i + p_2 \cdot \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} y_2^i + \ldots + p_{\underline{t}-1} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1} \cup \mathcal{I}_{\underline{t}-2}} y_{\underline{t}-1}^i + p_{\underline{t}} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1}} y_{\underline{t}}^i > \\
&p_1 \cdot \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x_1^i + p_2 \cdot \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} x_2^i + \ldots + p_{\underline{t}-1} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1} \cup \mathcal{I}_{\underline{t}-2}} x_{\underline{t}-1}^i + p_{\underline{t}} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1}} x_{\underline{t}}^i
\end{aligned}
$$

Since, $y^i_t = x^i_t$ for all $i \in \mathcal{I}_t$, $t \geq \underline{t}$, we get:

$$p_1 \cdot \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} y^i_1 + p_2 \cdot \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} y^i_2 + \ldots + p_{\underline{t}-1} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1} \cup \mathcal{I}_{\underline{t}-2}} y^i_{\underline{t}-1} + p_{\underline{t}} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1}} y^i_{\underline{t}} >$$
$$p_1 \cdot \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x^i_1 + p_2 \cdot \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} x^i_2 + \ldots + p_{\underline{t}-1} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1} \cup \mathcal{I}_{\underline{t}-2}} x^i_{\underline{t}-1} + p_{\underline{t}} \cdot \sum_{i \in \mathcal{I}_{\underline{t}-1} \cup \mathcal{I}_{\underline{t}}} x^i_{\underline{t}}$$

which is in contradiction with the feasibility of (y^i) implying $\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x^i_t = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} y^i_t = e_t$.

Conversely, let x be a WPO allocation. We first truncate the economy at a finite horizon t by considering the t first generations. Denote $\mathcal{J}_{t-1} = \prod_{\tau=0}^{t-1} \mathcal{I}_\tau$. We shall prove the result by induction on the truncation at t.

First, consider the truncated economy $\mathcal{E}_1$ at date $t = 1$, which consists of the generation 0, $\mathcal{I}_0$. From Lemma 4.1, $p_1 = \gamma^i(x^i)$ for any $i \in \mathcal{I}_0$ supports $(x^i)_{i \in \mathcal{I}_0}$.

Now, suppose that $(p_1, \ldots p_t)$ is supporting $(x^i)_{i \in \mathcal{J}_{t-1}}$, and let us prove that there is a unique $p_{t+1} \gg 0$ such that $(p_1, \ldots p_{t+1})$ supports $(x^i)_{i \in \mathcal{J}_t}$. From Lemma 4.1, for any $i_0 \in \mathcal{I}_t$, $\gamma^{i_0}(x^{i_0})$ supports x^i for all $i \in \mathcal{I}_t$. So, it suffices to prove that $\gamma^{i_0}_t(x^{i_0})$ is collinear to p_t and then to choose $p_{t+1} = \frac{\|p_t\|}{\|\gamma^{i_0}_t(x^{i_0})\|} \gamma^{i_0}_{t+1}(x^{i_0})$.

We consider a reduced economy with L_t commodities, the individuals in $\mathcal{I}_{t-1} \cup \mathcal{I}_t$ and the preferences defined by $Q^i(\xi^i) = P^i(x^i_{t-1}, \xi^i)$ for $i \in \mathcal{I}_{t-1}$ and $Q^i(\xi^i) = P^i(\xi^i, x^i_{t+1})$ for $i \in \mathcal{I}_t$. Since x is a WPO allocation, the allocation $(x^i_t)_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t}$ is Pareto optimal in this finite economy. We also remark that $-N_{Q^i(x^i_t)}(x^i_t) = \{\lambda \gamma^i_t(x^i_{t-1}, x^i_t) \mid \lambda \geq 0\}$ for $i \in \mathcal{I}_{t-1}$ and $-N_{Q^i(x^i_t)}(x^i_t) = \{\lambda \gamma^i_t(x^i_t, x^i_{t+1}) \mid \lambda \geq 0\}$ for $i \in \mathcal{I}_t$. So, using the same argument as in the proof of Lemma 4.1, we prove that the vectors $((\gamma^i_t(x^i_{t-1}, x^i_t))_{i \in \mathcal{I}_{t-1}}$ and $(\gamma^i_t(x^i_t, x^i_{t+1}))_{i \in \mathcal{I}_t})$ are colinear. Since (p_{t-1}, p_t) supports x^i for any $i \in \mathcal{I}_{t-1}$, p_t is colinear to $\gamma^i_t(x^i_{t-1}, x^i_t))$, so it is also colinear to $\gamma^i_t(x^i_t, x^i_{t+1})$ for all $i \in \mathcal{I}_t$. □

Definition of aggregate Pareto improving transfers

The proof of the main result studies the behavior of Pareto improving transfers that we now introduce. In the following, we distinguish transfers and aggregate transfers. For an allocation (x^i), for each generation t, we define an aggregate preferred set as follows:

$$\bar{P}_t((x^i)) := \sum_{i \in \mathcal{I}_t} \bar{P}^i(x^i)$$

From Assumption A, $\bar{P}_t((x^i))$ is a closed convex subset of $\mathbb{R}^{L_t}_+ \times \mathbb{R}^{L_{t+1}}_+$ (or $\mathbb{R}^{L_1}_+$ for the generation 0) and satisfies $\bar{P}_t((x^i)) + (\mathbb{R}^{L_t}_+ \times \mathbb{R}^{L_{t+1}}_+) \subset \bar{P}_t((x^i))$ (or $\bar{P}_t((x^i)) + \mathbb{R}^{L_1}_+ \subset$

$\bar{P}_t((x^i))$ for the generation 0). The closedness of $\bar{P}_t((x^i))$ is ensured by the fact that $\bar{P}^i(x^i)$ is a nonempty closed subset of $\mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$ for all $i \in \mathcal{I}_t$. Indeed, for all t and for all $i \in \mathcal{I}_t$, the asymptotic cones [5] $A\bar{P}^i(x^i) = \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$ are positively semi-independent [6], a sufficient condition for a finite sum of closed sets to be closed.

If $x = (x^i)$ is an interior allocation supported by the price $p \in \Delta$ and $x^t = \sum_{i \in \mathcal{I}_t} x^i$. Then one checks that for all $t \geq 0$, for all $i \in \mathcal{I}_t$,

$$N_{\bar{P}_t((x^i))}(\bar{x}^t) = \{\lambda \Pi_t \mid \lambda \geq 0\} = \{\lambda \gamma^i(x^i) \mid \lambda \geq 0\}$$

Definition 4.3.4. (a) For a given feasible allocation $x = (x^i)$, the sequence of commodity transfers $h = (h^i) \in \prod_{i \in \mathcal{I}_0} \mathbb{R}^{L_1} \times \prod_{t=1}^{\infty} \prod_{i \in \mathcal{I}_t} (\mathbb{R}^{L_t} \times \mathbb{R}^{L_{t+1}})$ is feasible if $(x^i + h^i)$ is feasible, which means that $(x^i + h^i)$ belongs to $\prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i$ and $\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} h_t^i = 0$.

