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Tame Harmonic Bundles on Punctured Riemann Surfaces

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A Abstract

A punctured Riemann surface is a compact Riemann surface with finitely many points removed. We will discuss an equivalence by [Sim90] between tame harmonic bundles, regular filtered stable Higgs bundles resp. $\mathscr{D}_X$-modules and regular filtered local systems over these surfaces.

Zusammenfassung

Eine punktierte Riemannsche Fläche ist eine kompakte Riemannsche Fläche ohne einer endlichen Anzahl ausgezeichneter Punkte. Wir zeigen eine Äquivalenz aus [Sim90] zwischen zahmen harmonischen Bündeln, regulär gefilterten Higgs Bündeln bzw. $\mathscr{D}_X$-Modulen und reguläre gefilterten lokalen Systemen über einer punktierten Riemannschen Fläche.

Contents

I Introduction

The roots of the so called Kobayashi-Hitchin correspondence go back to the 1960's, when M.S. Narasimhan and C.S. Seshardi [NS65] proved the correspondence between irreducible flat unitary bundles and stable vector bundles with degree 0, on a compact Riemann surfaces. Once the result was well-understood, the natural question arose how it might be extended beyond complex dimension 1 and beyond flat unitary bundles. One of the first extensions was done by Mehta and Ramanathan [MR84], who proved that flat unitary bundles in general correspond to stable vector bundles with trivial Chern class.
Then at the beginning of the 1980's Kobayashi [Kob80] (and independently Lübke [Lüb82]) proved that a holomorphic bundle on a Kähler manifold equipped with a Hermitian-Einstein metric is stable. This led Kobayashi and independently Hitchin (according to Donaldson [Don85]) to formulate the inverse problem, i.e. does every stable bundle possess a Hermitian-Einstein metric? Donaldson himself found a proof in the compact Riemann surfaces case in [Don87],[Don85] and extended it to algebraic surfaces. The general result over every Kähler manifold was finally proved by Uhlenbeck and Yau [UY86].
On the other hand Hitchin introduced in [Hit87] Higgs bundles, labeled like this because there are similarities to the mathematical description of physical particles like the "Higgs boson" in mathematical gauge theory. The term of a Higgs bundle soon proved very useful, since Hitchin was able to find a correspondence between the existence of a Hermitian-Einstein metric on a Higgs bundle over a compact Riemann surface and stability of the Higgs bundle.
When Carlos Simpson started his PhD thesis at end of the 1980's the non-compact case was still untouched or at least the correspondence could not be established. After proving the existence of a Hermitian-Einstein metric for every Higgs bundle on a (in general not-compact) Kähler manifold (satisfying certain conditions) in his [Sim88] article on "Constructing variations of Hodge structure", he turned to the complex curve case, i.e. to Riemann surfaces. There he was able to establish a correspondence between harmonic bundles and stable filtered regular Higgs

bundles, filtered local systems as well as flat regular bundles (cf. [Sim90]).[1]
The correspondence established by Simpson in [Sim90] connects various fields of mathematics:

Algebraic Geometry		**Differential Geometry**		**Topology**
stable filtered regular Higgs bundles with degree 0	↔	tame harmonic bundles	↔	stable filtered local systems eit degree 0

Furthermore there are connections to partial differential equations resp. analysis as well as physics.[2]
Therefore it seems worthy to take a closer look at the proof of the correspondence. As it turns out, the article [Sim90] flawlessly describes all major steps on the way to the equivalence of categories, but leaves out the calculations and some intermediate steps. This is one of the reasons that make it difficult to follow the central theme of the article. Another one is that the necessary background knowledge is either assumed or scattered around the various articles [Sim88], [Sim92].
So in our explanation we try to collect these definitions and results so that the work becomes self-contained as far as possible. Furthermore we will add the missing details or rewrite some of the proofs if we were not able to follow the original argument.
In view of these two tasks we start in the first chapter with some background knowledge on Algebraic and Differential Geometry as well as a small introduction to Hodge theory. Of particular importance are some consequences of Serre's GAGA and the result of Deligne on regular singularities establishing a relation between algebraic and holomorphic vector bundles. They enable us to choose one of the categories as long as we are on a compact surface. Moreover Deligne's article tells us about the requirements a holomorphic connection has to obey in order to be algebraic too - regularity. Consequently we will repeat the Riemann-Hilbert correspondence connecting flat holomorphic vector bundles, local system and representations of the fundamental group. The theorem allows us for example to understand the equivariance of the harmonic metric map or the correspondence of the flat bundles and local system in the context of filtered objects.
In the main text, we will often have to change between a vector bundle and its sheaf of sections. We will recall that this construction essentially commutes with tensor products in the differential, holomorphic resp. algebraic setting.
The Hodge theoretical part will explain the $*-$adjoint of a connection and we will have some basic Kähler identities in the hermitian case, i.e. not necessarily kählerian case. These identities will prove very useful later on when we consider a metric as a map into the positive-definite matrices. Another application is the proof of the Chern-Weil formula.

[1] For the last of these four categories this was done by Corlette [Cor88], too.
[2] We will see that the boundaries between the different fields become blurred and we may assign the objects to different fields.

In the next section we are going to start with the discussion of connections and metrics. In particular we explain how a metric connection on higher differential forms has to look like in our notation. The second part of this section will be devoted to the study of maps into the space $\mathbb{P}_n$ of positive-definite matrices. This space will be identified with the homogeneous space $\mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$. After identification of the space of hermitian matrices with a real vector space we get a differential structure on $\mathbb{H}_n$ and later, on $\mathbb{P}_n$ via the exponential map. The main result of this section is that $\mathbb{P}_n$ is a complete Riemannian manifold, which is negatively curved. Thus we are in the position to use some convexity properties of geodesics in negatively curved complete spaces.
In the proof of completeness we introduce a formula for the differential of a bounded linear operator on the space of hermitian matrices, which we call divided sums as [Bha06] . This concept is used by Simpson in his article [Sim88] to prove the existence of a harmonic metric.
Before we proceed with the description of our metric map $X \to \mathbb{P}_n$ we need to define the basis concepts of our thesis, namely Higgs bundles, $\mathscr{D}_X$-modules and harmonic bundles. A Higgs bundle is a holomorphic vector bundle E with the additional structure of a Higgs field θ, that fulfills some compatibility relation with the holomorphic structure $\overline{\partial}_E$. A $\mathscr{D}_X-$module is in fact a flat bundle in the sense of a vector bundle equipped with a flat connection. From the structure of the flat bundle we can construct a Higgs field θ and from a Higgs bundle we can construct a connection D. Harmonicity is now the statement that the Higgs bundle is flat, i.e. that D has vanishing curvature. Hence we get a flat bundle. On a flat bundle harmonicity means that θ fulfills the compatibility relation with the holomorphic structure $\overline{\partial}_E$, i.e. is a Higgs bundle. As a result we see that Higgs bundles and flat bundles are the same if they are harmonic. In order to justify the term "harmonic" bundle we will show, using the Kähler identities, that our metric map into $\mathbb{P}_n$ is harmonic iff the corresponding bundle is.
After defining tameness of a Higgs field and the notion of a local system we will turn to the regularity of a connection. In order to do so we define the pushforward sheaf $j_*(E)$ corresponding to the inclusion $j : X \hookrightarrow \overline{X}$. Then the regularity of ∇ may be described in terms of its connection matrix, i.e. we allow only poles of order one in some suitable frame.[3]
In the last section of this first chapter we start with the description of filtered vector bundles. We recall the original idea of Mehta and Seshadri (cf. [MS80]). A filtered vector bundle is then a decreasing filtration of $j_*(E)$ by coherent subsheaves, that fulfill certain relations. A filtered Higgs bundle is a filtered vector bundle with a Higgs field θ that respects the filtration. Analogously for a filtered $\mathscr{D}_X-$module, ∇ has to respect the filtration. We get a filtered local system if we have a filtration

[3] ∇ is the $(1,0)$-part of the flat connection.

L_α of the locally constant sheaf L, such that L_α is preserved by the monodromy μ of the local system.
The main result of the thesis will be a correspondence of categories. So we need to clarify how the morphisms in the different categories look like. However, the choice is natural, i.e. a map is a morphism of filtered objects if it preserves the filtration as well as all additional structure, such as the Higgs field θ. A short subsection about the concept of residues in our context closes the chapter.

The second chapter describes Simpson's main estimate. The aim is to reflect tameness, i.e. the growth of the eigenvalues, in terms of the Frobenius norm of θ itself and more important to find a bound of the curvature of the metric connection, often denoted R_h. Before we start with the actual proof we will first describe how we can simplify the matrix representation of our Higgs field θ in order to make it manageable. This can be done using an eigenspace decomposition of E. However our eigenvalues are unfortunately not single-valued. The problem can be overcome by transferring to a finite branched cover of X and the corresponding pullback of our bundle E. One advantage thereof is that we have a Laurent expansion there with order at most $-n$, where n is the dimension of our vector bundle. So we add to our eigenvalue λ a function α such that the order of $\lambda - \alpha$ is $-n+1$. Going back to our original bundle $\lambda - \alpha$ will be bounded in terms of $|z|^{-1+\varepsilon}, \varepsilon \geq 1/n$.[4] Schur decomposition will lead us to an upper triangular form of θ as well as ϕ, where ϕ comes form the α constructed before. It turns out that this is a block upper triangular matrix, that can be decomposed further into a block diagonal part $\sigma + \tau^0$ and a strictly block upper triangular part τ^+.[5] The main estimate will treat those parts separately.

After introducing some preliminary norm estimates involving θ and its $h-$conjugate $\theta^\dagger$, we describe the endomorphism bundle in detail. In the rest of the paper we will often profit from the interaction between the initial bundle E and the endomorphism bundle $\mathrm{End}(E)$. So we think it is worth taking a closer look. All differential operators on E induce operators on the endomorphism bundle. The induced operators inherit almost all properties of the original ones. In particular our Higgs field property $\overline{\partial}_E\theta + \theta\overline{\partial}_E = 0$ becomes $(\overline{\partial}_E)_{\mathrm{End}}(\theta) = 0$ for the induced operator $(\overline{\partial}_E)_{\mathrm{End}}$ on $\mathrm{End}(E)$. The metric h on E induces a metric $\mathrm{tr}(HB^*HA) = \langle A, B\rangle$ for $A, B \in \mathrm{End}(E)$, which becomes the Hilbert-Schmidt inner product if we choose an $h-$orthonormal frame. Here H is the corresponding map into $\mathbb{P}_n$.
Now we are in the position to use Griffith's and Harris' [GH78] statement, that

[4] z is a local coordinate vanishing at the puncture.
[5] Blocks with respect to the eigenvalues; σ diagonal.

curvature decreases in subbundles to get to our starting point:

$$-\Delta \log \|\theta_z\|_F \leq -\frac{\|[\theta_z, \theta_z^\dagger]\|_F^2}{\|\theta_z\|_F^2}, \tag{1}$$

where the index z indicates the connection matrix of θ resp. $\theta^\dagger$. Before we proceed with the estimate itself we add two examples which will come of importance when we try to construct a standard metric later on in the paper.
Step 1 of the main estimate is to find a bound of θ itself. The idea behind the calculation done in this subsection is to first distinct two cases (cf. 1.5.1), where one case is the desired result $\|\theta_z\| \leq \frac{c_3}{|z|}$, and then to show that the other case cannot occur. The second case is defined by some inequality: $\|\theta_z\|_F$ strictly bigger than some function m. Then we show that if this inequality holds on some non-empty set S_1 we get a contradiction. In order to do so prove that $\theta_z - m$ is subharmonic on S_1 (using (1)) and has its maximum $\theta_z = m$ on the boundary ∂S_1. Inside S_1 we get $\theta_z - m \leq 0 \Leftrightarrow \theta_z \leq m$ contradicting the definition of S_1.
At the beginning of step 2, we will first recall an easy estimate of the real logarithm function, namely that we always find a straight line through 0 that is everywhere greater than or equal the logarithm function. For a lower bound of the logarithm we will in general not find such a straight line strictly smaller than log. While Simpson writes no further details on this lower bound, Mochizuki [Moc07a] claims that it exists. The difference between those two different conclusions is that Mochizuki [Moc07a] expects a certain constant b to be always positive, while we are not so sure about that. So we describe a way around this problem in Step 3. The rest of Step 2 is another application of the idea of Step 1, although slightly more complicated. As a result we find a bound for τ^0.
The second part of the main estimate takes care about the strictly upper triangular part of θ. Therefore we need a modified version of (1), which we introduced in the context of the endomorphism bundle already, as well as some calculations involving the adjoint representation.[6] In particular we will show the invertibility of $\mathrm{ad}(\phi)$ on block upper triangular matrices. With the help of ad and some bounds of the adjoint representation, we will first show that the upper triangular parts of θ and ϕ are mutually bounded and hence get into the position to repeat the argument of Step 2. This leads us to a bound of τ^+ in terms of $|z|^{-1+\varepsilon}$. Using both estimates - of τ^0 and τ^+ - we can bound the norm of the curvature R_h of the metric connection by $R_h \leq \frac{c_D}{|z|^2 |\log|z||^2}$. Putting the previous results together will lead us to Simpson's theorem 1.
There are a few consequences of the main estimate described afterwards. The first one is that the norms of the flat sections increase at most polynomially. Our proof uses Gronwall's lemma to solve the differential equation mentioned by Simpson.

[6] We change from the commutator to the adjoint representation ad to simplify notation.

Using the polynomial bound of the flat sections we are able to describe tameness in terms of a $\mathscr{D}_X-$module: If the flat sections grow at most polynomially then the corresponding Higgs field is tame. The proof is somewhat lengthy. If H is our harmonic metric we construct a second metric K. Then we estimate K in term of H from above - from below we anyway have a bound by harmonicity. K is constructed as follows on an annulus $A_{\varepsilon,1}$: For a small $\sigma > 0$ let K be constant on $A_{\varepsilon+\sigma,1}$ and a geodesic along each ray out of the puncture on $A_{\varepsilon,\varepsilon+\sigma}$. Now we can use the completeness of $\mathbb{P}_n$ as well as the negative curvature to estimate the length of the geodesic. Thereafter Simpson uses, we guess, integration by parts - $\int_{D(0,1)} r^\nu \|\theta\|_F^2 = \nu \int_0^1 \varepsilon^{\nu-1} \, \mathrm{d}\, \nu \int_{D(\varepsilon,1)} \|\theta\|_F^2$. Instead of this formula we will use a weaker estimate still leading to the right conclusion, i.e. we show that if θ is not tame the estimates we got by constructing K are contradicted.

The last lemma in this chapter is of particular importance. It helps us to extend an inequality $-\Delta f \leq -b$ for $\frac{f}{\log|z|} \to 0$ and b positive, weakly over the puncture. The proof is more or less an application of the properties of the Green function $\frac{1}{2\pi} \log|z|$.

The third chapter deals with filtered objects. In the first part we construct our main functor Ξ from the category of tame harmonic bundles to the category of filtered regular Higgs bundles resp. filtered regular $\mathscr{D}_X-$modules. The functor will be a priori defined on acceptable bundles. These are, roughly speaking, bundles with a metric connection that is bounded in the same terms as the connection in the main estimate. For technical reasons we add a perturbation by a L^p-function. Ξ is in particularly compatible with taking duals, tensor products and determinants. The proofs are not included in Simpson's work [Sim90], but can be partly found in [Sim88]. After adding these compatibility properties, we show that when restricted to the subcategory of harmonic bundles, Ξ indeed maps E to a filtered regular Higgs bundle resp. $\mathscr{D}_X-$module: Since Ξ maps already into filtered vector bundles we only need to show that θ resp. ∇ respects the filtrations. For θ this is clear, for ∇ we get the result by using the weak extension described above. At the end of the section, we show that Ξ gives rise to a map of the corresponding morphisms, i.e. is a functor.
The second part of the chapter describes a functor Φ from filtered local systems to $\mathscr{D}_X-$modules. First we will recall the general Riemann-Hilbert correspondence and construct a meromorphic frame of the $\mathscr{D}_X-$module V using the flat sections in the corresponding local system L. In terms of this frame we can define the functor Φ. As a first step we show that Φ is well-defined. The second step is crucial, because it simplifies the rest of the proof: Φ is compatible with the decomposition into generalized $\mu-$eigenspaces, even more compatible with the decomposition into $\mu-$invariant subspaces (μ the monodromy). Thereafter we may restrict to local systems with only one eigenvalue to show compatibility with tensor products and

duals.
The functor Φ is furthermore essentially surjective: We construct an inverse Φ^{-1} that maps each filtered regular $\mathscr{D}_X-$module on a local system and is the inverse construction to Φ. Simpson constructs Φ^{-1} in a different way and concludes then, that his inverse is the same as ours.
Finally we show that the functor is fully faithful and therewith Φ establishes an equivalence of the categories of filtered local systems and filtered regular $\mathscr{D}_X-$modules.
The aim of chapter V is to show that two metrics that induce the same filtration under Ξ are mutually bounded and further that Ξ is fully faithful. The main point are two Weitzenböck formulas given at the beginning of the section. For the first part we need a lemma that tells us that if a holomorphic section e is bounded by $|z|^{-\varepsilon}$ for all $\varepsilon > 0$, then it is already bounded, if the curvature is L^p. The proof uses the L^p-integrability of the curvature to bound $\log \|e\|$ by a L^p-function f.[7] Then solve the Poisson equation $\Delta\tilde{u} = f$ and extend it weakly over the puncture. Subharmonicity will lead to a bound of e.
The lemma applied to the identity element of the endomorphism bundle is enough to show that two metrics that induce the same filtration are mutually bounded.
In order to show that Ξ is fully faithful we need to show that a Higgs bundle morphism on the by Ξ induced filtered regular Higgs bundle is a morphism of harmonic bundles, i.e. it is a morphism of filtered regular $\mathscr{D}_X-$modules. This is done by our Weitzenböck formula. The same holds for Ξ into filtered regular $\mathscr{D}_X-$modules.
The chapter titled "Residues and Standard Metrics" continues our two examples 1.4.17 and 1.4.18. It turns out that they are the basic building blocks to construct a "standard" metric on a Higgs bundle. We will describe how a "jump", i.e. a α in the filtration $\Xi(E)_\alpha$ such that $\mathrm{Gr}_\alpha(E) \neq 0$ transfers to the $\Xi(E)-\mathscr{D}_X-$module and further to the local system corresponding under Φ. From this observation we see that by considering tensor products of the bundles defined in the two examples will lead us to any residue and any residue homomorphism up to isomorphism. Now using the isomorphism between the residues we can state the main result 4.3.1 of the chapter, namely that every regular filtered Higgs bundle resp. $\mathscr{D}_X-$module resp. filtered local system can be equipped with an acceptable metric h that induces the filtrations under Ξ. This metric is not necessarily harmonic. In particular we show that it does not matter whether we consider the filtration on a local system directly induced by h via order of growth or the filtration coming from the $\mathscr{D}_X-$module $\Xi(E)$ via Φ.
The final chapter will lead to our main result. We first give the definition of the algebraic/parabolic degree of a filtered object and the analytic degree using Chern

[7] The Weitzenböck formula relates e and the curvature

classes. Then we show that both notions are actually the same for $E, \Xi(E)$. We will give an extended proof assuming less than necessary in order to treat subbundles, which will come up later, as well. One of the main topics of the chapter are the Chern-Weil formulas, that describe how the curvature of a subbundle looks like in terms of the original curvature. After defining stability for each of our objects we will be in the position to show that $\Xi(E)$ (for an irreducible tame harmonic bundle E) is stable and has degree 0 and that every harmonic bundle decomposes into irreducible ones.
The main existence theorem of [Sim88], will provide us with a harmonic metric, which is bounded with respect to our standard metric, i.e. induces the same filtrations under Ξ. Showing that every filtered bundle already comes from a tame harmonic one and showing that Φ preserves degree and stability will lead us to our main result:

Main theorem. The category of tame harmonic bundles is naturally equivalent via the functors Ξ, to the categories of direct sums of stable filtered regular Higgs bundles of degree zero, of direct sums of stable filtered regular $\mathscr{D}_X-$modules of degree zero, and of direct sums of stable filtered local system of degree zero.

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1 Main Estimate for Tame Harmonic Bundles

This first chapter starts with some definitions and preliminary results. The main part then contains the actual estimate of our Higgs field θ close to the punctures. Finally we add some conclusions as well as technical lemmas needed in the following chapters.

1.1. Harmonic Bundles

1.1.1. Higgs Bundles

In this first paragraph we will in short define the notion of a Higgs field and its basic properties. Most of the upcoming definitions and notational conventions are explained in the Appendix.
Hitchin introduced in [Hit87] the notion of a Higgs field:

Definition 1.1.1. $(E, \overline{\partial}_E, \theta)$ is a Higgs bundle if E is a (holomorphic or algebraic) vector bundle with holomorphic structure $\overline{\partial}_E$ and θ a bundle homomorphism

$$\theta : E \to E \otimes_{\mathbb{C}} \bigwedge\nolimits_X^{1,0}$$

which satisfies $\overline{\partial}_E \circ \theta + \theta \circ \overline{\partial}_E = 0$. Then θ is called Higgs field.

If we restrict to holomorphic sections, $\theta|_{\Gamma_{hol}(X,E)}$, then

$$\theta \circ \overline{\partial}_E(s) = 0, \quad \forall s \in \Gamma_{hol}(X, E),$$

i.e. the last property $\overline{\partial}_E \circ \theta + \theta \circ \overline{\partial}_E = 0$ simplifies to $\overline{\partial}_E \circ \theta = 0$.
Using A.1.24 and A.1.25 we can write for every open $U \subset X$

$$\begin{aligned} \theta|_U \in \quad & (\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U, E), \Gamma(U, E) \otimes_{\mathcal{E}_X} \Omega_U^{1,0}))_U \\ & \simeq \Gamma(U, \mathrm{Hom}_{\mathbb{C}}(E, E \otimes_{\mathbb{C}} \bigwedge^{1,0} U)) \end{aligned}$$

$$\begin{aligned}
&\simeq \Gamma(U, E^* \otimes_{\mathbb{C}} E \otimes_{\mathbb{C}} \bigwedge^{1,0} U) \\
&\simeq \Gamma(U, \mathrm{End}_{\mathbb{C}}(E) \otimes_{\mathbb{C}} \bigwedge^{1,0} U) \\
&\simeq \Gamma(U, \mathrm{End}_{\mathbb{C}}(E)) \otimes_{\mathcal{E}(U)} \Omega_U^{1,0}.
\end{aligned}$$

To find the equivalent to $\overline{\partial}_E \circ \theta + \theta \circ \overline{\partial}_E = 0$ we need the induced holomorphic structure on the endomorphism bundle, which is described in A.4:

$$\begin{aligned}
\overline{\partial}_{\mathrm{End}} &: \quad \Omega_X^{1,0}(\mathrm{End}(E)) \to \Omega_X^{1,1}(\mathrm{End}(E)) \\
\overline{\partial}_{\mathrm{End}}(\omega) &:= \quad \overline{\partial}_E \omega + \omega \overline{\partial}_E,
\end{aligned}$$

for any $(1,0)-$form ω. Therefore $\overline{\partial}_E \circ \theta + \theta \circ \overline{\partial}_E = 0 \Leftrightarrow \overline{\partial}_{\mathrm{End}}(\theta) = 0$.
But we have

$$\Gamma_{hol}(U, \mathrm{End}_{\mathbb{C}}(E) \otimes_{\mathbb{C}} \bigwedge_U^{1,0}) \simeq \{s \in \Gamma(U, \mathrm{End}_{\mathbb{C}}(E)) \otimes_{\mathcal{E}(U)} \Omega_U^{1,0} | \overline{\partial}_{\mathrm{End}}(s) = 0\}.$$ [1]

Thus θ can be understood as a holomorphic section into $\mathrm{End}_{\mathbb{C}}(E) \otimes_{\mathbb{C}} \bigwedge_X^{1,0}$. By some authors this discription is used as a definition for θ. However, we will stay with the definition chosen by Carlos Simpson.
Note that θ maps holomorphic sections s to holomorphic one-forms, since

$$0 \quad = \quad (\overline{\partial}_E \theta + \theta \overline{\partial}_E)(s) = \overline{\partial}_E(\theta s).$$

Thus in a suitable basis $\theta(s)$ has holomorphic coefficients.

Remark 1.1.2. For a Riemannian surface $\theta^2 = 0$:

$$\begin{aligned}
\theta(s) = s_\theta \otimes \omega_\theta \Rightarrow \theta^2(s) &= \theta(s_\theta \otimes \omega_\theta) \\
&:= \theta(s_\theta) \wedge \omega_\theta \\
&= (s_\theta)_\theta \otimes (\omega_{s_\theta} \wedge \omega_\theta).
\end{aligned}$$

But $\omega_\theta = f \,\mathrm{d}\, z, \omega_{s_\theta} = g \,\mathrm{d}\, z \Rightarrow f \,\mathrm{d}\, z \wedge g \,\mathrm{d}\, z = fg \;\mathrm{d}\, z \wedge \mathrm{d}\, z = 0$.

If our Higgs bundle with Higgs field θ is equipped with a hermitian structure h (see A.2), then let $\theta^\dagger : \Gamma(X, E) \to \Gamma(X, E) \otimes \Omega_X^{0,1} = \Omega_X^{0,1}(E)$[2] $\mathcal{E}_X-$sheaf homomorphism be the $h-$adjoint, i.e.

$$h(\xi, \theta\eta) = h(\theta^\dagger \xi, \eta), \quad \xi, \eta \in \Gamma(X, E).$$

[1] For a proof see [Huy05], p. 110 and p. 73 for $H^0(\cdot) = \Gamma_{hol}(\cdot)$.
[2] In [Sim90] $\theta^\dagger$ is denoted by $\overline{\theta}$.

Locally[3] for a Riemann surface this becomes $\theta\eta = \eta_\theta \otimes (1\,\mathrm{d}\,z), \theta^\dagger\xi = \xi_{\theta^\dagger} \otimes (1\,\mathrm{d}\,\overline{z})$ and

$$\begin{aligned}&h(\xi, \eta_\theta) \cdot (1\,\mathrm{d}\,\overline{z}) = h(\xi, \eta_\theta \otimes (1\,\mathrm{d}\,z)) = h(\xi_{\theta^\dagger} \otimes (1\,\mathrm{d}\,\overline{z}), \eta) = h(\xi_{\theta^\dagger}, \eta) \cdot (1\,\mathrm{d}\,\overline{z})\\ &\Leftrightarrow h(\xi, \eta_\theta) = h(\xi_{\theta^\dagger}, \eta).\end{aligned}$$

Thus $\theta^\dagger$ is uniquely determined by θ and h. It inherits the homomorphism property by

$$\begin{aligned}h(\theta^\dagger(\xi \otimes f), \eta) &=^4 h(\xi \otimes f, \theta(\eta)) = h(\xi, \theta(\eta) \wedge \overline{f})\\ &= h(\xi, \theta(\eta \otimes \overline{f})) = h(\theta^\dagger(\xi), \eta \otimes \overline{f})\\ &= h(\theta^\dagger(\xi) \wedge f, \eta), \quad \forall \eta, \xi \in \Gamma(U, E), f \in \mathcal{E}(U).\end{aligned}$$

Hence $\theta^\dagger(\xi \otimes f) = \theta^\dagger(\xi) \wedge f$. $\theta^\dagger(\xi)$ is smooth since h and θ are. Further we have $(\theta^\dagger)^2 = 0$ (see 1.1.2).

1.2. Harmonicity of Vector Bundles

In the first part of this section we will return to Higgs bundles and define harmonicity for Higgs bundles. We will see that harmonic Higgs bundles are just the harmonic $\mathscr{D}_X-$modules. Furthermore we will get relations between the defining operators. This will be the foundation of all our calculations. Please note that in comparison with Simpson we change $d' \leftrightarrow D'$ and $d'' \leftrightarrow D''$.
The second part considers our metric as a map into the space $\mathbb{P}_n$ of positive definite matrices. There we will use some perturbations in $\mathbb{P}_n$ to show that the metric is harmonic in the bundle case iff it minimizes the energy, i.e. is harmonic as a map into $\mathbb{P}_n$.

1.2.1. Higgs Bundles and Flat Vector Bundles

With the definitions of the last section we can introduce a new connection

$$\mathbb{D} = \partial_E + \overline{\partial}_E + \theta + \theta^\dagger$$

on our holomorphic Higgs bundle $(E, \overline{\partial}_E, \theta, h)$. We can check this is indeed another connection although it is in general no metric connection.

$$\mathbb{D} : \Omega^0_X(E) \to \Omega^1_X(E)$$

[3] $\theta, \theta^\dagger$ are sheaf homomorphisms and so well-defined on any $\Gamma(U, E), U$ open.

[4] At this point the $\otimes-$notion might be confusing. f is just a $\mathbb{C}-$valued function which acts by multiplication, i.e. here $\otimes = \cdot$. The hermitian structure of h enables us to put f on the side.

since all component maps go into $\Omega^1_X(E)$. The Leibniz Rule is satisfied for $D = \partial_E + \overline{\partial}_E$ since D is a connection. Since $\theta, \theta^\dagger$ are $\mathcal{E}(U)-$module homomorphisms Leibniz rule holds for $\mathbb{D}$: $\forall s \in \Gamma(U, E)$, $\xi \in \mathcal{E}(U) = \Omega^0_U$

$$\begin{aligned}
\mathbb{D}(s \otimes \xi) &= (\partial_E + \overline{\partial}_E + \theta + \theta^\dagger)(s \otimes \xi) \\
&= D(s) \wedge \xi + s \otimes \mathrm{d}\,\xi + \theta(s \otimes \xi) + \theta^\dagger(s \otimes \xi) \\
&= D(s) \wedge \xi + s \otimes \mathrm{d}\,\xi + \theta(s) \wedge \xi + \theta^\dagger(s) \wedge \xi \\
&= (D + \theta + \theta^\dagger)(s) \wedge \xi + s \otimes \mathrm{d}\,\xi \\
&= \mathbb{D}(s) \wedge \xi + s \otimes \mathrm{d}\,\xi.
\end{aligned}$$

Definition 1.2.1. A metric holomorphic Higgs bundle $(E, \overline{\partial}_E, h, \theta)$ is called harmonic bundle if $\mathbb{D}$ is flat, i.e. if $\mathbb{D} \circ \mathbb{D} = 0$.

We will not assume harmonicity for now.

Lemma 1.2.2.

$$\mathbb{D}^2 = \partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E + \theta\theta^\dagger + \theta^\dagger\theta$$

for Riemann surfaces.

Proof. This is because

$$\partial_E^2, \overline{\partial}_E^2, \theta^2, (\theta^\dagger)^2, \theta\overline{\partial}_E, \partial_E\theta, \theta^\dagger\overline{\partial}_E, \overline{\partial}_E\theta^\dagger,$$

vanish. Further $\overline{\partial}_E\theta + \theta\overline{\partial}_E = 0 \Rightarrow \partial_E\theta^\dagger + \theta^\dagger\partial_E = 0$. We could prove the last equation by using the compatibility with the metric of $\overline{\partial}_E + \partial_E$ to receive from the Higgs property an equation involving the term $\partial_E\theta^\dagger + \theta^\dagger\partial_E$. However, the aim of this section is to show that a harmonic bundle corresponds to a flat bundle with a certain property: vanishing Pseudo-curvature. The tools used in this section will therefore automatically lead to an easy explanation 1.2.6 of $F_h := \mathbb{D}^2 = \partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E + \theta\theta^\dagger + \theta^\dagger\theta$. □

Remark 1.2.3. (i) Note that $\overline{\partial}_E + \theta^\dagger$ is another holomorphic structure of E.

(ii) W.r.t. the holomorphic structure $\overline{\partial}_E + \theta^\dagger$ the operator $\partial_E + \theta$ becomes a holomorphic connection (cf. A.2.12) on a harmonic Higgs bundle, since

$$\begin{aligned}
&(\overline{\partial}_E + \theta^\dagger)(\partial_E + \theta) + (\partial_E + \theta)(\overline{\partial}_E + \theta^\dagger) \\
&= \underbrace{\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E + \theta\theta^\dagger + \theta^\dagger\theta}_{=\,0 \text{ harmonic}} + \underbrace{\overline{\partial}_E\theta + \theta\overline{\partial}_E}_{=\,0 \text{ Higgs}} + \underbrace{\partial_E\theta^\dagger + \theta^\dagger\partial_E}_{=\,0 \text{ Higgs}} \\
&= 0, \\
&(\partial_E + \theta)(\overline{\partial}_E + \theta^\dagger)|_{\Gamma_{hol}(X,E)} = 0,
\end{aligned}$$ [5]

$$\Rightarrow (\overline{\partial}_E + \theta^\dagger)(\partial_E + \theta) = 0,$$

i.e. holomorphic sections are mapped to holomorphic sections.

We want to start with a smooth vector bundle V with flat connection D, and a hermitian metric h_V. Then we can split up the connection into $(1,0)$ and $(0,1)$ part

$$D = D' + D''.$$

D is not necessarily metric, so we add an operator δ'' of type $(0,1)$ to D' and an operator δ' of type $(1,0)$ to D'' such that $D' + \delta''$ and $\delta' + D''$ become each metric connections.[6] Define $\overline{\partial}_V := \frac{D''+\delta''}{2}$. We want to show that the operator explains a holomorphic structure: Obviously $\overline{\partial}_V^2 = 0$ as $(0,2)-$form on a Riemann surface; the Leibniz rule follows from

$$\begin{aligned}
\overline{\partial}_V(s \otimes \omega) &= \frac{D'' + \delta''}{2}(s \otimes \omega) \\
&=^{7} \frac{1}{2}(D''(s) \wedge \omega + s \otimes \overline{\partial}\omega + \delta''(s) \wedge \omega + s \otimes \overline{\partial}\omega) \\
&= \left(\frac{D'' + \delta''}{2} s\right) \wedge \omega + s \otimes \overline{\partial}\omega \\
&= \left(\overline{\partial}_V s\right) \wedge \omega + s \otimes \overline{\partial}\omega, \quad s \in \Gamma(U,V),\, \omega \in \Omega_X^{p,q},
\end{aligned}$$

Thus $\overline{\partial}_V$ determines a holomorphic structure of V.
Similarly we get for $\partial_V := \frac{D'+\delta'}{2}$, that $\partial_V^2 = 0$ as $(2,0)-$form and the Leibniz rule holds:

$$\begin{aligned}
\partial_V(s \otimes \omega) &= \frac{D' + \delta'}{2}(s \otimes \omega) \\
&= \frac{1}{2}(D'(s) \wedge \omega + s \otimes \overline{\partial}\omega + \delta'(s) \wedge \omega + s \otimes \partial\omega) \\
&= \left(\frac{D' + \delta'}{2} s\right) \wedge \omega + s \otimes \partial\omega \\
&= (\partial_V s) \wedge \omega + s \otimes \partial\omega, \quad s \in \Gamma(U,V),\, \omega \in \Omega_X^{p,q},
\end{aligned}$$

In particular $\partial_V + \overline{\partial}_V$ is a connection. This connection respects the metric since

$$h_V(\overline{\partial}_V\xi, \eta) + h_V(\xi, \partial_V\eta) = \frac{1}{2}\left(h_V((D'' + \delta'')\xi, \eta) + h_V(\xi, (D' + \delta')\eta)\right)$$

[5] W.r.t. $\overline{\partial}_E + \theta^\dagger$ a section is holomorphic iff it is killed by $\overline{\partial}_E + \theta^\dagger$.
[6] cf. A.2.10 and the following remark.
[7] D'', δ'' are $(0,1)$ parts of a connection and satisfy therefore the Leibniz rule.

$$\begin{aligned}
&=^{8} \quad \frac{1}{2}\big(h_V(\delta''\xi,\eta)+\overline{\partial}_V h_V(\xi,\eta)-h_V(\xi,\delta'\eta)\\
&\qquad +h_V(\xi,\delta'\eta)+\overline{\partial}_V h_V(\xi,\eta)-h_V(\delta''\xi,\eta)\big)\\
&= \quad \partial_V h_V(\xi,\eta), \quad \xi,\eta\in\Gamma(U,V).
\end{aligned}$$

Moreover we can define $\theta := \frac{D'-\delta'}{2}$, $\theta^\dagger := \frac{D''-\delta''}{2}$. Again $\theta^2=(\theta^\dagger)^2=0$. Further the two operators are $\mathcal{E}(U)-$module sheaf homomorphisms

$$\begin{aligned}
\theta(s\otimes\omega) &= \frac{D'-\delta'}{2}(s\otimes\omega)\\
&= \frac{1}{2}(D'(s)\wedge\omega+s\otimes\overline{\partial}\omega-\delta'(s)\wedge\omega-s\otimes\partial\omega)\\
&= \left(\frac{D'-\delta'}{2}s\right)\wedge\omega\\
&= (\theta s)\wedge\omega, \quad s\in\Gamma(U,V),\omega\in\Omega_X^{p;q},\\
\theta^\dagger(s\otimes\omega) &= \frac{D''-\delta''}{2}(s\otimes\omega)\\
&= \frac{1}{2}(D''(s)\wedge\omega+s\otimes\overline{\partial}\omega-\delta''(s)\wedge\omega-s\otimes\partial\omega)\\
&= \left(\frac{D''-\delta''}{2}s\right)\wedge\omega\\
&= \big(\theta^\dagger s\big)\wedge\omega, \quad s\in\Gamma(U,V),\ \omega\in\Omega_X^{p;q}.
\end{aligned}$$

θ and $\theta^\dagger$ are adjoint:

$$\begin{aligned}
h_V(\theta^\dagger\xi,\eta) &= h_V\left(\frac{D''-\delta''}{2}\xi,\eta\right)\\
&= \frac{1}{2}\big(-h_V(\delta''\xi,\eta)+\overline{\partial}(\xi,\eta)-h_V(\xi,\delta'\eta)\big)\\
&= \frac{1}{2}\left(h_V(\xi,D'\eta)-h_V(\xi,\delta'\eta)\right)\\
&= h_V\left(\xi,\frac{D'-\delta'}{2}\eta\right)=h_V(\xi,\theta\eta), \quad \xi,\eta\in\Gamma(U,V).
\end{aligned}$$

Definition 1.2.4. The pseudo-curvature of the metric h_V is

$$G_h=(d'')^2=\overline{\partial}_V\theta+\theta\overline{\partial}_V,^{9}$$

where $d'':=\overline{\partial}_V+\theta, d'=\partial_V+\theta^\dagger$.
A flat bundle, $D^2=0$, is called harmonic if the pseudo-curvature vanishes.

[8] $D'+\delta'', D''+\delta'$ hermitian connections.

[9] $\overline{\partial}_V^2=\theta^2=0$ for a Riemann Surface.

Thus if the flat bundle is harmonic θ is a Higgs field and so we have constructed a harmonic bundle as defined for Higgs bundles, since the curvature F_h vanishes

$$F_h = (\partial_V + \overline{\partial}_V + \theta + \theta^\dagger)^2 = (D' + D'')^2 = D^2 = 0.$$

So starting with a harmonic (Higgs) bundle from which we constructed a flat connection D, we can get back to the same Higgs bundle by the scheme above, i.e. the two definitions of harmonicity are essentially the same.

Remark 1.2.5. Note that $\overline{\partial}_V + \partial_V + \theta^\dagger - \theta$ is another metric connection, since

$$\begin{aligned}
&h((\overline{\partial}_V + \partial_V + \theta^\dagger - \theta)\xi, \eta) + h(\xi, (\overline{\partial}_V + \partial_V + \theta^\dagger - \theta)\eta) \\
&= dh(\xi, \eta) + h((\theta^\dagger - \theta)\xi, \eta) + h(\xi, (\theta^\dagger - \theta)\eta) \\
&= dh(\xi, \eta) + h((\theta^\dagger - \theta)\xi, \eta) - h((\theta^\dagger - \theta)\xi, \eta) \\
&= dh(\xi, \eta).
\end{aligned}$$

In particular this connection respects the holomorphic structure $\overline{\partial}_V + \theta^\dagger$ from remark 1.2.3.

We want to add the missing part of the proof of $\mathbb{D}^2 = 0 \Leftrightarrow \partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E + \theta\theta^\dagger + \theta^\dagger\theta = 0$, i.e. to show that

Lemma 1.2.6. $\overline{\partial}_E\theta + \theta\overline{\partial}_E = 0 \Rightarrow \partial_E\theta^\dagger + \theta^\dagger_E\partial = 0.$ [10]

Proof. Start with a Higgs bundle (not necessarily harmonic). Note that for $\delta'' = \overline{\partial}_E - \theta^\dagger \Rightarrow D' + \delta'' = \partial_E + \theta + \overline{\partial}_E - \theta^\dagger$ is metric just as $D'' + \delta'$ for $\delta' = \partial_E - \theta$. Now we are in the same situation as in the flat bundle case. Next note that $(D')^2 = (D'')^2 = (\delta')^2 = (\delta'')^2 = 0$ by degree considerations. Then

$$\begin{aligned}
\overline{\partial}_E\theta + \theta\overline{\partial}_E &= (\overline{\partial}_E + \theta)^2 \\
&= \frac{1}{4}(D' + \delta' + D'' - \delta'')^2 \\
&= \frac{1}{4}(D'D'' + D''D' - D'\delta'' - \delta''D' \\
&\quad + D''\delta' + \delta'D'' - \delta'\delta'' - \delta''\delta') \\
&= -\frac{1}{4}(D' - \delta' + D'' + \delta'')^2 \\
&= -\left(\theta^\dagger + \partial_E\right)^2,
\end{aligned}$$

what proves the claim. ∎

[10] Even $\Leftrightarrow$.

Remark 1.2.7. Note that we didn't use harmonicity here. However, if D is flat then $D'D'' + D''D' = 0$. We further get

$$\begin{aligned}
&h((\delta' + \delta'')^2\xi, \eta) \\
&= 2dh((\delta' + \delta'')\xi, \eta) + h((\delta' + \delta'')\xi, (D' + D'')\eta) \\
&= 4d^2h(\xi, \eta) - 2dh(\xi, (D' + D'')) + 2dh(\xi, (D' + D'')\eta) - h(\xi, (D' + D'')^2\eta) \\
&= 4d^2h(\xi, \eta) - h(\xi, (D' + D'')^2\eta) \\
&= 0,
\end{aligned}$$

i.e. $\delta'\delta'' + \delta''\delta' = (\delta' + \delta'')^2 = 0$ and the Higgs field property becomes equivalent to $D''\delta' + \delta'D'' = D'\delta'' + \delta''D'$.

The main theorem is an equivalence of categories, so we must clarify the notion of a morphism.

Definition 1.2.8. Let (E, h_E, D_E, d''_E) be a harmonic vector bundle with flat connection $D_E = \overline{\partial}_E + \partial_E + \theta_E + \theta_E^\dagger$, $d''_E = \overline{\partial}_E + \theta$. Analogous define (F, h_F, D_F, d''_F). A map

$$\varphi : E \to F$$

is a morphism of harmonic vector bundles/a gauge transformation if

(i) $\varphi^*(D_F) = \varphi D_E$,

(ii) $\varphi^*(d''_F) = \varphi d''_E$,

(iii) $\|\varphi\|_{E\to F} \leq c_\varphi < \infty$ (cf. remark A.4.2)

Analogously in the language of sheaves.

1.2.2. Metric as a map

We proceed by considering metrics as maps into the positive-definite matrices (see A.2).

Let $\mathbb{D}$ be as before and $D_{\overline{\partial}_V + \theta^\dagger} = \overline{\partial}_V + \partial_V + \theta^\dagger - \theta$ the metric connection of the previous remark 1.2.5. Then $\mathbb{D} = D_{\overline{\partial}_V + \theta^\dagger} + 2\theta$. Let (s_i) be a $\mathbb{D}-$flat frame, i.e. $\mathbb{D}(s_i) = 0$.[11] Let H be the representation of the metric h in this frame. Then we have by remark 1.2.5

$$\begin{aligned}
dh_{ij} &= dh(s_j, s_i) = d(s_i^* H s_j) \\
&= ((\mathbb{D} - 2\theta)s_i)^* H s_j + s_i^* H (\mathbb{D} - 2\theta) s_j
\end{aligned}$$

[11] Existence by Kobayashi [Kob87], p. 5.

$$
\begin{aligned}
&= -2(\theta s_i)^* H s_j - 2 s_i^* H \theta s_i \\
&= -2\left(\left(\sum_{k=1}^{n} \theta_{ki} s_k\right)^* H s_j + s_i^* H \left(\sum_{k=1}^{n} \theta_{kj} s_k\right)\right) \\
&= -2\sum_{k=1}^{n} \left(\overline{\theta}_{ki} h_{kj} + h_{ik}\theta_{kj}\right) \\
&= -2(\theta^\dagger H + H\theta)_{ij}.
\end{aligned}
$$

and therefore $dH = -2(\theta^\dagger H + H\theta)$. Since every positive-definite matrix is invertible[12] and by degree considerations $dH = 0 \Leftrightarrow \theta = 0$.
Now use the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}_n,H}$ from A.2.20. Then we get

$$
\begin{aligned}
&\|dH\|^2_{\mathbb{H}_n,H} \\
&= \langle dH, dH\rangle_{\mathbb{H}_n,H} = 4\langle(\theta^\dagger H + H\theta), (\theta^\dagger H + H\theta)\rangle_{\mathbb{H}_n,H} \\
&= 4\mathrm{tr}(H^{-1}(\theta_z^\dagger H)H^{-1}\theta_z H)(\mathrm{d}\,\overline{z} \wedge *\overline{\mathrm{d}\,\overline{z}}) + 4\mathrm{tr}(H^{-1}(H\theta_z)H^{-1}\theta_z H)(\mathrm{d}\,z \wedge *\overline{\mathrm{d}\,\overline{z}}) \\
&+4\mathrm{tr}(H^{-1}(\theta_z^\dagger H)H^{-1}H\theta_z^\dagger)(\mathrm{d}\,\overline{z} \wedge *\overline{\mathrm{d}\,z}) + 4\mathrm{tr}(H^{-1}(H\theta_z)H^{-1}H\theta_z^\dagger)(\mathrm{d}\,z \wedge *\overline{\mathrm{d}\,z}) \\
&= 4\mathrm{tr}(H^{-1}\theta_z^\dagger\theta_z H)(-i\,\mathrm{d}\,\overline{z} \wedge \mathrm{d}\,z) + 4\mathrm{tr}(\theta_z\theta_z^\dagger)i\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} \\
&= 4i\mathrm{tr}(2\theta_z\theta^\dagger)\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} \\
&= 8i\|\theta\|_F^2\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} \\
&= 16\|\theta\|_F^2\,\mathrm{d}\,x \wedge \mathrm{d}\,y,
\end{aligned}
$$

where we used $\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} = -2i\,\mathrm{d}\,x \wedge \mathrm{d}\,y$.
Thus we have the energy of h as

$$
\begin{aligned}
\mathbb{E}_{\tilde{\varepsilon}}(h) &= \int_{A_{\tilde{\varepsilon},1}} \frac{\|dH\|^2_{\mathbb{H}_n,H}}{2} = 4i\int_{A_{\tilde{\varepsilon},1}} \langle\theta,\theta\rangle_{HS}\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} \\
&= 4i\int_{A_{\tilde{\varepsilon},1}} \|\theta\|_F^2\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}^{13}
\end{aligned}
$$

We want to show that h is harmonic, i.e. minimizes the energy functional.
Recall the construction of the previous subsection. We constructed first an operator δ' resp. δ'' (dependent on the metric) and defined θ with the help of these operators. We will redo the process for a variated metric.

Now consider an arbitrary variation $H+\varepsilon HU$, U any matrix-valued function which is at least twice differentiable on $\overline{B}$ and vanishes on the boundary of B^*. Note that

[12] similar to a diagonal matrix with positive diagonal entries.

[13] To ensure integrability use $A_{\tilde{\varepsilon},1} = \{z \in \mathbb{C} | 0 < \varepsilon_1 \leq |z| \leq 1\}$ closed and that we have no singularities outside the punctures. For simplicity we used that the boundary $\partial B_1 \subset X$ - possible after rescaling.

since det is polynomial (and therewith the invertible matrices are an open subset of $\mathbb{C}^{n\times n}$) $H+\varepsilon HU$ is invertible for ε small enough, $H\in \mathrm{Gl}_n(\mathbb{C})$. Let δ''_U denote the new extension of D' to a metric connection and $\delta''_\Delta + O(\varepsilon^2) = \delta''_U - \delta''$. We will show that $\delta''_\Delta = \varepsilon\delta'' U$.

$$\begin{aligned}\overline{\partial}(H+\varepsilon HU) &= (D'')^*(H+\varepsilon HU) + (H+\varepsilon HU)\delta''_U \\ \Rightarrow \varepsilon\overline{\partial}(HU) &= (H+\varepsilon HU)(\delta''_U - \delta''_U) \\ &= (H+\varepsilon HU)(\delta''_\Delta + O(\varepsilon^2)).\end{aligned}$$

In order to calculate $\overline{\partial}HU$ change to the endomorphism bundle. In A.4 we see that we get induced connections on $\mathrm{End}(E)$ and that $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ is $D_{\mathrm{End}}-$flat. The metric is $h_{\mathrm{End}}(A,B) = \mathrm{tr}(HAHB^*)$. We get

$$\begin{aligned}\overline{\partial}h_{\mathrm{End}}(U,E_{ij}) &= \overline{\partial}\mathrm{tr}(HUHE_{ji}) \\ &= \overline{\partial}\sum_{s,r,k,l=1}^{n} h_{sr}U_{rk}h_{kl}\delta_{lj}\delta_{si} \\ &= \overline{\partial}\sum_{r,k=1}^{n} h_{ir}U_{rk}h_{kj} \\ &= \overline{\partial}(HUH)_{ij}.\end{aligned}$$

and therefore

$$\begin{aligned}\overline{\partial}(HUH)_{ij} &= \overline{\partial}h_{\mathrm{End}}(U,E_{ij}) \\ &= h_{\mathrm{End}}(\delta''_{\mathrm{End}}U,E_{ij}) + h_{\mathrm{End}}(D''_{\mathrm{End}}U,E_{ij}) \\ &= h_{\mathrm{End}}(\delta''U - U\delta'', E_{ij}) \\ &= \mathrm{tr}(H\delta''UHE_{ji} - HU\delta''HE_{ji}) \\ &=^{14} \sum_{s,r,k,l,m=1}^{n} h_{sr}\delta''_{rk}U_{kl}h_{lm}\delta_{mj}\delta_{si} - h_{sr}U_{rk}\delta''_{kl}h_{lm}\delta_{mj}\delta_{si} \\ &= \sum_{r,k,l=1}^{n} h_{ir}\delta''_{rk}U_{kl}h_{lj} - h_{ir}U_{rk}\delta''_{kl}h_{lj} \\ &= (H\delta''UH)_{ij} - (HU\delta''H)_{ij}.\end{aligned}$$

Since H is invertible as a positive-definite hermitian matrix and $\overline{\partial}$ and δ'' contribute each one differential on each side of the equation, we get $\overline{\partial}(HU) = \delta''U - U\delta'' =$

[14] $\delta'' = \overline{\partial} + C, \delta''_{rk} = \overline{\partial} + C_{rk}$. Here $\overline{\partial}$ is the usual $\mathbb{C}-$operator, which can be informally written as $\mathrm{diag}(\overline{\partial},\ldots,\overline{\partial})$.

$\delta''_{\text{End}}(U)$.
Thus for $\delta''_\Delta = \varepsilon\delta''_{\text{End}}(U) + O(\varepsilon^2)$

$$\begin{aligned} \varepsilon\overline{\partial}(HU) &= H\delta''_{\text{End}}(U) = (H+\varepsilon HU)(\delta''_\Delta + O(\varepsilon^2)) \\ \Rightarrow 0 &= HO(\varepsilon^2) + H\varepsilon(\delta''_\Delta + O(\varepsilon^2)), \end{aligned}$$

and since $\varepsilon HU\delta''_\Delta$ is of order 2 in ε we can construct a term in $O(\varepsilon^2)$ to shorten $\varepsilon HU\delta''_\Delta$ producing a possibly higher order deviation, i.e. always $O(\varepsilon^2)$. Thus $\delta''_\Delta + O(\varepsilon^2) = \delta''_U - \delta''$.
Analogous $\delta'_U - \delta' = \delta'_\Delta + O(\varepsilon^2) = \varepsilon\delta'_{\text{End}}(U) + O(\varepsilon^2)$.
Recall $\theta = \frac{D'-\delta'}{2}, \theta^\dagger = \frac{D''-\delta''}{2} \Rightarrow \theta_U = \frac{D'-\delta'_U}{2} = \frac{D'-\delta'-\delta'_\Delta - O(\varepsilon^2)}{2} = \theta - \frac{\varepsilon}{2}\delta' U + O(\varepsilon^2)$ and $\theta^\dagger_U = \theta^\dagger - \frac{\varepsilon}{2}\delta' U + O(\varepsilon^2)$. Then proceed as in the unperturbed case to get[15]

$$\begin{aligned} &\mathbb{E}_{\tilde{\varepsilon}}(H+\varepsilon U) \\ &= 4\left(\int_{A_{\tilde{\varepsilon},1}} \langle\theta^\dagger,\theta^\dagger\rangle + \langle\theta,\theta\rangle - 2\langle\theta^\dagger, \frac{\varepsilon\delta'(U)}{2}\rangle - 2\langle\theta, \frac{\varepsilon\delta''(U)}{2}\rangle + O(\varepsilon^2)\right) \\ &= 4\left(\int_{A_{\tilde{\varepsilon},1}} \langle\theta^\dagger,\theta^\dagger\rangle + \langle\theta,\theta\rangle - \varepsilon\langle\theta^\dagger, \delta'(U)\rangle - \varepsilon\langle\theta, \delta''(U)\rangle + O(\varepsilon^2)\right). \end{aligned}$$ [16]

Since the integrand is $\varepsilon-$differentiable and the differential is integrable on $A_{\tilde{\varepsilon},1}$ we may interchange the $\varepsilon-$differentiation and integration

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\mathbb{E}_{\tilde{\varepsilon}}(H+\varepsilon U) &= -4\left(\int_{A_{\tilde{\varepsilon},1}} \langle\theta^\dagger, \delta'_{\text{End}}(U)\rangle - \langle\theta, \delta''_{\text{End}}(U)\rangle + O(\varepsilon)\right) \\ \Rightarrow \frac{\partial}{\partial\varepsilon}\mathbb{E}_{\tilde{\varepsilon}}(H+\varepsilon U)\Big|_{\varepsilon=0} &= -4\left(\int_{A_{\tilde{\varepsilon},1}} \langle\theta^\dagger, \delta'_{\text{End}}(U)\rangle - \langle\theta, \delta''_{\text{End}}(U)\rangle\right). \end{aligned}$$

Therefore we have a critical point iff the integral does vanish for all U, i.e. if the integrand vanishes.
Now we use Hodge theory (cf. A.1.34) to apply the $*-$adjoint:[17]

$$\begin{aligned} \langle\theta^\dagger, \delta''_{\text{End}}(U)\rangle + \langle\theta, \delta'_{\text{End}}(U)\rangle &= \langle(\delta'')^*_{\text{End}}\theta^\dagger, U\rangle + \langle(\delta')^*_{\text{End}}(\theta), U\rangle \\ &= \langle(\delta'')^*_{\text{End}}\theta^\dagger + (\delta')^*_{\text{End}}\theta, U\rangle. \end{aligned}$$

Note that $\delta''_{\text{End}}(U) = \delta'' U - U\delta''$ on a $0-$form U and $\delta''_{\text{End}}(\theta) = \delta''\theta - \theta\delta''$ on the $1-$forms θ.

[15] The inner product, i.e. the trace is defined since as mentioned above for ε small enough $H + \varepsilon HU$ invertible.
[16] Again integrability by differentiability of U, θ, δ outside the punctures.
[17] Note that the requirement $U|_{\partial A_{\tilde{\varepsilon},1}} = 0$ is satisfied.

In order to show $(\delta'')^*\theta^\dagger + (\delta')^*\theta$ we apply the Kähler identities from A.1.36 and receive $-i\Lambda((D'\theta^\dagger+\theta^\dagger D')+i\Lambda(D''\theta+\theta D'')$. Recall that $D' = \partial_V+\theta, D'' = \overline{\partial}_V+\theta^\dagger$. Therefore

$$\begin{aligned}
&i\Lambda D'_{\mathrm{End}}\theta^\dagger - i\Lambda D''_{\mathrm{End}}\theta \\
&= i\Lambda\left(\partial_V\theta^\dagger + \theta\theta^\dagger + \theta^\dagger\theta + \theta^\dagger\partial_V - \overline{\partial}_V\theta - \theta^\dagger\theta - \theta\theta^\dagger - \theta\overline{\partial}_V\right) \\
&= i\Lambda\left((\partial_V)_{\mathrm{End}}\theta^\dagger - (\overline{\partial}_V)_{\mathrm{End}}\theta\right).
\end{aligned}$$

Since we saw in the previous subsection that $(\partial_V)_{\mathrm{End}}\theta^\dagger = 0$ if $(\overline{\partial}_V)_{\mathrm{End}}\theta = 0$, the map h is harmonic if $(\overline{\partial}_V)_{\mathrm{End}} = 0$, i.e. if the flat bundle is harmonic on $A_{\tilde{\varepsilon},1}$ for all $\tilde{\varepsilon} > 0$.

1.3. Tameness

In this short section we will given the definition of tameness as well as some basic properties of multivalued eigenvalues.

Definition 1.3.1. Let $E \xrightarrow{\pi} X$ be a holomorphic vector bundle. $s : U \to \mathscr{P}(E)$, $U \subset X$[18] is a multivalued section iff $\exists \tilde{s} : \tilde{X} \to E$ holomorphic s.t. $s(z) = \tilde{s} \circ \tilde{\pi}^{-1}(\{z\})$ and $\pi\tilde{s}\tilde{\pi}^{-1}(\{z\}) = \{z\}$ for all $z \in U$.[19]
Note that on a (small enough) open set $U \subset X$, $\tilde{\pi}^{-1}$ decomposes into a disjoint union of sets U_j, each diffeomorphic to U. So we can identify (via $U_j \simeq U$) $\tilde{s}$ with a collection of sections s_j on U. Define $\theta(\tilde{s})$ as the unique holomorphic function on $\tilde{X}$ which is identified with $\theta(s_j)$ for all j.[20]

Remark 1.3.2. In literature (and in the following) one usually writes s as a map $s : U \to E$.

Definition 1.3.3. An eigenvalue of θ is a possibly multivalued holomorphic one-form λ such that

$$\theta s = s \otimes \lambda,$$

for a multivalued section s.
Locally we can write $\lambda \,\mathrm{d}\, z$ and then λ is a multivalued holomorphic function.

Remark 1.3.4. Locally (on a small enough contractible set U) θ can be written as a matrix $\Theta \ \mathrm{d}\, z$ w.r.t. the local frame field s_i and a local coordinate z. Then λ (in the eigenvalue $\lambda \,\mathrm{d}\, z$) is a solution of

$$\det(\Theta(x) - \lambda(x)E) = 0.$$

[18] $\mathscr{P}(E) = \bigcup_{x\in X} \mathscr{P}(E_x), E_x = \pi^{-1}(\{x\}), \mathscr{P}(E_x)$ power set of E_x.
[19] $\tilde{X}$ the universal cover of X and $\tilde{\pi} : \tilde{X} \to X$ the corresponding projection.
[20] As θ is a sheaf morphism, the $\theta(s_j)$ (as functions on U_j) glue together.

Since $\mathbb{C}$ is algebraically closed there are n (not necessarily disjoint) solutions on every fiber. But the entries of Θ are holomorphic and so is det as a polynomial, i.e. locally the solutions λ_i are meromorphic functions of order at least $\frac{1}{n}$ at the singularity.[21] Now if necessary minimize U such that there is at most one singularity s_i.[22] Then the λ_i are holomorphic on $U \setminus \{s\}$ (resp. U if there is no singularity) and they are eigenvalues. They can be multivalued but there are at most n branches since $\det(\Theta(x) - \lambda(x)E) = 0$ has at most n solutions on each fiber.
Note that around a puncture $s_i \in \overline{X} \setminus X$ there are still eigenvalues, but their order at the puncture (considered as a meromorphic extension) is no longer bounded by $\frac{1}{n}$ since θ resp. sections can in general not be extended holomorphically over the puncture.
As usual if two eigenvalues resp. sections on different open sets coincide on a connected open intersection of the sets, then they extend each other holomorphically.

Definition 1.3.5. A harmonic bundle is tame if for all eigenvalues λ_i of θ and all punctures s_j:

$$|\lambda_i| \leq C/|z_j|,$$

for a local coordinate z_j of a small enough neighbourhood of s_j, where C is any constant and $z_j(s_j) = 0$.
The absolute value (w.r.t. some local coordinate z) of a local one-form $\lambda_i = \mu_i \, \mathrm{d}\, z$ is defined as $|\lambda_i| := |\mu_i|$.

1.4. Decomposition of the Higgs Field

In the first subsection we will see how we may use finite branched covers to construct an operator ψ such that the eigenvalues of θ and ψ have the same residue. Then using Schur decomposition we will get the upper triangular matrices $\theta = \sigma + \tau$, $\psi = \alpha + q$, which we may further decompose into block upper triangular and block diagonal parts. The second part of this section will provide us with some basic estimates of these operators.
For the proof of the main estimate we will apply the fact that curvature decreases in subbundles to a section of the endomorphism bundle, namely θ. It will lead us to a first inequality, widely used in the following two sections on the actual main estimate.
Further we will add two examples that might help to illustrate the concept of a Higgs bundle. They will prove important in the following chapters.

[21] cf. the Main Estimate below.
[22] Possible since the subset of singularities has no accumulation point for a meromorphic function.

Please note that from now on we will usually work on a punctured disc. Since all our punctures are isolated this is certainly justified.

1.4.1. Eigenvalues, Eigenspaces and Finite branched Covers

As a model to describe the behaviour around a puncture we choose the unit disc $B^* = B\backslash\{0\}, B := \{z \in \mathbb{C} | |z| \leq 1\}$ with the euclidean metric $|z \, \mathrm{d}\, z|^2 = z\overline{z} \; \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}$; z the standard coordinate. The distance from the puncture is sometimes denoted $r = |z|$. Assume that there are no further punctures on the boundary $r = 1$; if necessary we rescale.
Further choose a frame $(s_i)_{1\leq i\leq n}$ for our bundle E on B^* and denote Θ the matrix representation of θ w.r.t. this frame. If we apply $\Theta(x)$ to a vector $e_x = \sum_{i=1}^n \alpha_i s_i(x)$ in E_x this can be understood as the evaluation of $\Theta(s), s = \sum_{i=1}^n \alpha_i s_i$ at x.
For a tame harmonic bundle $(E, \theta, \overline{\partial}_E, h)$ the eigenvalues of θ are bounded by definition: $|\lambda_i| \leq C/r$. They are holomorphic on B^* resp. on some smaller punctured disc around the puncture 0. If necessary we rescale the whole process to use B^*.

Lemma 1.4.1. $\exists \varepsilon > 0$ such that $\exists C > 0$ and $\forall \lambda_j$ eigenvalue $\exists a_i \in \mathbb{C}$

$$\left| \lambda_j^i - a_i \frac{\mathrm{d}\, z}{z} \right| \leq \frac{C}{|z|^{1-\varepsilon}}, \quad a_i \in \mathbb{C} \text{ pairwise disjoint.} \tag{1.4.1.1}$$

This holds independent of the branch of λ_j, i.e. for all values. In particular ε can be chosen greater or equal $1/n$.

In order to understand the claim we need to find a suitable description of multivalued functions. One way to look at multivalued eigenvalues is to use a branched cover. The open punctured unit disc B^* is covered by itself via the map $\pi_{B^*} : B^* \to B^*, z \mapsto z^k$ for some positive integer k. The map is obviously surjective and proper. Hence we have on B a finite branched cover $\pi_B : B \to B, z \mapsto z^N$. To clarify notation write $\pi_D : B \to B = D, z \mapsto z^N$. θ extends to an endomorphism θ_{D^*} on D by $\theta_{D^*}\pi_D = \theta$. Denote by u the local coordinate of D with $z = u^N$. The eigenvalue equation $\det(\Theta - \lambda E) = 0$ written as $\sum_{i=0}^n p_i(z)\lambda^i(z) = 0$ becomes $\sum_{i=0}^n p_i(u^N)\lambda^i(u^N) = \sum_{i=0}^n p_{D^*,i}(u)\lambda_{D^*}^i(u)$ where we used that the p_i are sums of products of entries of Θ and therefore transform as $\theta_{D^*}\pi_D = \theta$. Having a solution λ on $B_-^* = B^* \setminus (B^* \cap \mathbb{R}_-)$[23] there is a corresponding solution λ_{D^*} on a component $B_0 \subset \pi_D(B_-^*)$.[24] We can extend λ_{D^*} to the next component of $B_1 \subset \pi_D(B_-^*)$ along some path in D. After at most n such extension we find a component B_l such that $\lambda_{D^*}\pi_D|B_0 = \lambda_{D^*}\pi_D|B_l = \lambda$, $0 \leq l \leq n$. So if we choose N to be $n!$ we get for each

[23] Respectively any other branch cut.
[24] Choosing a component corresponds to choosing a branch of the Nth root.

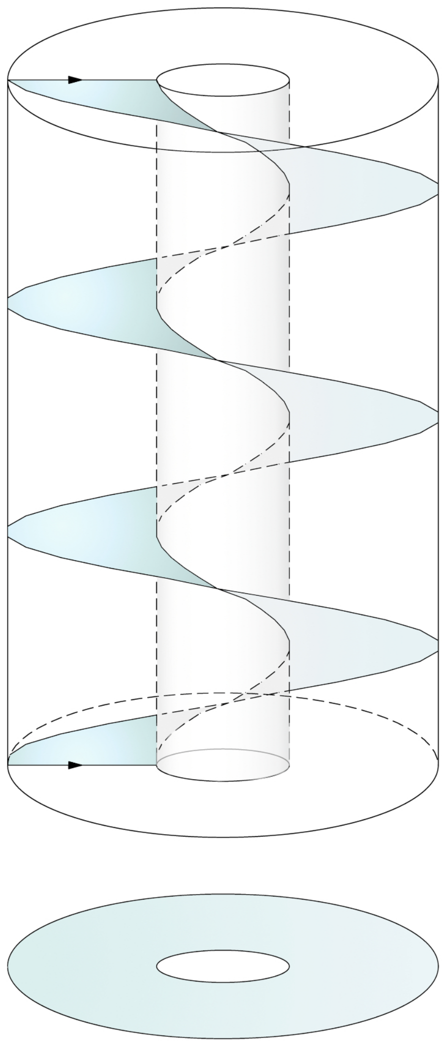

Figure 1.1: A multivalued section - possibly an eigensection of θ. It can be closed by gluing along the arrows.

solution λ on B^* a meromorphic function λ_{D^*} on D such that $\lambda_{D^*}\pi_D = \lambda$.
Over D^* we have the pullback bundle $E_{D^*} = \pi_D^* E = \{(x,e) \in D^* \times E | \pi_D(x) = \pi(e)\} \subset D \times E$.[25]

proof of 1.4.1. Note that for the Laurent series expansion $\lambda_{D^*} = \sum_{k=-m}^{\infty} q_k u^k$, for tame bundles with $m = N$, we have $q_k = 0$, for all $-N < k < \frac{N}{n}$. This is because $\lambda_{D^*}\pi_D$ shall be a solution of a degree n algebraic equation with holomorphic coefficients[26], i.e. the nth-power of $\lambda_{D^*}\pi_D$ has to be an integer, in particular $nk \geq -N \Rightarrow k \geq -\frac{N}{n}$. Of course this extends to $q_k = 0\ \forall -1 \leq l \forall k$ with $lN < k < l + \frac{N}{n}$. Still there could be a function with k branches $\gcd(k,n) < k$, i.e. a solution which has no single-valued holomorphic counterpart in D for $N = n$. To see this consider the algebraic equation $x\lambda + \lambda^3 = 0$ with solutions $\lambda_1 = 0$ and $\lambda_{2/3} = \pm\sqrt{x}$ (each 2 branches) and $\gcd(2,3) = 1 < 2$. However, for our purpose $N = n!$ is sufficient.
We can write $\lambda_{D^*} = \sum_{k=-m}^{\infty} q_k u^k$ as Laurent series and get $\lambda = \sum_{k=-m}^{\infty} q_k \exp(\frac{k\ln(z)+2\pi lk}{N})$, with $\exp(\frac{k\ln(z)+2\pi lk}{N})$ the lth branch of λ for $1 \leq l \leq n$ (or less than n in the case of less branches).[27] This shows in particular that $z^{-1} = u^{-N} = \exp(-\ln(z) + 2\pi l) = \exp(-\ln(z))$ independent of l, and therefore a_i (here q_{-N}) constant on all branches of a multivalued solution. □

By the lemma we may uniquely define a function $m : \{1,\dots,n\} \to \{1,\dots,\tilde{k}\}, 0 \leq \tilde{k} \leq n$ for which

$$\left|\lambda_j - a_{m(j)}\frac{\mathrm{d}\,z}{z}\right| \leq \frac{C}{|z|^{1-\varepsilon}}.$$

m becomes monotonically increasing after reordering the λ_j.

Remark 1.4.2. If in lemma 1.4.1 already $|\lambda_j| \leq \frac{C}{|z|^{1-\varepsilon}}$ then $a_{m(j)} = 0$.

Let

$$E_{a_i} := \mathrm{span}\{v | (\Theta - \lambda_j E)^k v = 0 \text{ for some } k \in \mathbb{N} \text{ and } m(j) = a_i\}$$

the union of the generalized eigenspaces which correspond to a_i. By the meromorphy of the eigenvalues we know that distinct eigenvalues don't coincide on a dense

[25] This is a holomorphic vector bundle on D with projection $\pi_{E_{D^*}}[x,e] = x$ and equipped with the subspace topology. Local trivializations and transition functions $(U_i, \varphi_{U_i}, g_{ij})$ become $(\pi_D^{-1}(U_i), \mathrm{id}_{D^*} \times \mathrm{pr}_2(\varphi_{U_i} \circ \mathrm{pr}_2), \pi_D^* g_{ij} = g_{ij} \circ \pi_D^*)$. Furthermore any section $s \in \Gamma(B^*, E)$ induces a section $\pi_D^* s \in \Gamma(D^*, E)$ by $\pi_D^* s(x) := (x, s \circ \pi_D(x))$ and a metric h becomes $h_{D^*}((x,e),(x,f)) := h(e,f)$; $\dim E_{D^*} = \dim E$. See for example Munteanu [Mun04], p. 22ff.
[26] holomorphic on the punctured disc, meromorphic on the whole disc.
[27] $j + 1 \leq l \leq n$ for $j \neq 0 \mod n$ will lead to another eigenvalue. For $l > n$ branches will repeat.

subset of an open connected neighbourhood of 0 by the identity theorem. Hence we may assume, after possible rescaling, that distinct eigenvalues do not intersect on our punctured neighbourhood, i.e. their algebraic multiplicity is constant. Therefore $\operatorname{rank}(\Theta - \lambda_j E)^k$ is constant for k big enough and thus we know that the generalized eigenspaces of λ_j span a holomorphic subbundle of E.[28] Construct a basis of generalized eigenvectors as usual, i.e. start with some (multivalued) eigenvector v_1 to the eigenvalue λ, add all linear independent eigenvectors to the same eigenvalue, say $v_2, \dots, v_r$, than solve $(\Theta - \lambda E)v_{r+1} = v_1$ to find a generalized eigenvector and so on until we have all generalized eigenvectors to λ. Then start over with the next eigenvalue. This construction ensures that

$$\Theta v_{r+1} = v_1 + \lambda v_{r+1} \subset \operatorname{span}\{v_j | 1 \leq j \leq r+1\},$$

i.e. $\Theta(\operatorname{span}\{v_j | 1 \leq l\}) \subset \operatorname{span}\{v_j | 1 \leq j \leq l\}$. Denote $V_l := \operatorname{span}\{v_l\}$ and

$$\mathcal{F}_l := \bigoplus_{j=1}^{l} V_j$$

the corresponding complete flag , i.e. a filtration with $\operatorname{Gr}_i(\mathcal{F}) = \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ of dimension 1. The $\operatorname{Gr}_i(\mathcal{F}) = \operatorname{span}\{v_i\} = V_i$ are not necessarily subbundles since the v_i are in general multivalued. However, $E_{a_i} = \bigoplus_{j,m(j)=i} \operatorname{span}\{v_j\} = \bigoplus_{j=1}^{n} V_j \delta_{m(j)i}$ is a subbundle as an union of generalized eigenbundles. In particular the classical direct sum decomposition into generalized eigenspaces ensures $E_{a_i} \cap E_{a_j} = \emptyset$ for $a_i \neq a_j$.

Definition 1.4.3. An endomorphism $f : V \to V$ is called semi-simple iff $\forall W \subset V$ $f-$invariant subspace, i.e. $f(W) \subset W$, $\exists W' \subset V$ $f-$invariant subspace such that $W \oplus W' = V$. A sheaf homomorphism $\Gamma(U, E) \to \Gamma(U, E) \otimes \Omega_U^{0,1}(E)$ is semi-simple if the restriction to each fiber resp. stalk is.

Remark 1.4.4. Every diagonalizable endomorphism of a finite-dimensional vector space is semi-simple.

Define a semi-simple endomorphism $\phi : \Gamma(B^*, E) \to \Gamma(B^*, E) \otimes \Omega_{B^*}^{0,1}(E)$ in the basis (v_i) by

$$\tilde{\Phi} = \begin{pmatrix} a_{m(1)} & 0 & \cdots & 0 \\ 0 & a_{m(2)} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{m(n)} \end{pmatrix} \frac{\mathrm{d}\, z}{z}.$$

[28] Here we consider θ as a bundle homomorphism.

Like Θ, $\tilde{\Phi}$ respects the filtration since it is diagonal (hence semi-simple) and Φ has eigenspaces V_j and eigenvalues $a_{m(j)}$. Transform $\tilde{\Phi}$ to Φ in the initial frame (s_i). Of course this does not change the eigenspaces of $\tilde{\Phi}$ resp. Φ. Later on we make some norm estimates and so we only want unitary transformations. So we stick with (s_i) for now.

Remark 1.4.5. In general for matrices A, B with the same generalized eigenspaces E_i and B with geometric equals algebraic multiplicity, i.e. B diagonalizable, we have A and B commuting. Since the vector space decomposes into the E_i and both A and B respect this decomposition we can restrict to one of the generalized eigenspaces. Choose $v_i \in E_i$ then $ABv_i = A\lambda_i^B v_i = \lambda_i^B Av_i$ and $BAv_i = \lambda_i^B Av_i$, since $Av_i \subset E_i$ eigenvalue to B. Thus $AB = BA$.
By definition θ and Φ commute and naturally a base transformation does not influence that, $T^{-1}ATT^{-1}BT = T^{-1}ABT = T^{-1}BAT = T^{-1}BTT^{-1}AT$.

Recall the Gram-Schmidt process to produce via $e_1 := \frac{v_1}{\|v_1\|_h}, \tilde{e}_k := v_k - \sum_{j=1}^{k-1} \frac{h(v_k,\tilde{e}_j)}{h(\tilde{e}_j,\tilde{e}_j)}\tilde{e}_j, e_j := \frac{\tilde{e}_j}{\|e_j\|_h}$ an orthonormal frame. This frame is smooth by construction and the smooth inner product. We know that $\mathcal{F}_i = \operatorname{span}\{e_j | 1 \leq j \leq i\}$. This frame is still multivalued in general. In fact as long as we require a continuous frame any non-subbundle V_i will lead to a multivalued basis element. Note that $E_i := \operatorname{span}\{e_i\} \neq V_i$ in general.[29]. Moreover every generalized eigenbundle is spanned by some e_i, in particular $E_{a_i} = \operatorname{span}\{e_j | m(j) = i\}$.
Now let us apply Schur decomposition. Since Φ and Θ commute we can construct the normal form simultaneously for both matrices. The $e_i(z)$ are eigenvectors (resp. generalized eigenvectors) so Θ and Φ have (Schur) normal forms in this frame, i.e. are upper triangular with eigenvalues on the diagonal. We write

$$\theta = \theta_z \,\mathrm{d}\, z = \sigma \,\mathrm{d}\, z + \tau \,\mathrm{d}\, z, \quad \phi = \phi_z \,\mathrm{d}\, z = \alpha \,\mathrm{d}\, z + q \,\mathrm{d}\, z,$$

where $\theta_z, \sigma, \tau, \phi_z, \alpha, q$ are the matrix representations:

$$\sigma \,\mathrm{d}\, z = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad \alpha = \operatorname{diag}(a_1, \ldots, a_{\tilde{k}}),$$

are the diagonal parts, while τ and q are strictly upper triangular.[30]

Remark 1.4.6. Note that $\tilde{\Phi}$ has eigenspaces V_i. Since $\tilde{\Phi}$ acts like $a_{m(i)}E$ on the generalized eigenspace of λ_i, $\tilde{\Phi}$ is single-valued and so ϕ is. The eigenspaces of α are just the E_i and α acts also like $a_{m(i)}E$ on the generalized eigenbundle of λ_i, i.e. α single-valued.[31] This is because the projections of two distinct v_i resp. e_i

[29] In Simpson E_i and V_i are interchanged.
[30] We will often drop "strictly", but we will always mean it.
[31] Single-valued as functions on B^*. On the cover this corresponds to periodicity. More precisely $q_k = \delta_{\mathbb{Z}N,k} q_k$ for the Laurent coefficients.

may differ but will still lie in the same generalized eigenspace by the subbundle property. Hence ϕ, α are invariant under the projection, i.e. single-valued.
Define $\beta := \sigma - \alpha$, than

$$\theta - \alpha = \sigma - \alpha + \tau = \beta + \tau, \quad \text{and} \quad \phi - \alpha = q,$$

are single-valued as differences of single-valued functions.

