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Nernst branes with Lifshitz asymptotics in $\mathcal{N} = 2$ gauged supergravity

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ABSTRACT: We discuss two classes of non-supersymmetric interpolating solutions in $\mathcal{N} = 2$, $D = 4$ gauged supergravity, that flow from either a $z = 2$ Lifshitz geometry or a conformal AdS background to the near-horizon geometry of a Nernst brane. We obtain these solutions by constructing a $z = 2$ supersymmetric Lifshitz solution in the STU model from a first-order rewriting of the action, then lifting it up to a five-dimensional background and subsequently modifying this five-dimensional solution by a two-parameter family of deformations. Under reduction, these give four-dimensional non-supersymmetric Nernst brane solutions. This is a step towards resolving the Lifshitz tidal force singularity in the context of $\mathcal{N} = 2$ gauged supergravity and suggests an approach to encoding the Nernst brane in terms of the Schrödinger symmetry group of the holographically dual field theory.

KEYWORDS: Black Holes in String Theory, Gauge-gravity correspondence, Holography and condensed matter physics (AdS/CMT), Supergravity Models

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1 Introduction

One of the most far-ranging and conceptually incisive insights to emerge from string theory is the celebrated AdS/CFT correspondence [1–3], which maps the Hilbert space and interactions of a lower dimensional conformal field theory to states and interactions in quantum gravity in asymptotically AdS backgrounds. This paves the way for a new template to conceive of and analyse the structure of quantum field theories grouped by their universality class. Within this framework, black holes offer a fecund laboratory to test and advance the conjectured holographic duality between gravity and gauge theories. Under this map, entropic black hole solutions in gravity map to thermal ensembles in the lower dimensional dual field theory. The gauge-gravity duality, being a strong-weak coupling correspondence, potentially offers, via this map, a radically new approach to address problems in ensembles

of strongly coupled field theories in terms of the dynamics of classical fields in the dual black hole backgrounds. Adopting this promising program involves pursuing a two-fold strategy: firstly to identify the universality classes of field theories that are dual to existing black holes, and secondly and of unequivocal significance to quantum field theorists, to identify and construct black solutions, dual to states in quantum field theories which exhibit non-perturbative phenomena such as phase transitions, cf. [4–6] and references therein. An exciting research avenue that has opened up, in this spirit, focuses on modelling different states of particular field theories which can undergo quantum critical phase transitions by their holographic duals which consist of extremal black backgrounds.

On the way to achieve this goal, solutions called Nernst branes were constructed in $\mathcal{N} = 2$ U(1) gauged supergravity theories in four dimensions [7]. They were shown in [8] to be the extremal limits of well behaved black brane solutions which obey the third law of thermodynamics (i.e. their entropy density approaches zero for vanishing temperature), a prerequisite for modelling condensed matter systems. The extremal Nernst branes in [7] interpolate between conformal AdS, and a conformal hyperscale violating Lifshitz near-horizon geometry. Embedding these solutions as states in a field theory dual to conformal AdS was naively unfeasible, as the asymptotic symmetry group is not known for this background. However, embedding Nernst branes in non-relativistic backgrounds such as Lifshitz backgrounds offers a way to achieve this, as follows. The Lifshitz spacetime is a non-relativistic background that preserves a sub-group of the Galilean symmetry group corresponding to broken boost symmetries [9], as opposed to the full Poincaré group. Hence, the Lifshitz spacetime is invariant only under spatial rotations, translations, and non-relativistic scaling. Further, it is not a smooth solution as it has a tidal force singularity at its center [4, 10–13]. The only Lifshitz spacetime for which the asymptotic symmetry group is known is the so-called $z = 2$ Lifshitz spacetime. In this case, the symmetry algebra governing the spectrum of the putative holographically dual field theory was identified as the Schrödinger symmetry group [14].¹ Hence, constructing Nernst branes in $z = 2$ Lifshitz backgrounds opens up a way to obtain a field theoretic understanding of Nernst branes while simultaneously resolving the Lifshitz tidal force singularity.

1.1 Computational setup

In classical supergravity, singular geometries such as Lifshitz space-times might, under deformation, flow to extremal black solutions that cloak the singularity or to smooth relativistic or non-relativistic BPS vacua. These vacua are attractors for the scalar flows and the resulting flow equations are of first order, cf. [18–21] for the ungauged case. Such solutions have been extensively studied to obtain relativistic flows in $\mathcal{N} = 2$, $D = 4$ gauged supergravity [7, 22–25], wherein gravity is coupled to massless gauge fields of the vector multiplets. In this paper, we expand this analysis to advance the study of non-relativistic solution spaces. Non-relativistic backgrounds are backgrounds with broken Poincaré symmetry. One way to achieve this breaking is via a massive U(1) gauge field [26, 27], and so

¹The $z = 2$ Lifshitz spacetime can be viewed as the Scherk-Schwarz reduction of a Poincaré-breaking deformation of AdS_5 [15–17]. The Schrödinger asymptotic symmetry group was established by analysing the effect of the deformation of AdS on its conformal symmetry algebra [14].

we allow for the gauging of isometries of the quaternionic Kähler manifold that the hypermultiplet scalars are coordinates on. The resulting minimal coupling generates an effective Proca term in the Lagrangian. Non-relativistic supersymmetric solutions in $\mathcal{N} = 2, D = 4$ gauged supergravity with hypermultiplets have been studied in [28–30].

The primary objective of our analysis in this note is to find a solution of supergravity that interpolates between an asymptotic $z = 2$ Lifshitz geometry and another geometry that is regular and, in particular, does not suffer from any tidal force singularity. If this other geometry was a vacuum solution (for example AdS_4 or $AdS_2 \times \mathbb{R}^2$), such an interpolating solution would be a solitonic solution. In the dual field theory, it would correspond to an RG flow from a Lifshitz fixed point in the UV to a stable IR fixed point. Alternatively, the other geometry could be a regular non-vacuum solution. Given these two distinct possibilities motivates us to pursue two different strategies when examining the solution space of the low-energy gauged supergravity Lagrangian (and in particular that part of the solution space with Lifshitz asymptotics).

In a first attempt we search for solitonic solutions. These are expected to satisfy first-order flow equations resulting from a BPS rewriting of the action. Therefore, in section 2 we perform a first-order rewriting of the Lagrangian (with purely electric gauging) to formulate first-order flow equations, which are in perfect accord with those derived in [31] by imposing supersymmetry conditions. In the case without hypermultiplets a similar rewriting was performed in [7]. However, given that in string theory compactifications with $\mathcal{N} = 2$ supersymmetry one always has at least the universal hypermultiplet, the inclusion of hypermultiplets is a crucial step towards making contact with actual string theory models and, thus, to embedding non-relativistic solutions into string theory. In addition, our first-order rewriting allows for a straightforward generalization that yields first-order flow equations for a particular class of non-supersymmetric solutions, as we discuss briefly at the end of section 2.1.

The solution space of these first-order flow equations is then shown in section 3 to support $z = 2$ Lifshitz geometries, which have a tidal force singularity. These solutions are written down for the STU model, whose cubic prepotential is more general than the t^3 -model studied in [28, 29] in which Lifshitz solutions have been previously obtained. The charge and flux quantum numbers that support our Lifshitz solutions are found to be incompatible with the usual relativistic vacua, such as AdS_4 or $AdS_2 \times \mathbb{R}^2$, preventing us from constructing an interpolating solution between a Lifshitz and an AdS geometry.

Not having found a solitonic solution with Lifshitz asymptotics, we change strategy according to the general twofold strategy outlined above, and we look for non-solitonic solutions interpolating between an asymptotic Lifshitz geometry and another regular solution in the center. This is done in section 4. We make use of the fact that the class of Lifshitz backgrounds found in section 3 can be obtained via a Scherk-Schwarz reduction of a Poincaré breaking background in five dimensions [17]. Next we study deformations of the five-dimensional line element that are consistent with the second order equations of motion of the five-dimensional theory, arriving at a 2-parameter family of solutions which was first obtained in [32]. We then perform a Scherk-Schwarz reduction of these solutions to obtain two classes of four-dimensional interpolating solutions. One class indeed flows

from a $z = 2$ Lifshitz background to the near-horizon geometry of the Nernst brane constructed in [7], while the second class flows from conformal AdS_4 to the same near-horizon Nernst brane geometry. These flows do not satisfy the first-order BPS equations, and hence are non-supersymmetric (they also do not obey the non-supersymmetric modification of the first-order flow equations mentioned above, but they do solve the second order equations of motion). They establish the path to resolve, within $\mathcal{N} = 2$ gauged supergravity, the tidal force singularity at the Lifshitz center, which is now replaced by a Nernst brane near-horizon geometry. Admittedly, the horizon of the Nernst brane also has a tidal force singularity. However, this can be resolved at the classical level by heating up the brane. This transfers the singularity to the Cauchy horizon cloaked by a smooth non-extremal event horizon [8, 33]. In this sense the singularity is of the “good” type in the Gubser classification [34]. We leave this final step for future work. Moreover, the flows from the asymptotic Lifshitz geometry to the near-horizon Nernst geometry might lead to a better understanding of the holographic embedding of this class of Nernst branes, which can now be seen as states in the Schrödinger group representation of the dual field theory.²

2 First-order flow equations for static charged configurations

In this section, we derive first-order flow equations for static charged configurations in $\mathcal{N} = 2$ gauged supergravity theories in four dimensions with n_V vector multiplets and n_H hypermultiplets. We denote the $4n_H$ real scalars residing in the hypermultiplets by q^α ($\alpha = 1, \dots, 4n_H$). These are local coordinates of a quaternionic Kähler manifold, and the gaugings we consider are gaugings of the Abelian isometries of this manifold. We denote the complex scalar fields residing in the vector multiplets by z^A ($A = 1, \dots, n_V$). However, rather than directly working with z^A , we work in the big moduli space with local complex coordinates X^I ($I = 0, \dots, n_V$), in terms of which $z^A = X^A/X^0$. The associated bosonic part of the Lagrangian is given in (A.1). We refer to appendix A for details on $\mathcal{N} = 2$ gauged supergravity Lagrangians.

2.1 Electrically charged configurations

In this subsection we will focus on static electrically charged configurations. The configurations we seek are specified by the following space-time line element

$$ds^2 = -e^{2U} dt^2 + e^{2A} (dr^2 + dx^2 + dy^2) , \tag{2.1}$$

with $U = U(r)$, $A = A(r)$. These configurations are supported by scalar fields $X^I(r)$, $q^\alpha(r)$ as well as electric fields $E^I(r) = -\partial_r A_t^I(r)$. Inserting this ansatz into the four-dimensional Lagrangian ($-\sqrt{-g}L$) given in (A.1) yields the following one-dimensional effective La-

²Given that the Nernst brane does not carry any transverse momentum, we do not expect the effects discussed in [35] to prevent a description of the Nernst brane using the boundary field theory.

grangian,

$$\mathcal{L}_{1d} = e^\psi \left[U'^2 - \psi'^2 + A'^2 + e^{-2A} N_{IJ} Y'^I \bar{Y}'^J + \Sigma_r^2 + \frac{1}{2} h_{\alpha\beta} q'^\alpha q'^\beta \right. \\ \left. - \frac{1}{2} e^{2(A-U)} h_{\alpha\beta} (k_I^\alpha A_t^I) (k_J^\beta A_t^J) + \frac{1}{2} e^{-2U} \text{Im} \mathcal{N}_{IJ} E^I E^J + e^{2A} V(X, \bar{X}, q) \right] + \text{T.D.}, \quad (2.2)$$

where $' = \partial_r$. The Y^I denote rescaled scalar fields X^I , cf. (B.5), and Σ_r is given in (A.8). We also introduced

$$\psi = A + U, \quad (2.3)$$

for convenience. T.D. denotes a total derivative. We refer to appendix B for details on the derivation of (2.2).