(b) The sequence of commodity transfers $h = (h^i)$ is Pareto improving upon $x = (x^i)$ if h is feasible and $x + h$ Pareto dominates x, that is for all $t \geq 1$ and all $i \in \mathcal{I}_t$, $x^i + h^i \in \bar{P}^i(x^i)$, with $x^i + h^i \in P^i(x^i)$ for at least one agent i.

(c) An aggregate transfer $\bar{h} \in \mathbb{R}^{L_1} \times \prod_{t=1}^{\infty} \mathbb{R}^{L_t} \times \mathbb{R}^{L_{t+1}}$ is feasible if $\bar{h}_t^{t-1} = -\bar{h}_t^t$ for all $t \geq 1$ and Pareto improving upon the allocation $x = (x^i)$ if:

i) for all t, $x^t = \sum_{i \in \mathcal{I}_t} x^i + \bar{h}^t \in \bar{P}_t((x^i))$

ii) there exists $\underline{t}$ such that $x^{\underline{t}} = \sum_{i \in \mathcal{I}_{\underline{t}}} x^i + \bar{h}^{\underline{t}} \in \operatorname{int}(\bar{P}_{\underline{t}}((x^i)))$

By the very definition of Pareto optimality, the allocation $x = (x^i)$ is Pareto optimal if and only if there exists no feasible Pareto improving transfer upon x. But, we also remark that the interior allocation $x = (x^i)$ is Pareto optimal if and only if there exists no feasible aggregate Pareto improving transfer upon x. So, in the next section, we will be able to work only on aggregate feasible transfers and not on feasible transfers, which will greatly simplify the notations and the formulas.

If there exists $\bar{h}$, an agregate Pareto improving transfer, then let $t \geq 0$, by definition of $\bar{P}_t((x^i))$, there exists $(\xi^i)_{i \in \mathcal{I}_t}$ in $\prod_{i \in \mathcal{I}_t} \bar{P}^i(x^i)$ such that $\sum_{i \in \mathcal{I}_t} x^i + \bar{h}^t = \sum_{i \in \mathcal{I}_t} \xi^i$. By letting $h^i = \xi^i - x^i$, we easily check that $x^i + h^i \in \bar{P}^i(x^i)$ for all $i \in \mathcal{I}_t$.

Furthermore, for $\underline{t}$, $\sum_{i \in \mathcal{I}_{\underline{t}}} x^i + \bar{h}^{\underline{t}} - \alpha(\mathbf{1}_{L_{\underline{t}}}, \mathbf{1}_{L_{\underline{t}+1}}) \in \bar{P}^{\underline{t}}((x^i))$, for some $\lambda > 0$ small enough. Then there exists $(\xi^i) \in \prod_{i \in \mathcal{I}_{\underline{t}}} \bar{P}^i(x^i)$ such that $\sum_{i \in \mathcal{I}_{\underline{t}}} x^i + \bar{h}^{\underline{t}} - \lambda(\mathbf{1}_{L_{\underline{t}}}, \mathbf{1}_{L_{\underline{t}+1}}) = \sum_{i \in \mathcal{I}_{\underline{t}}} \xi^i$.

---

[5] Asymptotic cones are a generalization of recession cones of convex sets as defined in Rockafellar [52]. Here, since the $\bar{P}^i(x^i)$ is closed and convex, its asymptotic cone coincides with its recession cone, which consists of the set of all directions in which $\bar{P}^i(x^i)$ is unbounded.

[6] In Debreu [19], two cones A and B of $\mathbb{R}^m$ are positively semi-independent if $x \in A$ and $y \in B$ such that $x + y = 0$ implies that $x = y = 0$.

Let us consider the individual $i_0 \in \mathcal{I}_{\underline{t}}$, we can then write $\bar{h}^{\underline{t}} = \sum_{i \neq i^0}(\xi^i - x^i) + (\xi^{i_0} - {x^i}_0) + \lambda(\mathbf{1}_{L_{\underline{t}}}, \mathbf{1}_{L_{\underline{t}+1}})$. Take $h^i = \xi^i - x^i$ and $h^{i_0} = \xi^{i_0} - x^{i_0} + \lambda(\mathbf{1}_{L_{\underline{t}}}, \mathbf{1}_{L_{\underline{t}+1}})$. We note that $x^{i_0} + h^{i_0} = \xi^{i_0} + \lambda(\mathbf{1}_{L_{\underline{t}}}, \mathbf{1}_{L_{\underline{t}+1}}) \in P^{i_0}((x^{i_0}))$. So, h is a Pareto improving transfer and x is not Pareto optimal.

Conversely, if x is not Pareto optimal, there exists h a Pareto improving transfer. From the convexity of $P^i(x^i)$ (Assumption A(a)), $(1/2)h$ is a Pareto improving transfer. We now check that $\bar{h}$ defined by $\bar{h}^t = (1/2)\sum_{i \in \mathcal{I}_t} h^i$ is an aggregate Pareto improving transfer. $\bar{h}$ is obviously feasible since h is feasible. For all t, since $(1/2)h$ is Pareto improving, $x^t + \bar{h}^t \in \bar{P}_t((x^i))$, with $x^t = \sum_{i \in \mathcal{I}_t} x^i$. Since x^i is an interior allocation, $x^i + (1/2)h^i$ belongs to the interior of X^i for all i. For the agent i_0 of generation t_0 such that $x^{i_0} + h^{i_0} \in P^{i_0}(x^{i_0})$, since $P^{i_0}(x^{i_0})$ is open (Assumption A(a)), one gets that there exists $\lambda > 0$ such that $x^{i_0} + (1/2)h^{i_0} - \lambda(\mathbf{1}_{L_{\underline{t}}}, \mathbf{1}_{L_{\underline{t}+1}}) \in P^{i_0}(x^{i_0})$. So, $\sum_{i \in \mathcal{I}_{t_0}} x^i + (1/2)h^i - \lambda(\mathbf{1}_{L_{\underline{t}}}, \mathbf{1}_{L_{\underline{t}+1}}) \in \sum_{i \in \mathcal{I}_{t_0}} \bar{P}^i(x^i) = \bar{P}^{t_0}((x^i))$, which implies that $\sum_{i \in \mathcal{I}_{t_0}} x^i + (1/2)h^i \in \text{int}(\bar{P}_{t_0}((x^i)))$.

4.4 Characterization of Pareto-optimal allocations

We now state the main result of the paper. It provides a condition on the supporting price of a weak Pareto optimal allocation, which is necessary and sufficient for the Pareto optimality of the given allocation.