Definition 1.4.7. Decompose further

$$\begin{aligned} M^+ &:= \bigoplus_{0 \le j < i \le n} \mathrm{Hom}(E_{a_i}, E_{a_j}) \subset \mathrm{End}(E) \\ M^0 &:= \bigoplus_{0 \le i \le n} \mathrm{End}(E_{a_i}) \subset \mathrm{End}(E) \\ M^- &:= \bigoplus_{0 \le i < j \le n} \mathrm{Hom}(E_{a_i}, E_{a_j}) \subset \mathrm{End}(E) \end{aligned}$$

$\bigoplus_{0 \le j < i \le n} \mathrm{Hom}(E_{a_i}, E_{a_j})$ is the set of endomorphisms $E \to E$ which restrict to homomorphisms E_{a_i} to $\bigoplus_{j<i}^n E_{a_j}$. Since the E_{a_j} are subbundles, the bundle resp. sheaf homomorphisms are defined as usual, in particular single-valued.
M^+ are the block upper triangular matrices, M^- the block lower triangular matrices, and M^0 the block diagonal matrices.

Remark 1.4.8. As described in remark 1.4.6 α has eigenspaces E_i and ϕ eigenspaces V_i. In particular both preserve the filtration $\mathcal{F}_i = \bigoplus_{j=1}^l V_j = \bigoplus_{j=1}^l E_j$. Let $w \in \mathcal{F}_i \setminus \mathcal{F}_j$, $m(i) = m(j) + 1$. Then $\exists w_v, w_e \in \mathcal{F}_j$ and c_{v_k}, c_{e_k} constants such that $w = w_v + \sum_{k=0}^n c_{v_k} v_k \delta_{m(k)i} = w_e + \sum_{k=0}^n c_{e_k} e_k \delta_{m(k)i}$ with $\mathcal{F}_j \ni w_e - w_v = \sum_{k=0}^n \delta_{m(k)i}(c_{v_k} v_k - c_{e_k} e_k)$ and

$$\begin{aligned} q(w) &= \phi(w) - \alpha(w) = \phi(w_v + \sum_{k=0}^n c_{v_k} v_k \delta_{m(k)i}) - \alpha(w_e + \sum_{k=0}^n c_{e_k} e_k \delta_{m(k)i}) \\ &= \phi(w_v) - \alpha(w_e) + \sum_{k=0}^n a_{m(i)} c_{v_k} v_k \delta_{m(k)i} - a_{m(i)} c_{e_k} e_k \delta_{m(k)i} \\ & \underbrace{\phi(w_v) - \alpha(w_e)}_{\in \mathcal{F}_j} + a_{m(i)} \sum_{k=0}^n \delta_{m(k)i} \underbrace{(c_{v_k} v_k - c_{e_k} e_k)}_{\in \mathcal{F}_j} \in \mathcal{F}_j, \end{aligned}$$

i.e. $q(\mathcal{F}_i) = \mathcal{F}_j$ or $q(E_{a_j}) \subset \bigcup_{i, m(i)<j} E_{a_i} \Rightarrow q \in M^+$.

Definition 1.4.9. Decompose $\tau = \tau^0 + \tau^+$ with τ^0 block diagonal and $\tau^+ \in M^+$.

Lemma 1.4.10. τ^+ is single-valued or equivalently the decomposition $\tau = \tau^0 + \tau^+$ exists; the decomposition into block diagonal and block upper diagonal part is obvious, but we claim further that $\tau^+ \in M^+$, i.e. single-valued.

Proof. $\beta + \tau$ is single-valued by 1.4.6, whereas β and τ could be multivalued. More precisely β is multivalued if and only if there is a multivalued eigenvalue λ_j. If there was no multivalued eigenvalue, everything would be single-valued and the lemma would follow trivially. Therefore assume β multivalued. Since θ and α preserve the generalized eigenbundles so does β. Then for every $w \in \mathcal{F}_i \setminus \mathcal{F}_j, m(i) = m(j) + 1$ resp. $\exists s \in E_{a_i}$ single-valued, e.g. a combination of the (s_i)[32], $\beta s \in E_{a_i}$ multivalued. But $\tau^0 s \in E_{a_i}$ and $\tau^+ s \in \mathcal{F}_j$ leads to $(\beta + \tau)s$ single-valued $\Rightarrow \underbrace{(\beta + \tau^0)s}_{\in \mathcal{F}_i \setminus \mathcal{F}_j} \oplus \underbrace{\tau^+ s}_{\in \mathcal{F}_j}$ single-valued, i.e. $\beta + \tau^0$ single-valued, τ^+ single-valued on each E_{a_i} and hence on E. □

1.4.2. Norm estimates

Remark 1.4.11. All estimates in the rest of the chapter are meant pointwise and branchwise, i.e. the calculation works if we compare corresponding branches, e.g. of σ and τ^0. Especially later on, when we differentiate w.r.t. z it might seem more comfortable to have no discontinuity at branch cuts, i.e. consider the whole process over D^*. Naturally we will use the norm $\mathrm{d}\, u \wedge \mathrm{d}\, \overline{u}$ on D^* as well. By the transformation formula we have $\mathrm{d}\, z = \pi_D^*(\mathrm{d}\, u)$ or informally

$$\mathrm{d}\, z = \frac{\partial u^N}{\partial u} \mathrm{d}\, u = (N) u^{N-1} \mathrm{d}\, u = N \frac{z}{u} \mathrm{d}\, u$$

resp. $\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z} = N^2 \frac{|z|^2}{|u|^2} \mathrm{d}\, u \wedge \mathrm{d}\, \overline{u}$. We will see in the subsection after the next, 1.4.16, that for Simpson's main estimate we consider only the matrix part and the form part shortens out. There is one more slightly different behaviour in the branched cover case, which we will mention at the point where is becomes important. Everywhere else we may just replace z by u, B by D and E by E_{D^*}.

In the following proof we will use the Frobenius norm $\|\cdot\|_F$, which is equivalent to all other norms on $\mathbb{C}^{n\times n}$, in particular $\|\cdot\|_2 \leq \|\cdot\|_F \leq \sqrt{n}\|\cdot\|$. This choice might need some explanation. On the one hand side after the transition to the $h-$orthonormal frame (e_i), $h = E$ in this frame, the adjoint of an operator in matrix form is just the adjoint matrix and the norm is just the euclidean norm. So it seems somehow natural to choose the induced euclidean norm. But this choice makes life rather difficult. The reason for our choice of the Frobenius norm will

[32] E_{a_i} subbundle guarantees the existence of a single-valued frame.

be given in the next subsection. The advantage of the Frobenius norm is that it is entry-wise and so we are able to split up our matrix θ:

$$\begin{aligned}\|\theta\|_F^2 &= \|\sigma+\tau\|_F^2 = \sum_{i,j=1}^{n} |\theta_{ij}|^2 = \sum_{i,j=1}^{n} |\sigma_{ij}\delta_{ij} + \tau_{ij}\delta_{i<j}|^2 \\ &= \sum_{i,j=1}^{n} |\sigma_{ij}|^2 + \sum_{i,j=1}^{n} |\tau_{ij}|^2 = \|\sigma\|_F^2 + \|\tau\|_F^2.\end{aligned}$$

Lemma 1.4.12. (i) $\|[\theta,\theta^\dagger]\|_F \geq c_1 \|[\tau,\tau^\dagger]\|_F$.

(ii) $\|[\tau,\tau^\dagger]\|_F \geq c_2 \|\tau\|_F^2$.

Proof. The purely diagonal part of the commutator $[\theta,\theta^\dagger]$ vanishes as diagonal matrices are commutative - $[\sigma,\overline{\sigma}] = 0$. $[\sigma,\tau^\dagger]$ is lower triangular and $[\overline{\sigma},\tau]$ upper triangular and therewith both do not contribute to the diagonal Then

$$\begin{aligned}(\theta\theta^\dagger - \theta^\dagger\theta)_{ii} &= \sum_{j=1}^{n} \tau_{ij}\tau_{ji}^\dagger - \sum_{j=1}^{n} \tau_{ij}^\dagger \tau_{ji} \\ &= \sum_{j=1}^{n} \tau_{ij}\overline{\tau}_{ij} - \sum_{j=1}^{n} \overline{\tau}_{ji}\tau_{ji} = \sum_{j=1}^{n} |\tau_{ij}|^2 - |\tau_{ji}|^2 \\ &= \sum_{i<j} |\tau_{ij}|^2 - \sum_{j<i} |\tau_{ji}|^2 \\ (\tau\tau^\dagger - \tau^\dagger\tau)_{ik} &= \sum_{j=1}^{n} \tau_{ij}\overline{\tau}_{kj} - \sum_{j=1}^{n} \overline{\tau}_{ji}\tau_{jk}.\end{aligned}$$

Obviously $\|[\theta,\theta^\dagger]\|_F^2 \geq \sum_{i,j=1}^n |\tau_{ij}|^2 - |\tau_{ji}|^2 \geq \sum_{i<j} |\tau_{ij}|^2 - \sum_{j<i} |\tau_{ji}|^2, \forall 1 \leq i \leq n$. For $i = 1$ this is just $\sum_{1<j} |\tau_{1j}|^2 - \sum_{j<1} |\tau_{j1}|^2 = \sum_{j=1}^n |\tau_{1j}|^2$ since $\tau_{11} = 0$. But then we have $|\tau_{1j}|^2 \leq \|[\theta,\theta^\dagger]\|_F^2$ for all $1 \leq j \leq n$. Inductively we get from $|\tau_{kj}|^2 \leq (k!)\|[\theta,\theta^\dagger]\|_F^2, \forall 1 \leq j \leq n, 1 \leq k \leq i$ just $|\tau_{(i+1)j}|^2 \leq (i+1)!\|[\theta,\theta^\dagger]\|_F^2, \forall 1 \leq j \leq n$ by

$$\begin{aligned}\|[\theta,\theta^\dagger]\|_F^2 &\geq \sum_{(i+1)<j} |\tau_{(i+1)j}|^2 - \underbrace{\sum_{j<(i+1)} |\tau_{j(i+1)}|^2}_{\leq i(i!)\|[\theta,\theta^\dagger]\|_F \text{ [33]}} \\ \Rightarrow \|[\theta,\theta^\dagger]\|_F^2 + i(i!)\|[\theta,\theta^\dagger]\|_F^2 &\geq \sum_{(i+1)<j} |\tau_{(i+1)j}|^2 - \underbrace{\sum_{j<(i+1)} |\tau_{j(i+1)}|^2 + \leq i(i!)\|[\theta,\theta^\dagger]\|_F}_{\geq 0} \\ &\geq \sum_{(i+1)<j} |\tau_{(i+1)j}|^2\end{aligned}$$

$$\Rightarrow (i+1)!\|[\theta,\theta^\dagger]\|_F^2 \quad \geq \quad \sum_{(i+1)<j} |\tau_{(i+1)j}|^2.$$

Thus $|\tau_{ij}|^2 \leq n!\|[\theta,\theta^\dagger]\|_F \Rightarrow \|[\theta,\theta^\dagger]\|_F^2 \geq \frac{n^2}{n!}\|\tau\|_F^2$ shows (i) although this estimate is generous.

For (ii) the argument is similar. Consider the two cases $\|\tau\|_F \leq 1$ and $a = 1$ as well as $\|\tau\|_F > 1$ and $a = 2$. Assume that $|(\tau\tau^\dagger - \tau^\dagger\tau)_{ii}|^a < \frac{1}{(n!)^{2a}\cdot n^{2a}}\|\tau\|_F^2 = \frac{1}{(n!)^{2a}\cdot n^{2a}}\sum_{ij}^n |\tau_{ij}|^2$. Then

$$|(\tau\tau^\dagger - \tau^\dagger\tau)_{11}| = \sum_{j=1}^n |\tau_{1j}|^2 < \frac{1}{(n!)^2\cdot n^2}\|\tau\|_F^{2/a}$$
$$\Rightarrow |\tau_{1j}|^2 < \frac{1}{(n!)^2\cdot n^2}\|\tau\|_F^{2/a}, \quad \forall 1 \leq j \leq n.$$

Again by induction from $|\tau_{kj}|^2 < \frac{k!}{n!\cdot n^2}\|\tau\|_F^{2/a}, \forall 1 \leq j \leq n, 1 \leq k \leq i$ to $|\tau_{(i+1)j}|^2 < \frac{(i+1)!}{n!\cdot n^2}\|\tau\|_F^{2/a}, \forall 1 \leq j \leq n$

$$\begin{aligned} \frac{1}{n!\cdot n^2}\|\tau\|_F^{2/a} &> |(\tau\tau^\dagger - \tau^\dagger\tau)_{(i+1)(i+1)}| \\ &= \sum_{(i+1)<j} |\tau_{(i+1)j}|^2 - \underbrace{\sum_{j<(i+1)} |\tau_{j(i+1)}|^2}_{<\frac{i(i!)}{n!\cdot n^2}\|\tau\|_F^{2/a}} {}^{34} \end{aligned}$$

and further we have

$$\begin{aligned} \Rightarrow \frac{1}{n!\cdot n^2}\|\tau\|_F^{2/a} + \frac{i(i!)}{n!\cdot n^2}\|\tau\|_F^{2/a} &= \sum_{(i+1)<j} |\tau_{(i+1)j}|^2 - \underbrace{\sum_{j<(i+1)} |\tau_{j(i+1)}|^2 + \frac{i!}{n!\cdot n^2}\|\tau\|_F^{2/a}}_{>0} \\ &> \sum_{(i+1)<j} |\tau_{(i+1)j}|^2 \\ \Rightarrow \frac{(i+1)!}{n!\cdot n^2}\|\tau\|_F^{2/a} &> |\tau_{(i+1)j}|^2, \quad \forall 1 \leq j \leq n. \end{aligned}$$

This leads to the contradiction

$$a = 1 : \|\tau\|_F^2 \quad = \quad \sum_{i,j=1}^n |\tau_{ij}|^2 < \sum_{i,j=1}^n \frac{n!}{n!\cdot n^2}\|\tau\|_F^{2/a} \leq \frac{n^2}{n^2}\|\tau\|_F^2 = \|\tau\|_F^2$$

[33] $\tau_{(i+1)(i+1)} = 0$.

[34] We can drop the absolute value, because if the first term is smaller than the second, we are done.

$$a = 2 : \|\tau\|_F^2 \quad = \quad \sum_{i,j=1}^{n} |\tau_{ij}|^2 < \sum_{i,j=1}^{n} \frac{n!}{n! \cdot n^2} \|\tau\|_F^{2/a} \leq \frac{n^2}{n^2} \|\tau\|_F^1 < \|\tau\|_F^2.$$

Thus $\exists 1 \leq i \leq n : |(\tau\tau^\dagger - \tau^\dagger\tau)_{ii}| \geq \frac{1}{(n!)\cdot n^2}\|\tau\|_F^{2/a} \Rightarrow \|[\tau,\tau^\dagger]\|_F^2 \geq |(\tau\tau^\dagger - \tau^\dagger\tau)_{ii}|^2 \geq \frac{1}{(n!)^{2/a}\cdot n^{4/a}}\|\tau\|_F^{2a} \geq \frac{1}{(n!)^2\cdot n^4}\|\tau\|_F^2$. This estimate is another quite generous one, but for us sufficient. □

Lemma 1.4.13. (i) $\|\sigma\|_F^2 \leq \frac{c_\sigma}{|z|^2}$

(ii) For $\beta = \sigma - \alpha$ we have $\|\beta\|_F \leq c_\beta |z|^{-1+\varepsilon}$.

(iii) $\|\alpha\|_F^2 = \frac{c_\alpha}{|z|^2}$.

Proof. By construction. □

1.4.3. Sections of the Endomorphism Bundle

It is natural to consider $\mathrm{End}(E)$ with the induced holomorphic structure $[\overline{\partial}_E, \cdot]$. D_{End} denotes the unique metric connection compatible with this holomorphic structure (see A.4).
Next we want to use that the curvature decreases in subbundles [Huy05] 4.3.18 or [GH78]:[35] Let $\mathrm{End}(E)_{\varphi_z}$ denote the subbundle of $\mathrm{End}(E)$ spanned by φ_z (cf. Def. 1.3.1). Here $\varphi = \varphi_z \,\mathrm{d}\, z$, i.e. φ_z is the endomorphism part of φ.[36] φ_z is a holomorphic section w.r.t. the holomorphic structure on $\mathrm{End}(E)$ and therefore spans indeed a subbundle. We have seen that the induced connection on $\mathrm{End}(E)_\varphi$ is again hermitian and compatible with the holomorphic structure. Since such a connection is unique (cf. theorem A.2.10.), it is just the canonical connection on a hermitian line bundle with curvature $D^2_{\mathrm{End},\chi}(\chi) = \overline{\partial}\partial \log h_{\mathrm{End}}(\chi,\chi), \chi \in \mathrm{End}(E)_\varphi$ non-vanishing. In particularly for the natural frame which extends φ_z we get $D^2_{\mathrm{End},\varphi}(\varphi_z) = \overline{\partial}\partial \log h_{\mathrm{End}}(\varphi_z,\varphi_z)$.[37] But

$$h_{\mathrm{End}}(\varphi_z,\varphi_z) \quad = \quad \sum_{i,j=1}^{n} |\varphi_{ij}|^2 = \|\varphi_z\|_F^2.$$

Then we have $D^2_{\mathrm{End},\varphi}(\varphi_z) = \overline{\partial}\partial \log \|\varphi_z\|_F^2$.

[35] We will more or less prove this theorem when we prove the Chern-Weil formula later on.
[36] $\varphi_z \in \Gamma(X, \mathrm{End}(E))$.
[37] $\overline{\partial}\partial \log h_{\mathrm{End}}(\varphi,\varphi) = \overline{\partial}(h_{\mathrm{End}}(\varphi,\varphi))^{-1}\partial h_{\mathrm{End}}(\varphi,\varphi)$, i.e. the curvature is invariant under multiplication by a scalar (here a smooth non-vanishing function) and therefore independent of the chosen frame.

The result of Griffith (cf. [GH78], p. 79.) applied to line subbundles reads (cf. [Huy05], rmk. 4.3.16 on p. 189):

$$h_{\mathrm{End}}(D^2_{\mathrm{End},\varphi}(\varphi_z)s,s)(v,\overline{v}) \leq h_{\mathrm{End}}((D^2_{\mathrm{End}}|_{\mathrm{End}(E)_{\varphi_z}})s,s)(v,\overline{v}), \quad \forall s \in \mathrm{End}(E)_{\varphi_z}.$$ [38]

If we apply this inequality to φ_z itself we get

$$h_{\mathrm{End}}((\overline{\partial}\partial \log \|\varphi_z\|_F^2)\varphi_z, \varphi_z)(v,\overline{v}) \leq h_{\mathrm{End}}([D^2,\varphi_z],\varphi_z)(v,\overline{v})$$

But $\mathbb{D}^2 = \partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E + \theta\theta^\dagger + \theta^\dagger\theta = D^2 + \theta\theta^\dagger + \theta^\dagger\theta = 0$ the curvature for a harmonic bundle, i.e. $D^2 = -\theta\theta^\dagger - \theta^\dagger\theta = (\theta_z\theta_z^\dagger - \theta_z^\dagger\theta_z)\,\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z$. Hence

$$\begin{aligned}&\left(\frac{\partial}{\partial\overline{z}}\frac{\partial}{\partial z}\log\|\varphi_z\|_F^2\right)\|\varphi_z\|_F^2(\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z)(v,\overline{v})\\ &\leq -h_{\mathrm{End}}([[\theta_z,\theta_z^\dagger],\varphi_z],\varphi_z)(\mathrm{d}\,z\wedge\mathrm{d}\,\overline{z})(v,\overline{v})\end{aligned}$$

Now evaluate for $v = \frac{\partial}{\partial z}, \overline{v} = \frac{\partial}{\partial\overline{z}}$. Then we have

$$\Rightarrow -\frac{\partial}{\partial\overline{z}}\frac{\partial}{\partial z}\log\|\varphi_z\|_F^2 \leq -\frac{h_{\mathrm{End}}([[\varphi_z,\varphi_z^\dagger],\varphi_z],\varphi_z)}{\|\varphi_z\|_F^2}.$$

Remark 1.4.14. Simpson rewrites both sides in terms of real differential forms via $\mathrm{d}\,z\wedge\mathrm{d}\,\overline{z} = -2i\,\mathrm{d}\,x\wedge\mathrm{d}\,y$ and $\frac{\partial}{\partial\overline{z}}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta$. Then note that for the Lefschetz operator Λ on the vector space T_xX with (almost) holomorphic structure I, $I\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, I\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}$, and a differential $2-$form ω we have $\Lambda\omega = \omega\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$.[39] This applied to a holomorphic tangent vector $v = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ leads to $\mathrm{d}\,z\wedge\mathrm{d}\,\overline{z}(v,\overline{v}) = -2i\,\mathrm{d}\,x\wedge\mathrm{d}\,y(v,\overline{v}) = -2i\frac{1}{2}i = 1 = \mathrm{d}\,x\wedge\mathrm{d}\,y\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) = \Lambda\,\mathrm{d}\,x\wedge\mathrm{d}\,y = \frac{i}{2}\Lambda\,\mathrm{d}\,z\wedge\mathrm{d}\,\overline{z}$. Now multiplying by the factor 4 from $\frac{1}{4}\Delta = \frac{\partial}{\partial\overline{z}}\frac{\partial}{\partial z}$ implies $2i\Lambda\,\mathrm{d}\,z\wedge\mathrm{d}\,\overline{z}$. Still this has value 4 for the chosen v and so the ratio between the two sides won't change.

Further for $\varphi_z = \theta_z$ we get

$$\begin{aligned}&h_{\mathrm{End}}([[\theta_z,\theta_z^\dagger],\theta_z],\theta_z) = h_{\mathrm{End}}([\theta_z\theta_z^\dagger - \theta_z^\dagger\theta_z,\theta_z],\theta_z)\\ =\ & h_{\mathrm{End}}(\theta_z\theta_z^\dagger\theta_z - \theta_z^\dagger\theta_z^2 - \theta_z^2\theta_z^\dagger + \theta_z\theta_z^\dagger\theta_z,\theta_z)\\ =\ & h_{\mathrm{End}}(\theta_z^\dagger\theta_z,\theta_z^\dagger\theta_z) - h_{\mathrm{End}}(\theta_z^2,\theta_z^2) - h_{\mathrm{End}}(\theta_z\theta_z^\dagger,\theta_z^\dagger\theta) + h_{\mathrm{End}}(\theta_z^\dagger\theta_z,\theta_z^\dagger\theta_z)\\ =\ & h_{\mathrm{End}}(\theta_z^\dagger\theta_z - \theta_z\theta_z^\dagger,\theta_z^\dagger\theta_z) - h_{\mathrm{End}}(\theta^\dagger\theta_z^2,\theta_z) + h_{\mathrm{End}}(\theta_z\theta_z^\dagger\theta_z,\theta_z)\end{aligned}$$

[38] A real $(1,1)-$form ω is (semi-)positive, iff for all holomorphic tangent vectors $v \in T^{1,0}(X)$: $-i\omega(v,\overline{v}) \geq 0$. An imaginary $(1,1)-$form ω' is positive if $i\omega$ is positive as a real form.

[39] See Huybrechts [Huy05], p. 41, Ex. 1.2.10.

$$
\begin{aligned}
&= h_{\mathrm{End}}(\theta_z^\dagger\theta_z - \theta_z\theta_z^\dagger, \theta_z^\dagger\theta_z) - h_{\mathrm{End}}((\theta_z^\dagger\theta_z - \theta_z\theta_z^\dagger)\theta_z, \theta_z) \\
&= {}^{[40]}h_{\mathrm{End}}(\theta_z^\dagger\theta_z - \theta_z\theta_z^\dagger, \theta_z^\dagger\theta_z) - h_{\mathrm{End}}((\theta_z^\dagger\theta_z - \theta_z\theta_z^\dagger), \theta_z\theta_z^\dagger) \\
&= h_{\mathrm{End}}([\theta_z, \theta_z^\dagger], [\theta_z, \theta_z^\dagger]) = \|[\theta_z, \theta_z^\dagger]\|_F^2.
\end{aligned}
$$

Another special case occurs if φ_z commutes with θ_z:

$$
\begin{aligned}
& h_{\mathrm{End}}([[\theta_z, \theta_z^\dagger], \varphi_z], \varphi_z) \\
&= h_{\mathrm{End}}([\theta_z\theta_z^\dagger - \theta_z^\dagger\theta_z, \varphi_z], \varphi_z) \\
&= h_{\mathrm{End}}(\theta_z\theta_z^\dagger\varphi_z - \theta_z^\dagger\theta_z\varphi_z - \varphi_z\theta_z\theta_z^\dagger + \varphi_z\theta_z^\dagger\theta_z, \varphi_z) \\
&= h_{\mathrm{End}}(\theta_z^\dagger\varphi_z, \theta_z^\dagger\varphi_z) - h_{\mathrm{End}}(\theta_z\varphi_z, \theta_z\varphi_z) \\
&\quad - h_{\mathrm{End}}(\varphi_z\theta_z^\dagger, \theta_z^\dagger\varphi_z) + h_{\mathrm{End}}(\varphi_z\theta_z^\dagger, \varphi_z\theta_z^\dagger) \\
&= h_{\mathrm{End}}(\theta_z^\dagger\varphi_z - \varphi_z\theta_z^\dagger, \theta_z^\dagger\varphi_z) - h_{\mathrm{End}}(\theta^\dagger\theta_z\varphi_z, \varphi_z) + h_{\mathrm{End}}(\varphi_z\theta_z^\dagger\theta_z, \varphi_z) \\
&= h_{\mathrm{End}}(\theta_z^\dagger\varphi_z - \varphi_z\theta_z^\dagger, \theta_z^\dagger\varphi_z) - h_{\mathrm{End}}(\theta^\dagger\varphi_z\theta_z, \varphi_z) + h_{\mathrm{End}}(\varphi_z\theta_z^\dagger\theta_z, \varphi_z) \\
&= h_{\mathrm{End}}(\theta_z^\dagger\varphi_z - \varphi_z\theta_z^\dagger, \theta_z^\dagger\varphi_z) - h_{\mathrm{End}}(\theta^\dagger\varphi_z - \varphi_z\theta_z^\dagger, \varphi_z\theta_z^\dagger) \\
&= h_{\mathrm{End}}([\theta_z^\dagger, \varphi_z], [\theta_z^\dagger, \varphi_z]) \\
&= \|[\theta_z^\dagger, \varphi_z]\|_F^2
\end{aligned}
$$

Remark 1.4.15. For θ with constant rank and $\theta \neq 0$ we get $\theta_z \neq 0 \Rightarrow \|\theta_z\|_F^2 \neq 0$. By holomorphy of θ this still holds on some neighbourhood of 0 even if we have no constant rank. Still the Frobenius norm squared is just a sum of absolute values $|f|^2$. If $f = 0$ this might be not differentiable. But the preimage of 0 under the smooth function f is closed in B^*. So it is the union of closed balls and of countably many isolated points. On the closed balls (apart from $0-\mathrm{dim}$) $|f|^2$ is differentiable and the countably many isolated points add nothing in the distributional sense (not measurable) - $\int \chi\Delta \log \|\theta_z\|_F^2 \leq \int \chi \frac{\|[\theta_z, \theta_z^\dagger]\|_F^2}{\|\theta_z\|_F^2}$ for all smooth, non-negative χ.

We have shown the following theorem:

Theorem 1.4.16. For the Higgs field θ

$$
-\Delta \log \|\theta_z\|_F^2 \leq -\frac{\|[\theta_z, \theta_z^\dagger]\|_F^2}{\|\theta_z\|_F^2},
$$

and more general

$$
-\Delta \log \|\varphi_z\|_F^2 \leq -\frac{h_{\mathrm{End}}([[\theta_z, \theta_z^\dagger], \varphi_z], \varphi_z)}{\|\varphi_z\|_F^2},
$$

[40] Cf. A.4.1.

for a non-vanishing $\mathcal{E}(U)-$module homomorphism $\Gamma(U,E) \to \Gamma(U,E) \otimes \Omega_X^{1,0}$. If θ_z and φ_z commute we get

$$-\Delta \log \|\phi_z\|_F^2 \leq -\frac{\|[\theta_z^\dagger, \varphi_z]\|_F^2}{\|\varphi_z\|_F^2} = -\frac{\|[\varphi_z, \theta_z^\dagger]\|_F^2}{\|\varphi_z\|_F^2}.$$

Example 1.4.17. Let $X = B^*$, $\overline{X} = B$, $E = X \times \mathbb{C}$ with $\pi = \mathrm{pr}_1$. Define $H = \left(|z|^{2\alpha}\right), \alpha \in \mathbb{R}$ and $\overline{\partial}_E e = 0$ w.r.t. the frame $e(z) := \left(z, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$. Further $\partial_E e := e \otimes \left(\frac{\alpha \,\mathrm{d}\, z}{z}\right), \theta(e) = e \otimes a\frac{\mathrm{d}\, z}{z}$, $a \in \mathcal{O}^{an}(X)$ and $\theta^\dagger = e \otimes \overline{a}\frac{\mathrm{d}\,\overline{z}}{\overline{z}}$ by

$$\begin{aligned}
\overline{\partial} h(\xi,\eta) &= \overline{\partial} h(e \otimes \xi_1, e \otimes \eta_1) = \overline{\partial}(|z|^{2\alpha} \xi_1 \overline{\eta}_1) \\
&= \underbrace{\alpha z^\alpha \overline{z}^{\alpha-1}}_{\frac{\alpha|z|^{2\alpha}}{\overline{z}}} \xi_1 \overline{\eta}_1 \,\mathrm{d}\,\overline{z} + |z|^{2\alpha}(\overline{\partial}\xi_1)\overline{\eta}_1 + |z|^{2\alpha}\xi_1 \underbrace{(\overline{\partial}\overline{\eta}_1)}_{\overline{\partial\eta_1}} \\
&= |z|^{2\alpha}(\overline{\partial}\xi)\overline{\eta}_1 + |z|^{2\alpha}\xi_1 \overline{(\partial\eta_1)} + |z|^{2\alpha}\xi_1\overline{\eta}_1 \left(\frac{\alpha \,\mathrm{d}\,\overline{z}}{\overline{z}}\right) \\
&= h(e \otimes \overline{\partial}\xi, \eta) + h(\xi, e \otimes \eta_1 \left(\frac{\alpha \,\mathrm{d}\, z}{z}\right) + e \otimes \partial\eta_1) \\
&= h(\overline{\partial}_E \xi, \eta) + h(\xi, \partial_E \eta); \\
h(\theta\xi,\eta) &= h(e \otimes \xi_1 a \frac{\mathrm{d}\, z}{z}, \eta) = |z|^{2\alpha}\xi_1 a \frac{\mathrm{d}\, z}{z}\overline{\eta}_1 \\
&= h(\xi, \overline{a}\frac{\mathrm{d}\,\overline{z}}{\overline{z}}\eta_1) = h(\xi, \theta^\dagger \eta),
\end{aligned}$$

for all $\xi, \eta \in \Gamma(X,E)$. Moreover θ is a Higgs field:

$$\begin{aligned}
& h((\overline{\partial}_E\theta + \theta\overline{\partial}_E)\xi, \eta) = h((\overline{\partial}_E\theta)\xi, \eta) + h((\theta\overline{\partial}_E)\xi, \eta) \\
&= h(\overline{\partial}_E e \otimes \xi_1 a \frac{\mathrm{d}\, z}{z}, \eta) + h(\theta e \otimes \overline{\partial}\xi_1, \eta) \\
&= h(e \otimes \overline{\partial}(\xi_1 a \frac{\mathrm{d}\, z}{z}), \eta) + h(e \otimes \xi_1 a \frac{\mathrm{d}\, z}{z} \wedge \overline{\partial}\xi_1, \eta) \\
&= h(e \otimes ((\overline{\partial}\xi_1) a \frac{\mathrm{d}\, z}{z} + \xi_1 \overline{\partial} a \frac{\mathrm{d}\, z}{z}), \eta) + h(e \otimes \xi_1 a \frac{\mathrm{d}\, z}{z} \wedge \overline{\partial}\xi_1, \eta) \\
&= h(e \otimes ((\overline{\partial}\xi_1) \wedge a \frac{\mathrm{d}\, z}{z} + \xi_1 \overline{\partial} a \frac{\mathrm{d}\, z}{z}), \eta) + h(e \otimes \xi_1 a \frac{\mathrm{d}\, z}{z} \wedge \overline{\partial}\xi_1, \eta) \\
&= h(e \otimes (\overline{\partial}\xi_1) \wedge a \frac{\mathrm{d}\, z}{z}, \eta) + h(e \otimes \xi_1 a \frac{\mathrm{d}\, z}{z} \wedge \overline{\partial}\xi_1, \eta) \\
&= 0,
\end{aligned}$$

by holomorphy of a and the alternation of differential forms. The bundle is harmonic:

$$\partial_E \overline{\partial}_E (\xi_1 e) = \partial_E (e \otimes \overline{\partial}\xi_1)$$

$$
\begin{aligned}
&= e \otimes \left(\frac{\alpha \,\mathrm{d}\, z}{z}\right) \wedge \overline{\partial}\xi_1 + (e \otimes \partial\overline{\partial}\xi_1),\\
\overline{\partial}_E\partial_E(\xi_1 e) &= \overline{\partial}_E(e \otimes \left(\frac{\alpha \,\mathrm{d}\, z}{z}\right) \wedge \xi_1 + e \otimes \partial\xi_1)\\
&= e \otimes \underbrace{\overline{\partial}\left(\frac{\alpha \,\mathrm{d}\, z}{z}\right)}_{=0} \wedge \xi_1 - e \otimes \left(\frac{\alpha \,\mathrm{d}\, z}{z}\right) \wedge \overline{\partial}\xi_1 + e \otimes \overline{\partial}\partial\xi_1\\
&= -e \otimes \left(\frac{\alpha \,\mathrm{d}\, z}{z}\right) \wedge \overline{\partial}\xi_1 + e \otimes \overline{\partial}\partial\xi_1,
\end{aligned}
$$

where we used $\overline{\partial}\left(\left(\frac{\alpha\,\mathrm{d}\,z}{z}\right) \wedge \xi_1\right) = \left(\overline{\partial}\left(\frac{\alpha\,\mathrm{d}\,z}{z}\right)\right) \wedge \xi_1 - \left(\frac{\alpha\,\mathrm{d}\,z}{z}\right) \wedge \overline{\partial}\xi_1$.
Hence $\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E = 0$. Moreover

$$
(\theta\theta^\dagger + \theta^\dagger\theta)(\xi) \quad = \quad \xi_1 e \otimes |a|^2 \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{|z|^2} + \xi_1 e \otimes |a|^2 \frac{\mathrm{d}\, \overline{z} \wedge \mathrm{d}\, z}{|z|^2} = 0.
$$

θ has a single eigenvalue $\lambda = a\frac{\mathrm{d}\,z}{z} = \lambda_0 \,\mathrm{d}\, z$. In this trivial case $\sigma = (\lambda)$ and $\tau = 0$, $a\frac{\mathrm{d}\,z}{z} = a_0 \,\mathrm{d}\, z$ as well as $E_{a_0} = \langle e \rangle = E$. Obviously $|\lambda_0 - a_0| \leq \frac{1}{|z|}$. The curvature D^2 for $D = \partial_E + \overline{\partial}_E$ is zero, i.e. the bundle is flat, by the calculation above. Then the curvature on the endomorphism bundle is flat and the equation $-\Delta \log |\theta|^2 \leq 0$ holds trivially by the concavity of the logarithm.

The following example has non trivial curvature:

Example 1.4.18. Let $X = B^*$, $\overline{X} = B$, $E = X \times \mathbb{C}^2$ with $\pi = \mathrm{pr}_1$. Define $H = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$, $y = -\log|z|^2 \geq 0$ and $\overline{\partial}_E e_1 = \overline{\partial}_E e_2 = 0 \,\mathrm{d}\, \overline{z}$ w.r.t. the frame $e_1(z) := \left(z, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $e_2(z) := \left(z, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. Further $\partial_E e_1 = -e_1 \otimes \frac{\mathrm{d}\,z}{zy}$, $\partial_E e_2 = e_2 \otimes \frac{\mathrm{d}\,z}{zy} \,\mathrm{d}\, z$, $\theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}\,z}{z}$, $\theta^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}\,\overline{z}}{\overline{z}y^2}$:

$$
\begin{aligned}
\overline{\partial}h(\xi, \eta) &= \overline{\partial}(\xi_1\overline{\eta}_1 y + \xi_2\overline{\eta}_2 y^{-1})\\
&= -\frac{d\overline{z}}{\overline{z}}\xi_1\overline{\eta}_1 + y\overline{\partial}(\xi_1\overline{\eta}_1) + \frac{d\overline{z}}{y^2\overline{z}} + y^{-1}\overline{\partial}(\xi_2\overline{\eta}_2)\\
&= -y\xi_1 \otimes \frac{\mathrm{d}\,\overline{z}}{\overline{z}y} \wedge \overline{\eta}_1 + y^{-1}\xi_2 \otimes \frac{\mathrm{d}\,\overline{z}}{\overline{z}y} \wedge \eta_2\\
&\quad + (\overline{\partial}\xi_1 \wedge \overline{\eta_1} + \xi_1 \wedge \overline{\partial\eta_1})y + (\overline{\partial}\xi_2 \wedge \overline{\eta_2} + \xi_2 \wedge \overline{\partial\eta_2})y^{-1}\\
&= h(\sum_{i=1}^{2} e_i \otimes \overline{\partial}\xi_i, \eta)
\end{aligned}
$$

$$
\begin{aligned}
& +h(\xi, -e_1 \otimes \frac{\mathrm{d}\, z}{zy} \wedge \eta_1 + e_2 \otimes \frac{\mathrm{d}\, z}{zy} \wedge \eta_2 + \sum_{i=1}^{2} e_i \otimes \partial \eta_i) \\
&= h(\overline{\partial}_E \xi, \eta) + h(\xi, \partial_E \eta), \\
h(\theta\xi, \eta) &= h(e_2 \otimes \frac{\mathrm{d}\, z}{z} \wedge \xi_1, \eta) = y^{-1} \frac{\mathrm{d}\, z}{z} \wedge \xi_1 \overline{\eta}_2 \\
&= \frac{y}{y^2} \frac{\mathrm{d}\, z}{z} \wedge \xi_1 \overline{\eta}_2 = h(\xi, e_1 \otimes \frac{\mathrm{d}\, \overline{z}}{\overline{z} y^2} \eta_2) \\
&= h(\xi, \theta^\dagger \eta),
\end{aligned}
$$

for $\xi, \eta \in \Gamma(X, E)$. Hence $D = \partial_E + \overline{\partial}_E$ is the unique hermitian connection, which is compatible with the holomorphic structure $\overline{\partial}_E$. Moreover θ is a Higgs field:

$$
\begin{aligned}
(\theta \overline{\partial}_E + \overline{\partial}_E \theta)\xi &= \theta(\sum_{i=1}^{2} e_i \otimes \overline{\partial} \xi_i) + \overline{\partial}_E (e_2 \otimes \frac{\mathrm{d}\, z}{z} \wedge \xi_1) \\
&= e_2 \otimes \frac{\mathrm{d}\, z}{z} \wedge \overline{\partial} \xi_1 + e_2 \otimes \overline{\partial} \left(\frac{\mathrm{d}\, z}{z} \wedge \xi_1 \right) \\
&= e_2 \otimes \frac{\mathrm{d}\, z}{z} \wedge \overline{\partial} \xi_1 - e_2 \otimes \frac{\mathrm{d}\, z}{z} \wedge \overline{\partial} \xi_1 \\
&= 0,
\end{aligned}
$$

where the minus in the last row comes from the usual Leibniz rule for $p-$forms (here a 1-form). The curvature of the unique hermitian connection for a metric H in a $\overline{\partial}_E-$holomorphic frame is just $\overline{\partial H}^{-1} \partial \overline{H}$, i.e.

$$
\begin{aligned}
D^2 &= \overline{\partial} H^{-1} \partial \overline{\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}} = \overline{\partial} \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} \frac{-1}{z} & 0 \\ 0 & \frac{1}{y^2 z} \end{pmatrix} \mathrm{d}\, z \\
&= \overline{\partial} \begin{pmatrix} -\frac{1}{yz} & 0 \\ 0 & \frac{1}{yz} \end{pmatrix} \mathrm{d}\, z = \begin{pmatrix} -\frac{1}{y^2|z|^2} & 0 \\ 0 & \frac{1}{y^2|z|^2} \end{pmatrix} \mathrm{d}\, \overline{z} \wedge \mathrm{d}\, z \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{y^2 |z|^2}.
\end{aligned}
$$

Further

$$
\begin{aligned}
\theta\theta^\dagger + \theta^\dagger\theta &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{|z|^2 y^2} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}\, \overline{z} \wedge \mathrm{d}\, z}{|z|^2 y^2} \\
&= \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{|z|^2 y^2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{|z|^2 y^2}.
\end{aligned}
$$

Thus $(E, h, \overline{\partial}_E, \theta)$ is a harmonic bundle.
The eigenvalues of θ are $\lambda_1 = \lambda_2 = 0$. The eigenspaces are spanned by the

orthogonal vectors e_1, e_2. Define $e_1' := e_2\sqrt{y}, e_2' := e_1\sqrt{y^{-1}}$. Then θ is upper triangular in this new basis,

$$\theta' = \begin{pmatrix} 0 & y^{-2} \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}\, z}{z}.$$

Hence

$$\sigma = 0, \quad \tau = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}\, z}{z},$$

and so on.
In the Frobenius norm we get

$$\begin{aligned} \|[\theta', \theta^{\dagger\prime}]\|_F^2 &= \left\| \begin{pmatrix} 0 & y/z \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \\ y/\overline{z} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \\ y/\overline{z} & 0 \end{pmatrix} \begin{pmatrix} 0 & y/z \\ 0 & 0 \end{pmatrix} \right\|_F^2 \\ &= \left\| \begin{pmatrix} y^{-2}/|z|^2 & \\ 0 & -y^{-2}/|z|^2 \end{pmatrix} \right\|_F^2 = 2y^{-4}/|z|^4. \end{aligned}$$

Moreover $\|\theta'\|_F^2 = y^{-2}/|z|^2$ and so

$$\frac{\|[\theta', \theta^{\dagger\prime}]\|_F^2}{\|\theta'\|_F^2} = \frac{2y^{-4}/|z|^4}{y^{-2}/|z|^2} = \frac{2}{y^2|z|^2}.$$

On the other hand

$$\overline{\partial}\partial \log \|\theta'\|_F^2 = \overline{\partial}\partial \log(y^{-2}/|z|^2) = \frac{2}{(\log(|z|^2))^2|z|^2}\,\mathrm{d}\,\overline{z} \wedge \mathrm{d}\, z = \frac{2}{y^2|z|^2}\,\mathrm{d}\,\overline{z} \wedge \mathrm{d}\, z.$$

Hence

$$-\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} \log \|\theta'\|_F^2 \leq -\frac{\|[\theta', \theta^{\dagger\prime}]\|_F^2}{\|\theta'\|_F^2},$$

and since we have even equality this bound is sharp.

The previous example fulfills as expected that we have

$$-\Delta \log \|\theta'\|_F^2 \leq -4\frac{\|[\theta', \theta^{\dagger\prime}]\|_F^2}{\|\theta'\|_F^2} \leq -\frac{\|[\theta', \theta^{\dagger\prime}]\|_F^2}{\|\theta'\|_F^2} \tag{1.4.18.1}$$

with $\Delta = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}$. If we omit the middle term we get the inequality of Simpson [Sim90], p. 728. By Lemma 1.4.12 we can write

$$-\Delta \log(\|\sigma\|_F^2 + \|\tau\|_F^2) \quad \leq \quad -\frac{\|[\theta', \theta^{\dagger\prime}]\|_F^2}{\|\sigma\|_F^2 + \|\tau\|_F^2}$$

$$\begin{aligned} &\leq -\frac{\|c_1[\tau', \tau^{\dagger\prime}]\|_F^2}{\|\sigma\|_F^2 + \|\tau\|_F^2} \\ &= -\frac{c_1^2 c_2^2 \|\tau\|_F^4}{\|\sigma\|_F^2 + \|\tau\|_F^2}. \end{aligned} \tag{1.4.18.2}$$

Remark 1.4.19. As promised we want to mention the differences in the branched cover case. In fact by the chain rule

$$\begin{aligned} \frac{\partial}{\partial u} f(\pi_D(u)) &= \frac{\partial \pi_D(u)}{\partial u} \cdot \frac{\partial}{\partial \pi_D(u)} f(\pi_D(u)) = \frac{\partial u^N}{\partial u} \cdot \frac{\partial}{\partial z} f(z) \\ &= N u^{N-1} \frac{\partial}{\partial z} f(z); \end{aligned}$$

and hence $4\Delta_u f(\pi_D(u)) = N^2 |u|^{2N-2} \frac{\partial^2}{\partial z \partial \bar{z}} f(z)$ for every differentiable function f.

1.5. Main Estimate, Part I

We subdivide the section again, similar to Simpson in [Sim90], p. 729ff. Note as well the rewritten version by [Moc07a], chapter 7.
As already described in the introduction we will distinguish two cases, and then use the maximum principle to show that one of the cases cannot occur. The second step follows the same principles, although we will need more intermediate steps until we may use the maximum principle. As mentioned before we will construct a constant b that is certainly bounded from below, but could be still negative. In order to find a lower bound for b and consequently for $\beta = \sigma - \alpha$ we add a third step, originally not included in Simpson or [Moc07a]. Thus we get a bound for τ^0.

1.5.1. Step 1

In what follows we will use the special form the Laplacian takes on rotationally symmetric functions in the two dimensional case, i.e.

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r}.$$

The following estimate should be understood pointwise.
Let B be the unit disc as before. Let B_R be the disc of radius $0 < R \leq 1$ and B_R with the same - euclidean - metric.

Lemma 1.5.1. Define

(i) $\|\theta_z\|_F^2 \leq c_3 |z|^{-2}$.

(ii) $-\Delta \log \|\theta_z\|_F^2 \leq -c_4 \|\theta_z\|_F^2$.

There are constants $c_3, c_4 \geq 0$ such that for all $z \in B_R^*$ either (i) or (ii) holds:

Proof. Choose $c_3 = 2c_\sigma$. Assume that (i) does not hold, i.e. $\|\theta_z\|_F^2 \geq 2c_\sigma |z|^{-2}$. Hence

$$\begin{aligned} \|\tau\|_F^2 &= \|\theta_z\|_F^2 - \|\sigma\|_F^2 \geq \|\theta_z\|_F^2 - \frac{c_\sigma}{|z|^{-2}} \geq 2c_\sigma |z|^{-2} - \frac{c_\sigma}{|z|^{-2}} \\ &= \frac{c_\sigma}{|z|^{-2}} \end{aligned}$$

and further

$$\|\tau\|_F^2 = \|\theta_z\|_F^2 - \frac{c_\sigma}{|z|^{-2}} \geq \|\theta_z\|_F^2 - \|\tau\|_F^2 \Rightarrow \|\tau\|_F^2 \geq \frac{\|\theta_z\|_F^2}{2}.$$

By 1.4.18.2

$$\begin{aligned} -\Delta \log \|\theta_z\|_F^2 &\leq -\frac{c_1^2 c_2^2 \|\tau\|_F^4}{\|\sigma\|_F^2 + \|\tau\|_F^2} \leq -\frac{c_1^2 c_2^2 \|\theta_z\|_F^4}{4\|\theta_z\|_F^2} \\ &\leq -\frac{c_1^2 c_2^2 \|\theta_z\|_F^2}{4}. \end{aligned}$$

With $c_4 := \frac{c_1^2 c_2^2}{4}$ the claim follows. □

Lemma 1.5.2. Let

$$m_{\varepsilon_1, c_m}(z) := \frac{c_m}{(|z| - \varepsilon_1)^2 (|z| - R)^2}, \quad B, \varepsilon_1 > 0, z \in \{\tilde{z} | \varepsilon_1 < \tilde{z} < R\} =: \mathring{A}_{\varepsilon_1, R}.^{41}$$

If

$$c_m \geq \frac{4R^2}{c_4}, \quad \text{and} \quad c_m \geq c_3 R^2,$$

then

(i) $-\Delta \log m_{\varepsilon_1, c_m}(z) \geq -c_4 m_{\varepsilon_1, c_m}(z)$ and

(ii) $m_{\varepsilon_1, c_m}(z) \geq c_3 |z|^{-2}$.

Proof. By direct calculation for $|z| = r$.

$$-\Delta \log m_{\varepsilon_1, c_m}(z) = -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \frac{c_m}{(r - \varepsilon_1)^2 (r - R)^2}$$

[41] Since R for "ring" is already taken we choose A for annulus to denote ring areas.

$$\begin{aligned}
&= \frac{2}{r}\frac{\partial}{\partial r} r \frac{2r-\varepsilon_1-R}{(r-\varepsilon_1)^3(r-R)^3} = \frac{2}{r}\frac{\partial}{\partial r}\frac{2r^2-\varepsilon_1 r - Rr}{(r-\varepsilon_1)^3(r-R)^3} \\
&= \frac{2}{r}\frac{(-r^2R - r^2\varepsilon_1 + 4r\varepsilon_1 R - \varepsilon_1^2 R - R^2\varepsilon_1)}{(r-\varepsilon_1)^4(r-R)^4} \\
&= \frac{2}{r}\frac{-R(r^2-2r\varepsilon_1+\varepsilon_1^2) - \varepsilon_1(r^2-2rR+R^2)}{(r-\varepsilon_1)^4(r-R)^4} \\
&= \frac{-2\varepsilon_1}{r(r-\varepsilon_1)^2} + \frac{-2R}{r(r-R)^2} \\
&= \frac{-2}{c_m}\left(\frac{\varepsilon_1(|z|-R)^2}{|z|} + \frac{-R(|z|-\varepsilon_1)^2}{|z|}\right) m_{\varepsilon_1,c_m}(z).
\end{aligned}$$

Moreover $\frac{\varepsilon_1}{|z|} < 1$ on $\mathring{A}_{\varepsilon_1,R}$, $(|z|-R)^2 = |z|^2 - 2R|z| + R^2 \leq |z|^2 - 2|z||z| + R^2 \leq R^2 - |z|^2 \leq R^2$ on $\mathring{A}_{\varepsilon_1,R}$ and

$$\frac{(|z|-\varepsilon_1)^2}{|z|} \leq \frac{(|z|-\varepsilon_1)^2}{\underbrace{|z|-\varepsilon_1}_{>0}} \leq |z| - \varepsilon_1 \leq |z| \leq R.$$

So

$$\underbrace{\frac{\varepsilon_1}{|z|}}_{<1}\underbrace{(|z|-R)^2}_{\leq R^2} + \underbrace{\frac{(|z|-\varepsilon_1)^2}{|z|}}_{\leq R} R \leq 2R^2$$

implies

$$\begin{aligned}
-\Delta \log m_{\varepsilon_1,c_m}(z) &= \frac{-2}{c_m}\underbrace{\left(\frac{\varepsilon_1(|z|-R)^2}{|z|} + \frac{-R(|z|-\varepsilon_1)^2}{|z|}\right)}_{\leq 2R^2}\underbrace{m_{\varepsilon_1,c_m}(z)}_{\geq 0} \\
&\geq \frac{-2\cdot 2R^2}{c_m} m_{\varepsilon_1,c_m}(z).
\end{aligned}$$

Now the condition $c_m \geq \frac{4R^2}{c_4}$ implies (i). The second inequality (ii) is the result of

$$\begin{aligned}
m_{\varepsilon_1,c_m}(z) &= \frac{c_m}{(|z|-\varepsilon_1)^2(|z|-R)^2} \geq \frac{c_3R^2}{(|z|-\varepsilon_1)^2\cdot(|z|-R)^2} \geq \frac{c_3}{(|z|-\varepsilon_1)^2} \\
&\geq \frac{c_3}{|z|^2},
\end{aligned}$$

where we used first $B > c_3R^2$ by assumption, then $R^2 \geq (|z|-R)^2$ on $\mathring{A}_{\varepsilon_1,R}$ and at last $(|z|-\varepsilon_1)^2 \leq |z|^2$ on $\mathring{A}_{\varepsilon_1,R}$. □

Before we come to the next lemma recall the (weak) maximum principle

Theorem 1.5.3. Let $U \subset \mathbb{R}^d$ be open and bounded and $u \in C^2(U) \cap C(\overline{U})$ subharmonic on U, i.e. $-\Delta u \leq 0$. Then $\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x)$.

Proof. Müller, [PDE09] p. 16 or any book on the topic. □

Remark 1.5.4. There is a stronger version of the Maximum principle (for example in [LiLo00] p. 244, theorem 9.4, or [Gar91] resp. [Dah77]) using upper semi-continuity requiring only $\limsup_{x\to\partial U} u(x) =: F < \infty$[42] instead of the continuity on the boundary. Then subharmonicity implies $u(x) \leq F$.
We will mainly use the classical version.

Lemma 1.5.5. The set $S_1 := \{z \in B_R^* | \|\theta_z\|_F^2 > m_{\varepsilon_1,c_m}(z)\}$ is empty.

Proof. Assume that S_1 is not empty. Then at least for one $z \in \mathring{A}_{\varepsilon_1,R}$: $\|\theta_z\|_F^2 > m_{\varepsilon_1,c_m}(z) \geq \frac{c_3}{z^2}$ by the previous lemma. Hence (ii) of 1.5.1 holds: $-\Delta \log \|\theta_z\|_F^2 \leq -c_4\|\theta_z(z)\|_F^2$,

$$\begin{aligned} \Rightarrow -\Delta \log\left(\|\theta_z\|_F^2/m_{\varepsilon_1,c_m}(z)\right) &= -\Delta \log \|\theta_z\|_F^2 + \Delta \log\left(\|\theta_z\|_F^2/m_{\varepsilon_1,c_m}(z)\right) \\ &\leq^{43} -c_4\|\theta_z(z)\|_F^2 + c_4 m_{\varepsilon_1,c_m}(z) \\ &\leq -c_4\left(\|\theta_z(z)\|_F^2 - m_{\varepsilon_1,c_m}(z)\right) \\ &\leq 0 \end{aligned}$$

Then $\log\left(\|\theta_z\|_F^2/m_{\varepsilon_1,c_m}(z)\right)$ is subharmonic and we can use the maximum principle, if the function is continuous on ∂S_1. The function is continuous on $\mathring{A}_{\varepsilon_1,R}$ and $\partial\mathring{A}_{\varepsilon_1,R} \cap \partial S_1 = \emptyset$ since $m_{\varepsilon_1,c_m}(z)$ is infinity on $\partial\mathring{A}_{\varepsilon_1,R} = \{\tilde{z}||\tilde{z}| = \varepsilon_1\} \cup \{\tilde{z}||\tilde{z}| = R\}$, i.e. $f \not> m_{\varepsilon_1,c_m}(z)$ there. The maximum principle tells us that the function has its maximum on the boundary of S_1, so the maximum is just obtained for $\|\theta_z\|_F^2 = m_{\varepsilon_1,c_m}(z)$. On S_1

$$\log\left(\|\theta_z\|_F^2/m_{\varepsilon_1,c_m}(z)\right) \leq \log(1) = 0 \Rightarrow \|\theta_z\|_F^2 \leq m_{\varepsilon_1,c_m} \text{ on } S_1.$$

But this is a contradiction to the definition of S_1 if $S_1 \neq \emptyset$. Hence $S_1 = \emptyset$. □

Conclusion 1.5.6. For $\varepsilon_1 \to 0$ we obtain

$$\|\theta_z\|_F^2 \leq \frac{c_m}{|z|^2(|z| - R)^2}$$

on B_R^*.

[42] For almost every x.
[43] (i) of 1.5.2.

Conclusion 1.5.7. For any R_2 with $0 < R_1 < R_2 < R$

$$\|\theta_z\|_F^2 \leq \frac{c_5}{|z|^2}$$

on $B^*_{R_2}$. Here $c_5 = \frac{c_m}{(R_2-R)^2}$.

Remark 1.5.8. Remember A.1.18 in the branched cover case. We get instead of c_4 now $\frac{c_4}{N|u|^{2N-2}}$ and so in the proof of lemma 1.5.5 we still have subharmonicity and so conclusion 1.5.7 still holds.

1.5.2. Step 2

For this paragraph assume $c_\alpha \neq 0$.[44]
Recall, $\beta = \sigma - \alpha$, $b := \|\sigma\|_F^2 - \|\alpha\|_F^2$. Note that since σ and α have non-trivial entries at the same position the Frobenius norm does not decompose; $\|\beta\|_F^2 + \|\alpha\|_F^2 \geq \|\sigma\|_F^2$. We saw already $\|\beta\| \leq c_\beta |z|^{-1+\varepsilon}$ and further $b \leq c_\beta |z|^{-2+\varepsilon}$.[45]

Lemma 1.5.9. Define

$$k(z) = \log\left(\frac{|z|^2(\|\alpha\|_F^2 + b + \|\tau\|_F^2)}{c_\alpha}\right).$$

Then there is a constant c_6 such that

$$\frac{c_6}{c_\alpha}\left(b|z|^2 + \|\tau\|_F^2|z|^2\right) \leq k(z), \quad \text{(left inequality)}$$

for $b|z|^2 + \|\tau\|_F^2|z|^2 \geq 0$

$$k(z) \leq \frac{1}{c_\alpha}\left(b|z|^2 + \|\tau\|_F^2|z|^2\right), \quad \text{(right inequality)}.$$

everywhere.

Proof. Rewrite k

$$k(z) = \log\left(\frac{|z|^2(\|\alpha\|_F^2 + b + \|\tau\|_F^2)}{c_\alpha}\right) \quad = \quad \log\left(1 + \frac{|z|^2(b + \|\tau\|_F^2)}{c_\alpha}\right).$$

Note that $\frac{b|z|^2}{c_\alpha} \leq \frac{c_b}{c_\alpha} z^{-2+\varepsilon+2} < \varepsilon_2 < 1$ for z small enough, i.e. $1 + \frac{|z|^2(b+\|\tau\|_F^2)}{c_\alpha} > \varepsilon_2 > 0$.

[44] c_α works analogously, although the case is less complicated.
[45] For λ_i eigenvalue of θ, $\lambda_i = \frac{a_i}{z} + \beta_i \Rightarrow \lambda_i^2 = \frac{a_i^2}{z^2} + O(z^{-2+\varepsilon})$.

Now the right inequality is obvious, for example by $\log(1+t)$ monotonically increasing, $\log(1+t)|_{t=0} = 0 = t|_{t=0}$ and $\frac{\partial}{\partial t}\log(t+1) = \frac{1}{t+1} \leq 1 = \frac{\partial t}{\partial t}$ for $t \geq 1$ and the other way round for $t \leq 1$. ($f(t) = t$ is a majorant function). For the left inequality use Conclusion 1.5.7

$$\|\tau\|_F^2 \leq \|\tau\|_F^2 + \|\sigma\|_F^2 = \|\theta_z\|_F^2 \leq \frac{c_5}{|z|^2} \Rightarrow \frac{|z|^2\|\tau\|_F^2}{c_\alpha} \leq \frac{c_5}{c_\alpha}$$
$$\Rightarrow \frac{|z|^2(b+\|\tau\|_F^2)}{c_\alpha} \leq \frac{c_b+c_5}{c_\alpha}.$$[46]

Choose $c_6 := \frac{c_\alpha}{c_b+c_5} \cdot \log\left(1+\frac{c_b+c_5}{c_\alpha}\right)$, i.e. use a the secant line through the smallest and the biggest value resp. upper limit of $k(z)$ and the concavity of the logarithm. □

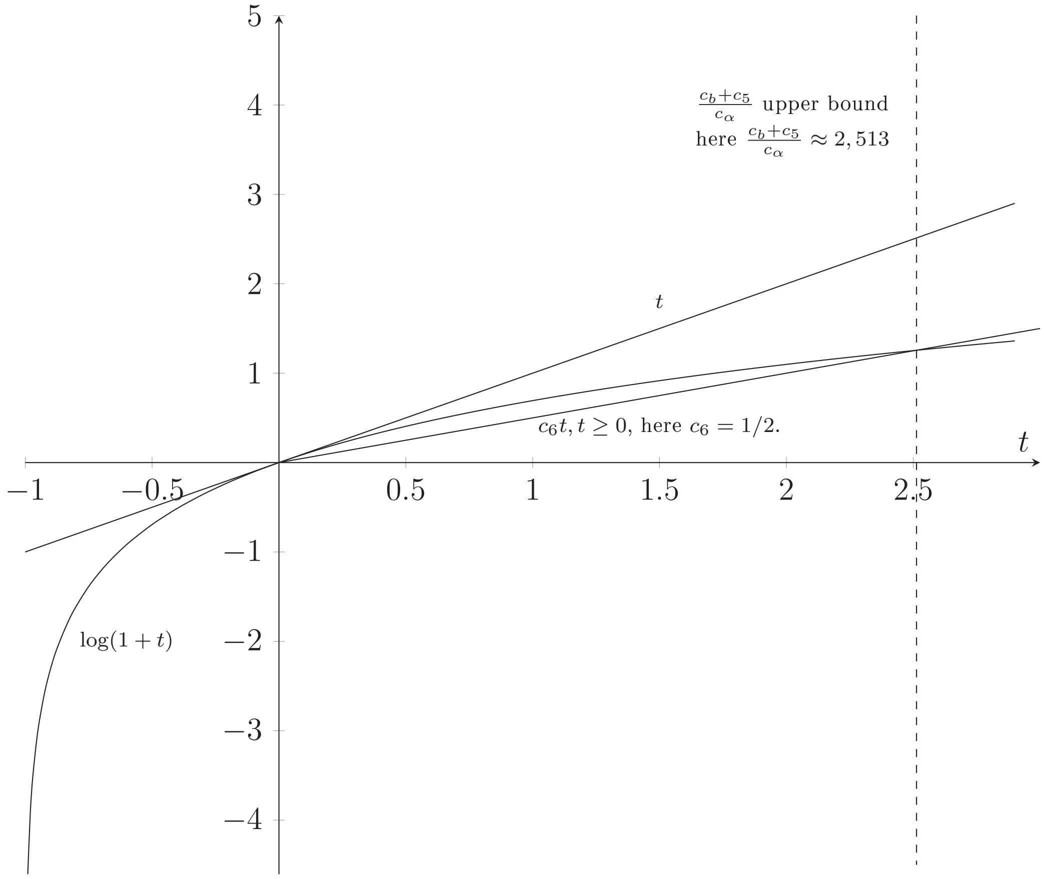

Figure 1.2: Lower and upper bound for a part of the logarithm; $t = \frac{|z|^2(b+\|\tau\|_F^2)}{c_\alpha}$.

[46] $|z|^2 b \leq c_b|z|^{-2+2\varepsilon+2} \leq c_b|z|^{2\varepsilon} \leq c_b$ for $|z| \leq 1$.

Lemma 1.5.10.

$$\frac{c_b}{c_\alpha}|z|^\varepsilon \leq \frac{1}{2}c_7\left(-\log|z|\right)^{-2}$$

for a suitable constant c_7.

Proof. To simplify calculation write $|z| = a_z 10^{-k} \geq 10^{-(l+1)}$ for a unique $a \in [1, 10[$ and $l \geq 1$. Then $\log(|z|) = \log(a_z) - l\log(10)$. By definition $\log a \in [0, 25]$:

$$\left(-\log|z|\right)^{-2} \geq \left((l+1)\log(10)\right)^{-2} = \frac{1}{(l+1)^2\log(10)^2},$$

On the other hand

$$|z|^\varepsilon = a^\varepsilon 10^{-\varepsilon l} \leq 10^{-\varepsilon(l-1)}.$$

The claim follows if we are able to show $\frac{c_7 c_\alpha}{2c_b(\log(10))^2} \geq (l+1)^2 10^{-\varepsilon(l-1)}$:

$$\begin{aligned}\frac{(l+1)^2 10^{-\varepsilon(l-1)}}{l^2 10^{-\varepsilon(l-2)}} &= \frac{(l+1)^2 10^{-\varepsilon}}{l^2} \leq 1\\ \Leftrightarrow 1 + \frac{1}{l} \leq 10^{\varepsilon/2} &\Leftrightarrow l \geq \frac{1}{10^{\varepsilon/2}-1}.\end{aligned}$$ [47]

So the right hand side decreases for $l \geq \frac{1}{10^{\varepsilon/2}-1}$. Hence for

$$c_7 = \frac{2c_b(\log(10))^2}{c_\alpha}\left(\frac{1}{10^{\varepsilon/2}-1}+1\right)10^{-\varepsilon\left(\frac{1}{10^{\varepsilon/2}-1}-1\right)}$$

we have the desired estimate. □

Choose $c_8 \geq 0$ with

$$c_8 < \frac{c_1^2 c_2^2 c_\alpha^2}{4c_5}, c_8 < \frac{6}{c_7}.$$

Similar to step 1 we want to get

Lemma 1.5.11. Either

(i) $k(z) < c_7\left(\log(|z|)\right)^{-2}$ or

(ii) $-\Delta k(z) < -c_8 k(z)^2|z|^{-2}$,

holds.

[47] The other branch of the square root leads to no solution within the $k \geq 1$.

Proof. Assume (i) does not hold. Then we get by 1.5.10

$$\begin{aligned} &\frac{1}{2}k(z) \geq \frac{1}{2}c_7\left(-\log|z|\right)^{-2} \geq \frac{c_b}{c_\alpha}|z|^{\varepsilon} \geq \frac{b|z|^2}{c_\alpha}, \\ &\Rightarrow k(z) - \frac{b|z|^2}{c_\alpha} \geq k(z) - \frac{1}{2}k(z) = \frac{1}{2}k(z) \end{aligned}$$

The right inequality in 1.5.9 leads to

$$\frac{1}{2}k(z) \leq k(z) - \frac{b|z|^2}{c_\alpha} \leq \frac{1}{c_\alpha}\left(b|z|^2 + \|\tau\|_F^2|z|^2\right) - \frac{b|z|^2}{c_\alpha} = \frac{\|\tau\|_F^2|z|^2}{c_\alpha}.$$

Negate and square the inequality:

$$-\|\tau\|_F^4 \leq \frac{-c_\alpha^2 k^2(z)}{4|z|^4}.$$

Recall Equation 1.4.18.2

$$\begin{aligned} -\Delta k(z) &= -\Delta\log(\|\theta_z\|_F^2) - \Delta\log\left(\frac{|z|^2}{c_\alpha}\right) \leq -\frac{c_1^2c_2^2\|\tau\|_F^4}{\|\theta_z\|_F^2} - \frac{2}{|z|^2} \\ &\overset{(1.5.7)}{\leq} -\frac{c_1^2c_2^2\|\tau\|_F^4|z|^2}{c_5} - \frac{2}{|z|^2} \leq -\frac{c_1^2c_2^2c_\alpha^2k^2(z)|z|^2}{4|z|^4c_5} - \frac{2}{|z|^2} \\ &\leq -\frac{c_8k^2(z)}{|z|^2} - \frac{2}{|z|^2} \leq -\frac{c_8k^2(z)}{|z|^2}. \end{aligned}$$

□

Lemma 1.5.12. Define

$$p_{\varepsilon_p,c_p}(z) \quad := \quad c_p\left(-\log|z|\right)^{-2} + \varepsilon_p\left(-\log|z|\right),$$

for $c_p = 6c_8^{-1}$. Then

(i) $-\Delta p_{\varepsilon_p,c_p}(z) \geq -c_8\frac{p_{\varepsilon_p,c_p}^2}{|z|^2}$ and

(ii) $p_{\varepsilon_p,c_p}(z) > c_7\left(-\log|z|\right)^{-2}$.

Proof.

$$\begin{aligned} -\Delta p_{\varepsilon_p,c_p}(z) &= -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}p_{\varepsilon_p,c_p}(z) \\ &= -\frac{1}{r}\frac{\partial}{\partial r}r\left(-\frac{2}{(\log(r))^3 r} - \frac{\varepsilon_p}{r}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{r}\frac{6c_p}{(\log(r))^4 r} = -\frac{6c_p}{(\log(r))^4 r^2} \\
&\geq -\frac{6c_p}{(\log(r))^4 r^2} + \underbrace{\frac{12\varepsilon_p}{r^2\log(r)}}_{\leq 0} - \underbrace{\frac{6\varepsilon_p^2(\log(r))^2}{c_p r^2}}_{\geq 0} \\
&= -\frac{6p^2_{\varepsilon_p,c_p}(z)}{c_p r^2} = -\frac{c_8 p^2_{\varepsilon_p,c_p}(z)}{r^2}.
\end{aligned}
$$

This shows (i). Recall that $c_8 < \frac{6}{c_7}$ by definition $\Rightarrow c_p = 6c_8^{-1} > c_7$. Thus

$$
\begin{aligned}
p_{\varepsilon_p,c_p}(z) &= c_p\left(-\log|z|\right)^{-2} + \underbrace{\varepsilon_p\left(-\log|z|\right)}_{\geq 0} \\
&\geq c_p\left(-\log|z|\right)^{-2} > c_7\left(-\log|z|\right)^{-2};
\end{aligned}
$$

the second inequality. □

We want to use again the maximum principle to show $p_{\varepsilon_p,c_p}(z) \geq k(z)$. Therefore set $S_2 := \{z|k(z) > p_{\varepsilon_p,c_p}(z)\}$.

Lemma 1.5.13. $S_2 = \emptyset$.

Proof. Assume $S_2 \neq \emptyset$, then $k(z) > p_{\varepsilon_p,c_p}(z) > c_7\left(-\log|z|\right)^{-2}$ for at least one z. 1.5.11 applied to the current situation implies

$$
-\Delta k(z) < -c_8 k^2(z)|z|^{-2},
$$

and further

$$
\begin{aligned}
-\Delta(k - p_{\varepsilon_p,c_p})(z) &= -\Delta(k) + \Delta p_{\varepsilon_p,c_p}(z) \overset{1.5.12}{<} -\frac{c_8 k^2(z)}{|z|^2} + \frac{c_8 p^2_{\varepsilon_p,c_p}}{|z|^2} \\
&= -c_8 \underbrace{\frac{k^2(z) - p^2_{\varepsilon_p,c_p}}{|z|^2}}_{>0} < 0.
\end{aligned}
$$

So the function $k - p_{\varepsilon_p,c_p}$ is subharmonic and since $k - p_{\varepsilon_p,c_p}$ is smooth on B_R^*, and it is continuous on ∂S_2 (as $\partial S_2 \cap \partial B_R^* = \emptyset$) - p_{ε_p,c_p} is infinite on $\partial B_R^* = \{\tilde{z}||\tilde{z}| = 0\} \cap \{\tilde{z}||\tilde{z}| = R\} \Rightarrow k \not> p_{\varepsilon_p,c_p}$ on ∂B_R^* - i.e. p_{ε_p,c_p} continuous on $\partial S_2 \subset B_R^*$. So we are in the position to use the maximum principle again and we get $(k - p_{\varepsilon_p,c_p})(z) \leq (k-k)(z) = 0 \Rightarrow k(z) \leq p_{\varepsilon_p,c_p}(z), \forall z \in S_2$, i.e. a contradiction to the definition of S_2. Hence $S_2 = \emptyset$. □

Conclusion 1.5.14. The previous lemma holds for all $\varepsilon_p > 0$ and yields for $\varepsilon_p \to 0$

$$k(z) \le c_p \left(-\log|z|\right)^{-2}.$$

Remark 1.5.15. Remember A.1.18 in the branched cover case. We get instead of c_8 now $\frac{c_8}{N|u|^{2N-2}}$ and so in the proof of lemma 1.5.13 we still have subharmonicity and so conclusion 1.5.14 still holds.

1.5.3. Step 3

$b = \|\sigma\|_F^2 - \|\alpha\|_F^2$ is in general not positive. We need to find a connection between $\|\beta\|_F^2$ and b. First note that both are not equal. For example for $\lambda(z) = \frac{1}{z} = \alpha(z), \beta(z) = -4$ at $z = \frac{1}{2}$: $\|\alpha\|_F^2 = \frac{1}{z^2} = 4, \|\lambda\|_F^2 = \left|\frac{1}{z^2} - 4\right|^2 = 0 \Rightarrow \|\beta\|_F^2 = 16, b = 0$.