We would like to rewrite the one-dimensional effective Lagrangian (2.2) as a sum/difference of squares that involve first-order derivatives. In the absence of gauging, the rewriting would proceed by introducing electric charges and expressing the electric fields E^I in terms of these charges. In the presence of gauging, we follow this procedure, introducing a set of real fields C_I and expressing the electric fields E^I in terms of them. In the absence of hypermultiplets, the C_I reduce to electric charges dressed by warp factors [7]. Furthermore, since the scalar fields Y^I are complex, the rewriting will be achieved by means of complex combinations that we denote by q_I ,³

$$q_I = e^{i\gamma(r)} (C_I - iP_I^3), \quad (2.4)$$

where γ is an r -dependent phase and P_I^3 denotes one of the three real Killing prepotentials appearing in the potential V given in (A.12). Here we have opted to select P_I^3 , but we could have equally selected either P_I^1 or P_I^2 . The significance of the phase γ is as follows. In the context of $\mathcal{N} = 2$ U(1) gauged supergravity (a gauged supergravity theory without hypermultiplets), the combination $q_I Y^I$ can be viewed as describing an effective complex superpotential whose phase is γ [23]. The first-order flow equations are then derived from this superpotential. Hence, it is important to allow for the presence of a phase γ , and this we have done in (2.4). Upon performing a first-order rewriting of (2.2), the phase γ will turn out to satisfy a first-order flow equation, given in (2.11) below.

The rewriting of the one-dimensional effective Lagrangian (2.2) as a sum/difference of squares of first-order flow equations is described in detail in appendix B. The rewriting results in

$$\mathcal{L}_{1d} = \text{T.D.} + \mathcal{L}_{\text{squares}} + \Delta, \quad (2.5)$$

where T.D. denotes a total derivative term, $\mathcal{L}_{\text{squares}}$ denotes a sum/difference of squares,

$$\mathcal{L}_{\text{squares}} = e^\psi \left[(U' + \text{Re}(Y^I q_I) + e^{A-U} A_t^I C_I)^2 - (\psi' + 2\text{Re}(Y^I q_I) + e^{A-U} A_t^I C_I)^2 \right. \\ + (A' + \text{Re}(Y^I q_I))^2 + e^{-2A} N_{IJ} (Y'^I - e^{2A} N^{IK} \bar{q}_K) (\bar{Y}'^J - e^{2A} N^{JL} q_L) + \Sigma_r^2 \\ + \frac{1}{2} h_{\alpha\beta} (q'^\alpha - 2h^{\alpha\gamma} \text{Im}(e^{i\gamma} Y^I) \nabla_\gamma P_I^3) (q'^\beta - 2h^{\beta\delta} \text{Im}(e^{i\gamma} Y^J) \nabla_\delta P_J^3) \\ \left. + \frac{1}{2} e^{-2U} \text{Im} \mathcal{N}_{IJ} (E^I + e^{U+A} (\text{Im} \mathcal{N})^{IK} C_K) (E^J + e^{U+A} (\text{Im} \mathcal{N})^{JL} C_L) \right], \quad (2.6)$$

³The quantity q_I should not be confused with the hyper scalars q^α .

while Δ is given by

$$\begin{aligned} \Delta = e^\psi & \left[2\Lambda^- (\gamma' + \Lambda^+) - \frac{1}{2} e^{2(A-U)} h_{\alpha\beta} (k_I^\alpha \tilde{A}_t^I) (k_J^\beta \tilde{A}_t^J) - e^{A-U} C_I' \tilde{A}_t^I \right. \\ & + 2e^{A-U} \left(h_{\alpha\beta} (k_I^\alpha \tilde{A}_t^I) k_J^\beta \operatorname{Re}(Y^J e^{i\gamma}) + C_I \tilde{A}_t^I \operatorname{Re}(q_J Y^J) \right) \\ & \left. + (e^{2A} N^{IJ} - 2Y^I \bar{Y}^J) (P_I^1 P_J^1 + P_I^2 P_J^2) \right], \end{aligned} \quad (2.7)$$

where we introduced the combinations

$$\begin{aligned} \tilde{A}_t^I &= A_t^I + 2e^{U-A} \operatorname{Re}(Y^I e^{i\gamma}), \\ \Lambda^\pm &= \operatorname{Im}(C_I Y^I e^{i\gamma}) \pm \operatorname{Re}(P_I^3 Y^I e^{i\gamma}). \end{aligned} \quad (2.8)$$

Let us now consider the variation of (2.5) with respect to the various fields and demand its vanishing. Demanding that the variation of $\mathcal{L}_{\text{squares}}$ vanishes can be achieved by setting each of the squares in $\mathcal{L}_{\text{squares}}$ to zero. This yields the first-order flow equations

$$\begin{aligned} U' &= -\operatorname{Re}(Y^I q_I) - e^{A-U} A_t^I C_I, \\ \psi' &= -2\operatorname{Re}(Y^I q_I) - e^{A-U} A_t^I C_I, \\ A' &= -\operatorname{Re}(Y^I q_I), \\ Y'^I &= e^{2A} N^{IK} \bar{q}_K, \\ q'^\alpha &= 2h^{\alpha\beta} \operatorname{Im}(e^{i\gamma} Y^I) \nabla_\beta P_I^3, \\ E^I &= -e^{U+A} (\operatorname{Im}\mathcal{N})^{IK} C_K, \\ \Sigma_r &= 0. \end{aligned} \quad (2.9)$$

Using (B.5), we infer the compatibility of the equations for A', Y'^I .

Next, we consider the variation of Δ with respect to the various fields. Since the Killing prepotentials P_I^1 and P_I^2 enter quadratically, the variation of the last line of (2.7) can be made to vanish by setting

$$P_I^1 = P_I^2 = 0. \quad (2.10)$$

This selects a submanifold of the quaternionic Kähler manifold. Note that on this submanifold $\nabla_\beta P_I^3 = \partial_\beta P_I^3$.

We now demand the vanishing of the variations of the first two lines of Δ in (2.7) with respect to the various fields. Varying with respect to C_I yields equations that are satisfied if we set

$$\begin{aligned} \Lambda^- &= 0, \\ \gamma' + \Lambda^+ &= 0, \\ \tilde{A}_t^I &= 0. \end{aligned} \quad (2.11)$$

Then, varying with respect to A_t^I yields

$$C_I' = 2h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Re}(Y^J e^{i\gamma}) + 2C_I \operatorname{Re}(q_J Y^J). \quad (2.12)$$

And finally, varying with respect to γ or any of the remaining fields, the resulting equations are satisfied provided we impose (2.11) and (2.12).

Note that (2.12) can also be written as

$$(C_I e^{2A})' = 2e^{2A} h_{\alpha\beta} k_I^\alpha k_J^\beta \text{Re}(Y^J e^{i\gamma}), \quad (2.13)$$

and that the consistency of $E^I = -\partial_r A_t^I$ with (2.9) imposes the following relation,

$$(\text{Im}\mathcal{N})^{IJ} C_J = -2e^{-2A} [(U' - A') \text{Re}(Y^I e^{i\gamma}) + \text{Re}(Y'^I e^{i\gamma}) - \gamma' \text{Im}(Y^I e^{i\gamma})]. \quad (2.14)$$

This has to be consistent with (2.12).

Next, let us analyze the condition $\Sigma_r = 0$ in (2.9). Using (A.8) and the flow equations for Y'^I and for A' in (2.9), we compute

$$\Sigma_r = -i \text{Im}(Y^I q_I) = -i \Lambda^-. \quad (2.15)$$

Thus, the condition $\Sigma_r = 0$ is equivalent to $\Lambda^- = 0$. We proceed to analyze the latter. We will now show that the condition $\Lambda^- = 0$ implies that the flow equation for γ is automatically satisfied, provided that $\text{Re}(q_I Y^I) \neq 0$. Namely, starting from $\Lambda^- = \text{Im}(q_I Y^I) = 0$ and differentiating with respect to r results in

$$[\text{Im}(q_I Y^I)]' = \text{Im}[i\gamma' q_I Y^I + e^{i\gamma} (C_I' Y^I - i(P_I^3)' Y^I) + q_I Y'^I] = 0. \quad (2.16)$$

The term $q_I Y'^I$ is real by virtue of the flow equation for Y'^I . Using (2.12), we obtain

$$\begin{aligned} [\text{Im}(q_I Y^I)]' &= [\gamma' + 2\text{Im}(C_I Y^I e^{i\gamma})] \text{Re}(q_I Y^I) \\ &\quad - (P_I^3)' \text{Re}(Y^I e^{i\gamma}) + 2h_{\alpha\beta} k_I^\alpha k_J^\beta \text{Im}(Y^I e^{i\gamma}) \text{Re}(Y^J e^{i\gamma}). \end{aligned} \quad (2.17)$$

Using $(P_I^3)' = q'^\alpha \nabla_\alpha P_I^3$, the flow equation for q^α and the relation (B.17), we obtain that the second line of (2.17) vanishes. It follows that

$$\gamma' + 2\text{Im}(C_I Y^I e^{i\gamma}) = 0, \quad (2.18)$$

provided $\text{Re}(q_I Y^I) \neq 0$. Using $\Lambda^- = 0$, we see that equation (2.18) is precisely the flow equation for γ derived above.

Using the expression for A_t^I in (2.8), we note that the flow equations for U and ψ in (2.9) can also be written as

$$\begin{aligned} U' &= \text{Re}(Y^I C_I e^{i\gamma}) - \text{Im}(Y^I P_I^3 e^{i\gamma}), \\ \psi' &= -2 \text{Im}(Y^I P_I^3 e^{i\gamma}). \end{aligned} \quad (2.19)$$

This concludes the discussion of the first-order flow equations derived from (2.5). However, in addition to analyzing the conditions arising from the one-dimensional effective Lagrangian (2.5), one has to make sure that also the Hamiltonian constraint is fulfilled. By using the line element (2.1), the Hamiltonian constraint turns out to coincide with the equation that is obtained by varying the effective Lagrangian with respect to U . Thus, the Hamiltonian constraint is satisfied.

The flow equations (2.9), the Maxwell equation (2.13) and conditions (2.11) were first derived in the BPS context in [31]. They can be generalized to first-order flow equations

for non-supersymmetric configurations in the following way. Inspection of (B.6) shows that if one rotates (E^I, C_I) by a constant invertible matrix M according to $(E^I, C_I) \rightarrow ((ME)^I, ((M^{-1})^T C)_I)$ such that $M^T \text{Im}\mathcal{N} M = \text{Im}\mathcal{N}$, one obtains a similar rewriting of the one-dimensional effective Lagrangian, but now based on $q_I = e^{i\gamma} (((M^{-1})^T C)_I - iP_I^3)$. This allows one to obtain non-supersymmetric configurations from supersymmetric ones, as first observed in [36]. We will return to this observation in section 4 when discussing interpolating solutions.

2.2 Adding magnetic charge density

Now we add magnetic charges to the configuration specified by the line element (2.1). This configurations will thus be supported by flowing scalar fields $Y^I(r), q^\alpha(r)$ and by electric fields $E^I(r) = F_{tr}^I = -\partial_r A_t^I$ as well as magnetic fields $F_{xy}^I = p^I$, where the p^I are constant parameters that correspond to magnetic charge densities. The associated non-vanishing gauge potentials are

$$\left(A_t^I, A_x^I = -\frac{1}{2} p^I y, A_y^I = \frac{1}{2} p^I x \right). \quad (2.20)$$

Using (2.1), we then obtain for (A.11),

$$\begin{aligned} h_{\alpha\beta} D_\mu q^\alpha D^\mu q^\beta &= e^{-2A} h_{\alpha\beta} q'^\alpha q'^\beta + e^{-2U} h_{\alpha\beta} (k_I^\alpha A_t^I)(k_J^\beta A_t^J) \\ &\quad + \frac{1}{4}(y^2 + x^2) e^{-2A} h_{\alpha\beta} (k_I^\alpha p^I)(k_J^\beta p^J), \end{aligned} \quad (2.21)$$

where $q'^\alpha = \partial q^\alpha / \partial r$. Observe that in the presence of the magnetic charge densities p^I , there are (x, y) -dependent terms in (2.21), which can be removed by imposing

$$p^I k_I^\alpha = 0. \quad (2.22)$$

We will thus impose this constraint in the following.