Proposition 4.4.1. Let $x = (x^i) \in \prod_{i \in \mathcal{I}} X^i$ be a WPO allocation supported by the price sequence $p = (p_1, p_2, \ldots p_t, \ldots)$. We suppose that:

Assumption B: there exist $\bar{\chi} > 0$ and $\underline{\chi} > 0$ such that for all $t \geq 1$, $\bar{e}_t \leq \bar{\chi}\mathbf{1}_{L_t}$, $\underline{\chi}(\mathbf{1}_{L_t}, \mathbf{1}_{L_{t+1}}) \leq x^i$ for all $i \in \mathcal{I}_t$ and $\underline{\chi}\mathbf{1}_{L_1} \leq x^i$ for all $i \in \mathcal{I}_0$;

Assumption C: there exists $\underline{r} > 0$ such that for all $i \in \mathcal{I}$, $B(x^i + \underline{r}\gamma^i(x^i), \underline{r}) \subset P^i(x^i)$;

Assumption C': there exists $\bar{r} > 0$ such that for all $i \in \mathcal{I}_{-0}$ (resp. $i \in \mathcal{I}_0$), for all $\xi^i \in \bar{P}^i(x^i)$, if $\xi^i \leq (e_t, e_{t+1})$, then $\xi^i \in \bar{B}(x^i + \bar{r}\gamma^i(x^i), \bar{r})$ (resp. if $\xi^i \leq e_1$, then $\xi^i \in \bar{B}(x^i + \bar{r}\gamma^i(x^i), \bar{r})$;

Assumption G: there exists $\overline{\nu} \geq \underline{\nu} > 0$ such that for all $t \geq 1$, $i \in \mathcal{I}_t$,

$$\underline{\nu} \leq \frac{\|\gamma_t^i(x^i)\|}{\|\gamma_{t+1}^i(x^i)\|} \leq \overline{\nu}$$

Then, x is Pareto optimal if and only if:

$$\sum_{t \in \mathbb{N}^*} \frac{1}{\|p_t\|} = +\infty.$$

Since x is supported by the price sequence $p = (p_1, p_2, \dots p_t, \dots)$, we recall that Lemma 4.1 implies that for all period t, for all $i \in \mathcal{I}_t$, $\frac{1}{\|\Pi_t\|}\Pi_t = \frac{1}{\|(p_t, p_{t+1})\|}(p_t, p_{t+1})$ is equal to $\gamma^i(x^i)$. We denote by γ^t the vector $\frac{1}{\|\Pi_t\|}\Pi_t$ and we let $x^t = \sum_{i \in \mathcal{I}_t} x^i$.

Remark 4.4.1. *i)* Assumptions C and C' mean that the boundaries of the preferred sets lie below small closed balls of radius $\underline{r}$ and above the comprehensive hull of bigger closed balls of radius $\bar{r}$. Assumptions C and C' means that the preferences are smooth and uniformly strictly convex. While Assumptions C and C' in [4] are stated in terms of curvature of the utility functions, we have chosen a more geometric approach because the preferences are not representable by a utility function.

For each individual i, let us consider the closed set F^i defined as the complementery of $P^i(x^i)$, $F^i := \complement P^i(x^i)$. Then Assumption C means that F^i is prox-regular at x^i. Indeed, let $r > 0$ and $\beta > 0$. The set F is called (r, β)- prox-regular at $x \in F$, if for any $y \in F \cap B(x, \beta)$ and any $v \in N^P(F, y)$ with $\|v\| \leq 1$, $y \in Proj_F(y + rv)$ where $N^P(F, y)$ is the proximal normal cone to F at y, that is:

$$N^P(F, y) = \{v \mid \exists \rho > 0, y \in Proj_F(y + \rho v)\}$$

In our framework, under Assumption A,

$$N^P(F, y) = \{\mu\gamma(y) \mid \mu \geq 0\}$$

The notion of prox-regularity was introduced by Poliquin and Rockafellar, as a new important regularity in variational analysis, see [49], and extensively studied by Colombo and Thibault in [21].

ii) Note that Assumption B implies that the number of individuals is uniformly bounded above at each generation. Indeed, if I_t is the number of individual of the generation t, and h is a commodity at period t, then $I_t \underline{\chi} \leq \sum_{i \in \mathcal{I}_t} x^i_h \leq e_{th} \leq \bar{\chi}$. We denote by $\bar{I}$, an upper bound of the number of individual at each generation.

iii) Assumptions C and C' still hold when we aggregate the finitely many consumers at each period by considering the set $\bar{P}^t((x^i))$. Assumption C implies $B(x^t + I_t\underline{r}\gamma^t, I_t\underline{r}) \subset \sum_{i \in \mathcal{I}_t} P^i(x^i)$, thus, whatever is the number of consumers of generation t, $B(x^t + \underline{r}\gamma^t, \underline{r}) \subset \mathrm{int}\bar{P}^t((x^i))$.

Let $\xi^t \in \bar{P}^t((x^i))$ such that $\xi^t \leq (e_t, e_{t+1})$. Then Assumption C' implies that ξ^t belongs to $\sum_{i \in \mathcal{I}_t} \bar{B}(x^i + \bar{r}\gamma^i(x^i), \bar{r})$, which is equal to $\bar{B}(x^t + I_t\bar{r}\gamma^i(x^i), I_t\bar{r})$. So uniformly in t, ξ^t belongs to $\bar{B}(x^i + \bar{I}\bar{r}\gamma^i(x^i), \bar{I}\bar{r})$.

For the coherence of the notations, we let $\underline{\rho} = \underline{r}$ and $\bar{\rho} = \bar{I}\bar{r}$.

iv) We remark that Assumption G is slightly weaker than Property G in Balasko and Shell in [4]. Indeed, Property G assumes that the ratio $\frac{p_{s\ell}}{\|\Pi_t\|}$ is uniformly bounded above and away from 0 for all period t, for $s = t, t+1$ and for all commodities ℓ at date t or $t+1$. Recalling that Π_t is positively collinear to γ^t, this clearly implies that the ratio $\frac{\|\gamma^t_t\|}{\|\gamma^t_{t+1}\|}$ is uniformly bounded above and away from 0 for all period t.

Proof of Proposition 4.4.1 . Since the allocation $x = (x^i)$ is Pareto optimal if and only if there exists no feasible aggregate Pareto improving transfer upon x, the proof will be established by constructing and characterizing a sequence of aggregate Pareto improving transfers.

Let $\bar{h}$ be a feasible aggregate transfer. Set $\eta^t := \Pi_t \cdot \bar{h}^t$, the net present value of the aggregate transfer $\bar{h}$ at each date t.