Lemma 1.5.16.

$$-c_{b,\beta}|z|^{-2+\varepsilon} \le b - \|\beta\|_F^2 \le c_{b,\beta}|z|^{-2+\varepsilon}.$$

Proof. We have

$$\begin{aligned}
b - \|\beta\|_F^2 &= \|\sigma\|_F^2 - \|\alpha\|_F^2 - \|\beta\|_F^2 \\
&= \sum_{i=1}^n \left|\frac{a_i}{z} + \beta_i\right|^2 - \left|\frac{a_i}{z}\right|^2 - |\beta_i|^2 \\
&= \sum_{i=1}^n \left|\frac{a_i}{z} + \beta_i\right|^2 - \left|\frac{a_i}{z}\right|^2 - |\beta_i|^2 \\
&\le \sum_{i=1}^n \left|\frac{a_i}{z}\right|^2 + 2\left|\frac{a_i}{z}\right||\beta_i| + |\beta_i|^2 - \left|\frac{a_i}{z}\right|^2 - |\beta_i|^2 \\
&\le \sum_{i=1}^n 2\left|\frac{a_i}{z}\right||\beta_i| \\
&\le 2nc_\alpha c_\beta |z|^{-2+\varepsilon}
\end{aligned}$$

and

$$\begin{aligned}
b - \|\beta\|_F^2 &= \|\sigma|_F^2 - \|\alpha|_F^2 - \|\beta\|_F^2 \\
&= \sum_{i=1}^n \left|\frac{a_i}{z} + \beta_i\right|^2 - \left|\frac{a_i}{z}\right|^2 - |\beta_i|^2 \\
&= \sum_{i=1}^n \left|\frac{a_i}{z} + \beta_i\right|^2 - \left|\frac{a_i}{z}\right|^2 - |\beta_i|^2
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{n} \left|\left|\frac{a_i}{z}\right| - |\beta_i|\right|^2 - \left|\frac{a_i}{z}\right|^2 - |\beta_i|^2 \\
&\geq \sum_{i=1}^{n} \left|\frac{a_i}{z}\right|^2 - 2\left|\frac{a_i}{z}\right| |\beta_i| + |\beta_i|^2 - \left|\frac{a_i}{z}\right|^2 - |\beta_i|^2 \\
&\geq -\sum_{i=1}^{n} 2\left|\frac{a_i}{z}\right| |\beta_i| \\
&\geq -2nc_\alpha c_\beta |z|^{-2+\varepsilon}.
\end{aligned}
$$

□

Conclusion 1.5.17. Lemma 1.5.10 tells us that we find another constant, say c_9 such that

$$\|\beta\|_F^2|z|^2 \leq b|z|^2 + c_{b,\beta}|z|^\varepsilon \leq b|z|^2 + c_9\left(-\log|z|\right)^{-2}.$$

We have seen in 1.5.9 that k is bounded from below for $b|z|^2 + \|\tau\|_F^2 \geq 0$. Our aim is to find a lower bound for all z. Therefore we construct a lower bound for $b|z|^2 + \|\tau\|_F^2 < 0$ and show that the sum of this bound with the one for $b|z|^2 + \|\tau\|_F^2 \geq 0$ from 1.5.9 bounds k everywhere.
Choose a radius R_5 such that $1 - \frac{c_b}{c_\alpha}|z|^\varepsilon \geq \varepsilon_4 > 0$ as well as $1 - \frac{c_b}{c_\alpha}c_{10}\left(-\log|z|\right)^{-2} \geq \varepsilon_5 > 0$ for some constants $\varepsilon_4, \varepsilon_5$. This is obviously possible as both terms converge to 1 for $z \to 0$. Then

$$1 + \frac{b|z|^2 + \|\tau\|_F^2}{c_\alpha} \geq 1 + \frac{b|z|^2}{c_\alpha} \geq 1 - \frac{c_b}{c_\alpha}|z|^\varepsilon,$$

and so $1 + \frac{b|z|^2+\|\tau\|_F^2}{c_\alpha} \geq 1 - \frac{c_b}{c_\alpha}|z|^\varepsilon \geq \varepsilon_4 > 0$ on $B^*_{R_5}$. Now we use 1.5.10 again to find more constants c_{10}, c_{11} such that for $\frac{b|z|^2+\|\tau\|_F^2}{c_\alpha} < 0$, $0 < |z| < R_5$

$$
\begin{aligned}
k(z) &= \log\left(1 + \frac{b|z|^2 + \|\tau\|_F^2}{c_\alpha}\right) \\
&\geq \log\left(1 + \frac{b|z|^2}{c_\alpha}\right) \geq \log\underbrace{\left(1 - \frac{c_b}{c_\alpha}|z|^\varepsilon\right)}_{\geq \varepsilon_4} \\
&\geq \log\left(1 - \underbrace{\frac{c_b}{c_\alpha}c_{10}\left(-\log|z|\right)^{-2}}_{=:-t\leq 1-\varepsilon_5}\right) \\
&\geq -c_{11}\frac{c_b}{c_\alpha}c_{10}\left(-\log|z|\right)^{-2}.
\end{aligned}
\tag{1.5.17.1}
$$

The last step is again using the concavity of the logarithm. Since z is bounded from above the logarithm is bounded from below and so the graph of $\log(1-t)$ is bound from below by $\log(\varepsilon_5)$, and above by 0. $c_{11}t$ intersects $\log(1-t)$ at zero for arbitrary c_{11}. Now choose $c_{11}t$ the line through the smallest value of $\log(1-t)$ and the biggest value of $\log(1-t)$, i.e. 0.
Note that $-(-\log|z|)^{-2}$ is negative everywhere and adding a negative function

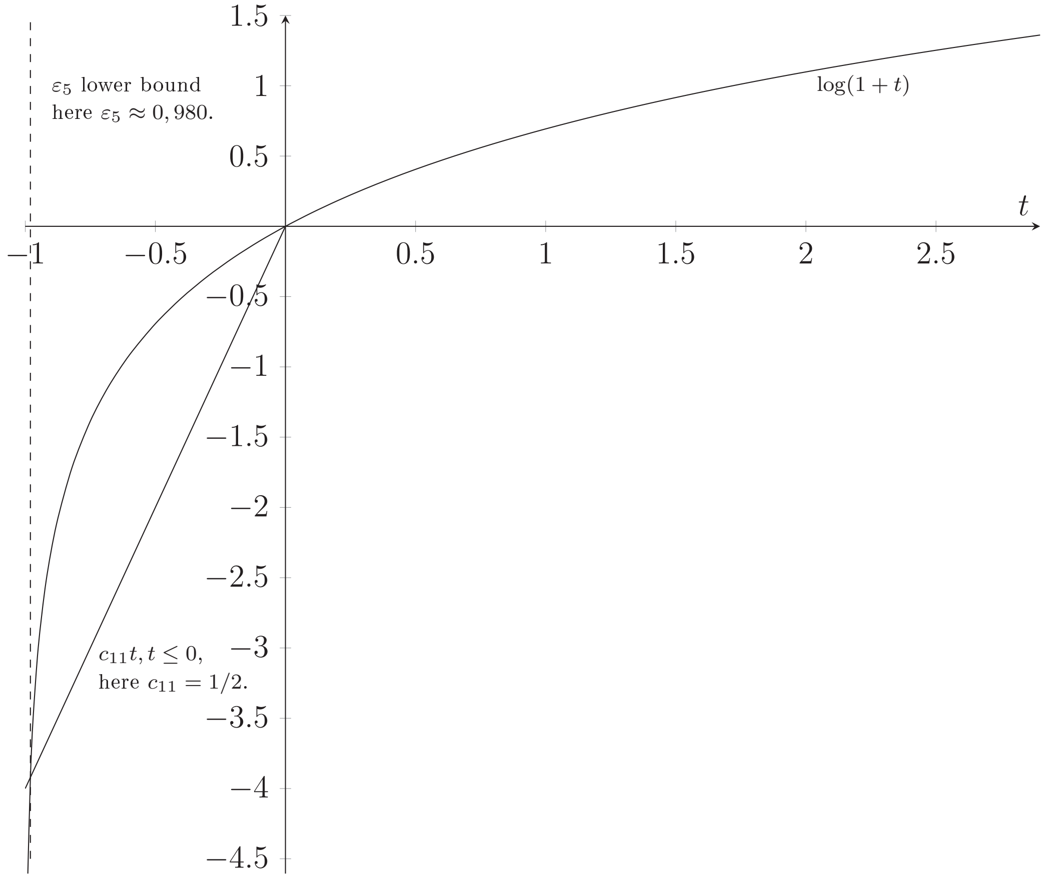

Figure 1.3: Lower bound for a (negative) part of the logarithm; $t = -\frac{c_b}{c_\alpha} c_{10} (-\log|z|)^{-2}$.

to the left side of the left inequality in 1.5.9 won't change the inequality. On the other hand adding $b|z|^2 + \|\tau\|_F^2 < 0$ on the right-hand side of equation 1.5.17.1 preserves the inequality as well. In formula

$$\begin{aligned} b|z|^2 + \|\tau\|_F^2|z|^2 \geq 0: &\quad \tfrac{c_6}{c_\alpha}\left(b|z|^2 + \|\tau\|_F^2|z|^2\right) - c_{11}\tfrac{c_b}{c_\alpha}c_{10}(-\log|z|)^{-2} \leq \tfrac{c_6}{c_\alpha}\left(b|z|^2 + \|\tau\|_F^2|z|^2\right) \leq k(z), \\ b|z|^2 + \|\tau\|_F^2|z|^2 < 0: &\quad \tfrac{c_6}{c_\alpha}\left(b|z|^2 + \|\tau\|_F^2|z|^2\right) - c_{11}\tfrac{c_b}{c_\alpha}c_{10}(-\log|z|)^{-2} \leq -c_{11}\tfrac{c_b}{c_\alpha}c_{10}(-\log|z|)^{-2} \leq k(z). \end{aligned}$$

Therefore we have by conclusion 1.5.17

$$\frac{c_6}{c_\alpha}\left(\|\beta\|_F^2|z|^2+\|\tau\|_F^2|z|^2\right)-\underbrace{\left(c_{11}\frac{c_b}{c_\alpha}c_{10}+c_9\frac{c_6}{c_\alpha}\right)}_{=:c_{12}}(-\log|z|)^{-2}$$
$$\le k(z). \tag{1.5.17.2}$$

This shows the first part of the main estimate:

Theorem 1.5.18.

$$\|\tau^0\|_F^2 \le \|\tau^0+\beta\|_F^2 \le \|\tau+\beta\|_F^2 \le c_{\tau^0}\left(-|z|\log|z|\right)^{-2}.$$

Proof. Use first that the Frobenius norm is entry-wise, than equation 1.5.17.2 and finally 1.5.14

$$\begin{aligned}\|\tau^0\|_F^2 &\le \|\tau+\beta\|_F^2 \le \|\beta\|_F^2+\|\tau\|_F^2 \le c_6^{-1}c_\alpha\frac{k(z)}{|z|^2}+c_6^{-1}c_\alpha c_{12}\left(-|z|\log|z|\right)^{-2}\\ &\le c_6^{-1}c_\alpha c_p\left(-|z|\log|z|\right)^{-2}+c_6^{-1}c_\alpha c_{12}\left(-\log|z|\right)^{-2}.\end{aligned}$$

Choose $c_{\tau^0}=c_6^{-1}c_\alpha(c_p+c_{12})$. □

Remark 1.5.19. We have seen that the results of Steps 1 and 2 hold for the branched cover case as well, and since we have no more differentiation in step 3 theorem 1.5.18 holds in the branched cover case as well.

1.6. Main Estimate, Part II

We will start this second part of the main estimate by recalling the adjoint representation. Then we will add a proof that $\mathrm{ad}(\phi)$ is invertible on the block upper-triangular matrices. Using consistency of the Frobenius norm with the original norm h (in an $h-$orthonormal frame) we may estimate ad in terms of its entries. In particular τ^+ is bounded in terms of q. Now we can construct a function k as in the second step of part I. Fortunately the problems that made us add step 3, will not occur in this case and the rest of the argument works as in step 2. Now we get a bound for τ^+ additionally to our bound for τ^0 from part I. Together we get the desired bound for the curvature of the metric connection.

1.6.1. The Adjoint Representation

Next we will estimate the block upper triangular part τ^+. We start with a general discussion of the Lie bracket of two block matrices.

Lemma 1.6.1. Let $1 = l_1 < l_2 < \ldots < l_{\tilde{k}} \leq n$, $\tilde{k} \leq n$ and $m : \{1, \ldots, n\} \to m(i) \in \{1, \ldots, \tilde{k}\}$ an increasing map. Define $l_{\tilde{k}+1} = n+1$. Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$ with $a_{ij} = \delta_{j \geq i} a_{ij}, a_{ii} = a_{m(i)} \delta_{ii}, a_{m(i)} \in \mathbb{C}$ and $m(i) = m(j) \Leftrightarrow \exists 1 \leq s \leq \tilde{k} : l_s \leq i, j < l_{s+1}$. In words the diagonal of the upper triangular A is $a_{m(i)} E$ on blocks. Then $\mathrm{ad}(A)$ is bijective on the set of block upper triangular matrices.

Proof. Let $C = (c_{ij}) \in \mathrm{M}_n(\mathbb{C})$ be block upper triangular w.r.t. the l_i, i.e. $c_{ij} = 0$ for all $j \leq l_{m(i)}$.

$$\begin{aligned}
(AC)_{ik} &= \sum_{j=1}^{n} a_{ij} c_{jk} = \sum_{j=1}^{n} \delta_{j \geq i} a_{ij} c_{jk} \delta_{k \geq l_{m(j)+1}}, \\
(CA)_{ik} &= \sum_{j=1}^{n} c_{ij} a_{jk} = \sum_{j=1}^{n} \delta_{k \geq j} c_{ij} a_{jk} \delta_{j \geq l_{m(i)+1}}, \\
(AC - CA)_{ik} &= \sum_{j=1}^{n} a_{ij} c_{jk} \delta_{\{j \geq i\} \cap \{k \geq l_{m(j)+1}\}} - c_{ij} a_{jk} \delta_{\{k \geq j\} \cap \{j \geq l_{m(i)+1}\}}.
\end{aligned}$$

Assume $AC - CA = D$, $D = (d_{ij})_{1 \leq i,j \leq n}$ strictly block upper triangular. Consider $(AC - CA)_{ik}$ for $l_{\tilde{k}} \leq i \leq n$. Then $l_{\tilde{k}} = l_{m(i)} \leq l_{m(j)} < l_{m(j)+1} \Rightarrow l_{m(j)+1} > n$ and thus $\{j \geq i\} \cap \{k \geq l_{m(j)+1}\} = \emptyset$. Similar $\{j \geq i\} \cap \{k \geq l_{m(j)+1}\} = \emptyset$. So this suits to $c_{ij} = 0, d_{ij} = 0, \forall l_{\tilde{k}} \leq i \leq n$. This is our base case for the induction A. Now we are going to show that stepwise for all $l_s, 1 \leq s \leq \tilde{k}, \forall l_s \leq i < l_{s+1}, \forall 1 \leq j \leq n : c_{ij}$ is determined by D. Assume this holds for s; $s \to s-1$:
Note that $j \geq l_s - 1 \Rightarrow m(j) \geq s-1$. Hence $\{j \geq l_s - 1\} \cap \{l_s \geq l_{m(j)+1}\} = \{j = l_s - 1\}$ and $\{l_s \geq j\} \cap \{j \geq l_{m(l_s-1)+1}\} = \{j = l_s\}$.

$$\begin{aligned}
&D_{l_s-1,l_s} = d_{l_s-1,l_s} = (AC - CA)_{l_s-1,l_s} \\
&= \sum_{j=1}^{n} a_{l_s-1,j} c_{j,l_s} \delta_{\{j \geq l_s-1\} \cap \{l_s \geq l_{m(j)+1}\}} - c_{l_s-1,j} a_{j,l_s} \delta_{\{l_s \geq j\} \cap \{j \geq l_{m(l_s-1)+1}\}} \\
&= a_{l_s-1,l_s-1} c_{l_s-1,l_s} - c_{l_s-1,l_s} a_{l_s,l_s} = c_{l_s-1,l_s}(a_{l_s-1,l_s-1} - a_{l_s,l_s}) \\
&= c_{l_s-1,l_s}(\underbrace{a_{m(l_s-1)} - a_{m(l_s)}}_{\neq 0}). \\
&\Rightarrow c_{l_s-1,l_s} = \frac{d_{l_s-1,l_s}}{a_{m(l_s-1)} - a_{m(l_s)}},
\end{aligned}$$

uniquely determined. Of course $c_{l_s-1,l_s} = 0$ for $D = 0$. This is just the base case for another induction B with induction step $c_{l_s-1,r}, \forall l_s \leq r \leq k \leq n$ uniquely determined $\Rightarrow c_{l_s-1,k+1}$, uniquely determined. It follows directly from

$$D_{l_s-1,k+1} = d_{l_s-1,k+1} = (AC - CA)_{l_s-1,k+1}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} a_{l_s-1,j} c_{j,k+1} \delta_{\{j \geq l_s - 1\} \cap \{k+1 \geq l_{m(j)+1}\}} - c_{l_s-1,j} a_{j,k+1} \delta_{\{k+1 \geq j\} \cap \{j \geq l_{m(l_s-1)+1}\}} \\
&= a_{l_s-1,l_s-1} c_{l_s-1,k+1} + \sum_{j=1}^{n} a_{l_s-1,j} c_{j,k+1} \delta_{\{j \geq l_s\} \cap \{k+1 \geq l_{m(j)+1}\}}{}^{48} \\
&\quad - c_{l_s-1,j} a_{j,k+1} \delta_{\{k+1 \geq j \geq l_s\}} \\
&= a_{l_s-1,l_s-1} c_{l_s-1,k+1} - c_{l_s-1,k+1} \underbrace{a_{k+1,k+1}}_{a_{m(l_s)}} \\
&\quad + \sum_{j=1}^{n} a_{l_s-1,j} c_{j,k+1} \delta_{\{j \geq l_s\} \cap \{k+1 \geq l_{m(j)+1}\}} - c_{l_s-1,j} a_{j,k+1} \delta_{\{k+1 > j \geq l_s\}} \\
&= c_{l_s-1,k+1} \big(\underbrace{a_{m(l_s-1)} - a_{m(l_s)}}_{\neq 0} \big) \\
&\quad + \sum_{j=1}^{n} a_{l_s-1,j} c_{j,k+1} \delta_{\{j \geq l_s\} \cap \{k+1 \geq l_{m(j)+1}\}} - c_{l_s-1,j} a_{j,k+1} \delta_{\{k+1 > j \geq l_s\}} \\
&\Rightarrow c_{l_s-1,k+1} = \frac{d_{l_s-1,k+1}}{a_{m(l_s-1)} - a_{m(l_s)}} - \sum_{j=1}^{n} a_{l_s-1,j} c_{j,k+1} \delta_{\frac{\{j \geq l_s\} \cap \{k+1 \geq l_{m(j)+1}\}}{a_{m(l_s-1)} - a_{m(l_s)}}} \\
&\qquad + \frac{c_{l_s-1,j} a_{j,k+1} \delta_{\{k+1 > j \geq l_s\}}}{a_{m(l_s-1)} - a_{m(l_s)}},
\end{aligned}
$$

where the second term is uniquely determined by D and thus $c_{l_s-1,k+1}$ is: $c_{j,k+1}\delta_{\{j \geq l_s\} \cap \{k+1 \geq l_{m(j)+1}\}}$ determined by Induction hypothesis A and $c_{l_s-1,j}\delta_{\{k+1 \geq j \geq l_s\}}$ by Hypothesis B. Note that the second term vanishes by hypothesis for $D = 0$.

This proves the induction B claim and therefore c_{l_s-1j} uniquely determined by D, $\forall 1 \leq j \leq n$.[49]

The claim of induction B could be seen as a starting point for another induction C. Before we start with this last induction we want to repeat what we are actually doing: Induction A shows that once we have one block of C already determined by D all other blocks are determined. Induction C induces the determination from one row up to another and induction B induces the determination from one entry to the one right of it.

For induction C the hypothesis is $c_{rk}, \forall 1 \leq k \leq n, \forall l_s - 1 \geq r \geq i \geq l_{s-1} + 1$

[48] $k \geq l_s$ by assumption implies that $\{j = l_s - 1\} \subset \{j \geq l_s - 1\} \cap \{k + 1 \geq l_{m(j)+1}\}$ and of course for $j = l_s - 1$ there is no further restriction on k since $k + 1 \geq l_{m(l_s-1)+1} = l_s$ is anyway fulfilled.

[49] For $j < l_s$ $c_{l_s-1j} = 0$ by the block upper triangular form. For $D = 0$ all c_{l_s-1j} vanish.

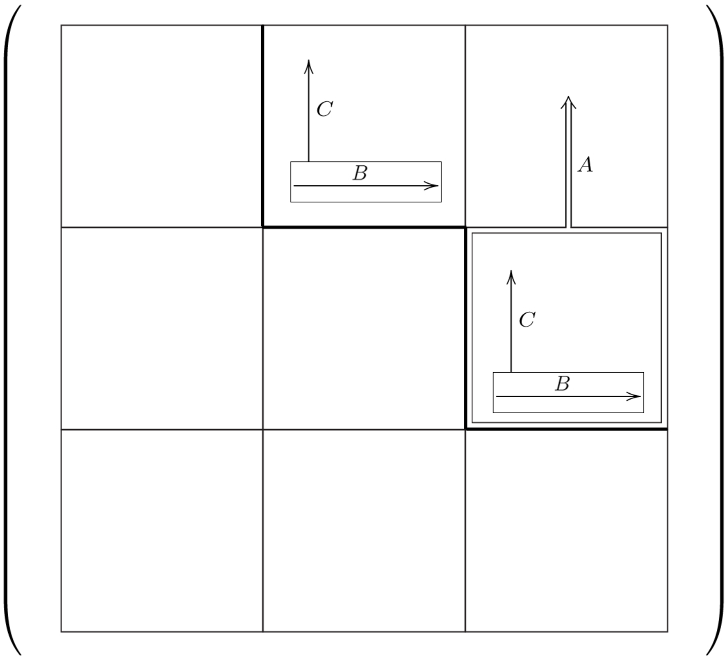

Figure 1.4: The arrows shall illustrate how the inductions A, B and C proceed.

determined by $D \Rightarrow c_{i-1,k}, \forall 1 \leq k \leq n$ determined. Again

$$\begin{aligned}
&D_{i-1,l_s} = d_{i-1,l_s} = (AC - CA)_{i-1,l_s} \\
&= \sum_{j=1}^{n} a_{i-1,j} c_{j,l_s} \delta_{\{j \geq i-1\} \cap \{l_s \geq l_{m(j)+1}\}} - c_{i-1,j} a_{j,l_s} \delta_{\{l_s \geq j\} \cap \{j \geq l_{m(i-1)+1}\}} \\
&= \sum_{j=1}^{n} a_{i-1,j} c_{j,l_s} \delta_{\{l_s > j \geq i-1\}} - c_{i-1,l_s} a_{l_s,l_s} \\
&= a_{i-1,i-1} c_{i-1,l_s} - c_{i-1,l_s} a_{l_s,l_s} + \sum_{j=1}^{n} a_{i-1,j} c_{j,l_s} \delta_{\{l_s > j \geq i\}} \\
&= c_{i-1,l_s} \underbrace{\left(a_{m(l_{s-1})} - a_{m(l_s)}\right)}_{\neq 0} + \sum_{j=1}^{n} a_{i-1,j} c_{j,l_s} \delta_{\{l_s > j \geq i\}} \\
&\Rightarrow c_{i-1,l_s} = \frac{d_{i-1,l_s} - \sum_{j=1}^{n} a_{i-1,j} c_{j,l_s} \delta_{\{l_s > j \geq i\}}}{a_{m(l_{s-1})} - a_{m(l_s)}},
\end{aligned}$$

uniquely determined by D. The second term is determined by hypothesis C. Then induction B implies that $c_{i-1,k}$ uniquely determined for all k. This shows the claim of induction C, which implies the claim of induction A and our lemma. □

Conclusion 1.6.2. $\mathrm{ad}(\phi)$ is invertible on the block upper triangular matrices M^+ for $\phi = \alpha + q$. Analogously α invertible on M^+.

Use the same notation as in the previous lemma. Define new matrices

$$A' = \mathrm{diag}(\alpha_{m(l_1)}, \dots, \alpha_{m(l_1)}, \alpha_{m(l_2)}, \dots, \alpha_{m(l_{\tilde{k}-1})}, \alpha_{m(l_{\tilde{k}})}, \dots, \alpha_{m(l_{\tilde{k}})}),$$

the diagonal part of A and $A'' := A - A'$. Finally let $A''' := (\tilde{a}_{ij})_{1\le i,j\le n}, \tilde{a}_{ij} = a_{ij}\delta_{j\ge l_{m(i)+1}}$ be the block upper triangular part of A.

Lemma 1.6.3. $\mathrm{ad}(A')(A'') = \mathrm{ad}(A')(A''')$, i.e. the adjoint representation with a matrix which acts as a dilation on the blocks depends only on the block upper triangular part.

Proof. By direct calculation

$$\begin{aligned}
(A'A'' - A''A')_{ik} &= \sum_{j=1}^{n} a_{m(i)}\delta_{ij}a_{jk}\delta_{k>j} - a_{ij}\delta_{j>i}a_{m(k)}\delta_{jk} \\
&= a_{m(i)}a_{ik}\delta_{k>i} - a_{ik}\delta_{k>i}a_{m(k)} = a_{ik}(a_{m(i)} - a_{m(k)})\delta_{k>i} \\
&= a_{ik}(a_{m(i)} - a_{m(k)})\delta_{k\ge l_{m(i)+1}} \\
&= a_{m(i)}a_{ik}\delta_{k\ge l_{m(i)+}} - a_{ik}\delta_{k\ge l_{m(i)}+1}a_{m(k)} \\
&= \sum_{j=1}^{n} a_{m(i)}\delta_{ij}a_{jk}\delta_{k\ge l_{m(j)+}} - a_{ij}\delta_{j\ge l_{m(i)}+1}a_{m(k)}\delta_{jk} \\
&= (A'A''' - A'''A')_{ik}.
\end{aligned}$$

□

1.6.2. An estimate for τ^+

Remember that $[\phi, \theta_z] = 0$ by definition. Then

$$\begin{aligned}
0 &= \mathrm{ad}(\theta_z)(\phi) = [\sigma + \tau, \alpha + q] \\
&= \underbrace{[\sigma, \alpha]}_{=0 \text{ diagonal}} + [\sigma, q] + [\tau, \alpha] + [\tau, q] \\
&= [\sigma + \tau, q] + [\tau, \alpha] = [\beta + \alpha + \tau, q] + [\tau, \alpha] \\
&= [\beta + \tau, q] + [\alpha, q] + [\tau, \alpha] \\
\Rightarrow \mathrm{ad}(\alpha)(\tau) &= \mathrm{ad}(\beta + \tau)(q) + \mathrm{ad}(\alpha)(q)
\end{aligned}$$

Lemma 1.6.3 tells us that $\mathrm{ad}(\alpha)(\tau) = \mathrm{ad}(\alpha)(\tau^+), \mathrm{ad}(\alpha)(q) \in M^+$ and thus $\mathrm{ad}(\beta + \tau)(q) \in M^+$, since $q \in M^+$.

Remark 1.6.4. (i) We have seen that our inner product h induces the Hilbert-Schmidt inner product h_{End} on the endomorphism bundle. To estimate maps from $M^+ \to M^+$ we use the induced norm

$$\|g\|_{M^+} := \sup_{\|B\|_F=1} \|g(B)\|_F = \sup_{B\neq 0} \frac{\|g(B)\|_F}{\|B\|_F}, \quad g \in \mathrm{End}(M^+).$$

If g is a bounded and invertible operator on a finite-dimensional normed vector space (hence a Banach space), so is g^{-1}. To estimate the norm of the inverse, recall the proof of lemma 1.6.1. Then for $AC - CA = D$ we saw that the c_{ij} depend on D and A naturally. On the diagonal part of A, i.e. A', however, the dependence is clearer: c_{ij} is proportional to the inverse of the product of differences between distinct entries of A'. But these differences are just $\frac{m_i}{z}$ for some constants m_i. So the finite product looks like $\frac{m_k}{z^r}, m_k$ constant, $r \in \mathbb{N} \setminus \{0\}$. Thus the inverse is proportional to $|z|^r \leq |z|$. Hence $c_{ij} \leq m_{ij}|z| \Rightarrow \|\mathrm{ad}(A')^{-1}(D)\|_F = \|C\|_F \leq c_{\mathrm{ad}(A')^{-1}}|z|, \forall \|D\|_F = 1$ and some constant $c_{\mathrm{ad}(A')^{-1}}$.
In our case $A' = \alpha$: $\|\mathrm{ad}(\alpha)^{-1}\|_{M^+} \leq c_{\mathrm{ad}(\alpha)^{-1}}|z|$.

(ii) For the Frobenius norm we have

$$\begin{aligned}
\|AB\|_F^2 &= \sum_{i,k=1}^{n} \left| \sum_{j=1}^{n} a_{ij} b_{jk} \right|^2 \leq \sum_{i,k=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right) \left(\sum_{j=1}^{n} |b_{jk}|^2 \right) \\
&= \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right) \left(\sum_{j,k=1}^{n} |b_{jk}|^2 \right) \\
&= \|A\|_F^2 \|B\|_F^2,
\end{aligned}$$

by Cauchy-Schwarz. Hence we have submultiplicity for the Frobenius norm. By the triangle inequality

$$\begin{aligned}
\|\mathrm{ad}(A)\|_{M^+} &= \sup_{\|B\|_F=1} \|AB - BA\|_F \leq \sup_{\|B\|_F=1} \|AB\|_F + \|BA\|_F \\
&\leq \sup_{\|B\|_F=1} 2\|A\|_F \|B\|_F = 2\|A\|_F.
\end{aligned}$$

The remark leads to

$$\begin{aligned}
\mathrm{ad}(\alpha)(\tau^+) &= \mathrm{ad}(\beta + \tau)(q) + \mathrm{ad}(\alpha)(q) \\
\Rightarrow \tau^+ &= (\mathrm{ad}(\alpha))^{-1} \mathrm{ad}(\alpha)(\tau^+) = \underbrace{(\mathrm{ad}(\alpha))^{-1} \mathrm{ad}(\beta + \tau)}_{=:f}(q) + q \\
&= (f + E)(q).
\end{aligned}$$

Note that by 1.5.18

$$\|\tau+\beta\|_F^2 \le c_{\tau^0}\left(-|z|\log|z|\right)^{-1}$$
$$\Rightarrow \|f\|_{M^+} \le {}^{50} 2|z|\underbrace{c_{\mathrm{ad}(\alpha)^{-1}}\cdot c_{\tau^0}}_{=:c_f}\left(-|z|\log|z|\right)^{-1} = \frac{c_f}{|\log|z||}.$$

After restricting, if necessary, to a smaller neighbourhood $B^*_{R_3}$ we have $\|f\|_{M^+} \le \frac{c_f}{|\log|z||} < C_f < 1$. Then the von Neumann series $\sum_{k=0}^{\infty}(-f)^k$ converges and $f+E$ is invertible. Moreover $\exists c_{13}$ independent of $|z|$ such that $\|(f+E)^{-1}\| < c_{13}$ by

$$\begin{aligned}
\frac{\|(f+E)^{-1}B\|_F}{\|B\|_F} &\le \frac{\|\sum_{k=0}^{\infty}(-f)^k B\|_F}{\|B\|_F} \le {}^{51} 1+\frac{\sum_{k=1}^{\infty}\|(-f)^k\|_{M^+}\|B\|_F}{\|B\|_F} \\
&\le \frac{1}{1-C_f} \\
\Rightarrow \|(f+E)^{-1}\|_{M^+} &\le \frac{1}{1-C_f}.
\end{aligned}$$

Remark 1.6.5. Using again 1.5.18 we get

$$\|q\|_F \le \|\tau^+\|_F\|(f+E)^{-1}\|_{M^+} \le \left(\frac{1}{1-C_f}\right)\frac{c_{\tau^0}}{-|z|\log|z|} \le \frac{c_{14}}{||z|\log|z||}.$$

In particular a bound for q leads to a bound for τ^+ which differs at most by a scalar multiplication, and vice versa.

Now

$$\begin{aligned}
\|\phi\|_F^2 &= \|\alpha\|_F^2+\|q\|_F^2 \le \frac{c_\alpha}{|z|^2}+\frac{c_{14}^2}{|z|^2}\cdot\underbrace{(\log|z|)^{-2}}_{<C_f^2<1} \\
&\le \frac{c_{15}}{|z|^2}.
\end{aligned}$$

Recall 1.4.16 for $\varphi=\phi$

$$-\Delta\log\|\phi_z\|_F^2 \le -\frac{\|[\theta_z^\dagger,\phi_z]\|_F^2}{\|\phi_z\|_F^2} = -\frac{\|[\phi_z,\theta_z^\dagger]\|_F^2}{\|\phi_z\|_F^2}.$$

As we have seen, the denominator is bounded by $\frac{c_{15}}{|z|^2}$. For $[\varphi_z,\theta_z^\dagger]$ we calculate

$$[\varphi_z,\theta_z^\dagger] = [\alpha+q,\theta_z^\dagger] = [\alpha,\theta_z^\dagger]+[q,\overline{\sigma}+\tau^\dagger] = [\alpha,\theta_z^\dagger]+[q,\overline{\alpha}+\overline{\beta}+\tau^\dagger]$$

[50] Use (ii) of the previous remark 1.6.4.
[51] For the operator norm we know that $\|g(A)\|_F \le \|g\|_{M^+}\|A\|_F$ (by definition).

$$= \quad [\alpha, \theta_z^\dagger] + [q, \overline{\alpha}] + [q, \overline{\beta} + \tau^\dagger]$$

The first term $[\alpha, \theta_z^\dagger]$ is not upper triangular, while the second term $[q, \overline{\alpha}]$ is. The third term $[q, \overline{\beta} + \tau^\dagger]$ is mixed. Since

$$\|[\varphi_z, \theta_z^\dagger]\|_F^2 = \|[\varphi_z, \theta_z^\dagger]_{bup}\|_F^2 + \|[\varphi_z, \theta_z^\dagger]_{bdiag,blow}\|_F^2,$$

with $[\varphi_z, \theta_z^\dagger]_{bup}$ the block upper triangular part of $[\varphi_z, \theta_z^\dagger]$ and $[\varphi_z, \theta_z^\dagger]_{bdiag,blow}$, i.e. the block lower triangular part including the block diagonal part, it will be enough to estimate one of the terms to find a lower bound. In fact again by lemma 1.6.1 and remark 1.6.4, (i) we know that $\overline{\alpha}$ is invertible on M^+ and bounded by $\|\mathrm{ad}(\overline{\alpha})^{-1}\|_{M^+} \leq c_{\mathrm{ad}(\overline{\alpha})^{-1}}|z|$. Furthermore remark 1.6.4, (ii) and 1.5.18 lead to

$$\begin{aligned} \sup_{B\neq 0} \|(\mathrm{ad}(\overline{\beta} + \tau^\dagger)(B))_{bup}\|_F &\leq \sup_{B\neq 0} \|(\mathrm{ad}(\overline{\beta} + \tau^\dagger)(B))\|_F \leq 2\|\overline{\beta} + \tau^\dagger\|_F \\ &\leq \frac{C_{\tau^0}}{-|z|\log|z|}. \end{aligned}$$

Hence on $B^*_{R_6}$ for some small enough radius R_6, $\tilde{f} = (\mathrm{ad}(\overline{\alpha}))^{-1}(\mathrm{ad}(\overline{\beta} + \tau^\dagger)(B))_{bup}$ satisfies $\|\tilde{f}\|_{M^+} \leq \frac{c_{\tilde{f}}}{|\log|z||} < C_{\tilde{f}} < 1$ for suitable constants as before. Thus $E + \tilde{f}$ invertible with upper bound $\|(E + f)^{-1}\|_{M^+} \leq c_{16}$,

$$\begin{aligned} \Rightarrow \|q\|_F &\leq c_{16}\|q + (\mathrm{ad}(\overline{\alpha}))^{-1}(\mathrm{ad}(\overline{\beta} + \tau^\dagger)(q))_{bup}\|_F \\ &\leq c_{16}\|(\mathrm{ad}(\overline{\alpha}))^{-1}\|_F\|\mathrm{ad}(\overline{\alpha})(q) + (\mathrm{ad}(\overline{\beta} + \tau^\dagger)(q))_{bup}\|_F \\ &= c_{16}\|(\mathrm{ad}(\overline{\alpha}))^{-1}\|_F\|[\varphi_z, \theta_z^\dagger]_{bup}\|_F. \end{aligned}$$

So we have shown

Lemma 1.6.6. There is a constant c_{17} such that

$$c_{17}\frac{\|q\|_F^2}{|z|^2} = \frac{\|q\|_F^2}{c_{16}c^2_{\mathrm{ad}(\overline{\alpha})^{-1}}|z|^2} \leq \|[\varphi_z, \theta_z^\dagger]_{bup}\|_F^2 \leq \|[\varphi_z, \theta_z^\dagger]\|_F^2.$$

and hence

$$-\Delta \log \|\phi_z\|_F^2 \leq -c_{17}\frac{\|q\|_F^2}{|z|^2\|\phi_z\|_F^2} \leq -\frac{\overbrace{c_{17}c_{15}}^{=:c_{18}}|z|^2\|q\|_F^2}{|z|^2} \leq -c_{18}\|q\|_F^2.$$

Now we are in the position to proceed similarly to part I. First

$$-c_{18}\|q\|_F^2 \quad \geq \quad -\Delta \log \|\phi_z\|_F^2 = -\Delta \log \left(\frac{c_\alpha}{|z|^2}\left(1 + \frac{|z|^2}{c_\alpha}\|q\|_F^2\right)\right)$$

$$\begin{aligned}
&= -\Delta \log\left(\frac{c_\alpha}{|z|^2}\right) - \Delta \log\left(1 + \frac{|z|^2}{c_\alpha}\|q\|_F^2\right) \\
&= -\frac{1}{r}\frac{\partial}{\partial r}\underbrace{r\frac{-2c_\alpha}{r^3}\cdot\frac{r^2}{c_\alpha}}_{=-2} - \Delta \log\left(1 + \frac{|z|^2}{c_\alpha}\|q\|_F^2\right) \\
&= -\Delta \underbrace{\log\left(1 + \frac{|z|^2}{c_\alpha}\|q\|_F^2\right)}_{=:k(z)} \\
&= -\Delta k(z). \qquad (1.6.6.1)
\end{aligned}$$

Lemma 1.6.7. There is a constant c_{19} such that

$$c_{19}\left(\frac{|z|^2}{c_\alpha}\|q\|_F^2\right) \leq k(z) \leq \frac{|z|^2}{c_\alpha}\|q\|_F^2.$$

Proof. Note that this is the analogon to lemma 1.5.9. Fortunately this case is easier as $\frac{|z|^2}{c_\alpha}\|q\|_F^2 \geq 0$. The right inequality holds in general for the logarithm (cf. proof 1.5.9). On the other hand

$$\frac{|z|^2}{c_\alpha}\|q\|_F^2 \leq \frac{|z|^2}{c_\alpha c_{14}^2 |z|^2 \underbrace{|\log|z||^2}_{\geq 1}} \leq \frac{1}{c_\alpha c_{14}^2} =: c_{20}.$$

Set $c_{19} = c_{20}^{-1} \cdot \log(1 + c_{20})$. Then concavity of the logarithm guarantees the claim. □

Lemma 1.6.8. $\exists c_{21}$ such that

$$-\Delta k(z) \leq -c_{21}\frac{k(z)}{|z|^2}.$$

Proof. From equation 1.6.6.1 and the right inequality in lemma 1.6.7

$$-\Delta k(z) \leq -c_{18}\|q\|_F^2 \leq -\frac{\overbrace{c_{18}c_\alpha}^{=:c_{21}}}{|z|^2}k(z) = -\frac{c_{21}k(z)}{|z|^2}.$$

□

Remark 1.6.9. In particular we have by 1.6.5

$$k(z) \leq \frac{|z|^2}{c_\alpha}\|q\|_F^2 \leq \frac{c_{14}^2}{c_\alpha \underbrace{(-\log|z|)^2}_{\geq 1}} \leq c_{22},$$

for a constant c_{22}.

Similar to 1.5.12

Lemma 1.6.10. Define

$$p_{\varepsilon_p,c_p,u_p}(z) \quad := \quad c_p|z|^{u_p} + \varepsilon_p\left(-\log|z|\right),$$

for $u^2 < c_{21}$, $c_p R_6^{u_p} > c_{22}$. Then

(i) $-\Delta p_{\varepsilon_p,c_p,u_p}(z) > -\frac{c_{21}c_p|z|^{u_p}}{|z|^2} > -c_{21}\frac{p_{\varepsilon_p,c_p,u_p}(z)}{|z|^2}$.

(ii) $p_{\varepsilon_p,c_p,u_p}(R_6) > c_{22} > k(R_6)$.

Proof.

$$\begin{aligned}
-\Delta p_{\varepsilon_p,c_p,u_p}(z) &= -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}p_{\varepsilon_p,c_p,u_p}(z) = -\frac{1}{r}\frac{\partial}{\partial r}r\left(u_p c_p r^{u_p-1} - \frac{\varepsilon_p}{r}\right) \\
&= -\frac{1}{r}\frac{\partial}{\partial r}\left(u_p c_p r^{u_p} - \varepsilon_p\right) = -\frac{1}{r}\left(u_p^2 c_p r^{u_p-1}\right) \\
&= -u_p^2 c_p r^{u_p-2} \geq -c_{21}c_p r^{u_p-2} \\
&\geq -c_{21}\frac{c_p r^{u_p}}{r^2} - \underbrace{c_{21}\varepsilon_p\left(-\log r\right)r^{-2}}_{\geq 0} \\
&= -c_{21}\frac{p_{\varepsilon_p,c_p,u_p}(z)}{|z|^2}.
\end{aligned}$$

This shows (i). For (ii) consider

$$p_{\varepsilon_p,c_p,u_p}(R_6) \quad = \quad c_p R_6^{u_p} + \underbrace{\varepsilon_p\left(-\log|R_6|\right)}_{\geq 0} \geq c_p R_6^{u_p} \geq c_2 2 > k(R_6).$$

□

Next we show the analogon to 1.5.13, i.e. use the maximum principle to show $p_{\varepsilon_p,c_p,u_p}(z) \geq k(z)$. Therefore set $S_3 := \{z|k(z) > p_{\varepsilon_p,c_p,u_p}(z)\}$.

Lemma 1.6.11. $S_3 = \emptyset$.

Proof. Assume $S_2 \neq \emptyset$, then $k(z) > p_{\varepsilon_p,c_p}(z)$ for at least one z. Use lemma 1.6.8 and lemma 1.6.10 to obtain

$$\begin{aligned}
-\Delta(k(z) - p_{\varepsilon_p,c_p,u_p}(z)) &< -c_{21}\frac{k(z)}{|z|^2} - c_{21}\frac{p_{\varepsilon_p,c_p,u_p}(z)}{|z|^2} \\
&= -\frac{c_{21}}{|z|^2}(k(z) - p_{\varepsilon_p,c_p,u_p}(z)) < 0.
\end{aligned}$$

Thus $k(z) - p_{\varepsilon_p,c_p,u_p}(z)$ is subharmonic and is continuous on ∂S_3 since it is continuous of $B^*_{R_6}$ and $\partial S_3 \cap \partial B^*_{R_6} = \emptyset$ by (ii) of the previous lemma 1.6.10 and since $\log(z)$ blows up for $z \to 0$. The requirements of the maximum principle 1.5.3 are fulfilled and hence $k(z) - p_{\varepsilon_p,c_p,u_p}(z) = 0$ is the maximal value of $k(z) - p_{\varepsilon_p,c_p,u_p}(z)$ on $\overline{S}_3$ and so $k(z) \leq p_{\varepsilon_p,c_p,u_p}(z)$ on S_3 - a contradiction. Therefore $S_3 = \emptyset$. □

Conclusion 1.6.12. Using the left inequality in lemma 1.6.7

$$\|q\|_F \quad \leq \quad \frac{c_{19}^{-1} c_\alpha}{|z|^2} k(z) \leq \frac{c_{19}^{-1} c_\alpha}{|z|^2} \left(c_p |z|^{u_p} + \varepsilon_p \left(-\log|z|\right)\right)$$

we get for $\varepsilon_p \to 0$

$$\|q\|_F^2 \quad \leq \quad \overbrace{c_{19}^{-1} c_\alpha c_p}^{=:c_q} |z|^{u_p - 2} = c_q^2 |z|^{u_p - 2}.$$

Finally set $\varepsilon = \min\{u_p/2, 1/n\}$ and use remark 1.6.5 to receive

Theorem 1.6.13. There is a constant c_{τ^+} such that

$$\|q\|_F \leq \frac{c_q}{|z|^{1-\varepsilon}} \quad \|\tau^+\|_F^2 \leq \frac{c_{\tau^+}}{|z|^{1-\varepsilon}}.$$

Remark 1.6.14. Again recall A.1.18 in the branched cover case. We get instead of c_{21} now $\frac{c_{21}}{N|u|^{2N-2}}$ and so in the proof of lemma 1.6.11 we still have subharmonicity and so conclusion 1.5.14 still holds.

In Simpson [Sim90], theorem 1.6.13 and theorem 1.5.18 is called "Theorem 1". Moreover "Theorem 1" additionally states

Lemma 1.6.15. For $D = \partial_E + \overline{\partial}_E$ and the metric connection $D_{\overline{\partial}_E + \theta^\dagger} := \partial_E + \overline{\partial}_E + \theta^\dagger - \theta$ (cf. 1.2.5) we have

$$\|D_z^2\|_F \leq \frac{c_D}{|z|^2 |\log|z||^2}, \quad \|D^2_{\overline{\partial}_E + \theta^\dagger, z}\|_F \leq \frac{c_{D_{\overline{\partial}_E + \theta^\dagger}}}{|z|^2 |\log|z||^2}.$$

Here the norm of the curvature D shall be understood as the norm of the corresponding matrix representation D_z w.r.t. z. In future we will often refer to the curvature of the unique metric connection by R_h.

Proof. We have already seen, that for the harmonic bundle $D^2 = \overline{\partial}_E \partial_E + \partial_E \overline{\partial}_E = -\theta\theta^\dagger - \theta^\dagger\theta$. Further

$$D^2_{\overline{\partial}_E + \theta^\dagger} \quad = \quad (\partial_E + \overline{\partial}_E + \theta^\dagger - \theta)(\partial_E + \overline{\partial}_E + \theta^\dagger - \theta)$$

$$
\begin{aligned}
&= \underbrace{\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E}_{=-\theta^\dagger\theta-\theta\theta^\dagger} - \theta^\dagger\theta - \theta\theta^\dagger + \partial_E\theta^\dagger + \theta^\dagger\partial_E - \overline{\partial}_E\theta - \theta\overline{\partial}_E \\
&= -2\theta^\dagger\theta - 2\theta\theta^\dagger,
\end{aligned}
$$

where all other terms shorten out by the holomorphy of the Higgs field or degree considerations. In matrices we have $D_z^2 = -\theta_z\theta_z^\dagger + \theta_z^\dagger\theta_z$, $D^2_{\overline{\partial}_E+\theta^\dagger,z} = -2(\theta_z\theta_z^\dagger - \theta_z^\dagger\theta_z)$. Moreover

$$
\begin{aligned}
\|[\theta,\theta^\dagger]\|_F &= \|[\sigma+\tau,\overline{\sigma}+\tau^\dagger]\|_F = \|[\beta+\alpha+\tau,\overline{\beta}+\overline{\alpha}+\tau^\dagger]\|_F \\
&= \|[\beta+\tau,\overline{\beta}+\tau^\dagger] + [\alpha,\overline{\beta}+(\tau^\dagger)^+] - [\overline{\alpha},\beta+\tau^+]\|_F \\
&\leq 2\|\beta+\tau\|_F\|\overline{\beta}+\tau^\dagger\|_F + 2\|\alpha\|_F\|\overline{\beta}+(\tau^\dagger)^+\|_F + 2\|\overline{\alpha}\|_F\|\beta+\tau^+\|_F \\
&= 2\|\beta+\tau\|_F^2 + 4\|\alpha\|_F\|\beta+\tau^+\|_F \\
&\leq \frac{c_D}{|z|^2|\log|z||^2},
\end{aligned}
$$

where we used in the last step lemma 1.5.10 and 1.6.13 to get

$$
\begin{aligned}
\|\alpha\|_F\|\overline{\beta}+(\tau^\dagger)^+\|_F &= \|\alpha\|_F(\|\beta\|_F + \|\tau^+\|_F) \leq \frac{c_\alpha}{|z|}\left(\frac{c_\beta}{|z|^{-1+\varepsilon}} + \frac{c_{\tau^+}}{|z|^{-1+\varepsilon}}\right) \\
&\leq \frac{c_\alpha(c_\beta + c_{\tau^+})}{|z|^{2-\varepsilon}} \leq \frac{c_{23}}{|z|^2|\log|z||^2},
\end{aligned}
$$

and 1.5.18 for

$$
\|\beta+\tau\|_F^2 \leq \frac{c_{24}}{|z|^2|\log|z||^2},
$$

with suitable constants c_{23}, c_{24}.
Before we used (to receive the second line) that α and $\overline{\alpha}$ commute as diagonal matrices and that α resp. $\overline{\alpha}$ commute with $(\tau^\dagger)^0$ resp. τ^0: Since $\alpha, \overline{\alpha}, \tau^0, (\tau^\dagger)^0$ are block diagonal it is enough to consider the multiplication of the blocks. But α acts on each block as a_iE and E commutes with every matrix. Therefore α commutes with $\tau^0, (\tau^\dagger)^0$ as well as $\overline{\alpha}$ commutes with $\tau^0, (\tau^\dagger)^0$.
For $c_{D_{\overline{\partial}_E+\theta^\dagger}} = 2c_D$ the second inequality holds, too. □

1.7. Consequences

As a direct consequence of the main estimate we get the following lemma as well as its inverse below. It tells us that the flat sections of a tame harmonic Higgs bundle grow at most polynomially. The inverse tells that polynomially bounded flat sections lead to tameness. Thus we have two different descriptions of tameness. At the end of the section we add a technical result that allows us to extend solutions of the Poisson equation weakly over a puncture.

Lemma 1.7.1. (i) Let v be a flat section, i.e. $\mathbb{D}v = 0$ then $\|v\|_h^2 \leq c_{26} r^{-c_{25}}$.[52]

(ii) Let w be a flat section of the dual of the determinant bundle, i.e. $(\det E)^*$ (cf. 5.1.6), then $\|w\|_h^2 \geq c_{30} r^{c_{27}}$. Hence these sections decrease at most polynomially along a ray into 0.

Proof. Ad (i): $D_{\overline{\partial}_E+\theta^\dagger} + 2\theta = \partial_E + \overline{\partial}_E + \theta^\dagger + \theta = \mathbb{D}$ is a flat connection by harmonicity of the bundle and $\|\theta_z\|_F^2 \leq \frac{c_5}{|z|^2}$ by conclusion 1.5.7.
Hence

$$\begin{aligned} dh(v,v) &= h(D_{\overline{\partial}_E+\theta^\dagger}v, v) + h(v, D_{\overline{\partial}_E+\theta^\dagger}v) \\ &= h((\mathbb{D}-2\theta)v, v) + h(v, (\mathbb{D}-2\theta)v) \\ &= -2h(\theta v, v) - 2h(v, \theta v). \end{aligned}$$

Now reduce the equality to the matrix part as in the section on the endomorphism bundle - A.4. We get

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right) h(v,v) = -2h(\theta_z v, v) - 2h(v, \theta_z v).$$

In order to prove the estimate recall that we have

$$\begin{aligned} r\frac{\partial}{\partial r} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ &= \frac{z+\overline{z}}{2}\cdot\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right) + \frac{z-\overline{z}}{2}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right) \\ &= z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}} \end{aligned}$$

and therefore

$$\begin{aligned} r\frac{\partial}{\partial r}h(v,v) &= \left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right) h(v,v) \\ &= -2zh(\theta_z v, v) - 2\overline{z}h(v, \theta_z v) \leq 4|z|\|\theta_z\|_F\|v\|_h^2 \text{[53]} \\ \Rightarrow -\frac{c_{25}\|v\|_h^2}{r} \leq \frac{\partial}{\partial r}\|v\|_h^2 &\leq 4\|v\|_h^2\frac{\sqrt{c_5}}{r} = \frac{c_{25}\|v\|_h^2}{r} \\ \Rightarrow \|v\|_h^2 &\leq c_{26} r^{-c_{25}} \end{aligned}$$

For the last line use Gronwall's lemma (A.1.35): Substitute $t(r) = 1 - r$, $\frac{\partial t}{\partial r} = -1$. Rescale if necessary to work on the unit disc again. Equally we could choose

[52] Simpson's missing a sign in the first remark to the main estimate in [Sim90]. Compare Deligne [Del70], p. 55, theorem 1.19.

[53] For $v = (v_1, \ldots, v_n)$ in our orthonormal basis, define $V := \operatorname{diag}(v_1, \ldots, v_n)$. Then $\|v\|_h^2 = \|V\|_F^2$ and by submultiplicity of the Frobenius norm and Cauchy-Schwarz's inequality $|h_{\mathrm{End}}(\theta_z V, V)| \leq \|\theta_z V\|_F \|V\|_F \leq= \|\theta_z\|_F \|V\|_F^2 = \|\theta_z\|_F \|v\|_h^2$.

$t(r) = R_7 - r$ on $B^*_{R_7}$. Then

$$\begin{aligned}
&\frac{\partial}{\partial r}\|v\|_h^2 \geq -\frac{c_{25}\|v\|_h^2}{r} \\
&\Rightarrow -\frac{\partial}{\partial t}\|v(1-t)\|_h^2 \geq -\frac{c_{25}\|v(1-t)\|_h^2}{1-t} \\
&\Rightarrow \frac{\partial}{\partial t}\|v(1-t)\|_h^2 \leq \frac{c_{25}\|v(1-t)\|_h^2}{1-t} \\
&\Rightarrow \|v(1-t)\|_h^2 \leq \|v(1-0)\|_h^2 \exp\left(\int_0^t \frac{c_{25}}{1-s}\,\mathrm{d}\,s\right) \\
&\qquad\qquad = \|v(1)\|_h^2 \exp\left(-c_{25}\ln(1-t)\right) \\
&\Rightarrow \|v(r)\|_h^2 \leq \|v(1)\|_h^2 r^{-c_{25}}.
\end{aligned}$$

As we have no punctures apart from 0 the smooth section v is bounded, i.e. $\|v(1)\|_h^2 \leq c_{26}$.
Ad (ii): Every section into $\bigwedge_{j=1}^n E$ looks like $f(x) = \alpha(x)e_1 \wedge \ldots \wedge e_n$. h induces a metric on $\bigwedge_{j=1}^n E$ by

$$h_{\bigwedge}(\alpha(x)e_1 \wedge \ldots \wedge e_n, \beta(x)e_1 \wedge \ldots \wedge e_n) = \alpha(x)\overline{\beta(x)}\det(H)^{54},$$

H w.r.t. the basis (e_i). For example for a $\mathbb{D}-$flat frame (e_i) we get from (i): $|h_{ij}| = |h(e_j, e_i)| \leq \sqrt{h(e_i,e_i)h(e_j,e_j)} \leq c_{26}r^{-c_{25}}$ by Cauchy-Schwarz. Then $|\det(H)| \leq c_{29}r^{-c_{27}}$ for suitable constants c_{29}, c_{30}, since it is a degree n polynomial in the entries.
In a $\mathbb{D}-$flat frame (e_i) $\alpha(x)e_1 \wedge \ldots \wedge e_n$ is flat if $d\alpha(x) = 0$ - locally constant. d is metric hence $\alpha(x)(e_1 \wedge \ldots \wedge e_n)^*$ is flat if $d\alpha(x) = 0$. For the metric on the dual bundle we got a formula in the context of A.4. If $e_1 \wedge \ldots \wedge e_n$ is our basis of $\bigwedge_{j=1}^n E$ then $h_{\bigwedge}(\cdot, e_1 \wedge \ldots \wedge e_n)\det(H)^{-1}$ is the dual basis. In general $\alpha(e_1 \wedge \ldots \wedge e_n)^* = h_{\bigwedge}(\cdot, \overline{\alpha}e_1 \wedge \ldots \wedge e_n)\det(H)^{-1}$ and now set

$$\begin{aligned}
h_{\bigwedge,*}(f^*\det(H)^{-1}, g^*\det(H)^{-1}) &= h_{\bigwedge,*}(f^*, g^*)\det(H)^{-2} \\
&= \overline{h_{\bigwedge}(f,g)}\det(H)^{-2} \\
&= \overline{\alpha(x)}\beta(x)\det(H)^{-1}
\end{aligned}$$

for $f = \alpha(x)e_1 \wedge \ldots \wedge e_n, g = \beta(x)e_1 \wedge \ldots \wedge e_n$ and $f^* = \overline{\alpha}(x)e_1 \wedge \ldots \wedge e_n, g^* = \overline{\beta}(x)e_1 \wedge \ldots \wedge e_n$. Hence a flat section into the dual of the determinant bundle is bounded from below by $\det(H)^{-1}$, i.e. by $c_{29}^{-1}r^{c_{27}} = c_{30}r^{c_{30}}$. The case $c_{29} = 0$ cannot occur since H is positive-definite, hence invertible. □

[54] Note that $\det(H) > 0$ by unitary similarity to a diagonal matrix with positive entries.

Proposition 1.7.2. Suppose V is a $\mathscr{D}_X-$module with harmonic metric and suppose that flat sections grow at most polynomially and that the determinant decreases at most polynomially, i.e. the bounds of the previous lemma 1.7.1 apply. Then the eigenvalues of θ are bounded by $\frac{C}{r}$, i.e. the harmonic metric is tame.

Proof. We will think of the harmonic metric as a harmonic map $H : X \to \mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$ as in the chapter about Harmonic bundles. Let S_ε denote a circle of radius ε around 0 and $S_1 = \partial B_1^*$. In the subsection on harmonic maps we have seen that our metric H has minimal energy on the annulus $A_{\varepsilon,1}$ under all metrics with the same boundary values, since H is harmonic. Hence for any other metric K with the same boundary values, $K|_{\partial A_{\varepsilon,1}} = H|_{\partial A_{\varepsilon,1}}$ we get

$$\int_{A_{\varepsilon,1}} \|dH\|^2_{\mathbb{H}_n} \leq \int_{A_{\varepsilon,1}} \|dK\|^2_{\mathbb{H}_n}.$$

We want to calculate the energy of K in order to estimate the energy of H.
Choose K as follows: Let $\rho_\phi : [0,1] \to X, r \mapsto re^{i\phi}, \phi \in [0, 2\pi[$ and $K_\phi : [\varepsilon, 1] \to \mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$ such that $K_\phi = g\rho_\phi, g : X \to \mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$ with

$$\begin{aligned} g|_{A_{\varepsilon+\sigma,1}} &\equiv H|_{S_1} \Rightarrow K_\phi|_{[\varepsilon+\sigma,1]} = (H|_{S_1\cap\rho_\phi}). \\ g|_{\rho_\phi \cap A_{\varepsilon,\varepsilon+\sigma}} &= \text{ geodesic from } H|_{S_\varepsilon\cap\rho_\phi} \text{ to } H|_{S_1\cap\rho_\phi} \end{aligned}$$

for $\varepsilon + \sigma \leq 1$.
Parametrize S_ε by $\gamma_\varepsilon(\phi) = \varepsilon e^{2\pi\phi i}, \phi \in [0,1]$ and S_1 by $\gamma_1(\phi) = e^{2\pi\phi i}$. Denote $v_\varepsilon(\phi) = \frac{\partial H(\gamma_\varepsilon(\phi))}{\partial\phi}, v_1(\phi) = \frac{\partial H(\gamma_1(\phi))}{\partial\phi}$ the tangent vectors of H on the boundary. Now consider the sector with corners $H(\gamma_\varepsilon(\phi)) := H_{\varepsilon,\phi}, H(\gamma_1(\phi)) := H_{1,\phi}, H_{\varepsilon,\phi} + \varepsilon v_\varepsilon(\phi)\Delta(\phi) + O(\Delta^2(\phi)), H_{\varepsilon,\phi} + v_1(\phi)\Delta(\phi) + O(\Delta^2(\phi))$, where $\Delta(\phi) \geq 0$ is a small perturbation in the angle ϕ (Taylor series). Note that $v_1(\phi)$ is bounded - $\|v_1(\phi)\|^2_{\mathbb{H}_n} \leq c_{31}, \forall\phi \in [0,1]$ - since H is smooth on $A_{\varepsilon,1}$ as is θ, and $\mathrm{d}\,H$ can be written in terms of H and θ, as we saw in the chapter on harmonic bundles.
In order to estimate the energy of K let us start with the $r-$integration on the interval $[\varepsilon + \sigma, 1]$:

$$\begin{aligned} &\int_{\varepsilon+\sigma}^1 \left| \left(\frac{\partial}{r} + \frac{\partial}{r\partial\varphi} \right) K_\phi \right|^2 r \,\mathrm{d}\, r \\ &= \int_{\varepsilon+\sigma}^1 \left| \left(\frac{\partial}{r\partial\varphi} \right) K_\phi \right|^2 r \,\mathrm{d}\, r \\ &= \int_{\varepsilon+\sigma}^1 \left| \frac{v_1 \Delta(\phi)}{r\partial\varphi} + O(\Delta(\phi)) \right|^2 r \,\mathrm{d}\, r \\ &\leq c_{31}(|\log(1) - \log(\varepsilon+\sigma)|)\Delta(\phi) + (|\log(1) - \log(\varepsilon+\sigma)|)(O(\Delta^2(\phi))) \end{aligned}$$

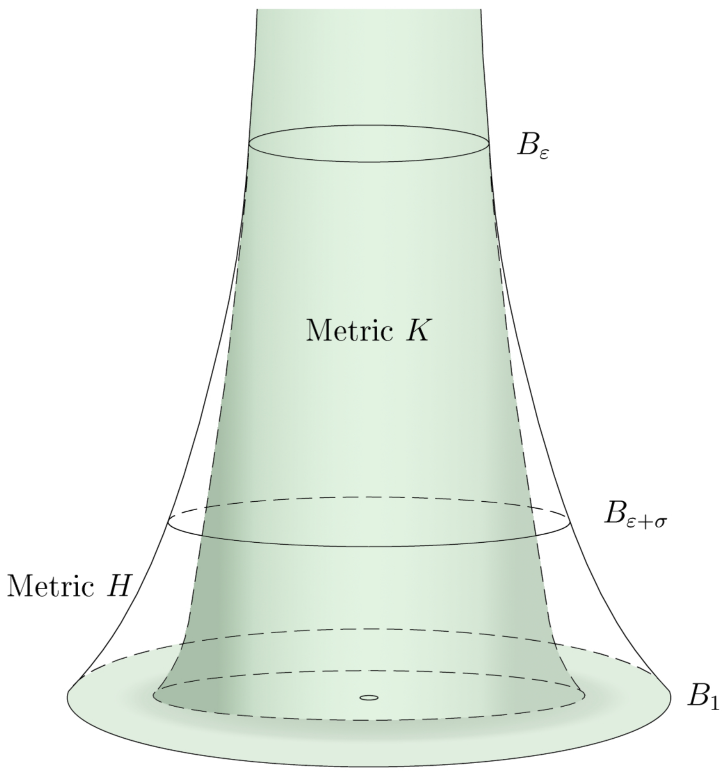

Figure 1.5: An example, how the metric K could look like. The metric K is marked by the green surface, H the outer transparent surface. K is first constant and increases than geodesically to the inner circle B_ε.

$$= c_{31}|\log(\varepsilon+\sigma)|d\phi.$$

Later on summing up over all sectors, i.e. integration over the angle ϕ the second order terms will drop out, so we could as well save some space and omit them from now on and simply write $\mathrm{d}\,\phi$ for $\Delta(\phi)$. The other equalities follow by the $r-$constancy of f on $[\varepsilon+\sigma,1]$ and since K coincides with H on the boundary. On the interval $[\varepsilon,\varepsilon+\sigma]$ the integration is slightly more complicated. Let $L = \int_\varepsilon^{\varepsilon+\sigma}\left\|\frac{\partial}{\partial r}K_\phi(r)\right\|_{\mathbb{H}_n,K_\phi(r)}$ be the length of the geodesic. Since a geodesic has by definition constant speed, the speed on the interval is constant $\frac{L}{\sigma}$[55]. Then the radial component is

$$\begin{aligned}
\int_\varepsilon^{\varepsilon+\sigma}\left\|\frac{\partial}{\partial r}K_\phi\right\|_{\mathbb{H}_n}^2 r\,\mathrm{d}\,r &= \int_\varepsilon^{\varepsilon+\sigma}\left|\frac{L}{\sigma}\right|^2 r\,\mathrm{d}\,r\\
&\le \left(\frac{L}{\sigma}\right)^2\int_\varepsilon^{\varepsilon+\sigma}(\varepsilon+\sigma)\,\mathrm{d}\,r\\
&= \frac{L^2(\varepsilon+\sigma)}{\sigma^2}(\varepsilon+\sigma-\varepsilon)\\
&\le \frac{L^2(\varepsilon+\sigma)}{\sigma}.
\end{aligned}$$

Finally the angular component: Recall that we showed that the space $\mathbb{P}_n$ is negatively curved. The field $J(r) = \left.\frac{\partial g_\varphi(r)}{\partial\varphi}\right|_{\varphi=\phi}$ for a parametrized family of geodesics g_φ is called Jacobi field. In non-positively curved spaces recall that $|J(r))|^2$ is convex.[56] Hence it is maximal on the boundary, i.e. $\left\|\frac{\partial}{\partial r}K_\phi\right\|_{\mathbb{H}_n}^2 \le \max\{\|v_\varepsilon\|_{\mathbb{H}_n}^2, \frac{\|v_1\|_{\mathbb{H}_n}^2}{(\varepsilon+\sigma)^2}\} \le \|v_\varepsilon\|_{\mathbb{H}_n}^2 + \frac{c_{31}}{(\varepsilon+\sigma)^2}$. Therefore we get

$$\begin{aligned}
\int_\varepsilon^{\varepsilon+\sigma}\left\|\frac{\partial}{\partial\phi}K_\phi\right\|_{\mathbb{H}_n}^2 r\,\mathrm{d}\,r &\le \int_\varepsilon^{\varepsilon+\sigma}\left(\|v_\varepsilon\|_{\mathbb{H}_n}^2+\frac{c_{31}}{(\varepsilon+\sigma)^2}\right)\mathrm{d}\,\phi r\,\mathrm{d}\,r\\
&\le \left(\|v_\varepsilon\|_{\mathbb{H}_n}^2+\frac{c_{31}}{(\varepsilon+\sigma)^2}\right)\int_\varepsilon^{\varepsilon+\sigma}\varepsilon+\sigma\,\mathrm{d}\,\phi\,\mathrm{d}\,r\\
&= \left(\|v_\varepsilon\|_{\mathbb{H}_n}^2(\varepsilon+\sigma)+\frac{c_{31}}{(\varepsilon+\sigma)}\right)(\varepsilon+\sigma-\varepsilon)\,\mathrm{d}\,\phi\\
&= \|v_\varepsilon\|_{\mathbb{H}_n}^2\sigma(\varepsilon+\sigma)\,\mathrm{d}\,\phi+\frac{c_{31}\sigma}{(\varepsilon+\sigma)}\,\mathrm{d}\,\phi.
\end{aligned}$$

Adding up the single terms and summing up over the sectors leads us to

$$\int_{A_{\varepsilon,1}}\|dK\|_{\mathbb{H}_n}^2 \le \int_0^1 c_{31}|\log(\varepsilon+\sigma)|\,\mathrm{d}\,\phi+\int_0^1\frac{L^2(\varepsilon+\sigma)}{\sigma}\,\mathrm{d}\,\phi$$

---

55 $\frac{\int_a^b\|\dot\gamma\|dt}{b-a} = \|\dot\gamma\|$ if $\dot\gamma$ independent of t.

56 Greene and Yau [GY93], p. 182. Use Proposition A.2.27 to guarantee completeness.

$$
\begin{aligned}
& + \int_0^1 \|v_\varepsilon\|^2_{\mathbb{H}_n} \sigma(\varepsilon+\sigma) + \frac{c_{31}\sigma}{(\varepsilon+\sigma)} \,\mathrm{d}\,\phi \\
\leq\; & c_{31}|\log(\varepsilon+\sigma)| + \frac{L^2(\varepsilon+\sigma)}{\sigma} + \frac{c_{31}\sigma}{(\varepsilon+\sigma)} \\
& + \sigma(\varepsilon+\sigma)\int_0^1 \|v_\varepsilon\|^2_{\mathbb{H}_n} \,\mathrm{d}\,\phi \\
=\; & c_{31}|\log(\varepsilon+\sigma)| + \frac{L^2(\varepsilon+\sigma)}{\sigma} + \frac{c_{31}\sigma}{(\varepsilon+\sigma)} \\
& + \frac{\sigma(\varepsilon+\sigma)}{\varepsilon}\int_0^1 \left\|\frac{\partial H(\gamma_\varepsilon(\phi))}{\partial\phi}\right\|^2_{\mathbb{H}_n} \mathrm{d}\,\phi^{57} \\
=\; & c_{31}|\log(\varepsilon+\sigma)| + \frac{L^2(\varepsilon+\sigma)}{\sigma} + \frac{c_{31}\sigma}{(\varepsilon+\sigma)} \\
& + \frac{\sigma(\varepsilon+\sigma)}{\varepsilon}\int_{S_\varepsilon} \|dH\|^2_{\mathbb{H}_n}
\end{aligned}
$$

where we used that only $\|v_\varepsilon\|^2_{\mathbb{H}_n}$ actually depends on the angle and that it is the derivation of the harmonic metric H along S_ε.

We want to estimate the length L. In A.2.27 we saw that $\mathbb{P}_n$ is a complete metric space and so a geodesic minimizes the length.[58] For any path $\gamma : [\varepsilon, \varepsilon+\sigma] \to \mathbb{P}_n$:

$$L(\gamma) \leq \int_\varepsilon^{\varepsilon+\sigma} \|\gamma'(t)\|_{\mathbb{H}_n,\gamma(t)}.$$

Define a path $\gamma(t) = \exp\left(\log(H(\varepsilon))t_\varepsilon + \log(H(1))(1-t_\varepsilon)\right)$ with $t_\varepsilon = \frac{1}{\sigma}(t-\varepsilon)$. Then

$$
\begin{aligned}
\gamma(\varepsilon) &= H(1), \quad \gamma(\varepsilon+\sigma) = H(\varepsilon), \\
(\gamma(t))^{-1} &= \exp\left(-\log(H(\varepsilon))t_\varepsilon - \log(H(1))(1-t_\varepsilon)\right) \\
\gamma'(t) &= \frac{\log(H(\varepsilon)) - \log(H(1))}{\sigma} \exp\left(\log(H(\varepsilon))t_\varepsilon + \log(H(1))(1-t_\varepsilon)\right) \\
(\gamma(t))^{-1}\gamma'(t) &= \frac{\log(H(\varepsilon)) - \log(H(1))}{\sigma}
\end{aligned}
$$

Then $\|(\gamma(t))^{-1}\gamma'(t)\|_{\mathbb{H}_n,E} = \frac{1}{\sigma}\left\|\log(H(\varepsilon)^{-1})\right\|_{\mathbb{H}_n,E} + \frac{1}{\sigma}\left\|\log(H(1)^{-1})\right\|_{\mathbb{H}_n,E}$.
We now want to use the polynomial bound (c.f. 1.7.1) of $H(\varepsilon)$ and $H(\varepsilon)^{-1}$.
As in the case of the exponential function we get $\log(U^*DU) = U^*\log(D)U$ from the power series expansion by

$$(U^*DU)^n = U^*DUU^*\dots UU^*DU = U^*DU.$$

[57] We have an additional factor $(\varepsilon^{-1})^2\varepsilon = \varepsilon^{-1}$ for polar coordinates.
[58] Geodesically complete follows from complete and path-connected.

Hence

$$\log(U_\varepsilon^* H(\varepsilon)U_\varepsilon) = U_\varepsilon^* \log D(\varepsilon)U_\varepsilon,$$
$$\Rightarrow \log D(\varepsilon) = \operatorname{diag}(\log(d_1(\varepsilon)), \dots, \log(d_n(\varepsilon))).$$

Moreover

$$\begin{aligned}\operatorname{tr}(\log(U_\varepsilon^* D(\varepsilon)U_\varepsilon)\log(U_\varepsilon^* D(\varepsilon)U_\varepsilon)^*) &= \operatorname{tr}(U_\varepsilon^* \log(D(\varepsilon))U_\varepsilon U_\varepsilon^* \log(D(\varepsilon))U_\varepsilon) \\ &= \operatorname{tr}(\log(D(\varepsilon))^2).\end{aligned}$$

Analogous for $H(\varepsilon)^{-1} \Rightarrow \|\log(H(\varepsilon)^{-1})\|^2_{\mathbb{H}_n,E} = \operatorname{tr}(\log(D(\varepsilon))^2)$ since $\log(d_i^{-1}) = -\log(d_i)$.
Therefore we have $\|D(\varepsilon)\|_{\mathbb{H}_n,E} = \|U^* D(\varepsilon)U\|_{\mathbb{H}_n,E} = \|H(\varepsilon)\|_{\mathbb{H}_n,E} \le c_{26}\varepsilon^{-c_{25}}$ and $\|D(\varepsilon)^{-1}\|_{\mathbb{H}_n,E} = \|U^* D(\varepsilon)^{-1}U\|_{\mathbb{H}_n,E} = \|H(\varepsilon)^{-1}\|_{\mathbb{H}_n,E} \ge c_{26}^{-1}\varepsilon^{c_{25}}$.

Remark 1.7.3. We only know up to now that sections into the dual of the determinant bundle satisfy such a bound. However in the next chapter we will construct a functor Ξ which describes the growth of sections close to the puncture. We will see that this functor is compatible with taking duals and determinants, i.e. the growth of a section into the dual of the determinant bundles determines the growth of its corresponding section into E uniquely. For now it is only important that such a bound exists.[59]

Hence the entries $d_i(\varepsilon), d_i^{-1}(\varepsilon)$ are bounded by the same bounds and therefore by monotony of the real logarithm

$$\begin{aligned}c_{25}\log(\varepsilon) - \log(c_{26}) &= \log(c_{26}^{-1}\varepsilon c^{25}) \le \log(d_i^{-1}(\varepsilon)), \log(d_i(\varepsilon)) \\ &\le \log(c_{26}\varepsilon - c^{25}) = -c_{25}\log(\varepsilon) + \log(c_{26}).\end{aligned}$$

For $d_i \le 1$ the logarithm will be negative and we may multiply by -1 and then take the absolute value: $-c_{25}\log(\varepsilon) + \log(c_{26}) \ge -\log(d_i^{-1}(\varepsilon)) \Rightarrow |c_{25}\log(\varepsilon) - \log(c_{26})| \ge |\log(d_i^{-1}(\varepsilon))|$. For $d_i \ge 1$:

$$|\log(d_i(\varepsilon))| \le |c_{25}\log(\varepsilon) - \log(c_{26})|.$$

For ε small enough we may collect $\log(c_{26})$ into $c_{42}\log(\varepsilon)$ for some $c_{42} > nc_{25}$. Then square and sum up to receive the trace and take the root again: $\|\log(H(\varepsilon))\| \le c_{42}|\log(\varepsilon)|$. $\log(H(1))$ is constant in ε, so increase c_{42} to c_{43} to bound the integrand by $\frac{c_{43}}{\sigma}|\log(\varepsilon)|$. Finally integration from ε to $\varepsilon + \sigma$ leads to a bound $c_{43}|\log(\varepsilon)|$.[60]

$$\Rightarrow L \le c_{43}|\log(\varepsilon)|.$$

[59] We stick with c_{26}, although Ξ might change the constant.

[60] After moving the supremum out of the integral, we integrate over 1 from ε to $\varepsilon + \sigma \Rightarrow$ the integral has the value $\varepsilon + \sigma - \varepsilon = \sigma$.

Therefore we get using $|\log(\varepsilon)| \leq \log(\varepsilon)^2$ for ε small enough, $\varepsilon \leq \sigma + \varepsilon \leq 1$ and $c_{35} = 2\max\{c_{31}, c_{43}\}$

$$c_{35}\frac{(\varepsilon+\sigma)}{2\sigma}|\log(\varepsilon)|^2 + \frac{\sigma(\varepsilon+\sigma)}{\varepsilon}\int_{S_\varepsilon}\|dH\|^2_{\mathbb{H}_n}$$

as an upper bound for the energy of K. Further choose $\sigma = \varepsilon$ and the harmonicity of H leads to

$$\int_{A_{\varepsilon,1}}\|dH\|^2_{\mathbb{H}_n} \leq c_{35}|\log(\varepsilon)|^2 + 2\varepsilon\int_{S_\varepsilon}\|dH\|^2_{\mathbb{H}_n}. \tag{1.7.3.1}$$

Lemma 1.7.4. $\int_{A_{\varepsilon,1}}\|dH\|^2_{\mathbb{H}_n}$ is $\varepsilon-$integrable (from 0 to 1).

Proof. The right-hand side of the last inequality is $\varepsilon-$integrable: For $|\log(\varepsilon)|^2$ this is clear. The second term is slightly more involved. Our map H is harmonic, hence $H : \varepsilon S^1 \to \mathbb{P}_n$ harmonic[61] tells us that $H(\gamma_\varepsilon)$ is a closed geodesic:[62]

Theorem 1.7.5. Let N be a complete Riemannian manifold of non-negative sectional curvature, $p, q \in N$. Then in any homotopy class of curves from p to q, there is precisely one geodesic arc from p to q, and this arc minimizes length in its class.

Proof. Jost [Jos05], p. 217, theorem 4.8.1. □

In the section on maps into $\mathbb{P}_n$ we have seen, that our situation satisfies the requirements made by the theorem. Moreover H is continuous and for a path $\tilde{\gamma} : I \to X$ homotopic to γ_ε the induced path $H(\tilde{\gamma})$ is homotopic to $H(\gamma_\varepsilon)$. Choose an arbitrary point $x_\varepsilon \in S_\varepsilon$ as base point, for example $\gamma_\varepsilon(0) = x_\varepsilon$. Let $\tilde{\gamma}(t)$ be the path which for $0 \leq t \leq \frac{1}{4}$ is the geodesic between x_ε and $y \in S_{2/3} \cap \rho_{x_\varepsilon}$ unique. Here ρ_ε is the ray out of the puncture through x_ε and $S_{2/3}$ the circle with radius $2/3$. For $\frac{1}{4} \leq t \leq \frac{3}{4}$ $\tilde{\gamma}(t)$ is the circle $S_{2/3}$, and for $\frac{3}{4} \leq t \leq 1$ we require $\tilde{\gamma}_t = \tilde{\gamma}(1-t)$, i.e. we go back along the geodesic. In order to have a smooth path, we may smooth out the "edges" by minor modifications of $\tilde{\gamma}$ in $A_{1/2,3/4}$.[63]
The length of the circle part is bounded by $2\pi\sup_{A_{1/2,1}}\|dH\| = c_{\gamma,1}$ finite by the smoothness of H. The geodesic part we have estimated before by $c_{43}|\log(\varepsilon)|^2$. Adding up we get

$$L(H(\gamma_\varepsilon)) \leq L(H(\tilde{\gamma})) \leq c_{\tilde{\gamma}}|\log(\varepsilon)|^2,$$

for a suitable constant $c_{\tilde{\gamma}}$. Another lemma:

[61] S_ε is a degenerate closed annulus and H is harmonic on all annuli.
[62] Reparametrize if necessary to have constant speed.
[63] A possible non-differentiability at x_ε is of no importance. We may choose piecewise smooth functions as well.

Lemma 1.7.6. For each smooth curve $\gamma : [a, b] \to M$ (for example $M = \mathbb{P}_n$):

$$L(\gamma)^2 \le 2(b-a)E(\gamma)$$

and equality if and only if γ has constant speed. Here $E(\gamma)$ is the energy.