Let us denote the combination $C_I e^{2A}$ appearing on the left hand side of (2.13) by Q_I . It is well known [20] that in the presence of magnetic charge densities, the term $-\frac{1}{2} Q_I (\text{Im}\mathcal{N})^{IJ} Q_J$ in the Maxwell Lagrangian (B.6) gets extended to

$$V_{\text{BH}} = g^{A\bar{B}} \mathcal{D}_A Z \bar{\mathcal{D}}_{\bar{B}} \bar{Z} + |Z|^2 = (N^{IJ} + 2X^I \bar{X}^J) \hat{Q}_I \bar{\hat{Q}}_J, \quad (2.23)$$

where

$$Z = -\hat{Q}_I X^I, \quad \hat{Q}_I = Q_I - F_{IJ} p^J. \quad (2.24)$$

Hence, we may readily incorporate magnetic charge densities by replacing C_I in (2.4) with

$$\hat{C}_I = C_I - F_{IJ} p^J e^{-2A}. \quad (2.25)$$

The rewriting of the one-dimensional effective action proceeds as before, but now in terms of

$$q_I = e^{i\gamma} \left(\hat{C}_I - iP_I^3 \right). \quad (2.26)$$

The electric field E^I becomes expressed in terms of C_I and p^I as

$$E^I = -e^{U+A} (\text{Im}\mathcal{N})^{IJ} (C_J + \text{Re}\mathcal{N}_{JK} p^K e^{-2A}), \quad (2.27)$$

cf. appendix C for a few more details on the derivation. The quantity Δ acquires an additional term, namely $-e^\psi P_I^3 p^I$. Varying this term with respect to ψ yields the constraint

$$P_I^3 p^I = 0. \quad (2.28)$$

The quantities \tilde{A}_t^I and Λ^\pm are given by (2.8), with C_I replaced by \hat{C}_I .

The resulting flow equations and constraints are summarized as follows,

$$\begin{aligned} U' &= \text{Re} \left(Y^I \hat{C}_I e^{i\gamma} \right) - \text{Im} \left(Y^I P_I^3 e^{i\gamma} \right), \\ \psi' &= -2 \text{Im} \left(Y^I P_I^3 e^{i\gamma} \right), \\ A' &= -\text{Re} \left(Y^I q_I \right), \\ Y'^I &= e^{2A} N^{IK} \bar{q}_K, \\ q'^\alpha &= 2 h^{\alpha\beta} \text{Im} \left(e^{i\gamma} Y^I \right) \nabla_\beta P_I^3, \\ E^I &= -e^{U+A} (\text{Im} \mathcal{N})^{IJ} \left(C_J + \text{Re} \mathcal{N}_{JK} p^K e^{-2A} \right), \\ (C_I e^{2A})' &= 2e^{2A} h_{\alpha\beta} k_I^\alpha k_J^\beta \text{Re} \left(Y^J e^{i\gamma} \right), \\ A_t^I &= -2e^{U-A} \text{Re} \left(Y^I e^{i\gamma} \right), \\ \gamma' &= -\Lambda^+, \\ \Sigma_r &= \Lambda^- = 0, \\ p^I k_I^\alpha &= 0, \\ p^I P_I^3 &= 0, \\ P_I^1 &= P_I^2 = 0, \end{aligned} \quad (2.29)$$

with q_I given by (2.26), and with Λ^\pm given by (2.8), with C_I replaced by \hat{C}_I . As before, the conditions $\Sigma_r = 0$ and $\Lambda^- = 0$ are equivalent, and the latter implies the flow equation for γ provided $\text{Re}(q_I Y^I) \neq 0$. The consistency of $E^I = -\partial_r A_t^I$ now imposes the following relation,

$$\begin{aligned} (\text{Im} \mathcal{N})^{IJ} \left(C_J + \text{Re} \mathcal{N}_{JK} p^K e^{-2A} \right) &= -2e^{-2A} \left[(U' - A') \text{Re} \left(Y^I e^{i\gamma} \right) \right. \\ &\quad \left. + \text{Re} \left(Y'^I e^{i\gamma} \right) - \gamma' \text{Im} \left(Y^I e^{i\gamma} \right) \right]. \end{aligned} \quad (2.30)$$

The above equations were first derived in [31] in the BPS context.

3 Backgrounds

Next, we consider solutions to the equations (2.29). We focus on solutions that are supported by constant scalars \tilde{X}^I and q^α . The scalar \tilde{X}^I is related to X^I by (A.6).

3.1 Lifshitz backgrounds

The Lifshitz line element is specified by

$$e^U = r^{-z}, \quad e^A = r^{-1}, \quad (3.1)$$

such that the center is at $r \rightarrow \infty$ and the asymptotic region is at $r \rightarrow 0$. Below we will show that $z \geq 1$.

We consider backgrounds that are supported by electric fields ($p^I = 0$). Inspection of (2.8) yields

$$A_t^I = -2r^{-z} \operatorname{Re} \left(\tilde{X}^I e^{i\gamma} \right). \quad (3.2)$$

Inserting these expressions into the flow equation for U yields

$$z = \operatorname{Re} \left(q_I \tilde{X}^I \right) - 2\operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) = -\operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) + \operatorname{Im} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right). \quad (3.3)$$

The flow equation for A yields

$$\operatorname{Re} \left(q_I \tilde{X}^I \right) = 1, \quad (3.4)$$

and hence

$$z = 1 - 2\operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right). \quad (3.5)$$

Since z is constant, we also take C_I and γ to be constant, in which case q_I is also constant. This is consistent with the flow equation for Y^I ,

$$\tilde{X}^I = -N^{IJ} \bar{q}_J. \quad (3.6)$$

Taking the real part of $\tilde{X}^I e^{i\gamma}$ in (3.6) gives

$$\operatorname{Re} \left(\tilde{X}^I e^{i\gamma} \right) = -N^{IJ} C_J, \quad (3.7)$$

and hence, using (3.2), we obtain

$$A_t^I = 2r^{-z} N^{IJ} C_J. \quad (3.8)$$

Since we take γ to be constant, we have $\Lambda_+ = 0$ which, together with $\Lambda^- = 0$, results in

$$\begin{aligned} \operatorname{Im} \left(C_I \tilde{X}^I e^{i\gamma} \right) &= 0, \\ \operatorname{Re} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) &= 0. \end{aligned} \quad (3.9)$$

Inspection of (2.30) then yields

$$(\operatorname{Im} \mathcal{N})^{IJ} C_J = 2z \operatorname{Re} \left(\tilde{X}^I e^{i\gamma} \right), \quad (3.10)$$

Contracting with P_I^3 we obtain,

$$P_I^3 (\operatorname{Im} \mathcal{N})^{IJ} C_J = 0 \quad (3.11)$$

due to (3.9).

Using the equation for $(C_I e^{2A})'$ in (2.29) we obtain the relation

$$C_I = -h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Re} \left(\tilde{X}^J e^{i\gamma} \right). \quad (3.12)$$

Contracting (3.12) with $\tilde{X}^I e^{i\gamma}$ results in

$$\begin{aligned} h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Im} \left(\tilde{X}^I e^{i\gamma} \right) \operatorname{Re} \left(\tilde{X}^J e^{i\gamma} \right) &= 0, \\ \operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) &= -h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Re} \left(\tilde{X}^I e^{i\gamma} \right) \operatorname{Re} \left(\tilde{X}^J e^{i\gamma} \right), \end{aligned} \quad (3.13)$$

where we used (3.9). Combining with (3.5), we obtain

$$z = 1 + 2h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Re} \left(\tilde{X}^I e^{i\gamma} \right) \operatorname{Re} \left(\tilde{X}^J e^{i\gamma} \right) \geq 1. \quad (3.14)$$

Contracting (3.6) with C_I we obtain

$$C_I \tilde{X}^I e^{i\gamma} = -C_I N^{IJ} (C_J + iP_J^3). \quad (3.15)$$

Using (3.9), (3.11) and (A.4), we obtain

$$\begin{aligned} C_I N^{IJ} P_J^3 &= - \left(C_I \tilde{X}^I P_J^3 \tilde{X}^{\bar{J}} + P_I^3 \tilde{X}^I C_J \tilde{X}^{\bar{J}} \right) \\ &= -\operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) \left[\operatorname{Im} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) + \operatorname{Im} \left(P_I^3 \tilde{X}^I e^{-i\gamma} \right) \right] = 0. \end{aligned} \quad (3.16)$$

Hence

$$\operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) = -C_I N^{IJ} C_J. \quad (3.17)$$

Similarly, we infer

$$\operatorname{Im} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) = -P_I^3 N^{IJ} P_J^3. \quad (3.18)$$

Thus, we may write (3.4) as

$$\operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) + \operatorname{Im} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) = -C_I N^{IJ} C_J - P_I^3 N^{IJ} P_J^3 = 1. \quad (3.19)$$

And finally, combining (3.7) and (3.10) gives

$$(z-1) N^{IJ} C_J = \tilde{X}^I \left(C_J \tilde{X}^{\bar{J}} \right) + \tilde{X}^{\bar{I}} \left(C_J \tilde{X}^J \right). \quad (3.20)$$

The constancy of q^α requires

$$\operatorname{Im} \left(\tilde{X}^I e^{i\gamma} \right) \frac{\partial P_I^3}{\partial q^\beta} = 0. \quad (3.21)$$

Summarizing, the Lifshitz backgrounds we constructed are described by the following conditions,

$$\begin{aligned} \tilde{X}^I &= \text{const}, & q^\alpha &= \text{const}, & C_I &= \text{const}, & p^I &= 0, & \gamma &= \text{const}, \\ e^U &= r^{-z}, & e^A &= r^{-1}, \\ \tilde{X}^I &= -N^{IJ} \bar{q}_J, \\ C_I &= -h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Re} \left(\tilde{X}^J e^{i\gamma} \right), \\ A_t^I &= 2r^{-z} N^{IJ} C_J, \\ z &= 1 - 2 \operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) = 1 + 2h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Re} \left(\tilde{X}^I e^{i\gamma} \right) \operatorname{Re} \left(\tilde{X}^J e^{i\gamma} \right) \geq 1, \\ (z-1) N^{IJ} C_J &= \tilde{X}^I \left(C_J \tilde{X}^{\bar{J}} \right) + \tilde{X}^{\bar{I}} \left(C_J \tilde{X}^J \right), \\ 0 &= h_{\alpha\beta} k_I^\alpha k_J^\beta \operatorname{Im} \left(\tilde{X}^I e^{i\gamma} \right) \operatorname{Re} \left(\tilde{X}^J e^{i\gamma} \right), \\ C_I N^{IJ} P_J^3 &= 0, \quad \operatorname{Im} \left(C_I \tilde{X}^I e^{i\gamma} \right) = 0, \quad \operatorname{Re} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) = 0, \quad \operatorname{Im} \left(\tilde{X}^I e^{i\gamma} \right) \frac{\partial P_I^3}{\partial q^\alpha} = 0, \\ \operatorname{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) + \operatorname{Im} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) &= -C_I N^{IJ} C_J - P_I^3 N^{IJ} P_J^3 = 1. \end{aligned} \quad (3.22)$$

In the BPS case, conditions for Lifshitz backgrounds have been discussed before in [28, 29].

Note that while there is a lower bound on the allowed value of z , namely $z \geq 1$, there does not appear to be an upper bound. In the following, we turn to specific examples with $z = 2$.