Remark 4.4.2. If $\bar{h}$ is a feasible aggregate Pareto improving transfer upon (x^i), then by Definition 4.3.4, $x^t + \bar{h}^t \in \bar{P}^t((x^i))$, so thanks to Assumption C', $\Pi_t \cdot \bar{h}^t > 0$, thus $\eta^t > 0$.

Let us define the sequence α by:

$$\alpha^t := \frac{\|\bar{h}^t\|^2 \|\Pi_t\|}{\Pi_t \cdot \bar{h}^t} = \frac{\|\bar{h}^t\|^2 \|\Pi_t\|}{\eta^t}$$

Remark 4.4.3. *i)* Note that $\|\bar{h}^t\|^2 = \frac{\alpha^t \eta^t}{\|\Pi_t\|} \leq \frac{\alpha^t \|\Pi_t\| \|\bar{h}^t\|}{\|\Pi_t\|}$, thus $\|\bar{h}^t\| \leq \alpha^t$. Thus, if α is bounded then $\bar{h}$ is also bounded.

ii) $\frac{\alpha^t}{2}$ actually represents the radius of the sphere $S(x^t + \frac{\alpha^t}{2}\gamma^t, \frac{\alpha^t}{2})$ which is tangent to $\bar{P}^t((x^i))$ at x^t, and contains $x^t + \bar{h}^t$. Indeed, we easily check that:

$$\left\| x^t + \bar{h}^t - \left(x^t + \frac{\alpha^t}{2}\gamma^t \right) \right\| = \frac{\alpha^t}{2}$$

We prepare the proof by three lemmas and then we prove the necessary and the sufficient condition in two additional lemmas.

Lemma 4.3. *If the feasible aggregate transfer $\bar{h}$ is Pareto improving upon $x = (x^i)$, and if Assumption C' holds, then the sequence α is bounded from above.*

Proof. Since $\bar{h}$ is a feasible aggregate Pareto improving transfer, $0 \leq \xi^t = x^t + \bar{h}^t \leq (e_t, e_{t+1})$. So, from Assumption C' and Remark 4.4.1, $\xi^t \in \bar{B}(x^t + \bar{\rho}\gamma^t, \bar{\rho})$. Hence, from Remark 4.4.3 (ii), the sphere $S(x^t + \frac{\alpha^t}{2}\gamma^t, \frac{\alpha^t}{2})$, which is tangent to $\bar{P}^t((x^i))$ at x^t, and contains ξ^t is included in $\bar{B}(x^t + \bar{\rho}\frac{\Pi_t}{\|\Pi_t\|}, \bar{\rho})$. So $\frac{\alpha^t}{2} \leq \bar{\rho}$, which shows that α is bounded from above. □

Lemma 4.4. *If $\bar{h}$ is a feasible Pareto improving aggregate transfer, then:*

$$\alpha^t = \frac{\|\Pi_t\|}{\eta^t}\|\bar{h}^t\|^2 \geq \frac{\|\Pi_t\|}{\eta^t}\left[\frac{1}{\|p_t\|^2}(\eta^0 + \ldots + \eta^{t-1})^2 + \frac{1}{\|p_{t+1}\|^2}(\eta^0 + \ldots + \eta^t)^2\right]$$

Proof. Indeed, the construction of η and the feasibility of the transfer h allow us to write:

$\eta^0 = p_1 \cdot \bar{h}^0 = p_1 \cdot \bar{h}_1^0 = -p_1 \cdot \bar{h}_1^1$

$\eta^1 = \Pi_1 \cdot \bar{h}^1 = p_1 \cdot \bar{h}_1^1 + p_2 \cdot \bar{h}_2^1 = -p_1 \cdot \bar{h}_1^0 + p_2 \cdot \bar{h}_2^1$

...

$\eta^t = \Pi_t \cdot \bar{h}^t = p_t \cdot \bar{h}_t^t + p_{t+1} \cdot \bar{h}_{t+1}^t = -p_t \cdot \bar{h}_t^{t-1} + p_{t+1} \cdot \bar{h}_{t+1}^t$

By summing up, we obtain that: $\eta^0 + \eta^1 + \ldots + \eta^t = p_{t+1} \cdot \bar{h}_{t+1}^t$, with $\bar{h}_t^t = -\bar{h}_t^{t-1}$

By definition,

$$\alpha^t = \frac{\|\Pi_t\|}{\eta^t}\left(\|\bar{h}_t^t\|^2 + \|\bar{h}_{t+1}^t\|^2\right)$$

By Schwarz inequality, $(p_t \cdot \bar{h}_t^t)^2 \leq \|p_t\|^2\|\bar{h}_t^t\|^2$, that is: $\left(\frac{p_t \cdot \bar{h}_t^t}{\|p_t\|}\right)^2 \leq \|\bar{h}_t^t\|^2$, we then obtain that:

$$\alpha^t \geq \frac{\|\Pi_t\|}{\eta^t}\left[\frac{(p_t \cdot \bar{h}_t^t)^2}{\|p_t\|^2} + \frac{(p_{t+1} \cdot \bar{h}_{t+1}^t)^2}{\|p_{t+1}\|^2}\right]$$

Hence,

$$\alpha^t \geq \frac{\|\Pi_t\|}{\eta^t}\left[\frac{1}{\|p_t\|^2}(\eta^0 + \ldots + \eta^{t-1})^2 + \frac{1}{\|p_{t+1}\|^2}(\eta^0 + \ldots + \eta^t)^2\right]$$

□

Lemma 4.5. *Let η be a positive sequence in $\mathbb{R}$. Let us define an aggregate transfer $\bar{h}$ in $\mathbb{R}^{L_1} \times \prod_{t=1}^{\infty}(\mathbb{R}^{L_t} \times \mathbb{R}^{L_{t+1}})$ and the associated α respectively by:*

$$\bar{h}_{t+1}^t = \left(\eta^0 + \eta^1 + \ldots + \eta^t\right)\frac{p_{t+1}}{\|p_{t+1}\|^2} \text{ and } \bar{h}_t^t = -\bar{h}_t^{t-1}$$

$$\alpha^t = \frac{\|\Pi_t\|}{\eta^t}\|\bar{h}^t\|^2 = \frac{\|\Pi_t\|}{\eta^t}\left[\frac{1}{\|p_t\|^2}(\eta^0 + \ldots + \eta^{t-1})^2 + \frac{1}{\|p_{t+1}\|^2}(\eta^0 + \ldots + \eta^t)^2\right]$$

Under Assumptions B and C, if α is bounded then there exists $\mu > 0$ such that $\mu\bar{h}$ is a feasible Pareto improving aggregate transfer.

Remark 4.4.4. In Lemma 4.5, $\bar{h}$ is computed in such a way that $\Pi_t \cdot \bar{h}^t = \eta^t$ and the associated α^t is the smallest possible one.