Proof. Jost, [Jos05] p. 27 lemma 1.4.2. □

We assumed constant speed so we may estimate the energy $\int_{S_\varepsilon} \|\mathrm{d}\, H\|^2_{\mathbb{H}_n} \le L(H(\tilde{\gamma})) \le c_{\tilde{\gamma}} |\log(\varepsilon)|^4$. But then $\int_{S_\varepsilon} \|\mathrm{d}\, H\|^2_{\mathbb{H}_n}$ is $\varepsilon-$integrable as claimed. □

Remark 1.7.7. (i) Note that by using the estimate for the length of a geodesic again, we used the polynomial growth of the metric again.

(ii) $\lim_{\varepsilon \to 0} \varepsilon^\nu \int_{A_{\varepsilon,1}} \|dH\|^2_{\mathbb{H}_n} \le \varepsilon^\nu |\log(\varepsilon)|^2 = 0$ for any $\nu > 0$.

Let $\nu > 0$ small. Apply integration by parts (PI) on the $g(\varepsilon) = \frac{\partial}{\partial \varepsilon}\varepsilon^\nu, f(\varepsilon) = \int_{A_{\varepsilon,1}} \|dH\|^2_{\mathbb{H}_n}$:

$$\begin{aligned}
\int_0^1 \varepsilon^\nu \frac{\partial}{\partial \varepsilon} f(\varepsilon)\,\mathrm{d}\,\varepsilon &= \int_0^1 \varepsilon^\nu \frac{\partial}{\partial \varepsilon} \int_{A_{\varepsilon,1}} \left\| \left(\frac{\partial}{\partial r} + \frac{\partial}{r \partial \phi} \right) H \right\|^2_{\mathbb{H}_n} r \,\mathrm{d}\,\phi \,\mathrm{d}\, r \\
&= - \int_{A_{0,1}} \varepsilon^\nu \|dH\|^2_{\mathbb{H}_n} \\
&\overset{PI}{=} [\underbrace{\varepsilon^\nu f(\varepsilon)}_{g \equiv 0 \text{ for } \varepsilon = 0^{64}, f \equiv 0 \text{ for } \varepsilon = 1}]_0^1 - \int_0^1 \nu \varepsilon^{\nu-1} f(\varepsilon)\,\mathrm{d}\,\varepsilon \\
&= - \int_0^1 \nu \varepsilon^{\nu-1} \int_{A_{\varepsilon,1}} \|dH\|_{\mathbb{H}_n} \,\mathrm{d}\,\varepsilon,
\end{aligned}$$

or $\int_{A_{0,1}} r^\nu \|dH\|_{\mathbb{H}_n} = \int_0^1 \nu \varepsilon^{\nu-1} \int_{A_{\varepsilon,1}} \|dH\|_{\mathbb{H}_n} \,\mathrm{d}\,\varepsilon$. Proceed with 1.7.3.1

$$\begin{aligned}
\int_{A_{\varepsilon,1}} r^\nu \|dH\|^2_{\mathbb{H}_n} &= \int_0^1 \nu \varepsilon^{\nu-1} \int_{A_{\varepsilon,1}} \|dH\|_{\mathbb{H}_n} \,\mathrm{d}\,\varepsilon \\
&\le \int_0^1 \nu \varepsilon^{\nu-1} c_{35} |\log(\varepsilon)| \,\mathrm{d}\,\varepsilon + 2\nu \int_0^1 \varepsilon^\nu \int_{S_\varepsilon} \|dH\|^2_{\mathbb{H}_n} \,\mathrm{d}\,\varepsilon \\
&\le \int_0^1 \nu \varepsilon^{\nu-1} c_{35} |\log(\varepsilon)| \,\mathrm{d}\,\varepsilon + 2\nu \int_{D_{0,1}} r^\nu \|dH\|^2_{\mathbb{H}_n}
\end{aligned}$$

[67] By the previous remark, (ii).

Now subtract the finite second term:

$$(1-2\nu)\int_{A_{\varepsilon,1}} r^{\nu}\|dH\|^2_{\mathbb{H}_n} \quad\leq\quad \int_0^1 \nu\varepsilon^{\nu-1}c_{35}|\log(\varepsilon)|\,\mathrm{d}\,\varepsilon$$

In the previous remark we saw that for ε small enough we have $|\log(\varepsilon)| \leq c_\mu \varepsilon^{-\mu}$ for any $\mu > 0$. Hence the right-hand side is still finite for $\mu < \nu$. Choose $\mu = \frac{\nu}{2}$ and $\nu < \frac{1}{2}$ in order to make the left-hand side positive.

$$\begin{aligned}\int_{A_{\varepsilon,1}} r^{\nu}\|dH\|^2_{\mathbb{H}_n} &\leq (1-2\nu)^{-1}\int_0^1 \nu\varepsilon^{\nu-1}c_{35}|\log(\varepsilon)|\,\mathrm{d}\,\varepsilon \\ &\leq (1-2\nu)^{-1}\nu c_{35}\int_0^1 \varepsilon^{\nu/2-1}\,\mathrm{d}\,\varepsilon \\ &\leq c_\nu.\end{aligned}$$

We are still on our way to prove that the eigenvalues of θ have no singularities of order greater than 1. In fact we are almost done: $\|dH\|^2_{\mathbb{H}_n} = 8\|\theta\|^2_F$. Assume that the eigenvalues of θ were not bounded by $\frac{C}{r}$ for some constant C, then there is a λ_i eigenvalue with $\|\lambda_i\| \geq r^{-1-\mu}$. After transforming unitarily to the Schur normal form we see, if it wasn't obvious, that the Frobenius norm of θ is bounded from below by λ_i. On the other hand our bound of $\|\,\mathrm{d}\,H\|$ implies that $z\lambda_i$ is L^2-integrable ($\nu = 1$). [68]

Lemma 1.7.8. Every $L^p(U)-$integrable function f, U some compact neighbourhood of the origin, $p \geq 1$, holomorphic on $U \setminus \{0\}$ has no essential singularities.

Proof. Every $f \in L^p(U)-$integrable function is integrable, i.e. in $L^1(U)$, for example by $|f| \leq \max\{1, |f|^p\} \leq 1 + |f|^p$ and 1 in $L^p(U)$. By the holomorphy of f we find a Laurent expansion $f = \sum_{k=-\infty}^{\infty} f_k z^k$. W.l.o.g. assume $U = B_1$. By Cauchy's integral theorem $f_k = \frac{1}{2\pi}\oint_{S_r}\frac{f(z)}{z^{k+1}}\,\mathrm{d}\,z, 0 < r \leq 1$

$$\begin{aligned}\Rightarrow |f_k| &= \left|\frac{1}{2\pi}\int_0^1 2\pi i r f(re^{2\pi i t})r^{-k-1}e^{-2\pi i t(k+1)}\,\mathrm{d}\,t\right| \\ &\leq \int_0^1 \left|f(re^{2\pi i t})r^{-k}e^{-2\pi i t(k+1)}\right|\mathrm{d}\,t \\ &= r^{-k}\int_0^1 \left|f(re^{2\pi i t})\right|\mathrm{d}\,t.\end{aligned}$$

Hence

$$\infty > \int_U |f(z)|\,\mathrm{d}\,z \quad=\quad \int_0^1\int_0^1 \left|f(re^{2\pi i t})\right|\mathrm{d}\,tr\,\mathrm{d}\,r$$

[68] The choice $\nu < \frac{1}{2}$ was for convenience of calculation. By the trivial estimate for $\nu' > \nu$: $r^{\nu'-\nu} \leq 1 \Rightarrow \int\int_{A_{\varepsilon,1}} r^{\nu'}\|dH\|^2_{\mathbb{H}_n} \leq \int_{A_{\varepsilon,1}} r^{\nu}\|dH\|^2_{\mathbb{H}_n}$.

$$\geq \int_0^1 |f_k| r^{k+1} \,\mathrm{d}\, r = |f_k| \int_0^1 r^{k+1} \,\mathrm{d}\, r.$$

But the last integral is only finite if $k > -2$, i.e. f is meromorphic with a pole of order at most 1 at 0. □

This shows that $z\lambda_i$ is meromorphic and hence that λ_i is meromorphic.
Then $\|dH\|^2 \geq r^{-2-2\mu}$ and moreover $r^\nu \|dH\|^2 \geq r^{-2-2\mu-\nu} > r^{-2}$ for the choice $\nu < 2\mu$, for example $\mu = \nu$. But r^{-2} is not integrable over a two dimensional space - a contradiction to $\int_{A_{\varepsilon,1}} r^\nu \|dH\|^2_{\mathbb{H}_n} \leq c_\nu$. Hence all eigenvalues of θ have no singularity of order greater than -1. □

While the previous lemma and proposition provide us with a final result, namely that we have two equivalent ways of describing tameness (one is by the order of growth of the eigenvalues, the other one is by order of growth of flat sections), the following lemma is a technical lemma. We will use it several times to expand an estimate weakly over our puncture.

Lemma 1.7.9. Suppose $f : X \to \mathbb{R}$ is a function smooth away from the origin and $\left|\frac{f(z)}{\log|z|}\right| \to 0$ for $z \to 0$. Suppose further that $-\Delta f \leq -b$ away from the origin for a non-negative function $b : X \to \mathbb{R}_+$. Then $\int_X b < \infty$ and the estimate holds weakly over the origin.
Furthermore if b is not non-negative but any L^1−function with $-\Delta f \leq -b$ then $\int_X b < \infty$ and the estimate holds weakly over the origin.

Proof. We may restrict to some small neighbourhood $U \subset \overline{X}$ of the origin, since $-\Delta f \leq -b$ holds outside U and b integrable on the complement of U. Let $\Psi(z) = -\frac{1}{2\pi}\log|z|$ be Green's function on U.[69] If U small enough, i.e. $U \subset B_1(0)$: $\Psi(z) \geq 0$. Note that Ψ is harmonic on X (without 0).[70] Further let $\Psi_N = \min\{\Psi, N\}$ for $N \in \mathbb{N}$. Hence $\Psi_N = N$ close to 0. Therefore

$$\int_X |\Delta\Psi_N(z)| \,\mathrm{d}\, z < c_{37}, \quad \text{and} \quad \int_X |\nabla\Psi_N(z)| \,\mathrm{d}\, z < c_{37},$$

for a suitable constant c_{37}. Consider any compactly supported positive smooth function η on U, $\operatorname{supp}\left\{\frac{\partial^k}{\partial z^k}\eta\right\} \cap \partial U = \emptyset, \forall k \in \mathbb{N}$. Let $b_N := \min\{N, b\} \Rightarrow -\Delta f \leq -b_N, b_N$ integrable and receive by integration by parts

$$\int_U b_N(z)\eta(z) \,\mathrm{d}\, z \quad \leq \quad \int_U (\Delta f(z))\eta(z) \,\mathrm{d}\, z$$

[69] Müller [PDE09], Sheet 3, Ex.8 or Lieb and Loss [LiLo00].
[70] Müller [PDE09], Sheet 3, Ex.9. or [LiLo00].

$$
\begin{aligned}
&= -\int_U (\nabla f(z))\nabla\eta(z)\,\mathrm{d}\,z + \int_{\partial U} (\nabla f(z)) \underbrace{\eta(z)}_{=0 \text{ on } \partial U} \,\mathrm{d}\,z \\
&= \int_U f(z) \underbrace{\Delta\eta(z)}_{\leq c_{38}} \mathrm{d}\,z - \int_{\partial U} f(z) \underbrace{\nabla\eta(z)}_{=0 \text{ on } \partial U} \,\mathrm{d}\,z \\
&\leq c_{38} \int_U f(z)\,\mathrm{d}\,z \leq c_{38} \int_U |\log|z||\,\mathrm{d}\,z \leq c_{39}.
\end{aligned}
$$

So $\int_U b_N(z)\eta(z)\,\mathrm{d}\,z < \infty$ and we may interchange integration and the limit $N \to \infty$. So $\int_U b(z)\eta(z)\,\mathrm{d}\,z$ for all η and hence b integrable.
From now on use only that b is L^1-integrable and $-\Delta f \leq -b$.
With $\Delta\eta \leq c_{38}$ we conclude

$$
\begin{aligned}
\left|\int_X f\left(1 - \frac{\Psi_N}{N}\right)\right| \Delta\eta &\leq c_{38} \int_X |\log|z|| \left(1 + \frac{|\log|z|| + N}{N}\right) \mathrm{d}\,z \\
&\leq 2c_{38} \underbrace{\int_X |\log|z||\,\mathrm{d}\,z}_{<\infty} + c_{38} \underbrace{\int_X |\log|z||^2\,\mathrm{d}\,z}_{<\infty}{}^{71} \\
&\leq c_{39}.
\end{aligned}
$$

Then we may interchange integration and the limit in

$$
\begin{aligned}
\lim_{N\to\infty} \int_U f\left(1 - \frac{\Psi_N}{N}\right)(-\Delta)\eta &= \int_U f\left(1 - \lim_{N\to\infty}\frac{\Psi_N}{N}\right)(-\Delta)\eta \\
&= \int_U f\left(1 - \lim_{N\to\infty}\frac{\log|z|}{N}\right)(-\Delta)\eta \\
&= \int_U f(-\Delta)\eta \overset{\text{Partial Integration}}{=} \int_U (-\Delta) f\eta.
\end{aligned}
$$

On the other hand integration by parts leads to[72]

$$
\begin{aligned}
&\lim_{N\to\infty} \int_U f(-\Delta)\left(1 - \frac{1}{N}\Psi_N\right)\eta \\
&= \lim_{N\to\infty} \int_U (\nabla f)\,\nabla\left(1 - \frac{1}{N}\Psi_N\right)\eta - \lim_{N\to\infty}\int_{\partial U} f \underbrace{\nabla\left(1 - \frac{1}{N\Psi_N}\right)\eta}_{=0 \text{ on } \partial U^{73}} \\
&= \lim_{N\to\infty} \int_U (-\Delta f)\left(1 - \frac{1}{N}\Psi_N\right)\eta + \lim_{N\to\infty}\int_{\partial U} (\nabla f) \underbrace{\left(1 - \frac{1}{N}\Psi_N\right)\eta}_{=0 \text{ on } \partial U}
\end{aligned}
$$

[71] For any branch of the logarithm.
[72] The Laplacian acts on η as well.

$$
\begin{aligned}
&= \lim_{N\to\infty} \int_U (-\Delta f)\left(1 - \frac{1}{N}\Psi_N\right)\eta \\
&\le \lim_{N\to\infty} \int_U b\left(1 - \frac{1}{N}\Psi_N\right)\eta \\
&\le \lim_{N\to\infty} \int_{U\setminus B_{e^{-2\pi N}}} b\eta\left(1 + \underbrace{\frac{\log|z|}{N}}_{\le 1}\right) + \lim_{N\to\infty}\int_{B_{e^{-2\pi N}}} \underbrace{b\eta}_{L^1} \\
&\le \underbrace{\lim_{N\to\infty} \int_{U\setminus B_{e^{-2\pi N}}} 2b\eta}_{\to\int_U 2b\eta} + \underbrace{\lim_{N\to\infty}\int_{B_{e^{-2\pi N}}} b\,\eta}_{\to 0} \\
&< \infty,
\end{aligned}
$$

since b is integrable (B_R ball with radius R). Moving the limit under the integration leads to

$$
\lim_{N\to\infty}\int_U f(-\Delta)\left(1 - \frac{1}{N}\Psi_N\right)\eta \le \int_U b\eta.
$$

Moreover

$$
(-\Delta)\left(1 - \frac{1}{N}\Psi_N\right)\eta = \left(1 - \frac{1}{N}\Psi_N\right)(-\Delta)\eta + \left(\frac{\Delta\Psi_N}{N}\right)\eta + 2\left(\frac{1}{N}\nabla\Psi_N\right)\nabla\eta.
$$

Therefore

$$
\begin{aligned}
&\int_U b\eta - \int_U (-\Delta) f\eta \\
&= \int_U b\eta - \int_U f(-\Delta)\eta \\
&\ge \lim_{N\to\infty}\int_U f(-\Delta)\left(1 - \frac{1}{N}\Psi_N\right)\eta - \int_U f\left(1 - \frac{\Psi_N}{N}\right)(-\Delta)\eta \\
&= \lim_{N\to\infty}\int_U f\left(\frac{\Delta\Psi_N}{N}\right)\eta + 2\left(\frac{f}{N}\nabla\Psi_N\right)\nabla\eta \\
&\to 0.
\end{aligned}
$$

We see that the last term does indeed vanish for f bounded.
In our case, i.e. $\left|\frac{f(z)}{\log|z|}\right| \to 0$, the last term vanishes by

$$
\lim_{N\to\infty}\left|\int_U f\left(\frac{\Delta\Psi_N}{N}\right)\right|\eta \quad = \quad \lim_{N\to\infty}\left|\int_{\mathrm{supp}\{\Delta\Psi_N\}} f\left(\frac{\Delta\Psi_N}{N}\right)\right|\eta
$$

[73] Ψ_N, $\nabla\Psi_N$ is bounded on the boundary and $\eta, \nabla\eta$ vanish.

$$
\begin{aligned}
&\le c_{40} \lim_{N\to\infty} \left(\sup_{\mathrm{supp}\{\Delta\Psi_N\}} \left|\frac{f}{N}\right| \right) \left| \int_{\mathrm{supp}\{\Delta\Psi_N\}} \Delta\Psi_N \right| \\
&\le c_{40}c_{37} \lim_{N\to\infty} \left(\sup_{\{x\in U|\Psi(x)\le 2N\}} \left|\frac{f}{N}\right| \right) \\
&\le c_{40}c_{37} \lim_{N\to\infty} \lim_{\substack{x\to x_N \\ \Psi(x_N)=2N^{74}}} \left|\frac{f}{N}\right| \\
&\le 2c_{40}c_{37} \lim_{N\to\infty} \lim_{\substack{x\to x_N \\ \Psi(x_N)=2N}} \left|\frac{f}{\Psi}\right| \\
&\le 2c_{40}c_{37} \lim_{x\to 0} \left|\frac{f}{\Psi}\right| \\
&= 0.
\end{aligned}
$$

Define $c_{40} > 0$ such that $\eta, \nabla\eta \le c_{40}$ on U. To move the supremum out of the integral is justified since the supremum is finite by our assumption. Analogous we get

$$
\begin{aligned}
\lim_{N\to\infty} \left| \int_U f \left(\frac{\nabla\Psi_N}{N}\right) \nabla\eta \right| &= \lim_{N\to\infty} \left| \int_{\mathrm{supp}\{\nabla\Psi_N\}} f \left(\frac{\nabla\Psi_N}{N}\right) \nabla\eta \right| \\
&\le c_{40} \lim_{N\to\infty} \left(\sup_{\mathrm{supp}\{\nabla\Psi_N\}} \left|\frac{f}{N}\right| \right) \left| \int_{\mathrm{supp}\{\nabla\Psi_N\}} \nabla\Psi_N \right| \\
&\le c_{40}c_{37} \lim_{N\to\infty} \left(\sup_{\{x\in U|\Psi(x)\le 2N\}} \left|\frac{f}{N}\right| \right) \\
&\le c_{40}c_{37} \lim_{N\to\infty} \lim_{\substack{x\to x_N \\ \Psi(x_N)=2N^{56}}} \left|\frac{f}{N}\right| \\
&\le 2c_{40}c_{37} \lim_{N\to\infty} \lim_{\substack{x\to x_N \\ \Psi(x_N)=2N}} \left|\frac{f}{\Psi}\right| \\
&\le 2c_{40}c_{37} \lim_{x\to 0} \left|\frac{f}{\Psi}\right| \\
&= 0.
\end{aligned}
$$

Hence we have proved the lemma. □

[74] f has no singularities outside 0. For f bounded we know that the claim holds, so assume f unbounded at 0, i.e. for a big enough N f takes its maximum on $\partial\{x \in U|\Psi(x) \le 2N\}$. Call x_N the point where f becomes maximal. In particular $x_N \to 0$ for $N \to \infty$.

2 Filtered Objects

2.1. Filtered Vector Bundles

There are two different concepts of a parabolic vector bundle. We will define both and shortly explain why they are in fact the same. We will usually work with the second concept (Simpson). However, in the language of Mehta and Seshadri (first concept) residues are easier to be described. These residues are defined in the last part of the section. Before we will define morphisms of filtered objects.

Remark 2.1.1. Most of the following definitions make sense if we replace locally free sheaves with coherent or even with quasi-coherent sheaves.

In 1980 Mehta and Seshadri ([MS80]) established the notation of parabolic structure on a Riemann surface.

Definition 2.1.2 (Parabolic Structure)**.** Let $\overline{X}$ be a compact Riemannian surface and S a finite set of points in $\overline{X}$, $X = \overline{X} \setminus S$. Let $\pi : E \to \overline{X}$ be a holomorphic vector bundle and $E_s = \pi^{-1}(\{s\})$. A parabolic structure on $\overline{X}$ is given by a filtration

$$E_s = E_{\alpha_1,s} \supset E_{\alpha_2,s} \ldots \supset E_{\alpha_r,s},$$

and weights $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_r < 1$.
Equivalently we can choose $E_{\alpha,s}$ with $\alpha \in [0,1]$, $E_{\alpha,s} \supset E_{\beta,s}$ for $\alpha \leq \beta, \forall s \in S$ and

$$\mathrm{Gr}^s_\alpha(E) := E_{\alpha,s} / \bigcup_{\beta > \alpha} E_{\beta,s}$$

has finite support, i.e. only for finitely many α it is non-zero.

Remark 2.1.3. Note that some authors like Takuro Mochizuki write the filtration increasing with α_i. We will stay with the convention used by Simpson and originally in [MS80].

Since all punctures are isolated we will usually work on some neighbourhood around a puncture s which contains no other punctures. Here the definition of a filtered regular bundle in this case:

Definition 2.1.4 (Filtered Vector bundle)**.** Let X be a non-compact curve, $X = \overline{X} \setminus \{s\}$ as above and E an algebraic vector bundle on X. $(E, (E_\alpha)_{\alpha\in\mathbb{R}})$ is a filtered vector bundle, if

(i) $\exists$ a filtration E_α of coherent sheaves, i.e. $E_\alpha \supset E_\beta$ for $\alpha \leq \beta$.

(ii) $\bigcup_{\alpha\in\mathbb{R}} E_\alpha = j_* E$.

(iii) $E_\alpha = \bigcap_{\beta<\alpha} E_\beta$ (left-continuity).

(iv) for $\alpha' = \alpha + n_s, n_s \in \mathbb{Z}; \alpha, \alpha' \in \mathbb{R}$ we have $E_{\alpha'} = E_\alpha \otimes_{\mathcal{O}_X} \mathscr{O}(-n_s s)$.

For the general case of more than one puncture we add a filtration for each puncture.

Remark 2.1.5. (1) Note that in (iii) we used the equivalence between bundles and locally free sheaves. We can define filtered vector bundles or more accurately filtered sheaves by requiring E to be a locally free sheaves of $\mathcal{O}_X-$modules, and the E_α to be submodules with the properties mentioned above.

(2) A filtered bundle induces a parabolic bundle with the filtrations

$$F_\alpha := \bigcup_{\beta\geq\alpha} E_\beta, \quad 0 \leq \alpha < 1.$$

The weights are just the elements of $\{\alpha \in [0,1[| \mathrm{Gr}^s_\alpha(E) \neq 0\}$. On the other hand given a parabolic vector bundle with weights $0 \leq \alpha_1 < \ldots \alpha_r < 1$ set $E_{\alpha,s} = E_{\alpha_i,s}$ for all $\alpha \in]\alpha_i, \alpha_{i+1}], 0 \leq i \leq r, \alpha_0 = 0,\ \alpha_{r+1} = 1$ and extend it to a filtered vector bundle using (iv).
For example let (e_i) be a holomorphic frame of E, then we may assume (by (iv)) that all e_i are in E_0 but not in E_1. By the left-continuity we even find a $\gamma_i \in [0,1[$ such that $e_i \in E_\gamma \setminus E_{\gamma+\varepsilon}, \forall\varepsilon > 0$. Then the γ_i are our n weights and the e_i $\mathcal{O}_{\overline{X}}-$span E_0. The $(E_{\gamma_i})_s$ have all different dimension as vector subspaces - we will often describe them as "jump" since they add a dimension.

(3) If the original bundle E is equipped with a hermitian structure h, $E_{\alpha,s}$ has a hermitian structure outside the singularities, i.e. we may identify each $s \in E_{\alpha,s}$ with a section of E and then evaluate the inner product.

Using A.1.11 we can rewrite the definition in the form chosen by Simpson. Therefore we replace (iii) and (iv) by

(iii') the filtration is left-continuous in α: $\exists \varepsilon > 0$ such that for all $\alpha \in \mathbb{R}$: $E_{\alpha,s} = E_{\alpha-\varepsilon,s}$;

(iv') for a coordinate neighbourhood U_s of s and a coordinate vanishing to order one $E_{a+1,s} = E_{a,s} \otimes_{\mathcal{O}_X} \mathscr{O}(-s) =: zE_{a,s}$ holds.

Further note that by (iv) resp. (iv') the filtration is already uniquely determined by the data between zero and one resp. on any compact interval of length 1.

2.1.1. Regular Filtered Higgs Bundles and Regular Filtered $\mathscr{D}_X$–modules

We will usually treat only one puncture at a time. Hence we will sometimes drop the additional index s in $E_{\alpha,s}$. Recall A.3.9, (iv).

Definition 2.1.6. A filtered regular Higgs bundle (E, E_α, θ) is a Higgs bundle (E, θ), a corresponding filtered vector bundle (E_α) and

$$\theta_*|_{E_\alpha} =: \theta_\alpha : \Gamma(X, E_\alpha) \to \Gamma(X, E_\alpha) \otimes \Omega^{1,0}_X(\log s).$$

Then $\theta_\alpha|_{\Gamma(X,E)} = \theta$. Here $\Omega^1_X(\log s)$ is the set of logarithmic one-forms, i.e. one forms which can be locally written as $\alpha \frac{\mathrm{d}z}{z}$, α smooth.

Definition 2.1.7. A filtered regular $\mathscr{D}_X$–bundle is a flat vector bundle (V, ∇, D''_V)[1] a corresponding filtered vector bundle (V_α) and

$$\nabla_*|_{V_\alpha} =: \theta_\alpha : \Gamma_{hol}(X, V_\alpha) \to \Gamma_{hol}(X, V_\alpha) \otimes \Omega^{1,0}_X(\log s).$$

Then $\nabla_\alpha|_{\Gamma_{hol}(X,E)} = \nabla$.

In order to work with categories and functors later, define morphisms of filtered vector bundles:

Definition 2.1.8. (i) Let $((E, \overline{\partial}_E), E_\alpha), ((F, \overline{\partial}_F), F_\alpha)$ be filtered vector bundles. A family $(\varphi_\alpha) : E_\alpha \to F_\alpha$ and $\varphi : E \to F$ of sheaf morphisms is called morphism of filtered objects if

(I) $\varphi^* \overline{\partial}_F = \varphi \overline{\partial}_E$.

(ii) Let $((E, \overline{\partial}_E, \theta_E), E_\alpha), ((F, \overline{\partial}_F, \theta_F), F_\alpha)$ be filtered regular Higgs bundles. A family $(\varphi_\alpha) : E_\alpha \to F_\alpha$ and $\varphi : E \to F$ of sheaf morphisms is called morphism of regular filtered Higgs bundles if $\varphi^* \overline{\partial}_F = \varphi \overline{\partial}_E$ and

[1] D'_V holomorphic structure, $D''_V + \nabla$ flat.

(IIa) $\varphi^*\theta_F = \varphi\theta_E$.

(iii) Let $((V, \nabla_V, D''_V), V_\alpha), ((W, \nabla_W, D''_W), W_\alpha)$ be filtered regular $\mathscr{D}_X-$bundles, in particular $\nabla_V + D''_V, \nabla_W + D''_W$ flat. A family $(\varphi_\alpha) : V_\alpha \to W_\alpha$ and $\varphi : V \to W$ of sheaf morphisms is called morphism of regular filtered $\mathscr{D}_X-$modules if $\varphi^* D''_W = \varphi D''_V$ and

(IIb) $\varphi^*\nabla_W = \varphi\nabla_V$.

2.1.2. Filtered local Systems

Definition 2.1.9. Let L be a local system (a locally constant sheaf) over X, $X = \overline{X} \setminus \{s\}$. A filtered local system $(L, L_\alpha)_{\alpha\in\mathbb{R}}$ is defined by

(i) $L_\alpha \subset L_\beta$ if $\beta \le \alpha$.

(ii) $L_\alpha \subset L_s$ a filtration (by vector spaces) of the stalk at s, $L_s = \bigcup_{\alpha\in\mathbb{R}} L_\alpha, \forall x \in X$.

(iii) $\bigcap_{\beta<\alpha} L_\beta = L_\alpha$.

(iv) L_α invariant under monodromy.[2]

Definition 2.1.10. A morphism of filter local systems is a family $(\varphi, \varphi_\alpha)$ with $\varphi : L_1 \to L_2$ sheaf homomorphism and $\varphi_\alpha : (L_1)_\alpha \to (L_2)_\alpha$ vector space homomorphism.

2.1.3. Residue

Definition 2.1.11. (i) Let (E, E_α, θ) be a filtered regular Higgs bundle, $\mathrm{Gr}_\beta(E) = E_\beta/E_{\beta+\varepsilon}, \varepsilon > 0$. Then the residue is defined to be $(\mathrm{res}(E), \mathrm{res}(\theta))$ with

$$\mathrm{res}(\theta) : \Gamma(X, E) \xrightarrow{\theta} \Gamma(X, E) \otimes \Omega^1_{\overline{X}}(\log s) \to \Gamma(X, E) \otimes \Omega^1_{\overline{X}},$$

where the second arrow maps $e \otimes \omega$ with $\omega = \frac{\xi}{z}\,\mathrm{d}\,z + \eta\,\mathrm{d}\,z, \xi, \eta \in \mathcal{O}_{\overline{X}}$ to $e \otimes \xi\,\mathrm{d}\,z$. Further $\mathrm{res}(E) = \bigoplus_{0\le\beta<1} \mathrm{Gr}_\beta(E)$.

(ii) Let (V, V_α) be a filtered regular $\mathscr{D}_X-$module, $\mathrm{Gr}_\beta(V) = V_\beta/V_{\beta+\varepsilon}, \varepsilon > 0$. Then the residue is defined to be $(\mathrm{res}(V), \mathrm{res}(\nabla))$ with

$$\mathrm{res}(\nabla) : \Gamma(X, E) \xrightarrow{\nabla} \Gamma(X, E) \otimes \Omega^1_{\overline{X}}(\log s) \to \Gamma(X, E) \otimes \Omega^1_{\overline{X}},$$

[2] We will define monodromy when we construct a functor Φ from local systems to $\mathscr{D}_X-$modules in the chapter on filtered objects.

where the second arrow is the same as in (i). Moreover $\mathrm{res}(V) = \bigoplus_{0\leq\beta<1}\mathrm{Gr}_\beta(V)$.

(iii) Let (L, L_α) be a filtered local system, $\mathrm{Gr}_\beta(L) = L_\beta/L_{\beta+\varepsilon}, \varepsilon > 0$. Then the residue is defined to be $(\mathrm{res}(L), \mu)$ with μ the monodromy around s and $\mathrm{res}(L) = \bigoplus_{\beta\in\mathbb{R}}\mathrm{Gr}_\beta(L)$. When we talk of the residue map of a local system we mean μ.

2.2. The Functor Ξ

The second section of this chapter treats a functor Ξ from the category of acceptable bundles (below) to the category of filtered vector bundles. We will show that the functor is compatible with determinants, duals and tensor products. This will help us when we stepwise reduce the category of acceptable bundles to the category of tame harmonic vector bundles as well as reduce the category of filtered vector bundles to the category of stable filtered regular Higgs bundles with degree zero resp. stable filtered regular $\mathscr{D}_X-$modules with degree zero.
In this section the first part is done, i.e. we show that Ξ maps tame harmonic bundles into regular filtered Higgs bundles resp. $\mathscr{D}_X-$modules.

Definition 2.2.1. A bundle (E, h) is called L^p-acceptable if the unique metric connection D which is compatible with the holomorphic structure on E has curvature bounded by $f + \frac{c_{acc}}{|z|^2|\log|z||^2}$ for some $c_{acc} > 0$, $f \in L^p$ for some $p > 1$. We call it acceptable if $f = 0$.

Remark 2.2.2. In most of the cases f may be chosen 0. In the main estimate we actually saw that in most of our cases we have acceptability. The exception will be subbundles. The already used fact that curvature increases in subbundles will lead us to an estimate of the curvature of the subbundle, that differs from the original curvature by a L^p-term.

Definition 2.2.3. Let (E, h) be a metric holomorphic vector bundle on $X = \overline{X}\setminus\{s\}, S := \overline{X}\setminus X$. Denote $\mathcal{E}$ the category of L^p-acceptable vector bundles and $\mathcal{F}$ the category of filtered vector bundles. Define a functor $\Xi : \mathcal{E} \to \mathcal{F}$ by $\Xi(E) = (E, \Xi(E)_\alpha)$ with

$$\Xi(E)_\alpha = \{e \in \Gamma_{hol}(X, E)|\forall s \in S\ \exists C > 0\ \forall\varepsilon > 0 : \|e\|_h \leq Cr_s^{\alpha-\varepsilon}\},$$

with r_s the distance to the puncture s w.r.t. some coordinate vanishing at s. Equivalently $\Xi(E)$ is E plus the information of the filtration of the stalk at s given above.

Remark 2.2.4. As before we may w.l.o.g. restrict to a punctured open disc with the euclidean metric. In particular we may treat only one puncture at a time.
That Ξ truly is a functor is justified by the following proposition and 2.2.11 below.

Proposition 2.2.5. If E is L^p-acceptable on X, then $(E, \Xi(E)_\alpha$ is a filtered vector bundle, i.e. $\Xi(E)_\alpha$ is coherent. The construction Ξ on the class of L^p-acceptable bundles is compatible with the operations of taking determinants, duals, and tensor products.

Proof. For the algebraicity (coherence) of the sheaves Ξ_α we may use the proof of [CG75], p. 23, theorem I in a modified version following the remark by Simpson [Sim88], p. 910. While Cornalba and Griffith modify their metric by $e^{\log(-\log|z|)}$ in order to get negative curvature, we need to add e^σ for σ the bounded solution of the Laplace equation[3] $-\Delta\sigma = f$ in order to compensated the additional f. $e^{\log(-\log|z|)}$ will than compensate the remaining $\frac{c_{acc}}{|z|^2|\log|z||^2}$ (cf. 3.3 on p. 10 of [CG75]).[4] Then we may use the proof of theorem I, p. 23. [5] in the same matter as done there. Note that Cornalba and Griffith give the bound of the curvature in terms of a Poincaré metric $ds = \frac{4}{|z|^2|\log|z|^2|^2}\,\mathrm{d}\,z\wedge\mathrm{d}\,\overline{z} = \frac{1}{|z|^2|\log|z||^2}\,\mathrm{d}\,z\wedge\mathrm{d}\,\overline{z}$.
The other properties of a filtered vector bundle are fulfilled:

(i) $\Xi(E)_\alpha \subset \Xi(E)_\beta$ for $\alpha \geq \beta$ follows clearly from $|z|^\alpha \leq |z|^\beta$ for z small enough, i.e. on the unit disc.

(ii) $j_*(E) = \bigcup_{\alpha\in\mathbb{R}} \Xi(E)_\alpha$ since in the limit there are no growth restrictions on the section, i.e. we get any section holomorphic outside the puncture, i.e. every section of E.[6]

(iii) $\bigcap_{\alpha<\beta} \Xi(E)_\alpha = \Xi(E)_\beta$ because if $\|e\| \leq c_e|z|^{\alpha-\varepsilon}, \forall\alpha < \beta, \forall\varepsilon > 0 \Rightarrow \|e\| \leq c_e|z|^{\beta-(\beta-\alpha)-\varepsilon} = c_e|z|^{\beta-\varepsilon'}, \forall\varepsilon' > 0,\ \varepsilon' = \beta-\alpha+\varepsilon$.

(iv) $z\Xi(E)_\alpha = \Xi(E)_{\alpha+1}$ by $|z|z|^{\alpha-\varepsilon}| = |z|^{\alpha+1-\varepsilon}, \forall\varepsilon > 0$.

The underlying spaces are invariant under Ξ, i.e. compatible with duals, tensor products and determinants. In the following we will show that this holds for the filtrations as well.

Behauptung. Ξ is compatible with taking tensor products.

[3][LiLo00] or [PDE09].

[4]We will not go into further details here. The same modification however, will be used in more detail when we prove the compatibility with taking determinants.

[5]proof on p. 35ff in [CG75]; see as well paragraph 9 and p. 29, remark, (ii).

[6]From now on $j^*(E)$ denotes the pushforward sheaf of the sheaf of holomorphic sections into E.

Proof. Let $(E, h_E, \overline{\partial}_E), (F, h_F, \overline{\partial}_F)$ be two L^p−acceptable bundles. We have seen in the section on endomorphism bundles that the metric on $E\otimes F$ is $h_{E\otimes F}(e_1\otimes f_1, e_2\otimes f_2) = h_E(e_1, e_2)h_F(f_1, f_2)$. Therefore $\|e \otimes f\|_{E\otimes F} = \|e\|_E\|f\|_F \leq |z|^{\alpha-\varepsilon} \Leftrightarrow \|e\|_E \leq |z|^{\beta}, \|f\|_F \leq |z|^{\alpha-\varepsilon-\beta}$, $\beta \in \mathbb{R}$. Therefore and since $\Gamma(X, E \otimes F) = \Gamma(X, E) \otimes \Gamma(X, F)$:

$$\Xi(E \otimes F)_\alpha = \sum_{\beta\in\mathbb{R}} \Xi(E)_\beta \otimes \Xi(F)_{\alpha-\beta}.$$

□

Behauptung. Ξ is compatible with taking duals.

Proof. The metric on the dual bundle is $h_{E^*}(e_1^*\|e_1\|_E^2, e_2^*\|e_2\|_E^2) = h_E(e_1, e_2)$. Then $\|e^*\|_{E^*} = \|e\|_E^{-2}\|e\|_E \geq |z|^{-\alpha+\varepsilon}$ for $e \in \Xi(E)_\alpha$. Further $\|e\|_E \leq |z|^{\alpha-\varepsilon} \Leftrightarrow e \in E_\alpha \Leftrightarrow e^* \in E_\alpha^* = (E^*)_{-\alpha} \Leftrightarrow \|e^*\|_{E^*} \leq |z|^{-\alpha-\varepsilon}$. Together we get for $e \in E_\alpha \setminus E_{\alpha+\varepsilon}, \forall\varepsilon > 0$ that $\|e^*\|_{E^*} \sim |z|^{-\alpha}$. Therefore

$$\begin{aligned}
(\Xi(E)^*)_\beta \setminus (\Xi(E)^*)_{\beta+\varepsilon} &= (\Xi(E)^*_{-\beta}) \setminus (\Xi(E)^*_{-\beta-\varepsilon}) \\
&= \{e^* : X \to E^* | \|e\|_E \sim |z|^{-\beta}\} \\
&= \{e^* : X \to E^* | \|e^*\|_E \sim |z|^{\beta}\} \\
&= \Xi(E^*)_\beta \setminus \Xi(E^*)_{\beta+\varepsilon}, \forall\varepsilon > 0.
\end{aligned}$$

But $(\Xi(E)^*)_\alpha = \bigcup_{\beta\geq\alpha}(\Xi(E)^*)_\beta\setminus(\Xi(E)^*)_{\beta+\varepsilon}$ and $\Xi(E^*)_\alpha = \bigcup_{\beta\geq\alpha} \Xi(E^*)_\beta\setminus\Xi(E^*)_{\beta+\varepsilon}$ and hence $\Xi(E^*)_\alpha = (\Xi(E)^*)_\alpha$, i.e. Ξ compatible with taking duals. □

Behauptung. Ξ is compatible with determinants.

Proof. E induces on the determinant bundle the inner product

$$h_{\bigwedge}(e_1 \wedge \ldots \wedge e_n, f_1 \wedge \ldots \wedge f_n) = \det(h_E(e_i, f_j)).$$

By the alternating and multilinear character of det this really defines a positive-definite inner product. We get $\det(\Xi(E))_\alpha = \left(\bigwedge_{i=1}^n \Xi(E)\right)_\alpha \subset \Xi\left(\bigwedge_{i=1}^n E\right)_\alpha = \Xi(\det(E))_\alpha$ by

$$\begin{aligned}
\|e_1 \wedge \ldots \wedge e_n\| &= \det(h(e_i, e_j)) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \prod_{j=1}^n h(e_j, e_{\sigma(j)}) \\
&\leq \sum_{\sigma\in S_n} \prod_{j=1}^n |h(e_j, e_{\sigma(j)})| \leq^7 \sum_{\sigma\in S_n} \prod_{j=1}^n \|e_j\|_E\|e_{\sigma(j)}\|_E \\
&\leq \sum_{\sigma\in S_n} \prod_{j=1}^n c_j c_{\sigma(j)}|z|^{i_j+i_\sigma} = \sum_{\sigma\in S_n} |z|^{\sum_{j=1}^n i_j+i_\sigma} \prod_{j=1}^n c_j c_{\sigma(j)}
\end{aligned}$$

$$\leq \quad c_{\bigwedge}|z|^{2\alpha-2\varepsilon}, \forall \varepsilon > 0,$$

for $e_1 \wedge \ldots \wedge e_n \in \left(\bigwedge_{i=1}^n \Xi(E)\right)_\alpha = \det(\Xi(E))_\alpha$. The other direction is slightly more advanced. We use the following lemma by Simpson:

Lemma 2.2.6. Let E be an L^p−acceptable vector bundle on X - curvature bounded by $f + \frac{c_{acc}}{|z|^2|\log(z)|^2}$ for some $c_{acc} > 0$ - with $\det(H) = \det(h(e_i, e_j)) \leq c_h|z|$ for a frame (e_i) of E. Further choose (e_i) such that $\|H\|_F \leq 1$. Then there is a section $e = \sum_i^n a_i e_i$ with a_i constant such that $\|e\|_E \leq c_e|z|^{1/2nN_R}$ for some constant $N_R > 0$.

Proof. Define a new metric h' on E by $h'(e,f) = \exp(4\sigma + 8c_{acc}\log(-\log|z|) + \varepsilon\log|z|)h(e,f)$, for σ the bounded solution of the Laplace equation $-\Delta\sigma = f$. E with the metric h' is still L^p−acceptable since

$$\begin{aligned}
R_{h'} &= \overline{\partial}\partial\log(\overline{H}') = \overline{\partial}\partial(\log(4\sigma + e^{8c_{acc}\log(-\log|z|)+\varepsilon\log|z|}E) + \log(\overline{H})) \\
&= \Delta\sigma + \overline{\partial}\partial 8c_{acc}\log(-\log|z|)E + \overline{\partial}\partial\varepsilon|z|E + \overline{\partial}\partial\log(\overline{H}) \\
&= -f + 8c_{acc}\overline{\partial}\partial\log(-\log|z|)E + \overline{\partial}\partial\varepsilon|z|E + R_h \\
&= -f\frac{8c_{acc}\,\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z}{4}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\log(-\log r) + \frac{\varepsilon\,\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z}{4}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}r + R_h \\
&= -f2c_{acc}\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z\frac{1}{r}\frac{\partial}{\partial r}\frac{-1}{-\log r} + \frac{\varepsilon\,\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z}{4r} + R_h \\
&= -f - 2c_{acc}\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z\frac{1}{r^2(\log r)^2} + \frac{\varepsilon\,\mathrm{d}\,\overline{z}\wedge\mathrm{d}\,z}{4r} + R_h
\end{aligned}$$

Now let e be any constant section in terms of our basis (e_i) and E_e the subbundle spanned by e. The unique metric connection compatible with the induced holomorphic structure on E_e is calculated as usual by $\overline{\partial}\partial\log\|s\|_{h'}$ for every $s \in E_e \Rightarrow s = fe, f \in \mathcal{O}_X$. Use that the curvature decreases in subbundles (cf. [GH78] p. 79) on E_e to get for $s = e$

$$\begin{aligned}
\Lambda\overline{\partial}\partial\log\|e\|_{h'} &= \Lambda h'(R_{h'}e, e) \\
-\Delta\log\|e\|_{h'} &= -2i\Lambda h'(R_{h'}e, e) \\
&= -2i\Lambda h'(R_h e, e) + \frac{\varepsilon}{|z|}h'(e,e) - f - 4\cdot 2c_{acc}\frac{1}{|z|^2(\log|z|)^2}h'(e,e) \\
&\leq^{8} \left(4c_{acc}\frac{1}{|z|^2(\log|z|)^2} + f - f - 4\cdot 2c_{acc}\frac{1}{|z|^2(\log|z|)^2} + \frac{\varepsilon}{|z|}\right)\|e\|_{h'} \\
&= \left(-4c_{acc}\frac{1}{r^2(\log r)^2} + \frac{\varepsilon}{|z|}\right)\|e\|_{h'} \\
&\leq 0,
\end{aligned}$$

[7] Cauchy-Schwarz inequality.

since $\frac{1}{r^2(\log r)^2}$ grows faster than r^{-1}. On the other hand we still have $\det(H') \leq c_h|z|$ since $-(\log|z|)|z|^\varepsilon = \exp(\varepsilon \log|z| + \log(-\log|z|)) \to 0$ for small ε and z. Analogously we ensure that we have $\|H'\|_F \leq 1$.
W.l.o.g. we restrict again to a disc around the puncture.
Choose a sequence r_j converging to 0. We want to find a section e^j for each r_j such that $\exists N_R \in \mathbb{N} : \|e^j(z)\| \leq r_j^{1/2n}, \forall |z| \leq r_i^{N_R}$ and for $e^j = \sum_{j=1}^n a_i e_i$ we have $\sum_{j=1}^n a_j = 1$.[9]
We know for our frame (e_i) that $\det(H) = \det(h(e_i, e_j)) \neq 0$. Assuming $\|e_i(z)\| > |z|^{1/2n}$, then the non-vanishing of the determinant tells us that $\det(h(e_i, e_j)) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^n h(e_i, e_{\sigma(i)}) > |z|^{2n/2n} = |z|^1$ since every summand is bounded by Cauchy-Schwarz by $|z|^{2n/2n}$ - a contradiction to our assumption on the determinant of K. Hence $\forall z \in X \exists\ i \in \{1, \ldots, n\}$ such that $\|e_i(z)\|_h \leq |z|^{1/2n}$. Let e^j denote the e_i that fulfills the bound at $z = 0$. Since e^j is continuous we find a disc B_R around 0 where $\|e^j(z)\|_h \leq r_j^{1/2n}$ holds. Further $r_j < 1 \Rightarrow \exists N_R : r_j^{N_R} \leq R$. Note that N_R only depends on r_1. In particular when we replace r_j by $r_j^{N_R}$ we have $\|e^j(z)\|_h \leq r_j^{1/2nN_R}$. Since the set where the inequality holds is closed the bound still holds on ∂B_R. The complement in B_1 however is open and since by $\|H'\|_F \leq 1$, $e^j \leq 1$ uniformly we can apply the maximum principle on any open Annulus $A_{R_1,R_2}, r_j \leq R_1 < R_2 \leq 1$ around 0. The maximum on the boundary is bounded by $(R_2)_j^{1/2nN_R}$. Since R_1 and R_2 were chosen arbitrary we get $\|e^j(z)\|_h \leq |z|_j^{1/2nN_R}$ for all $r_j \leq |z| \leq 1$. Now let $r_j \to 0$ to receive the claim. □

Remark 2.2.7. Simpson proves the lemma for $N_R = 3$. Still for what follows we only need that there is any $N_R > 0$.

Remember the discussion of finite branched covers at the beginning of the chapter on the Main Estimate. Let us consider such a cover $u^N = z$, u local coordinate on the covering space. If π_u is the corresponding projection the pullback bundle $\pi^*(E) = \bigoplus_{i=0}^{N-1} u^i E$ splits. The filtration of the pullback bundle is defined as $\pi^*(E)_\alpha = \sum_{N\beta+i\geq\alpha} u^i \pi^*(E_\beta)$. The properties of a filtration follow directly from the definition - only note that $u\pi^*(E)_\alpha = \sum_{N\beta+i\geq\alpha} u^{i+1}\pi^*(E_\beta) = \sum_{N\beta+j-1\geq\alpha} u^j \pi^*(E_\beta) = \sum_{N\beta+j\geq\alpha+1} u^j\pi^*(E_\beta) = \pi^*(E)_{\alpha+1}$.
Now let $e_u = \sum_{i=1}^{N-1} u^i e_u^{(i)}$ be a section in $\Xi(\pi^*(E))_\alpha$ then $\|e_u\|_{\pi^*(E)} = \sum_{i=1}^N |u|^i \|e^{(i)}\|_h \leq c_e|u|^{\alpha-\varepsilon}, \forall \varepsilon > 0 \Leftrightarrow e_u^{(i)} \leq c_e|z|^{\alpha/N-i/N} \Leftrightarrow e_u \in \sum_{N\beta+i\geq\alpha} u^i\pi^*(E_\beta)$ since for $\beta = \alpha/N - i/N \Rightarrow N\beta + i = \alpha - i + i = \alpha$. Therefore we have $\Xi(\pi^*(E))_\alpha = \pi^*(\Xi(E)_\alpha)$.

[8] By Cauchy-Schwarz and the consistency of the matrix norms $\|\cdot\|_H^2 = \operatorname{tr}(AHA^*H)$: $\|Av\|_h \leq \frac{\|A\|_H}{\|E\|_H}\|v\|_h$ for the unit matrix E and $v \in E$. Hence the scalar factor $\exp(8c_{acc}\log(-\log|z|) + \varepsilon\log|z|)$ drops out.

[9] This will guarantee that when $j \to \infty$, e^j converges to a non-trivial section.

Now assume that $\Xi(\det(E))_\alpha \not\subset \det(\Xi(E)_\alpha)$ for at least one α. The idea of the ramified cover is to enlarge this gap so much that we can use the previous estimate. By left-continuity we find a biggest index $\gamma < \alpha$ for which we have $\Xi(\det(E))_\alpha \supset \det(\Xi(E)_\gamma)$. Using the ramified cover for sufficiently big N we get

(i) $\Xi(\det(E))_0 \supset \det(\Xi(E)_2)$: As long as N is big for example $N(\alpha-\gamma) > 3$ we may shift the gap by property (iv) of a filtered bundle such that the claim holds.

(ii) Further choose N such that all weight are only slightly bigger than various integers, for example differ not more than $\frac{1}{2nN_R+1}$ from the next smaller integer[10], then there is no jump between $\frac{1}{2nN_R}$ and 1, i.e. $\Xi(\pi^*(E))_{1/2nN_R} = \Xi(\pi^*(E))_1$.

Now (i) and (ii) guarantee that the conditions of the lemma are satisfied for $\pi^*(E)_0$, i.e. we get a non-trivial section in $\Xi(\pi^*(E))_{1/2nN_R}$. But there are no weights between $\frac{1}{2nN_R+1}$ and 1, in particularly there is no $\mathcal{O}_{\overline{X}}-$holomorphic section apart from the trivial one. This contradiction implies $\Xi(\det(E))_\alpha \subset \det(\Xi(E)_\alpha) \Rightarrow \Xi(\det(E))_\alpha = \det(\Xi(E)_\alpha)$. □

□

Remark 2.2.8. Recall that our "inner product" on the extension $\Xi(E)_\alpha$ is the inner product of the corresponding elements of $\Gamma(\overline{X}\setminus\{s\}, E) = \Gamma(X, E)$. Since not all sections contribute to the stalk of E_α at s this is in general only a monomorphism. It is in fact the monomorphism ω Deligne uses in [Del70] 2.15.2, p. 66, where $\Gamma(X\setminus\{s\}, E)$ and $\mathcal{O}_X$ are naturally identified (map an $h-$orthonormal basis on the standard basis.) If our metric grows at most polynomially, i.e. the flat sections do, than these flat sections are meromorphic. Hence ∇e[11] of a holomorphic, moderate section e is meromorphic, since we may write e in terms of a flat basis and then ∇ acts just as the usual differential on the coefficient functions, which are meromorphic by the meromorphy of e and the frame (both moderate).

When we start with a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ as in the previous section we associate a flat bundle $(V, \mathbb{D})$ with holomorphic connection $\nabla := \partial_E + \theta$ - 1.2.3 - and holomorphic structure $\overline{\partial}_E + \theta^\dagger$. Further we get a local system by the locally constant sheaf of $\overline{\partial}_V$-holomorphic sections killed by ∇ $-V^\nabla := \{s \in \Gamma_{\text{hol}}(X, V) | \nabla(s) = 0\}$

Remark 2.2.9. A $\nabla-$flat and $\overline{\partial}_V + \theta^\dagger-$holomorphic section is $\mathbb{D}-$flat and vice versa[12], but not $\overline{\partial}_V-$holomorphic. While we find a $\mathbb{D}-$flat single-valued frame,

[10] Possible since there are only finitely many weights between two integers.
[11] ∇ the $(1,0)$ part of the flat connection. See below.
[12] $\mathbb{D}(s) = \nabla(s) + \overline{\partial}_V(s) = 0 \Leftrightarrow \nabla(s) = 0$ and $\overline{\partial}_V(s) = 0$ - by degree considerations.

elements of V^∇ are in general multivalued. For $\theta = 0$, i.e. if we find a frame within V^∇, everything becomes trivial.

Theorem 2.2.10. (a) Let (E, h) be a tame harmonic bundle over X with two holomorphic structures $\overline{\partial}_E$ and $\overline{\partial}_E + \theta^\dagger$. Then E is acceptable w.r.t. both holomorphic structures.

(b) The endomorphism-valued one-form θ and the connection $\mathbb{D} = \overline{\partial}_E + \partial_E + \theta + \theta^\dagger$ satisfy the regularity condition making $(\Xi(E, \overline{\partial}_E), \theta)$ into a filtered regular Higgs bundle, and $(\Xi(E, \overline{\partial}_E + \theta^\dagger), \nabla), \nabla = \partial_E + \theta$ into a filtered regular $\mathscr{D}_X-$module. These constructions are compatible with the operations of taking determinants, duals, and tensor products.

Proof. (a) The main estimate, namely lemma 1.6.15 bounds the curvature of both metric connections by $\frac{C}{|z|^2|\log|z||^2}$ for some C, i.e. the bundles are both acceptable.

(b) Again by the main estimate 1.5.7 $\|\theta_z\|_F^2 \leq \frac{\sqrt{c_5}}{|z|}$ and so the picture under θ has a pole of order at most 1 and hence the induced map on $\Xi(E)_\alpha$ maps into $\Xi(E)_\alpha \otimes \Omega^1_X(\log s)$. Obviously $\Xi(E)_\alpha \subset \Xi(E)_\beta, \alpha \geq \beta$ and $\Xi(E)_\alpha = \bigcup_{\beta<\alpha} \Xi(E)_\beta$. Finally $\Xi(E)_{\alpha+1} = \Xi(E)_\alpha \otimes_{\mathcal{O}_X} \mathscr{O}(-s)$ directly by splitting up $r^{\alpha-\varepsilon} = r^{\alpha+1-\varepsilon} \cdot \frac{1}{r}$. Hence $(\Xi(E, \overline{\partial}_E), \theta)$ is a filtered regular Higgs bundle.
To receive the properties of a filtered regular $\mathscr{D}_X-$module we need to show regularity of the holomorphic connection ∇. Let e be a $\overline{\partial} + \theta^\dagger-$holomorphic section, i.e. $(\overline{\partial} + \theta^\dagger)e = 0$ and $e \in E_\alpha$: $\|e\|_h \leq c_e|z|^{\alpha-\varepsilon}$. We need to show that $\nabla(e)$ is bounded in terms of $|z|^{\alpha-\varepsilon'-1}$ for all $\varepsilon' > 0$, in order to be in $\Xi(E)_\alpha \otimes \Omega^1_X(\log s)$ (the $|z|^{-1}$ is taken care of by the logarithmic $1-$form). By Leibniz rule $\nabla(z^\beta e) = \alpha z^{\beta-1}e + z^\beta e$. So it will be enough to show the bound of $\nabla(e)$ for $\|e\|_h \leq |z|^\mu, \mu > 0$ small.
Let $D_{\overline{\partial}+\theta^\dagger} = R_{\overline{\partial}+\theta^\dagger}\, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}$ denote the curvature of the metric connection $\partial_E - \theta + \overline{\partial}_E + \theta^\dagger$ compatible with $\overline{\partial} + \theta^\dagger$.

$$\begin{aligned}
\Rightarrow \overline{\partial}\partial h(e,e) &= \overline{\partial} h((\partial_E - \theta)e, e) + \overline{\partial} h(e, (\overline{\partial} + \theta^\dagger)e) \\
&= \overline{\partial} h((\partial_E - \theta)e, e) \\
&= h((\overline{\partial}_E + \theta^\dagger)(\partial_E - \theta)e, e) - h((\partial_E - \theta)e, (\partial_E - \theta)e) \\
\Rightarrow \partial\overline{\partial} h(e,e) &= \partial h((\overline{\partial}_E + \theta^\dagger)e, e) + \partial h(e, (\partial_E - \theta)e) \\
&= \partial h(e, (\partial_E - \theta)e) \\
&= -h(e, (\overline{\partial} + \theta^\dagger)(\partial_E - \theta)e) + h((\partial_E - \theta)e, (\partial_E - \theta)e).
\end{aligned}$$

Further note that $(\overline{\partial} + \theta^\dagger)(\partial_E - \theta)e = (\overline{\partial} + \theta^\dagger)(\partial_E - \theta)e + (\partial_E - \theta)(\overline{\partial} + \theta^\dagger)e = D^2_{\overline{\partial}_E+\theta^\dagger}e$. In order to get rid of the differential forms, evaluate at a holomorphic non-zero vector as in A.4

$$\begin{aligned}
&\left(\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} - \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\right)\|e\|_h^2 \,\mathrm{d}\, z \wedge \mathrm{d}\, z \\
&= -h(e, R_{\overline{\partial}+\theta^\dagger} e)\,\mathrm{d}\, z \wedge \mathrm{d}\,\overline{z} + h((\partial_E - \theta)e, (\partial_E - \theta)e) \\
&\quad -\overline{h(e, R_{\overline{\partial}+\theta^\dagger} e)\,\mathrm{d}\, z \wedge \mathrm{d}\,\overline{z}} + h((\partial_E - \theta)e, (\partial_E - \theta)e) \\
&= 2\|(\partial_E - \theta)e\|_h^2 - 2\Re h(e, R_{\overline{\partial}+\theta^\dagger} e)\,\mathrm{d}\, z \wedge \mathrm{d}\,\overline{z},
\end{aligned}$$

and hence

$$\begin{aligned}
\Delta\|e\|_h^2 &= 4\|(\partial_E - \theta)_z e\|_h^2 - 4\Re h(e, R_{\overline{\partial}+\theta^\dagger} e) \\
\Rightarrow -\Delta\|e\|_h^2 &= -4\|(\partial_E - \theta)_z e\|_h^2 + 4\Re h(e, R_{\overline{\partial}+\theta^\dagger} e).
\end{aligned}$$

Then by Cauchy-Schwarz

$$\begin{aligned}
\Re h(e, R_{\overline{\partial}+\theta^\dagger} e) &\leq \|e\|\|R_{\overline{\partial}+\theta^\dagger} e\| \leq \|e\|_h\|R_{\overline{\partial}+\theta^\dagger} e\|_h \\
&\leq \|e\|_h^2\|R_{\overline{\partial}+\theta^\dagger}\|_F \leq c_{D_{\overline{\partial}+\theta^\dagger}} c_e^2 \frac{|z|^{2\alpha-2\varepsilon}4}{|z|^2(\log|z|)^2}
\end{aligned}$$

Note that

$$\begin{aligned}
\Delta \log|\log r| &= \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} \log|\log r| \\
&= \frac{1}{r}\frac{\partial}{\partial r} r \cdot \frac{1}{r\log(r)} \\
&= -\frac{1}{r} \cdot \frac{1}{r(\log(r))^2} \\
&= -\frac{1}{r^2(\log(r))^2}.
\end{aligned}$$

and thus

$$-\Delta(\|e\|_h^2 - 4c_{D_{\overline{\partial}+\theta^\dagger}} c_e^2 \log(\log|z|)) \leq -4\|(\partial_E - \theta)_z e\|_h^2.$$

By lemma 1.7.9 $4\|(\partial_E - \theta)_z e\|_h^2$ is integrable and the inequality holds weakly over the puncture. The lemma can be applied since $\log(|\log(|z|)|)$ is dominated by $\log|z|$ (concave) and $\|e\|_h \leq |z|^\mu$ holds obviously, too. Moreover $\nabla = \partial_E + \theta = \partial_E - \theta + 2\theta$ and by the main estimate $\|\theta(e)\|_h \leq \|\theta\|_F\|e\|_h \leq \sqrt{c_5}|z|^{-1+\mu}$ is L^2-integrable. Hence

$$\begin{aligned}
4|z|^{2\mu}\|(\nabla)_z e\|_h^2 &\leq 4|z|^{2\mu}\|(\partial_E - \theta + 2\theta)_z e\|_h^2 \\
&\leq 4|z|^{2\mu}\|(\partial_E - \theta)_z e\|_h^2 + 16|z|^{2\mu}\|\theta_z e\|_h^2
\end{aligned}$$

is integrable, i.e. $\nabla(e)$ in L^2. By 2.2.8 $\nabla(e)$ is meromorphic, i.e. $\|\nabla(e)\|_h \leq |z|^{\alpha'-\varepsilon}, \forall\varepsilon > 0$ for some $\alpha' \in \mathbb{R}$. But $\nabla(e)$ in L^2 tells us that $\|\nabla(e)\|_h^2$ must grow with order less than real dimension: 2. Then $\alpha' = -1$ and $\|\nabla(e)\|_h^2 \leq |z|^{-1-\varepsilon}, \forall\varepsilon > 0$, which is our claim.

□

Lemma 2.2.11. Ξ is a functor, where Ξ of a gauge transformation φ is the morphism of filtered bundles, which is induced by the vector bundle homomorphism φ.

Proof. Recall the definitions 1.2.8 and 2.1.8. First note that (iii) of 1.2.8 implies that for every $e \in E_\alpha$, i.e. $\|e\|_E \leq c_e|z|^{\alpha-\varepsilon}, \forall\varepsilon > 0$

$$\begin{aligned} &\|\varphi(e)\|_F \leq \|\varphi\|_{E\to F}\|e\|_E \leq c_e|z|^{\alpha-\varepsilon}, \forall\varepsilon > 0 \\ &\Rightarrow \varphi(e) \in F_\alpha. \end{aligned}$$

Hence the filtration is preserved by φ.
Use (i) and (ii) in 1.2.8.

$$\begin{aligned} &\partial_E + \theta_E^\dagger = D_E - d_E'' \\ &\Rightarrow \varphi^*(\partial_E + \theta_E^\dagger) = \varphi^*(D_E - d_E'') = \varphi(D_E - d_E'') = \varphi(\partial_E + \theta_E^\dagger) \\ &\Rightarrow \varphi^*(\partial_E) = \varphi(\partial_E), \quad \varphi^*(\theta_E^\dagger) = \varphi(\theta_E^\dagger) \end{aligned}$$

since ∂_E and $\theta^\dagger$ have different degree. We already know that $\varphi^* D_F = \varphi D_E \Rightarrow \varphi^*\nabla_F = \varphi\nabla_E, \varphi^*(\overline{\partial}_F + \theta_F^\dagger) = \varphi(\overline{\partial}_F + \theta_F^\dagger)$ as well as $\varphi^* d_F'' = \varphi d_E'' \Rightarrow \varphi^*(\overline{\partial}_F) = \varphi(\overline{\partial}_E), \varphi^*(\theta_F) = \varphi(\theta_E)$ by degree considerations. Moreover note that

$$\begin{aligned} &\partial_E + \theta_E^\dagger = D_E - d_E'' \\ &\Rightarrow \varphi^*(\partial_F + \theta_F^\dagger) = \varphi^*(D_F - d_F'') = \varphi(D_E - d_E'') = \varphi(\partial_E + \theta_E^\dagger) \\ &\Rightarrow \varphi^*(\partial_F) = \varphi(\partial_E), \quad \varphi^*(\theta_F^\dagger) = \varphi(\theta_E^\dagger). \end{aligned}$$

But then all operators of any importance to our Higgs resp. $\mathscr{D}_X-$bundle commute with φ as demanded by 2.1.8.

□

2.3. Local Systems

After constructing a functor between harmonic bundles, regular filtered Higgs bundles and $\mathscr{D}_X-$modules in the last section, which will later on lead to an invertible functor, i.e. to an equivalence of categories, we now want to bring the filtered local systems into play.

In contrast to the functor Ξ, the functor Φ that maps filtered local system to filtered regular $\mathscr{D}_X-$modules, fulfills not only compatibility with tensor products, determinants and duals, but additionally with the decomposition into monodromy invariant subsystems. Furthermore we may directly prove fully faithfulness and essential surjectivity, i.e. at the end of the chapter we will know that filtered local systems and filtered regular $\mathscr{D}_X-$modules are essentially the same.

Proposition 2.3.1. A local system gives rise to a $\mathscr{D}_X-$module with regular singularities.

Proof. By Proposition A.3.2 every local system induces a flat vector bundle and to a flat vector bundle $(V, \nabla, \overline{\partial}_V)$ the sheaf of multivalued horizontal sections is a local system $L = V^{\nabla}$. Moreover proving the proposition every flat vector bundle $(V, \nabla, \overline{\partial}_V)$ induces a representation of the fundamental group as follows. For each path γ into X there is a unique horizontal $\overline{\partial}_V-$holomorphic section s such that $\nabla_{\dot{\gamma}(t)} s = 0$[13] through each point v_0 - $s(\gamma(0)) = v_0$ - in the fiber over x. By flatness this section is invariant under homotopy. s is in general not single-valued. Hence moving around a puncture by a path shifts an element v_0 of a fiber V_x to an element gv_0, $g \in \mathrm{Gl}_n(\mathbb{C})$. In the proof of A.3.2 it is also shown, that up to $\mathrm{Gl}_n(\mathbb{C})-$conjugation g is independent of the chosen base point v_0. This leads us to the corresponding $\pi_1(X)-$representation.
If we started with the corresponding local system L and a locally constant cover U_i of L, any section of $s_i \in L(U_i)$ can be uniquely identified with a section $s_j \in U_j, U_{ij} := U_i \cap U_j \neq \emptyset$: The restrictions to U_{ij} are isomorphisms for a locally constant sheaf, i.e. elements of $\mathrm{Gl}_n(\mathbb{C})$. Hence $s_j = g_j^{-1} g_i s_i$, if we identify $L(U_i) \simeq \mathbb{C}^n$ [14] and g_i, g_j are the restrictions. Going once around the puncture will lead us to an element $\mu \in \mathrm{Gl}_n(\mathbb{C})$ - the monodromy - which is well-defined by the sheaf axioms.
Further we may identify each stalk with its surrounding constant sheaf by definition of the stalk. For example let ρ be a ray emitting from the puncture and U_ρ an open set over which the local system is constant and for which $s \in \overline{U_\rho}$ and $\rho \cap U_\rho \neq \emptyset$, i.e. the/an element of the covering of X closest to the puncture intersecting the ray. By the local constancy we may identify L_x with U_ρ for all $x \in U_\rho \cap \rho$. For $L(\rho) := \bigcup_{x \in \rho} L_x$, the monodromy maps $\mu : L(\rho) \to L(\rho), l \mapsto \mu l, \mu \in \mathrm{Gl}_n(\mathbb{C})$.
Proposition A.3.2 states that we find a sheaf isomorphism from L into the sheaf of multivalued horizontal sections of some $\mathscr{D}_X-$module (V, ∇).
We want to produce a single-valued holomorphic section of V. Choose a matrix M such that $e^{-2\pi i M} = \mu$,[15] i.e. $M = \frac{\log(\mu)}{-2\pi i}$. We have worked with the matrix logarithm

[13] The evaluation of $\nabla s(\gamma(t))$ at $\dot{\gamma}(t)$.

[14] Fix a basis and identify it with the standard basis on $\mathbb{C}^n$. Then the g_i are transformation matrices from one basis to another.

[15] Over $\mathbb{C}$ the matrix exponential is surjective into $\mathrm{Gl}_n(\mathbb{C})$.

before, but on hermitian matrices, where it is single-valued. μ is in general only invertible, so the logarithm is not unique and M corresponds to the choice of a branch of the logarithm for each eigenvalue.[16] Denote by $\bar{l}$ the multivalued flat section which coincides with some l in $L(\rho)$ on U_ρ. The section

$$h(z) = e^{M \log z} l(z)^{17}$$

is single-valued, since continuing around the puncture once, l goes to μl (another branch of $\bar{l}$) and $e^{M \log z}$ goes to $e^{M \log z} e^{2\pi i M}$ by the $2\pi i$ monodromy of the logarithm. Thus

$$h(z) \to e^{M \log z} e^{2\pi i M} \mu l(z) = e^{-M \log z} \mu^{-1} \mu l(z) = h(z),$$

i.e. h has only one branch, hence is single-valued. Similarly define $h_i(z) = e^{M \log z} l_i(z)$ for a basis $(l_i)_{1 \leq i \leq n}$ of $L(\rho)$. Since $e^{M \log z}$ does not depend on the flat section, the h_i become linearly independent like the l_i, and form therefore a holomorphic frame of V. Then ∇ has the following connection matrix

$$\nabla h = d\left(e^{M \log z}\right) l + e^{M \log z} \underbrace{\nabla l}_{=0} = M \frac{\mathrm{d}\, z}{z} h.$$

Define an extension $\overline{V} = \mathrm{span}\{\mathcal{O}_X h_i\}$. Then ∇ has a pole of order 1 in the (meromorphic) frame (h_i) and hence is regular. So we constructed a regular $\mathscr{D}_X-$module. $\square$

Theorem 2.3.2. Denote by $\mathcal{L}$ the category of local systems and by $\mathscr{D}$ the category of filtered regular $\mathscr{D}_X-$modules. $\Phi : \mathcal{L} \to \mathscr{D}$ with

(i) $\Phi(L)$ is the regular $\mathscr{D}_X-$module $\mathrm{span}\{\mathcal{O}_X h_i\}$ given by 2.3.1. $\nabla = \partial_V + M \frac{\mathrm{d}\, z}{z}$ in the frame (h_i).

(ii)

$$\begin{aligned} \Phi(L)_\alpha = \{\mathcal{O}_{\overline{X}} h_{l,M} \quad : \quad & h_{l,M} = e^{M \log(z)} l(z), l \in L_\beta, \\ & \lambda \text{ eigenvalues of } M \Rightarrow \Re(\lambda) \geq \alpha - \beta\}. \end{aligned}$$

is an equivalence of categories and is compatible with determinants, duals, and tensor products. (Denote by λ_M the eigenvalue with the smallest real part.)

[16] Write μ in Jordan normal form. For a Jordan block $J = \lambda E + N = \lambda(E + \lambda^{-1} N)$ by invertibility, λ eigenvalue. Then $\log(J) = \log(\lambda E) + \log(E + \lambda^{-1} N) = \log(\lambda) E + \sum_{k=0}^{\infty} (-1)^k \frac{(\lambda^{-1} N)^{k+1}}{k+1}$ and the second term is finite by N nilpotent.

[17] Choose the principal branch of the logarithm defined on the whole ray $\rho \cap U_\rho$.

Note that the monodromy of M is purely real and so (ii) reacts sensitive on monodromy change of M.

Proof. By $L = \bigcup_\alpha L_\alpha$ we will always find a basis (l_i) of $L_x, x \in X$ such that $\{l_i\} \subset L_\beta$ for β small enough.[18] Fix a matrix M_0 as above. Then we have a frame $h_i = e^{M_0 \log z} l_i$ as above. Note that the space spanned by the h_i is independent of the chosen basis l_i: Choose another basis (l'_i),$l'_i = \sum_{k=1}^n a_i l_i$, then $h'_i = e^{M_0 \log z} l'_i = \sum_{k=1}^n e^{M_0 \log z} a_k l_k = \sum_{k=1}^n a_k h_k$ for $a_i \in \mathcal{O}_X$. It depends not on M: Let $M_k = \frac{\log(\mu)+2\pi ik}{-2\pi} = M_0 - k$ be a different logarithm, then $h_i^{(k)} = z^{-k} h_i$ and $z^{-k} \in \mathcal{O}_X$. For the $\Phi(L)_\alpha(U)$ the same holds as long as $\overline{U}$ contains no puncture.

In order to get a filtered regular $\mathscr{D}_X$−module the filtration has to satisfy the conditions (i) to (iv) stated in definition 2.1.4 and the conditions of 2.1.7:

ad (iv): We start with the last property because it will simplify showing the rest of the properties. Let $v \in \Phi(L)_\alpha(U)$ then $\exists M_v \in \mathrm{Gl}_n(\mathbb{C}) \exists k \in \mathbb{N}, \forall 1 \le j \le k\ \exists \beta_j \in \mathbb{R}, \exists l_{\beta_j} \in L_{\beta_j} \exists a_j \in \mathcal{O}_{\overline{X}} : \Re(\lambda_M) \ge \alpha - \beta_j, v = \sum_{j=1}^k a_j e^{M_v \log(z)} l_{\beta_j}$. It looks worse than it is! Let $M'_v = M_v + 1$. Then $\Re(\lambda_{M'_v}) - 1 = \Re(\lambda_{M_v}) \ge \alpha - \beta_j \Rightarrow \Re(\lambda_{M'_v}) \ge \alpha + 1 - \beta_j \Rightarrow \Phi(L)_{\alpha+1}(U) \ni \sum_{j=1}^k a_j e^{M'_v \log(z)} l_{\beta_j} = zv \Rightarrow z\Phi(L)_\alpha(U) \subset \Phi(L)_{\alpha+1}(U)$. On the other hand for $v \in \Phi(L)_\alpha(U)$ $\exists M_v \in \mathrm{Gl}_n(\mathbb{C}) \exists k \in \mathbb{N}, \forall 1 \le j \le k\ \exists \beta_j \in \mathbb{R}, \exists l_{\beta_j} \in L_{\beta_j} \exists a_j \in \mathcal{O}_{\overline{X}}$[19]: $\Re(\lambda_{M_v}) \ge \alpha + 1 - \beta_j, v = \sum_{j=1}^k a_j e^{M_v \log(z)} l_{\beta_j}$. Define $M'_v = M_v - 1$. Then $\Re(\lambda_{M'_v}) + 1 = \Re(\lambda_{M_v}) \ge \alpha + 1 - \beta_j \Rightarrow \Re(\lambda_{M'_v}) \ge \alpha - \beta_j \Rightarrow \Phi(L)_\alpha(U) \ni \sum_{j=1}^k a_j e^{M'_v \log(z)} l_{\beta_j} = z^{-1} v \Rightarrow z\Phi(L)_\alpha(U) \supset \Phi(L)_{\alpha+1}(U)$. This shows (iv) of 2.1.4.

ad (ii): For a basis (l_i) of L we get our frame $(h_i) = (e^{M_0 \log(z)} l_i)$ of holomorphic sections. For α small enough all l_i (resp. n linearly independent vectors) will be in L_β. For $\alpha \to -\infty$ the eigenvalues of M_0 are unrestricted and hence $M_k = \frac{\log(\mu)+2\pi ik}{-2\pi} = M_0 - k$ leads to $h_i z^{-k}$ for each $k \in \mathbb{N}$, i.e. in the limit every f with a Laurent series expansion around 0 is in $\bigcup_{\alpha \in \mathbb{R}} \Phi(L)(U)_\alpha$ $\Rightarrow j_* \Phi(L) = \bigcup_{\alpha \in \mathbb{R}} \Phi(L)(U)_\alpha$.

ad (i)a: $\Phi(L)(U)_\alpha \supset \Phi(L)(U)_\beta$ for $\alpha \le \beta$: By ad (iv) we may assume that M is fixed - a change of M results in the desired inclusion. But then all $h = e^{M \log z} l$ differ from l by the same function and so $L_\alpha \supset L_\beta$ for $\alpha \le \beta$ implies the claim.

ad (iii): $\bigcap_{\beta < \alpha} \Phi(L)(U)_\beta = \Phi(L)(U)_\alpha$. Analogous to the previous item we may fix M by ad (iv) and use $\bigcap_{\beta < \alpha} L_\beta = L_\alpha$.

[18] Or the vector space spanned by L_α has always dimension less L, and so the union over all α has too small dimension.