3.1.1 Examples

On the hypermultiplet side, we now restrict to the universal hypermultiplet. The associated scalars parametrize the coset space $SU(2, 1)/SU(2) \times U(1)$. This is a Kähler manifold, and we employ two standard parametrizations of this manifold in terms of complex coordinates, which we summarize in appendix D. We denote these complex coordinates by $q^i, \bar{q}^i, i = 1, 2$. We focus on solutions that are supported by constant scalars \tilde{X}^I and q^i .

In the first example, we choose the parametrization (D.1), and consider the gauging of the following two isometries [29, 37],

$$k_I = ia_I \left(q^1 \frac{\partial}{\partial q^1} - \bar{q}^1 \frac{\partial}{\partial \bar{q}^1} \right) + ib_I \left(q^2 \frac{\partial}{\partial q^2} - \bar{q}^2 \frac{\partial}{\partial \bar{q}^2} \right), \quad (3.23)$$

whose associated Killing prepotentials are $P_I^x = a_I P_a^x + b_I P_b^x$, with P_a, P_b given by

$$P_a = \frac{-2}{(|q^1|^2 + |q^2|^2)\sqrt{1 - |q^1|^2 - |q^2|^2}} \begin{pmatrix} -\text{Im}(q^1 q^2) \\ \text{Re}(q^1 q^2) \\ \frac{-|q^2|^4 + |q^2|^2 - |q^1|^2(1 + |q^2|^2)}{2\sqrt{1 - |q^1|^2 - |q^2|^2}} \end{pmatrix} \quad (3.24)$$

and

$$P_b = \frac{-2}{(|q^1|^2 + |q^2|^2)\sqrt{1 - |q^1|^2 - |q^2|^2}} \begin{pmatrix} \text{Im}(q^1 q^2) \\ -\text{Re}(q^1 q^2) \\ \frac{-|q^1|^4 + |q^1|^2 - |q^2|^2(1 + |q^1|^2)}{2\sqrt{1 - |q^1|^2 - |q^2|^2}} \end{pmatrix}. \quad (3.25)$$

We keep q^2 fixed at $q^2 = 0$, so that $P_I^1 = P_I^2 = 0$. At $q^2 = 0$, P_I^3 is then given by

$$P_I^3 = \frac{a_I}{1 - |q^1|^2} - b_I. \quad (3.26)$$

On the vector multiplet side, we consider the STU-model with prepotential $F(X) = -X^1 X^2 X^3 / X^0$ with X^0 real and X^A imaginary ($A = 1, 2, 3$). For this case, the fields $S = -iX^1/X^0, T = -iX^2/X^0, U = -iX^3/X^0$ are all real, and the matrix N^{IJ} is given by

$$N^{IJ} = \frac{1}{4STU} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & S^2 & -ST & -SU \\ 0 & -ST & T^2 & -TU \\ 0 & -SU & -TU & U^2 \end{pmatrix}. \quad (3.27)$$

We set $\gamma = 0$ and take non-vanishing charges and fluxes $(C_0; a_0, b_0, b_1, b_2, b_3)$. From $C_I N^{IJ} P_J^3 = 0$ we obtain

$$C_0 N^{00} P_0^3 = 0 \rightarrow P_0^3 = \frac{a_0}{1 - |q^1|^2} - b_0 = 0, \quad (3.28)$$

which implies $b_0 > a_0$ and expresses the value of $|q^1|$ in terms of fluxes a_0, b_0 . Since $k_I^i \text{Im}(\tilde{X}^I e^{i\gamma}) = 0$, the equation $h_{\alpha\beta} k_I^\alpha k_J^\beta \text{Im}(\tilde{X}^I e^{i\gamma}) \text{Re}(\tilde{X}^J e^{i\gamma}) = 0$ is satisfied, and so are the equations

$$\text{Im}(\tilde{X}^I e^{i\gamma}) \frac{\partial P_I^3}{\partial q^i} = 0, \quad \text{Im}(C_I \tilde{X}^I e^{i\gamma}) = 0, \quad \text{Re}(P_I^3 \tilde{X}^I e^{i\gamma}) = 0. \quad (3.29)$$

The equation

$$\tilde{X}^0 = -N^{0J} \bar{q}_J = -N^{00} C_0, \quad (3.30)$$

when combined with $C_I = -h_{1\bar{1}} |q^1|^2 a_I a_J \text{Re}(\tilde{X}^J e^{i\gamma})$, results in

$$h_{1\bar{1}} |q^1|^2 a_0^2 N^{00} = 1, \quad (3.31)$$

which relates the value of $|q^1|$ to the values of the scalar fields of the vector multiplets.

The value of z is given by

$$z = 1 + 2C_0^2 N^{00}. \quad (3.32)$$

This satisfies equation (3.20).

The equation

$$\tilde{X}^A = -N^{AJ} \bar{q}_J = iN^{AB} b_B, \quad A = 1, 2, 3. \quad (3.33)$$

when combined with (3.30), results in

$$T^A = -i \frac{\tilde{X}^A}{\tilde{X}^0} = -\frac{N^{AB} b_B}{N^{00} C_0}. \quad (3.34)$$

This yields

$$S = \frac{C_0}{b_1}, \quad T = \frac{C_0}{b_2}, \quad U = \frac{C_0}{b_3}. \quad (3.35)$$

Requiring $S, T, U > 0$ leads to

$$\frac{C_0}{b_A} > 0. \quad (3.36)$$

The equation

$$\text{Re}(C_I \tilde{X}^I e^{i\gamma}) + \text{Im}(P_I^3 \tilde{X}^I e^{i\gamma}) = -C_I N^{IJ} C_J - P_I^3 N^{IJ} P_J^3 = 1 \quad (3.37)$$

gives

$$C_0^2 N^{00} = -1 - b_A N^{AB} b_B. \quad (3.38)$$

Inserting (3.35) into (3.38) gives

$$C_0 = \frac{b_1 b_2 b_3}{2}, \quad (3.39)$$

so that

$$C_0^2 N^{00} = \frac{1}{2}, \quad (3.40)$$

and hence⁴

$$z = 2. \quad (3.41)$$

⁴We have checked that $z = 2$ also holds when all the fluxes (a_I, b_I) are turned on.

Using the metric (D.1), the relations (3.28) and (3.31) result in the relation

$$b_0(b_0 - a_0) = 4 \frac{C_0^3}{b_1 b_2 b_3}, \quad (3.42)$$

which is consistent with the requirements $b_0 > a_0$ and (3.36). Thus, we have constructed a new $z = 2$ Lifshitz background.

Next, we consider a different example, based on the parametrization (D.4) (with $q^i = S, \xi$), and we gauge the following two isometries [29],

$$k_I = a_I i \left(\frac{\partial}{\partial S} - \frac{\partial}{\partial \bar{S}} \right) + b_I i \left(\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right). \quad (3.43)$$

The associated Killing prepotentials are $P_I^x = a_I P_a^x + b_I P_b^x$ with

$$P_a^x = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2\rho} \end{pmatrix}, \quad P_b^x = \begin{pmatrix} \frac{\xi + \bar{\xi}}{\sqrt{\rho}} \\ -i \frac{\xi - \bar{\xi}}{\sqrt{\rho}} \\ \frac{|\xi|^2}{\rho} - 1 \end{pmatrix}. \quad (3.44)$$

We keep $q^2 = \xi$ fixed at $q^2 = 0$, so that $P_I^1 = P_I^2 = 0$. At $q^2 = 0$, P_I^3 is then given by

$$P_I^3 = -\frac{a_I}{2\rho} - b_I. \quad (3.45)$$

On the vector multiplet side, we consider the t^3 -model with prepotential $F(X) = -(X^1)^3/X^0$. We set $t = X^1/X^0 = t_1 + it_2$. The matrix N^{IJ} is given by

$$N^{IJ} = \frac{1}{4t_2^3} \begin{pmatrix} 1 & t_1 \\ t_1 & t_1^2 - \frac{1}{3}t_2^2 \end{pmatrix}. \quad (3.46)$$

We set γ to a constant, but arbitrary, value, and we take non-vanishing charges and fluxes $(C_0; a_0, b_1)$. The equation $\tilde{X}^I = -N^{IJ} \bar{q}_J$ results in

$$\begin{aligned} e^{i\gamma} \tilde{X}^0 &= -\frac{1}{4t_2^3} \left(C_0 - i \frac{a_0}{2\rho} - it_1 b_1 \right), \\ e^{i\gamma} \tilde{X}^1 &= -\frac{1}{4t_2^3} \left(t_1 \left(C_0 - i \frac{a_0}{2\rho} \right) - i \left(t_1^2 - \frac{1}{3}t_2^2 \right) b_1 \right). \end{aligned} \quad (3.47)$$

Imposing $\text{Im} \left(C_I \tilde{X}^I e^{i\gamma} \right) = 0$, we obtain

$$\frac{a_0}{2\rho} + t_1 b_1 = 0, \quad (3.48)$$

which determines the value of ρt_1 in terms of the fluxes. Thus, $e^{i\gamma} \tilde{X}^0$ is real. Using (3.48), we determine from (3.47)

$$t = \frac{t_1 \left(C_0 - i \frac{a_0}{2\rho} \right) - i \left(t_1^2 - \frac{1}{3}t_2^2 \right) b_1}{C_0} = t_1 + i \frac{t_2^2 b_1}{3C_0}, \quad (3.49)$$

from which we conclude

$$t_2 = \frac{3C_0}{b_1}, \tag{3.50}$$

which we demand to be positive, and hence

$$\frac{C_0}{b_1} > 0. \tag{3.51}$$

The condition $\text{Re} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) = 0$ is satisfied by virtue of (3.48), and so are the conditions $C_I N^{IJ} P_J^3 = 0$, $k_I^\alpha \text{Im} \left(\tilde{X}^I e^{i\gamma} \right) = 0$ and $\text{Im} \left(\tilde{X}^I e^{i\gamma} \right) \partial P_I^3 / \partial q^i = 0$. The condition $\text{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) + \text{Im} \left(P_I^3 \tilde{X}^I e^{i\gamma} \right) = 1$ gives

$$t_2^3 = \frac{1}{2} C_0^2, \tag{3.52}$$

where we used (3.50).

The equation for C_0 results in

$$C_0 = -h_{S\bar{S}} a_0^2 \text{Re} \left(\tilde{X}^0 e^{i\gamma} \right). \tag{3.53}$$

Using $h_{S\bar{S}} = 1/(4\rho^2)$ this results in

$$\rho^2 = \frac{a_0^2}{16t_2^3}, \tag{3.54}$$

which combined with (3.52) gives

$$\rho^2 = \frac{a_0^2}{8C_0^2}. \tag{3.55}$$

Thus, the only scalar whose value remains unfixed is the hyper scalar $S - \bar{S}$.

Finally, we compute the value of z ,

$$z = 1 - 2 \text{Re} \left(C_I \tilde{X}^I e^{i\gamma} \right) = 1 + \frac{C_0^2}{2t_2^3} = 1 + 1 = 2, \tag{3.56}$$

and we check that the condition (3.20) is satisfied.

This reproduces the BPS background constructed in [28].

3.2 AdS backgrounds

A possible way for resolving the tidal force singularity consists in searching for solutions that flow from AdS_4 to $AdS_2 \times \mathbb{R}^2$ passing through an intermediate Lifshitz region. Numerical flows of this kind have, for instance, been studied in the context of Einstein-Maxwell-dilaton systems with loop or Weyl-squared corrections in [38, 39]. Thus, it is of interest to investigate the construction of AdS backgrounds in the models at hand. To this end, we return to the gauging (3.23) with Killing prepotential (3.26). On the vector multiplet side, we again consider the STU-model with prepotential $F(X) = -X^1 X^2 X^3 / X^0$.