Proof. Let $\mu > 0$ be taken small enough so that $\mu\alpha^t < 2\underline{\rho}$, for all t, which is feasible since $\alpha = (\alpha^t)$ is bounded.

From the formula above and Remark 4.4.3 *(ii)*, since $\frac{\mu\alpha^t}{2} \leq \underline{\rho}$, $x^t + \mu\bar{h}^t$ belongs to the sphere $S(x^t + \frac{\mu\alpha^t}{2}\gamma^t, \frac{\mu\alpha^t}{2}) \subset B(x^t + \underline{\rho}\gamma^t, \underline{\rho}) \cup \{x^t\}$. Since $\bar{h}^t \neq 0$, $x^t + \mu\bar{h}^t \in B(x^t + \underline{\rho}\gamma^t, \underline{\rho}) \setminus \{x^t\}$. From Assumption C and Remark 4.4.1 *(iii)*, $x^t + \mu\bar{h}^t \in \mathrm{int}\bar{P}^t((x^i))$, that is $\mu\bar{h}$ is a feasible aggregate Pareto improving transfer upon the allocation (x^i). □

Lemma 4.6. *Given the positive price sequence p, if x is not PO, then under Assumption C′, $\sum_{t\in\mathbb{N}^*} \frac{1}{\|p_t\|} < +\infty$.*

Proof. Since x is not PO, there exists a Pareto improving aggregate transfer $\bar{h}$. From Lemma 4.3, the associated α is bounded from above by $2\bar{\rho}$, thus, from Lemma 4.4:

$$2\bar{\rho} \geq \alpha^t \geq \frac{\|\Pi_t\|}{\eta^t\|p_{t+1}\|^2}[\eta^0 + \ldots + \eta^t]^2 \geq \frac{1}{\eta^t\|p_{t+1}\|}[\eta^0 + \ldots + \eta^t]^2$$

since $\frac{\|\Pi_t\|}{\|p_{t+1}\|} = \sqrt{1 + \frac{\|p_t\|^2}{\|p_{t+1}\|^2}} \geq 1$. Thus:

$$\frac{1}{\|p_{t+1}\|} \leq \frac{2\bar{\rho}\eta^t}{[\eta^0 + \ldots + \eta^t]^2}$$

But, we notice that:

$$\frac{\eta^t}{[\eta^0 + \ldots \eta^t]^2} \leq \frac{1}{\eta^0 + \ldots + \eta^{t-1}} - \frac{1}{\eta^0 + \ldots + \eta^t}$$

This implies that:

$$\sum_{t=1}^{\infty} \frac{\eta^t}{[\eta^0 + \ldots + \eta^t]^2} \leq \frac{1}{\eta^0}$$

Hence

$$\sum_{t=1}^{\infty} \frac{1}{\|p_t\|} = \frac{2\bar{\rho}}{\eta^0} < +\infty$$

□

Remark 4.4.5. The following example shows that if preferences are flat at the given allocation, so not satisfying Assumption C', then the allocation could be not Pareto optimal even if $\sum_t \frac{1}{\|p_t\|} = +\infty$. Let us consider an OLG economy with one commodity per period, one consumer per generation, the allocation $(1,1)$ for all generation and the preferred set $\{\xi \in \mathbb{R}^2_+ \mid t\xi_t + (t+1)\xi_{t+1} > 2t+1\}$. Then, one easily check that this allocation is not Pareto optimal because the allocation $(1/2, 3/2)$ for each generation is Pareto dominating But, one also checks that the price $(1, 2, \ldots, t, \ldots)$ supports this allocation and $\sum_t \frac{1}{\|p_t\|} = \sum_t \frac{1}{t} = +\infty$.

The last step of the proof of Proposition 4.4.1 is given by the following lemma.

Lemma 4.7. *Under Assumptions B, C and G, if the positive price sequence p satisfies $\sum_t \frac{1}{\|p_t\|} < +\infty$, then x is not PO.*

Proof. Consider $\eta^t = \frac{1}{\|p_t\|}$ for all t, and denote by $\bar{\eta}^t := \sum_{s=1}^t \eta^t$. Note that $(\bar{\eta}^t)$ is bounded. Let the corresponding agregate transfer $\bar{h}$ and the associated sequence α defined by the formula of Lemma 4.5. From this Lemma, it remains to show that α is bounded.

From Lemma 4.5, we have: $\bar{h}^t = (\bar{\eta}^{t-1}\frac{p_t}{\|p_t\|^2}, \bar{\eta}^t \frac{p_{t+1}}{\|p_{t+1}\|^2})$. Consequently, since $\bar{\eta}^{t-1} \le \bar{\eta}^t$, $\|\bar{h}^t\|^2 = (\bar{\eta}^{t-1})^2 \frac{1}{\|p_t\|^2} + (\bar{\eta}^t)^2 \frac{1}{\|p_{t+1}\|^2} \le (\bar{\eta}^t)^2 \left(\frac{1}{\|p_t\|^2} + \frac{1}{\|p_{t+1}\|^2}\right)$. Since $\eta^t = \frac{1}{\|p_t\|}$, the associated α defined in Lemma 4.5 satisfies:

$$\begin{aligned}
\alpha^t &= \frac{\|\Pi_t\|}{\eta^t}\|\bar{h}^t\|^2 \le \|\Pi_t\|\|p_t\|(\bar{\eta}^t)^2\left(\frac{1}{\|p_t\|^2} + \frac{1}{\|p_{t+1}\|^2}\right) \\
&= \sqrt{\|p_t\|^2 + \|p_{t+1}\|^2}\frac{(\bar{\eta}^t)^2}{\|p_t\|}\left(1 + \frac{\|p_t\|^2}{\|p_{t+1}\|^2}\right) \\
&= (\bar{\eta}^t)^2\sqrt{1 + \frac{\|p_{t+1}\|^2}{\|p_t\|^2}}\left(1 + \frac{\|p_t\|^2}{\|p_{t+1}\|^2}\right)
\end{aligned}$$

Assumption G implies that $\frac{\|p_t\|}{\|p_{t+1}\|}$ and $\frac{\|p_{t+1}\|}{\|p_t\|}$ are bounded. Since $\bar{\eta}^t$ is bounded, α is so. □

Remark 4.4.6. The following example shows that if preferences exhibit a kink at the given allocation, so not satisfying Assumption C, then the allocation could be Pareto optimal even if $\sum_t \frac{1}{\|p_t\|} < +\infty$. Let us consider an OLG economy with one commodity per period, one consumer per generation, the allocation $(1,1)$ for all generation and the same preferred set $\{(1,1)\} + \mathbb{R}^2_{++}$. Then, one easily checks that this allocation is Pareto optimal because to strictly improve the welfare of an agent, we need to reduce the allocation of the agent of the next generation when she is young so that her welfare strictly decreases. But, one also checks that the price $(1, 2, \ldots, 2^t, \ldots)$ supports this allocation and $\sum_t \frac{1}{\|p_t\|}$ is finite.