[19] locally constant, see below!

ad (i)b The $\Phi(L)(U)_\beta$ are coherent: Although it is clear if the locally constant sheaf takes values in vector bundles, recall that the coefficients a_i in $l = \sum_{i=1}^n a_i l_i$, $l \in L$ are constant - $\nabla l = 0 \Leftrightarrow \sum_{i=1}^n l_i \otimes \partial \alpha_i = 0 \Leftrightarrow \partial a_i = 0, 1 \leq i \leq n$ by linear independence; analogous $\overline{\partial} a_i = 0$ by holomorphy. For every $\alpha \in \mathbb{R}$ we will always find a β small enough such that all (l_i) (resp. n-linear independent sections) are in L_β and a $k \in \mathbb{Z}$ such that $\Re\lambda_{M_0} - k \geq \alpha - \beta$. Choose k minimal. Then $h_i^{(k)} = e^{(M_0 - k\log(z))} l_i$ is a frame of $\Phi(L)(U)_\alpha$ by the linear independence and the minimality of k. Hence $\Phi(L)(U)_\alpha$ isomorphic to $\mathcal{O}^n_{\overline{X}}$.

Reg: Regularity of ∇: We have seen above that for $h \in \Phi(L)_\alpha$[20]

$$\begin{aligned}\nabla h = \nabla e^{M\log(z)} l &= \frac{M\,\mathrm{d}\,z}{z} e^{M\log(z)} l \\ &\in^{21}\ z^{-1}\Phi(L)_\alpha\,\mathrm{d}\,z = \Phi(L)_\alpha \otimes \Omega^{1,0}_{\overline{X}}(\log(s)).\end{aligned}$$

Hence we get a filtered regular $\mathscr{D}_X$-module $\Phi(L)$.

Let $L = \bigoplus_{i=1}^k L^{\mu_i}, k \leq n$ be the decomposition of L into generalized eigenspaces L^{μ_i} of μ to the eigenvalue μ_i. Remember that L_α was μ-invariant, $\mu : L_\alpha \to L_\alpha$ induces an eigenvalue decomposition $L_\alpha = \bigoplus_{i=1}^k L_\alpha^{\mu_i}$. Since $l \in L_\alpha^{\mu_i} \subset L \Rightarrow \exists r \in \mathbb{N} : (\mu - \mu_i E)^r l = 0 \Rightarrow L_\alpha^{\mu_i} \subset L^{\mu_i}$.

Lemma 2.3.3. $\Phi(\bigoplus_{i=1}^k L^{\mu_i}) = \bigoplus_{i=1}^k \Phi(L^{\mu_i})$.

Proof. Decompose $M = \bigoplus_{i=1}^k M^{\mu_i}$ as follows: Let $J = P^{-1}MP$ be the Jordan normal form and J^{μ_i} the block diagonal matrix consisting of all Jordan blocks to the eigenvalue μ_i. Then

$$J = \begin{pmatrix} J^{\mu_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J^{\mu_k} \end{pmatrix} \quad M^{\mu_i} := P^{-1} \begin{pmatrix} J^{\mu_1}\delta_{1i} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J^{\mu_k}\delta_{ki} \end{pmatrix} P,$$

i.e. the matrix constructed from only one Jordan block. The J^{μ_1} are just $\ln(\mu_i)E + \sum_{j=0}^\infty (-1)^j \frac{(\mu_i^j N)^{j+1}}{j+1}$, N nilpotent. Since M^{μ_i} is block diagonal we get (here for $i = 1$)

$$e^{M^{\mu_i}\log(z)} \quad = \quad P^{-1} \exp \begin{pmatrix} J^{\mu_1}\delta_{1i}\log(z) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J^{\mu_k}\delta_{ki}\log(z) \end{pmatrix} P$$

[20] The $\mathcal{O}_{\overline{X}}$-coefficients f have holomorphic differentials, i.e. $\partial f \in \mathcal{O}_{\overline{X}}, h \otimes \partial f \in \Phi(L)_\alpha$.

[21] $\log \mu$ is a (finite) sum of potentials of μ, i.e. Ml in L_β if $l \in L_\beta$. When we construct Ξ^{-1} we have to be careful with an inverse statement.

$$= \quad P^{-1} \begin{pmatrix} \exp(J^{\mu_1}\log(z)) & 0 & \dots & 0 \\ 0 & E & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & E \end{pmatrix} P.$$

Then $e^{M^{\mu_i}\log(z)}$ preserve L^{μ_i} and $e^{M^{\mu_i}\log(z)}l = 0$ for $l \in L^{\mu_i}, i \neq j$ - $e^{M^{\mu_j}\log(z)}l^{\mu_i} = e^{M^{\mu_j}\log(z)}l^{\mu_i}\delta_{ij}$ for $l^{\mu_i} \in L^{\mu_i}$. The eigenvalues of M^{μ_i} are just $\frac{\ln(\mu_i)}{-2\pi i}$. Now a choice of a M amounts to one unique choice of the logarithm in M^{μ_i} by construction. Of course a different branch of the logarithm in M^{μ_i} will change M. Finally using that M^{μ_i} and M^{μ_j} commute for $i \neq j$ since they are block diagonal and have different non-zero blocks we may decompose the matrix exponential: For every $l = \sum_i^k l^{\mu_i} \in \bigoplus_{i=1}^k L^{\mu_i}$

$$\begin{aligned} e^{M\log(z)}\left(\sum_i^k l^{\mu_i}\right) &= \sum_i^k \prod_{j=1}^k e^{M^{\mu_j}\log(z)}l^{\mu_i} \\ &= \sum_i^k \prod_{j=1}^k e^{M^{\mu_j}\log(z)}l^{\mu_i}\delta_{ij} \\ &= \sum_i^k e^{M^{\mu_i}\log(z)}l^{\mu_i}. \end{aligned}$$

Therefore $\Phi(\bigoplus_{i=1}^k L^{\mu_i}) = \bigoplus_{i=1}^k \Phi(L^{\mu_i})$ for the underlying space of the filtered bundle.

For the filtration we need to show that $\Phi(L)_\alpha = \bigoplus_{i=1}^k \Phi(L^{\mu_i})_\alpha$, where

$$\begin{aligned} \Phi(L^{\mu_i})_\alpha(U) = \{\mathcal{O}_{\overline{X}}|_U h : h = e^{M^{\mu_i}\log(z)}l^{\mu_i}(z), l^{\mu_i} \in L^{\mu_i}_\beta, \\ \lambda \text{ eigenvalues of } M^{\mu_i} \Rightarrow \Re(\lambda_{M^{\mu_i}}) \geq \alpha - \beta\}. \end{aligned}$$

But

$$\begin{aligned} \Phi(L)_\alpha(U) &= \{\mathcal{O}_{\overline{X}}|_U h : h = e^{M\log(z)}l(z), l \in L_\beta, \\ &\qquad \lambda \text{ eigenvalues of } M \Rightarrow \Re(\lambda_M) \geq \alpha - \beta\} \\ &= \{\mathcal{O}_{\overline{X}}|_U h : h = \sum_{i=1}^k e^{M^{\mu_i}\log(z)}l^{\mu_i}(z), l^{\mu_i} \in L^{\mu_i}_\beta, \\ &\qquad \lambda \text{ eigenvalues of } M^{\mu_i} \Rightarrow \Re(\lambda_{M^{\mu_i}}) \geq \alpha - \beta\} \\ &= \bigoplus_{i=1}^k \{\mathcal{O}_{\overline{X}}|_U h : h = e^{M^{\mu_i}\log(z)}l^{\mu_i}(z), l^{\mu_i} \in L^{\mu_i}_\beta, \\ &\qquad \lambda \text{ eigenvalues of } M^{\mu_i} \Rightarrow \Re(\lambda_{M^{\mu_i}}) \geq \alpha - \beta\} \end{aligned}$$

where the last step is due to the fact that a choice of M corresponds to a unique choice of a tuple $(M^{\mu_1}, \ldots, M^{\mu_k})$.

□

Conclusion 2.3.4. Note that we even proved the stronger result that Φ is compatible with a decomposition into $\mu-$invariant subspaces: The proof above obviously works for the decomposition into spaces corresponding to Jordan blocks. Let $L = W \oplus W'$ into invariant subspaces and J the space corresponding to a Jordan block, than $W \cap J$ and $W' \cap J$ are invariant subspace, i.e. $J = (W \cap J) \oplus (W' \cap J)$. But J does not decompose into non-trivial invariant subspaces (can be read of the form of a Jordan block). So $J \subset W$ or $J \subset W'$, i.e. the decomposition into Jordan blocks is finer and therefore Φ preserves the decomposition $L = W \oplus W'$.

The lemma allows us to consider only local systems with one eigenvalue for the rest of the proof. Let $\tilde{\lambda}$ be the unique eigenvalue of μ and let $\lambda_0 = \frac{\ln(\tilde{\lambda})}{-2\pi}$ an eigenvalue to a specific choice M_0. Define a vector subspace of $\Phi(L)$ by

$$H = e^{M \log(z)} L$$

and

$$H_\gamma = e^{M \log(z)} L_{\gamma - \Re(\lambda_0)}.$$

Then

$$\begin{aligned}
&\operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{z^k H_{\alpha-k} | k \in \mathbb{Z}\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{z^k e^{M_0 \log(z)} L_{\alpha-k-\Re(\lambda_0)} | k \in \mathbb{Z}\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{e^{(M_0+k)\log(z)} L_{\alpha-(k+\Re(\lambda_0))} | k \in \mathbb{Z}\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{e^M \log(z) l | l \in L_\beta \\
&= \bigcap_{\gamma<\beta} L_\gamma, M = \log(\mu)/(-2\pi i), \Re(\lambda_M) = \alpha - \beta\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{e^M \log(z) l | l \in L_\gamma, M = \log(\mu)/(-2\pi i), \Re(\lambda_M) \geq \alpha - \gamma\} \\
&= \Phi(L)_\alpha(U).
\end{aligned}$$

Instead of working with our original definition of the filtration $\Phi(L)_\alpha$ we may use the more compact one $\operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{z^k H_{\alpha-k} | k \in \mathbb{Z}\}$.

Lemma 2.3.5. Φ is compatible with taking tensor products.

Proof. Let L_1 be a filtered local system with monodromy μ_{L_1} with the eigenvalue $\tilde{\lambda}_{L_1}$; let L_2 be a filtered local system with monodromy μ_{L_2} with the eigenvalue $\tilde{\lambda}_{L_2}$. Let $L = L_1 \otimes L_2$ and the monodromy is $\mu = \mu_1 \otimes \mu_2$.[22] The eigenvalue of $\mu_1 \otimes \mu_2$ is $\tilde{\lambda} = \tilde{\lambda}_1 \tilde{\lambda}_2$[23] and we get $\lambda_0 = \log(\tilde{\lambda}) = \log(\lambda_1) + \log(\lambda_2)$. Recall the Kronecker sum $A \oplus B = A \otimes E + E \otimes B$ and $\exp(A \oplus B) = \exp(A) \otimes \exp(B)$ or

$$\log(\mu_1 \otimes \mu_2) \quad = \quad \log(\mu_1) \oplus \log(\mu_2) = M_1 \otimes E + E \otimes M_2,$$

with $M_1 = \log(\mu_1), M_2 = \log(\mu_2)$ as before. In particular

$$\exp(M \log z) \quad = \quad \exp(M_1 \log(z)) \otimes \exp(M_2 \log(z)).$$

Using $H = \exp(M \log z)L, H_1 = \exp(M_1 \log z)L_1, H_2 = \exp(M_2 \log z)L_2$ we get

$$H = H_1 \otimes H_2 \Rightarrow \Phi(L) = \Phi(L_1) \otimes \Phi(L_2).$$

for the underlying systems. Now consider the filtrations: Let $L_\alpha := \sum_{\beta \in \mathbb{R}} (L_1)_\beta \otimes (L_2)_{\alpha-\beta} = \bigcup_{\beta \in \mathbb{R}} (L_1)_\beta \otimes (L_2)_{\alpha-\beta}$.[24] This is in fact a filtration of the filtered local system L:

(i) $L_\alpha = \bigcup_{\beta \in \mathbb{R}} (L_1)_\beta \otimes (L_2)_{\alpha-\beta} \subset L_\gamma = \bigcup_{\beta \in \mathbb{R}} (L_1)_\beta \otimes (L_2)_{\gamma-\beta}$ for $\gamma \leq \alpha$ since $(L_2)_{\alpha-\beta} \subset (L_2)_{\gamma-\beta}$ for all β.

(ii) The tensor product of vector spaces is a vector space and since $(L_1)_{\max\{\beta,\alpha-\beta\}} \otimes (L_2)_{\max\{\beta,\alpha-\beta\}} \subset (L_1)_\beta \otimes (L_2)_{\alpha-\beta} \subset (L_1)_{\min\{\beta,\alpha-\beta\}} \otimes (L_2)_{\min\{\beta,\alpha-\beta\}}$ we get $L = \bigcup_{\alpha \in \mathbb{R}} L_\alpha = \bigcup_{\alpha \in \mathbb{R}} \bigcup_{\beta \in \mathbb{R}} (L_1)_\beta \otimes (L_2)_{\alpha-\beta}$.

(iii) For the intersection we have $\bigcap_{\alpha<\gamma} L_\gamma = \bigcap_{\alpha<\gamma} \bigcup_{\beta \in \mathbb{R}} (L_1)_\beta \otimes (L_2)_{\gamma-\beta} = \bigcup_{\beta \in \mathbb{R}} \bigcap_{\alpha<\gamma} (L_1)_\beta \otimes (L_2)_{\gamma-\beta} = \bigcup_{\beta \in \mathbb{R}} (L_1)_\beta \otimes (L_2)_{\alpha-\beta} = L_\alpha$.

(iv) all $L_\beta \otimes L_{\alpha-\beta}$ are $\mu_{L_1} \otimes \mu_{L_2}-$invariant and hence the union is.

Hence

$$\begin{aligned} H_\gamma &= e^{M \log(z)} L_{\gamma - \Re(\lambda_M)} = e^{M \log(z)} \sum_{\beta \in \mathbb{R}} (L_1)_\beta \otimes L_{\gamma - \Re(\lambda_M) - \beta} \\ &= e^{M_1 \log(z)} \otimes e^{M_2 \log(z)} \cdot \\ &\quad \cdot \sum_{\beta - \Re(\lambda_{M_1}) \in \mathbb{R}} (L_1)_{\beta - \Re(\lambda_{M_1})} \otimes L_{\gamma - \Re(\lambda_{M_1}) - \Re(\lambda_{M_2}) - (\beta - \Re(\lambda_{M_1}))} \\ &= \sum_{\beta \in \mathbb{R}} \left(e^{M_1 \log(z)} L_{\beta - \Re(\lambda_{M_1})} \right) \otimes \left(e^{M_2 \log(z)} \otimes L_{\gamma - \Re(\lambda_{M_2} - \beta)} \right) \end{aligned}$$

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[22] If γ_1 circle in L_1, γ_2 in L_2 then $\gamma_1 \otimes \gamma_2$ is circle in L.
[23] cf. Kronecker Product of Matrices and Applications [Ste91].
[24] Sum is justified by the finite dimension of the vector space.

$$= \sum_{\beta\in\mathbb{R}} (H_1)_\beta \otimes (H_2)_{\gamma-\beta}.$$

This implies the claim by

$$\begin{aligned}
\Phi(L)_\alpha(U) &= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{z^k H_{\alpha-k} | k\in\mathbb{Z}\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{z^k \sum_{\beta\in\mathbb{R}} (H_1)_\beta \otimes (H_2)_{\alpha-k-\beta} | k\in\mathbb{Z}\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{ \sum_{\beta-p\in\mathbb{R}} (z^p H_1)_{\beta-p} \otimes (z^{k-p} H_2)_{\alpha-k-(\beta-p)} | k\in\mathbb{Z}\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}|U}\{ \sum_{\beta\in\mathbb{R}} (z^p H_1)_{\beta-p} \otimes (z^{k-p} H_2)_{\alpha-\beta-(k-p)} | k\in\mathbb{Z}\} \\
&= \sum_{\beta\in\mathbb{R}} \Phi(L_1)_\beta(U) \otimes \Phi(L_1)_{\alpha-\beta}(U).
\end{aligned}$$

□

Conclusion 2.3.6. Φ is compatible with tensor products $\Phi(L)_\alpha = \bigoplus_{i=1}^k \Phi(L^{\mu_i})_\alpha = \bigoplus_{i=1}^k \sum_{\beta\in\mathbb{R}} \Phi(L_1^{\mu_i})_\beta \otimes \Phi(L_2^{\mu_i})_{\alpha-\beta} = \sum_{\beta\in\mathbb{R}} \Phi(L_1)_\beta \otimes \Phi(L_2)_{\alpha-\beta}$.

Lemma 2.3.7. Φ is compatible with taking the determinant.

Proof. Let (l_i) be a basis of L and $(l_{i_1} \otimes \ldots \otimes l_{i_n}), (i_1, \ldots, i_n) \in \{1, \ldots, n\}^n$ be the corresponding basis of L^n. Let $\mathcal{I}(L)$ denote the ideal generated by those basis element with $\exists j, k \in \{1, \ldots, n\} : i_j = i_k$. Analogously let $h_i = e^{M\log(z)} l_i$ be the corresponding basis of $\Phi(L)$, $(h_{i_1} \otimes \ldots \otimes h_{i_n}), (i_1, \ldots, i_n) \in \{1, \ldots, n\}^n$ the basis of $\Phi(L)^n$, and $\mathcal{I}(\Phi(L))$ the ideal generated by the elements with $\exists j, k \in \{1, \ldots, n\}$: $i_j = i_k$. We want to show that $\Phi(\mathcal{I}(L)) = \mathcal{I}(\Phi(L))$, but for $(i_1, \ldots, i_n) \in I$ index set, the compatibility of Φ with tensor products leads us to

$$\begin{aligned}
\Phi(\operatorname{span}_{\mathcal{O}_X,I}\{(l_{i_1} \otimes \ldots \otimes l_{i_n})\}) &= \operatorname{span}_{\mathcal{O}_X,I}\{e^{\log(z)M\oplus\ldots\oplus M} l_{i_1} \otimes \ldots \otimes l_{i_n}\} \\
&= \operatorname{span}_{\mathcal{O}_X,I}\{e^{\log(z)M} l_{i_1} \otimes \ldots \otimes e^{\log(z)M} l_{i_n}\} \\
&= \operatorname{span}_{\mathcal{O}_X,I}\{h_{i_1} \otimes \ldots \otimes h_{i_n}\}.
\end{aligned}$$

Now if I is the set of all families with at least $i_j = i_k$ for some $j \neq k$ this reads $\Phi(\mathcal{I}(L)) = \mathcal{I}(\Phi(L))$ (underlying bundles). Remember that at the beginning of the proof of compatibility with tensor products, we saw that $\Phi(L)$ is generated by the h_i independent of the chosen basis or matrix M.
For the filtrations this works analogously: Let $\mathcal{I}(L)_\alpha = L^n_\alpha \cap \mathcal{I}(L)$ (vector space)[25] be the induced filtration on $\mathcal{I}(L)$. Then for $l = (l_1, \ldots, l_n) \in (L^n)_\beta =$

[25]monodromy invariant: $\mu \otimes \ldots \otimes \mu$ maps a tensor with two equal entries to another one with this property.

$\sum_{r_1+\ldots+r_n=\alpha} L_{r_1} \otimes \ldots\ldots \otimes L_{r_n}$

$$\begin{aligned}
\Phi(\mathcal{I}(L))_\alpha &= \operatorname{span}_{\mathcal{O}_{\overline{X}}}\{\bigotimes_{i=1}^{n} e^{M_i \log(z)} l_i | l \in L_\gamma^n \cap \mathcal{I}(L), M_i = \log(\mu)/(-2\pi i), \\
&\quad \Re(\lambda_{M_i}) \geq \alpha - \gamma\} \\
&= \operatorname{span}_{\mathcal{O}_{\overline{X}}}\{\bigotimes_{i=1}^{n} e^{M_i \log(z)} l_i | l \in L_\gamma^n, M_i = \log(\mu)/(-2\pi i), \\
&\quad \Re(\lambda_{M_i}) \geq \alpha - \gamma\} \cap \operatorname{span}_{\mathcal{O}_X}\{\bigotimes_{i=1}^{n} e^{M_i \log(z)} l_i | l \in \mathcal{I}(L), \\
&\quad M_i = \log(\mu)/(-2\pi i)\} \\
&= \Phi(L^n)_\alpha \cap j_* \Phi(\mathcal{I}(L)) = \Phi(L^n)_\alpha \cap j_* \mathcal{I}\Phi(L) =: \mathcal{I}(\Phi(L))_\alpha.
\end{aligned}$$

The exterior product is $\bigwedge_{i=1}^n L = L^n/\mathcal{I}(L), \bigwedge_{i=1}^n \Phi(L) = (\Phi(L))^n/\mathcal{I}(\Phi(L)) = \Phi(L^n)/\Phi(\mathcal{I}(L)) = \Phi(\bigwedge_{i=1}^n L) \Rightarrow \Phi(\det(L)) = \det(\Phi(L))$, where we used that $\Phi(\bigwedge_{i=1}^n L \otimes \mathcal{I}(L)) = \Phi(\bigwedge_{i=1}^n L) \otimes \Phi(\mathcal{I}(L))$ is guaranteed by 2.3.4 and the μ invariance of $I(L)_\alpha$ and $\bigwedge_{i=1}^n L$ (μ invertible $\Rightarrow \{\mu l_{i_1} = \mu l_{i_2} \Leftrightarrow l_{i_1} = l_{i_2}\}$). □

Lemma 2.3.8. Φ is compatible with taking duals.

Proof. Since duals and direct sums commute, we may restrict again to local systems with one eigenvalue $\tilde{\lambda}$. The filtration of L^* is $(L^*)_\beta = (L_{-\beta})^*$. Check the properties of a filtration:

(i) Let $L_\beta \subset L_\alpha, \beta \geq \alpha$. Then every linear form $f : L_\alpha \to \mathbb{C}$ restricts to a linear form $L_\beta \to \mathbb{C} \Rightarrow (L^*)_{-\alpha} = L_\alpha^* \subset L_\beta^* = (L^*)_{-\beta}, -\alpha \geq -\beta$.

(ii) $L = \bigcup_{\beta\in\mathbb{R}} L_\beta \Rightarrow L^* = \left(\bigcup_{\beta\in\mathbb{R}} L_\beta\right)^* = \bigcup_{\beta\in\mathbb{R}} L_\beta^* = \bigcup_{\beta\in\mathbb{R}} (L^*)_{-\beta} = \bigcup_{\beta\in\mathbb{R}} (L^*)_\beta$.

(iii) $(L^*)_\alpha = L_{-\alpha}^* = \left(\bigcap_{\beta<-\alpha} L_\beta\right)^* = \bigcap_{\beta>-\alpha} (L_\beta)^* = \bigcap_{-\beta>-\alpha} L_{-\beta}^* = \bigcap_{\beta<\alpha} L_{-\beta}^* = \bigcap_{\beta<\alpha} (L^*)_\beta$.

(iv) The transition functions of the dual bundle are $(\psi_{ji}^{-1})_{ij}$ of those of the original bundle - ψ_{ij}; hence $(\mu^T)^{-1}$ is the monodromy of the dual bundle if μ was the monodromy of the original bundle.
The dual bundle $(\Phi(L)^*)$ is the bundle where each stalk is the dual of the corresponding stalk of $\Phi(L)$. Hence we may as well define $(\Phi(L)^*)_\beta = (\Phi(L)_{-\beta})^*$. The properties follow as above[26], only note that $z(\Phi(L)^*)_\beta =$

[26] That the dual of a coherent sheaf is coherent follows directly from the definition of a coherent sheaf.

$(\Phi(L)^*)_{\beta+1}$ since if $f^*(f) = 1 \Rightarrow f^*(zf) = z \Rightarrow z^{-1}f^*(fz) = 1 \Rightarrow z^{-1}f^* = (zf)^* \Rightarrow z(\Phi(L)^*)_\beta = (z^{-1}\Phi(L)_{-\beta})^* = (\Phi(L)_{-\beta-1})* = (\Phi(L)^*)_{\beta+1}$.
Further note that $(B^T)^{-1}(l_i)^* = (Bl_i)^*$: Let $l'_i = Bl_i = \sum_{j=1}^n B_{ji}l_j$ be the new basis and $(B^T)^{-1}l_i = \sum_{j=1}^n (B^{-1})_{ij}l_j$. Hence $B(l_i)^*(l_j) = \sum_{k=1}^n B_{ki}l_k^*(l_j) = B_{ji}$ and $((B^T)^{-1}l_i)^*(l_j) = ((B^T)^{-1}l_i)^*((B^T)(B^T)^{-1}l_j) = (b'_i)^*(B^Tb'_j) = (b'_i)^*(\sum_{k=1}^n B_{jk}b'_k) = \sum_{k=1}^n B_{jk}(b'_i)^*(b'_k) = B_{ji}$.

The logarithm of $(\mu^T)^{-1}$ is $M = \frac{\log((\mu^T)^{-1})}{-2\pi i} = -\frac{\log(\mu)^T}{-2\pi i}$. Let (l_i) be a basis of L and (l_i^*) be the dual basis of L^*. Then $h_i^* = e^{-M^T\log(z)}l_i^*$ is the basis of $\Phi(L^*)$. On the other hand $(h_i)^* = (e^{M\log(z)}l_i)^* = e^{-M^T\log(z)}l_i^* = h_i^*$. Therefore $\Phi(L^*) = \Phi(L)^*$ for the base spaces.
Let $l \in L_\beta$, then $h = e^{M\log(z)}l \in \Phi(L)_{\Re(\lambda_M)+\beta}$ by left-continuity. For l^* we know $l^* \in (L^*)_{-\beta}$ and $h^* = e^{-M^T\log(z)}l \in \Phi(L)_{-\Re(\lambda_{M^T})-\beta} = ((\Phi(L))^*)_{\Re(\lambda_{M^T})+\beta}$. Thus the dual filtration is induced by the dual sections, i.e. Φ compatible with taking dual. □

Lemma 2.3.9. Φ is invertible/essentially surjective.

Proof. We will proceed in a different fashion than Simpson does. Although his construction leads to the same result, the bundles constructed there are not so easy to work with a priori, i.e. before we find a good-looking representation.
Let M denote a logarithm of the monodromy corresponding to ∇ divided by $-2\pi i$.

$$\mathscr{E}_\gamma = \mathrm{span}_{\mathcal{O}_{\overline{X}}}\{f = e^{-M(\log z+2\pi ik)}h | k \in \mathbb{Z}, h \in E_\alpha, -\Re\lambda_M \geq \gamma - \alpha\}.$$

We want to show that $(E, \mathscr{E}_\gamma)$ is a filtered regular $\mathscr{D}_X$−module. But this is just the first part of the proof of our theorem 2.3.2. There we did the proof for a particular branch of $\log(z)$[27] and not for all of them. However, the union over all k will not interfere with the union over all indices, the inclusion, the coherence or $z\mathscr{E}_\gamma = \mathscr{E}_{\gamma+1}$. Same holds for the left-continuity, i.e. the intersection, since k and γ are independent of each other. We still need to show that ∇ preserves the filtration. Note that the proof above uses $\nabla l = 0$ and is not applicable here. However, we know that ∇ preserves E_α, i.e. for $f = e^{-M\log z}h \in \mathscr{E}_\gamma$:

$$\begin{aligned}
\nabla f &= \underbrace{(\nabla h)}_{\in z^{-1}E_\alpha} \otimes e^{-M(\log z+2\pi ik)} + h \otimes \frac{-M\,\mathrm{d}\,z e^{-M(\log z+2\pi ik)}}{z} \\
&= e^{-M(\log z+2\pi ik)} \underbrace{z\nabla h}_{\in E_\alpha} \otimes z^{-1} + e^{-M(\log z+2\pi ik)} h \otimes \frac{-M\,\mathrm{d}\,z}{z} \\
&\in \mathscr{E}_\gamma \otimes \Omega^{1,0}_{\overline{X}}(\log(s)).
\end{aligned}$$

[27] $\log z + 2\pi ik$ the k-th branch of the logarithm.

Define $\Phi^{-1}(E, E_\alpha) = (E^\nabla, \mathscr{E}_\alpha^\nabla)$ the corresponding sheaves of $\nabla-$flat holomorphic sections. We know already that E^∇ is a local system. The other properties of a filtered local system follow directly from the filtration of $\mathscr{E}_\gamma$:

(a) $\mathscr{E}_\gamma^\nabla := E^\nabla \cap \mathscr{E}_\gamma \subset E^\nabla \cap \mathscr{E}_\beta = \mathscr{E}_\beta^\nabla$ for $\beta \leq \gamma$.

(b)

$$\begin{aligned}\bigcup_{\gamma\in\mathbb{R}} \mathscr{E}_\gamma^\nabla &= \bigcup_{\gamma\in\mathbb{R}} E^\nabla \cap \mathscr{E}_\gamma \\ &= E^\nabla \cap \bigcup_{\gamma\in\mathbb{R}} \mathscr{E}_\gamma = E^\nabla \cap E \\ &= E^\nabla.\end{aligned}$$

(c)

$$\begin{aligned}\bigcap_{\beta<\gamma} \mathscr{E}_\beta^\nabla &= \bigcap_{\beta<\gamma} E^\nabla \cap \mathscr{E}_\beta \\ &= E^\nabla \cap \bigcap_{\beta\gamma} \mathscr{E}_\beta = E^\nabla \cap \mathscr{E}_\gamma \\ &= E_\gamma^\nabla.\end{aligned}$$

(d) The monodromy invariance is the only say interesting point, because we can use that we allowed any branch of the logarithm. Assume that $f \in E_\gamma^\nabla$. Then $\mu f = \mu e^{-M(\log(z)+2\pi k)} h = e^{-2\pi M} e^{-M(\log(z)+2\pi k)} h = e^{-M(\log(z)+2\pi(k+1))} h = f \in E_\gamma^\nabla$.

So we know now that $\Phi^{-1}(E)$ is a filtered local system. We still need to show that the two constructions are inverse to each other.
Let $E = \Phi(L)$: $f = e^{-M\log z} h \in \mathscr{E}_\gamma \Rightarrow \exists l \in L_\beta, M$ and $\tilde{M}$ logarithm of the monodromy μ such that

$$f = e^{-M(\log(z)+2\pi k i)} e^{\tilde{M}\log z} l, \Re\lambda_{\tilde{M}} \geq \alpha - \beta \geq \gamma + \Re\lambda_M - \beta.$$

But two branches of M differ by an integer j: $\tilde{M} = jE + M \Rightarrow j + \beta \geq \gamma$[28] and hence

$$\nabla f = \nabla e^{jE\log(z)} e^{-2\pi i k M} l = \frac{jE \,\mathrm{d}\, z}{z} f \in \mathscr{E}_\gamma \otimes \Omega_{\overline{X}}^{1,0}(\log(z)),$$

[28] In particular $\tilde{M}M = (jE+M)M = jM + M^2 = M(j+M) = M\tilde{M} \Rightarrow M, \tilde{M}$ commute.

where we used that $e^{-2\pi ikM}l = \mu^k l$ still flat. So f is flat if and only if $j = 0$ or $f = 0$ (trivial case). For $j = 0$

$$\Phi^{-1}(\Phi(L)) = \text{span}_{\mathbb{C}}\{l | l \in L_\beta : \beta \geq \gamma\} = L_\gamma,$$

as $L_\beta \subset L_\gamma$ for $\beta \geq \gamma$. We get $\Phi^{-1}(\Phi(L)) = L$. In particular Φ^{-1} surjective.
On the other hand let $L = \Phi^{-1}(E)$. Then for $h \in \Phi(L)_\alpha$ $\exists M, \tilde{M}$ logarithm of $\mu, k \in \mathbb{Z}$ $\tilde{h} \in E_\beta$:

$$h = e^{M\log(z)}e^{-\tilde{M}(\log(z)+2\pi ki)}\tilde{h}, \Re\lambda_M \geq \alpha - \gamma \geq \alpha + \Re\lambda_{\tilde{M}} - \beta.$$

We know that $\nabla e^{-M\log(z)}h = \nabla e^{-\tilde{M}(\log(z)+2\pi ki)}\tilde{h} = 0 \Rightarrow e^{-M\log(z)}h \in L_{\beta-\Re\lambda_{\tilde{M}}}$. The last step becomes

$$\begin{aligned}
e^{M\log(z)}\mathscr{E}_\gamma &= \text{span}_{\mathcal{O}_{\overline{X}}}\{e^{-\tilde{M}\log(z)-2\pi ik\tilde{M}}\tilde{h} | \Re\lambda_{\tilde{M}} \geq \gamma - \beta, \tilde{h} \in E_\beta\} \\
&= \text{span}_{\mathcal{O}_{\overline{X}}}\{e^{jE\log(z)-2\pi il\tilde{M}}\tilde{h} | \Re\lambda_{\tilde{M}} \geq \gamma - \beta, \tilde{h} \in E_\beta\} \\
&= \text{span}_{\mathcal{O}_{\overline{X}}}\{e^{-2\pi il\tilde{M}}z^j E\tilde{h} | -\Re\lambda_{\tilde{M}} \geq \gamma - \beta, \tilde{h} \in E_\beta\} \\
&= e^{-2\pi il\tilde{M}}\text{span}_{\mathcal{O}_{\overline{X}}}\{\tilde{h} \in E_{\beta+j} | -\Re\lambda_{\tilde{M}} \geq \gamma - \beta - j + j \\
&\qquad\qquad = \gamma - \beta + \Re\lambda_M - \Re\lambda_{\tilde{M}}\} \\
&= e^{-2\pi il\tilde{M}}\text{span}_{\mathcal{O}_{\overline{X}}}\{\tilde{h} | -\Re\lambda_M \geq \gamma - \beta, \tilde{h} \in E_\beta\} \\
&= e^{-2\pi il\tilde{M}}E_{\gamma+\Re\lambda_M}.
\end{aligned}$$

Note l variable, i.e. the union over all l here. Thus

$$\begin{aligned}
h &\in e^{-2\pi il\tilde{M}}E_{\beta-\Re\lambda_{\tilde{M}}+\Re\lambda_M} \cap e^{-2\pi ik\tilde{M}}e^{-\tilde{M}\log(z)}e^{M\log(z)}E_\beta \\
&= e^{-2\pi il\tilde{M}}E_{\beta-\Re\lambda_{\tilde{M}}+\Re\lambda_M} \cap e^{-2\pi ik\tilde{M}}z^j E_\beta \\
&= e^{-2\pi il\tilde{M}}E_{\beta+j} \cap e^{-2\pi ik\tilde{M}}E_{\beta+j} \\
\Rightarrow h &= \mu^k E_{\beta+j}.
\end{aligned}$$

We get $\Phi(\Phi^{-1}(E))_\alpha$ is $\mu-$invariant, since $\Phi^{-1}(E)_\beta$ as filtered local system is, i.e. for every $h \in \Phi(\Phi^{-1}(E))_\alpha$ $\exists\hat{h} \in \Phi(\Phi^{-1}(E))_\alpha, \exists k_h \in \mathbb{Z} : \hat{h} = \mu^{-k_h}h \in E_{\beta+j}$

$$\begin{aligned}
\mu^l\Phi(\Phi^{-1}(E)) &= \{\hat{h} \in E_{\beta-\Re\lambda_{\tilde{M}}+\Re\lambda_M} : \Re\lambda_M \geq \alpha + \Re\lambda_{\tilde{M}} - \beta\} \\
&= \{\hat{h} \in E_\beta : \beta \geq \alpha\} = E_\alpha.
\end{aligned}$$

Hence $\Phi(\Phi^{-1}(E)) = E$ for every filtered $\mathscr{D}_X-$module. This finishes our proof.

□

Remark 2.3.10. Note that for a μ with only one eigenvalue, this implies that $l \in L_\beta$ if and only if $e^{M\log z}l \in \Phi(L)_{\beta+\Re(\lambda_M)}$ for some M.

Lemma 2.3.11. Φ is fully faithful.

Proof. Let $\varphi : L_1 \to L_2$ be a morphism of filtered local systems. Then $\Phi(\varphi) : \Phi(L_1) \to \Phi(L_2)$ the $\mathcal{O}_X-$linear extension of φ.[29] $\Phi(\varphi)$ preserves the filtration: Since $\varphi(L_1)_\beta \subset (L_2)_\beta \Rightarrow \varphi(e^{M\log(z)}L_1)_\beta = e^{M\log(z)}\varphi(L_1)_\beta \subset e^{M\log(z)}(L_2)_\beta \Rightarrow \Phi(L_1)_\alpha \subset \Phi(L_2)$. Further

$$\begin{aligned}
(\nabla_{\Phi(L_1)}\varphi)(l\otimes\alpha) &= \nabla_{\Phi(L_1)}\varphi(l)\otimes\alpha \\
&= \underbrace{(\nabla_{\Phi(L_1)}\varphi(l))}_{=0}\otimes\alpha + \varphi(l)\otimes\partial\alpha \\
&= \varphi(l)\otimes\partial\alpha \\
&= \varphi(l\otimes\partial\alpha) \\
&= \varphi(\underbrace{\nabla_{\Phi(L_2)}l}_{=0}\otimes\alpha + l\otimes\partial\alpha) \\
&= (\varphi\nabla_{\Phi(L_2)})(l\otimes\alpha) \\
\Rightarrow \nabla_{\Phi(L_1)}\varphi &= \varphi\nabla_{\Phi(L_2)}.
\end{aligned}$$

and by D''-holomorphy of the flat sections

$$\begin{aligned}
(D''_{\Phi(L_1)}\varphi)(l\otimes\alpha) &= D''_{\Phi(L_1)}\varphi(l)\otimes\alpha \\
&= \underbrace{(D''_{\Phi(L_1)}\varphi(l))}_{=0}\otimes\alpha + \varphi(l)\otimes\partial\alpha \\
&= \varphi(l)\otimes\partial\alpha \\
&= \varphi(l\otimes\partial\alpha) \\
&= \varphi(\underbrace{D''_{\Phi(L_2)}l}_{=0}\otimes\alpha + l\otimes\partial\alpha) \\
&= (\varphi D''_{\Phi(L_2)})(l\otimes\alpha) \\
\Rightarrow D''_{\Phi(L_1)}\varphi &= \varphi D''_{\Phi(L_2)}.
\end{aligned}$$

Hence φ is a morphism of filtered regular $\mathscr{D}_X-$modules. Analogously if we start with a morphism of filtered regular $\mathscr{D}_X$-bundles $\psi : \Phi(L_1) \to \Phi(L_2)$, ψ restricts to the subset of flat sections. Then ψ is $\mathcal{O}_X-$linear as a morphism of coherent sheaves. Therefore for $l \in (L_1)_\beta$ we get $\nabla_{\Phi(L_2)}\psi(l) = \psi\nabla_{\Phi(L_1)}l = 0, D''_{\Phi(L_2)}\psi(l) = \psi D''_{\Phi(L_1)}l = 0$, i.e. $\psi(l)$ flat. Furthermore we now that from the invertibility

$$\begin{aligned}
\psi(l) &= \psi(e^{-M\log(z)}e^{M\log(z)}l) \\
&= e^{-M\log(z)}\psi(\underbrace{e^{M\log(z)}l}_{\Phi(L_1)_{\Re\lambda_M+\beta}})
\end{aligned}$$

[29] There is a flat frame, so φ is fully determined.

$$\in \quad \Phi^{-1}(\Phi(L))_{-\Re\lambda_M+\Re\lambda_M+\beta} = \Phi^{-1}(\Phi(L))_\beta = L_\beta.$$

Thus ψ preserves the filtration. □

Φ is a fully-faithful essentially surjective functor, i.e. an equivalence of categories. Hence theorem 2.3.2 is shown. □

3 Sections and Morphisms

The aim of this chapter is to show that Ξ is fully faithful. In order to prove this we start with a few technical lemmas, resulting in two Weitzenböck formulas, i.e. a relation between pseudo-curvature G_h and the curvatures F_h resp. R_h. The second step will provide the mutual boundedness of metrics that induce the same filtration under Ξ. Combining the two results shows Ξ fully faithful. So for the last two chapters we are left with the task to show essential surjectivity.

Lemma 3.0.1. Let $(E, h, \theta, \overline{\partial}_E)$ be a holomorphic bundle (not necessarily harmonic resp. the Higgs field not necessarily holomorphic). Write $D = \partial_E + \theta + \overline{\partial}_E + \theta^\dagger = D' + D''$ a metric connection, $d'' := \overline{\partial}_E + \theta$.

(i) For a $D-$flat section e, $D(e) = 0$:

$$-\Delta \log \|e\|_h^2 \ \leq \ -\frac{h(e, 2i\Lambda(F_h - 2G_h)e)}{\|e\|_h^2} \leq 2\|\Lambda(F_h - 2G_h)\|_F,$$

for $F_h = D^2$ the curvature of D and $G_h = (d'')^2 = \theta\overline{\partial}_E + \overline{\partial}_E\theta$ the pseudo-curvature.

(ii) For $d''-$flat section e, $d''(e) = 0$:

$$-\Delta \log \|e\|_h^2 \ \leq \ -\frac{h(e, 2i\Lambda(F_h - G_h - \overline{G}_k)e)}{\|e\|_h^2} \leq 2\|\Lambda(F_h - G_h - \overline{G}_k)\|_F$$

Proof. Ad (i): $D(e) = 0 \Rightarrow D'(e) = 0, D''(e) = 0.$

$$\begin{aligned}
\partial\overline{\partial}\|e\|_h^2 &= \partial\overline{\partial}h(e,e) \\
&= \partial\left(h(D''e, e) + h(e, \delta'(e))\right) \\
&= h(\delta'(e), \delta'(e)) + h(e, D''\delta'(e)) \\
&= h(\delta'(e), \delta'(e)) + h(e, (D''\delta' + \delta' D'')e)
\end{aligned}$$

Remember $\delta' = D' - 2\theta$ and hence

$$\begin{aligned} D''\delta' + \delta' D'' &= (D'D'' + D''D') - 2(\theta D'' + D''\theta) \\ &= D^2 - 2(\theta\overline{\partial}_E + \overline{\partial}_E\theta) - 2(\theta^\dagger\theta + \theta\theta^\dagger) \\ &= F_h - 2G_h - 2(\theta^\dagger\theta + \theta\theta^\dagger). \end{aligned}$$

Moreover $\delta'(e) = D'(e) - 2\theta e = -2\theta e$:

$$\begin{aligned} \Rightarrow \partial\overline{\partial}\|e\|_h^2 &= h(\delta'(e), \delta'(e)) + h(e, (D''\delta' + \delta' D'')e) \\ &= 4h(\theta(e), \theta(e)) + h(e, (F_h - 2G_h)e) - 2h(e, \theta\theta^\dagger e) - 2h(e, \theta^\dagger\theta e) \\ &= 4h(\theta(e), \theta(e)) + h(e, (F_h - 2G_h)e) - 2h(\theta^\dagger e, \theta^\dagger e) - 2h(\theta e, \theta e) \\ &= 2h(\theta_z(e), \theta_z(e))\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} + h(e, (F_h - 2G_h)e) \\ &\quad -2h(\theta_z^\dagger e, \theta_z^\dagger e)\,\mathrm{d}\,\overline{z} \wedge \mathrm{d}\,z \\ &= 2\|\theta_z(e)\|_h^2\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} + h(e, (F_h - 2G_h)e) + 2\|\theta_z^\dagger e\|_h^2\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}. \end{aligned}$$

Using 1.4.14 we may write

$$\begin{aligned} -\Delta\|e\|_h^2 &= -2i\Lambda h(e, (F_h - 2G_h)e) - 8\|\theta_z e\|_h^2 - 8\|\theta_z^\dagger e\|_h^2 \\ &= -h(e, 2i\Lambda(F_h - 2G_h)e) - 8\|\theta_z e\|_h^2 - 8\|\theta_z^\dagger e\|_h^2. \end{aligned} \tag{W1}$$

Remark 3.0.2. This is a Weitzenböck formula: In general a relation between two second order differential operators. See [GH78].

Now the logarithm comes into play:

$$\begin{aligned} \Delta \log \|e\|_h^2 &= 4\partial_{\overline{z}}\left(\frac{\partial_z(\|e\|_h^2)}{\|e\|_h^2}\right) \\ &= \left(\frac{\|e\|_h^2\Delta(\|e\|_h^2) - 4\partial_z\|e\|_h^2\partial_{\overline{z}}\|e\|_h^2}{\|e\|_h^4}\right) \\ &= \frac{\Delta(\|e\|_h^2)}{\|e\|_h^2} - 4\frac{\partial_z\|e\|_h^2\partial_{\overline{z}}\|e\|_h^2}{\|e\|_h^4} \\ &= \frac{\Delta(\|e\|_h^2)}{\|e\|_h^2} - 4\frac{i}{2}\Lambda\frac{\partial\|e\|_h^2 \wedge \overline{\partial}\|e\|_h^2}{\|e\|_h^4}. \end{aligned} \tag{3.0.2.1}$$

Use again that $\partial h(e,e) = h(\delta' e, e) + \underline{h(e, D''e)} = h(\delta' e, e)$ and $\overline{\partial} h(e,e) = h(e, \delta' e) + h(D''e, e) = h(e, \delta' e)$ and $h(e, \delta' e) = \overline{h(\delta' e, e)}$:

$$\begin{aligned} \frac{\partial\|e\|_h^2 \wedge \overline{\partial}\|e\|_h^2}{\|e\|_h^4} &= \frac{|h(\delta' e, e)|^2}{\|e\|_h^4} \\ &= \frac{4|h(\theta e, e)|^2}{\|e\|_h^4} \end{aligned}$$

$$= \frac{2|h(\theta e, e)|^2 + 2|h(\theta^\dagger e, e)|^2}{\|e\|_h^4}.$$

by $|h(\theta e, e)|^2 = |h(e, \theta^\dagger)|^2 = |\overline{h(\theta^\dagger e, e)}|^2 = |h(\theta^\dagger e, e)|^2$. By Cauchy-Schwarz $|h(\theta e, e)|^2 \leq |h(\theta e, \theta e)|^2 |h(e, e)|^2 = \|\theta e\|_h^2 \|e\|_h^2$ and further

$$\begin{aligned}
\Delta \log \|e\|_h^2 &\geq \frac{\Delta(\|e\|_h^2)}{\|e\|_h^2} - 4\frac{i}{2}\Lambda \frac{2\|e\|_h^2(\|\theta e\|_h^2 + \|\theta^\dagger e\|_h^2)}{\|e\|_h^4} \\
&= \frac{h(e, 2i\Lambda(F_h - 2G_h)e) + 8\|\theta_z e\|_h^2 + 8\|\theta_z^\dagger e\|_h^2}{\|e\|_h^2} \\
&\quad - \frac{8(\|\theta_z e\|_h^2 + 8\|\theta_z^\dagger e\|_h^2)}{\|e\|_h^2} \\
&= \frac{h(e, 2i\Lambda(F_h - 2G_h)e)}{\|e\|_h^2}.
\end{aligned}$$

Negating the inequality and applying Cauchy-Schwarz again

$$\begin{aligned}
-\Delta \log \|e\|_h^2 &\leq -\frac{h(e, 2i\Lambda(F_h - 2G_h)e)}{\|e\|_h^2} \leq \left| \frac{h(e, -2i\Lambda(F_h - 2G_h)e)}{\|e\|_h^2} \right| \\
&\leq \frac{\|e\|_h \|2\Lambda(F_h - 2G_h)e\|_h}{\|e\|_h^2} \leq \frac{\|e\|_h \|2\Lambda(F_h - 2G_h)\|_F \|e\|_h}{\|e\|_h^2} \\
&= 2\|\Lambda(F_h - 2G_h)\|_F.
\end{aligned}$$

Ad (ii): The proof is similar to the one of (i). Start with $d''(e) = (\overline{\partial}_E + \theta)(e) = 0 \Rightarrow \overline{\partial}_E(e) = \theta(e) = 0$ by degree considerations. Then

$$\begin{aligned}
\partial\overline{\partial}\|e\|_h^2 &= \partial\overline{\partial} h(e, e) \\
&= \partial\left(h(\overline{\partial}_E e, e) + h(e, \partial_E(e))\right) \\
&= h(e, \overline{\partial}_E \partial_E(e)) + h(\partial_E(e), \partial_E(e)) \\
&= h(e, (\overline{\partial}_E \partial_E + \partial_E \overline{\partial}_E)e) + h(\partial_E(e), \partial_E(e)).
\end{aligned}$$

Next

$$\begin{aligned}
\overline{\partial}_E \partial_E + \partial_E \overline{\partial}_E &= D^2 - (\theta\theta^\dagger + \theta^\dagger\theta + \overline{\partial}_E\theta + \theta\overline{\partial}_E + \theta^\dagger\partial_E + \partial_E\theta^\dagger) \\
&= F_h - G_h - \overline{G}_h - \theta\theta^\dagger - \theta^\dagger\theta.
\end{aligned}$$

implies

$$\begin{aligned}
h(e, (\overline{\partial}_E \partial_E + \partial_E \overline{\partial}_E)e) &= h(e, (F_h - G_h - \overline{G}_h - \theta\theta^\dagger + \theta^\dagger\theta)e) \\
&= h(e, (F_h - G_h - \overline{G}_h)e) - h(e, \theta\theta^\dagger e) - h(e, \theta^\dagger \underbrace{\theta e}_{=0})
\end{aligned}$$

$$= h(e, (F_h - G_h - \overline{G}_h)e) - h(\theta^\dagger e, \theta^\dagger e).$$

and hence

$$\begin{aligned}
\Rightarrow -\Delta\|e\|_h^2 &= -2i\Lambda\left(h(\partial_E(e), \partial_E(e)) + h(e, (F_h - G_h - \overline{G}_h)e) - h(\theta^\dagger e, \theta^\dagger e)\right)\\
&= -2i\Lambda\left(h((\partial_E + \theta^\dagger)e, (\partial_E + \theta^\dagger)e) + h(e, (F_h - G_h - \overline{G}_h)e)\right)\\
&= -2i\Lambda\|(\partial_E + \theta^\dagger)e\|_h^2 + h(e, -2i\Lambda(F_h - G_h - \overline{G}_h)e). \qquad \text{(W2)}
\end{aligned}$$

where we used $\partial_E\theta^\dagger = 0$ of degree $(2,0)$. Again we can use 3.0.2.1 and $\partial h(e,e) = h(\partial_E e, e) + h(e, \overline{\partial}_E e) = h(\partial_E e, e)$ resp. $\overline{\partial} h(e,e) = h(e, \partial_E e) = \overline{h(\partial_E e, e)}$:

$$\begin{aligned}
2i\Lambda\frac{\partial\|e\|_h^2 \wedge \overline{\partial}\|e\|_h^2}{\|e\|_h^4} &= 2i\Lambda\frac{|h(\partial_E e, e)|^2}{\|e\|_h^4}\\
&\le 2i\Lambda\frac{\|\partial_E e\|_h^2\|e\|_h^2}{\|e\|_h^4} = 2i\Lambda\frac{\|\partial_E e\|_h^2}{\|e\|_h^2}\\
&\le 2i\Lambda\frac{\|\partial_E e\|_h^2 + \|\theta e\|_h^2}{\|e\|_h^2} = 2i\Lambda\frac{\|(\partial_E + \theta)e\|_h^2}{\|e\|_h^2}\\
\Rightarrow \Delta\log\|e\|_h^2 &= \frac{\Delta(\|e\|_h^2)}{\|e\|_h^2} - 4\frac{i}{2}\Lambda\frac{\partial\|e\|_h^2 \wedge \overline{\partial}\|e\|_h^2}{\|e\|_h^4}\\
&\ge \frac{2i\Lambda\|(\partial_E + \theta^\dagger)e\|_h^2 + h(e, 2i\Lambda(F_h - G_h - \overline{G}_h)e)}{\|e\|_h^2}\\
&\quad -2i\Lambda\frac{\|(\partial_E + \theta)e\|_h^2}{\|e\|_h^2}\\
&\ge \frac{h(e, 2i\Lambda(F_h - G_h - \overline{G}_h)e)}{\|e\|_h^2}\\
\Rightarrow -\Delta\log\|e\|_h^2 &\le -\frac{h(e, 2i\Lambda(F_h - G_h - \overline{G}_h)e)}{\|e\|_h^2}\\
&\le \left|\frac{h(e, 2i\Lambda(F_h - G_h - \overline{G}_h)e)}{\|e\|_h^2}\right|\\
&\le \frac{2\|e\|_h\|\Lambda(F_h - G_h - \overline{G}_h)e\|_h}{\|e\|_h^2}\\
&\le \frac{2\|e\|_h^2\|\Lambda(F_h - G_h - \overline{G}_h)\|_F}{\|e\|_h^2}\\
&= 2\|e\|_h^2\|\Lambda(F_h - G_h - \overline{G}_h)\|_F.
\end{aligned}$$

again by Cauchy-Schwarz and the consistency of the Frobenius norm with h. □

Proposition 3.0.3. Let $(E, h, \theta, \overline{\partial}_E)$ a holomorphic bundle as before. Suppose $\|F_h\|_F, \|G_h\|_F \in L^p$ for some $p > 1$.[1] Let e be a section which is $D-$flat or $d''-$flat,

[1] No singularity outside the origin and smooth on any annulus $A_{\tilde{\varepsilon},\hat{\varepsilon}}, 1 \ge \hat{\varepsilon} > \tilde{\varepsilon} > 0$.

and additionally $\|e\|_h \leq c_e|z|^{-\varepsilon}$ for all $\varepsilon > 0$ ans some $c_e > 0$. Then e is bounded.

Before we start with the actual proof, we will need the following lemma

Lemma 3.0.4. $\|e\|_h \leq c_e|z|^{-\varepsilon}, \forall\varepsilon > 0$ implies $\frac{\|e\|_h}{\log|z|} \to 0$.

Proof. We know that

$$\frac{\|e\|_h}{|\log|z||} \leq \frac{c_e|z|^{-\varepsilon}}{|\log|z||}, \forall\varepsilon > 0.$$

Differentiating the right-hand side once (close to $r = 0$) leads us to

$$\frac{\mathrm{d}}{\mathrm{d}\,r}\frac{c_e r^{-\varepsilon}}{|\log r|} = \frac{-c_e\varepsilon r^{-\varepsilon-1}|\log r| + c_e r^{-\varepsilon-1}}{|\log r|^2} = -c_e r^{-\varepsilon-1}\frac{(\varepsilon|\log r| - 1)}{|\log r|^2}.$$

We see that the function has an extrema at $\varepsilon|\log r| = 1 \Rightarrow r = \exp(-1/\varepsilon)$ and it is the unique extrema for $r < 1$. This is a minimum by

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}\,r}c_e r^{-\varepsilon-1}\frac{(\varepsilon|\log r| - 1)}{|\log r|^2}\bigg|_{r=\exp(-1/\varepsilon)}\\
&= -c_e\frac{(-\varepsilon-1)r^{-\varepsilon-2}(\varepsilon|\log r| - 1 - \frac{\varepsilon}{-1-\varepsilon})|\log r|^2}{|\log r|^4}\\
&\quad - c_e\frac{r^{-\varepsilon-1}(\varepsilon|\log r| - 1)(2|\log r|r^{-1})}{|\log r|^4}\bigg|_{r=\exp(-1/\varepsilon)}\\
&= -c_e\frac{(-\varepsilon-1)r^{-\varepsilon-2}(1 - 1 + \frac{\varepsilon}{1+\varepsilon})|\log r|^2 + r^{-\varepsilon-1}(1-1)(2|\log r|r^{-1})}{|\log r|^4}\bigg|_{r=e^{\frac{-1}{\varepsilon}}}\\
&= c_e\frac{(\varepsilon+1)r^{-\varepsilon-2}(\varepsilon)|\log r|^2}{|\log r|^4}\bigg|_{r=\exp(-1/\varepsilon)} > 0
\end{aligned}$$

as the remaining parts are positive for all r. Now for $\varepsilon \to 0 \Rightarrow \exp(-1/\varepsilon) \to 0$, i.e. in the limit

$$\frac{c_e|z|^{-\varepsilon}}{|\log|z||}\bigg|_{r=\exp(-1/\varepsilon)} = \frac{c_e e^{\frac{\varepsilon}{\varepsilon}}}{\varepsilon^{-1}} = c_e\varepsilon e \to 0.$$

□

proof of lemma 3.0.3. By the previous lemma $-\Delta\log\|e\|_h^2 \leq 2\|\Lambda(F_h - 2G_h)\|_F \leq 2\|\Lambda F_h\|_F + 4\|\Lambda G_h\|_F$ resp. $-\Delta\log\|e\|_h^2 \leq 2\|\Lambda(F_h - G_h - \overline{G}_k)\|_F \leq 2\|\Lambda F_h\|_F + 4\|\Lambda G_h\|_F$ is in L^p. In particular let f be a L^p-majorant - $-\Delta\log\|e\|_h^2 \leq f \Rightarrow -\Delta\log\|e\|_h^2 - f \leq 0$.Let $\tilde{u} : B^1 \to \mathbb{R}$ be a solution of the Poisson equation

$-\Delta\tilde{u} = f$: For the existence see Lieb & Loss [LiLo00], p. 157, theorem 6.21.[2] In particular [LiLo00], theorem 10.2 tells us that for $p > 1$ u is Hölder continuous with positive exponent, in particular bounded. The bound $\frac{\|e\|_h}{\log|z|} \to 0$ from the previous lemma leads to lemma 1.7.9: $\Rightarrow -\Delta \log\|e\|_h^2 - f = -\Delta(\log\|e\|_h^2 - \tilde{u}) \leq 0$ weakly over the origin, i.e. $\log\|e\|_h^2 - \tilde{u}$ subharmonic.
Let $u := \tilde{u} + \|u\|_\infty + c_e$ then u is a solution of $-\Delta u = f$, i.e. we still have subharmonicity, and $u \geq c_e \geq c_e\varepsilon \log(|z|)\log\|e\|_h^2 \Rightarrow \limsup_{z\to\partial B_1} \log\|e(z)\|_h^2 - u(z) \leq 0$. By the maximum principle (+ remark 1.5.4) $\log\|e\|_h^2 - u \leq 0 \Rightarrow \log\|e\|_h^2 \leq u < \infty \Rightarrow \|e\|_h^2 \leq e^u$. □

Corollary 3.0.5. Let E be a vector bundle with either a flat connection D or a (Higgs) operator d'', and two metrics h, k. Suppose that the curvatures F_h, F_k, G_h, G_k are in L^p and that $\Xi(E)_\alpha$ is the same for both metric bundles. Then h and k are mutually bounded.

Proof. Let id_E be the identity automorphism of the bundle E; $\mathrm{id}_{h,k}$ if it is considered as a map from the metric bundle $(E,h) \to (E,k)$ and $\mathrm{id}_{k,h}$ as the inverse map. Then $\mathrm{id}_E \in \Gamma(X, \mathrm{End}(E))$ under the usual identification. In the section on the endomorphism we have seen, that D, d'' induces operators $D \cdot - \cdot D, d'' \cdot - \cdot d''$ on $\mathrm{End}(E)$. But then

$$D\mathrm{id}_E(e) - \mathrm{id}_E D(e) = 0 = d''\mathrm{id}_E(e) - \mathrm{id}_E d''(e).$$

There we saw as well (remark A.4.2) that the inner product for maps $\psi, \varphi : (E,h) \to (E,k)$ is

$$\begin{aligned} &h_{\mathrm{End},h\to k}(\varphi,\psi) = \mathrm{tr}(K\psi^* H\varphi) \\ &\Rightarrow \|\varphi\|_{\mathrm{End},h\to k} = \mathrm{tr}(K\varphi^* H\varphi) = \langle H\varphi, \varphi K\rangle_{HS}, \end{aligned}$$

w.r.t. a suitable basis.
By construction this operator norm is consistent with the norms h and k

$$\|e\|_k = \|\mathrm{id}_{h,k} e\|_k \leq \|\mathrm{id}_{h,k}\|_{\mathrm{End},h\to k}\|e\|_h.$$

From the definition of $e \in \Xi(E)_\alpha$ we read of that $\exists c_{e,k} \geq 0 : \|e\|_k \leq c_{e,k} r^{\alpha-\varepsilon}, \forall\varepsilon > 0 \Leftrightarrow \exists c_{e,h} \geq 0 : \|e\|_h \leq c_{e,k} r^{\alpha-\varepsilon}$. Therefore $\|\mathrm{id}_{h,k}\|_{\mathrm{End},h\to k} \leq c_{h,k} r^{-\varepsilon}, \forall\varepsilon > 0$ for a suitable constant $c_{h,k}$ or else the consistency of the norm gives a contradiction, because every section s can be identified with an element of $j_*(E) = \bigcup_{\alpha\in\mathbb{R}} \Xi(E)_\alpha$, i.e. if $\|\mathrm{id}_{h,k}\|_{\mathrm{End},h\to k} > c_{h,k} r^{-\tilde{\varepsilon}}$ for one $\tilde{\varepsilon} > 0$ we find a section s such that the consistency is violated.
Further recall from remark A.4.2 that the (pseudo-)curvature operators are L^p on

[2] Our L^p–function is L^1, in particular L^1_{loc} on a bounded neighbourhood like B^1.

the endomorphism bundle, too. Then we may apply Proposition 3.0.3 and get $\|\mathrm{id}_{h,k}\|_{\mathrm{End},h\to k} = \langle H, K\rangle_{HS}$ bounded. Analogously for $\mathrm{id}_{k,h}$. So the norms are mutually bounded, i.e. $\|e\|_h < \infty \Leftrightarrow \|e\|_k < \infty$ for every section e. □

Theorem 3.0.6. The functor Ξ from the category of tame harmonic bundles to the category of filtered regular Higgs bundles and filtered regular $\mathscr{D}_X-$modules is fully faithful, i.e. the induced function on the set of homomorphisms is a bijection.

Proof. We have already seen, how Ξ maps a morphism of harmonic bundles to a morphism of filtered objects, simply by considering it as a map on the bundle underlying the filtered object. On the other hand every morphism of filtered objects φ restricts to a bundle morphism of the underlying bundle, this will be $\Xi^{-1}\varphi$. The two constructions are obviously inverse. So we only need to show that Ξ is surjective or in other words, that Ξ^{-1} maps every morphism of filtered bundles onto a morphism of tame harmonic bundles.
More explicitly: Let $\varphi : \Xi(E) \to \Xi(F)$ be a morphism of the filtered regular Higgs resp. $\mathscr{D}_X-$bundles corresponding to two Higgs resp. $\mathscr{D}_X-$bundles E, F. We have seen in 2.2.5, that Ξ commutes with tensors and duals. So we may consider a homomorphism as an element of $\mathrm{Hom}(\Xi(E), \Xi(F)) \simeq \Xi(E)^* \otimes \Xi(F) = \Xi(E^*) \otimes \Xi(F) = \Xi(E^* \otimes F) = \Xi(\mathrm{Hom}(E, F))$ as vector bundle homomorphism. Hence it will be enough to show that φ is harmonic on the underlying bundles.

H(ii) For the Higgs bundle case we have $\varphi^*\overline{\partial}_F = \varphi\overline{\partial}_E, \varphi^*\theta_F = \varphi\theta_E$:

$$\Rightarrow \varphi^* d''_F = \varphi^*(\overline{\partial}_F + \theta_F) = \varphi^*(\overline{\partial}_F) + \varphi^*(\theta_F) = \varphi(\overline{\partial}_E + \theta_E)\varphi d''_E.$$

Therefore the induced homomorphism on the harmonic bundle fulfills (ii) of 1.2.8 - (i), (iii) are missing.

D(i) For the $\mathscr{D}_X-$module case we know $\varphi^* D''_F = \varphi D''_E, \varphi^*\nabla_F = \varphi\nabla_E$

$$\Rightarrow \varphi^* D_F = \varphi^*(D''_F) + \varphi^*(\nabla_F) = \varphi D''_E + \varphi\nabla_E = \varphi D_E.$$

The induced homomorphism on the harmonic bundle fulfills (i) of 1.2.8 - (ii), (iii) are missing.

First we treat (iii) simultaneously for both cases: Then $\varphi : E \to F$ may be identified with $\varphi \in E^* \otimes F$ with $D\varphi - \varphi D = D\varphi - D\varphi = 0$ and $D''\varphi - \varphi D'' = 0$ by the definition 1.2.8. In the proof to the previous corollary we saw (at the example of id) that preserving the filtration implies $\|\varphi\|_{E\to F} \leq c_\varphi r^{-\varepsilon}$. Analogously to the proof of the corollary $\|\varphi\|_{E\to F} \leq c'_\varphi < \infty$ bounded.

H(i) In the Higgs bundle case in order to prove (i) we need to show that $\varphi^* d'_F = \varphi d'_E, d'_E = \partial_E + \theta^\dagger_E$ and d'_F analogously. We know already that the pseudo curvature $G_E = (d''_E)^2 = \overline{\partial}_E \theta_E - \theta_E \overline{\partial}_E = 0$ by the Higgs field property. Same in the bundle F and in the homomorphism bundle $G_{E\to F} = G_F \cdot + \cdot G_E = 0$. But we have $F_E = 0$ as well: $\varphi^* \tilde{D}^F = \varphi \tilde{D}^E$ for any operator $\tilde{D}$ extends to higher $(p,q)-$forms $\tilde{\varphi}$ as $\tilde{\varphi}^* \tilde{D}^F = (-1)^{p+q+1} \tilde{\varphi}\tilde{D}^E$, because for $\tilde{\varphi} = \varphi \otimes \omega$

$$\begin{aligned}
\Rightarrow \quad & \tilde{D}^F(\varphi\otimes\omega) - (\varphi\otimes\omega)\tilde{D}^E \\
& = (\tilde{D}^F_0 \varphi)\otimes\omega + \varphi\otimes d\omega - (-1)^{p+q}(\varphi\tilde{D}^E_0 \otimes \omega)^3 \\
& = (\tilde{D}^F_0 \varphi - (-1)^{p+q}\varphi\tilde{D}^E_0)\otimes\omega + \varphi\otimes d\omega \\
& = (\tilde{D}^F_0 \cdot - (-1)^{p+q}(\cdot)\tilde{D}^E_0)(\varphi\otimes\omega) \\
\Rightarrow \quad & \tilde{D}^F_0 \varphi - \varphi\tilde{D}^E_0 = 0 \Rightarrow \tilde{D}^F(\varphi\otimes\omega) - (-1)^{p,q}(\varphi\otimes\omega)\tilde{D}^E = 0.
\end{aligned}$$

Then

$$\begin{aligned}
F_F\varphi + \varphi F_E &= \varphi(d''_E)^2 + (d''_F)^2\varphi + \varphi(d'_E)^2 + (d'_F)^2\varphi \\
&\quad + \varphi d''_E d'_E + d''_F d'_F \varphi + \varphi d'_E d''_E + d'_F d''_F \varphi \\
&= \varphi(d'_E)^2 + (d'_F)^2\varphi + d''_F\varphi d'_E - d'_F\varphi d''_F - d''_F\varphi d'_E + d'_F\varphi d''_E \\
&= \varphi(\underbrace{\partial_E\theta^\dagger_E + \theta^\dagger_E\partial_E}_{=}) + (\underbrace{\partial_F\theta^\dagger_F + \theta^\dagger_F\partial_F}_{=0})\varphi \\
&= 0,
\end{aligned}$$

by the Higgs field property (resp. 1.2.6). Our Weitzenböck formula W2 becomes

$$\begin{aligned}
-\Delta\|\varphi\|^2_{E\to F} &= -2i\Lambda\|(\partial_E + \theta^\dagger)_{\mathrm{Hom}}\varphi\|^2_{E\to F} \\
&\quad + h_{E\to F}(e, -2i\Lambda(F_h - G_h - \overline{G}_h)_{\mathrm{Hom}}\varphi) \\
&= -2\|\Lambda(\partial_E + \theta^\dagger)_{\mathrm{Hom}}\varphi\|^2_{E\to F}
\end{aligned}$$

The equality holds on X and holds weakly over all of $\overline{X}$: Again by lemma 1.7.9, $\|\varphi\|^2_{E\to F} \le -2\|\Lambda(\partial_E + \theta^\dagger)_{\mathrm{Hom}}\varphi\|^2_{E\to F}$, $\|\Lambda(\partial_E + \theta^\dagger)_{\mathrm{Hom}}\varphi\|^2_{E\to F}$ positive and since f bounded $\Rightarrow \left|\frac{\varphi(z)}{\log|z|}\right| \to 0$, we get $\|\Lambda(\partial_E + \theta^\dagger)_{\mathrm{Hom}}\varphi\|^2_{E\to F}$ is integrable. But the integral of the Laplacian of a bounded function on any ball is zero. [4] So

$$0 = \int_{B^1} \Delta\|\varphi(z)\|^2_{E\to F}\,\mathrm{d}\,z = \int_{B^1} 2\|\Lambda(\partial_E + \theta^\dagger)_{\mathrm{Hom}}\varphi(z)\|^2_{E\to F}\,\mathrm{d}\,z,$$

and by the positivity of the integrand $\|\Lambda(\partial_E + \theta^\dagger)_{\mathrm{Hom}}\varphi(z)\|^2_{E\to F} = 0 \Rightarrow d'(\varphi) - \varphi d' = (\partial_E + \theta^\dagger)\varphi - \varphi(\partial_E + \theta^\dagger) = 0$. Finally $D(\varphi) - \varphi D = d'(\varphi) -$

[3] D^E_0 is a $1-$form, so $D^E_0 \wedge \omega = (-1)^{p+q}\omega \wedge D^E_0$ for deg $(\omega) = p+q$.
[4] cf. Müller, [PDE09], p. 14, lemma 2.6, (radius r fixed).

$\varphi d' + d''(\varphi) - \varphi d'' = 0$ shows the missing part (i). So Ξ is fully faithful into the filtered regular Higgs bundles.

D(ii) Similarly in the $\mathscr{D}_X$−module case (ii) will follow from $\varphi^* d' = \varphi d'$. It is enough to show $\varphi^*\theta_F = \varphi\theta_E, \varphi^*\theta_F^\dagger = \varphi\theta_E^\dagger$ or equivalently the induced operators on the homomorphism bundle vanish, since $d'' = D'' - \theta^\dagger + \theta$ and $\varphi^* D''_F = \varphi^* D''_E$ by degree considerations from $\varphi^* D_F = \varphi D_E$. We know that the curvature $F_E = F_F = F_{E\to F} = 0$ vanishes. For the pseudo-curvature we get

$$\begin{aligned}
&G_F\varphi + \varphi G_E \\
&= (d''_F)^2\varphi + \varphi(d''_E)^2 = (D_F - d'_F)^2\varphi + \varphi(D_E - d'_e)^2 \\
&= ((D_F)^2 - D_F d'_F - d'_F D_F + (d'_F)^2)\varphi \\
&\quad + \varphi((D_E)^2 - D_E d'_E - d'_E D_E + (d'_E)^2) \\
&= d'_F\varphi D_E - d'_F\varphi D_E + (d'_F)^2\varphi + \varphi(D_E)^2 - D_F\varphi d'_E + D_F\varphi d'_E + \varphi(d'_E)^2 \\
&= (\partial_F + \theta_F^\dagger)^2\varphi + \varphi(\partial_E + \theta_E^\dagger)^2 \\
&= (\partial_F\theta_F^\dagger + \theta_F^\dagger\partial_F)\varphi + \varphi(\partial_E\theta_E^\dagger + \theta_E^\dagger\partial_E) \\
&= \overline{G}_F\varphi + \varphi\overline{G}_E = 0.
\end{aligned}$$

by the vanishing of the pseudo-curvature (resp. 1.2.6). Our Weitzenböck formula W1 becomes

$$\begin{aligned}
-\Delta\|\varphi\|^2_{E\to F} &= -h_{E\to F}(\varphi, 2i\Lambda(F_h - 2G_h)_{\mathrm{Hom}}\varphi) \\
&\quad -8\|\Lambda(\theta)_{\mathrm{End}}\varphi\|^2_{E\to F} - 8\|\Lambda(\theta^\dagger)_{\mathrm{End}}\varphi\|^2_{E\to F} \\
&= -8\|\Lambda(\theta)_{\mathrm{End}}\varphi\|^2_{E\to F} - 8\|\Lambda(\theta^\dagger)_{\mathrm{End}}\varphi\|^2_{E\to F}.
\end{aligned}$$

Like in the Higgs bundle case lemma 1.7.9 guarantees the integrability over the origin. Again the integral over the Laplacian does vanish and so the same holds for the integrand $8\|\Lambda(\theta)_{\mathrm{End}}\varphi\|^2_{E\to F} + 8\|\Lambda(\theta^\dagger)_{\mathrm{End}}\varphi\|^2_{E\to F} = 0$. Both summands are non-negative and therefore both have to vanish, i.e. $\theta\varphi - \varphi\theta = \theta^\dagger\varphi - \varphi\theta^\dagger = 0$. This was the missing part of the proof of (i). So Ξ is fully faithful into the filtered regular $\mathscr{D}_X$−modules.

□

4 Residues and Standard Metrics

On our way to essential surjectivity, we need to construct a metric h on our bundle E that induces a given filtration under Ξ. This standard metric will not be our harmonic metric, but we will be able to construct a harmonic metric that is mutually bounded with respect to h.
Recall our two examples 1.4.17 and 1.4.18. The bundles constructed are our smallest building blocks in the following sense: By tensoring these building block we may construct to each residue (V_α, N) a bundle $\tilde{E}$, that has the residue (V_α, N), up to isomorphism. We will then show that $\tilde{E}$ is isomorphic as a filtered vector bundle to our initial bundle E. Then define a new metric h on E as the pullback of the harmonic metric $\tilde{h}$ on $\tilde{E}$. It turns out that h makes E into an acceptable bundle, the new curvatures are $L^p, p > 1$ and h induces the initial filtrations on E under Ξ.
We get a nice corollary at the end of the chapter, telling us that the filtration induced on a filtered local system by order of growth, is the same as the filtration provided by our functor Φ.

4.1. Jumps and Residues of Line Bundles

Let $\alpha \in \mathbb{R}$ be arbitrary for now and let X be the unit disc again. Let E be a line bundle over X with trivialization $\varphi : E_X \to X \times \mathbb{C}$. Choose $e(z) = \varphi^{-1}(z, 1)$. Define an inner product h on E by requiring $h(e, e) = |z|^{2\alpha}$ and extending sesquilinearly. If $\overline{\partial}_E$ denotes the holomorphic structure then the holomorphy of φ implies those of e: $\overline{\partial}_E e = 0$. Let $\theta e = e \otimes \frac{a}{z} \operatorname{d} z$. In example 1.4.17 we saw that $\partial_E e = e \otimes \frac{\alpha \operatorname{d} z}{z}, \theta e = e \otimes \frac{\overline{a}}{z} \operatorname{d} z$[1], that the bundle is harmonic and that the metric connection $D = \partial_E + \overline{\partial}_E$ is flat.
The harmonicity of E leads to a filtered regular Higgs bundles under $\Xi : \Xi(E)_\alpha =$

[1]We could write $\partial_E = \partial + \frac{\overline{a}}{z} \operatorname{d} z$ for the usual decomposition into differential and connection matrix.