We first consider AdS_4 backgrounds. These are backgrounds satisfying (3.22) with $z = 1$ and $C_I = 0$, and supported by arbitrary fluxes (a_I, b_I) . The condition $\tilde{X}^I = -N^{IJ} \bar{q}_J$

then gives $\text{Re}(\tilde{X}^I e^{i\gamma}) = 0$. This, in turn, implies that the physical moduli S, T, U are all purely imaginary, which does not correspond to a sensible solution in supergravity. Thus, we cannot construct AdS_4 backgrounds in this model.⁵

Next, let us consider $AdS_2 \times \mathbb{R}^2$ backgrounds. These are solutions to the equations (2.29) with constant scalar fields Y^I and q^α . The associated line element is specified by

$$U = r, \quad A = \text{constant}. \quad (3.57)$$

Setting $Y'^I = 0$ gives $q_I = 0$, which results in $\hat{C}_I = iP_I^3$, and hence

$$\begin{aligned} C_I e^{2A} &= \frac{1}{2} (F_{IJ} + \bar{F}_{IJ}) p^J, \\ P_I^3 e^{2A} &= -\frac{1}{2} N_{IJ} p^J. \end{aligned} \quad (3.58)$$

The value of e^{2A} is given by

$$e^{2A} = -N_{IJ} Y^I \bar{Y}^J. \quad (3.59)$$

The C_I are constant. Since $q_I = 0$, the condition $\Lambda^- = \text{Im}(q_I Y^I) = 0$ is automatically satisfied. Demanding $\gamma' = 0$, the condition $\Lambda^+ = 0$ yields

$$\text{Re}(e^{i\gamma} Y^I P_I^3) = 0. \quad (3.60)$$

The flow equation for ψ yields

$$\text{Im}(e^{i\gamma} Y^I P_I^3) = -\frac{1}{2}. \quad (3.61)$$

The constancy of q^α implies (3.21).

For the model at hand, we take the complex hyper scalars q^i to have values $q^2 = 0, q^1 \neq 0$. We seek axion free solutions, i.e. solutions for which Y^0 is real and Y^A are imaginary ($A = 1, 2, 3$), so that $S = -iY^1/Y^0, T = -iY^2/Y^0, U = -iY^3/Y^0$ are real. We choose non-vanishing charges (C_0, p^1, p^2, p^3) . Then, the attractor equations (3.58) yield

$$\begin{aligned} C_0 e^{2A} &= -(TU p^1 + SU p^2 + ST p^3), \\ P_A^3 e^{2A} &= -\frac{1}{2} N_{AB} p^B, \quad A, B = 1, 2, 3, \\ P_0^3 &= 0. \end{aligned} \quad (3.62)$$

Taking $q^1 \neq 0$, the condition $p^I k_I^\alpha = 0$ yields $p^I a_I = 0$. The condition $p^I P_I^3 = 0$ then gives $p^I b_I = 0$. The condition (3.21) results in

$$a_I \text{Im}(e^{i\gamma} Y^I) = 0. \quad (3.63)$$

Requiring $C'_I = 0$ gives

$$a_I \text{Re}(e^{i\gamma} Y^I) = 0, \quad (3.64)$$

⁵This is similar to the findings of [29] where also no AdS_4 solutions were found with only electric gaugings. Note, however, that there are AdS_4 solutions with purely electric gaugings in a duality frame with a square root prepotential, cf. [31, 40].

cf. (2.12) with $q_J = 0$. We take $\gamma = 0$ in the following. Then, since the Y^A are imaginary, the condition (3.64) is satisfied provided $a_0 = 0$. It follows from $P_0^3 = 0$ that then also $b_0 = 0$. Then, the conditions $p^I a_I = p^I b_I = 0$ and equations (3.60), (3.61) and (2.30) result in

$$\begin{aligned} p^1 a_1 + p^2 a_2 + p^3 a_3 &= 0, \\ p^1 b_1 + p^2 b_2 + p^3 b_3 &= 0, \\ \text{Re}(Y^A b_A) &= 0, \\ \text{Im}(Y^A b_A) &= \frac{1}{2}, \\ (\text{Im}\mathcal{N})^{IJ} (C_J e^{2A} + \text{Re}\mathcal{N}_{JK} p^K) &= -2 \text{Re}(Y^I e^{i\gamma}). \end{aligned} \tag{3.65}$$

These equations, combined with (3.62) and (3.63), can be consistently solved. For instance, taking $a_2 = a_3 = -1, b_2 = -1, b_3 = 1, p^3 = 0.111$ and setting $Y^0 = 1$, we obtain

$$a_1 \approx 0.996 b_1, \quad p^1 \approx -56.270/b_1, \quad p^2 \approx -56.159 \tag{3.66}$$

and

$$|q^1| \approx 0.044, \quad S \approx 42.328/b_1, \quad T \approx 41.995, \quad U \approx 0.167, \quad C_0 = \frac{1}{4}, \tag{3.67}$$

where b_1 remains unfixed. Hence, we have constructed a new background.

Thus, we see that in the model at hand, when $\gamma = 0$, it is not possible to construct an $AdS_2 \times \mathbb{R}^2$ background carrying the same fluxes as the Lifshitz backgrounds constructed above which necessarily have $a_0 \neq 0$, cf. (3.31). This does not exclude the existence of both types of backgrounds but with different values of γ . An interpolating solution between two such vacua would then require a varying $\gamma(r)$.⁶ Rather than pursuing this question, we turn to a different mechanism for cloaking the singularity at the center of the Lifshitz background.

4 Interpolating between $z = 2$ Lifshitz and Nernst geometries

Lifshitz backgrounds with $z = 2$ can be obtained from five dimensions by dimensional Scherk-Schwarz reduction [17]. Let us thus consider the $z = 2$ Lifshitz background of the STU-model described above (3.41) and lift it to five dimensions. For convenience we set $b_1 b_2 b_3 = 2$, so that $C_0 = 1$.⁷ Then, the four-dimensional line element and the gauge potential 1-form $A^0 = A_t^0 dt$ read

$$\begin{aligned} ds_4^2 &= -\frac{dt^2}{r^4} + \frac{dr^2 + dx^2 + dy^2}{r^2}, \\ A_t^0 &= \frac{1}{r^2}. \end{aligned} \tag{4.1}$$

⁶A non-supersymmetric solution in 5d gauged supergravity interpolating between asymptotic AdS_5 and Lif_3 in the center was given in [41].

⁷The following analysis can, however, easily be generalized to other values of the fluxes b_1, b_2, b_3 .

This lifts to a solution in five dimensions with line element [17]

$$ds_5^2 = ds_4^2 + (du - A^0)^2 = \frac{dr^2}{r^2} + \frac{(-2dt du + dx^2 + dy^2)}{r^2} + du^2, \quad (4.2)$$

where u denotes the coordinate of the S^1 along which the Scherk-Schwarz reduction is performed. The five-dimensional vector scalars X_5^A supporting the solution are constant, and their values are given by $X_5^A = -iY^A/Y^0$ given in (3.35). The Proca mass term for A^0 in four dimensions arises by Scherk-Schwarz reduction of one of the hyper scalars in five dimension. Furthermore, the solution is uncharged in five dimensions.

We now deform the line element in five dimensions as follows,

$$ds_5^2 = \frac{dr^2}{r^2} + \frac{(-2dt du + dx^2 + dy^2)}{r^2} + f^{-2}(r) du^2, \quad (4.3)$$

which we write as

$$ds_5^2 = f \left(-\frac{f dt^2}{r^4} + \frac{dr^2 + dx^2 + dy^2}{f r^2} \right) + f^{-2} \left(du - \frac{f^2}{r^2} dt \right)^2. \quad (4.4)$$

Then, dimensionally reducing along the S^1 parametrized by u , we obtain

$$ds_4^2 = -\frac{f dt^2}{r^4} + \frac{dr^2 + dx^2 + dy^2}{f r^2}, \quad (4.5)$$

$$A_t^0 = \frac{f^2}{r^2}, \quad (4.6)$$

with the four-dimensional vector scalars $z^A = Y^A/Y^0$ now given by

$$z^A = z_u^A / f, \quad (4.7)$$

where z_u^A denote the values of the vector scalars in the undeformed case corresponding to (3.35). The values of the four-dimensional hyper scalars q^i are not deformed. Then the four-dimensional equations of motion are solved provided that f is given by the following two-parameter family of deformations,

$$f^2(r) = \frac{r^2}{r^2 + \alpha r^4 + \beta}, \quad \alpha, \beta \in \mathbb{R}. \quad (4.8)$$

Note that this four-dimensional solution does not satisfy the first-order flow equations (2.29), and hence, is not supersymmetric. The five-dimensional metric (4.3) with f given in (4.8) was first written down in [32].⁸ Here we would like to investigate what the reduced four-dimensional metric (4.5) implies for our goal to cloak the Lifshitz tidal force singularity within $\mathcal{N} = 2$ gauged supergravity.

To this end, let us consider the behaviour of the deformed solution (4.5). First, consider the case $\alpha \neq 0, \beta = 0$. At $r \rightarrow 0$ (i.e. asymptotically), we have $f \rightarrow 1$, and the solution approaches the $z = 2$ Lifshitz solution described above (3.41). On the other hand, as

⁸Metrics with a similar structure also appear in [16, 42], where, however, the corresponding function f does not depend on r , but on the analog of our coordinate u along which one performs the reduction.

$r \rightarrow \infty$ (i.e. at the center), we have $f^2 \rightarrow (\alpha r^2)^{-1}$, and hence α must be taken to be positive. As $r \rightarrow \infty$, the line element and the vector scalars z^A asymptote to

$$\begin{aligned} ds_4^2 &= -\frac{dt^2}{\sqrt{\alpha} r^5} + \sqrt{\alpha} \frac{dr^2 + dx^2 + dy^2}{r}, \\ z^A &= z_u^A \sqrt{\alpha} r. \end{aligned} \tag{4.9}$$

By changing the radial coordinate, $r = c/\sqrt{\tilde{r}}$ with $c = 4/(\alpha)^{1/2}$, and by rescaling dt, dx and dy , the deformed line element is seen to describe the near-horizon geometry of the Nernst brane constructed in [7],

$$\begin{aligned} ds_4^2 &= -\tilde{r}^{5/2} dt^2 + \frac{d\tilde{r}^2}{\tilde{r}^{5/2}} + \tilde{r}^{1/2} (dx^2 + dy^2), \\ z^A &= 4z_u^A \tilde{r}^{-1/2}. \end{aligned} \tag{4.10}$$

Next, let us consider the case $\alpha = 0, \beta \neq 0$. When $r \rightarrow \infty$, we have $f \rightarrow 1$, and hence we recover the Lifshitz background described above (3.41). However, this is where the Lifshitz metric exhibits a tidal force singularity, and hence the interpolating solution does not help in addressing this issue. We will thus discard this case.

Finally, consider the case $\alpha \neq 0, \beta \neq 0$. As $r \rightarrow \infty$, the solution has the Nernst behavior (4.10). When $r \rightarrow 0$, f approaches $f^2 \rightarrow r^2/\beta$, and hence β has to be positive. In this limit we then obtain

$$\begin{aligned} ds_4^2 &= r^{-3} \left(-\frac{dt^2}{\sqrt{\beta}} + \sqrt{\beta} (dr^2 + dx^2 + dy^2) \right), \\ z^A &= z_u^A \sqrt{\beta} r^{-1}. \end{aligned} \tag{4.11}$$

This line element describes a conformal AdS geometry. Thus, we have an interpolating solution that interpolates between a conformal AdS geometry and the near-horizon geometry of a Nernst brane.