4.5 Conclusion

Providing a simple set-theoretic and geometric version of the proof of Balasko and Shell [4] has allowed us to directly consider the multi-goods case [7], while in [4], the proof needs to first go through the one-commodity case. We have encompassed the case of non-complete and non-transitive preferences where basic assumptions are made on the preference sets and the associated normal cones. We provide a characterization of Pareto optimality that is established following a simple and geometric version of the proof of Balasko and Shell [4]. Moreover, we provide an explicit expression of improving transfers h^t at each date t, that do not satisfy the criterion. Indeed, this is easily obtained thanks to the minimal value of the parameter α that is also used in [4].

A natural continuation to this work would be the extension to the case of OLG models with heterogeneous longevities within each generation. In this case, we would need to review some hypotheses concerning for instance the initial endowments and strict monotony of utility functions. We also propose to extend this work to the case of OLG models with durable commodities. As established in the previous chapter 3, Section 3.3, such an extension calls for an optimality characterization to the non conventional case where the consumption sets do not consist of the positive orthant anymore.

[7] Note that in [14], we have considered the particular case where $\mathcal{L}_t = \mathcal{L}$ for all t.

General Conclusion

In this thesis, we have considered OLG models with production, where we go beyond the classical hypothesis of constant returns. We have then provided a formalization of incresing returns by allowing firms to behave in a more general way through the notion of pricing rule. Although losses may occur, we focus on loss-free pricing rules that are relevant in unregulated markets and can be applied to firms whose increasing returns are due to fixed cost or associated to a S-shape production function, which means that the technology displays first increasing returns to scale until some level of production then decreasing returns to scale. Since we assume that firms are privately owned and inaction is possible, this principle instructs firms to produce only when the profits are non-negative, otherwise they close down. These pricing rules allow us to give also account of some models that treat growth theory and provide structures that permit to reconcile competitive behavior of firms and increasing returns by associating them to imperfect competition. These structures have the advantage to show how firms can exploit their market power in presence of increasing returns by charging prices that will bring them positive profits. This approach clearly constitutes a variant of loss-free pricing rules.

Beyond the existence result, we are led to study the mechanism of transfer between generations in order to analyze the possible perpetuation of firms. We then incorporate durable goods which may be stored from one period to a successive period through a linear technology. In this model, we establish not only the existence of an equilibrium but also highlight features of durable goods that entitle consumers, the roles of producers, lenders and borrowers, even at the end of their lifetime. Allowing agents to hold assets at the end of their lifetime is relevant when their lifetimes are actually uncertain. This extension is easily obtained from our model. A natural continuation to this work is the introduction of a production, where durable goods are involved and increasing returns are allowed.

Another important result on the relation between prices allows us to make a link with the Pareto efficiency of equilibrium, confirming their role in restoring the market failure in OLG economies. We review the characterization of Pareto optimal allocations, in the line of Balasko and Shell [4], but in addition we allow for multiple agents and multiple goods per period. Our approach is set-theoretic and geometrical. The consumers characteristics are described by their consumption sets, their preference sets and the associated normal cones. We give conditions of Pareto optimality, under very basic assumptions, by providing a simple and geometric version of the proof of Balasko and Shell [4], encompassing the case of non-complete and non-transitive preferences. As part of the motivations of this chapter, a future work will consist of including durable goods and characterizing optimal allocations in a framework where the consumption sets do not consist of the positive orthant anymore.

Throughout this thesis, we have raised aditionnal issues but we have also accumulated further tools and intuitions that will be important and helpful for further studies, especially in production economies where increasing returns are allowed and thus growth can be expected. Indeed, increasing returns may induce losses, but in the long run, the economy may grow to an efficient allocation, where there are no losses. This meets the notion of dynamical efficiency, an issue that deserves a deeper consideration, and constitutes a natural perspective to this thesis.

Appendix A

On the equivalent economy with“no” durable goods

Boundedness of $\mathcal{A}(\tilde{\mathcal{E}})$

Let $e \in \prod_{t\in\mathbb{N}^*} \mathbb{R}^L_+$ be defined by $\tilde{e}_t = \sum_{i\in\mathcal{I}_t\cup\mathcal{I}_{t-1}} \tilde{e}^i_t$ and $e' \in \prod_{t\in\mathbb{N}^*} \mathbb{R}^L_+$ such that $e' \geq \tilde{e}$. Let $(\chi^i) \in \mathcal{A}(\tilde{\mathcal{E}})$, with $\chi^i = (x^i_t, \xi^i_{t+1}, \zeta^i_{t+2})$, then for all $t = 1, 2 \ldots$,

$$\sum_{i\in\mathcal{I}_0} \xi^i_1 + \sum_{i\in\mathcal{I}_1} x^i_1 \leq e'_1$$

$$\sum_{i\in\mathcal{I}_{t-2}} \zeta^i_t + \sum_{i\in\mathcal{I}_{t-1}} \xi^i_t + \sum_{i\in\mathcal{I}_t} x^i_t \leq e'_t,\ t > 1$$

For the first period, define $M_1 := e'_1$. Since $\xi^i_1 \geq 0$ and $x^i_1 \geq 0$, we have $0 \leq x^i_1 \leq M_1$ for $i \in \mathcal{I}_0$, and $0 \leq \xi^i_1 \leq M_1$ for $i \in \mathcal{I}_1$.
For the second period, let $M_2 := e'_2 + \Gamma^1(M_1)$.
For $i \in \mathcal{I}_0$, since $\zeta^i_2 = -\Gamma^1(\xi^i_1)$, we have: $-\Gamma^1(M_1) \leq \zeta^i_2 \leq 0$.
By definition, we know that:

$$\sum_{i\in\mathcal{I}_0} \zeta^i_2 + \sum_{i\in\mathcal{I}_1} \xi^i_2 \geq \sum_{i\in\mathcal{I}_0} -\Gamma^1(\xi^i_1) - \sum_{i\in\mathcal{I}_1} \Gamma^1(x^i_1) \geq -\Gamma^1(e'_1)$$

Thus, for $i \in \mathcal{I}_2$, $0 \leq x^i_2 \leq \sum_{i\in\mathcal{I}_2} x^i_2 \leq e'_2 - \sum_{i\in\mathcal{I}_0} \zeta^i_2 - \sum_{i\in\mathcal{I}_1} \xi^i_2 \leq e'_2 + \Gamma^1(e'_1)$, that is: $0 \leq x^i_2 \leq M_2$.