$\{e : X \to E | \exists C > 0 \forall \varepsilon > 0 : \|e\| \leq C r^{\alpha-\varepsilon}\}$. Then e is only in level α and lower. We get $\Xi(E)_{\alpha+\varepsilon} \neq \Xi(E)_\alpha$ for all $\varepsilon > 0$. Further note that the map $e \mapsto 1$ from $E_\alpha \to \mathcal{O}_{\overline{X}}$ is an $\mathcal{O}_{\overline{X}}-$isomorphism, which preserves holomorphy.
On the other hand when we start with a regular filtered Higgs line bundle $(E, E_\beta)_{\beta\in\mathbb{R}}$ with $\Xi(E)_{\alpha+\varepsilon} \neq \Xi(E)_\alpha$ for all $\varepsilon > 0$ and $\psi : E_\alpha \to \mathcal{O}_{\overline{X}}$ holomorphic isomorphism. Then we find an element $e \in \Xi(E)_\alpha \setminus \Xi(E)_{\alpha+\varepsilon}$. e is holomorphic as preimage of a holomorphic function under ψ and the required compatibility of ψ with the holomorphic structure $\overline{\partial}_E$. Then we may define $h(e,e) = |z|^{2\alpha}, \theta e = e \otimes \frac{a}{z} \mathrm{d}\, z$ and construct $\partial_E, \theta_E^\dagger$ as in 1.4.17 receiving harmonicity and flatness of the metric connection. In particular note that e is a frame of the base space E, i.e. E is holomorphic.
We want to understand how the "jump" in the filtration of E shifts when considering associated filtered regular $\mathscr{D}_X-$modules and later filtered local systems. Therefore we construct a $D'' = \overline{\partial}_E + \theta^\dagger-$holomorphic frame

$$\begin{aligned}
v := e^{-2\overline{a}\log|z|}e &= e^{-\overline{a}\log(z\overline{z})}e \\
\Rightarrow (\overline{\partial}_E + \theta^\dagger)v &= \underbrace{\overline{\partial}_E e}_{=0} \otimes \exp(-\overline{a}\log(z\overline{z})) + e \otimes \frac{-\overline{a}z\,\mathrm{d}\,\overline{z}}{z\overline{z}}\exp(-\overline{a}\log(z\overline{z})) \\
&\quad +\theta^\dagger e \otimes \exp(-\overline{a}\log(z\overline{z})) \\
&= \frac{-\overline{a}\,\mathrm{d}\,\overline{z}}{\overline{z}}v + \frac{\overline{a}\,\mathrm{d}\,\overline{z}}{\overline{z}} \\
&= 0
\end{aligned}$$

Moreover

$$\begin{aligned}
\|v\|_{h_e}^2 &= \exp(-2\overline{a}\log|z|)\exp(-2a\log|z|)h(e,e) \\
&= \exp(-2(\overline{a}+a)\log|z|)|z|^{2\alpha} \\
&= \exp(-4\Re(a)\log|z|)|z|^{2\alpha} = |z|^{2\alpha-2(\overline{a}+a)}.
\end{aligned}$$

Hence $v \in \Xi(E)_{\alpha-(a+\overline{a})}$. Since $\|e\|_h^2 > |z|^{2\alpha+\varepsilon}, \forall\varepsilon > 0$ for z small enough, we see that $v \notin \Xi(E)_{\alpha-(a+\overline{a})+\varepsilon}, \forall\varepsilon > 0$, i.e. we get a "jump" at $\alpha-(a+\overline{a}) = \alpha - 2\Re(a)$ in the filtration of the $\Xi-$corresponding regular filtered $\mathscr{D}_X-$module. Furthermore

$$\begin{aligned}
\nabla v &= \partial_E v + \theta v \\
&= \partial_E e \otimes \exp(-2\overline{a}\log|z|) + e \otimes \partial\exp(-\overline{a}\log(z\overline{z})) + \frac{a\,\mathrm{d}\,z}{z}e \otimes \exp(-2\overline{a}\log|z|) \\
&= \frac{\alpha\,\mathrm{d}\,z}{z}e \otimes \exp(-2\overline{a}\log|z|) + e \otimes \frac{-\overline{a}}{z}\exp(-\overline{a}\log) + \frac{a\,\mathrm{d}\,z}{z}e \otimes \exp(-2\overline{a}\log|z|) \\
&= \frac{\alpha-\overline{a}+a}{z}v\,\mathrm{d}\,z.
\end{aligned}$$

Then our matrix M from the previous section is $(\alpha - \overline{a} + a)$ and the monodromy of the corresponding system is $\mu = e^{-2\pi i(\alpha-\overline{a}+a)}$. By the equivalence Φ given in the chapter on Filtered Objects:

$$\begin{aligned} V_{\alpha-\Re(a-\overline{a})} &= \{v = e^{M\log(z)}l | l \in L_\beta : \Re(\lambda_M) = \alpha > \alpha - (a+\overline{a}) - \beta\} \\ &= \{v = e^{M\log(z)}l | l \in L_\beta : 2\Re(a) > -\beta\}. \end{aligned}$$

So the "jump" in the filtration of the local system occurs at $-2\Re(a)$.

Remark 4.1.1. Recall 2.1.11. Then θ has residue map $\operatorname{res}(\theta)(e) = e \otimes a \,\mathrm{d}\, z$, ∇ has residue map $\operatorname{res}(\nabla)(e) = e \otimes (\alpha - \overline{a} + a)\,\mathrm{d}\, z = e \otimes (\alpha + 2i\Im(a))\,\mathrm{d}\, z$ and the local system has residue map $\mu = e^{2\pi i(\alpha - \overline{a} + a)} = e^{-2\pi i\alpha + 4\pi\Im(a)}$.

Remark 4.1.2. Note that we get a flat section $e^{-(\alpha-\overline{a}+a)\log(z)}v$. This section $\|e^{-(\alpha-\overline{a}+a)\log(z)}v\|_h = |z|^{\alpha-\overline{a}-3a}$ is polynomially bounded.

4.2. Residues of Vector Bundles

Let us proceed with example 1.4.18. There we had $E = X \times \mathbb{C}^2$ or analogously $E \simeq X \times \mathbb{C}^2$ with a trivialization φ and a frame $(\varphi^{-1}e_1, \varphi^{-1}e_2)$. The rest of the calculation is the same, so we may stay with e_i. Further $E = E_1 \otimes E_2$ with $E_i = \operatorname{span}_{\mathcal{O}_X}\{e_i\}, 1 \leq i \leq 2$. Our metric was defined by $\|e_1\|_h^2 = -\log|z|^2 =: y$ and $\|e_2\|_h^2 = (-\log|z|^2)^{-1} =: y^{-1}$. The other operators are

$$\begin{aligned} &\overline{\partial}_E e_1 = \overline{\partial}_E e_2 = 0, \quad \partial_E e_1 = -e_1 \otimes \frac{\mathrm{d}\, z}{zy}, \partial_E e_2 = e_2 \otimes \frac{\mathrm{d}\, z}{zy}, \\ &\theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}\, z}{z}, \quad \theta^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}\, \overline{z}}{\overline{z}y^2}. \end{aligned}$$

Again we want to consider the associated $\mathscr{D}_X-$module. Therefore construct a $\overline{\partial}_E + \theta^\dagger-$holomorphic frame. e_1 is already $\overline{\partial}_E + \theta^\dagger-$holomorphic:

$$(\overline{\partial}_E + \theta^\dagger)e_1 = \theta^\dagger e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}\, \overline{z}}{\overline{z}y^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

For e_2 this is not the case:

$$\begin{aligned} (\overline{\partial}_E + \theta^\dagger)e_2 &= \theta^\dagger e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}\, \overline{z}}{\overline{z}y^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\mathrm{d}\, \overline{z}}{\overline{z}y^2} = e_1 \otimes \frac{\mathrm{d}\, \overline{z}}{\overline{z}y^2}. \end{aligned}$$

Therefore $v_2 := e_2 - e_1 \otimes y^{-1}$ is flat:

$$(\overline{\partial}_E + \theta^\dagger)v_2 = -e_1 \otimes \overline{\partial} y^{-1} + \theta^\dagger e_2$$

$$\begin{aligned}
&= -e_1 \otimes \overline{\partial}(-\log(z\overline{z}))^{-1} + e_1 \otimes \frac{\mathrm{d}\,\overline{z}}{\overline{z}y^2} \\
&= -e_1 \otimes \frac{z\,\mathrm{d}\,\overline{z}}{z\overline{z}} \underbrace{(-\log(z\overline{z}))^{-2}}_{=y^{-2}} + e_1 \otimes \frac{\mathrm{d}\,\overline{z}}{\overline{z}y^2} \\
&= 0.
\end{aligned}$$

The norm of this new basis element is $\|v_2\|_h^2 = \|e_2\|_h^2 + \|e_1\|_h^2 y^2 = y^{-1} + yy^{-2} = 2y^{-1}$.

Remark 4.2.1. v_2, e_1 is not an $h-$orthonormal frame: $h(v_2, e_1) = h(e_2, e_1) - h(e_1, e_1)y^{-1} = -1$.

Again we want to consider the corresponding local system, too. We can further modify to get a $D-$flat frame. v_2 is already flat

$$\begin{aligned}
\nabla v_2 &= (\partial_E + \theta)v_2 = (\partial_E + \theta)(e_2 - e_1 \otimes y^{-1}) \\
&= e_2 \otimes \frac{\mathrm{d}\,z}{zy} + 0 - \left(-e_1 \otimes \frac{\mathrm{d}\,z}{zy^2}\right) - e_1 \otimes \partial y^{-1} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{\mathrm{d}\,z}{zy} \\
&= e_2 \otimes \frac{\mathrm{d}\,z}{zy} + e_1 \otimes \frac{\mathrm{d}\,z}{zy^2} - e_1 \otimes \frac{\overline{z}\,\mathrm{d}\,z}{z\overline{z}y^2} - e_2 \otimes \frac{\mathrm{d}\,z}{zy} \\
&= 0.
\end{aligned}$$

The first half of the calculation leads to

$$\begin{aligned}
\nabla e_1 &= (\partial_E + \theta)(e_1) = -e_1 \otimes \frac{\mathrm{d}\,z}{zy} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{\mathrm{d}\,z}{z} \\
&= (e_2 - e_1 y^{-1}) \otimes \frac{\mathrm{d}\,z}{z} = v_2 \otimes \frac{\mathrm{d}\,z}{z}.
\end{aligned}$$

Then define $v_1 = e_1 - v_2 \log(z)$. Note that this section is in general multivalued.

$$\nabla v_2 = v_2 \otimes \frac{\mathrm{d}\,z}{z} - v_2 \otimes \partial \log(z) = v_2 \otimes \left(z^{-1} - z^{-1}\right) \mathrm{d}\,z = 0.$$

Let $v_1^{(k)} = e_1 - v_2(\log(z) + 2\pi k)$ be an arbitrary branch. Then

$$\begin{aligned}
\|v_1^{(k)}\|_h^2 &= \|e_1\|_h^2 - 2\Re(h(e_1, v_2)(\log(z) + 2\pi ik)) + \|v_2\|^2 |\log(z) + 2\pi ik|^2 \\
&= y + 2\Re((\log(z) + 2\pi ik)) + 2y^{-1} |\log(z) + 2\pi ik|^2 \\
&= y + 2\Re(\log(z)) + 2y^{-1} |\log(z)|^2 = y|1 + y^{-1}(\log(z) + 2\pi ik)|^2 \\
&\sim y.
\end{aligned}$$

The monodromy of the corresponding local system can be read of

$$0 = \nabla v_2 = \frac{M}{z}\,\mathrm{d}\,z v_2 \quad \text{resp. } v_2 \otimes \frac{\mathrm{d}\,z}{z} = \nabla e_1 = \frac{M}{z}\,\mathrm{d}\,z e_1$$

$$\Rightarrow M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ [2]

in the holomorphic frame of the $\mathscr{D}_X-$module (v_2, e_1) (in this order). The frame of L obtained in the way of chapter IV is just

$$e^{-M\log(z)} = E - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \log(z) \qquad \Rightarrow e^{-M\log(z)} v_2 = v_2$$

$$\Rightarrow e^{-M\log(z)} e_1 = e_1 - v_2 \log(z) = v_1.$$

Remark 4.2.2. Again we want to take a look at the residue maps. For the local system we just saw that the residue map is $\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For the Higgs bundle we had $\mathrm{res}(\theta) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathrm{d}\, z$ w.r.t. (e_1, e_2) and for the $\mathscr{D}_X-$module $\mathrm{res}(\nabla) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathrm{d}\, z$ w.r.t. (e_1, v_2), i.e. the same representation as for the Higgs bundle.

Remark 4.2.3. We gave some bounds on the growth of the $\nabla-$flat holomorphic sections here. v_1 and v_2 are both bounded in terms of the logarithm, in particular have polynomial growth.

Lemma 4.2.4. Given a vector space $V = \bigoplus_{0 \leq \alpha < 1} V_\alpha$ with and endomorphism N compatible with the decomposition, there is always a harmonic bundle E such that $\mathrm{res}(\Xi(E), \theta) \simeq (V, N)$ or $\mathrm{res}(\Xi(E), \nabla) \simeq (V, N)$. Here an isomorphism of residues is a vector space homomorphism $\varphi : \Xi(E) \to V$ which is compatible with restriction to the subspaces of the grading and compatible with the endomorphism: $\mathrm{res}(\theta) = \varphi^{-1} N \varphi$.

Proof. We will use the two constructions from 1.4.17 and 1.4.18 to construct any vector space V resp. a V' in the isomorphism class of V, with grading, and a compatible homomorphism. By the compatibility of Ξ with tensor products, combinations of those two concepts using tensor products will lead to the corresponding connection between the filtrations.
Our second example 1.4.18 and its conclusions above have a natural extension to higher symmetric powers than 2. For power n our resulting residue of the Higgs

[2] Check by power series of log.

resp. $\mathscr{D}_X$ bundle will be the matrix

$$\mathrm{res}(\theta) = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

or $(\mathrm{res}(\theta))_{kl} = \delta_{k+1,l}$.

Next we use the one-dimensional line bundles from the previous section. We constructed harmonic line bundles with arbitrary "jump" at α and arbitrary eigenvalue a. In the section on the endomorphism bundle we saw as well that a connection in the tensor product of (E, ∇_E) and (F, ∇_F) is chosen as $\nabla_E \otimes \mathrm{id}_F + \mathrm{id}_E \otimes \nabla_F$. Analogous for homomorphism. So if N was our nilpotent matrix from the second example and (a) was our one dimensional from the first, then $N \otimes 1 + E \otimes a$ is a Jordan block to the eigenvalue a. Now taking the direct product leads us to an arbitrary Jordan normal form, i.e. we may construct any homomorphism. This homomorphism will respect the filtration by construction.

Further note that for the filtration of tensor products $(E \otimes F)_\alpha = \sum_{\gamma \in \mathbb{R}} E_\gamma \otimes F_{\alpha-\gamma}$ the "jump" of the second bundle induces a "jump" here. But the Gr_β are only non-zero at a "jump". Since our "jump" in the line bundle was one-dimensional[3], we can get any dimensional "jump" by repeatedly tensoring with line bundles with the "jump" at α.

Hence we get any grading, i.e. a vector space $\tilde{V} \simeq V$ and subspaces $V'_\alpha \simeq V_\alpha$ by restriction of the general isomorphism. Together with our homomorphism we can construct any $V = \bigoplus_{0 \le \alpha < 1} V_\alpha$ with any endomorphism N from a harmonic bundle (up to isomorphism).[4] Thus for any (V, V_α, N) exists a harmonic bundle E such that $\mathrm{res}(\Xi(E), \theta) \simeq (V, N)$.

The proof for $\mathscr{D}_X-$modules is the same only that we use now $\alpha' = \alpha - 2\Re(a)$ instead of α' and $\alpha + 2\Im(a)i$ instead of a. It does not restrict our choice: Choose first α as the real part of our eigenvalue and $2\Im(a)$ as the imaginary part, then a $\Re(a)$ suitable to get a desired value $\alpha' \in \mathbb{R}$. □

Remark 4.2.5. The harmonic bundle is not uniquely determined by the data (V, V_α, N).

Lemma 4.2.6. Let E be a filtered regular Higgs bundles resp. filtered regular $\mathscr{D}_X-$module. There is a tame harmonic bundle $(\tilde{E}, \overline{\partial}_{\tilde{E}}, \theta_{\tilde{E}}, \tilde{h})$ such that $\mathrm{res}(E, \theta) \simeq$

[3] As a vector subspace of all Laurent series.

[4] Note that the construction of the spaces V_α only depended on the choice of α and e, and did not interfere with the construction of N dependent on a.

$\operatorname{res}(\tilde{E}, \theta_{\tilde{E}})$ resp. $\operatorname{res}(E, \nabla) \simeq \operatorname{res}(\tilde{E}, \tilde{\nabla})$ and an isomorphism of filtered vector bundles between E and $\tilde{E}$.

Proof. We have already proved the lemma above - more or less. Note first that all eigenvalues above where tame in 1.4.17, 0 in 1.4.18, hence the eigenvalues of $\theta_{\tilde{E}}$ are tame.
Consider the stalk - a vector space - at the puncture s. By $zE_\alpha = E_{\alpha+1}$ all data of the filtration is encoded in $\operatorname{res}(E) = \bigoplus_{0\leq\beta<1} \operatorname{Gr}_\beta(E)$. We required the isomorphism of residue $\psi : \operatorname{res}(E, \theta) \to \operatorname{res}(\tilde{E}, \theta_{\tilde{E}})$ to restrict to $\operatorname{res}(E) = \bigoplus_{0\leq\beta<\gamma} \operatorname{Gr}_\beta(E)$. Hence ψ induces a map $\psi_\beta : E_\beta \to \Xi(\tilde{E})_\beta$. On the other hand ψ extends to a vector space homomorphism on $j_*(E)_s$. So close enough to the puncture ψ maps a frame (e_i) of E to a frame $(\tilde{e}_i)$ of $\tilde{E}$.
As vector bundles of the same rank on a punctured disc ($\Rightarrow$ trivial) we may define an extension of ψ, called ψ as well, to be the $\mathcal{O}_X-$linear extension of $\psi(e_i) = (\tilde{e}_i)$. There is one more degree of freedom, namely we may require ψ to be compatible with the holomorphic structures $\varphi\overline{\partial}_E = \varphi^*\overline{\partial}_{\tilde{E}}$. The last property is a choice which does not influence the residue of the harmonic bundle, since we never specified a particular holomorphic structure by construction of the two examples. We rather started already with the choice of a holomorphic section.
So (ψ, ψ_β) is an isomorphism of filtered vector bundles. □

Remark 4.2.7. This is not an isomorphism of regular filtered Higgs or $\mathscr{D}_X-$bundles. In fact we will need further restrictions, namely stability, to get an equivalence of categories.

4.3. Existence of a Standard Metric

Now we are in the position to state the main theorem of the chapter:

Theorem 4.3.1. Given a filtered regular Higgs bundle, $\mathscr{D}_X-$module $(E, h_E, \overline{\partial}_E, \theta)$ or local system L, there is a metric h making E into an acceptable bundle, such that the curvatures F_h resp. G_h are in L^p for some $p > 1$ and such that Ξ induces the original filtration on E resp. L and the dual filtration on E^* resp. L^*.

Proof. Let $\operatorname{res}(E) = (E, E_\alpha, \operatorname{res}(\theta))$. By the last lemma there is a harmonic bundle $(\tilde{E}, \tilde{h}, \overline{\partial}_{\tilde{E}}, \tilde{\theta})$ with $\operatorname{res}(E) \simeq \operatorname{res}(\tilde{E})$. Let $\varphi : E \to \tilde{E}$ be the corresponding isomorphism of filtered vector bundles provided by the previous lemmas. Let $\tilde{\theta}$ denote the Higgs field of $\tilde{E}$ and $\operatorname{res}(\tilde{\theta})$ the residue. We get $\varphi\operatorname{res}(\theta) = \varphi^*\operatorname{res}(\tilde{\theta})$. Analogously $\varphi\operatorname{res}(\nabla) = \varphi^*\operatorname{res}(\tilde{\nabla})$.
Define a new metric $h(e, f) := \tilde{h}(\varphi(e), \varphi(f)), e, f \in \Gamma(X, E)$. This new metric is

clearly harmonic with respect to $\tilde\theta := \varphi^{-1}\theta_{\tilde E}\varphi$: φ is an isomorphism of filtered vector bundles, in particular we have $\varphi^*\overline\partial_{\tilde E} = \varphi\overline\partial_E$.

$$\begin{aligned}
\tilde h(\varphi(\partial_E e),\varphi(f)) &= h(\partial_E e, f) = \partial h(e,f) - h(e,\overline\partial_E f)\\
&= \partial\tilde h(\varphi(e),\varphi(f)) - \tilde h(\varphi(e),\varphi(\overline\partial_E f))\\
&= \partial\tilde h(\varphi(e),\varphi(f)) - \tilde h(\varphi(e),\overline\partial_{\tilde E}\varphi(f))\\
&= \tilde h(\partial_{\tilde E}\varphi(e),\varphi(f)).\\
\Rightarrow \varphi\partial_E &= \varphi^*\partial_{\tilde E}.
\end{aligned}$$

Hence $\|R_h\|_{\varphi^* H\varphi} = \|R_{\tilde h}\|_{\tilde H}$ acceptable, since the $\|R_{\tilde h}\|$ must satisfy the necessary bound by harmonicity of $\tilde E$ and then by the main estimate. In general we have $\varphi^* R_{\tilde h} = \varphi R_h$.
Moreover

$$\begin{aligned}
\overline\partial_E\tilde\theta + \tilde\theta\overline\partial_E &= \varphi^{-1}\overline\partial_{\tilde E}\varphi\varphi^{-1}\theta_{\tilde E}\varphi + \varphi^{-1}\theta_{\tilde E}\varphi\varphi^{-1}\overline\partial_{\tilde E}\varphi\\
&= \varphi^{-1}(\overline\partial_{\tilde E}\theta_{\tilde E} + \theta_{\tilde E}\overline\partial_{\tilde E})\varphi\\
&= 0.
\end{aligned}$$

is a Higgs field. The adjoint of $\tilde\theta$ is $\tilde\theta^\dagger = \varphi^{-1}\theta^\dagger_{\tilde E}\varphi$ since

$$\begin{aligned}
h(\tilde\theta e, f) &= \tilde h(\theta_{\tilde E}\varphi(e),\varphi(f)) = \tilde h(\varphi(e),\theta_{\tilde E}\varphi(f))\\
&= \tilde h(\varphi(e),\varphi\tilde\theta^\dagger(f)) = h(e,\theta^\dagger(f)),\quad \forall e,f\in\Gamma(X,E).
\end{aligned}$$

Finally

$$\begin{aligned}
(\partial_E + \overline\partial_E + \tilde\theta + \tilde\theta^\dagger)^2 &= \varphi^{-1}(\partial_{\tilde E} + \overline\partial_{\tilde E} + \theta_{\tilde E} + \theta^\dagger_{\tilde E})\varphi\\
&= 0,
\end{aligned}$$

harmonic. So we can work on E now.
Define $\theta_\Delta = \theta - \tilde\theta$, i.e. θ_Δ has no longer a $\frac{dz}{z}$−part by equality of the residues: $\exists\varepsilon, c_\Delta > 0 : \|\theta_\Delta\|_H \le c_\Delta|z|^{-1+\varepsilon}$. The norm is the one we now from the endomorphism bundle, i.e. $\|A\|^2_H = \mathrm{tr}(A^*HAH)$.
Again by the harmonicity of $\tilde E$ we know that $0 = F_{h,\tilde\theta} = R_h + \theta_{\tilde E}\theta^\dagger_{\tilde E} + \theta^\dagger_{\tilde E}\theta_{\tilde E}$

$$\begin{aligned}
\Rightarrow F_h &= R_h + \theta_E\theta_E^\dagger + \theta_E^\dagger\theta_E\\
&= R_h + \tilde\theta\tilde\theta^\dagger + \tilde\theta^\dagger\tilde\theta + \theta_\Delta\theta_\Delta^\dagger + \theta_\Delta^\dagger\theta_\Delta + \tilde\theta^\dagger\theta_\Delta + \theta_\Delta\tilde\theta^\dagger + \tilde\theta\theta_\Delta + \theta_\Delta\tilde\theta\\
&= F_{h,\tilde\theta} + \theta_\Delta\theta_\Delta^\dagger + \theta_\Delta^\dagger\theta_\Delta + \tilde\theta^\dagger\theta_\Delta + \theta_\Delta\tilde\theta^\dagger + \tilde\theta\theta_\Delta + \theta_\Delta\tilde\theta\\
&= \theta_\Delta\theta_\Delta^\dagger + \theta_\Delta^\dagger\theta_\Delta + \tilde\theta^\dagger\theta_\Delta + \theta_\Delta\tilde\theta^\dagger + \tilde\theta\theta_\Delta + \theta_\Delta\tilde\theta.
\end{aligned}$$

Thus $\|F_h\|_H \le 2\|\theta_\Delta\|^2_H + 4\|\theta_\Delta\|_H\|\tilde\theta\|_H \le 2c^2_\Delta|z|^{-2+2\varepsilon} + 4c_\Delta\sqrt{c_5}|z|^{-2+\varepsilon} \le c_{44}|z|^{-2+\varepsilon}$ by the main estimate 1.5.7. For $p = 1 + \frac{\varepsilon}{2} \Rightarrow p(-2+\varepsilon) = -2 + \varepsilon - \frac{2}{2}\varepsilon + \frac{\varepsilon^2}{4} =$

$-2+\frac{\varepsilon^2}{4} > -2 \Rightarrow F_h$ is L^p-integrable.
The proof for $\mathscr{D}_X-$modules works similarly: The previous lemma now applied to E with holomorphic structure $D''_E = \overline{\partial}_E + \theta_E$ leads to a harmonic $\mathscr{D}_X-$module $\tilde{E}$ and an isomorphism of filtered vector bundles φ such that $\varphi D''_E = D''_{\tilde{E}}\varphi$. Furthermore

$$\begin{aligned}
\tilde{h}(\varphi(\delta'_E e), \varphi(f)) &= h(\delta'_E e, f) = \partial h(e,f) - h(e, D''_E(f)) \\
&= \partial\tilde{h}(\varphi(e),\varphi(f)) - \tilde{h}(\varphi(e), \varphi D''_E(f)) \\
&= \partial\tilde{h}(\varphi(e),\varphi(f)) - \tilde{h}(\varphi(e), D''_{\tilde{E}}\varphi(f)) \\
&= \tilde{h}(\delta'_{\tilde{E}}\varphi(e), \varphi(f)) \\
\Rightarrow \delta'_{\tilde{E}}\varphi &= \varphi\delta'_E.
\end{aligned}$$

Construct $\tilde{\nabla} = \varphi^{-1}\nabla_{\tilde{E}}\varphi$ and $\nabla_\Delta = \nabla - \tilde{\nabla} \Rightarrow \exists\varepsilon > 0 : \|\nabla_\Delta\|_H \le c_\Delta |z|^{-1+\varepsilon}$. As above $(E, \tilde{\nabla}, \overline{\partial}_E + \tilde{\theta})$ is harmonic. The last calculation guarantees that $\varphi^* R_{h,\overline{\partial}_{\tilde{E}}+\theta^\dagger_{\tilde{E}}} = \varphi^*(\delta'_{\tilde{E}} + D''_{\tilde{E}})^2 = \varphi(\delta'_E + D''_E)^2 = \varphi R_{h,\overline{\partial}_E+\theta^\dagger_E}$ resp. $R_{h,\overline{\partial}_E+\theta^\dagger_E} = R_{h,\overline{\partial}_E+(\tilde{\theta})^\dagger}$. Further $\theta_E = \frac{\nabla_E - \delta'_E}{2} = \frac{\tilde{\nabla}+\nabla_\Delta - \varphi^{-1}\delta'_{\tilde{E}}\varphi}{2} = \varphi^{-1}\frac{\nabla_{\tilde{E}}-\delta'_{\tilde{E}}}{2}\varphi + \frac{\nabla_\Delta}{2} = \varphi^{-1}\theta_{\tilde{E}}\varphi + \frac{\nabla_\Delta}{2} = \tilde{\theta} + \frac{\nabla_\Delta}{2}$.
In the general $\mathscr{D}_X-$module case (not necessarily vanishing Pseudo-curvature) we already know that

$$\begin{aligned}
&R_{h,\overline{\partial}_E+\theta^\dagger_E} + 2(\theta_E\theta^\dagger_E + \theta^\dagger_E\theta) \\
&= (\partial_E - \tilde{\theta} + \overline{\partial}_E + \tilde{\theta}^\dagger) + 2(\theta^\dagger_E\theta_E + \theta^\dagger_E\theta_E) \\
&= \underbrace{F_h}_{=0 \text{ flat}} -2(\theta_E\overline{\partial}_E + \overline{\partial}_E\theta_E) + \theta^\dagger\tilde{\theta} + (-2+2)(\theta_E\theta^\dagger_E + \theta^\dagger_E\theta_E) \\
&= -2(\theta_E\overline{\partial}_E + \overline{\partial}_E\theta_E) = -2G_h.
\end{aligned}$$

Since $G_{h,\tilde{\nabla}} = 0$ by harmonicity of $\tilde{E}$ this yields

$$\begin{aligned}
-2G_h &= R_{h,\overline{\partial}_E+\theta^\dagger_E} + 2(\theta_E\theta^\dagger_E + \theta^\dagger_E\theta) \\
&= R_{h,\overline{\partial}_E+(\tilde{\theta})^\dagger} + 2\Bigg(\tilde{\theta}(\tilde{\theta})^\dagger + (\tilde{\theta})^\dagger\tilde{\theta} + \frac{\nabla_\Delta}{2}(\tilde{\theta})^\dagger \\
&\quad +(\tilde{\theta})^\dagger\frac{\nabla_\Delta}{2} + \frac{(\nabla_\Delta)^\dagger}{2}\theta + \theta\frac{(\nabla_\Delta)^\dagger}{2} + \frac{(\nabla_\Delta)^\dagger\nabla_\Delta\nabla_\Delta(\nabla_\Delta)^\dagger}{4}\Bigg) \\
&= \underbrace{G_{h,\tilde{\nabla}}}_{=0} + \nabla_\Delta(\tilde{\theta})^\dagger + (\tilde{\theta})^\dagger\nabla_\Delta + (\nabla_\Delta)^\dagger\theta + \theta(\nabla_\Delta)^\dagger \\
&\quad + \frac{(\nabla_\Delta)^\dagger\nabla_\Delta\nabla_\Delta(\nabla_\Delta)^\dagger}{2} \\
\Rightarrow \|G_h\|_H &\le \|\nabla_\Delta\|^2 + 4\|\nabla_\Delta\|\|\tilde{\theta}\| \le c^2_\Delta|z|^{-2+2\varepsilon} + c_\Delta\sqrt{c_5}|z|^{-2+\varepsilon}.
\end{aligned}$$

For $p = 1 + \frac{\varepsilon}{2}$ we get: G_h is L^p-integrable.
Therefore we already have that the curvatures F_h and G_h are L^p and that the bundle is acceptable. That $\tilde{E}$ induces the correct filtration was already proved in the previous lemmas: $e \in E_\beta \Leftrightarrow \varphi(e) \in \Xi(\tilde{E})_\beta \Leftrightarrow \|e\|_h = h(e,e) = \tilde{h}(\varphi(e), \varphi(e)) \leq c_e|z|^{\alpha-\varepsilon}, \forall \varepsilon > 0$.

We are left with the local system case: Analogously to the procedure in chapter IV we consider the eigenspace L^{μ_i} of the local system L, then construct $\Phi(L)$ and equip it with a harmonic metric. When summing up over all eigenvalues the compatibility of Φ with the eigenvalue decomposition allows us to sum up over the $\mathscr{D}_X-$modules with harmonic metric, and we get a harmonic metric on $\Phi(L)$. Therefore it will be enough to consider a local system with one eigenvalue $\tilde{\lambda}$ of the monodromy μ and $\lambda = \frac{\log(\tilde{\lambda})}{-2\pi i}$ the eigenvalue corresponding to a choice of $M = \frac{\log(\mu)}{-2\pi i}$.
Consider the harmonic metric h induced on the corresponding $\mathscr{D}_X-$module. Then we want to show that $l \in L_\beta$, where L_β is now the filtration induced by Φ, if and only if $\|l\|_h \leq c_l r^{\beta-\varepsilon}, \forall \varepsilon > 0$. Let us start with $l \in L_\beta \setminus L_{\beta+\varepsilon} \forall \varepsilon > 0 \Rightarrow k = e^{M\log(z)} l \in \Phi(L)_\alpha \setminus \Phi(L)_{\alpha+\varepsilon}, \forall \varepsilon > 0$ for $\Re(\lambda_M) > \alpha + \beta \Rightarrow k \in \Phi(L)_{\beta+\Re(\lambda)} \setminus \Phi(L)_{\beta+\Re(\lambda)+\varepsilon}$ by left continuity. In the previous part of the proof we constructed a harmonic metric such that $\|k\|_h \sim |z|^{\beta+\Re(\lambda)}$. We have seen before that the Frobenius norm of matrices in an $h-$orthonormal frame is consistent with the $h-$norm - so use the Frobenius norm here. Therefore

$$\begin{aligned} \|l\|_h \leq \|ke^{-M\log(z)}\|_h &\leq \|e^{-M\log(z)}\|_H \|k\|_h \leq c_{48}|z|^{-\Re(\lambda)}|z|^{\beta+\Re(\lambda)-\varepsilon} \\ &= c_{48}|z|^{\beta-\varepsilon}. \end{aligned}$$

where we used that $M \log z = P^{-1}JP$, J Jordan normal form of $M\log(z)$, for example

$$J_{\text{example}} = \begin{pmatrix} \lambda\log(z) & 1 & 0 & 0 & 0 \\ 0 & \lambda\log(z) & 0 & 0 & 0 \\ 0 & 0 & \lambda\log(z) & 1 & 0 \\ 0 & 0 & 0 & \lambda\log(z) & 1 \\ 0 & 0 & 0 & 0 & \lambda\log(z) \end{pmatrix}.$$

Thus $\exp(-M\log(z)) = P^{-1}\exp(-J)P \Rightarrow \|\exp(-M\log(z))\|_H \leq \|P^{-1}\|_H \|\exp(-J\log(z))\|_H \|P\|_H \leq c_{45}\|\exp(-J\log(z))\|_H$ since P is constant. For $J = \lambda\log(z)E + N$, N nilpotent, $N^j = 0$:[5]

$$\|\exp(-J)\|_H \leq \|\exp(-\lambda\log zE)\|_H \|\exp(-N)\|_H$$

[5] J and N commute.

$$\begin{aligned} &\leq c_{46}\|\exp(-\lambda \log zE)\|_H \\ &= c_{46}\,|\exp(-\lambda\log z)|\,\|E\|_H \leq c_{46}\exp(-\Re(\lambda\log z)) \\ &\leq c_{47}\exp(-\lambda\log|z|) = c_{47}|z|^{-\lambda}. \end{aligned}$$

The last step follows by $\log(z) = \log|z| + i\varphi_z, \lambda = \Re(\lambda) + i\Im(\lambda) \Rightarrow \Re(\lambda\log z) = \Re(\lambda)\log|z| - \varphi_z\Im(\lambda)$ and $\varphi_z \in [0, 2\pi]$ bounds $\varphi_z\Im(\lambda)$ independently of z. This amounts to a possibly bigger constant c_{47}.
Recall that Φ was compatible with taking duals. Hence we can repeat the procedure above to estimate $\|l^*\| \leq c_{49}|z|^{-\beta-\varepsilon}$ for $l^* \in (L^*)_{-\beta}$. But $l \in L_\beta$ $\Rightarrow l^* \in (L_\beta)^* = (L^*)_{-\beta}$

$$\begin{aligned} \Rightarrow 1 = l^*(l) \leq \|l^*\|_{h^*}\|l\|_h \leq c_{49}|z|^{-\beta-\varepsilon}\|l\|_h, \forall\varepsilon > 0 \\ \Rightarrow |z|^{\beta+\varepsilon} \geq \|l\|_h, \forall\varepsilon > 0 \end{aligned}$$

Both estimates together lead to $l \sim |z|^\beta$. This is our claim.[6]
We have seen above that we find a metric with acceptable curvature. Not only Ξ is well defined, it is as well compatible with duals, determinants and tensor products. So we get the dual filtration on the dual bundles E^*. Furthermore Φ fulfills the same compatibility properties so we get as well the dual filtration on the dual local system. □

Conclusion 4.3.2. Let E be a $\mathscr{D}_X-$module with a tame harmonic metric k. Then the metric induces filtrations on the corresponding local system L by $l \in L'_\alpha \Leftrightarrow l \in L, \|l\|_k \leq c_l r^{\alpha-\varepsilon}, \forall\varepsilon > 0$. On the other hand k induces a filtration E_α on E and Φ generates a filtration L_α on L. Then $L_\alpha = L'_\alpha$.

Proof. Let E_α be the filtration on E induced by k and L_α the by Φ induced filtered local system. Then by the previous theorem 4.3.1 we find a possibly different metric h on E that induces E_α, too. Since all requirements of 3.0.5 are fulfilled we may conclude that h and k are mutually bounded. But h induces the filtration L'_α on L, so does k. Hence $L_\alpha = L'_\alpha$. □

Remark 4.3.3. Remembering 4.1.2 and 4.2.3 we conclude that the flat section of the $\mathscr{D}_X-$module defined by Ξ w.r.t. the new metric are polynomially bounded.

[6] We used that $l \notin L_{\beta+\varepsilon}, \forall\varepsilon > 0$ in this last conclusion, else $l \leq |z|^{\beta+\varepsilon}$ for some ε is possible, i.e. we could not specify if l really appears at level β in the filtration which is induced by the metric.

5 Main equivalence

5.1. Analytic and Parabolic Degree

The first part of this chapter introduces the notions of degree and stability. We will get three different forms of the Chern-Weil formula, connecting the curvature of a subbundle with the curvature of the original bundle. The formula will particularly help us prove that the analytic degree defined as the integral over the Chern form is the same as the parabolic degree counting "jumps". Same holds for stability. Then we may show that irreducible tame harmonic bundles are mapped onto stable filtered regular Higgs bundle of degree 0 resp. stable filtered regular $\mathscr{D}_X-$modules of degree 0, and that every tame harmonic bundles decomposes into a direct sum of irreducible tame harmonic bundles.

Definition 5.1.1 (Chern form)**.** Let P_n be the homogeneous polynomials in $\det(E + A) = 1 + P_1(A) + \ldots P_n(A)$. For the characteristic polynomial we know that

$$\det(\lambda E - A) = \lambda^n - \operatorname{tr}(A)\lambda^{n-1} + \ldots + (-1)^n \det(A).^1$$

In particular $c_1 = \operatorname{tr}(A)$. Define the $k-$th Chern form as

$$c_k = P_k \left(\frac{i}{2\pi} R_h \right),$$

for the curvature of the unique metric connection R_h compatible with the holomorphic structure of the hermitian vector bundle $(E, h, \overline{\partial}_E)$. In particular $c_1 = \operatorname{tr}(R_h)$.

We want to add some properties of Chern forms. A proof may be found in Huybrechts ([Huy05]), Complex Geometry, p. 197.

[1] Most books on linear algebra.

Lemma 5.1.2. (i) Let E_1 and E_2 be two hermitian vector bundles. Then $E_1 \oplus E_2$ has an induced metric and an induced metric connection. We have $c_1(E_1 \oplus E_2) = c_1(E_1) + c_2(E_2)$. This is a special case of the Whitney sum. In particular higher Chern forms decompose not trivially.

(ii) Let E^* be the dual bundle to E with the induced dual metric and connection. Then $c_k(E^*) = (-1)^k c_k(E)$. In particular $c_1(E^*) = -c_1(E)$.

(iii) For the tensor product of two vector bundles E_1, E_2 - again with induced metric and connection -

$$c_1(E_1 \otimes E_2) = \dim(E_1)c_1(E_2) + \dim(E_2)c_1(E_1).$$

Higher Chern classes are calculated by the product of the total Chern characters.

(iv) Since $e^{\mathrm{tr}X} = \det e^X$ we get for $X = \log(\overline{H})$:

$$\begin{aligned} \mathrm{tr}(\log \overline{H}) &= \log(e^{\mathrm{tr}(\log \overline{H})}) = \log\det e^{\log \overline{H}} = \log\det(\overline{H}) \\ \Rightarrow \overline{\partial}\partial \log\det(\overline{H}) &= \overline{\partial}\partial \mathrm{tr}(\log \overline{H}) = \mathrm{tr}(\overline{\partial}\partial \log \overline{H}) = \mathrm{tr}(R_h) \end{aligned}$$

by linearity of the trace, where R_h is the curvature of the metric connection on the original bundle. Hence the first Chern form of a bundle E coincides with the first Chern form of the determinant bundle of E.

Definition 5.1.3. The analytic degree of a hermitian vector bundle E over X is defined as

$$\deg(E) \;:=\; \pi\Lambda \int_{\overline{X}} c_1(E) = \int_{\overline{X}} (i\Lambda \mathrm{tr}(R_h))\, \mathrm{d}\, x \wedge \mathrm{d}\, y,$$

with R_h the curvature of the unique metric connection compatible with the holomorphic structure.

Remark 5.1.4. By the previous lemma 5.1.2 (iv) we see that the degree is invariant under the transition to the determinant bundle. By (i) we get $\deg(E_1 \oplus E_2) = \deg(E_1) + \deg(E_2)$ and by (iii) $\deg(E_1 \otimes E_2) = \deg(E_1)\dim(E_2) + \deg(E_2)\dim(E_1)$. For the dual bundle we get $\deg(E^*) = -\deg(E)$.

For filtered vector bundles there is as well a notion of degree, which we call parabolic or algebraic degree.

Definition 5.1.5. Let (E, E_α) be a filtered vector bundle. The algebraic or parabolic degree is defined as

$$\text{par-deg}(E, E_\alpha) \;:=\; \pi \deg(E_0) + \pi \sum_{s \in \overline{X}\setminus X} \sum_{0 \le \alpha < 1} \alpha \dim(\mathrm{Gr}_\alpha(E_s)).$$

Here $\deg(E_0)$ is the usual degree of the determinant line bundle:

Definition 5.1.6. The degree of a line bundle is the degree of the corresponding divisor, i.e. for $D = \sum_{s\in S} n_s s \Rightarrow \deg D = \sum_{s\in S} n_s$.
For a vector bundle E the degree is the degree of the corresponding determinant bundle - $\deg(E) := \deg(\det(E))$.

We added here a scalar factor π in comparison with Simpson, because we have the same factor for the analytic degree. In [Sim88] (explanation before lemma 10.5) he adds 2π - the factor two corresponds to $i\Lambda$ instead of $\frac{i}{2}\Lambda$.

Remark 5.1.7. The $\deg(E_0)$ part fulfills the compatibility described above. So does the second part:

(i) The determinant of E has the fibers $\bigwedge_{k=1}^n E_x$ and $\det(E_x)_\alpha = \sum_{r_1+\ldots+r_n=\alpha} \bigwedge_{k=1}^n (E_{r_k})_x$. In order to get a non-vanishing Gr_α in this filtration we need a basis (e_k) of E such that $e_k \in E_{r_k}$ and r_k a jump, i.e. $\alpha = r_1 + \ldots + r_n$ and r_k the jumps of the original filtration of E $\Rightarrow$ par-deg(E) = par-deg$(\det(E))$.

(ii) For $E_1 \oplus E_2$ we have $(E_1 \oplus E_2)_\alpha = (E_1)_\alpha \oplus (E_2)_\alpha$ and therewith $\dim \mathrm{Gr}_\alpha(E_1 \oplus E_2) = \dim \mathrm{Gr}_\alpha(E_1) + \dim \mathrm{Gr}_\alpha(E_2)$.

(iii) For E^* we have $(E^*)_\alpha = E^*_{-\alpha}$ and hence the non-vanishing quotients occur at $-\alpha$ instead of α. The dimension is the same by $E \simeq E^*$. Hence the degree changes by a sign.

(iv) For the tensor product $(E_1 \otimes E_2)_\alpha = \sum_{r+r_2=\alpha} (E_1)_{r_1} \otimes (E_2)_{r_2}$. Thus we will get a non-trivial contribution to the degree only if r_1 and r_2 are each jumps. Note that the weight is $r_1 + r_2$, i.e. it "counts for both jumps" (we are not counting twice). Now we may combine a jump α_1 with each jump of E_2, i.e. get $\dim(E_2)$par-deg(E_1) and analogous $\dim(E_1)$par-deg(E_2) $\Rightarrow$ par-deg$(E_1 \otimes E_2)$ = par-deg$(E_1) \dim(E_2)$ + par-deg$(E_2) \dim(E_1)$.

Lemma 5.1.8. If (E, h) is a holomorphic vector bundle with acceptable metric, then the degree is convergent. If E_α is the filtration induced by Ξ then $\deg(E, h)$ = par-deg(E, E_α).

Proof. If (E, h) is acceptable then the curvature R_h of the metric connection is bounded by $\frac{c_{acc}}{|z|^2 |\log |z||^2}$, and is in particular L^1-integrable. So the degree is absolutely integrable: $\mathrm{tr}(R_h) = \langle R_h, E\rangle_F \leq \|R_h\|_F \|E\|_F = \sqrt{n}\|R_h\|_F$ by Cauchy-Schwarz.
As usual restrict to the punctured unit disc. For more singularities we only need to sum up as done in the definition of the parabolic degree.

Further we know that Ξ is compatible with taking the determinant and that both notions of degree are invariant under the transition to the determinant bundle. Hence it will be enough to prove the lemma for every determinant bundle, i.e. for every line bundle. Let L be our line bundle with filtration $L_\gamma := \Xi(L)_\gamma$, i.e. the sections in L_0 are the $h-$bounded sections. Hence we may identify L_0 with $\mathcal{O}_{\overline{X}}$. In particular the corresponding divisor is 0, i.e. $\deg(L_0) = 0$. Moreover the single jump α, $0 \le \alpha < 1$ in the filtration of L is of dimension 1 for a line bundle. Hence $\deg(L, L_\gamma) = \pi\alpha$. Furthermore we know that $L = L_\alpha$ as $\mathcal{O}_X-$bundles, i.e. we find a $\mathcal{O}_{\overline{X}}$-frame e of L in L_α - $e \sim |z|^\alpha$ in the sense

$$|\log \|e\|_h - \log r^\alpha| \le \tilde{\varepsilon} |\log r|, \forall \varepsilon > 0.$$

In order to show this, note first that $e \in L_\alpha = \Xi(L)_\alpha \Rightarrow \|e\|_h \le c_e r^{\alpha - \tilde{\varepsilon}}, \forall \tilde{\varepsilon} > 0$. We may assume $c_e = 1$ by rescaling e. Obviously this is still in $L_\alpha \Rightarrow \log \|e\|_h \le \log r^\alpha - \tilde{\varepsilon} \log r \Rightarrow \log \|e\|_h - \log r^\alpha \le \tilde{\varepsilon} |\log r|$.
For the other direction use that Ξ induces the dual connection on the dual bundle. Take $e^* \in L^*_\alpha = (L^*)_{-\alpha}$ the dual section - $e^*(e) = 1$. Remember as well that $\|e^*\|_{h^*} = \|e\|_h^{-1}$

$$\begin{aligned}
&\Rightarrow \|e^*\|_{h^*} \le r^{-\alpha - \tilde{\varepsilon}}, \forall \tilde{\varepsilon} > 0 \Rightarrow -\log \|e\|_h \le -\log r^\alpha - \tilde{\varepsilon} \log r, \forall \tilde{\varepsilon} > 0 \\
&\Rightarrow \log \|e\|_h - \log r^\alpha \ge \tilde{\varepsilon} \log r = -\tilde{\varepsilon} |\log r|, \forall \tilde{\varepsilon} > 0 \\
&\Rightarrow |\log \|e\|_h - \log r^\alpha| \le \tilde{\varepsilon} |\log r|.
\end{aligned}$$

Remark 5.1.9. Note that we didn't use any properties of a line bundle here, so the estimate holds in general for every $\alpha \in \Xi(E)_\alpha \setminus \Xi(E)_{\alpha + \tilde{\varepsilon}}$ for a general acceptable bundle E.

Now the curvature of the unique metric connection R_h of L is given by $R_h = \frac{\mathrm{d}\,\overline{z} \wedge \mathrm{d}\,z}{4} \Delta_X \log \|e\|_h$. On the completion however, we get $\Rightarrow \frac{\mathrm{d}\,\overline{z} \wedge \mathrm{d}\,z}{4} \Delta_{\overline{X}} \log \|e\|_h = R_h - 2\pi\alpha\delta_0 \,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}$[2]: First note $\log r^\alpha = \alpha \log r$. But $\frac{1}{2\pi} \log r$ is the Green's function, i.e. a δ-distribution with weight at 0. Therefore $\Delta_{\overline{X}} \log r^\alpha = \Delta_X \log r^\alpha + 2\pi\alpha\delta_0$. We have seen that $\|e\|_h$ differs from $\log r^\alpha$ at most $\tilde{\varepsilon} \log r$. So we still need to show that the Laplacian of $\log \|e\|_h - \log r^\alpha$ does vanish. But

$$\begin{aligned}
&\frac{|\log \|e\|_h - \log r^\alpha|}{\log r} \le \frac{\tilde{\varepsilon} |\log r|}{\log r} = \tilde{\varepsilon}, \quad \forall \tilde{\varepsilon} > 0, \\
\Rightarrow &\frac{|\log \|e\|_h - \log r^\alpha|}{\log r} \to 0, \quad \text{for } r \to 0.
\end{aligned}$$

Let $\Delta|_X$ denote the trivial extension of Δ_X to $\overline{X}$. Then $-\Delta_X(\log \|e\|_h - \log r^\alpha) = -\Delta|_X(\log \|e\|_h - \log r^\alpha) =: -b$. b is a L^1-function since $\Delta_X \log \|e\| = 4R_h$ and $\Delta \log r^\alpha$ are L^1-integrable.

[2] δ_0 is the $\delta-$distribution with weight at 0.

Remark 5.1.10. Note that R_h is L^1 since it is acceptable. Later on we will treat subbundles, where we have an additional perturbation term which is L^1. So the theory applies there too.

By lemma 1.7.9 we get $\Delta_{\overline{X}}(\log \|e\|_h - \log r^\alpha) = \Delta|_X(\log \|e\|_h - \log r^\alpha)$.[3] Now the integral of the Laplacian over a compact domain has to vanish[4] $\Rightarrow 0 = \int_{\overline{X}} \Delta_{\overline{X}}(\log \|e\|_h - \log r^\alpha) = \int_{\overline{X}} \Delta|_X(\log \|e\|_h - \log r^\alpha) = \int_X \Delta_X(\log \|e\|_h - \log r^\alpha)$. This shows the formula ahead.

□

5.2. Chern-Weil formula

In this section we will show that we may calculate the analytic degree of a Higgs resp. $\mathscr{D}_X$-bundle using the curvature of the connection F_h or the pseudo-curvature G_h instead of the curvature of the metric connection R_h.

Proposition 5.2.1 (Chern-Weil)**.** Let E be a metric bundle and F be a holomorphic subbundle, then h induces a metric on F and we get

$$\deg(F, h_F) = i \int_X \operatorname{tr}(\pi \Lambda R_h) - \int_X \|(\overline{\partial}_E)_{\mathrm{End}}(\pi)\|_h^2,$$

with R_h the metric connection on E and π the $h-$orthogonal projection on F.

(i) If moreover E is a Higgs bundle with Higgs field θ and F a Higgs subbundle, i.e. F is preserved by θ, then

$$\deg(F, h_F) = i \int_X \operatorname{tr}(\pi \Lambda F_h) - \int_X \|(d''_E)_{\mathrm{End}}(\pi)\|_h^2,$$

where F_h is the curvature of $\overline{\partial}_E + \partial_E + \theta + \theta^\dagger$, $d''_V = \overline{\partial}_V + \theta$.

(ii) If V is a $\mathscr{D}_X-$module, i.e. $D_V = D'_V + D''_V$ flat, F a $\mathscr{D}_X-$subbundle, i.e. preserved by D'_V and D''_V, then

$$\deg(W, h_W) = i \int_X \operatorname{tr}(-2\pi \Lambda G_h) - \frac{1}{2} \int_X \|(D_V)_{\mathrm{End}}(\pi)\|_h^2,$$

where G_h is the pseudo curvature: $\overline{\partial}_V \theta + \theta \overline{\partial}_V$.

---

[3] Equality since $-\Delta f = -b$ on $X \Rightarrow -\Delta f + b = 0, \Delta f - b = 0$ on $X \Rightarrow -\Delta f \leq -b, \Delta f \leq b$ on $\overline{X} \Rightarrow b \leq \Delta f \leq b$.

[4] cf. Müller, [PDE09], p. 14, lemma 2.6.

Proof. The proof is similar to the one in [Sim90], p. 752f. The general idea can be found in [GH78], p. 78.
Note that

(i) For a Higgs bundle $F_h = \overline{\partial}_E \partial_E + \partial_E \overline{\partial}_E + \theta\theta^\dagger + \theta^\dagger\theta + \overline{\partial}_E\theta + \theta\overline{\partial}_E + \partial_E\theta^\dagger + \theta^\dagger\partial_E = \overline{\partial}_E \partial_E + \partial_E \overline{\partial}_E + \theta\theta^\dagger + \theta^\dagger\theta = R_h + \theta\theta^\dagger + \theta^\dagger\theta$ by the holomorphy of the Higgs field. When calculating the degree we use the trace, but $\mathrm{tr}(\theta\theta^\dagger + \theta^\dagger\theta) = \mathrm{tr}(\theta\theta^\dagger - \theta\theta^\dagger) = 0$, where the "-" is due to the fact that we work with one-forms. We have

$$\deg(E,h) \;=\; i\int_X \Lambda\mathrm{tr}(R_h) = i\int_X \Lambda\mathrm{tr}(F_h).$$

As every Higgs subbundle is a Higgs bundle itself, this notion of degree is available in subbundles too.

(ii) For a $\mathscr{D}_X-$module V we get by direct calculation using A.1.37

$$\begin{aligned} R_h &= (\overline{\partial}_V + \theta^\dagger)(\partial_V - \theta) + (\partial_V - \theta)(\overline{\partial}_V + \theta^\dagger) \\ &= \overline{\partial}_V\partial_V + \partial_V\overline{\partial}_V - (\theta\theta^\dagger + \theta^\dagger\theta) + (\partial_V\theta^\dagger + \theta^\dagger\partial_V) - (\overline{\partial}_V\theta + \theta\overline{\partial}_V) \\ &= \overline{\partial}_V\partial_V + \partial_V\overline{\partial}_V + \theta\theta^\dagger + \theta^\dagger\theta + G_h - G_h \\ &\quad -2(\theta\theta^\dagger + \theta^\dagger\theta) - G_h - G_h \\ &= D^2 - 2(\theta\theta^\dagger + \theta^\dagger\theta) - 2G_h \\ &= -2(\theta\theta^\dagger + \theta^\dagger\theta) - 2G_h. \end{aligned}$$

Again the trace of $-2(\theta\theta^\dagger + \theta^\dagger\theta)$ does vanish:

$$\deg(V,h) \;=\; i\int_X \Lambda\mathrm{tr}(R_h) = i\int_X \Lambda\mathrm{tr}(-2G_h).$$

Of course the same holds for a subbundle.

Let us start with the actual proof. Let $D_E = \partial_E + \overline{\partial}_E$ be our metric connection. We use the Higgs bundle notation here but ∂_E can be any $(1,0)$ part of a connection, such as $\partial_V - \theta$ in the $\mathscr{D}_X-$module notation. Let $F \subset E$ be a holomorphic subbundle. Since F is $\overline{\partial}_E-$invariant, the metric connection is $D_F = \pi\overline{\partial}_E\pi + \pi\partial_E\pi = \overline{\partial}_E\pi + \pi\partial_E\pi$.
Then by $\pi^2 = \pi$ for the orthogonal projection

$$\begin{aligned} R_h^F &= \pi\partial_E\pi\overline{\partial}\pi + \pi\overline{\partial}_E\pi\partial_E\pi \\ &= \pi\partial_E\overline{\partial}\pi + \pi\overline{\partial}_E\partial_E\pi + \pi(\overline{\partial}_E\pi - \pi\overline{\partial}_E)\partial_E\pi \\ &= \pi R_h^E\pi + (\overline{\partial}_E\pi - \pi\overline{\partial}_E)(\partial_E\pi - \pi\partial_E) \end{aligned}$$

$$= \pi R_h^E \pi + (\overline{\partial}_E)_{\mathrm{End}}(\pi)(\partial_E)_{\mathrm{End}}(\pi),$$

where we used in the line before the last $(\overline{\partial}_E\pi - \pi\overline{\partial}_E)\pi = \overline{\partial}_E\pi - \overline{\partial}_E\pi = 0$. We now have

$$\begin{aligned} i\mathrm{tr}(\Lambda R_h^F) &= i\Lambda\mathrm{tr}(\pi R_h^E\pi) + i\Lambda\mathrm{tr}((\overline{\partial}_E)_{\mathrm{End}}(\pi)(\partial_E)_{\mathrm{End}}(\pi)) \\ &= i\Lambda\mathrm{tr}(\pi^2 R_h^E) - i\Lambda\mathrm{tr}((\partial_E)_{\mathrm{End}}(\pi)(\overline{\partial}_E)_{\mathrm{End}}(\pi)) \\ &= i\Lambda\mathrm{tr}(\pi R_h^E) - i\mathrm{tr}(\Lambda(\partial_E)_{\mathrm{End}}(\pi)(\overline{\partial}_E)_{\mathrm{End}}(\pi)). \end{aligned}$$

Now using our Kähler identities (applied to the connection on the endomorphism bundle) we further conclude

$$\begin{aligned} -\mathrm{tr}(i\Lambda(\partial_E)_{\mathrm{End}}(\pi)(\overline{\partial}_E)_{\mathrm{End}}(\pi)) &= -\mathrm{tr}(-((-\overline{\partial}_E)_{\mathrm{End}}(\pi))^*(\overline{\partial}_E)_{\mathrm{End}}(\pi)) \\ &= +\langle(\overline{\partial}_E)_{\mathrm{End}}(\pi), -(\overline{\partial}_E)_{\mathrm{End}}(\pi)\rangle_{HS} \\ &= -\|\overline{\partial}_E)_{\mathrm{End}}(\pi)\|_F^2, \end{aligned}$$

where we used that $(\overline{\partial}_E\pi - \pi\overline{\partial}_E)^* = \pi^*\overline{\partial}_E^* - \overline{\partial}_E^*\pi^* = -(\overline{\partial}_E)_{\mathrm{End}}(\pi)$ as the orthogonal projection π is hermitian. Putting all together we get our Chern-Weil formula

$$\begin{aligned} \int_X i\mathrm{tr}(\Lambda R_h^F) &= \int_X i\Lambda\mathrm{tr}(\pi R_h^E) - \int_X \|(\overline{\partial}_E)_{\mathrm{End}}(\pi)\|_F^2 \\ \Rightarrow \deg(F,h) &= \int_X i\Lambda\mathrm{tr}(\pi R_h^E) - \int_X \|(\overline{\partial}_E)_{\mathrm{End}}(\pi)\|_F^2. \end{aligned} \tag{CW1}$$

For the curvature F_h and G_h the procedure works essentially the same:

(i) Let $D_E = \partial_E + \overline{\partial}_E + \theta + \theta^\dagger = d'_E + d''_E; d'' = \overline{\partial}_V + \theta$ be our connection with curvature F_h^E and F a sub-Higgs bundle. Since F is $\overline{\partial}_E-$invariant as well as $\theta-$invariant we have $D_F = \pi d'_E\pi + \pi d''_E\pi = \pi d'_E\pi + d''_E\pi$.
Note that for a Higgs bundle $0 = \overline{\partial}_E\theta + \theta\overline{\partial}_E = (d''_E)^2 = -(d')^2 = -\partial_E\theta^\dagger - \theta^\dagger\partial_E$ $\Rightarrow F_h^E = d'_E d''_E + d''_E d'_E$. Again using $\pi^2 = \pi$ for the orthogonal projection

$$\begin{aligned} F_h^F &= \pi d'_E\pi\overline{\partial}\pi + \pi d''_E\pi d'_E\pi \\ &= \pi d'_E\overline{\partial}\pi + \pi d''_E d'_E\pi + \pi(d''_E\pi - \pi d''_E)d'_E\pi \\ &= \pi F_h^E\pi + (d''_E\pi - \pi d''_E)(d'_E\pi - \pi d'_E) \\ &= \pi F_h^E\pi + (d''_E)_{\mathrm{End}}(\pi)(d'_E)_{\mathrm{End}}(\pi), \end{aligned}$$

where we used $(d''_E\pi - \pi d''_E)\pi = d''_E\pi - d''_E\pi = 0$. We now have

$$\begin{aligned} i\mathrm{tr}(\Lambda F_h^F) &= i\Lambda\mathrm{tr}(\pi F_h^E\pi) + i\Lambda\mathrm{tr}((d''_E)_{\mathrm{End}}(\pi)(d'_E)_{\mathrm{End}}(\pi)) \\ &= i\Lambda\mathrm{tr}(\pi^2 F_h^E) - i\Lambda\mathrm{tr}((d'_E)_{\mathrm{End}}(\pi)(d''_E)_{\mathrm{End}}(\pi)) \\ &= i\Lambda\mathrm{tr}(\pi F_h^E) - i\mathrm{tr}(\Lambda(d'_E)_{\mathrm{End}}(\pi)(d''_E)_{\mathrm{End}}(\pi)). \end{aligned}$$

Applying our Kähler identities (applied to the connection on the endomorphism bundle) we get

$$\begin{aligned}
-\mathrm{tr}(i\Lambda(d'_E)_{\mathrm{End}}(\pi)(d''_E)_{\mathrm{End}}(\pi)) &= -\mathrm{tr}(-((-d''_E)_{\mathrm{End}}(\pi))^*(d''_E)_{\mathrm{End}}(\pi)) \\
&= \langle (d''_E)_{\mathrm{End}}(\pi), -(d''_E)_{\mathrm{End}}(\pi)\rangle_{HS} \\
&= -\|d''_E)_{\mathrm{End}}(\pi)\|_F^2,
\end{aligned}$$

where we used again that the orthogonal projection π is hermitian (cf. the metrized bundle case). Putting all together we get our Chern-Weil formula

$$\begin{aligned}
\int_{\overline{X}} i\mathrm{tr}(\Lambda F_h^F) &= \int_{\overline{X}} i\Lambda\mathrm{tr}(\pi F_h^E) - \int_{\overline{X}} \|(d''_E)_{\mathrm{End}}(\pi)\|_F^2 \\
\Rightarrow \deg(F,h) &= \int_{\overline{X}} i\Lambda\mathrm{tr}(\pi F_h^E) - \int_{\overline{X}} \|(d''_E)_{\mathrm{End}}(\pi)\|_F^2. \qquad \text{(CW2)}
\end{aligned}$$

(ii) The third case of interest is the degree in terms of the pseudo-curvature G_h. Let $W \subset V$ be a sub$-\mathscr{D}_X-$ module: $\nabla_V = \partial_V + \theta$ and $D''_V = \overline{\partial}_V + \theta^\dagger$ preserve $W \Rightarrow D_V = \nabla_V + D''_V$ preserves W. Then the by D_V induced connection D_W satisfies

$$D_W = \pi D_V \pi = D_V \pi.$$

Now we may write the pseudo-curvature as

$$\begin{aligned}
4G_h^V &= D_V(d''_V - d'_V) + (d''_V - d'_V)D_V \\
&= \partial_V\overline{\partial}_V - \partial_V\theta^\dagger - \overline{\partial}_V\partial_V + \overline{\partial}_V\theta + \theta\overline{\partial}_V - \theta\theta^\dagger + \theta^\dagger\theta - \theta^\dagger\partial_V \\
&\quad + \overline{\partial}_V\partial_V + \overline{\partial}_V\theta - \partial_V\theta^\dagger - \partial_V\overline{\partial}_V + \theta\theta^\dagger + \theta\overline{\partial}_V - \theta^\dagger\partial_V - \theta^\dagger\theta \\
&= 2(\overline{\partial}_V\theta + \theta\overline{\partial}_V) - 2(\partial_V\theta^\dagger + \theta^\dagger\partial_V) \\
&= 4(\overline{\partial}_V\theta + \theta\overline{\partial}_V)
\end{aligned}$$

using the proof of lemma 1.2.6. Let $D_V^c := d''_V - d'_V$. Thus

$$\begin{aligned}
4G_h^W &= D_W D_W^c + D_W^c D_W \\
&= \pi D_V \pi D_V^c \pi + \pi D_V^c \pi D_V \pi \\
&= \pi D_V D_V^c \pi + \pi(D_V\pi - \pi D_V)D_V^c\pi + \pi D_V^c D_V \pi \\
&= 4\pi G_h^V \pi + (D_V\pi - \pi D_V)(D_V^c\pi - \pi D_V^c) \\
&= 4\pi G_h^V \pi + (D_V)_{\mathrm{End}}(\pi)(D_V^c)_{\mathrm{End}}(\pi)
\end{aligned}$$

since $D_V\pi^2 - \pi D_V \pi = D_V\pi - D_V\pi = 0$. Further use that

$$\delta''_V - \delta'_V = \left(\frac{D''_V + \delta''_V}{2}\right) + \left(\frac{D'_V - \delta'_V}{2}\right) - \left(\frac{D'_V + \delta'_V}{2}\right) - \left(\frac{D''_V - \delta''_V}{2}\right)$$

$$= d''_V - d'_V.$$

and therefore $D_V D^c_V = D'_V \delta''_V - D''_V \delta'_V$ by degree considerations.

$$\begin{aligned}
& i\mathrm{tr}(\Lambda(4G^W_h)) \\
&= i\Lambda\mathrm{tr}(\pi(4G^V_h)\pi) + i\Lambda\mathrm{tr}((D'_V)_{\mathrm{End}}(\pi)(\delta''_V)_{\mathrm{End}}(\pi) - (D''_V)_{\mathrm{End}}(\pi)(\delta'_V)_{\mathrm{End}}(\pi)) \\
&= i\Lambda\mathrm{tr}(\pi^2(4G^V_h)) - i\Lambda\mathrm{tr}((\delta''_V)_{\mathrm{End}}(\pi)(D'_V)_{\mathrm{End}}(\pi) - (\delta'_V)_{\mathrm{End}}(\pi)(D''_V)_{\mathrm{End}}(\pi)) \\
&= i\Lambda\mathrm{tr}(\pi(4G^V_h)) - i\mathrm{tr}(\Lambda(\delta''_V)_{\mathrm{End}}(\pi)(D'_V)_{\mathrm{End}}(\pi) - \Lambda(\delta'_V)_{\mathrm{End}}(\pi)(D''_V)_{\mathrm{End}}(\pi))
\end{aligned}$$

Applying our Kähler identities we get this time

$$\begin{aligned}
& -i\mathrm{tr}(\Lambda(\delta''_V)_{\mathrm{End}}(\pi)(D'_V)_{\mathrm{End}}(\pi) - \Lambda(\delta'_V)_{\mathrm{End}}(\pi)(D''_V)_{\mathrm{End}}(\pi)) \\
&= -i\mathrm{tr}(((-D'_V)_{\mathrm{End}}(\pi))^*(D'_V)_{\mathrm{End}}(\pi) + ((-D''_V)_{\mathrm{End}}(\pi))^*(D''_V)_{\mathrm{End}}(\pi)) \\
&= \langle (D'_V + D''_V)_{\mathrm{End}}(\pi), (D'_V + D''_V)_{\mathrm{End}}(\pi)\rangle_{HS} \\
&= \|(D_V)_{\mathrm{End}}(\pi)\|^2_F,
\end{aligned}$$

where we used the hermitian property of the orthogonal projection π as before. Putting all together we get our Chern-Weil formula

$$\begin{aligned}
\int_{\overline{X}} i\mathrm{tr}(\Lambda(-2G^W_h)) &= \int_{\overline{X}} i\Lambda\mathrm{tr}(\pi(-2G^V_h)) - \frac{1}{2}\int_{\overline{X}} \|(D_V)_{\mathrm{End}}(\pi)\|^2_F \\
\Rightarrow \deg(W,h) &= \int_{\overline{X}} i\Lambda\mathrm{tr}(\pi(-2G^V_h)) - \frac{1}{2}\int_{\overline{X}} \|(D_V)_{\mathrm{End}}(\pi)\|^2_F. \quad \text{(CW3)}
\end{aligned}$$

□

Lemma 5.2.2. Let E be a metrized vector bundle with acceptable metric h and let $F \subset E$ be a holomorphic subbundle with induced metric h_F. Then either

(1) $\deg(F,h) = -\infty$ or,

(2) $\deg(F,h)$ finite and F extends to a filtered subbundle with $\deg(F, F_\alpha) = \deg(F)$. Here F_α is induced by Ξ.

(3) If $\deg(F,h)$ is finite and $\overline{E}$ is a meromorphic completion over $\overline{X}$, i.e. h has meromorphic growth w.r.t. a basis of $\overline{E}$, then F extends to a subsheaf $\overline{F} \subset \overline{E}$.

Proof. Since R_h is acceptable we know that $\mathrm{tr}(R_h)$ is L^1-integrable. So the first term in the Chern-Weil formula (CW1) is finite. The second is negative and therefore $\deg(F,h) = -\infty$ or finite. So let us assume that the degree is finite. Then the curvature of the subbundle is L^p-acceptable and hence $\Xi(F)$ is a filtered vector bundle. The proof that the degrees coincide for the subbundle F is the same as for the general bundle E using remark 5.1.10. For the third property see [Sim88], p. 915, lemma 10.6.[5] □

Remark 5.2.3. Note as well that for an analytic degree of $-\infty$ the parabolic degree will be $-\infty$ too, since $\deg(L_0) = -\infty$.

5.2.1. Slope

Definition 5.2.4. (i) Let (E, h) be a metrized bundle. E is called analytically stable if for all proper subbundles F

$$\frac{\deg(F,h)}{\operatorname{rank}(F)} < \frac{\deg(E,h)}{\operatorname{rank}(E)}.$$

$\frac{\deg(E,h)}{\operatorname{rank}(E)}$ is called slope of E.

(ii) If moreover E is a Higgs bundle, E is analytically stable if for all proper Higgs subbundles F preserved by θ

$$\frac{\deg(F,h)}{\operatorname{rank}(F)} < \frac{\deg(E,h)}{\operatorname{rank}(E)}.$$

(iii) If V is a $\mathscr{D}_X-$module it is analytically stable if for all proper sub-$\mathscr{D}_X-$modules W preserved by ∇

$$\frac{\deg(W,h)}{\operatorname{rank}(W)} < \frac{\deg(V,h)}{\operatorname{rank}(V)}.$$

Definition 5.2.5. (i) Let (E, h) be a metrized bundle. E is called algebraically stable if for all proper subbundles F

$$\frac{\deg(F,F_\alpha)}{\operatorname{rank}(F)} < \frac{\deg(E,E_\alpha)}{\operatorname{rank}(E)}.$$

(ii) If moreover E is a Higgs bundle, E is stable if for all proper Higgs subbundles F preserved by θ

$$\frac{\deg(F,F_\alpha)}{\operatorname{rank}(F)} < \frac{\deg(E,E_\alpha)}{\operatorname{rank}(E)}.$$

[5] Use the Plücker embedding to reduce to the line bundle case. Then modify the metric as before to apply an estimate by Aubin, giving us a L^2-bounded section, holomorphic outside the puncture, i.e. holomorphic everywhere. This section extends the line bundle to a meromorphic subsheaf.

(iii) If V is a $\mathscr{D}_X$−module it is stable if for all proper sub-$\mathscr{D}_X$−modules W preserved by ∇

$$\frac{\deg(W, W_\alpha)}{\operatorname{rank}(W)} < \frac{\deg(V, V_\alpha)}{\operatorname{rank}(V)}.$$

Remark 5.2.6. (i) The Chern-Weil formulas $(CW1), (CW2), (CW3)$ guarantee that this definition makes sense - for $-\infty$ the condition is trivially satisfied.

(ii) We have semi-stability if we replace $<$ with $\leq$ in the definitions.

(iii) If (E, θ) is stable as a Higgs bundle, E is not necessarily a stable vector bundle, since there might be a subbundle, not preserved by θ that contradicts stability in the vector bundle case.

Conclusion 5.2.7. If E is an acceptable vector/Higgs/$\mathscr{D}_X$−bundle and $E_\alpha = \Xi(E)_\alpha$ is the induced filtered object. Then (E, h) is analytically stable, if and only if (E, E_α) is algebraically stable.

Proof. Directly by 5.2.2. □

Definition 5.2.8. A vector/Higgs/$\mathscr{D}_X$−bundle is called irreducible, if it has no non-trivial holomorphic/Higgs subbundles resp. sub-$\mathscr{D}_X$−modules.

Remark 5.2.9. Every irreducible bundle is a priori stable.

Theorem 5.2.10. Suppose $(E, \overline{\partial}_E, h, \theta)$ is an irreducible tame harmonic bundle. Then the filtered Higgs resp. $\mathscr{D}_X$−bundle induced by Ξ has degree 0 and is stable. Furthermore every tame harmonic bundle is the direct sum of irreducible ones.

Proof. We will treat the Higgs bundle case first, then the $\mathscr{D}_X$−module case. The proofs are the same up to renaming.

(i) By harmonicity we have $F_h^E = 0$, i.e. the analytic and therefore the parabolic degree vanishes. Now assume that E was not stable, i.e. that we find a Higgs subbundle F of E with $\deg(F, h) \geq 0$. Then the Chern-Weil formula (CW2) tells us that

$$-\int_X |d''_{\text{End}}(\pi)|_F^2 = \deg(F, h) \leq 0 \Rightarrow (d''_E)_{\text{End}}(\pi) = 0.$$

Now apply the proof of theorem 3.0.6, namely H(i). Note that we have π bounded.[6] There we proved that $(d''_E\pi - \pi d''_E) \Rightarrow D_E\pi - \pi D_E = 0$, i.e. F is a

[6] It obviously preserves the filtration.

harmonic subbundle of E. Thus E was either not irreducible - then we may repeat the procedure to gain an irreducible subbundle - or if E was already irreducible we have a contradiction for non-trivial $F \Rightarrow F = 0$. Moreover $d''_E\pi - \pi d''_E = 0 \Rightarrow \overline{\partial}_E\pi = \pi\overline{\partial}_E$ and so E/F is a holomorphic subbundle, too. The projection onto E/F is $(1-\pi)$.[7] This yields $d''_E\pi = \pi d''_E \Rightarrow d''_E(1-\pi) = (1-\pi)d''_E$ and analogous $D_E(1-\pi) = (1-\pi)D_E$, i.e. E/F becomes a harmonic bundle, too. Thus we found a direct sum decomposition into harmonic subbundles - repetition leads to a direct sum decomposition into irreducible tame harmonic subbundles.[8]

(ii) By harmonicity we have $G_h^E = 0$, i.e. the analytic and therefore the parabolic degree vanishes. Now assume that E was not stable, i.e. that we find a sub-$\mathscr{D}_X$-module W of E with $\deg(W,h) \geq 0$. Then the Chern-Weil formula (CW2) tells us that

$$-\frac{1}{2}\int_X |D_{\text{End}}(\pi)|_F^2 = \deg(W,h) \leq 0 \Rightarrow (D_E)_{\text{End}}(\pi) = 0.$$

Now apply the proof of theorem 3.0.6, namely D(ii). Note that we have π bounded. There we proved that $(D_E\pi - \pi D_E) \Rightarrow d''_E\pi - \pi d''_E = 0$, i.e. W is a harmonic sub-bundle of E. Thus E was either not irreducible - then we may repeat the procedure to gain an irreducible subbundle - or if E was already irreducible we have a contradiction for non-trivial $W \Rightarrow W = 0$. From here on we may as well imply that we find a direct sum decomposition, since we are now in the same situation as in (i), namely W is a Higgs subbundle too.

□

5.3. Main Existence theorem

As the title might suggest, we will finally get a harmonic metric. The proof is not a part of Simpson's [Sim90]-article and we originally planned to simply cite the result from his previous work. However, for the convenience of the reader we will cite the most of the intermediate steps with some further explanations.
In [Sim88] Simpson proves much more general the existence of a Hermitian-Einstein-metric for every non-compact Kähler manifold, that fulfills certain assumptions described below. We are not sure if there is an essentially shorter proof if one only searches for harmonic metrics. However, here is the theorem:

[7] Here 1 is the identity.
[8] The subbundles are tame since the restriction of θ has the same eigenvalues (or less) than θ on E.

Theorem 5.3.1. (1) Let E be a Higgs bundle on X and h a metric on E. If F_h is in $L^p, p > 1$ with analytic degree 0 and stable, then there is a harmonic metric k which is bounded with respect to h.

(2) Suppose V is an analytic $\mathscr{D}_X-$module on X with a metric h on V. If the pseudo-curvature G_h is in L^p for some $p > 1$, V has degree 0 and is stable, then there is a harmonic metric k on V bounded in terms of h.

We will start with some preliminaries, since they suit our results on hermitian matrices of chapter one. We will namely extend the concept of divided sums. The idea is to cover the space of positive-definite matrices (with L^p-curvature) $\mathscr{P}$ by another space $\mathcal{P}_{\tilde{D}}(\mathcal{S}_k)$, with a local diffeomorphism Ke^S for $S \in \mathcal{P}_{\tilde{D}}(\mathcal{S}_k)$ and K our initial metric in matrix form. If our bundle is stable, we can use a result of Uhlenbeck-Yau [UY86] and Donaldson's functional to bounded S.
The rest of the proof consists of technical details, how to extend the solution H_t of a heat equation involving the traceless part of our curvature F_h resp. G_h. A proof is included in [Sim88], but the proofs of [Don85] resp. the methods of [Ham75] work with some modifications.
The last step is to use $H_t = Ke^S$ for some S and our bound on S to bound H_t, and hence find a weakly convergent subsequence. After restriction to a further subsequence $t \to \infty$, H_t converges to H_∞ with vanishing trace-free curvature. Our degree assumption implies $F_{H_\infty} = 0$, i.e. harmonicity.
Note that in this section we usually work on some endomorphism bundle. Hence we may omit the additional index End, adding it only if we want to use special properties of the connection on the endomorphism bundle, like $D(E) = DE - ED = 0$ for the identity matrix E.