To summarize, (4.5)–(4.8) constitute a two-parameter class of Nernst brane backgrounds (for $\alpha \neq 0$) which do not obey the first-order flow equations (2.29), and hence are non-supersymmetric. In fact, they also do not solve the non-supersymmetric flow equations based on a constant matrix M mentioned at the end of section 2.1. We show this in appendix E. The near-horizon geometry for these non-supersymmetric Nernst branes is exactly the same as the one of the supersymmetric Nernst brane of [7], indicating that this might be a universal feature of Nernst backgrounds, at least of those arising from gauged supergravity. We note that also the η geometries of [43] can be embedded in gauged supergravity, cf. [44]. For $\eta > 0$ they also have the feature that the entropy vanishes in the zero temperature limit, according to $s \sim T^\eta$. However, in contrast to our Nernst brane solutions which have vanishing Ricci and Kretschmann scalar at the horizon but suffer from a tidal force singularity, η geometries with $\eta > 0$ have curvature singularities at the horizon.⁹

⁹For example, the near-horizon behavior of the $\eta = 1$ geometry is schematically given by $ds^2 = -r^a dt^2 + r^{-a} dr^2 + r^{1/2} (dx^2 + dy^2)$ with $a = 3/2$ instead of the $a = 5/2$ behavior of the Nernst brane, cf. (4.10). It is easy to show that for this metric the Ricci and Kretschmann scalars diverge at $r = 0$ for $a < 2$ (i.e. for the $\eta = 1$ geometry) whereas they vanish for $a > 2$ (i.e. for the Nernst brane geometry).

Thus, the η geometries for $\eta > 0$ behave more like small black holes which also have a curvature singularity at the horizon. In the non-supersymmetric case discussed here, the near-horizon Nernst background turns out to be the common IR fixed point of flows with two different types of UV behaviors corresponding to Lifshitz ($\beta = 0$) and conformal AdS ($\beta \neq 0$) backgrounds.¹⁰

Both these backgrounds have divergent scalar fields (in addition to the tidal force singularity at the center), and this indicates that the correct frame to view them is in five dimensions, as was also the case for the supersymmetric Nernst brane [7] whose five-dimensional uplift was discussed in [45]. When lifted up to five dimensions, the Lifshitz background becomes a Poincaré breaking solution, while the conformal AdS background lifts up to AdS_5 , cf. eq. (4.4). We expect that the scalar fields can be regularized by heating up the system, as was the case for the non-extremal version of the supersymmetric Nernst brane derived in [8].

5 Conclusions

Lifshitz geometries are plagued with a tidal force singularity at their center, which needs to be dealt with, for instance by constructing interpolating solutions that flow away from Lifshitz to other solutions. Hence, the primary goal of this paper was to examine the space of solutions interpolating between non-relativistic vacua such as $z = 2$ Lifshitz and other solutions of supergravity. A special class of interpolating solutions, namely solitonic solutions that interpolate between a Lifshitz background and another *vacuum* solution, would be an ideal way to holographically model the RG flow from a Lifshitz fixed point in the UV of the dual field theory to a well-behaved IR fixed point. In order to look for such solitonic solutions we performed a first-order rewriting of the one-dimensional action, (B.3), with purely electric gaugings, in terms of a sum of squares. This is a generalization of a similar rewriting performed in [7] to the case with hypermultiplets. This constitutes an important step towards making contact with actual string theory embeddings where hypermultiplets in $\mathcal{N} = 2$ compactifications are omnipresent and $\mathcal{N} = 2$ gauged supergravities (with hypermultiplets) can be obtained in a straightforward manner via flux compactifications. Moreover, even though our first-order flow equations (2.29) are in complete agreement with the ones obtained in [31] via analyzing the supersymmetry conditions, our rewriting has the advantage to allow for a straightforward generalization to non-supersymmetric first-order flows, as briefly discussed at the end of section 2.1. It would be worthwhile exploring this further.

The first-order flow equations allow for Lifshitz backgrounds that are characterized by the set of equations given in (3.22). While there is a lower bound on the allowed value of z ($z \geq 1$), the equations do not appear to set an upper bound on z . The explicit examples we constructed have $z = 2$, but it would be interesting to see if examples with $z \neq 1, 2$ can be

¹⁰In five dimensions, the constant β can be set to zero by the coordinate transformation $t \rightarrow t + \beta u/2$, as was also noticed in [32]. However, this is not a viable coordinate transformation in the four-dimensional effective theory obtained by keeping only the Kaluza-Klein zero modes. Thus, different values for β do lead to different physics in the four-dimensional effective theory.

constructed as well. We note that non-supersymmetric flows in four dimensions between Lifshitz geometries with different values of z have been considered in [46, 47] and a non-supersymmetric solution of a gauged $\mathcal{N} = 2$ supergravity in four dimensions exhibiting $z \approx 39$ was found in [48].

Unfortunately, the charge and flux quantum numbers supporting our Lifshitz solutions do not allow for the usual relativistic vacua AdS_4 or $AdS_2 \times \mathbb{R}^2$ in any straightforward way and, thus, we did not obtain any solitonic solutions. Therefore, we henceforth adopted a completely different strategy to find interpolating flows from Lifshitz to a black background which could cloak the singularity at the center of Lifshitz. Generalizing the way in which Lifshitz backgrounds with $z = 2$ can be obtained via Scherk-Schwarz reduction from five dimensions (cf. [17]) led us to write down two classes of four-dimensional non-supersymmetric Nernst brane solutions, one of them indeed having Lifshitz asymptotics (and the other one having conformal AdS asymptotics).¹¹ Like their supersymmetric cousins found in [7] they also suffer from a tidal force singularity and diverging scalars. However, these can be regulated by heating them up, as in [8, 33].¹² This in turn might help to shed light on the embedding of the Nernst brane in the Hilbert space of the holographically dual field theory. We leave these questions for future work.

Ultimately, adopting the first strategy of exploring the solution space of the first-order flow equations (2.29) should provide us with a classification of extremal relativistic and non-relativistic solutions of $\mathcal{N} = 2$ gauged supergravity and, thus, with a systematic understanding of the zero temperature ground states of the dual field theories. The second strategy, however, opens up distinct possibilities of constructing a different (non-solitonic) class of interpolating solutions in four dimensional gauged supergravity. It also motivates a thorough exploration of the space of solitonic solutions in the corresponding five-dimensional gauged supergravity which can then be used to generate interpolating four-dimensional solutions via Scherk-Schwarz reduction. This avenue of research is currently being pursued in [51].

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¹¹Given that the solutions are non-supersymmetric, it would be interesting to perform a stability analysis as performed for example in [41].

¹²Alternatively, one could analyze how quantum corrections in the $\mathcal{N} = 2$ gauged supergravity action affect the near-horizon geometry, along the lines of [49, 50].

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A $\mathcal{N} = 2$ gauged supergravity theories in four dimensions

We consider the bosonic part of the Lagrangian of $\mathcal{N} = 2$ supergravity theories with n_V physical vector multiplets and n_H hypermultiplets, in the presence of gauging of Abelian isometries of the quaternionic Kähler manifold [52] (we use the conventions of [53]),

$$L = \frac{1}{2} R - N_{IJ} \mathcal{D}_\mu X^I \mathcal{D}^\mu \bar{X}^J - \frac{1}{2} h_{\alpha\beta} D_\mu q^\alpha D^\mu q^\beta + \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F_{\mu\nu}^I \tilde{F}^{\mu\nu J} - V(X, \bar{X}, q). \quad (\text{A.1})$$

The complex scalar fields X^I ($I = 0, \dots, n_V$) satisfy the constraint

$$N_{IJ} X^I \bar{X}^J = -1, \quad (\text{A.2})$$

where $N_{IJ} = -i(F_{IJ} - \bar{F}_{IJ})$, with $F_{IJ} = \partial^2 F(X) / \partial X^I \partial X^J$. The holomorphic function $F(X)$ is called the prepotential. The gauge kinetic couplings are given by

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i \frac{N_{IK} X^K N_{JL} X^L}{X^M N_{MN} X^N}. \quad (\text{A.3})$$

We note the following useful identity for the inverse of $\text{Im} \mathcal{N}_{IJ}$,

$$-\frac{1}{2} (\text{Im} \mathcal{N})^{IJ} = N^{IJ} + X^I \bar{X}^J + X^J \bar{X}^I. \quad (\text{A.4})$$

The X^I carry a U(1) charge,

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + i \mathcal{A}_\mu X^I. \quad (\text{A.5})$$

It will be convenient to introduce U(1) invariant variables $\tilde{X}^I = e^{i\phi} X^I$ by means of a compensating phase ϕ , so that

$$\mathcal{D}_\mu X^I = e^{-i\phi} \left(\partial_\mu + i \tilde{\mathcal{A}}_\mu \right) \tilde{X}^I, \quad \tilde{\mathcal{A}}_\mu = \mathcal{A}_\mu - \partial_\mu \phi, \quad (\text{A.6})$$

and

$$N_{IJ} \mathcal{D}_\mu X^I \mathcal{D}^\mu \bar{X}^J = N_{IJ} \partial_\mu \tilde{X}^I \partial^\mu \bar{\tilde{X}}^J + i \tilde{\mathcal{A}}_\mu \left(\tilde{X}^I N_{IJ} \partial^\mu \bar{\tilde{X}}^J - \bar{\tilde{X}}^I N_{IJ} \partial^\mu \tilde{X}^J \right) - \tilde{\mathcal{A}}_\mu \tilde{\mathcal{A}}^\mu. \quad (\text{A.7})$$

Eliminating $\tilde{\mathcal{A}}_\mu$ using its equation of motion yields $\tilde{\mathcal{A}}_\mu = \Sigma_\mu$ with

$$\Sigma_\mu = \frac{1}{2} i \left(\tilde{X}^I N_{IJ} \partial_\mu \bar{\tilde{X}}^J - \bar{\tilde{X}}^I N_{IJ} \partial_\mu \tilde{X}^J \right) = i \tilde{X}^I N_{IJ} \partial_\mu \bar{\tilde{X}}^J = -i \bar{\tilde{X}}^I N_{IJ} \partial_\mu \tilde{X}^J \quad (\text{A.8})$$

by means of (A.2). Hence

$$N_{IJ} \mathcal{D}_\mu X^I \mathcal{D}^\mu \bar{X}^J = N_{IJ} \partial_\mu \tilde{X}^I \partial^\mu \bar{\tilde{X}}^J + \Sigma_\mu \Sigma^\mu. \quad (\text{A.9})$$

The q^α ($\alpha = 1, \dots, 4n_H$) denote local real coordinates of the quaternionic Kähler manifold. Gauging its Abelian isometries results in

$$D_\mu q^\alpha = \partial_\mu q^\alpha + k_I^\alpha A_\mu^I, \quad (\text{A.10})$$

and hence

$$h_{\alpha\beta} D_\mu q^\alpha D^\mu q^\beta = h_{\alpha\beta} \partial_\mu q^\alpha \partial^\mu q^\beta + 2h_{\alpha\beta} \partial_\mu q^\alpha k_I^\beta A^{I\mu} + h_{\alpha\beta} (k_I^\alpha A_\mu^I) (k_J^\beta A^{J\mu}). \quad (\text{A.11})$$

Here, $k_I = k_I^\alpha \partial / \partial q^\alpha$ denote Killing vectors associated with the isometries that are being gauged. The scalar potential V reads (see (21.30) in [53])

$$V(X, \bar{X}, q) = 2h_{\alpha\beta} (k_I^\alpha X^I) (k_J^\beta \bar{X}^J) + (N^{IJ} - 2X^I \bar{X}^J) P_I^x P_J^x, \quad (\text{A.12})$$

where the triplet of Killing prepotentials $P_I^x(q)$ ($x = 1, 2, 3$) satisfies (see (20.174) in [53])

$$\nabla_\beta P_I^x = -k_I^\alpha \Omega_{\alpha\beta}^x, \quad (\text{A.13})$$

with the $SU(2)$ curvature Ω^x satisfying (see (20.24) in [53])

$$\Omega_{\alpha\gamma}^x \Omega^{y\gamma} = -\delta^{xy} h_{\alpha\beta} + \epsilon^{xyz} \Omega_{\alpha\beta}^z. \quad (\text{A.14})$$