For $i \in \mathcal{I}_1$, it is clear that $\zeta_2^i \geq -\Gamma^1(e_1')$. Futhermore,

$$\zeta_2^i + \sum_{i' \in \mathcal{I}_1, i' \neq i} \zeta_2^{i'} \leq e_2' - \sum_{i \in \mathcal{I}_0} \zeta_2^i$$

But,

$$\sum_{i' \in \mathcal{I}_1, i' \neq i} \zeta_2^{i'} + \sum_{i \in \mathcal{I}_0} \zeta_2^i \geq -\Gamma(\sum_{i' \in \mathcal{I}_1, i' \neq i} x_1^i) - \Gamma(\sum_{i \in \mathcal{I}_0} \xi_1^i) \geq -\Gamma(e_1')$$

Thus, for $i \in \mathcal{I}_1$, $-\Gamma^1(M_1) \leq \zeta_2^i \leq e_2' + \Gamma(e_1') = M_2$.
For period $t \geq 3$, we recursively proceed with the same reasoning to prove that the sequence of nonnegative vectors $(M_t)_{t \geq 1}$, defined by $M_t = e_t' + \Gamma^{t-1}(M_{t-1} + \Gamma^{t-2}(M_{t-2}) \cdots + \Gamma^1(M_1))$, where $M_1 = e_1'$satisfies the desired inequalities.
□

Proof of Lemma 3.1

There is no modification concerning the budget constraints feasibility, we just have to look at Condition (b) for the period $\tau - 1$ in the definition of a pseudo-equilibrium. Since $(\bar{p}^*, (\bar{a}^{i*}))$ is a pseudo-equilibrium in the economy $\mathcal{E}_{\bar{\tau}}$ and $\bar{\tau} - 1 > \tau - 1$, one has:

$$\sum_{i \in \mathcal{I}_0^{\bar{\tau}-2}} \bar{a}_{\tau-1}^{i*} = \sum_{i \in \mathcal{I}_0^{\bar{\tau}-2}} \tilde{e}_{\tau-1}^{\bar{\tau}i}$$

Considering the definition of $X^{\bar{\tau}i}$, for all $i \in \cup_{t=\tau}^{\bar{\tau}-1} \mathcal{I}_t$, $\bar{a}_{\tau-1}^{i*} = 0$. From the definition of $\tilde{e}^{\bar{\tau}i}$, for all $i \in \cup_{t=\tau}^{\bar{\tau}-1} \mathcal{I}_t$, $\tilde{e}_{\tau-1}^{\bar{\tau}i} = 0$. So, one deduces that:

$$\sum_{i \in \mathcal{I}_0^{\bar{\tau}-2}} \bar{a}_{\tau-1}^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-2}} \bar{a}_{\tau-1}^{i*} + \sum_{i \in \mathcal{I}_{\tau-1}} \bar{a}_{\tau-1}^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}_{\tau-1}^{\bar{\tau}i} + \sum_{i \in \mathcal{I}_{\tau-1}} \tilde{e}_{\tau-1}$$

and since $\sum_{i \in \mathcal{I}_{\tau-1}} \bar{a}_{\tau-1}^{i*} \geq 0$, we have:

$$\sum_{i \in \mathcal{I}_0^{\tau-2}} \bar{a}_{\tau-1}^{i*} \leq \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}_{\tau-1}^{\bar{\tau}i} + \sum_{i \in \mathcal{I}_{\tau-1}} \tilde{e}_{\tau-1}$$

So we get Condition (b) for the period $\tau - 1$ since $\bar{a}_{\tau-1}^{i*} = \hat{a}_{\tau-1}^{i*}$ and $\tilde{e}_{\tau-1}^{\bar{\tau}i} = \tilde{e}_{\tau-1}^{\tau i}$ for all $i \in \mathcal{I}_0^{\bar{\tau}-2}$ and $\tilde{e}_{\tau-1}^{\bar{\tau}i} = \tilde{e}_{\tau-1}^{i}$ for all $i \in \mathcal{I}_{\tau-1}$.
□

Proof of Lemma 3.2

The first part comes from the strict monotonicity of the utility of agent $i \in \mathcal{I}_t$, for all $t = 1, 2 \ldots \tau - 1$, as mentionned in Assumption C.

We normalize a non zero equilibrium price p^* so that $\sum_{t=1}^{\tau} \sum_{\ell \in \mathcal{L}} p^*_{t\ell} = 1$.

Let us consider a sequence of pseudo-equilibria $(p^\nu, (a^{i\nu}))$ that converges to $(\bar{p}, (\bar{a}^i))$. We prove that $(\bar{p}, (\bar{a}^i))$ is also a pseudo-equilibrium.

We easily establish that $(\bar{p}, (\bar{a}^i))$ satisfies Condition (b) in Definition 3 of pseudo-equilibrium. So it remains to show that Condition (a) is also satisfied.

Denote by $(w^{i\nu})$ the associated wealth sequence and by $\bar{w}^i$ its limit. One easily shows that the budget constraint is satisfied by $\bar{a}^i$. If $\bar{p} \cdot a^i < \bar{w}^i$, then for ν large enough, $p^\nu \cdot a^i \leq w^{i\nu}$. But this implies that $\tilde{u}^i(a^i) \leq \tilde{u}^i(a^{i\nu})$, and by the continuity of $\tilde{u}^i$, $\tilde{u}^i(a^i) \leq \tilde{u}^i(\bar{a}^i)$. Thus $(\bar{p}, (\bar{a}^i))$ satisfies Condition (a) in Definition 3.4.2 of a quasi-equilibrium. Thus $(\bar{p}, (\bar{a}^i))$ is actually a "pseudo-quasi-equilibrium". But thanks to the irreducibility condition, we can discard the possibility of minimal wealth at any quasi-equilibrium price. Thus each agent is an utility maximizer at any quasi-equilibrium price. □

Proof of Lemma 3.3

We have established that for all $T \geq 2$, there exists a pseudo-equilibrium $(p^T, (a^{iT}))$ of the truncated economy $\tilde{\mathcal{E}}_T$. Since $p^T_1 \neq 0$, we normalize p^T so that $\sum_{\ell \in \mathcal{L}} p^T_{1\ell} = 1$.

We extend the price and the allocations to the whole space $\prod_{t=1}^{\infty} \mathbb{R}^L$ by adding zeros for the missing components without modifying the notations. So, now the sequences (p^T), (a^{iT}) are in $\prod_{t=1}^{\infty} \mathbb{R}^L$.

If Lemma 3.3 is not true, then there exist $\bar{t}$ and an increasing sequence (T^ν) such that $p^{T^\nu}_{\bar{t}} \geq \nu \mathbf{1}$. Let $\tau > \bar{t} + 3$. We assume without any loss of generality that $T^\nu > \tau$ for all ν.