5.3.1. Further properties of divided sums

The concept of divided sums introduced when we first talked about harmonic maps, can be generalized to other functions than exp. This works in exactly the same way as for exp. Even further we may extend this notion to any differential operator D that respects the Leibniz rule, since this is the only property explicitly used above. Of course a different connection matrix than E will lead to

$$Df(H)(B) = f^{\Delta}(H) \bullet (DB).$$

Note that above we chose an orthonormal frame. However if our function has a power series expansion we may move a basis transformation matrix inside - $P^{-1}f(H)P = f(P^{-1}HP)$. Hence the property holds for other operators, too. If we further extend to sections into $\mathbb{H}_n$ this will add a differential ∂ or $\overline{\partial}$ on both sides. Still this looks like

$$Df(H)(B) = f^{\Delta}(H) \bullet (DB).$$

Since self-adjointness depends on a chosen inner product we might consider $\mathbb{H}_n$ as the space $\mathcal{S}_k$ of hermitian operators w.r.t. some initial metric k on E. k induces as norm on the endomorphism bundle $\langle A, B\rangle_K = \mathrm{tr}(KBKA)$.
Now we need to describe how this construction does work on Sobolev spaces $H^{p,q}$. Let's denote by $H^{p,1}(\mathcal{S}_k)$ the space of sections $H : X \to \mathbb{H}_n$ such that $H \in L^p(\mathcal{S}_k)$ and $d''(H) \in L^p(\mathcal{S}_k)$. For $\mathscr{D}_X-$module denote by $H^{p,1}(\mathcal{S}_k)$ the space of sections $H : X \to \mathbb{H}_n$ such that $H \in L^p(\mathcal{S}_k)$ and $DH \in L^p(\mathcal{S}_k)$. The Sobolev space has the usual Sobolev norm $\|f\|_{H^{p,q}} = \left(\sum_{i=0}^q \|f^{(i)}\|_{L^p}^p\right)^{1/p}$. Then for $b > 0$:

(i) $H_{d''}^{p,0,b} = \{H \in H_{d''}^p \| \|H\|_K \leq b\}$ resp. $H_D^{p,0,b} = \{H \in H_D^p \| \|H\|_K \leq b\}$.

(ii) $H_{d''}^{p,1,b} = \{H \in H_{d''}^p \| \|H\|_K, \|d''(H)\|_K \leq b\}$ resp. $H_D^{p,1,b} = \{H \in H_D^p \| \|H\|_K, \|D(H)\|_K \leq b\}$.

Finally define the subspace of smooth section with finite norm by $\mathcal{P}_{d''}(\mathcal{S}_k)$ and $\mathcal{P}_{d''}(\mathcal{S}_k)$. Here the norms are $\|H\|_{d''} = \sup_X \|H\|_F + \|d''(H)\|_{L^2} + \|i\Lambda d''d'(H)\|_{L^1}$ and $\|H\|_D = \sup_X \|H\|_F + \|D(H)\|_{L^2} + \|i\Lambda(D''\delta' - D'\delta'')(H)\|_{L^1}$. The L^2- resp. L^1-norms shall be understood as $\int_X \|\cdot\|_K^2$ resp. $\int_X \|\cdot\|_K$.

Remark 5.3.2. We will often omit the indicies for the connection - d'' or D. It will either be clear which one is meant or we may treat both cases in one. Usually we will use $\tilde{D}$ as operator if both choices work the same way.

Remark 5.3.3. Let φ[9] be a smooth function $\mathbb{R} \to \mathbb{R}$ and $\varphi : \mathcal{S}_k \to \mathcal{S}_k$ the bounded linear operator given by the continuous functional calculus. Let e_i be an orthonormal frame of (E, k) consisting of eigenvectors of $H \in \mathcal{S}_k$.[10] Then $\varphi(H)(e_i) = f(\lambda_i)e_i$. In the same matter let Ψ be a function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and let $\mathcal{S}(\mathrm{End}, k)$ denote the space of hermitian operators from $\mathrm{End}(E) \to \mathrm{End}(E)$. Here hermitian is w.r.t. the inner product $\mathrm{tr}(K \cdot K \cdot)$. Choose again an $H-$orthonormal frame (and the induced frame e_{ij}^{End} on End). Define $\Psi(H)(A) = \Psi(\lambda_i, \lambda_j)A_{ij}e_{ij}^{\mathrm{End}}$. At least linearity in A and continuity in H can be read of this formula.

Proposition 5.3.4. Let φ, Ψ be the smooth functions constructed ahead.

(i) The map φ extends to a continuous map $\varphi : H^{p,0,b}(\mathcal{S}_k) \to H^{p,0,b'}(\mathcal{S}_k)$ for some b'.

(ii) The map Ψ extends to a map

$$\Psi : H^{p,0,b}(\mathcal{S}_k) \to \mathrm{Hom}(L^p(\mathrm{End}(E)), L^q(\mathrm{End}E)),$$

for $q \leq p$. If we can choose $q < p$, Ψ is continuous w.r.t. the operator norm.

[9] We changed f to φ to stay comparable with [Sim88].
[10] H hermitian, hence diagonalizable, hence such a basis exists (in particular every eigenspace is one-dimensional, i.e. the eigenvectors are single-valued).

(iii) The map φ extends to a map

$$\varphi : H^{p,1,b}(\mathcal{S}_k) \to H^{p,0,b'}(\mathcal{S}_k)$$

for $q \leq p$. For $q < p$ we get again continuity. The formula $d''\varphi(H)(H) = \varphi^{\Delta}(H) \bullet (d''H)$ resp. $D\varphi(H)(H) = \varphi^{\Delta}(H) \bullet (DH)$ is still well-defined.

(iv) If φ and Ψ have a power series expansion with infinity radius of convergence, the

$$\begin{aligned} \varphi &: \mathcal{P}_{\tilde{D}}(\mathcal{S}_k) \to \mathcal{P}_{\tilde{D}}(\mathcal{S}_k) \\ \Psi &: \mathcal{P}_{\tilde{D}}(\mathcal{S}_k) \to \mathcal{P}_{\tilde{D}}(\mathcal{S}(\mathrm{End}, k)), \end{aligned}$$

are analytic, too. Here $\tilde{D}$ can either be d'' or D.

Proof. (i) φ is continuous by the functional calculus. Then $\|\varphi(H_1)-\varphi(H_2)\|_F \leq c_{50}\|s_1 - s_2\|_F \leq 2bc_{50} = b'$ and $\|\Psi(H_1)(A) - \Psi(H_2)(A)\|_F \leq \|\Psi(H_1) - \Psi(H_2)\|_F\|A\|_F \leq c_{51}\|s_1-s_2\|_F\|A\|_F \leq 2b\|A\|_F$. Integration over some power p (of the first inequality) leads to (i) in the L^p-norm.

(ii) Let $H \in H^{p,0,b}(\mathcal{S}_k), A \in L^p(\mathrm{End}E)$. We have to show that $\Psi(H)(A) \in L^q(\mathrm{End}E)$: $\|H_1 - H_2\| \leq 2b$, in particular $\|H_1 - H_2\|$ in every L^p-space. Thus we will always find a constant c_{52} depending on b and the area of integration such that $\|H_1 - H_2\|_{L^r} \leq c_{52}\|H_1 - H_2\|_{L^p}$ for every $r, p \geq 1$. Let $q < p$ and $\frac{1}{r} + \frac{1}{p} = \frac{1}{q} \Rightarrow r = \frac{pq}{p-q}$. Then by Hölder inequality[11]

$$\begin{aligned} \|\Psi(H_1)(A) - \Psi(H_2)(A)\|_{L^q} &\leq \|\Psi(H_1) - \Psi(H_2)\|_{L^r}\|A\|_{L^p} \\ &\leq c_{53}\|H_1 - H_2\|_{L^r}\|A\|_{L^p} \\ &\leq c_{52}c_{53}\|H_1 - H_2\|_{L^p}\|A\|_{L^p}. \end{aligned}$$

Thus we have continuity and well-definiteness. For $p = q$ directly by $\|\Psi(H_1)(A) - \Psi(H_2)(A)\|_{L^q} \leq \|H_1 - H_2\|_{L^p}\|A\|_{L^p}$.

(iii) The same calculation as in (ii) can be applied to φ^{Δ} instead of Ψ. Then $\|\varphi^{\Delta}(H_1)-\varphi^{\Delta}(H_2)\|_{L^q} \leq c_{54}\|H_1-H_2\|_{L^p}$ and for $\tilde{D}\varphi(H)(H) = \varphi^{\Delta}(H)\bullet(\tilde{D}H)$ we get

$$\begin{aligned} &\|\tilde{D}\varphi(H_1) - \varphi(H_2)\|_{L^q} \\ &= \|\varphi^{\Delta}(H_1) \bullet (\tilde{D}(H_1 - H_2)) + (\varphi^{\Delta}(H_1) - \varphi^{\Delta}(H_2)) \bullet (\tilde{D}(H_2))\|_{L^q} \\ &\leq \|\varphi^{\Delta}(H_1)\|_{L^p} \underbrace{\|\tilde{D}(H_1 - H_2)\|_{L^p}}_{\in H^{p,1}} + \|\varphi^{\Delta}(H_1) - \varphi^{\Delta}(H_2)\|_{L^p}\|\tilde{D}(H_2)\|_{L^p} \end{aligned}$$

[11] $\Psi(H)$ acts by entry-wise multiplication on the matrix valued function A.

$$\leq b'\|H_1 - H_2\|_{H^{p,1}} + \|\varphi^\Delta(H_1) - \varphi^\Delta(H_2)\|_{L^p}\|H_2\|_{H^{p,1}}.$$

This shows the claim for $p < q$. For $p = q$ we get the same result as in (ii). This holds for $\tilde{D} = d''$ as well as $\tilde{D} = D$.

(iv) Let $\varphi(H) = \sum_{k=0}^{\infty}\varphi_k H^k$ and $\Psi(H)(A) = \sum_{j,k=0}^{\infty}\Psi_{jk}H^j A H^k$. It will be enough to show that $\|H^m\|_\infty \leq c_m\|H\|_\infty^m$, i.e. use a $k-$orthonormal frame and the Frobenius norm. Then the operators on $\mathcal{P}(\mathcal{S}_k)$ will have a convergent power series extension. Pointwise by chain rule:

$$\|\tilde{D}H^m\|_F \leq m\|H\|_F\|\tilde{D}H\|_F, \quad \tilde{D} = d'' \text{ or } \tilde{D} = D$$

and further

$$\begin{aligned}
&\|i\Lambda d''d'(H^m)\|_F \\
&= \|i\Lambda(d''(mH^{m-1}d'(H)))\|_F \\
&= \|i\Lambda(d''(mH^{m-1})d'(H) + mH^{m-1}d''d'(H))\| \\
&= \|i\Lambda \mathrm{tr}(m(m-1)H^{m-2}(d''(H))d'(H) + mH^{m-1}d''d'(H))\|_F \\
&\leq m\|H^{m-1}\|_F\|i\Lambda d''d'(H)\|_F + m(m-1)\|H^{m-2}\|_F\|d''(H)\|_F\|d'(H)\|_F \\
&\leq m\|H\|_F^{m-1}\|i\Lambda d''d'(H)\|_F + m(m-1)\|H\|_F^{m-2}\|d''(H)\|_F^2.^{12}
\end{aligned}$$

Here $\|d'(H)\|_F = \|d''(H)\|_F$ for H hermitian[13]. Then the first term will be compensated by $c_m\|i\Lambda d''d'(H)\|_{L^1}^m$ within $c_m\|H\|_\infty^m$ and the second term by $c_m\|d''(H)\|_{L^2}^m$.
The same calculation shall be done for D:

$$\begin{aligned}
&\|i\Lambda(\delta''D' - D'\delta'')(H^m)\|_F \\
&= \|i\Lambda(\delta''(mH^{m-1}D'(H)) - \delta'(mH^{m-1}D''(H)))\|_F \\
&= \|i\Lambda \mathrm{tr}(m(m-1)H^{m-2}(\delta''(H))D'(H) + mH^{m-1}\delta''D'(H) \\
&\quad -m(m-1)H^{m-2}(\delta'(H))D''(H) - mH^{m-1}\delta'D''(H))\|_F \\
&\leq m\|H\|_F^{m-1}\|i\Lambda(\delta''D' - \delta'D'')(H)\|_F + m(m-1)\|H\|_F^{m-2}\|D(H)\|_F^2,
\end{aligned}$$

where we used similar to the calculation of the Chern-Weil formula (CW3) that for H hermitian $i\Lambda\delta'(H)D''(H) = (D''(H))^*D''(H)$ as well as $i\Lambda\delta''(H)D'(H) = -(D'(H))^*D'(H)$. This shows the claim.

□

5.3.2. Donaldson's functional

Before we start with the main trick let us cite the conditions imposed by Simpson

[12] Submultiplicity of the Frobenius norm.
[13] Compare with the calculation of the Chern-Weil formula, where π was hermitian.

Assumption 1. *X has a Kähler metric ω and finite volume.*

Assumption 2. *There exists a smooth function $\phi : X \to \mathbb{R}_{\geq 0}$ with $\{x \in X | \phi(x) \leq a\}, \forall a \in \mathbb{R}$ as well as $0 \leq i\partial\overline{\partial}\phi \leq C\omega$ for some $C > 0$.*

Assumption 3. *There is an increasing function $a : [0, \infty[\to [0, \infty[$ with $a(0) = 0$ and $a(x) = x$ for $x > 1$, such that if f is a bounded positive function on X with $\Delta(f) \leq b$ for some $b \in L^p, p > 1$ then*

$$\sup_X |f| \leq c_b a \left(\int_X |f| \right).$$

Furthermore, if $\Delta(f) \leq 0$ then $\Delta(f) = 0$.

However in future we don't have to care about these conditions any more:

Remark 5.3.5. Every Riemannian metric on a Riemann surface is a Kähler metric. In complex dimension one every open domain is pseudoconvex, i.e. there is a continuous pluriharmonic function resp. a bounded pluriharmonic function ϕ (Simpson gives a more general result in [Sim88], 2.2.) To prove assumption 3 use our weak extension from lemma 1.7.9. Then we may consider the compact case, proved by Donaldson [Don87] or Simpson 2.1.
On can extend further to arbitrary metrics $v(z)\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}$ on the unit disc if v is in L^p for some $p > 1$.[14] Then the Laplacian $\Delta_0 = v\Delta$ with Δ_0 the Laplacian w.r.t. the euclidean metric. Hence for $\Delta f \leq B \Rightarrow \Delta_0 f \leq Bv$ on X is still L^1 and we may extend again by lemma 1.7.9.

Remark 5.3.6. Simpson states the assumption with b constant, however proves it for $b \in L^p, p > 1$. Further he always requires the curvature to be uniformly bounded in order to use the version of assumption 3 stated in [Sim88]. Since our existence theorem expects a curvature in $L^p, p > 1$ we will use the more general version.

Remember that the exponential map was a map from the space of hermitian matrices to the space of positive-definite ones. We may us it again to define a map $\mathcal{P}_{\tilde{D}}(\mathcal{S}_k) \to \mathscr{P}_k$, to the space of positive-definite matrices w.r.t. k that satisfy $\int_X \|\Lambda F_k\|_K < \infty$. The map $S \mapsto Ke^S$ is a diffeomorphism around 0 and K into this space, which is even analytic by the previous section (5.3.4, part (iv)). Note that the curvature $F_{ke^S} = (d'')^2 + (d'_k)^2 + (d''d'_k + d'_k d'')$ is integrable if and only if $\|S\|_{d''}$ is finite. Same holds for F replaced by the pseudo-curvature G_{ke^S}.
This is a consequence of [Sim88], 3.1 (c) below and $\mathrm{tr}(d''(e^S)d'(e^S)) = \|d''(e^S)\|_F = \|d'(e^S)\|_F$ as well as the assumption that k has already L^p-curvature.

[14] cf. [Sim88], p. 875, proposition 2.4.

Lemma 5.3.7. If d'_k is the usual $d'-$operator constructed with respect to the initial metric k (and analogously $\theta^\dagger_k$ and so on), then for a metric h with $H = Ke^S$:

$$\begin{aligned} d'_h &= d'_k + S^{-1}d'_k(S) \\ d''(d'_k(e^S)) &= e^S(F_h - F_k) + d''(e^S)e^{-S}d'_k(e^S); \end{aligned}$$

and for $\mathscr{D}_X-$modules:

$$(D''\delta'_k - D'\delta''_k)(e^S) = 4e^S(G_h - G_k) + D''(e^S)e^{-S}\delta'_k(e^S) - D'(e^S)e^{-S}\delta''_k(e^S).$$

Remark 5.3.8. The lemma holds for e^S replaced by any hermitian matrix say A, too.

proof of lemma 5.3.7. First we know that

$$\begin{aligned} h(\theta\xi, \eta) &= (\eta)^* H\theta\xi = (\eta)^* K e^S \theta\xi \\ &= k(e^S\theta\xi, \eta) = k(\xi, \theta^\dagger_k e^S \eta) \\ &= (\theta^\dagger_k e^S\eta)^* K\xi = (\theta^\dagger_k e^S \eta)^* He^{-S}\xi \\ &= h(e^{-S}\xi, \theta^\dagger_k e^S\eta) = h(\xi, e^{-S}\theta^\dagger_k e^S \eta), \end{aligned}$$

i.e. $\theta^\dagger_h = e^{-S}\theta^\dagger_k e^S = \theta^\dagger_k + e^{-S}(\theta^\dagger_k e^S - e^S\theta^\dagger_k)$. Hence $d'_h = d'_k + e^{-S}(d'_k(e^S))$, i.e. the first equality holds. Further by $F_h = d''d'_h + d'_h d''$ we see

$$\begin{aligned} d''(d'_k(e^S)) &= d''d'_k(e^S) + d'_k(e^S)d'' \\ &= d''e^S(d'_h - d'_k) + e^S(d'_h - d'_k)d'' \\ &= e^S(F_h - F_k) - e^S(F_h - F_k) + d''e^Sd'_h + e^S(d'_hd'') - d''e^Sd'_k - e^S(d'_kd'') \\ &= e^S(F_h - F_k) + d''e^Sd'_h - e^S(F_h - d'_hd'') - d''e^Sd'_k + e^S(F_k - d'_kd'') \\ &= e^S(F_h - F_k) + d''e^Sd'_h - e^Sd''d'_h - d''e^Sd'_k + e^Sd''d'_k \\ &= e^S(F_h - F_k) + d''(e^S)d'_h - d''(e^S)d'_k \\ &= e^S(F_h - F_k) + d''d'_ke^S - d''e^Sd'_k - e^Sd''e^{-S}d'_he^S + e^Sd''d'_k \\ &= e^S(F_h - F_k) + d''d'_ke^S - d''e^Sd'_k - e^Sd''e^{-S}d'_ke^S + e^Sd''d'_k \\ &= e^S(F_h - F_k) + d''(e^S)e^{-S}d'_k(e^S). \end{aligned}$$

Analogously for the pseudo-curvature: Replace $d''d'_k$ by $(D''\delta'_k - D'\delta''_k)$. With the transformation rule derived for θ above we get the same transformation for δ'' resp. δ'.[15] We have

$$\begin{aligned} &D''(\delta'(e^S)) - (D'(\delta''(e^S))) \\ &= 4e^S(G_h - G_k) + D''(e^S)e^{-S}\delta'_k(e^s) - D'(e^S)e^{-S}\delta''_k(e^S). \end{aligned}$$

□

[15] Now the data D resp. D' and D'' are fixed and δ', δ'', as the missing parts to the metric connections, transform under the transition $k \leftrightarrow h$.

Let $\mathscr{P}_k^0$ be the component covered be the chart $S \mapsto Ke^S$ and $H \in \mathscr{P}_k^0$, i.e. we find a S such that $H = Ke^S$.
In order to define Donaldson's functional, we stick closer with the original definition by Donaldson [Don85] and follow the description by Mochizuki [Moc02a]. Mochizuki proves that

Lemma 5.3.9. For $S \in T_k\mathcal{P}_k$ - tangent space - $\Phi_k(S) := \int_X \mathrm{tr}(Si\Lambda G_k)$ is a closed one-form of $\mathcal{P}_k$.

In order to prove the lemma he uses the following theorem by Simpson:

Theorem 5.3.10 (Stoke)**.** Suppose X has an exhaustion function ϕ with $\int_X |\Delta\phi| < \infty$ and suppose η is a L^2-integrable $1-$form. If $d\eta$ is integrable, then $\int_X d\eta = 0$.

Proof. A standard argument shows the claim. See Simpson [Sim88], p. 884, lemma 5.2. □

Definition 5.3.11. Let γ be a path connecting two metrics K and H in $\mathcal{P}_h$ then $M(K,H) := \int_\gamma \Phi$. In particular $M(K,H) + M(H,J) = M(K,J)$ by construction.

For $H = Ke^S$ this is equivalent to the definition

$$M'(K,H) \quad = \quad \int_X \mathrm{tr}(Si\Lambda F_h) + \int_X \langle \Psi(S)(\tilde{D}S), \tilde{D}S\rangle_K,$$

with Ψ the operator coming from the smooth function

$$\Psi(v,w) \quad = \quad \frac{e^{w-v} - (w-v) - 1}{(w-v)^2},$$

given by Simpson. We will shortly repeat the idea, since we need some of the formulas later on: We have

$$\begin{aligned}
&\frac{\partial}{\partial u} M'(Ke^{tS}, Ke^{(t+u)S})\Big|_{u=0} \\
&= \frac{\partial}{\partial u}\int_X \mathrm{tr}(uSi\Lambda F_{Ke^{tS}}) + \int_X \langle \Psi(uS)(\tilde{D}uS), \tilde{D}uS\rangle_{Ke^{tS}}\Big|_{u=0} \\
&= \int_X \mathrm{tr}(Si\Lambda F_{Ke^{tS}}) + \frac{\partial}{\partial u} u^2 \int_X \langle \Psi(uS)(\tilde{D}S), \tilde{D}S\rangle_{Ke^{tS}}\Big|_{u=0} \\
&= \int_X \mathrm{tr}(Si\Lambda F_{Ke^{tS}}).^{16}
\end{aligned}$$

Furthermore

$$\frac{\partial^2}{\partial t \partial u} M'(Ke^{tS}, Ke^{(t+u)S})\bigg|_{u=0} - \frac{\partial^2}{\partial t \partial u} M'(K, Ke^{(t+u)S})\bigg|_{u=0}.$$

The formula is shown by [Sim88], lemma 5.1: With the help of a transformation rule for the curvature derived in 5.3.7 we may show that both sides differ by a function which has to vanish by the Stokes theorem above.
We have further

$$\frac{\partial}{\partial u} M'(Ke^{tS}, Ke^{(t+u)S})\bigg|_{u=t=0} = \frac{\partial}{\partial u} M'(K, Ke^{(t+u)S})\bigg|_{u=t=0}.$$

But then we have equality at one point and the differential is the same, i.e. the functions are equal:

$$\begin{aligned} M'(K,H) &= M(K, Ke^S) - M(K,K) \\ &= \int_0^1 \frac{\partial}{\partial t} M'(K, Ke^{(t+u)S})\bigg|_{u=0} \mathrm{d}\, t \\ &= \int_0^1 \frac{\partial}{\partial u} M'(Ke^{tS}, Ke^{(t+u)S})\bigg|_{u=0} \mathrm{d}\, t \\ &= \int_0^1 \int_X \mathrm{tr}(Si\Lambda F_{Ke^{tS}})\, \mathrm{d}\, t \\ &= M(K,H). \end{aligned}$$

Remark 5.3.12. Note that by smoothness of the metric $M(K,H)$ is smooth as well.

One of the major steps in order to prove the existence result of a harmonic metric is the following proposition by Simpson:

Proposition 5.3.13. Fix a $b \in L^p, p > 1$. Let E be our stable Higgs resp. $\mathscr{D}_X-$bundle with $\|i\Lambda F_k\|_K \leq b$ resp. $\|i\Lambda G_k\|_K \leq b$. Then there are constants $\tilde{C}_1, \tilde{C}_2$ such that

$$\sup_X \|S\|_K \leq \tilde{C}_1 + \tilde{C}_2 M(K, Ke^S).$$

for any $\theta-$resp. $\nabla-$invariant element $S \in \Gamma_{\tilde{D}}(X, \mathbb{H}_n)$ with $\mathrm{tr}(S) = 0$, $\sup_X \|S\|_K < \infty$ and $\|i\Lambda F_{Ke^S}\|_K \leq b$.

[16] For $\mathscr{D}_X-$modules replace F by the pseudo-curvature G.

Proof. First note that by the positivity of Ψ

$$M(K, Ke^S) \geq \int_X \operatorname{tr}(si\Lambda F_k) \geq \|b\|_{L^1}\|S\|_{L^\infty}$$[17]

resp.

$$M(K, Ke^S) \geq \int_X \operatorname{tr}(si\Lambda G_k) \geq \|b\|_{L^1}\|S\|_{L^\infty}.$$

Thus it will be enough to show that

$$\sup_X \|S\|_K \leq \tilde{C}_1 + \tilde{C}_2 \max\{0, M(K, Ke^S)\}.$$

Furthermore we have for $H = Ke^S$:

$$\Delta \log \operatorname{tr}(e^S) \leq 2(\|i\Lambda F_h\|_H + \|i\Lambda F_k\|_K) \leq 2b\tilde{C}_3$$
$$\Delta \log \operatorname{tr}(e^S) \leq 8(\|i\Lambda G_h\|_H + \|i\Lambda G_k\|_K) \leq 2b\tilde{C}_3$$

for a big enough $\tilde{C}_3$. For a proof see [Sim88], lemma 3.1 (d). The version of the proof there comes from Y.T. Sui and is a consequence of 5.3.7 resp. [Sim88], 3.1 (c). In particular once we have 5.3.7 the generalization to $\mathscr{D}_X-$modules is clear. By Assumption 3 we may further conclude that

$$\sup_X \|S\|_K \leq \tilde{C}_4 + \tilde{C}_5\|S\|_{L^1}.$$

Now let us assume that the claim does not hold. Then we have one of the following cases

Case 1: There is a sequence $S_i \in \mathcal{P}(S_h), \operatorname{tr}(S_i) = 0$ with $\sup_X \|S_i\|_K \to \infty$ and $M(K, Ke^{S_i}) \leq 0$.

Case 2: There is a sequence $S_i \in \mathcal{P}_{\tilde{D}}(\mathcal{S}_k), \operatorname{tr}(S_i) = 0$ with $\sup_X \|S_i\|_K \to \infty$ and $M(K, Ke^{S_i}) > 0$. Then we further find a series $C_i \geq 0$ such $C_i \to \infty$ and $\sup_X \|S_i\|_K \geq C_i M(K, Ke^{S_i})$.

In both cases, $\|S_i\|_{L^1} \to \infty$. Set $l_i := \|S_i\|_{L^1}$ and $u_i := l_i^{-1}S_i$, so that $\|u_i\|_{L^1} = 1$.[18] Further $\sup_X \|u_i\|_K \leq \tilde{C}_6$. Note that $\operatorname{tr}(u_i) = 0$ by the same property of the S_i.

Lemma 5.3.14. After going to a subsequence $u_i \to u_\infty$ weakly in $H^{2,1}(S)$ the limit is nontrivial. If $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a positive smooth function such that $\Phi(v, w) < (v - w)^{-1}$ whenever $v > w$, then

$$\int_X \operatorname{tr}(u_\infty i\Lambda F_k) + \int_X \langle \Phi(u_\infty)(d''u_\infty), d''u_\infty\rangle_K \leq 0.$$

[17] Hölder inequality.

[18] W.l.o.g. $S_i \neq 0$.

Proof. If necessary replace Φ with $\Phi - \varepsilon$ to justify the assumption $\Phi(v,w) < (v-w)^{-1}$ for $v > w$. From the proof of the equivalence of the two definitions of $M(K, Ke^S)$ resp. lemma 5.2 in [Sim88], we know the formula

$$\frac{\partial^2}{\partial t \partial u} M(He^{tS}, He^{(t+u)S})\bigg|_{u=0} = \frac{\partial^2}{\partial t \partial u} M(H, He^{(t+u)S})\bigg|_{u=0}.$$

Hence

$$l_i \int_X \operatorname{tr}(u_i i \Lambda F_k) + l_i^2 \int_X \langle \Psi(l_i u_i)(\tilde{D} u_i), u_i \rangle_K \le \tilde{C}_7 C_i^{-1} l_i.$$ [19]

The u_i are by construction uniformly bounded, thus we can cutoff Ψ, i.e. assume that Ψ has compact support. Furthermore for $v \le w$

$$\begin{aligned} l\Psi(lv, lw) &= l\frac{e^{lw-lv} - (lw - lv) - 1}{(lw-lv)^2} \le \frac{e^{l(w-v)}}{l(w-v)^2} \\ &\to \infty. \end{aligned}$$

and for $v > w$: $l\Phi(lv, lw) \to \frac{1}{v-w}$ and monotone increasing in l. Hence for a big enough l we may assume that

$$\Phi(v,w) < l\Psi(lv, lw).$$

This yields

$$l_i \int_X \operatorname{tr}(u_i i \Lambda F_k) + l_i^2 \int_X \langle \Phi(u_i)(\tilde{D} u_i), \tilde{D} u_i \rangle_K \le \frac{\tilde{C}_7}{C_i}^{-1} l_i.$$

The condition $\sup \|u_i\|_K \le \tilde{C}_6$ implies that $\tilde{D} u_i$ is L^2-integrable, i.e. $u_i \in H^{2,1}$. But any bounded sequence has a weakly convergent subsequence[20] in $H^{2,1}$. In abuse of notation call this subsequence as well u_i and the weak limit u_∞. Now take Z a compact subset of X. Then the $u_i \overset{L^2}{\to} u_\infty$ resp. $\int_Z \|u_i\|_K \to \int_Z \|u_\infty\|_K$.[21] By assumption 1 (finite volume) and $\sup_X \|u_i\|_K \le \tilde{C}_6$

$$\begin{aligned} &1 = \|u_i\|_{L^1} \le \int_{X-Z} \|u_i\|_K + \int_Z \|u_i\|_K \le \tilde{C}_6 \int_{X-Z} + \int_Z \|u_i\|_K \\ &\Rightarrow \int_Z \|u_i\|_K \ge 1 - \tilde{C}_6 \underbrace{\int_{X-Z}}_{=:\varepsilon} = 1 - \varepsilon. \end{aligned}$$

[19] For the first case choose any sequence $C_i \to \infty$.
[20] Functional Analysis Course [FA08].
[21] $H^{2,1} \subset L^1(Z)$ compact by Rellich-Kondrachov, Müller [PDE09], p. 125.

Thus $\|u_\infty\|_{L^1} \geq 1 - \varepsilon > 0$ for $X - Z$ with small enough volume. Since the trace is continuous in the u_i we get additionally that $\int_X \operatorname{tr}(u_i i\Lambda F_k) \to \int_X \operatorname{tr}(u_\infty i\Lambda F_k)$ resp. $\int_X \operatorname{tr}(u_i i\Lambda G_k) \to \int_X \operatorname{tr}(u_\infty i\Lambda G_k)$. Therefore we have $u_i \overset{L^2}{\to} u_\infty$ as well as $\tilde{D}u_i \overset{L^2}{\to} \tilde{D}u_\infty$. By continuity of Ψ: $\Phi(u_i)\tilde{D}u_i \overset{L^2}{\to} \Phi(u_\infty)\tilde{D}u_\infty$. Now L^p-convergence implies convergence in measure and convergence in measure allows us to apply the lemma of Fatou:

$$\int_X \langle \Phi(u_\infty)\tilde{D}u_\infty, u_\infty\rangle_K \leq \liminf_{i\to\infty} \int_X \langle \Phi(u_i)\tilde{D}u_i, u_i\rangle_K.$$

□

Remark 5.3.15. The idea to use Fatou's lemma is taken from [Moc02a] 13, p. 21. Note that L^p-convergence does in general not imply almost everywhere convergence (as claimed in [Moc02a], p. 21), but anyway convergence in measure is enough for Fatou's lemma.

Lemma 5.3.16. The eigenvalues of u_∞ are constant, in other words there are $\lambda_1, \ldots, \lambda_n$, which are the eigenvalues of $u_\infty(x)$ for almost all $x \in X$. There are at least two distinct eigenvalues.

Proof. If we are able to show that for all smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\operatorname{tr}(\varphi(u_\infty))$ is constant, then all eigenvalues are constant. Choose the smooth function given by the functional calculus belonging to the projection onto a generalized eigenspace. The projection is hermitian, thus this is justified. Since the trace is only the sum over the eigenvalues, we get constancy of each eigenvalue.
We have

$$\begin{aligned} \overline{\partial}\operatorname{tr}(\varphi(u_\infty)) &= \operatorname{tr}(Ed''_{\text{End}}\varphi(u_\infty)) + \operatorname{tr}((\underbrace{d'_{\text{End}}E}_{d'-d'=0})^*\varphi(u_\infty)) \\ &= \operatorname{tr}(Ed''_{\text{End}}\varphi(u_\infty)) = \operatorname{tr}(\varphi^\Delta(u_\infty)(d''u_\infty)) \end{aligned}$$

as well as

$$\begin{aligned} (\overline{\partial}+\partial)\operatorname{tr}(\varphi(u_\infty)) &= \operatorname{tr}(ED_{\text{End}}\varphi(u_\infty)) + \operatorname{tr}((\underbrace{D_{\text{End}}E}_{D-D=0})^*\varphi(u_\infty)) \\ &= \operatorname{tr}(ED_{\text{End}}\varphi(u_\infty)) = \operatorname{tr}(\varphi^\Delta(u_\infty)(Du_\infty)). \end{aligned}$$

Furthermore remember that $\varphi^\Delta(v,v)$ is the differential of φ at v. Construct a function Φ such that $\Phi(v,v) = \varphi^\Delta(v,v)$ and that for N large enough $N\Phi^2(v,w) < (v-w)^{-1}$ for $v > w$.[22] Hence $\operatorname{tr}(\varphi^\Delta(u_\infty)(\tilde{D}u_\infty)) = \operatorname{tr}(\Phi(u_\infty)\tilde{D}u_\infty)$. By the

[22] For $v \to w$ the condition holds trivially for any N. Thus the two conditions don't interact. For example take a cutoff function, that has support in some tubular neighbourhood of $v = w$ in $\mathbb{R}^2$.

previous lemma (Φ^2 is positive.)

$$\begin{aligned}\int_X \|\Phi(u_\infty)(\tilde{D}u_\infty)\|_K^2 &= \int_X \langle \Phi(u_\infty)(\tilde{D}u_\infty), \Phi(u_\infty)(\tilde{D}u_\infty)\rangle_K \\ &= \int_X \langle \Phi^2(u_\infty)(\tilde{D}u_\infty), \tilde{D}u_\infty\rangle_K \\ &\leq \frac{1}{N}\int_X \mathrm{tr}(u_\infty i\Lambda F_k),\end{aligned}$$

or replace F_k by G_k for the $\mathscr{D}_X-$module case. By our assumptions on the curvatures F_k resp. G_k and u_∞ (by construction of the u_i) the right-hand side is bounded by $\frac{\tilde{C}_8}{N}$. Together we get

$$\|(ov\partial \mathrm{tr}\varphi(u_\infty)\|_{L^2} \leq \frac{C}{N} \to 0, \quad N \to \infty,$$

or

$$\|(\overline{\partial} + \partial)\mathrm{tr}\varphi(u_\infty)\|_{L^2} \leq \frac{C}{N} \to 0, \quad N \to \infty.$$

Since u_∞ is hermitian $\mathrm{tr}(\varphi(u_\infty))$ is real-valued, i.e. $\overline{\partial}\mathrm{tr}\varphi(u_\infty) = 0 \Rightarrow \mathrm{tr}(\varphi(u_\infty))$ constant, $\forall\varphi$ smooth.
On the other hand if u_∞ had only one eigenvalue, then $\mathrm{tr}(Eu_\infty) = \mathrm{tr}(u_\infty) = 0$ would imply that u_∞ had only eigenvalue 0. But a hermitian matrix with only zero eigenvalues is 0, hence we get a contradiction to the non-triviality of u_∞ from the previous lemma. Therefore there are at least to distinct eigenvalues. □

In abuse of notation let $\lambda_1 < \ldots < \lambda_k$ be the distinct eigenvalues of u_∞. By definition of φ and Φ the operators $\varphi(u_\infty)$ and $\Phi(u_\infty)$ depend only on $\varphi(\lambda_i)$ resp. $\Phi(\lambda_i, \lambda_j), 1 \leq i, j \leq k$.

Lemma 5.3.17. If $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function with $\Phi(\lambda_i, \lambda_j) = 0$ for $\lambda_i > \lambda_j, 1 \leq i, j \leq k$. Then $\Phi(u_\infty)(\tilde{D}u_\infty) = 0$.

Proof. Since Φ depends only on the eigenvalues we may replace it (as in the previous proof) with Φ_1 defined by

$$\Phi_1(\lambda_i, \lambda_j) = 0 \text{ and } N(\Phi_1)^2(v, w) < (v - w)^{-1} \text{ for } v > w.$$

Again we get $\|\Phi_1(u_\infty)(\tilde{D}u_\infty)\|_{L^2}^2 \leq \frac{\tilde{C}_8}{N} \to 0$ for $N \to \infty$. Then $\Phi(u_\infty)(\tilde{D}u_\infty) = \Phi_1(u_\infty)(\tilde{D}u_\infty) = 0$. □

Let $I_i :=]\lambda_i, \lambda_{i+1}[, 1 \leq i \leq k-1$. $\forall 0 \leq i \leq k$.[23] Let $\rho_i : \mathbb{R} \to \mathbb{R}$ be any decreasing function such that $\rho_i(x) = 1$ for $x \leq \lambda_i$ and $\rho_i(x) = 0$ for $x \geq \lambda_{i+1}$. Denote further $\pi_i := \rho_i(u_\infty)$. By 5.3.4 the π_i are in $H^{2,1}$. Again since our operator only depends on the eigenvalues and $\rho_i(\lambda_j)^2 = \rho_i(\lambda_j)$ for all $1 \leq i, j \leq k$.
Moreover

$$\tilde{D}\pi_i = \tilde{D}\rho_i(u_\infty) = \rho_i^\Delta(u_\infty)(\tilde{D}u_\infty)$$

and define $\Phi_i(v,w) = (1-\rho_i)(w) \cdot \rho_i^\Delta(v,w)$. We conclude

$$\begin{aligned}(1-\pi_i)\tilde{D}\pi_i &= (1-\pi_i)\rho_i^\Delta(u_\infty)(\tilde{D}u_\infty) \\ &= \Phi_i(u_\infty)(\tilde{D}u_\infty).\end{aligned}$$

Since we have $\Phi_i(\lambda_i, \lambda_j) = (1-1)(\lambda_j) \cdot \rho_i^\Delta(v,w) = 0$ for $i > j$.
The previous lemma 5.3.17 tells us $\Phi(u_\infty)(\tilde{D}u_\infty) = 0$, i.e. $(1-\pi_i)\tilde{D}\pi_i = \Phi_i(u_\infty)(\tilde{D}u_\infty) = 0$. Thus we have a $H^{2,1}-$projection.
We are now in the position to state a result of Uhlenbeck and Yau, that will help us to contradict our assumption - $\sup_X \|S_i\|_K \to \infty$ for some sequence S_i- made far back:

Proposition 5.3.18. If E is a Higgs bundle resp. $\mathscr{D}_X-$module with metric k and if π is $H^{2,1}-$subbundle, then there is saturated sub-Higgs-sheaf resp. sub-$\mathscr{D}_X-$module $V \subset W$ such that π is a projection onto V, defined where V is a subbundle.

Proof. See Uhlenbeck-Yau [UY86] for the vector bundle case. Thus we only need to understand why the subbundles are preserved by θ resp. ∇. By degree considerations $(1-\pi)d''\pi = 0$ and $d''\pi \in L^2 \Rightarrow (1-\pi)\overline{\partial}_E\pi = 0, (1-\pi)\theta\pi = 0$ and $\overline{\partial}\pi \in L^2$. Then V is a saturated subsheaf by [UY86] and it is preserved by θ. Analogously for the $\mathscr{D}_X-$module: $(1-\pi)D\pi = 0$ and $D\pi \in L^2 \Rightarrow (1-\pi)\nabla\pi = 0, (1-\pi)D''\pi = 0$ and $(\overline{\partial} + \theta^\dagger)\pi \in L^2$. □

Thus if V_i is the image of π_i, it is already a subbundle preserved by either θ or ∇. Hence if one of the π_i satisfies $\frac{\deg(V_i)}{\dim(V_i)} \geq \frac{\deg(E,k)}{\dim(E)}$, we have a contradiction to stability.
But we can use our Chern-Weil formulas (CW2) resp. (CW3) now. Write

$$\begin{aligned}u_\infty &= \sum_{i=1}^k \lambda_i(\pi_i - \pi_{i-1}) \\ &= \lambda_k\pi_k - (\lambda_k - \lambda_{k-1})\pi_{k-1} - \ldots - (\lambda_2 - \lambda_1)\pi_1 - \lambda_1\pi_0\end{aligned}$$

[23] Note as well that the following construction makes sense since we have at least two distinct eigenvalues, in particular we find i, j such that $(\lambda_i - \lambda_j)^{-1}$ is well-defined.

$$= \lambda_k E - \sum_{i=1}^{k-1} a_i \pi_i$$

for $a_i = \lambda_{i+1} - \lambda_i$.[24] Using $\tilde{D}\pi_i = \rho_i^{\Delta}(u_\infty)(\tilde{D}u_\infty)$ again,

$$\begin{aligned} W &:= \lambda_k \deg(E) - \sum_{i=1}^{k-1} a_i \deg(V_i) \\ &= \int_X \operatorname{tr}((\lambda_k E - \sum_{i=1}^{k-1} a_i \pi_i) i\Lambda F_k) + \int_X \sum_{i=1}^{k-1} a_i \|d''(\pi_i)\|_K^2 \\ &= \int_X \operatorname{tr}(u_\infty i\Lambda F_k) + \int_X \sum_{i=1}^{k-1} \langle a_i \rho_i^{\Delta}(u_\infty)^2 (d'' u_\infty), d'' u_\infty \rangle_K, \end{aligned}$$

and

$$\begin{aligned} W &:= \lambda_k \deg(E) - \sum_{i=1}^{k-1} a_i \deg(V_i) \\ &= \int_X \operatorname{tr}((\lambda_k E - \sum_{i=1}^{k-1} a_i \pi_i) i\Lambda G_k) + \frac{1}{2}\int_X \sum_{i=1}^{k-1} a_i \|D(\pi_i)\|_K^2 \\ &= \int_X \operatorname{tr}(u_\infty i\Lambda G_k) + \frac{1}{2}\int_X \sum_{i=1}^{k-1} \langle a_i \rho_i^{\Delta}(u_\infty)^2 (D u_\infty), D u_\infty \rangle_K. \end{aligned}$$

For $\lambda_m > \lambda_j$ we have $\rho_i^{\Delta}(\lambda_m, \lambda_j) = \frac{\rho_i(\lambda_m) - \rho_i(\lambda_j)}{\lambda_i - \lambda_j} = \frac{\delta_{im} - \delta_{ij}}{\lambda_m - \lambda_j} \leq \frac{1}{\lambda_m - \lambda_j}$ since $\lambda_m \neq \lambda_j$;

$$\begin{aligned} \Rightarrow \sum_{i=1}^{k-1} a_i \left(\rho_i^{\Delta}(\lambda_m, \lambda_j)\right)^2 &= \frac{\lambda_m - \lambda_j}{(\lambda_m - \lambda_j)^2} \\ &= \frac{1}{\lambda_m - \lambda_j}. \end{aligned}$$

Now by Lemma 5.3.14 $W \leq 0$. We get $\lambda_k \deg(E) \leq \sum_{i=1}^{k-1} a_i \deg(V_i)$. Furthermore the trace is the sum of the λ_i times their multiplicity, i.e. $0 = \operatorname{tr}(u_\infty) = \sum_{i=1}^{k} \lambda_i \dim(V_i \setminus V_{i-1}) = \lambda_k \dim(E) - \sum_{i=1}^{k-1} a_i \dim(V_i) \Rightarrow \lambda_k \dim(E) = \sum_{i=1}^{k-1} a_i \dim(V_i)$. This contradicts stability: Assuming otherwise, i.e. $\frac{\deg(E,k)}{\dim(E)} > \frac{\deg(V_i)}{\dim(V_i)}$ leads to the contradiction

$$\frac{\deg(E,k)\dim(V_i)}{\dim(E)} > \deg(V_i) \qquad , \forall 1 \leq i \leq k-1.$$

[24] $\pi_i - \pi_{i-1}$ is the projection onto the generalized eigenspace to λ_i. In particular $\pi_k = E$.

$$\Rightarrow \sum_{i=1}^{k-1} a_i \frac{\deg(E,k)\dim(V_i)}{\dim(E)} = \lambda_k \deg(E,k) \frac{\sum_{i=1}^{k-1} a_i \dim(V_i)}{\sum_{i=1}^{k-1} a_i \dim(V_i)} = \lambda_k \deg(E,k)$$
$$> \sum_{i=1}^{k-1} a_i \deg(V_i).$$

Hence we find at least on V_i contradicting stability. □

5.3.3. Heat equation and Existence Result

We will shortly repeat the contents of chapter 6 in [Sim88]. There Simpson proves the existence of a global solution of the heat equation stated below. The proof is a modification (using 5.3.7 and the assumptions 1, 2, 3) of the one in [Don85] using further results of [Ham75] and additionally 5.3.7.

$$H_t^{-1}\frac{\mathrm{d}\,H_t}{\mathrm{d}\,t} = -i\Lambda F_{H_t}^{\perp} \quad \text{or}$$
$$H_t^{-1}\frac{\mathrm{d}\,H_t}{\mathrm{d}\,t} = -i\Lambda G_{H_t}^{\perp}$$

where $F_{H(t)}^{\perp} =: F_t^{\perp}$ resp. $G_{H(t)}^{\perp} =: G_t^{\perp}$ denotes the trace free part of $F_{H(t)}$ resp. $G_{H(t)}$.[25] Before we impose some boundary conditions remember that we wanted to bound our harmonic metric with respect to some initial metric say $H_0, H_t = H_0 f_t$. Then the Heat equation becomes

$$H_t^{-1}\frac{\mathrm{d}\,H_t}{\mathrm{d}\,t} = H_t^{-1}H_0 \mathrm{d}\,\overline{f}_t dt = f_t^{-1}\mathrm{d}\,\overline{f}_t dt$$
$$\Rightarrow \frac{\mathrm{d}\,f_t}{\mathrm{d}\,t} = -if_t\Lambda F_t^{\perp}$$
$$= -if_t\Lambda(F_0^{\perp}) - if_t\Lambda(F_t^{\perp} - F_0^{\perp})$$
$$\Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}\,t} + (d''d_0')_{\mathrm{End}}\right) f_t = -if_t\Lambda(F_0^{\perp}) + i\Lambda(d''(f_t)f_t^{-1}d_0'(f_t)).$$

where we used 5.3.7. For $\mathscr{D}_X-$modules this looks similar

$$\Rightarrow \quad \left(\frac{\mathrm{d}}{\mathrm{d}\,t} + (D_0''\delta' - D_0'\delta'')_{\mathrm{End}}\right) f_t$$
$$= -if_t\Lambda(G_0^{\perp}) + i\Lambda(D''(f_t)f_t^{-1}\delta_0'(f_t) - D'(f_t)f_t^{-1}\delta_0''(f_t)).$$

Remember the exhaustion ϕ function given by assumption 2. Then $X_\varphi = \{x \in X | \phi(x) \leq \varphi\}$ is compact. Denote $Y_\varphi = \partial X_\varphi$. If $\frac{\partial}{\partial \nu}$ denotes the differentiation in

[25] $F^{\perp} = F - \mathrm{tr}(F)\mathrm{id}$.

direction perpendicular (w.r.t. the unique $k-$metric connection compatible with ∂_E) to the boundary. Now we may state the boundary conditions

(1) Neumann boundary condition:

$$\frac{\partial}{\partial \nu} H|_{Y_\varphi} = 0.$$

(2) Dirichlet boundary condition

$$H|_{Y_\varphi} = K|_{Y_\varphi}.$$

Then

Proposition 5.3.19 ([Sim88], 6.6, p. 892)**.** Let $b \in L^p, p > 1$ and let X satisfy the assumptions 1, 2, 3 and let E be a Higgs bundle resp. $\mathscr{D}_X-$module over X. Suppose k is a metric satisfying the assumption that sup $\|i\Lambda F_k\|_K \leq b$.

(i) Then there is a unique solution H to the heat equation with $\det(H) = \det(K)$, with $H_0 = K$, and such that $\sup_X \|H\|_K < \infty$ on each finite interval of time. For this solution, $\|i\Lambda F_h\|_H \leq b$.

(ii) Let H_t be the solution of (i). Then there is a $S \in \mathcal{P}_{\tilde{D}}(\mathcal{S}_k)$ such that $K = H_t e^S$ and $M(K, H_t)$ is continuously differentiable in t with

$$\frac{\mathrm{d}}{\mathrm{d}\, t} M(K, H_t) = -\int_X \|i\Lambda F_t^\perp\|^2_{H_t}$$

resp.

$$\frac{\mathrm{d}}{\mathrm{d}\, t} M(K, H_t) = -\int_X \|i\Lambda G_t^\perp\|^2_{H_t}.$$

(iii) If $H_t \to H_\infty$ in C^0 then the H_t are bounded in $H^{p,2}_{loc}$.

Proof. Ad (i): Simpson uses methods and results described by Donaldson [Don85] as well as Hamilton [Ham75] as well as lemma 5.3.7.
Ad (ii): The first part is an application of 5.3.7 and the results derived in the proof of (i). The differentiability uses additionally $M(K, H) + M(H, J) = M(K, J)$ (by definition of M) to reduce the problem to differentiability at $t = 0$.
Ad (iii) is the remark to lemma 6.4 on page 891 of [Sim88]. □

The proposition and the previous subsection finally lead to the main existence theorem:

Theorem 5.3.20. Let $b \in L^p, p > 1$ and let X satisfy assumptions 1, 2, 3. Let E be a Higgs bundle resp. $\mathscr{D}_X-$module over X. Suppose k is a metric satisfying the assumption that $\|i\Lambda F_k\| \leq b$. Suppose further that (E, k) is stable. Then there is a metric h with $\det(H) = \det(K)$, H and K mutually bounded and $\tilde{D}(K^{-1}H) \in L^2$. Moreover $i\Lambda F_h^\perp = 0$ resp. $i\Lambda G_h^\perp = 0$.

Remark 5.3.21. A metric with $i\Lambda F_h^\perp = 0 \Leftrightarrow i\Lambda F = \operatorname{tr}(i\Lambda F)\mathrm{id}$ is called Hermitian-Yang-Mills metric or Hermitian-Einstein metric.

Proof. We only want to describe how our estimate on S given in proposition 5.3.13 comes into play. First note that most of the conditions of proposition 5.3.13 are clear for $H_t = Kf_t$ the solution given by the previous proposition. It remains to show that $\log(f_t)$ is traceless. But $\det(H_t) = \det(K)\det(f_t) \Rightarrow 1 = \det(f_t) = \det(e^{\log(f_t)}) = e^{\operatorname{tr}(\log(f_t))} \Rightarrow 0 = \log(1) = \operatorname{tr}(\log(f_t))$. Applying proposition 5.3.13 now, we get independently of $t < \infty$

$$\sup_X \|\log f_t\|_K \leq \tilde{C}_1 + \tilde{C}_2 M(K, Kf_t) \Rightarrow \sup_X \|f_t\|_K \leq \tilde{C}_9,$$

since $\|e^{\log f_t}\| \leq e^{\|\log f_t\|}$, the monotone behaviour of the real exponential function and the decreasing character of $M(K, Kf_t)$ (previous proposition (ii)). In particular since H_0 is already bounded uniformly in the $K-$norm by part (i), H_t is bounded for all t, too.
Furthermore $M(K, H_t)$ is bounded from below[26] and monotone decreasing, i.e. we find a subsequence $t_i \to \infty$ such that $M(K, H_{t_i})$ converges, $\|\Lambda F_i^\perp\|_{L^2}^2 \to 0$. Moreover

$$\int_X \langle \Psi(\log f_t)\tilde{D}(\log f_t), \tilde{D}(\log f_t)\rangle_K \leq \tilde{C}_{10}\|\tilde{D}(f_t)f_t^{-1/2}\|_{L^2}^2 \leq \tilde{C}_9\|\tilde{D}(f_t)\|_{L^2}^2.$$

This shows $\tilde{D}(K^{-1}H) \in L^2$.
Furthermore we may find by boundedness again a weakly convergent subsequence $H \to H_\infty$ - weakly in $H^{2,1}$. In particular $H_i \to H_\infty$ (strong) in L^2 on every bounded open set.[27] Additionally we know that the H_i are uniformly bounded, so they are a L^2-Cauchy sequence.
Now apply lemma 3.1 (d), [Sim88], as in the proof of proposition 5.3.13 to get first $\Delta \log \operatorname{tr}(H_i^{-1}H_j) \leq 2b$ and then by assumption 3 that $\sup_X \|\log \operatorname{tr}(H_i^{-1}H_j)\|_K \leq c_b\|\log \operatorname{tr}(H_i^{-1}H_j)\|_{L^1} \to 0$ (Cauchy). Thus $H_i \to H_\infty$ in C^0. Part (iii) of the previous proposition tells us that the H_i are bounded in $H^{p,2}_{loc}$. Therefore we find a further subsequence $H_i \to H_\infty$ weakly in $H^{p,2}_{loc}$, i.e. the limit F_{H_∞} exists and $i\Lambda F_{H_\infty}^\perp = 0$.

[26] The first term is bounded by our assumption on F_k resp. G_k, the second one is positive.
[27] Rellich-Kondrachov as in [LiLo00], p. 214, 8.9 (ii).

Finally use elliptic regularity and the standard boot strapping argument using 5.3.7 to conclude H_∞ smooth.[28] Thus we are done. □

We didn't use yet that we have degree 0. But $\det(H) = \det(K) \Rightarrow \deg(E, h) = \deg(E, k) = 0 \Rightarrow 0 = \int_X \operatorname{tr}(i\Lambda F_h) \Rightarrow i\Lambda F_h = i\Lambda F_h^\perp = 0$, i.e. h is harmonic. Thus the main existence result is proved.

5.4. Main result

There is not much left to do now. A first theorem will show that every filtered regular Higgs bundle resp. $\mathscr{D}_X-$module, that comes from a harmonic bundle has in fact degree 0 and is stable. Since Φ preserves the notion of degree and stability too, we are finally in the position to prove the main theorem.

Theorem 5.4.1. The Higgs bundles resp. $\mathscr{D}_X-$modules E, which come from tame harmonic bundles by the functors Ξ, are exactly those objects which are direct sums of stable objects of degree 0.

Proof. (i) Start with a stable filtered regular Higgs bundle E of degree 0. By theorem 4.3.1 we find an acceptable metric h such that F_h is in L^p, $p > 1$, which induces the original filtration E_α. Since the algebraic and the analytic degree coincide (conclusion 5.2.7) we can apply the main existence theorem 5.3.1. The harmonic metric k (provided by the theorem) and the original metric h are mutually bounded $\Rightarrow$ k induces the filtration E_α, i.e. the filtered bundle (E, E_α) comes from a harmonic bundle.
The bundle is tame, since θ is regular, i.e. preserves the filtration - assuming θ had an eigenvalue with pole of order greater than 1 and e an eigensection in some E_α, then $\theta e \notin \Gamma(X, E) \otimes \Omega^{1,0}_{\overline{X}}(\log s)$.[29]. This shows the first direction.

The other direction was shown by theorem 5.2.10.

(ii) Again the case of a $\mathscr{D}_X-$module is just the same, since all previous results used in (1) were formulated for $\mathscr{D}_X-$modules too. So we only have to care about tameness: Use 4.3.3, where we described the flat sections belonging to the construction of 4.3.1. They were polynomially bounded w.r.t. the metric h. Again by the mutual boundedness they grow at most polynomially w.r.t k. Now we are in the situation of proposition 1.7.2, which tells us that polynomial growth of the flat sections implies tameness.

□

[28] "boot strapping" : Once we have it for one differential, then the formula 5.3.7 lifts it to a higher differential; then apply the formula to this higher differential, and so on.

[29] We may multiply any eigensection with a scalar function to guarantee that it is in a certain E_α over some open U and at least $s \in \partial U$.

Definition 5.4.2. The degree of a filtered local system is defined as

$$\deg(L, L_\alpha) = \sum_{s\in\overline{X}\setminus X}\sum_{\beta\in\mathbb{R}} \dim \mathrm{Gr}_\beta(L_s).$$

A local system L is called stable if for all $\mu-$invariant subsystems $U \subset L$, $U_\alpha := L_\alpha \cap U$

$$\frac{\deg(U, U_\alpha)}{\dim(U)} < \frac{\deg(L, L_\alpha)}{\dim(L)},$$

holds. It is called semi-stable if $<$ is replaced with $\leq$.

Remark 5.4.3. The degree of a filtered local system underlies the same rules as the second term of the parabolic degree of filtered regular Higgs bundles resp. $\mathscr{D}_X-$modules. In particular it is compatible with taking duals, determinants and tensor products in the same way as described above.

Lemma 5.4.4. The degree of a filtered local system is the same as the degree of the $\Phi-$corresponding filtered regular $\mathscr{D}_X-$module.

Proof. By the previous remark we may again assume that L comes from a line bundle. Remember the previous chapter and our two examples 1.4.17 and 1.4.18. Let α be a jump in the filtration of L, $l \in L_\alpha \setminus L_{\alpha+\varepsilon}$. Then by the monodromy of M, $h = e^{M\log z} l$, we get a jump at $\Phi(L)_{\alpha+\Re(\lambda_M)}$ for every choice of logarithm M. In particular all jumps (in the line bundle case only 1) in the filtration of L_β have exactly one corresponding jump in $\Phi(L)_\gamma, 0 \leq \gamma < 1$. So it will be enough to show that $\Phi(L)_0$ has exactly degree $-\Re(\lambda_M)$ for one choice of M. So fix M. However we have seen in conclusion 4.3.1 that the filtration of the L_β is just the filtration induced by the standard metric k constructed there.[30] In order to calculate the degree of L_0 we have to find a holomorphic section in L_0. But $\tilde{h} = e^{-M\log z} h = l \in \Phi(L)_0 \Leftrightarrow \tilde{h} \in \Phi(L)_{\Re(\lambda_M)}$ and therefore $\|\tilde{h}\|_k \|l\|_h \leq c_l |z|^{\Re(\lambda_M)-\varepsilon}, \forall \varepsilon > 0$. Hence $\pi\deg(\Phi(L)_0) = -\pi\Re(\lambda_M)$ as claimed. □

Lemma 5.4.5. Φ preserves the notions of stability and semi-stability.

Proof. Let $U \subset L$ be a subsystem, i.e. a subspace of L preserved by the monodromy μ with filtrations $U_\alpha = U \cap L_\alpha$. Since Φ is compatible with the decomposition into $\mu-$invariant subspaces and U is such a subspace, we may restrict to generalized eigenspaces, i.e. assume that M has only one eigenvalue λ_M.
When we introduced the functor Φ we saw as well that the image of a $\mu-$invariant

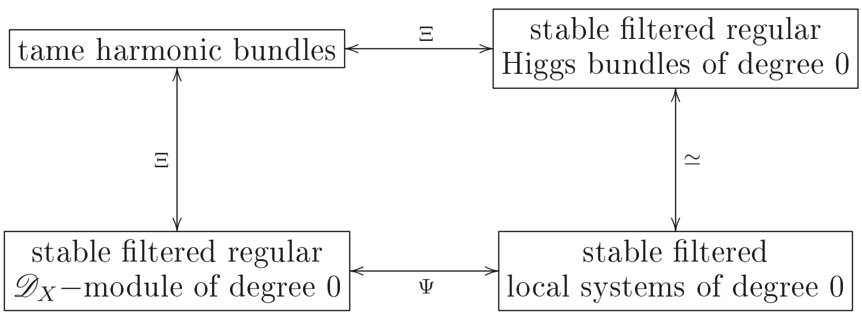
[30] If we take the main existence theorem in count, we get a harmonic metric witch is bounded w.r.t. k and then 4.3.2 tells us that the filtrations are the same.

space is preserved by ∇ and vice versa under Φ^{-1}. Thus $\Phi(U)$ is a sub-$\mathscr{D}_X-$module of $\Phi(L)$. But for any local system we now that $l \in L_\beta$ if and only if $e^{M \log z} l \in \Phi(L)_{\Re(\lambda_M)+\beta}$. Then $u \in U_\beta$ if and only if $e^{M \log z} l \in \Phi(U)_{\Re(\lambda_M)+\beta}$. On the other hand $u \in U_\beta$ implies that $e^{M \log z} u \in \Phi(L)_{\Re(\lambda_M)+\beta} \cap j_*\Phi(U)$ and $e^{M \log z} u \in \Phi(L)_{\Re(\lambda_M)+\beta} \cap j_*\Phi(U)$ implies $u \in L_\beta \cap U$. Thus $\Phi(U)_{\Re(\lambda_M)+\beta} = \Phi(L)_{\Re(\lambda_M)+\beta} \cap j_*\Phi(U)$, i.e. every μ invariant subsystem corresponds to a sub-$\mathscr{D}_X-$module of same rank. The degrees are anyway the same by 5.4.4. Thus the slopes are the same, i.e. stability and semi-stability is preserved. □

Remark 5.4.6. In particular the decomposition of the $\mathscr{D}_X-$module $\Phi(L)_\alpha$ into a direct sum of stable filtered regular $\mathscr{D}_X-$modules of degree 0 induces a decomposition of L into a direct sum of stabled filtered local systems of degree 0. Since every summand is $\nabla-$invariant, Φ respects invariant subspaces.

Main theorem. The category of tame harmonic bundles is naturally equivalent via the functors Ξ, to the categories of direct sums of stable filtered regular Higgs bundles of degree zero, of direct sums of stable filtered regular $\mathscr{D}_X-$modules of degree zero, and of direct sums of stable filtered local system of degree zero.

Proof. We have seen, that Φ is an equivalence of the categories of filtered local system and filtered regular $\mathscr{D}_X-$modules. The previous two lemmas 5.4.4 and 5.4.5 ensure that Φ respects degrees and stability. Already before we saw that Φ is compatible with direct sums of preserved subbundles/subsystems. A morphism of stable filtered regular $\mathscr{D}_X-$modules of degree zero resp. direct sums of stable filtered local system of degree zero is just an usual morphism of filtered regular $\mathscr{D}_X-$modules resp. filtered local systems. Hence Φ is still fully faithful on the restricted categories. Thus Φ is an equivalence of categories.
We know already that Ξ is fully faithful on the named categories. As for Φ this still holds for the restricted categories. Further Ξ maps harmonic bundles to filtered regular Higgs bundles resp. $\mathscr{D}_X-$modules. Essential surjectivity is the result of theorem 5.4.1. Thus Ξ establishes an equivalence of categories.



□

6 Some Further Developments

6.1. Analytic and Parabolic Degree

In the last twenty years there were several new developments, too many to mention them all. Here we will shortly review the different fields influenced by the main result resp. its generalizations.
First let us start with reviewing the last two chapters in [Sim90]. While the first part (chapter 7) stays an attempt to simplify Schmid's norm estimates not using Schmid's SL_2-orbit theorem (cf. [Sch73]), the last chapter leads back to Simpson's original area of interest. There he rewrites the main theorem in terms of variations of Hodge structure.
The Hodge theoretical point is one of the main developments proceeding [Sim90]. Simpson himself proves for example that a rigid discrete subgroup of a real algebraic group, which is not of Hodge type, cannot split quotient of the fundamental group of a smooth irreducible projective variety (cf. [Sim92]). He further describes the relation between Mixed twistor structures and variation of Hodge structure in [Sim97]. The topic is treated in great generality by Mochizuki, for example in the books [Moc07a] and [Moc07b]. See as well Sabbah [Sab05].

On the other hand there was a number of authors describing the moduli spaces of our objects, in particular Higgs bundles. To name only a few: Mumford, Gieseker, Maruyama, Mehta, Ramanathan or Yokagawa. A nice overview is given in Simpson [Sim94a] and [Sim94b]. Furthermore interpreting the moduli schemes as non-abelian cohomology we get a correspondence between the moduli scheme for principal $G-$modules with integrable connection and the moduli scheme for semi-stable principal Higgs bundles with vanishing rational Chern class. For further details and a general overview see for example [Sim97].

Maybe the most expectable development is the one generalizing Simpson's main theorem above. This has been done stepwise for higher dimensional quasiprojective manifolds by Corlette [Cor88], Biquard (for divisors smooth at infinity) [Biq97]

and Just-Zuo [JZ97] proving the existence of a tame pluri-harmonic metric on any semi-simple local system over a quasi-projective variety. Their metric is nowadays known as Corlette-Just-Zuo or only Just-Zuo metric. Furthermore there have been several extensions to $p-$adic harmonic metrics, i.e. replacing the field $\mathbb{C}$ by the $p-$adic field K_p. To mention only a few people involved in this field: Deligne and Husemöller [DH87], Goldman and Iwahori [GI63], Jost-Zuo [JZ87],[JZ97] and Gromov-Schoen [GS92]. An overview about the construction of finite energy harmonic maps in both cases gives Zuo [Zuo99].
Finally we want to add an obvious extension: P. Deligne suggested in a letter to Carlos Simpson (cf. [Sim94a]) that the equivalence constructed above is a special case of a more general correspondence, namely Higgs bundles and $\mathscr{D}_X-$modules are a special case of bundles equipped with a $\lambda-$connection. A $\lambda-$connection has to fulfill the twisted Leibniz rule $D^\lambda(fs) = fD^\lambda + \lambda s \otimes d(f)$. Then the special case $\lambda = 0$ is the Higgs bundle case, while for $\lambda = 1$ we have the $\mathscr{D}_X-$module case. We get a holomorphic structure $\overline{\partial}_E + \lambda\theta$ and all other operators are constructed as for the $\mathscr{D}_X$ bundle case. In particular this is the foundation of the considerations by Simpson [Sim97] and Mochizuki.

A Appendix

A.1. Basics

This first section contains basic definitions as well as some well-known results, mainly consequences of Serre's [GAGA] and the Riemann-Hilbert correspondence. The results from algebraic or differential geometry can be seen as a general starting point on the way of constructing our equivalence, while the subsection on the Hodge theory contains some technical results used later on.

A.1.1. Definitions (Algebraic Geometry)

Definition A.1.1. A non-compact curve is a compact Riemann surface deleting finitely many points, i.e. more precisely $X = M \setminus S$ with M compact Riemann surface and $S = \{a_1, \ldots, a_n\}$.
The completion $\overline{X}$ in the analytic topology is just M.

Remark A.1.2. The name curve comes from the well-known equivalence between compact Riemann surfaces and smooth projective complex algebraic curves, e.g. in [Sza09].

Definition A.1.3 (Algebraic Variety)**.** Let k be an algebraically closed field, $V(I) = \{v \in \mathbb{V}^n | f(v) = 0 \ \forall f \in S\}$, $\mathbb{V}^n \in \{\mathbb{A}^n, \mathbb{P}^n\}$. If $\mathbb{V}^n = \mathbb{A}^n$ then $S \subset k[x_1, \ldots, x_n]$; if $\mathbb{V}^n = \mathbb{P}^n$ then S is a subset of the set of homogeneous polynomials. $X \subset \mathbb{V}^n$ is an algebraic variety if $V(I) = X$ for some prime I. Further X is called affine if $V^n = \mathbb{A}^n := k^n$ and projective if $\mathbb{V}^n = \mathbb{P}^n = k^{n+1} \setminus \{0\}/\sim$ with the usual equivalence relation.[1]

Definition A.1.4. A map $f : D(g) \to k$ is regular if it is in $k[x_1, \ldots, x_n][1/g]$, i.e. a rational function with g in the denominator.

[1]More general one can define quasi-projective (resp. quasi-affine) varieties as locally closed subsets of a projective (resp. affine) variety.

A map $f : U \to X, U$ open is regular if $f|_{D(g)\cap V}$ is regular for all $g \in I$.
The ring of all regular maps on $U \subset X$ is denoted by $\mathcal{O}_X(U)$ resp. $\mathcal{O}_X(X) = \mathcal{O}_X$.

Definition A.1.5. Let X be an algebraic variety. The following definitions are equivalent:

(i) The Zariski topology is the weakest topology such that regular maps are continuous and points are closed.

(ii) The Zariski topology is the topology generated by the basis $D(g) = X - V(g), g \in k[x_1, \dots, x_n]$.

To avoid confusion we denote the sheaf of holomorphic functions by $\mathcal{O}_X^{an}$.

Remark A.1.6. $\mathcal{O}_X$ (with the usual restriction of functions) forms a sheaf with values in the category of rings. Further $\mathcal{O}_X = k[x_1, \dots, x_n]/I$ for $X = V(I)$.

Definition A.1.7. Let $\mathcal{F}$ be a sheaf on a manifold M. The stalk $\mathcal{F}_x, x \in X$ of $\mathcal{F}$ at M is the disjoint union of the $\mathcal{F}(U)$, $x \in U \subset M$ open, modulo the following equivalence relation for $s \in \mathcal{F}(U), t \in \mathcal{F}(V)$

$$s \sim t :\Leftrightarrow \exists W \subset U \cap V, x \in W : s|_W = t|_W.$$

In symbols

$$\mathcal{F}_x = \dot{\bigcup_{x \in U \text{ open}}} \mathcal{F}(U)/\sim .$$

Remark A.1.8. (i) Equivalently $\mathcal{F}_x$ is the direct limit of $\mathcal{F}(U)$ for $x \in U$.

(a) If $\mathcal{F}$ is a ring so is $\mathcal{F}_x$.

Definition A.1.9. A point $x \in X$ is called non-singular if the stalk $\mathcal{O}_{X,x}$ is a regular local ring, i.e. if $\dim_k T_x X$[2] equals the Krull dimension of $\mathcal{O}_{X,x}$. For $k = \mathbb{C}$ this is equivalent to $\exists U \subset \mathbb{C}^n$ open, $\exists B \subset \mathbb{C}^n$ ball, $\exists \varphi : U \to B$ biholomorphic such that $\varphi(U \cap X) = B \cap V$ for some subspace $V \subset \mathbb{C}^n$.[3]
A variety is called non-singular or smooth if all points are non-singular.

Remark A.1.10. For $k = \mathbb{C}$ a non-singular variety is a complex submanifold of $\mathbb{A}^n$ or $\mathbb{P}^n$.

[2] $T_x X$ tangent space at $x \in X$.
[3] A proof can be found in [Ara12].

Definition A.1.11. Suppose $D = \sum_{s \in S} n_s s$ is a divisor, $S \subset \overline{X}$ finite, $\overline{X}$ Riemann surfaces. Let $\mathcal{M}(U)$ denote the vector space of meromorphic functions (resp. rational functions) on U open. Define $\mathscr{O}(D)|_U$ as the vector space of all $f \in \mathcal{M}(U)$ which have at $s \in U$ order at least $-n_s$. In particular for $D = s$ these are just the functions with a pole of order at most 1. $\mathscr{O}(D)$ is a sheaf of $\mathcal{O}_X^{an}-$modules (resp. $\mathcal{O}_X-$modules), since multiplication with a holomorphic function preserves this property.
Now consider a coordinate neighbourhood of $s \in S$ with the coordinate z, then every function in $\mathscr{O}(D)|_U$ has a Laurent expansion of the form $z^{-n_s} g$ with some holomorphic function g.

Remark A.1.12. (i) The last property shows that $\mathscr{O}(D)$ is locally isomorphic to $\mathcal{O}_X$, i.e. is an invertible sheaf.

(ii) Further $\mathscr{O}(D) \otimes_{\mathcal{O}_X} \mathscr{O}(D') \simeq \mathscr{O}(D + D')$.

A.1.2. Definitions (Differential Geometry)

Definition A.1.13 (Vector bundle)**.** Let M be a complex manifold (resp. topological space resp. algebraic variety), E the total space (another manifold/topological space/algebraic variety) and $\pi : E \to M$ a holomorphic (resp. continuous resp. regular) map. (E, M, π) is called a holomorphic (resp. topological resp. algebraic) vector bundle if

(1) $\pi^{-1}(x)$ is a finite-dimensional $\mathbb{C}-$vector space.

(2) $\exists k \in \mathbb{N}$, $\exists (U_j)$ open covering (in the analytic resp. Zariski topology) with local trivializations

$$\varphi_j : E_{U_j} := \pi^{-1}(U_j) \to U_j \times \mathbb{K}^k \qquad \text{biholomorphic (resp. homeomorph resp. biregular)},$$

such that $\pi|_{E_{U_j}} = \mathrm{pr}_1 \circ \varphi_j$ and $\varphi_a = \varphi_j|_{E_a} : E_a \to \{a\} \times \mathbb{K}^k$ is an isomorphism.

Definition A.1.14. (i) A bundle morphism $\sigma : E \to F$ between two smooth (resp. holomorphic, resp. algebraic) vector bundles $\pi_E : E \to X, \pi_F : F \to X$ is a smooth (resp. holomorphic, resp. algebraic) map such that $\pi_E = \pi_F \sigma$ and σ is $\mathbb{C}-$linear on the fibres.

(ii) For a bundle morphism $\sigma : E \to F$ the kernel $\ker \sigma = \bigcup_{x \in X} \ker \sigma|_{E_x}$ and the range $\mathrm{im}\, \sigma = \bigcup_{x \in X} \mathrm{im}\, \sigma|_{E_x}$ are subbundles if and only if $\mathrm{rk}(\sigma|_{E_x})$ is independent of x, i.e. constant in x.[4]

[4] See Griffith and Harris [GH78] p. 68

Example A.1.15. (i) does not imply (ii): Take the trivial bundle E with $\pi = \mathrm{pr}_1 : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and the bundle morphism, $E \to E$, $\sigma(x, y) = (x, xy)$, then σ is obviously smooth (resp. holomorphic resp. regular) and $\mathrm{rk}(\sigma(1, \cdot)) = 1 \neq \mathrm{rk}(\sigma(0, \cdot)) = 0$.

Remark A.1.16. (1) For smooth manifolds replace holomorphic by smooth and biholomorphic by diffeomorphic in definition A.1.13.

(2) $(\varphi_j^{-1}(x, e_i))_{1 \leq i \leq k}$ for the standard basis $(e_i)_{1 \leq i \leq k} \in \mathbb{K}^k$ is sometimes called local frame field (over U_j). In particular we will always find such a local frame, which is holomorphic, smooth or regular dependening on the kind of bundle we are working with.

Theorem A.1.17. Let X be a smooth manifold (resp. complex manifold resp. algebraic variety) then there is a one-to-one correspondence between the categories of smooth (resp. holomorphic resp. algebraic) vector bundles of rank r and the category of isomorphism classes of locally free sheaves of $\mathcal{E}_X-$ (resp. $\mathcal{O}_X^{an}-$ resp. $\mathcal{O}_X$))-modules of rank r by assigning to every bundle its sheaf of smooth/holomorphic/regular sections.[5] Particularly the equivalence respects sums, tensor products, dualizing, etc.

Proof. This follows from Serre's [GAGA]. □

Remark A.1.18. (i) The theorem says nothing about morphisms. Note that vector bundles homomorphisms are often defined to have constant rank[6], while for $\mathcal{O}_X^{an}$ (resp. $\mathcal{O}_X$)-modules this is usually not the case.[7]

(ii) If we drop the restriction on constant rank homomorphisms there is an equivalence. To see this consider

$$\begin{aligned} \Psi \quad : \quad & \mathrm{Vect}(E, F) \to (\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U, E), \Gamma(U, F)))_U \\ & \sigma \to \{s \mapsto \{x \mapsto \sigma(x, s(x))\}\} \end{aligned}$$

where $(\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U, E), \Gamma(U, F)))_U$ is the set of $\mathcal{E}(U)-$sheaf homomorphisms and $\mathrm{Vect}(E, F)$ the set of smooth vector bundle homomorphisms with not necessarily constant rank. An inverse map is given by

$$\begin{aligned} \Phi \quad : \quad & (\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U, E), \Gamma(U, F)))_U \to \mathrm{Vect}(E, F) \\ & \varphi \to \Phi(x, s(x)) := (\varphi(s))(x), \quad x \in V \text{ open}, \forall s \in \Gamma(V, E).^8 \end{aligned}$$

[5] see A.1.20.
[6] A.1.14, (ii).
[7] See Huybrechts [Huy05], p. 72.