B First-order rewriting of one-dimensional effective Lagrangian

We consider static electrically charged configurations, specified by the line element¹³

$$ds^2 = -e^{2U(r)} dt^2 + e^{2A(r)} (dr^2 + dx^2 + dy^2), \quad (\text{B.1})$$

and supported by scalar fields $X^I(r), q^\alpha(r)$ as well as electric fields $E^I(r) = -\partial_r A_t^I(r)$. For these configurations, we obtain

$$\frac{1}{2} \sqrt{-g} R = -[e^{U+A} (2A' + U')] + e^{U+A} ((A' + U')^2 - U'^2), \quad (\text{B.2})$$

where $' = \partial_r$. Inserting (B.2) into the four-dimensional Lagrangian ($-\sqrt{-g}L$) given in (A.1) yields the following one-dimensional effective Lagrangian,

$$\begin{aligned} \mathcal{L}_{1d} = e^\psi & \left[U'^2 - \psi'^2 + N_{IJ} \tilde{X}^I \tilde{X}^{\prime J} + \Sigma_r^2 + \frac{1}{2} h_{\alpha\beta} q'^\alpha q'^\beta \right. \\ & \left. - \frac{1}{2} e^{2(A-U)} h_{\alpha\beta} (k_I^\alpha A_t^I) (k_J^\beta A_t^J) + \frac{1}{2} e^{-2U} \text{Im} \mathcal{N}_{IJ} E^I E^J + e^{2A} V(\tilde{X}, \bar{\tilde{X}}, q) \right] + \text{T.D.}, \end{aligned} \quad (\text{B.3})$$

where we introduced $\psi = A + U$ for convenience, and where T.D. denotes a total derivative,

$$\text{T.D.} = \left[e^\psi (2\psi' - U') \right]'. \quad (\text{B.4})$$

In obtaining (B.3), we used (A.9) and (A.11), restricted to electric gaugings. Moreover, Σ_r is given in (A.8).

¹³We reproduce some of the formulae of the main text for the convenience of the reader.

It will be convenient to work with rescaled scalar fields $Y^I = e^A \tilde{X}^I$. Using (A.2) we obtain

$$\begin{aligned} e^{2A} &= -N_{IJ} Y^I \bar{Y}^J, \\ N_{IJ} \tilde{X}^I \bar{\tilde{X}}^J &= e^{-2A} N_{IJ} Y^I \bar{Y}^J + A'^2. \end{aligned} \quad (\text{B.5})$$

We also introduce a new set of real fields C_I and rewrite

$$\begin{aligned} \frac{1}{2} e^{-2U} \text{Im} \mathcal{N}_{IJ} E^I E^J &= \frac{1}{2} e^{-2U} \text{Im} \mathcal{N}_{IJ} (E^I + e^{U+A} (\text{Im} \mathcal{N})^{IK} C_K) (E^J + e^{U+A} (\text{Im} \mathcal{N})^{JL} C_L) \\ &\quad - \frac{1}{2} e^{2A} (\text{Im} \mathcal{N})^{IJ} C_I C_J - e^{A-U} C_I E^I, \end{aligned} \quad (\text{B.6})$$

where $(\text{Im} \mathcal{N})^{IJ}$ denotes the inverse of $(\text{Im} \mathcal{N})_{IJ}$. Below we will use the identity (A.4) to rewrite terms involving $(\text{Im} \mathcal{N})^{IJ}$, such as $(\text{Im} \mathcal{N})^{IJ} C_I C_J$.

Next, we introduce the combination

$$q_I = e^{i\gamma(r)} (C_I - iP_I^3), \quad (\text{B.7})$$

where γ is an r -dependent phase and P_I^3 denotes one of the three real Killing prepotentials appearing in the potential V given in (A.12). Using (B.7), we rewrite

$$\begin{aligned} e^{-2A} N_{IJ} Y^I \bar{Y}^J &= e^{-2A} N_{IJ} (Y^I - e^{2A} N^{IK} \bar{q}_K) (\bar{Y}^J - e^{2A} N^{JL} q_L) - e^{2A} q_I N^{IJ} \bar{q}_J \\ &\quad + 2 \text{Re} (Y^I q_I). \end{aligned} \quad (\text{B.8})$$

Next, performing partial integrations on the last terms of (B.6) and (B.8) we obtain

$$\begin{aligned} e^\psi [-e^{A-U} C_I E^I + 2 \text{Re} (Y^I q_I)] &= \left[e^{\psi+A-U} A_t^I C_I + 2e^\psi \text{Re} (Y^I q_I) \right]' \\ &\quad - e^\psi [2\psi' \text{Re} (Y^I q_I) + 2 \text{Re} (Y^I q_I')] \\ &\quad + (\psi' + A' - U') e^{A-U} A_t^I C_I + e^{A-U} C_I' A_t^I. \end{aligned} \quad (\text{B.9})$$

Using $\psi' = 2\psi' - U' - A'$ and $\psi' + A' - U' = 2(\psi' - U')$ we obtain

$$\begin{aligned} e^\psi [-e^{A-U} C_I E^I + 2 \text{Re} (Y^I q_I)] &= \text{T.D.} - e^\psi [2(2\psi' - U' - A') \text{Re} (Y^I q_I) \\ &\quad - 2\gamma' (\text{Im} (e^{i\gamma} Y^I C_I) - \text{Re} (e^{i\gamma} Y^I P_I^3)) \\ &\quad + 2 \text{Re} (e^{i\gamma} Y^I C_I') + 2 \text{Im} (e^{i\gamma} Y^I) (P_I^3)' \\ &\quad + 2(\psi' - U') e^{A-U} A_t^I C_I + e^{A-U} C_I' A_t^I]. \end{aligned} \quad (\text{B.10})$$

Collecting all the terms involving derivatives of U, ψ and A , we obtain

$$\begin{aligned} U'^2 - \psi'^2 + A'^2 - 2(2\psi' - U' - A') \text{Re} (Y^I q_I) &- 2(\psi' - U') e^{A-U} A_t^I C_I = \\ (U' + \text{Re} (Y^I q_I) + e^{A-U} A_t^I C_I)^2 &- (\psi' + 2 \text{Re} (Y^I q_I) + e^{A-U} A_t^I C_I)^2 + (A' + \text{Re} (Y^I q_I))^2 \\ + 2 (\text{Re} (Y^I q_I))^2 + 2e^{A-U} A_t^I C_I \text{Re} (Y^J q_J). \end{aligned} \quad (\text{B.11})$$

Next, combining the term proportional to $(P_I^3)'$ in (B.10), with the kinetic term for the hypermultiplet scalar fields in (B.3) yields

$$\begin{aligned} & \frac{1}{2} h_{\alpha\beta} q'^{\alpha} q'^{\beta} - 2 \operatorname{Im} (e^{i\gamma} Y^I) q'^{\gamma} \nabla_{\gamma} P_I^3 = \\ & \frac{1}{2} h_{\alpha\beta} (q'^{\alpha} - 2 h^{\alpha\gamma} \operatorname{Im} (e^{i\gamma} Y^I) \nabla_{\gamma} P_I^3) (q'^{\beta} - 2 h^{\beta\delta} \operatorname{Im} (e^{i\gamma} Y^J) \nabla_{\delta} P_J^3) \\ & - 2 \operatorname{Im} (e^{i\gamma} Y^I) \operatorname{Im} (e^{i\gamma} Y^J) \nabla_{\alpha} P_I^3 h^{\alpha\beta} \nabla_{\beta} P_J^3. \end{aligned} \quad (\text{B.12})$$

Thus, the one-dimensional effective Lagrangian (B.3) can be written as

$$\mathcal{L}_{1d} = \text{T.D.} + \mathcal{L}_{\text{squares}} + \Delta, \quad (\text{B.13})$$

where T.D. denotes a total derivative term, $\mathcal{L}_{\text{squares}}$ denotes a sum of squares,

$$\begin{aligned} \mathcal{L}_{\text{squares}} = e^{\psi} & \left[(U' + \operatorname{Re} (Y^I q_I) + e^{A-U} A_t^I C_I)^2 - (\psi' + 2 \operatorname{Re} (Y^I q_I) + e^{A-U} A_t^I C_I)^2 \right. \\ & + (A' + \operatorname{Re} (Y^I q_I))^2 + e^{-2A} N_{IJ} (Y'^I - e^{2A} N^{IK} \bar{q}_K) (\bar{Y}'^J - e^{2A} N^{JL} q_L) + \Sigma_r^2 \\ & + \frac{1}{2} h_{\alpha\beta} (q'^{\alpha} - 2 h^{\alpha\gamma} \operatorname{Im} (e^{i\gamma} Y^I) \nabla_{\gamma} P_I^3) (q'^{\beta} - 2 h^{\beta\delta} \operatorname{Im} (e^{i\gamma} Y^J) \nabla_{\delta} P_J^3) \\ & \left. + \frac{1}{2} e^{-2U} \operatorname{Im} \mathcal{N}_{IJ} (E^I + e^{U+A} (\operatorname{Im} \mathcal{N})^{IK} C_K) (E^J + e^{U+A} (\operatorname{Im} \mathcal{N})^{JL} C_L) \right], \end{aligned} \quad (\text{B.14})$$

while Δ is given by

$$\begin{aligned} \Delta = e^{\psi} & \left[2\Lambda^- (\gamma' + \Lambda^+) - \frac{1}{2} e^{2(A-U)} h_{\alpha\beta} (k_I^{\alpha} \tilde{A}_t^I) (k_J^{\beta} \tilde{A}_t^J) - e^{A-U} C_t^I \tilde{A}_t^I \right. \\ & + 2e^{A-U} \left(h_{\alpha\beta} (k_I^{\alpha} \tilde{A}_t^I) k_J^{\beta} \operatorname{Re} (Y^J e^{i\gamma}) + C_t^I \tilde{A}_t^I \operatorname{Re} (q_J Y^J) \right) \\ & + 2 \operatorname{Im} (e^{i\gamma} Y^I) \operatorname{Im} (e^{i\gamma} Y^J) \left(k_I^{\alpha} h_{\alpha\beta} k_J^{\beta} - \nabla_{\alpha} P_I^3 h^{\alpha\beta} \nabla_{\beta} P_J^3 \right) \\ & \left. + (e^{2A} N^{IJ} - 2Y^I \bar{Y}^J) (P_I^1 P_J^1 + P_I^2 P_J^2) \right], \end{aligned} \quad (\text{B.15})$$

where we introduced the combinations

$$\begin{aligned} \tilde{A}_t^I &= A_t^I + 2e^{U-A} \operatorname{Re} (Y^I e^{i\gamma}), \\ \Lambda^{\pm} &= \operatorname{Im} (C_t Y^I e^{i\gamma}) \pm \operatorname{Re} (P_t^3 Y^I e^{i\gamma}). \end{aligned} \quad (\text{B.16})$$

On the quaternionic Kähler manifold, the relations (A.13) and (A.14) imply the identity

$$\nabla_{\alpha} P_I^3 h^{\alpha\beta} \nabla_{\beta} P_J^3 = k_I^{\alpha} h_{\alpha\beta} k_J^{\beta}. \quad (\text{B.17})$$

This eliminates the third line of Δ in (B.15), leading to

$$\begin{aligned} \Delta = e^{\psi} & \left[2\Lambda^- (\gamma' + \Lambda^+) - \frac{1}{2} e^{2(A-U)} h_{\alpha\beta} (k_I^{\alpha} \tilde{A}_t^I) (k_J^{\beta} \tilde{A}_t^J) - e^{A-U} C_t^I \tilde{A}_t^I \right. \\ & + 2e^{A-U} \left(h_{\alpha\beta} (k_I^{\alpha} \tilde{A}_t^I) k_J^{\beta} \operatorname{Re} (Y^J e^{i\gamma}) + C_t^I \tilde{A}_t^I \operatorname{Re} (q_J Y^J) \right) \\ & \left. + (e^{2A} N^{IJ} - 2Y^I \bar{Y}^J) (P_I^1 P_J^1 + P_I^2 P_J^2) \right]. \end{aligned} \quad (\text{B.18})$$

C Some details concerning adding magnetic charge densities

We begin by recalling how to incorporate magnetic charge densities p^I in the absence of mass terms for the Abelian gauge fields. We will denote the electric charges by Q_I for the time being.