Now we consider the restriction to the τ first period of the T^ν-equilibrium $(p^{T^\nu}, (a^{iT^\nu}))$:

- for all $i \in \mathcal{I}_0^{\tau-2}$, $a^{i\nu}$ is the restriction of a^{iT^ν} to $\prod_{t=1}^{\tau} \mathbb{R}^L$;

- p^ν is the restriction of p^{T^ν} to $\prod_{t=1}^{\tau} \mathbb{R}^L$.

From Lemma 3.1 in the previous section, $(p^\nu, (x^{i\nu})))$ is a pseudo-equilibrium of the truncated economy $\mathcal{E}_\tau$. We now renormalize the price p^ν as follows:

$$\pi^\nu = \frac{1}{\sum_{t=1}^{\tau} \sum_{\ell \in \mathcal{L}} p_{t\ell}^\nu} p^\nu$$

Since π^ν is nonnegative, the sequence π^ν remains in the simplex of $\prod_{t=1}^{\tau} \mathbb{R}^L$, which is compact. From the boundedness of $\mathcal{A}(\tilde{\mathcal{E}}_\tau(e))$, the sequence $(a^{i\nu})$ remains in the compact subset $\mathcal{A}(\tilde{\mathcal{E}}_\tau(e))$. So the sequence $(\pi^\nu, (a^{i\nu})$ has a cluster point $(\bar{\pi}, (\bar{a}^i))$. From Lemma 3.2, $(\bar{\pi}, (\bar{a}^i))$ is also a pseudo-equilibrium of the truncated economy $\tilde{\mathcal{E}}_\tau$. But $\bar{\pi}_1 = 0$ since

$$\sum_{t=1}^{\tau} \sum_{\ell \in \mathcal{L}} p_{t\ell}^\nu \geq \sum_{\ell \in \mathcal{L}} p_{t\ell}^\nu \geq \nu L$$

converges to $+\infty$ and $0 \leq p_{1\ell}^\nu \leq 1$ for all $\ell \in L$. Hence we get a contradiction since Lemma 3.2 shows that for all $t = 1, \ldots, \tau$, $\bar{\pi}_t \neq 0$.

□

Appendix B

OLG with multiple goods: on the existence result

Sketch of the proof of the existence of an equilibrium

We will proceed as in exchange economies (see Balasko et al. [4]) to establish the existence of equilibrium in $\mathcal{E}$.

Step 1: Truncate the economy at a finite horizon τ: we thus consider individuals born up to period $\tau - 1$ and group them all together into $\mathcal{I}_0^{\tau-1}$. The truncated economy with a finite horizon $\mathcal{E}_\tau = \left(X^{\tau i}, P^{\tau i}, e^{\tau i}\right)_{i \in \mathcal{I}_0^{\tau-1}}$ where, for each $t = 1, \ldots, \tau - 1$, for each $i \in \mathcal{I}_t$,

$X^{\tau i} = \{x \in \prod_{t=1}^{\tau} \mathbb{R}_+^L \mid x_{t'} = 0, \forall t' \neq t, t+1\}$,

$P^{\tau i}(x) = P^i(x^i)$,

$e^{\tau i} = (e_{t'}^{\tau i})_{t'=1}^{\tau}$ such that $e_t^{\tau i} = e_t^i$, $e_{t+1}^{\tau i} = e_{t+1}^i$ and $e_{t'}^{\tau i} = 0$ if $t' \neq t, t+1$ satisfies the sufficient conditions for the existence of pseudo-equilibrium in finite dimension of Gale and Mas-Colell [31], [32], where the production set is $Y = \{0\}$. Indeed, the $X^{\tau i}$'s are closed (as a sum of closed subsets of $\mathbb{R}_+^{2L}$), convex, non-empty and bounded below, the $P^{\tau i}(x)$'s are open in X and convex, and the income functions $\alpha^{\tau i}$'s defined by $\alpha^{\tau i}(p) = p \cdot e^{\tau i}$ satisfy $\alpha^{\tau i}(p) > \inf p \cdot X^{\tau i} = 0$. The difference between a pseudo-equilibrium and an equilibrium is that we do not require the market clearing condition at the last period τ and we artificially increase the initial endowments by adding those of the consumers of the generation τ. We also recall that if $\tau' > \tau$, then the restriction of a pseudo-equilibrium of $\mathcal{E}_{\tau'}$ to the $\tau - 1$ first generations is a pseudo-equilibrium of $\mathcal{E}_\tau$.

Step 2: We prove that prices and allocations remain in a compact space of a suitable linear space and we finally show that an equilibrium of the OLG economy is a limit point of pseudo-equilibria for the esquence of truncated economies.

Relation between the aggregate normal cone and the normal cones to the individual preferred sets [1]

$$N_{\bar{P}_t((x^i))}(\bar{x}^t) = \bigcap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$$

Proof. Indeed, let $q \in \cap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$, then $q \cdot (z^i - x^i) \leq 0$ for all $z^i \in \bar{P}^i(x^i)$, $i \in \mathcal{I}_t$. Thus by summing up, $q \cdot \sum_{i \in \mathcal{I}_t}(z^i - x^i) = q \cdot (\sum_{i \in \mathcal{I}_t} z^i - \sum_{i \in \mathcal{I}_t} x^i) \leq 0$, where $\sum_{i \in \mathcal{I}_t} z^i \in \bar{P}_t((x^i))$. Consequently, $\cap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i) \subset N_{\bar{P}_t((x^i))}(\bar{x}^t)$.

Conversely, let $q \in N_{\bar{P}_t((x^i))}(\bar{x}^t)$, then for all $z \in \bar{P}_t((x^i))$, $q \cdot (z - \bar{x}^t) \leq 0$. Since $P_t((x^i)) = \sum_{i \in \mathcal{I}_t} \bar{P}^i(x^i)$, for all $i \in \mathcal{I}_t$, there exists $z^i \in \bar{P}^i(x^i)$ such that $z = \sum_{i \in \mathcal{I}_t} z^i$. By linearity of the scalar product, $\sum_{i \in \mathcal{I}_t} q \cdot (z^i - x^i) \leq 0$. By taking $z^i = x^i$ for all $i \neq 1, i \in \mathcal{I}_t = \{1, 2, \ldots I_t\}$ and $z^1 \in \bar{P}^1(x^1)$, then $q \cdot (z^1 - x^1) \leq 0$ which means that $q \in N_{\bar{P}^i(x^1)}(x^1)$. By repeating the same reasoning for $i = 2, \ldots I_t$, we obtain that $q \cdot (z^i - x^i) \leq 0$ for all $i \in \mathcal{I}_t$, thus $N_{\bar{P}_t((x^i))}(\bar{x}^t) \subset \cap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$. □

[1] This relation was also provided in [52] p. 230, as an exercice

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