Let us consider the Maxwell part of the Lagrangian (A.1),

$$L_{\text{Maxwell}} = \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F_{\mu\nu}^I \tilde{F}^{\mu\nu J}. \quad (\text{C.1})$$

Consider non-vanishing electric fields $E^I = F_{tr}^I$ and constant magnetic fields $F_{xy}^I = p^I$. Inserting these into L_{Maxwell} yields

$$-\sqrt{-g} L_{\text{Maxwell}} = \frac{1}{2} e^{A-U} \left[E^I \mu_{IJ} E^J - p^I \mu_{IJ} p^J e^{2(U-A)} + 2E^I \nu_{IJ} p^J e^{U-A} \right], \quad (\text{C.2})$$

where $\mu_{IJ} = \text{Im} \mathcal{N}_{IJ}$ and $\nu_{IJ} = \text{Re} \mathcal{N}_{IJ}$. Adding and subtracting a total derivative term $Q_I E^I$ yields

$$\begin{aligned} -\sqrt{-g} L_{\text{Maxwell}} &= \frac{1}{2} e^{A-U} \left[(E + \mu^{-1}(Q + \nu p) e^{U-A})^T \mu (E + \mu^{-1}(Q + \nu p) e^{U-A}) \right. \\ &\quad \left. - (Q + \nu p)^T \mu^{-1} (Q + \nu p) e^{2(U-A)} - p^T \mu p e^{2(U-A)} \right] - Q_I E^I \\ &= \frac{1}{2} e^{A-U} (E + \mu^{-1}(Q + \nu p) e^{U-A})^T \mu (E + \mu^{-1}(Q + \nu p) e^{U-A}) \\ &\quad + e^{U-A} V_{\text{BH}} - Q_I E^I, \end{aligned} \quad (\text{C.3})$$

with the potential V_{BH} given by

$$V_{\text{BH}} = -\frac{1}{2} \begin{pmatrix} p & Q \end{pmatrix} \begin{pmatrix} \mu + \nu \mu^{-1} \nu & \nu \mu^{-1} \\ \mu^{-1} \nu & \mu^{-1} \end{pmatrix} \begin{pmatrix} p \\ Q \end{pmatrix}. \quad (\text{C.4})$$

This can also be expressed as

$$V_{\text{BH}} = g^{i\bar{j}} \mathcal{D}_i Z \bar{\mathcal{D}}_{\bar{j}} \bar{Z} + |Z|^2 = (N^{IJ} + 2X^I \bar{X}^J) \hat{Q}_I \bar{\hat{Q}}_J, \quad (\text{C.5})$$

where

$$Z = -\hat{Q}_I X^I, \quad \hat{Q}_I = Q_I - F_{IJ} p^J. \quad (\text{C.6})$$

Now let us return to the models considered here, where we used C_I rather than Q_I . Both are related by $Q_I = e^{2A} C_I$, and hence we may readily incorporate magnetic charge densities by replacing C_I in (2.4) with

$$\hat{C}_I = C_I - F_{IJ} p^J e^{-2A}, \quad (\text{C.7})$$

as discussed in the main text, cf. (2.25). From (C.3) we also infer the form of the electric field given in (2.27).

D Parametrizations of the coset space $SU(2, 1)/SU(2) \times U(1)$

We use the following two standard parametrizations of the coset space $SU(2, 1)/SU(2) \times U(1)$ in terms of complex coordinates q^i ($i = 1, 2$) [29]. In the first parametrization, the coset metric is given by

$$ds^2 = h_{i\bar{j}} dq^i d\bar{q}^{\bar{j}} = \frac{dq^1 d\bar{q}^{\bar{1}} + dq^2 d\bar{q}^{\bar{2}}}{1 - |q^1|^2 - |q^2|^2} + \frac{(q^1 d\bar{q}^{\bar{1}} + q^2 d\bar{q}^{\bar{2}})(\bar{q}^{\bar{1}} dq^1 + \bar{q}^{\bar{2}} dq^2)}{(1 - |q^1|^2 - |q^2|^2)^2}. \quad (\text{D.1})$$

The second parametrization uses the Kähler potential

$$K = -\ln \left(\frac{1}{2} (S + \bar{S} - 2\xi\bar{\xi}) \right). \quad (\text{D.2})$$

The resulting metric reads (with $q^i = S, \xi$)

$$ds^2 = h_{i\bar{j}} dq^i d\bar{q}^{\bar{j}} = \frac{1}{(S + \bar{S} - 2\xi\bar{\xi})^2} [dS d\bar{S} - 2\xi dS d\bar{\xi} - 2\bar{\xi} d\xi d\bar{S} + 2(S + \bar{S}) d\xi d\bar{\xi}]. \quad (\text{D.3})$$

Introducing real coordinates $S = D + i\sigma$ and $\xi = \chi e^{i\theta}$, the metric becomes

$$ds^2 = \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2} (d\sigma - i(\xi d\bar{\xi} - \bar{\xi} d\xi))^2 + \frac{1}{\rho} d\xi d\bar{\xi}, \quad (\text{D.4})$$

where $\rho = D - \chi^2$.

E Proof that the solutions of section 4 do not solve the first-order flow equations

First of all, it is easy to see that the solution (4.5)–(4.8) does not solve the first-order flow equations (2.29). It suffices to consider the equations for Y^I . The first equation in (B.5) fixes the Y^I up to an overall phase, which in general could be r -dependent, i.e.

$$Y^0 = -\frac{1}{2\sqrt{2}} \frac{\sqrt{b_1 b_2 b_3}}{\sqrt{r^2 + \alpha r^4 + \beta} C_0^{3/2}} e^{i\delta(r)}, \quad (\text{E.1})$$

$$Y^1 = -\frac{i}{2\sqrt{2}} \frac{\sqrt{b_2 b_3}}{\sqrt{b_1} C_0 r} e^{i\delta(r)}, \quad Y^2 = -\frac{i}{2\sqrt{2}} \frac{\sqrt{b_1 b_3}}{\sqrt{b_2} C_0 r} e^{i\delta(r)}, \quad Y^3 = -\frac{i}{2\sqrt{2}} \frac{\sqrt{b_1 b_2}}{\sqrt{b_3} C_0 r} e^{i\delta(r)}. \quad (\text{E.2})$$

The overall r -dependent phase in Y^I drops out in the physical scalar fields $z^A = Y^A/Y^0$.

Plugging (E.1) into the Y^0 equation in (2.29) and (E.2) into the Y^A equations, one obtains

$$-\frac{i(\alpha r^4 + r^2 + \beta)\delta'(r) - 2\alpha r^3 - r}{(\alpha r^4 + r^2 + \beta)^{1/2}} e^{i\delta(r)} = e^{-i\gamma(r)}, \quad (\text{E.3})$$

$$(-i\delta'(r)r + 1)e^{i\delta(r)} = e^{-i\gamma(r)}. \quad (\text{E.4})$$

This implies

$$\delta'(r) = i \frac{-2\alpha r^3 - r + \sqrt{\alpha r^4 + r^2 + \beta}}{\alpha r^4 + r^2 + \beta - \sqrt{\alpha r^4 + r^2 + \beta} r}. \quad (\text{E.5})$$

This does not have any real solution for non-vanishing α and/or β .¹⁴

¹⁴Obviously, for $\alpha = \beta = 0$, (E.3) and (E.4) are solved by $\delta = \gamma = 0$.

This state of affairs does not change when introducing a matrix M as in the discussion at the end of section 2.1. To see this, let us look at the Y^I equations of (2.29) again, i.e.

$$Y'^I = e^{2A} N^{IK} \bar{q}_K \tag{E.6}$$

with N^{IK} given in (3.27) and

$$\bar{q}_K = e^{-i\gamma(r)} [((M^{-1})^T C)_K + iP_K^3] = e^{-i\gamma(r)} [((M^{-1})^T)_K{}^0 C_0 + iP_K^3] \equiv e^{-i\gamma(r)} [t_K + iP_K^3], \tag{E.7}$$

where $P_0^3 = 0$, $P_A^3 = -b_A$ and t_K are real. The equations for Y^0 and Y^A become more complicated but one can go through similar steps as before for the Y^0 and Y^1 equations in order to obtain (for $b_1 = b_2 = 1$ and $b_3 = 2$ for concreteness)

$$\delta'(r) = \frac{-2i\sqrt{\alpha r^4 + r^2 + \beta} t_0 + 4(\alpha r^2 + \frac{1}{2})(i + t_1 - t_2 - \frac{t_3}{2})r}{2\sqrt{\alpha r^4 + r^2 + \beta} t_0 r + 2(it_1 - it_2 - i\frac{t_3}{2} - 1)(\alpha r^4 + r^2 + \beta)}. \tag{E.8}$$

If one instead takes the Y^0 and Y^2 or Y^0 and Y^3 equations one obtains similar equations just with the combination $t_1 - t_2 - \frac{t_3}{2}$ replaced by $-t_1 + t_2 - \frac{t_3}{2}$ and $-t_1 - t_2 + \frac{t_3}{2}$, respectively. Thus, in order to get a unique equation for δ' , one has to have the relations $t_1 = t_2$ and $t_3 = 2t_2$ (the asymmetry between t_1, t_2 and t_3 arises from our asymmetric choice of the b_A). Substituting this into (E.8) gives

$$\delta'(r) = \frac{-2i\sqrt{\alpha r^4 + r^2 + \beta} t_0 + 4(\alpha r^2 + \frac{1}{2})(i - t_1)r}{2\sqrt{\alpha r^4 + r^2 + \beta} t_0 r - 2(it_1 + 1)(\alpha r^4 + r^2 + \beta)}. \tag{E.9}$$

The right hand side has a non-trivial imaginary part given by

$$\frac{-3\left(\alpha r^4 + \frac{2}{3}r^2 + \frac{1}{3}\beta\right)t_0\sqrt{\alpha r^4 + r^2 + \beta} + 2r\left(\alpha(t_1^2 + 1)r^2 + \frac{1}{2}t_0^2 + \frac{1}{2}t_1^2 + \frac{1}{2}\right)(\alpha r^4 + r^2 + \beta)}{\left(-\alpha r^4 t_1^2 - \alpha r^4 - t_0^2 r^2 - r^2 t_1^2 + 2\sqrt{\alpha r^4 + r^2 + \beta} t_0 r - t_1^2 \beta - r^2 - \beta\right)(\alpha r^4 + r^2 + \beta)}. \tag{E.10}$$

This is non-trivial for arbitrary choices of t_0 and t_1 if α and/or β are non-vanishing. For instance, choosing $\alpha = 1$ and $\beta = 0$, (E.10) becomes

$$\frac{(-3r^2 t_0 - 2t_0)\sqrt{r^4 + r^2} + 2r(r^2 + 1)\left((t_1^2 + 1)r^2 + \frac{1}{2}t_0^2 + \frac{1}{2}t_1^2 + \frac{1}{2}\right)}{r(r^2 + 1)(-r^3 t_1^2 - r^3 - r t_0^2 - r t_1^2 + 2t_0\sqrt{r^4 + r^2} - r)} \tag{E.11}$$

There is no choice for t_0 and t_1 for which this vanishes and, thus, for which δ is purely real.

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