Φ is well-defined since sheaf homomorphisms respect restrictions, i.e. we need to consider only a small neighbourhood of x and hence there is always a local section given by the inverse of the local trivialization for some fixed second component ($\forall e \in E_y \subset E_V \exists \tilde{e} \in \mathbb{C}^n : \varphi_V^{-1}(y, \tilde{e}) = e$). Further take a local smooth frame field e_i and two sections $s, t \in \Gamma(V, E)$ with $s(y) = t(y) = e$. Then there are $s_i, t_i \in \mathcal{E}(V)$ such that $s(x) = \sum_{i=1}^n s_i(x)e_i(x)$, $t(x) = \sum_{i=1}^n t_i(x)e_i(x)$ and $\exists \alpha_i, \beta_i \in \mathcal{E}(V)$ such $\alpha(x)s(x) := \sum_{i=1}^n \alpha_i(x)s_i(x)e_i(x) = \sum_{i=1}^n \beta_i(x)t_i(x)e_i(x) = \beta_i(x), \forall x \in V$ and $\alpha_i(y) = \beta_i(y)$ $\forall \leq i \leq n$. Thus

$$\begin{aligned}
\sum_{i=1}^n \alpha_i \Phi\left(s_i e_i\right) &= \Phi\left(\sum_{i=1}^n \alpha_i s_i e_i\right) = \Phi\left(\sum_{i=1}^n \beta_i t_i e_i\right) = \sum_{i=1}^n \beta_i \Phi\left(t_i e_i\right) \\
\overset{x=y}{\Longrightarrow} (\varphi(s))(y) &= \left(\Phi\left(\sum_{i=1}^n s_i e_i\right)\right)(y) = \left(\Phi\left(\sum_{i=1}^n t_i e_i\right)\right)(y) \\
&= (\varphi(t))(y).
\end{aligned}$$

Hence Ψ is independent of the chosen section. It is a homomorphism on the fibers since

$$\begin{aligned}
\Phi(x, a(s(x) + t(x))) &= (\varphi(a(s+t)))(x) = a\varphi(s)(x) + a\varphi(t)(x) \\
&= a\Phi(x, s(x)) + a\Phi(x, t(x)),
\end{aligned}$$

where we associated a with the constant (smooth) map $x \mapsto a$.
The images of Ψ and Φ are smooth by definition of a smooth bundle resp. a smooth section.
Finally

$$\begin{aligned}
\Psi \circ \Phi(\varphi) &= \Psi((\varphi(s))(\cdot)) = s \mapsto \{x \mapsto \varphi(s(x))\} = \varphi \text{ and} \\
\Phi \circ \Psi(\sigma)(x, s(x)) &= \Phi\left(\{s \mapsto \{x \mapsto \sigma(x, s(x))\}\}\right) = \sigma(x, s(x))
\end{aligned}$$

for all $s \in \Gamma(U, E), x \in U$, i.e. Φ and Ψ are inverse to each other. As we have additionally

$$\begin{aligned}
&\Psi(\sigma + \tau) = \Psi(\sigma) + \Psi(\tau) \text{ and } \Phi(\varphi + \phi) = \Phi(\varphi) + \Phi(\phi), \\
&\Psi(\mathrm{id}_{\mathrm{Vect}(E,F)}) = \mathrm{id}_{(\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U,E),\Gamma(U,F)))_U} \text{ and} \\
&\Phi(\mathrm{id}_{(\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U,E),\Gamma(U,F)))_U}) = \mathrm{id}_{\mathrm{Vect}(E,F)}.
\end{aligned}$$

These are inverse functors making the two categories equivalent. As usual we can replace smooth by holomorphic resp. regular and so on.

[8] Of course we wrote a value on the right hand side which is formally not correct. However it should be clear, which map we mean.

On the other hand if we denote now by $(\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U,E),\Gamma(U,F)))_U$ the set of $\mathcal{E}(U)-$sheaf homomorphisms of constant rank[9] and $\mathrm{Vect}(E,F)$ the set of smooth vector bundle homomorphisms (of constant rank), then the functors Φ and Ψ establish an equivalence as well. For a rank k vector bundle homomorphism σ and a local holomorphic frame $(s_i)_{1\leq i\leq n}$, $\Psi(\sigma)(s_i) = \{x \mapsto \sigma(x, s_i(x)) = \sum_{i=1}^{k} \alpha_i j(x) s_j(x)\}, \forall 1 \leq i \leq n$ for some holomorphic functions s_j.[10] Hence $\Psi(\sigma)$ has rank k as a map from $\Gamma(U,E)_x \to \Gamma(U,F)_x$. On the other hand for $\mathrm{rk}\varphi = k, \varphi \in (\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U,E),\Gamma(U,F)))_U$, and for a basis $s_i(x) \in E_x$ we get $\Phi(\varphi)(s_i(x)) = \varphi(s_i)(x) = \sum_{j=1}^{k} \alpha_{ij}(x) s_j(x)$, i.e. $\mathrm{rk}\Phi(\varphi) = k$.

(iii) In algebraic geometry it is more reasonable to enlarge the category of locally free $\mathcal{O}_X-$modules rather than to restrict it. The reason is that one needs the cokernel of those morphisms to be in the category again. For locally free $\mathcal{O}_X-$modules the cokernel of a homomorphism with non-constant rank no longer is a locally free $\mathcal{O}_X-$module, but a coherent sheaf. In fact the category of coherent sheaves is the smallest category enlarging the category of locally free $\mathcal{O}_X-$modules (with not necessarily constant rank homomorphisms) where the cokernel of each (local) morphism is again a coherent sheaf. More precisely any sheaf $\mathcal{F}$ of $\mathcal{O}_X-$modules with a local presentation $\mathcal{O}_X^n \to \mathcal{O}_X^m \to \mathcal{F} \to 0, m, n \in \mathbb{N}$ (exact, i.e. $\mathcal{F}$ isomorphic to the cokernel) is coherent (and vice-versa); quasi-coherent if n, m are allowed to be not finite.

Theorem A.1.19. (1) Over a compact Riemann surface the algebraic and holomorphic vector bundles are the same.

(2) In the non-compact case (i) no longer holds.

(3) However, every algebraic vector bundle is holomorphic w.r.t. the Zariski topology.

(4) In fact every holomorphic vector bundle on a non-compact Riemann surfaces is trivial.

(5) There is as well an equivalence between the category of algebraic vector bundles with flat regular connection and holomorphic vector bundles with flat connection.

(6) There is a bijection between Gauge equivalence classes of flat vector bundles and equivalence classes of vector bundles defined by constant transition

[9] Restricted to the stalks the map has constant rank.

[10] W.l.o.g. we choose the image to be spanned by the first k basis elements s_i.

functions g_{ij} for $\varphi_i\varphi_j^{-1}(x,g) = (x, g_{ij}g)$ and $\varphi_i : \pi^{-1}(U_i) \to X \times \mathbb{C}^n$ local trivializations. Two cocycles are equivalent if they differ by a coboundary, i.e. $g_{ij} \equiv h_{ij} \Leftrightarrow \forall (i,j) \in I \times I\ \exists a_i, a_j \in \mathrm{Gl}_n(\mathbb{C})$ such that $g_{ij} = a_i^{-1} h_{ij} a_j$.

Proof. (1) Again by Serre's famous paper [GAGA]. For (2) and (3) see as well 16 in [CG75]. (4) by 30.4 in "Lectures on Riemann surfaces" [For81] by Otto Forster. (5) Deligne [Del70], p. 97, 5.9. (6) Szamuely [Sza09] for the vector bundle case or Reisert, [Rei10] for the principal $G-$bundle case. □

Definition A.1.20. Let $\pi : E \to X$ be a smooth projection, E, X smooth manifolds, then $\Gamma(X,E) := \{s : X \to E | s \text{ smooth}, \pi \circ s = \mathrm{id}_X\}$ is the set of smooth sections into E. In the same way $\Gamma_{hol}(X,E) := \{s : X \to E | s \text{ holomorphic}, \pi \circ s = \mathrm{id}_X\}$ or $\Gamma_{reg}(X,E) := \{s : X \to E | s \text{ regular}, \pi \circ s = \mathrm{id}_X\}$ are defined. Further denote $\mathcal{E}_X = \{f : X \to \mathbb{C} | s \text{ smooth}\}$. Then $\mathcal{E}_X$ acts on $\Gamma(X,E)$ by multiplication, i.e. $f \cdot s : x \mapsto f(x)s(x), f \in \mathcal{E}_X, s \in \Gamma(X,E)$.
Define $\bigwedge_a^p := \{\alpha : T_a M^p \to \mathbb{C} | \alpha \ \ \mathbb{C} - \text{multilinear and alternating}\}, a \in X$ and $\bigwedge_X^p = \bigcup_{a \in X} \bigwedge_a^p$. Then

$$\Omega_X^p := \Gamma(X, \textstyle\bigwedge_X^p)$$

is the set of differential $p-$forms. If X is a complex manifold and $z_i, \overline{z}_i, 1 \le i \le r$ are local holomorphic coordinates then $\Omega_X^{p,q}$ is the subspace of Ω_X^k locally spanned by forms

$$\omega(z) = \eta(z)\, \mathrm{d}\, z_{i_1} \wedge \ldots \mathrm{d}\, z_{i_p} \wedge \mathrm{d}\, \overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\, \overline{z}_{i_q}, \quad \eta \text{ smooth into } \mathbb{C}$$

(with pairwise different indices). Then

$$\Omega_X^k = \bigoplus_{p+q=k} \Omega_X^{p,q}.$$

Equivalently $\Omega_X^{p,q} = \Gamma(X, \bigwedge_X^{p,q})$ with $\bigwedge_X^{p,q} = \bigcup_{a \in X} \bigwedge_a^{p,q}, \bigwedge_a^{p,q} = \{\alpha : T_a X^{p,q} \to \mathbb{C} | \alpha \ \ \mathbb{C} - \text{multilinear and alternating}\}$ and $T_a X^{p,q} = T_a' X^p \oplus T_a'' X^p$, where $T_a' X = \mathbb{C}\langle \frac{\partial}{\partial z_i} | 1 \le i \le n\rangle, n = \dim X$ is the holomorphic and $T_a'' X = \mathbb{C}\langle \frac{\partial}{\partial \overline{z}_i} | 1 \le i \le n\rangle$ the antiholomorphic tangent space.[11]
On Ω_X^p an exterior derivative $d : \Omega_X^p \to \Omega_X^{p+1}$ can be defined by the $\mathbb{C}-$linear extension of

$$d\omega(x) = d\eta(x)\, \mathrm{d}\, x_{i_1} \wedge \ldots \mathrm{d}\, x_{i_p} := \sum_{j \in \{i_1, \ldots, i_p\}} \frac{\partial \eta(x)}{\partial x_j} \mathrm{d}\, x_j \wedge \mathrm{d}\, x_{i_1} \wedge \ldots \mathrm{d}\, x_{i_p}.$$

[11] These are subspaces of the complexified tangent space $T_a X = \mathbb{C}\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} | 1 \le i \le n\rangle$.

For a complex manifold there is a splitting $d = \partial + \overline{\partial}$ such that

$$\begin{aligned}
&\partial\eta(z)\,\mathrm{d}\,z_{i_1} \wedge \ldots \wedge \mathrm{d}\,z_{i_p} \wedge \mathrm{d}\,\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\,\overline{z}_{j_q} \\
&= \sum_{k\in\{i_1,\ldots,i_p\}} \frac{\partial\eta}{\partial z_k}\,\mathrm{d}\,z_k \wedge \mathrm{d}\,z_{i_1} \wedge \ldots \mathrm{d}\,z_{i_p} \wedge \mathrm{d}\,\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\,\overline{z}_{j_q} \\
&\overline{\partial}\eta(z)\,\mathrm{d}\,z_{i_1} \wedge \ldots \wedge \mathrm{d}\,z_{i_p} \wedge \mathrm{d}\,\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\,\overline{z}_{j_q} \\
&= \sum_{k\in\{j_1,\ldots,j_q\}} \frac{\partial\eta}{\partial \overline{z}_k}\,\mathrm{d}\,\overline{z}_k \wedge \mathrm{d}\,z_{i_1} \wedge \ldots \mathrm{d}\,z_{i_p} \wedge \mathrm{d}\,\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\,\overline{z}_{j_q}.
\end{aligned}$$

Remark A.1.21. (1) $\bigwedge_X^1 = T^*X$ the cotangent bundle.

(2) $d^2 = \partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$.

(3) $\partial, \overline{\partial}$ satisfy the Leibniz rule, i.e.

$$\begin{aligned}
\partial(\alpha \wedge \beta) &= \partial\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \partial\beta \\
\overline{\partial}(\alpha \wedge \beta) &= \overline{\partial}\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial}\beta,
\end{aligned}$$

for $\alpha \in \Omega_X^{p,q}, \beta \in \Omega_X^{s,r}$.[12]

(4) The holomorphic tangent bundle is naturally isomorphic to the holomorphic vector bundle $\mathcal{T}_X$ given by the transition functions $\psi_{ij}(z) = J(\varphi_{ij}(\varphi_j(z)))$ for the Jacobian $J(\varphi_{ij})(\varphi_j(z)) := \left(\frac{\partial\varphi_{ij}^k}{\partial z_l}(\varphi_j(z))\right)_{k,l}$ of the local trivializations φ_i of the bundle $\pi : E \to X$.[13]

There is an extension of $\overline{\partial}$ to the case of $E-$valued differential forms:

Theorem A.1.22. (i) If E is a holomorphic vector bundle then there exists a natural sheaf homomorphism of $\mathbb{C}-$vector spaces.

$$\begin{aligned}
\overline{\partial}_E : \Omega_X^{p,q}(E) \to \Omega_X^{p,q+1}(E) &:= \Gamma(X,E) \otimes_{\mathcal{E}_X} \Gamma(X, \textstyle\bigwedge_X^{p,q+1}) \\
&= \Gamma(X,E) \otimes_{\mathcal{E}_X} \Omega_X^{p,q+1}
\end{aligned}$$ [14]

which obeys $\overline{\partial}_E^2 = 0$ and the Leibniz rule $\overline{\partial}_E(\eta \otimes \omega) = \eta \otimes \overline{\partial}(\omega) + \overline{\partial}_E(\eta) \wedge \omega$ for $\eta \in \Omega_X^0(E), \omega \in \Omega_X^{p,q}$. Here the $\wedge-$product is defined as: Let $\eta \in \Omega_X^{0,0}(E), \omega_1, \omega_2 \in \Omega_X^{p,q}$ then $(\eta \otimes \omega_1) \wedge \omega_2 := \eta \otimes (\omega_1 \wedge \omega_2)$.

[12] See [Huy05] lemma 1.3.6, p. 44.
[13] Again in Huybrechts [Huy05], p. 71.
[14] $\mathcal{E}_X$ acts as $f \cdot s : x \mapsto f(x)s(x)$ for $f \in \mathcal{E}_X$ and $s \in \Omega_X^{p,q}$.

(ii) Let $(s_i)_{1\leq i\leq r}$ be a holomorphic local frame field of E, for example $s_i(x) = \varphi_U^{-1}(x, e_i), e_i$ standard basis element of $\mathbb{C}^r$, φ_U local trivialization, then we can write for all $s' \in \Omega_U^{p,q}(E) : s' = s \otimes \omega$ with $s = \sum_{i=1}^r \alpha_i s_i, \alpha_i \in \mathcal{E}(U)$. $\overline{\partial}_E$ acts on s' as

$$\overline{\partial}_E(s) = \sum_{i=1}^{r} s_i \otimes \overline{\partial}(\alpha_i \omega).$$

This description of $\overline{\partial}_E$ is independent of the chosen holomorphic frame field since any transition matrix to another holomorphic frame field is holomorphic in each component and makes therewith by Leibniz rule no contribution to the result.[15]

(iii) The holomorphic structure of a holomorphic vector bundle is uniquely determined by an operator $\overline{\partial}_E : \Omega_X^0(E) \to \Omega_X^{0,1}(E)$ with the properties named in (i).

Proof. Huybrechts [Huy05], p. 109f. □

Remark A.1.23. (i) In future it may be often enough to consider only local sections since a global section restricts to sections on open subsets and local section can be glued together to a global section if they coincide on some open set[16]; a sheaf homomorphism φ commutes with the restrictions and is thus defined by local sections, since for an open covering $(U_i)_{i\in I}$, e.g. those with local trivializations, $s_i|_{U_{ij}} = s_j|_{U_{ij}} \Rightarrow \varphi(s_i)|_{U_i\cap U_j} = \varphi(s_i|_{U_i\cap U_j}) = \varphi(s_j|_{U_i\cap U_j}) = \varphi(s_j)|_{U_i\cap U_j}$.
For example $\overline{\partial}_E^2 = 0$ follows by (ii) locally out of $\overline{\partial}^2 = 0$ and this follows from $d^2 = 0$ by degree considerations. However, for a Riemann surface this is obvious since there are no non-trivial $(p, q)-$forms for $p > 1$ or $q > 1$.

(ii) For smooth vector bundles an operator $\overline{\partial}_E$ does in general not exist.

In the following we will denote a holomorphic vector bundle by $(E, \overline{\partial}_E)$.

Remark A.1.24. Recall that the topology on $E \otimes_{\mathbb{C}} F := \dot{\bigcup}_{x\in X} E_x \otimes_{\mathbb{C}} F_x$ resp. $E_U \otimes_{\mathbb{C}} F_U := \dot{\bigcup}_{x\in U} E_x \otimes_{\mathbb{C}} F_x, U$ open, is the unique topology such that

$$\begin{aligned} \varphi_{\otimes,U} \quad : \quad & E_U \otimes F_U \to U \times (\mathbb{C}^e \otimes_{\mathbb{C}} \mathbb{C}^f), \quad e = \mathrm{rk}(E), f = \mathrm{rk}(F) \\ & x_E \otimes x_F \mapsto (x, \varphi_{E_U}(x_E) \otimes \varphi_{F_U}(x_F)), \quad x_E \otimes x_F \in E_x \otimes F_x \end{aligned}$$

[15] Huybrechts [Huy05], p. 109f.
[16] Differences to a presheaf.

is homeomorphic, where φ_{E_U} resp. φ_{F_U} are the local trivializations over U.[17] Note that $\mathbb{C}^e \otimes_{\mathbb{C}} \mathbb{C}^f \simeq \mathbb{C}^{ef}$, i.e. the $\varphi_{\otimes,U}$ can be identified with local trivializations of the $ef-$dimensional tensor product bundle.
For smooth resp. regular resp. holomorphic vector bundles we want the maps to be diffeomorphic resp. biregular resp. biholomorphic. [18]
Let $(e_i)_{1\leq i\leq e}$ be the standard basis of $\mathbb{C}^e$ and $(f_i)_{1\leq j\leq f}$ the standard basis of $\mathbb{C}^f$ then $e_i \otimes f_j(x) := \varphi^{-1}_{\otimes,U}\left(x, (e_i \otimes f_j)_{\substack{1\leq i\leq e\\ 1\leq j\leq f}}\right)$ is a local frame field since local trivializations are $\mathbb{C}-$isomorphisms for x fixed. Let $e_i(x) := \varphi^{-1}_{E_U}(x, (e_i)_{1\leq i\leq e})$ and $f_j(x) := \varphi^{-1}_{F_U}(x, (f_j)_{1\leq j\leq f})$ be the corresponding frame fields on E resp. F. Then define a map $\varphi : \Gamma(U, E \otimes_{\mathbb{C}} F) \to \Gamma(U, E) \otimes_{\mathcal{E}(U)} \Gamma(U, F)$ by linear extension of

$$\varphi(x \mapsto e_i \otimes f_j(x)) = \{x \mapsto e_i(x)\} \otimes \{x \mapsto f_j(x)\}.$$

i.e.

$$\varphi\left(x \mapsto \sum_{\substack{1\leq i\leq e\\ 1\leq j\leq f}} \alpha_{ij}(x) e_i \otimes f_j(x)\right) := \sum_{\substack{1\leq i\leq e\\ 1\leq j\leq f}} \alpha_{ij}\varphi(x \mapsto e_i \otimes f_j(x))$$

for some maps $\alpha_{ij} : X \to \mathbb{C}$. Of course the α_{ij} cannot be chosen arbitrary under all functions if φ is defined on $\Gamma(U, E \otimes_{\mathbb{C}} F)$. More precisely

$$s_\alpha := \left\{x \mapsto \sum_{\substack{1\leq i\leq e\\ 1\leq j\leq f}} \alpha_{ij}(x) e_i \otimes f_j(x)\right\}$$

is smooth iff

$$\varphi_{\otimes,U} \circ s_\alpha(x) = \left\{x \mapsto \left(x, \sum_{\substack{1\leq i\leq e\\ 1\leq j\leq f}} \alpha_{ij}(x) e_i \otimes f_j\right)\right\}$$

is smooth. But this is only the case iff all component functions are smooth, i.e. iff α_{ij} is smooth for all $1 \leq i \leq e, 1 \leq j \leq f$. In particular φ well-defined.
Since $\mathcal{E}(U)$ acts on $\Gamma(U, E \otimes_{\mathbb{C}} F)$ by multiplication we have $\varphi(\alpha s) = \alpha\varphi(s)$ for

[17] Use a common refinement to find a covering which suits both vector bundles. The map is well-defined in the first component since for local trivializations $\mathrm{pr}_1\varphi_{E_U}(x_E) = \pi|_{E_U}(x_E) = x = \pi|_{F_U}(x_F) = \mathrm{pr}_1\varphi_{F_U}$ for all $x_E \otimes x_F \in E_x \otimes F_x$.

[18] See e.g. Hatcher [Hat03], p. 9.

$\alpha \in \mathcal{E}(U)$, $s \in \Gamma(U, E \otimes_{\mathbb{C}} F)$, i.e. φ is a $\mathcal{E}(U)-$homomorphism.
On the other hand

$$\varphi^{-1}\left(\sum_{\substack{1\le i\le e\\1\le j\le f}} \alpha_{ij}\{x \mapsto e_i(x)\} \otimes \{x \mapsto f_j(x)\}\right)$$
$$:= \left\{ x \mapsto \sum_{\substack{1\le i\le e\\1\le j\le f}} \alpha_{ij}(x) e_i \otimes f_j(x) \right\}$$

inverts φ. Thus φ is a $\mathcal{E}(U)-$isomorphism. Since the restriction maps in the sheaves $\Gamma(U, E \otimes_{\mathbb{C}} F)$ and $\Gamma(U, E) \otimes_{\mathcal{E}(U)} \Gamma(U, F)$ are just the usual restrictions for maps, i.e. $s|_U : x \mapsto s(x), \forall x \in U$, φ is an isomorphism of sheaves with values in $\mathcal{E}(U)-$modules.
The same construction works if we replace smooth with holomorphic resp. regular, diffeomorphic with biholomorphic resp. biregular, $\mathcal{E}(U)$ with $\mathcal{O}_X^{an}(U)$ resp. $\mathcal{O}_X(U)$ and Γ with Γ_{hol} resp. Γ_{reg}. Therefore we have the following canonical sheaf-isomorphisms

$$\begin{aligned}
&\Gamma(U, E \otimes_{\mathbb{C}} F) \simeq \Gamma(U, E) \otimes_{\mathcal{E}(U)} \Gamma(U, F) \quad \text{resp.}\\
&\Gamma_{hol}(U, E \otimes_{\mathbb{C}} F) \simeq \Gamma(U, E)_{hol} \otimes_{\mathcal{O}_X^{an}(U)} \Gamma(U, F)_{hol} \quad \text{resp.}\\
&\Gamma_{reg}(U, E \otimes_{\mathbb{C}} F) \simeq \Gamma(U, E)_{reg} \otimes_{\mathcal{O}(U)_X} \Gamma(U, F)_{reg}.
\end{aligned}$$

Remark A.1.25. For vector spaces E_x, F_x we have $\mathrm{Hom}_{\mathbb{C}}(E_x, F_x) \simeq E_x^* \otimes_{\mathbb{C}} F_x$. So we can define the homomorphism bundle $\mathrm{Hom}_{\mathbb{C}}(E, F) = \bigcup_{x\in X} \mathrm{Hom}_{\mathbb{C}}(E_x, F_x) \simeq \bigcup_{x\in X} E_x^* \otimes_{\mathbb{C}} F = E^* \otimes_{\mathbb{C}} F$.[19] We have

$$\begin{aligned}
&\Gamma(U, \mathrm{Hom}_{\mathbb{C}}(E, F)) \simeq (\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U, E), \Gamma(X, F)))_U \quad \text{resp.}\\
&\Gamma_{hol}(U, \mathrm{Hom}_{\mathbb{C}}(E, F)) \simeq (\mathrm{Hom}_{\mathcal{O}_X^{an}(U)}(\Gamma_{hol}(U, E), \Gamma_{hol}(U, F)))_U \quad \text{resp.}\\
&\Gamma_{reg}(U, \mathrm{Hom}_{\mathbb{C}}(E, F)) \simeq (\mathrm{Hom}_{\mathcal{O}_X(U)}(\Gamma_{reg}(U, E), \Gamma_{reg}(U, F)))_U.
\end{aligned}$$

where $(\mathrm{Hom}_{\mathcal{E}(U)}(\Gamma(U, E), \Gamma(U, F)))_U$ is the set of $\mathcal{E}(U)-$sheaves homomorphisms or analogously in the other cases. For the last step we use that

$$\sigma(x, e) := \{e \mapsto (s(x))(e)\}, \quad e \in E_x$$

is a vector bundle homomorphism with not necessarily constant rank for all $s \in \Gamma(U, \mathrm{Hom}_{\mathbb{C}}(E, F))$: σ is smooth iff $\sigma(x, f(x)) = \{x \mapsto (s(x))(f(x)) \in \Gamma(U, E), \forall f \in \Gamma(U, E)$ iff s is smooth. σ is a $\mathbb{C}-$homomorphism since $s(x)$ is.

[19] Differential Forms and Connections, Darling [Dar94], p. 122, or [Rei09], p. 27.

On the other hand if we have a vector bundle homomorphism σ we get a smooth section of the homomorphism bundle (without rank restrictions) by

$$s(x) := \sigma(x, \cdot).$$

The two constructions are obviously inverse. Thus $\Gamma(U, \mathrm{Hom}_{\mathbb{C}}(E, F))$ is isomorphic to the vector bundle homomorphisms with not necessarily constant rank. Further A.1.18, (ii) shows the connection to sheaf isomorphisms.

At the end of this subsection we want to add two well-known theorems:

Theorem A.1.26. The following categories are equivalent

(i) Smooth projective algebraic curves (i.e. one dimensional projective algebraic varieties) with regular maps.

(ii) Compact Riemann surfaces with holomorphic maps.

Remark A.1.27. The analogue to punctured surfaces are quasi-projective curves. We will mainly work in the holomorphic category.

Theorem A.1.27. There is a one-to-one correspondence between the isomorphism classes of invertible sheaves, divisors and line bundles[20] on a Riemann surface X. To a line bundle L the corresponding invertible sheaf is $\Gamma(U, L)$, $U \subset X$ open and the corresponding divisor is defined as $\sum \mathrm{ord}_s(f)s$ for any non-zero meromorphic section $f : X \to L$.[21]

Proof. A proof can be found in Miranda [Mir95], chapter XI. □

A.1.3. Definitions (Hodge Theory)

Let V be a real vector space then $I : V \to V, I^2 = -\mathrm{id}$ is called almost complex structure. Extend $I : V_{\mathbb{C}} \to V_{\mathbb{C}}$ the complexification of V, $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Then I has eigenvalues $\pm i$ and eigenspaces $V^{1,0} = \{v \in V_{\mathbb{C}} | I(v) = iv\}, V^{0,1} = \{v \in V_{\mathbb{C}} | I(v) = -iv\}, V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Complex conjugation induces $V^{1,0} \simeq V^{0,1}$ real isomorphism. On V^* we have an induced almost complex structure $I(f)(v) := f(I(v))$ and an induced decomposition $(V^{1,0})^* = (V^*)^{1,0}, (V^{0,1})^* = (V^*)^{0,1}$. The exterior algebra is $\bigwedge^* V = \bigoplus_{k=0}^n \bigwedge^k V, n = \dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V_{\mathbb{C}}$ resp. $\bigwedge^* V_{\mathbb{C}} = \bigoplus_{k=0}^n \bigwedge^k V_{\mathbb{C}}, \bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$ for $\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1}$. For $(z_i, \overline{z}_i)$ the complex coordinates, $(x_1, y_1, \ldots)$ real coordinates, we have the volume form $(2i)^m \bigwedge_{k=1}^m (z_i \wedge \overline{z}_i) = \bigwedge_{k=1}^m (x_i \wedge y_i)$. We define the fundamental form to I on an

[20] See A.1.11 for a definition of a line bundle, i.e. a rank one vector bundle.
[21] This is well-defined up to some equivalence relation. See [Mir95].

euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ as $\omega := \langle I(\cdot), (\cdot) \rangle = -\langle (\cdot), I(\cdot) \rangle$ for I compatible with $\langle \cdot, \cdot \rangle$, i.e. $\langle I(\cdot), I(\cdot) \rangle = \langle (\cdot), (\cdot) \rangle$.[22] Moreover $(\cdot, \cdot) := \langle \cdot, \cdot \rangle - i\omega$ defines a positive hermitian form and $V_{\mathbb{C}} = V^{1,0} \otimes V^{0,1}$ is an orthogonal decomposition w.r.t. to $\langle v \otimes \lambda, w \otimes \mu \rangle := (\lambda \overline{\mu}) \langle v, w \rangle$ on $V \otimes_{\mathbb{R}} \mathbb{C}$ for $v, w \in V, \lambda, \mu \in \mathbb{C}$.

Remark A.1.28. An example is $V = T_x X$ a tangent space. Then $\Lambda^{p,q} V^* = \Omega_X^{p,q}$.

Definition A.1.29. Let V be an euclidean vector space with a compatible complex structure and e_i a basis of V with volume form $e_1 \wedge \ldots \wedge e_n$.

(i) $L : \bigwedge^* V_{\mathbb{C}}^* \to \bigwedge^* V_{\mathbb{C}}^*, \alpha \mapsto \omega \wedge \alpha$ is called Lefschetz operator. We have $L\left(\bigwedge^{p,q} V^*\right) \subset \bigwedge^{p+1,q+1} V^*$ of bidegree $(1, 1)$.

(ii) The Hodge $*-$operator $\bigwedge^k V \to \bigwedge^{n-k} V$ is defined as $\alpha \wedge * \beta = \langle \alpha_0, \beta_0 \rangle \mathrm{vol}$ for all $\alpha, \beta \in \bigwedge^* V, \alpha := \alpha_0 \bigwedge_{i \in I} e_i, \beta := \beta_0 \bigwedge_{i \in I} e_i, I \subset \{1, \ldots, n\}, |I| = k$ and extend $\mathbb{C}$-linearly on complex forms. In particular $*\mathrm{vol} = 1, *(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \mathrm{sign}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$.
We get a non-degenerated pairing $\bigwedge^k V \times \bigwedge^{n-k} V \to \bigwedge^n V, (\alpha, \beta) \mapsto \alpha \wedge * \beta$ that we will denote as well with $\langle \cdot, \cdot \rangle$.[23] Then $\langle \alpha, * \beta \rangle = (-1)^{k(n-k)} \langle * \alpha, \beta \rangle$, i.e. $*$ self-adjoint up to sign.
$*$ maps $\bigwedge^{p,q} V^* \subset \bigwedge^{m-p,n-q} V^*$. Note that $(-1)^{k(n-k)} *$ is the inverse to $*$ on $\bigwedge^k V$. In the manifold setting which we will treat next, a complex manifold will have even $\mathbb{R}-$dimension, i.e. $(-1)^{k(n-k)} = (-1)^{k^2} = (-1)^k$.

(iii) The dual Lefschetz operator is defined as $\Lambda = *^{-1} \circ L \circ *$. It is the unique operator $\Lambda : \bigwedge^* V^* \to \bigwedge^* V^*$ with this property. We have $\Lambda\left(\bigwedge^{p,q} V^*\right) \subset \bigwedge^{p-1,q-1} V^*$ of bidegree $(-1, -1)$.

For details see Huybrechts [Huy05] p. 33ff.
Next consider the hermitian manifolds (X, g), i.e. g_x on T_x is compatible with the almost complex structure I_x. In this context we call $\omega := g(I(\cdot), (\cdot))$ fundamental form. Locally $\omega = \frac{i}{2} \sum_{i,j=1}^n h_{ij} \, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}$, $H = h_{ij}$ positive-definite hermitian. We will mainly use $h_{ij} = E$ the euclidean metric. Note the g is uniquely defined by ω and I via $g(\cdot, \cdot) = g(I(\cdot), I(\cdot)) = \omega(\cdot, I(\cdot))$. Therefore we might sometimes use ω instead of g.

Remark A.1.30. (i) From a Riemannian metric g and the almost complex structure I we can construct a hermitian metric $g'(\cdot, \cdot) = \frac{1}{2}(g(\cdot, \cdot) + g(I(\cdot), I(\cdot)))$ or as in the vector space case above $(\cdot, \cdot) = \langle \cdot, \cdot \rangle - i\omega$ is a hermitian form for any fundamental form.

[22] I is orthogonal.
[23] For $n = 0$ we get our inner product back.

(ii) In the Kähler case ω has to be closed. For the euclidean metric on the unit disc the volume form is just $\frac{i}{2}\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}$. We will mainly use this metric in our one-dimensional case.

Recall that for an open subset in $U \subset \mathbb{C}$ we get a basis of T_xU: $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and a complex structure $I(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}, I(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$ and analogous on $(T_xU)^*$ the dual basis $\mathrm{d}\,x, \mathrm{d}\,y$ and the complex structure $I(\mathrm{d}\,x) = \mathrm{d}\,y, I(\mathrm{d}\,y) = -\mathrm{d}\,x$. Usually we use the complex coordinates on the complexified tangent space: $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ and the dual basis $\mathrm{d}\,z = \mathrm{d}\,x + i\,\mathrm{d}\,y, \mathrm{d}\,\overline{z} = \mathrm{d}\,x - i\,\mathrm{d}\,y$. On T_x^*U we may use the euclidean metric $\frac{i}{2}\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}$.

Definition A.1.31. (i) $L : \bigwedge^k T^*X \to \bigwedge^{k+2} T^*X, \alpha \mapsto \alpha \wedge \omega$.

(ii) $* : \bigwedge^k T^*X \to \bigwedge^{2n-k} T^*X$ as before using the metric g as inner product and the natural orientation $\mathrm{d}\,x \wedge \mathrm{d}\,y$ resp. $\frac{i}{2}\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}$ of the complex manifold X. Here $n = \dim_{\mathbb{C}}(X)$.[24]

(iii) $\Lambda : \bigwedge^k T^*X \to \bigwedge^{k-2} T^*X, \Lambda = *^{-1} \circ L \circ *$.

(iv) For any $d, \partial, \overline{\partial}$ define $d^*, \partial^*, \overline{\partial}^*$ as the $*$-adjoint. For example $d^* = - * \circ d \circ *$. Sometimes d^* is denoted as codifferential δ.

Extend the formalism to $E-$valued differential form for (E, h) a hermitian vector bundle. We may interpret h as a map $\varphi : E \to E^*, e \mapsto h(\cdot, e)$. This bijection is $\mathbb{C}-$antilinear.

Definition A.1.32. The Hodge $*-$operator on E is defined as

$$\overline{*}_E \ : \ E \otimes \bigwedge^{p,q} X \to E^* \otimes \bigwedge^{1-p,1-q} X$$

with $\overline{*}_E(s \otimes \omega) := h(\cdot, s) \otimes \overline{*(\omega)} = h(\cdot, s) \otimes *(\overline{\omega})$ by $\mathbb{C}-$linearity of $*$.

Again we get a pairing

$$\begin{aligned} \bigwedge^{p,q} X \times \bigwedge^{p,q} X &\to \bigwedge^{n,n} X, \quad n = \dim X \\ (\alpha, \beta) &\mapsto \alpha \wedge \overline{*}_E(\beta) =: (\alpha, \beta) \end{aligned}$$

where "$\wedge$" is the evaluation in the bundle part and the exterior product in the form part. Replacing E by E^* we get $\overline{*}_{E^*}$ and $\overline{*}_{E^*} \circ \overline{*}_E = (-1)^{p+q}$, since we have in the bundle part the identity and in the form part the conjugation drops out and it remains the classical $*-$operator which leads for even real dimension (in the complex manifold case) just $* \circ * = (-1)^k = (-1)^{p+q}$.
As before we define

[24] For the euclidean metric and for all other metrics by multiplying with the corresponding h_{11}.

Definition A.1.33. (i) For any connection $D = D' + D''$ define $(D')^* = -\overline{*}_{E^*} \circ D'_{E^*} \circ \overline{*}_E$ and $(D'')^* = -\overline{*}_{E^*} \circ D''_{E^*} \circ \overline{*}_E$. Let $D^* = (D')^* + (D'')^*$.

(ii) Analogous for $L_{E^*}(h(\cdot, s) \otimes \omega) = h(\cdot, s) \otimes L(\omega)$ define $\Lambda = -\overline{*}_{E^*} \circ L_{E^*} \overline{*}_E$.

Remark A.1.34. In general a connection D on a vector bundle E induces a connection on the dual bundle E^* by $D_{E^*}(f)(r) = df(r) - f(D(r)), f \in E^*, \forall r \in E$. This becomes $D'_{E^*} h(r, s) = \partial h(r, s) - h(D'r, s) = (-1)^{\deg r} h(r, D''s)$ for a metric connection - $f = h(\cdot, s)$ - or more general for any two operators D', D'' such that $D' + D'' = D$ a connection and $h(D'r, s) + h(r, D''s) = \partial h(r, s)$.[25] Analogously $D''_{E^*} h(r, s) = (-1)^{\deg r} h(r, D's)$.

Lemma A.1.35.

$$\int_X (\alpha, D\beta) = \int_X (D^*\alpha, \beta), \quad \beta \in E \otimes \bigwedge^{p,q} X, \alpha \in E \otimes \bigwedge^{p-1,q} X \otimes \bigwedge^{p,q-1} X$$

if both sides are well-defined, i.e. finite, and if β vanishes on the boundary.

Proof. See Huybrechts, [Huy05] lemma 4.1.12, p. 169f. Huybrechts assumes X compact, which is only used to ensure finiteness of the integral and in order to use Stoke's theorem in the proof we need β to vanish on the boundary. Further he takes $\overline{\partial}_E$ instead of an arbitrary (part of a) connection, but the only property used is the Leibniz rule, which by the way changes slightly in notation, since we redefined $\wedge$ as well. □

Remark A.1.36. For example $\Delta_E := \overline{\partial}_E^* \overline{\partial}_E + \overline{\partial}_E \overline{\partial}_E^*$ is the Laplace operator and a section of E is called harmonic if it is killed by Δ_E.

Lemma A.1.37. (Kähler Identity) Let $D' + D''$ be a connection and $\partial h(\cdot, \cdot) = h(D'\cdot, \cdot) + h(\cdot, D''\cdot)$[26], then $\forall s = s_0 \otimes \mathrm{d}\, z \in E \otimes \Omega_X^{1,0}, \tilde{s} = \tilde{s}_0 \otimes \mathrm{d}\, \overline{z} \in E \otimes \Omega_X^{0,1}$

$$i[\Lambda, D'']s = i\Lambda D''s = (D')^*s \quad \text{and} \quad i[\Lambda, D']\tilde{s} = i\Lambda D'\tilde{s} = -(D'')^*\tilde{s}.$$

Proof. We show the case of a metric connection. If the degree of D' is not fixed we need to apply the following calculation to $\tilde{s}$ too. By direct calculation

$$\begin{aligned} (D')^*s &= -\overline{*}_{E^*} \circ D'_{E^*} \circ \overline{*}_E s \\ &= -\overline{*}_{E^*} \circ D'_{E^*} \circ h(\cdot, s_0) \otimes *(\mathrm{d}\, \overline{z}) \\ &= -\overline{*}_{E^*} \circ h(\cdot, (-1)^{\deg(i\, \mathrm{d}\, \overline{z})} \underbrace{D''s_0}_{=:s_z \, \mathrm{d}\, \overline{z}}) \otimes (i)(\mathrm{d}\, \overline{z}) \end{aligned}$$

[25] The generalization lies in the fact that we don't require D' to be of degree $(1,0)$ resp. D'' of degree $(0,1)$. Furthermore $\deg r$ becomes of importance if we extend to higher order forms r.

[26] e.g. a metric connection.

$$\begin{aligned}
&= -\overline{*}_{E^*} \circ h(\cdot, s_z)(-i)\, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z} \\
&= -\overline{*}_{E^*} \circ h(\cdot, s_z) \cdot (-2)L(1) = -\overline{*}_{E^*} \circ L_{E^*} \circ h(\cdot, s_z) \cdot (-2) \\
&= -\overline{*}_{E^*} \circ L_{E^*} \circ h(\cdot, s_z) \cdot \frac{-2}{2i} *(\mathrm{d}\, \overline{z} \wedge \mathrm{d}\, z) = -i\, \underbrace{\overline{*}_{E^*} \circ L_{E^*} \overline{*}_E}_{-\Lambda} \underbrace{s_z\, \mathrm{d}\, \overline{z} \wedge \mathrm{d}\, z}_{D''s} \\
&= i\Lambda D'' s.
\end{aligned}$$

and since Λ maps 0-forms to 0 we have as well $i[\Lambda, D'']s = i\Lambda D''s$. The second equality follows analogously:

$$\begin{aligned}
(D'')^* \tilde{s} &= -\overline{*}_{E^*} \circ D''_{E^*} \circ \overline{*}_E \tilde{s} \\
&= -\overline{*}_{E^*} \circ D''_{E^*} \circ h(\cdot, \tilde{s}_0) \otimes *(\mathrm{d}\, z) \\
&= -\overline{*}_{E^*} \circ h(\cdot, (-1)^{\deg(-i\, \mathrm{d}\, z)} \underbrace{D'\tilde{s}_0}_{=:\tilde{s}_z\, \mathrm{d}\, z}) \otimes (-i)(\mathrm{d}\, z) \\
&= -\overline{*}_{E^*} \circ h(\cdot, \tilde{s}_z)(-i)\, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z} \\
&= -\overline{*}_{E^*} \circ h(\cdot, \tilde{s}_z) \cdot (-2)L(1) = -\overline{*}_{E^*} \circ L_{E^*} \circ h(\cdot, \tilde{s}_z) \cdot (-2) \\
&= -\overline{*}_{E^*} \circ L_{E^*} \circ h(\cdot, \tilde{s}_z) \cdot \frac{2}{2i} *(\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}) \\
&= -(-i)\, \underbrace{\overline{*}_{E^*} \circ L_{E^*} \overline{*}_E}_{-\Lambda} (\underbrace{\tilde{s}_z\, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}_{D'\tilde{s}} \\
&= -i\Lambda D' \tilde{s}.
\end{aligned}$$

□

Remark A.1.38. The general case of the Kähler identities can be found in [Huy05], p. 120, 3.1.2 or [GH78], p. 80ff.

A.2. Metric Bundles and Connections

This section is divided into two parts. The first one is still a general discussion of metric bundles and metric connections. It also includes an extension of a metric to differential forms. The second part will treat a hermitian inner product as a map into the space of positive-definite matrices $\mathbb{P}_n$. We will endow it with a real manifold structure using the exponential map and the correspondence between hermitian matrices and the euclidean space $\mathbb{R}^n$. In particular $\mathbb{H}_n$ may be equipped with an inner product. Furthermore we will show that $\mathbb{P}_n$ is a complete, non-negatively curved Riemannian manifold. In order to do so we will use the continuous functional calculus and "divided sums" to find a formula for the differential of a matrix. The "divided sum" will be used later when constructing a harmonic metric, the completeness and non-negative curvature will be used to work with geodesics.

A.2.1. Fiber-wise metrics and Connections

Definition A.2.1. A holomorphic bundle is called a metric bundle if for each $x \in X$ there is a hermitian inner product h_x on $E_x = \pi^{-1}(\{x\})$ such that for $s, \tilde{s} : U \to E$ smooth sections, U open neighbourhood of x: $h_x(s(x), \tilde{s}(x))$ is smooth.

Remark A.2.2. Each smooth vector bundle can be equipped with a hermitian metric.

Remark A.2.3. For an arbitrary function $f : X \to \mathbb{C}$ and smooth sections ξ and η: $h(f\xi, \eta) = fh(\xi, \eta) = h(\xi, \overline{f}\eta)$.

Definition A.2.4 (Connection form)**.** A connection form on a vector bundle $\pi : E \to M$ is a sheaf homomorphism of $\mathbb{C}-$vector spaces[27]

$$\begin{aligned} D_0 : \quad \Gamma(U, E) = \Omega^0_U(E) \quad & \to \Omega^1_U(E) = \Gamma(U, T^*U \otimes E) \\ s \quad & \mapsto D(s), \end{aligned}$$

which obeys

(I) $D_0(s \otimes \xi) = D_0(s) \wedge \xi + s \otimes d\xi \ \forall \xi \in \mathcal{E}(U),\ s \in \Gamma(U, E)$ (Leibniz Rule),

where d denotes the exterior derivative.
We can extend D_0 to

$$\begin{aligned} D_p : \quad \Omega^p_U(E) \quad & \to \Omega^{p+1}_U(E) \\ s \otimes \omega \quad & \mapsto D_p(s \otimes \omega) = D_0(s) \wedge \omega + s \otimes d\omega, \quad s \in \Omega^0_U(E), \omega \in \Omega^p_U. \end{aligned}$$

The curvature R is then

$$R = D_1 \circ D_0 : \Omega^0_U(E) \to \Omega^2_U(E).$$

A connection is called flat if $R = 0$.

Definition A.2.5 (Connection (local))**.** Let $s = (s_i)_{1 \le i \le r}, s_i \in \Omega^0_U(E)$ a basis of E_x for each $x \in U$, i.e. s a frame field. Then we can write $Ds_i = \sum_{j=1}^r s_j \otimes \omega_{ji} = s\omega$, where $\omega_{ji} \in \Omega^1_U$. The matrix $\omega := (\omega_{ji})_{1 \le i,j \le r}$ is called connection form of D. Further for a section $\xi = \sum_i^r s_i \otimes \xi_i, \xi_i \in \Omega^0_U$

$$D\xi \quad = \quad D\left(\sum_{i=1}^r s_i \otimes \xi_i\right) = \sum_{i=1}^r D(s_i \otimes \xi_i) = \sum_{i=1}^r (D(s_i) \wedge \xi_i + s_i \otimes d\xi_i)$$

[27] $\mathbb{C}$ acts on $\Omega^p_U(E)$ by multiplication with the value. Note that this is weaker than the $\mathcal{E}_X-$linearity; more precisely: it corresponds to the linearity w.r.t. constant $\mathbb{C}-$valued functions.

$$\begin{aligned}
&= \sum_{i=1}^{r}\left(\sum_{j=1}^{r}(s_j \otimes \omega_{ji}) \wedge \xi_i + s_i \otimes d\xi_i\right)\\
&= \sum_{i=1}^{r}\left(\sum_{j=1}^{r} s_j \otimes (\omega_{ji} \wedge \xi_i) + s_i \otimes d\xi_i\right)\\
&= \sum_{i=1}^{r}\left(\sum_{j=1}^{r} s_i \otimes (\omega_{ij} \wedge \xi_j) + s_i \otimes d\xi_i\right)\\
&= \sum_{i=1}^{r} s_i \otimes \left(d\xi_i + \sum_{j=1}^{r} \omega_{ij} \wedge \xi_j\right).
\end{aligned}$$

Writing $\xi = ((\xi_i)_{1\le i\le r})^t$ as a column vector this becomes

$$D\xi = d\xi + \omega\xi.$$

The curvature form Ω is

$$\Omega = d\omega + \omega \wedge \omega.$$

Remark A.2.6. Since $d : \Omega_X^p \to \Omega_X^{p+1}$ and $\Omega_X^k = \bigoplus_{p+q=k} \Omega_X^{p,q}$ (resp. $\Omega_X^k(E) = \bigoplus_{p+q=k} \Omega_X^{p,q}(E)$) we write $\partial : \bigoplus_{p+q=k} \Omega_X^{p,q} \to \bigoplus_{p+q=k} \Omega_X^{p+1,q}$ and $\overline{\partial} : \bigoplus_{p+q=k} \Omega_X^{p,q} \to \bigoplus_{p+q=k} \Omega_X^{p,q+1}$ such that $d = \partial + \overline{\partial}$. The same can be done for a connection D, i.e. $D_k = D_k' + D_k''$, $k = p+q$ with

$$D_k' : \bigoplus_{p+q=k} \Omega_X^{p,q}(E) \to \bigoplus_{p+q=k} \Omega_X^{p+1,q}(E), \quad D_k'' : \bigoplus_{p+q=k} \Omega_X^{p,q}(E) \to \bigoplus_{p+q=k} \Omega_X^{p,q+1}(E).$$

Further

$$D_k'(s \otimes \omega) = D_k'(s) \wedge \omega + s \otimes \partial\omega, \quad D_k''(s \otimes \omega) = D_k''(s) \wedge \omega + s \otimes \overline{\partial}\omega,$$

for $s \in \Omega_U^0(E), \omega \in \Omega_U^{p,q}$.[28]

Definition A.2.7. Let $s_U = (s_i)_{1\le i\le r}$ be a frame field over U and h a metric on our vector bundle E then define $h_{ij} := h(s_i, s_j)$ and $H_U = (h_{ij})_{1\le i,j\le r}$.

(i) s_U is called unitary at a point x_0 if $H_U = E$ is the identity matrix at x_0.

(ii) s_U is called normal at x_0 if $H_U = E$ and $w_{ij} = \sum_{k=1}^{r} h_{ik} \wedge \partial h_{jk} = 0$ at x_0.

[28] With the natural inclusion into $\bigoplus_{p+q=k} \Omega_U^{p,q}$.

(iii) A connection D on (E, h) is called metric connection if

$$d(h(\xi,\eta)) = h(D\xi,\eta) + (-1)^{\deg(\xi)}h(\xi, D\eta) \quad \xi,\eta \in \Omega^0_U(E).^{29} \quad \text{(A.2.7.1)}$$

This means that for each $Y \in \mathfrak{W}(X) = \Gamma(X, TX)$

$$Yh(\xi,\eta) = h(D_Y\xi,\eta) + h(\xi, D_Y\eta).^{30}$$

$Xh = L_X h = dh(X)$ is the Lie derivative: Let $\gamma(t) : I =]-\varepsilon,\varepsilon[\to X$ be any smooth path in X, then $h(\xi(\gamma(t)),\eta(\gamma(t)))$ for sections $\xi,\eta \in \Gamma(X,E)$ is a path in $\mathbb{C}$, i.e. $L_Y h(\xi(a),\eta(a)) = \frac{\mathrm{d}}{\mathrm{d}\,t}h(\xi(\gamma(t)),\eta(\gamma(t)))\big|_{t=0}$, where $a = \gamma(0)$ and $Y(a) = [\gamma]_a \in T_aX$ ($Y(a)$ tangent vector to the curve γ).
Another way to read A.2.7.1 is: $h \circ (\xi,\eta) : X \to \mathbb{C}$ and thus $h \circ (\xi,\eta) \in \Omega^0_X$, $D\xi(x) \in E_x \otimes T^*_x X$

$$\begin{aligned} h(D\xi,\eta) &= h(s_\xi \otimes \omega_\xi,\eta) := h(s_\xi,\eta)\omega_\xi \\ h(\xi, D\eta) &= h(\xi, s_\eta \otimes \omega_\eta) := h(\xi, s_\eta)\overline{\omega_\eta}^{31} \\ \Rightarrow d(h(\xi,\eta)) &= h(s_\xi,\eta)\omega_\xi + h(\xi, s_\eta)\overline{\omega_\eta}. \end{aligned}$$

Remark A.2.8. The last reformulation of h on higher forms corresponds to the choice of the euclidean metric on X. Usually the induced metric on a tensor product $E \otimes F$, where E is equipped with h_E, F with h_F, is defined as $h_{E\otimes F}(e_1 \otimes f_1, e_2 \otimes f_2) := h_E(e_1,e_2)h_F(f_1,f_2)$. Since we are on a one-dimensional complex manifold we only need the hermitian extension of $h_{\Lambda^1_X}(\mathrm{d}\,z,\mathrm{d}\,z) = h_{\Lambda^1_X}(\mathrm{d}\,\overline{z},\mathrm{d}\,\overline{z}) = 2, h_{\Lambda^2_X}(\mathrm{d}\,z \wedge \overline{z}, \mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}) = 4$ for the hermitian metric. Applying these to A.2.7.1 resp. decomposing the right hand side $h_E(\cdot,\cdot)h_{\Lambda^{1 \text{ or } 2}_X}(\cdot,\cdot)$ preserves the equality.

Remark A.2.9. (i) For a holomorphic hermitian vector bundle over X and $x_0 \in X$ there is a local normal frame field s_U at x_0.[32]

(ii) For a metric connection ω and Ω are skew-hermitian, i.e. $\omega, \Omega \in \mathfrak{o}(r)$ the Lie algebra to the Lie group of orthogonal $r \times r$ matrices.

[29] Although we consider for now only degree 0 forms, i.e. $(-1)^{\deg(\xi)} = 1$, $(-1)^{\deg(\xi)}$ indicates how the condition can be generalized. This will be important when we consider Higgs fields.

[30] $D\xi = s \otimes \omega, \omega(x) : T^*X \to \mathbb{C}$. Define $\omega_Y : x \mapsto (\omega(x))(Y(x))$, then $\tilde{D}_Y\xi := s \otimes \omega_Y = \omega_Y \cdot s \otimes 1, D_Y\xi := \omega_Y \cdot s$.

[31] Well-defined, i.e. compatible with the equivalence relations defining the tensor product, since the hermitian metric is $\mathbb{C}-$linear in the first component. Same holds with conjugated $\mathbb{C}-$linearity and the tensor product in the second component. Further note that $\overline{f\,\mathrm{d}\,z + g\,\mathrm{d}\,\overline{z}} := \overline{f}\,\mathrm{d}\,\overline{z} + \overline{g}\,\mathrm{d}\,z$ for a $1-$form.

[32] Kobayashi [Kob87], p. 13.

Theorem A.2.10. Given a hermitian structure h on a holomorphic vector bundle $(E, \overline{\partial}_E)$, there is a unique metric connection D such that $D'' = \overline{\partial}_E$.

Proof. A proof can be found in [Kob87], p. 11. or [Huy05], p. 177, 4.2.14. □

Remark A.2.11. (i) The curvature of a hermitian connection is of degree $(1,1)$ and ω of type $(1,0)$.

(ii) For a metric connection parallel transport is an isometry.

(iii) There is a (slightly more) general version of A.2.10 which states that for any given $(1,0)$ resp. $(0,1)$ part $\tilde{d}$ we find a unique $(0,1)$ resp. $(1,0)$ operator $\hat{d}$ such that $\tilde{d} + \hat{d}$ is a hermitian connection.[33]

Definition A.2.12 (Holomorphic Connection)**.** A holomorphic connection is a map

$$\mathscr{D}_0 : \Gamma_{hol}(X, E) \to \Gamma_{hol}(X, E) \otimes \Omega_X^{1,0}.$$

such that

(I) $\mathscr{D}(s \otimes \xi) = \mathscr{D}_0(s) \wedge \xi + s \otimes \partial\xi \ \forall \xi \in \Gamma(U, \mathbb{C})$, $s \in \Gamma(U, E)$ (Leibniz Rule),

In abuse of notation this becomes

$$E \to E \otimes \Omega_X^{1,0},$$

where E is now the associated locally free $\mathcal{O}^{an}-$module (i.e. $\Gamma_{hol}(X, E)$).
An extension to higher forms is done as in the case of an ordinary connection.

Remark A.2.13. (i) There is a decomposition of $\mathscr{D} = \partial + \omega$ as for general connections.

(ii) $D = \mathscr{D} + \overline{\partial}$ is an ordinary connection on E.

(iii) Not every $(1,0)-$part of an arbitrary connection is holomorphic, since D could send holomorphic sections to non-holomorphic elements of $\Omega_X^{1,0}(E)$, i.e. $f \,\mathrm{d}\, z, f$ not holomorphic.

Remark A.2.14. On a flat holomorphic vector bundle exists a holomorphic connection.

[33] See e.g. Bedford [Bed91] p. 96. The proof is essentially the same as the one of the theorem.

Consider the unique metric connection $D = \partial_E + \overline{\partial}_E$ given by theorem A.2.10. Then

$$\begin{aligned} d(h(\xi,\eta)) &= \underbrace{\partial h(\xi,\eta)}_{\in\Omega^{1,0}_X} + \underbrace{\overline{\partial} h(\xi,\eta)}_{\in\Omega^{0,1}_X} \\ &= h((\partial_E + \overline{\partial}_E)\xi,\eta) + h(\xi,(\partial_E + \overline{\partial}_E)\eta) \\ &= h(\partial_E\xi,\eta) + h(\overline{\partial}_E\xi,\eta) + h(\xi,\partial_E\eta) + h(\xi,\overline{\partial}_E\eta) \\ &= h(s^{1,0}_\xi \otimes \omega^{1,0}_\xi,\eta) + h(s^{0,1}_\xi \otimes \omega^{0,1}_\xi,\eta) \\ &\quad + h(\xi, s^{1,0}_\eta \otimes \omega^{1,0}_\eta) + h(\xi, s^{0,1}_\eta \otimes \omega^{0,1}_\eta) \\ &= \underbrace{h(s^{1,0}_\xi,\eta)\omega^{1,0}_\xi}_{\in\Omega^{1,0}_X} + \underbrace{h(s^{0,1}_\xi,\eta)\omega^{0,1}_\xi}_{\in\Omega^{0,1}_X} + \underbrace{h(\xi,s^{1,0}_\eta)\overline{\omega^{1,0}_\eta}}_{\in\Omega^{0,1}_X} + \underbrace{h(\xi,s^{0,1}_\eta)\overline{\omega^{0,1}_\eta}}_{\in\Omega^{1,0}_X}. \end{aligned}$$

and therefore

$$\begin{aligned} \partial(h(\xi,\eta)) &= h(\partial_E\xi,\eta) + h(\xi,\overline{\partial}_E\eta) \\ \overline{\partial}(h(\xi,\eta)) &= h(\overline{\partial}_E\xi,\eta) + h(\xi,\partial_E\eta). \end{aligned}$$

Of course $\partial_E^2 = 0$ for a Riemann surface since there are no non-trivial $(p,q)-$forms for $p > 1$ or $q > 1$ ($\mathrm{d}\, z \wedge \mathrm{d}\, z = \mathrm{d}\, \overline{z} \wedge \mathrm{d}\, \overline{z} = 0$).

A.2.2. Metric as a map

There is an equivalence between $\mathrm{Gl}_n(\mathbb{C})-$representation of the fundamental group (up to $\mathrm{Gl}_n(\mathbb{C})-$conjugation) and flat smooth vector bundles. In fact every flat regular smooth vector bundle is equivalent to $\tilde{X} \times \mathbb{C}^n/\sim$ with $(p,g) \sim (p\delta, f(\delta)g)$ for $\delta \in \pi_1(X)$. Here $\tilde{X}$ can be understood as a principal $\pi_1(X)-$bundle. f is the corresponding $\mathrm{Gl}_n(\mathbb{C})-$representation of $\pi_1(X)$. For more details see Reisert [Rei10].
So if we consider $E = \tilde{X} \times \mathbb{C}^n/\sim$ a metric is a map

$$h : (\tilde{X} \times \mathbb{C}^n/\sim) \times (\tilde{X} \times \mathbb{C}^n/\sim) \to \mathbb{C},$$

which rises to a map $\tilde{h} = h\mathrm{pr}$

$$\tilde{h} : (\tilde{X} \times \mathbb{C}^n) \times (\tilde{X} \times \mathbb{C}^n) \to \mathbb{C}.$$

By the equivalence relation a frame (s_i) can be written $s_i(x) = \tilde{x} \times s_i^g$ with $\tilde{x}$ in a fixed (basis) leaf of $\tilde{X}$, i.e. as a map of sets $X \to X \times \mathbb{C}^n$. Then a metric (which is always fiber-wise) acts only on the vector space part $\mathbb{C}^n$. Further a matrix is uniquely defined on the basis, i.e. by $h(s_i, s_j)$. As before write $H =$

$(h_{ij})_{1\le i,j\le n} = h(s_j, s_i))_{1\le i,j\le n}$ as a matrix and thus $h(v,w) = w^* H v$.[34] H has to be positive-definite and hermitian as representation of an inner product.

Remark A.2.15. Note that the unique hermitian connection compatible with the holomorphic structure $\overline{\partial}$ is

$$D_{un} := d + \overline{H}^{-1}\partial\overline{H},$$

where $d, \overline{\partial}$ act as the usual exterior derivative on each component. Hence the curvature is $D_{un}^2 = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$. In the line bundle case the curvature becomes $\overline{\partial}\partial\log(H)$.[35]

Let $\mathbb{P}_n$ denote the set of all positive-definite hermitian matrices.[36] Then $\mathrm{Gl}_n(\mathbb{C})$ acts on $\mathbb{P}_n$ by

$$\begin{aligned} \rho_{\mathrm{Gl},\mathbb{P}_n} : \quad \mathrm{Gl}_n(\mathbb{C}) \times \mathbb{P}_n &\to \mathbb{P}_n \\ (G, H) &\mapsto GHG^*.^{37} \end{aligned}$$

Note that the isotropy group of this action at E is just $\mathrm{U}(n)$.
This action is transitive since by Cholesky decomposition $H = L_H L_H^*, H' = L_{H'} L_{H'}^*$ for unique invertible lower triangular matrices L_H and $L_{H'}$. Hence

$$H' = L_{H'} L_{H'}^* = \underbrace{L_{H'} L_H^{-1}}_{=G_{H,H'}} H (L_H^*)^{-1} L_{H'}^* = G_{H,H'} H G_{H,H'}^*, \quad G_{H,H'} \in \mathrm{Gl}_n(\mathbb{C}).$$

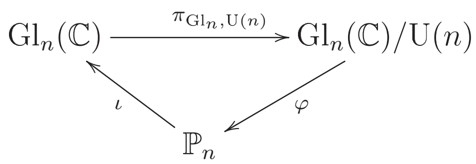
Remark A.2.16. Since $\mathrm{U}(n)$ is a closed subgroup of $\mathrm{Gl}_n(\mathbb{C})$[38] the action (right-translation) of $\mathrm{U}(n)$ on $\mathrm{Gl}_n(\mathbb{C})$ is smooth, proper and free and $\mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$ is a smooth manifold with smooth submersion $\pi_{\mathrm{Gl}_n,\mathrm{U}(n)} : \mathrm{Gl}_n(\mathbb{C}) \to \mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$.[39] Thus we may identify $T_{\pi_{\mathrm{Gl}_n,\mathrm{U}(n)}(G)}(\mathrm{Gl}_n(\mathbb{C})) \simeq T_G(\mathrm{Gl}_n(\mathbb{C}))/T_G(G\mathrm{U}(n))$ by the fundamental theorem on homomorphisms and $\ker(d\pi_{\mathrm{Gl}_n,\mathrm{U}(n)})_G = T_G(G\mathrm{U}(n))$.

Another Proposition of Lee [Lee00] p. 175, 7.21, tells us that for a set on which a Lie Group G acts such that the isotropy group at one point is a (closed) Lie subgroup of G, there is a unique manifold topology and a unique smooth structure such that the given $G-$action is smooth.

[34] v, w written in the basis s_i.
[35] A proof can be found p. 185f [Huy05].
[36] This is not a subgroup of $\mathrm{Gl}_n(\mathbb{C})$, since for hermitian matrices $H, H', (HH')^* = (H')^* H^* = H'H \neq HH'$ in general.
[37] $\rho_{\mathrm{Gl},\mathbb{P}_n}(E,H) = H, \rho_{\mathrm{Gl},\mathbb{P}_n}(F, \rho_{\mathrm{Gl},\mathbb{P}_n}(G,H)) = \rho_{\mathrm{Gl},\mathbb{P}_n}(F, GHG^*) = FGHG^*F* = (FG)H(FG)^* = \rho_{\mathrm{Gl},\mathbb{P}_n}(FG,H)$.
[38] Lee [Lee00] p. 174, proof of theorem 7.19. Even compact by Heine-Borel.
[39] Again [Lee00] p. 172, proof of theorem 7.15.

Then our action is smooth and therefore we find an $\mathrm{Gl}_n(\mathbb{C})-$equivariant diffeomorphism $\varphi(G\mathrm{U}(n)) = \rho_{\mathrm{Gl},\mathbb{P}_n}(G, E)$[40] from $\mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n) \to \mathbb{P}_n$.[41] In particular the smooth structure on $\mathbb{P}_n$ is just the unique smooth structure such that φ is a diffeomorphism. Moreover note that $\mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$ is not a Lie Group, as $\mathrm{U}(n) \subset \mathrm{Gl}_n(\mathbb{C})$ not normal.
As a result h can be written as a map $X \to \mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$.

Remark A.2.17. The singular value decomposition leads to the similar result $\mathrm{Gl}_n(\mathbb{C}) = \mathbb{P}_n\mathrm{U}(n)$, i.e. every invertible matrix can be uniquely written as the product of a positive-definite hermitian and an unitary matrix.[42] But then



with

$$\varphi \circ \pi_{\mathrm{Gl}_n,\mathrm{U}(n)} \circ \iota(H) = \varphi \circ \pi_{\mathrm{Gl}_n,\mathrm{U}(n)}(HE) = \varphi(H\mathrm{U}(n)) = H.$$

In particular we have the induced $\mathbb{C}^{n\times n}-$standard topology on $\mathbb{P}_n$.

Now lifting h to $\tilde{h}$ results in a map $\tilde{h} : \tilde{X} \overset{\mathrm{pr}}{\to} X \overset{h}{\to} \mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$ with

$$\tilde{h}(\tilde{x}\delta) = f^*(\delta)\tilde{h}(\tilde{x})f(\delta)$$

since for $s_i^\delta(x) = (\tilde{x}\delta, \mathrm{pr}_2 s_i(x)) \Rightarrow \mathrm{pr} s_i^\delta = [\tilde{x}\delta, \mathrm{pr}_2 s_i(x)] = [\tilde{x}, f(\delta)\mathrm{pr}_2 s_i(x)]$,

$$\begin{aligned} \tilde{h}_{ji} &= \tilde{h}(s_i^\delta(x), s_j^{\delta'}(x)) \\ &= h_x(f(\delta)s_i, f(\delta')s_j) = h_x\left(\sum_{k=1}^n f(\delta)_{ki}s_k, \sum_{l=1}^n f(\delta')_{lj}s_l\right) \\ &= \sum_{k,l=1}^n f(\delta)_{ki}\overline{f(\delta')_{lj}}h_{lk}|_x = \sum_{k=1}^n \overline{f(\delta')_{lj}}(H(f(\delta)))_{li}|_x \\ &= (f(\delta')^* H(f(\delta)))_{ji}|_x. \end{aligned}$$

This is just $f-$equivariance of h.

[40] $\mathrm{U}(n)$ acts by multiplication from the right.
[41] For a proof see [Lee00] p. 174, theorem 7.19. Choose the isotropy group at $p = E$. In fact we only map every H on its Cholesky lower triangular matrix.
[42] For example in Chevalley [Che46] p. 14, §5 Prop. 1.

Remark A.2.18. If $f : \pi_1(X) \to \mathrm{U}(n)$ than h is invariant under the action of f (by equivariance of φ). This is just Kobayashi's [Kob87], Proposition (4.21) on p. 14:
The existence of a flat metric connection E is equivalent to E defined by a homomorphism $f : \pi_1(X) \to \mathrm{U}(n)$.

Further note that for matrix groups G: $\exp(\mathfrak{g}) = G$. Under exp hermitian matrices are mapped injectively into the set of positive-definite hermitian matrices ($x \neq 0$), $\forall t \in \mathbb{R}$:

$$\begin{aligned}
\exp(tH)^* &= \exp(tH^*) = \exp(tH) \Leftrightarrow H = H^* \\
x^* \exp(H)x &= x^* \sum_{k=0}^{\infty} \frac{(tH)^k}{k!} x = \sum_{k=0}^{\infty} \frac{(tx^*Hx)^k}{k!\|x\|^{2k}} \|x\|^2 \\
&= \|x\|^2 \exp\left(\frac{tx^*Hx}{\|x\|^2}\right) \\
\Rightarrow x^* \exp(tH)x &> 0 \Leftrightarrow x^*Hx \in \mathbb{R},
\end{aligned}$$

and $x^*Hx \in \mathbb{R}$ already fulfilled for H hermitian. Moreover this is a homeomorphism w.r.t. the by $\mathrm{Gl}_n(\mathbb{C})$ induced topology.[43] The tangent space to $\mathbb{P}_n$ at the identity E is the space of hermitian matrices $\mathbb{H}_n$. This can be seen either by using the restriction of the exponential map on $\mathfrak{gl}_n$ and the previous calculation or by the decomposition

$$T_E\mathbb{P}_n \simeq T_E(\mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)) \simeq T_E(\mathrm{Gl}_n(\mathbb{C}))/T_E(\mathrm{U}(n)) = \mathfrak{gl}_n/\mathfrak{u}_n,$$

where the unitary Lie algebra is just the set of skew-hermitian matrices $\mathbb{SH}_n$: $\forall t \in \mathbb{R} : \exp(tS)^* \exp(tS) = \exp(t(S^* + S)) = E \Leftrightarrow S^* + S = 0$. But any $n \times n-$matrix can be decomposed

$$M_n(\mathbb{C}) = \mathbb{H}_n \otimes \mathbb{SH}_n.$$

For any $A \in M_n(\mathbb{C})$ set $A_{\mathbb{H}_n} = \frac{1}{2}(A + A^*) \in \mathbb{H}_n, A_{\mathbb{SH}_n} = \frac{1}{2}(A - A^*) \in \mathbb{SH}_n$. Uniqueness by $A \in \mathbb{H}_n \cap \mathbb{SH}_n \Rightarrow A = A^* = -A \Rightarrow A = 0$. Hence

$$T_E\mathbb{P}_n \simeq \mathfrak{gl}_n/\mathfrak{u}_n = M_n(\mathbb{C})/\mathbb{SH}_n \simeq \mathbb{H}_n.$$

Remark A.2.19. $\mathbb{P}_n$ is homeomorphic to $\mathbb{H}_n$ under exp. Further $\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n \times n}$:[44] Obviously all entries under the diagonal are uniquely determined by those over the diagonal. The diagonal itself is real, i.e. $\mathbb{H}_n \simeq \mathbb{R}^n \times \mathbb{C}^{n^2/2-n/2} \simeq$

[43] Chevalley [Che46] page 14, §4 Prop. 6.
[44] [Che46] , p. 14, §4 Prop. 6.

$\mathbb{R}^{n-2n^2/2-2n/2} = \mathbb{R}^{n\times n}$. Explicitly we have the splitting $\mathbb{H}_n = \mathbb{S}_n \otimes i\mathbb{S}_n^-$, $\mathbb{S}_n$ real symmetric $n-$dimensional matrices, $\mathbb{S}_n^-$ real skew-symmetric $n-$dimensional matrices and the $\mathbb{R}-$linear homeomorphism

$$\begin{array}{rrcll}
\varphi_{H_n,\mathbb{R}}: & \mathbb{S}_n \otimes i\mathbb{S}_n & \to & \mathbb{R}^{n^2} & \\
 & (S,T) & \mapsto & R_k & = 2\sum_{i,j=1}^n \delta_{k\le n(n-1)/2}\delta_{k,in+j-n}S_{ij} \\
 & & & & + \sum_i^n \delta_{n(n-1)/2\ge k>n(n-1)/2}\delta_{k,in+i-n}S_{ii} \\
 & & & & +2\sum_{i,j=1}^n \delta_{k>n(n-1)/2}\delta_{k,in+j-n}T_{ij} \\
\varphi_{H_n,\mathbb{R}}^{-1}: & \mathbb{R}^{n^2} & \to & \mathbb{H}_n & \\
 & R & \mapsto & 2H_{lm} = & \sum_{k=1}^{n^2} \delta_{k\le n(n+1)/2}\delta_{ln+m-n,k}\delta_{l\le m}R_k \\
 & & & & + \sum_{k=1}^{n^2} \delta_{k\le n(n+1)/2}\delta_{mn+l-n,k}\delta_{l\ge m}R_k \\
 & & & & +i\sum_{k=1}^{n^2} \delta_{k>n(n+1)/2}\delta_{ln+m+\frac{n^2-n}{2},k}\delta_{l>m}R_k \\
 & & & & +i\sum_{k=1}^{n^2} \delta_{k>n(n+1)/2}\delta_{mn+l+\frac{n^2-n}{2},k}\delta_{l<m}R_k{}^{45}
\end{array}$$

Now we find ourselves in the position to define a Riemannian metric, namely the standard inner product on $\mathbb{R}^{n^2}$ and pull it back. Thought the calculation might seem tiring, Q^TP for $Q,P \in \mathbb{R}^{n^2}, Q = \varphi_{H_n,\mathbb{R}}({}^QS + i{}^QT), P = \varphi_{H_n,\mathbb{R}}({}^PS + i{}^PT)$ becomes

$$\begin{aligned}
Q^TP &= \sum_{k=1}^{n^2} Q_kP_k \\
&= 2\sum_{i,jj=1}^{n} \delta_{i<j}{}^QS_{ij}{}^PS_{ij} + {}^QS_{ii}{}^PS_{ii} + 2\delta_{i>j}{}^QT_{ij}{}^PT_{ij} \\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{i<j}{}^QS_{ji}{}^PS_{ji} + {}^QS_{ii}{}^PS_{ii} + 2\sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{i>j}{}^QT_{ji}{}^PT_{ji} \\
&=^{46} \sum_{i=1}^{n}\sum_{j=1}^{n} {}^QS_{ji}{}^PS_{ji} + \sum_{i=1}^{n}\sum_{j=1}^{n} {}^QT_{ji}{}^PT_{ji} \\
&= \mathrm{tr}({}^QS^T \cdot {}^PS) + \mathrm{tr}({}^QT^T, {}^PT) \\
&= \mathrm{tr}({}^QS^T \cdot {}^PS) + \mathrm{tr}((-i)\cdot {}^QT^T \cdot i \cdot {}^PT) \\
&= \mathrm{tr}({}^QS^T \cdot {}^PS) + i\cdot \mathrm{tr}({}^QS^T \cdot {}^PT) \\
&\quad -i\cdot \mathrm{tr}({}^QT^T \cdot {}^PS) + \mathrm{tr}(i\cdot {}^QT^T \cdot i \cdot {}^PT) \\
&= \mathrm{tr}(({}^QS - i{}^QT)^T({}^PS + i{}^PT)) \\
&= \mathrm{tr}(({}^QS + i{}^QT)^*({}^PS + i{}^PT))
\end{aligned}$$

[45]Note as well that all component functions are in fact smooth. However, since $\mathbb{H}_n$ is already a tangent space, continuous will be enough for us.

where we used first the symmetry resp. skew-symmetry of ${}^QS, {}^PS$ resp. ${}^QT, {}^PT$ and further that the trace of the product of a symmetric and a skew-symmetric matrix vanishes.[47] But the last term is just the Hilbert-Schmidt norm $\langle A, B\rangle_{HS} = \operatorname{tr}(B^* A)$.

Remark A.2.20. (i) This is a symmetric inner product - not hermitian - and as such it only works on hermitian matrices. Write $\langle A, B\rangle_{HS} = \operatorname{tr}(BA) = \operatorname{tr}(AB), A, B \in \mathbb{H}_n$ to point out the symmetric character; $\langle A, B\rangle_{HS,\mathbb{C}} = \operatorname{tr}(B^* A)$ is more familiar and is directly recognizable as a positive-definite inner product. In fact:

(ii) We are used to considering the complexified tangent spaces $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ of a manifold M. Therefore $T_{\mathbb{C}}\mathbb{P}_n = \mathbb{H}_n \oplus i\mathbb{H}_n =: \mathbb{H}_n^{\mathbb{C}}$. The inner products extends trivially via $\langle A+iB, C+iD\rangle = \langle A, C\rangle + \langle B, D\rangle + i(\langle B, C\rangle - \langle A, D\rangle)$ to a hermitian inner product w.r.t. the usual scalar multiplication $(a+ib)(A+iB) = aA - bB + i(bA + aB)$, $a, b \in \mathbb{R}, A, B, C, D \in \mathbb{H}_n$. We will henceforth use this complexified inner product when working with complex differential forms:

(iii) When extending the metric to differential forms, for example to calculate the derivative of our metric map $h : \tilde{X} \to \mathbb{P}_n \simeq \mathrm{Gl}_n(\mathbb{C})/\mathrm{U}(n)$ we extend the metric on maps $TX \to \mathbb{H}_n^{\mathbb{C}}$. In the previous subsection A.2.8 we mentioned how the metric splits up under the tensor product. There we saw that for $\alpha(z)\,\mathrm{d}\,z = \alpha(z) \otimes (1\,\mathrm{d}\,z), \beta(z)\,\mathrm{d}\,z = \alpha(z) \otimes (1\,\mathrm{d}\,z)$ we have $h_{F \otimes \Lambda_X^{1,0} \otimes \Lambda_X^{0,1}}(\alpha(z) \otimes (1\,\mathrm{d}\,z), \beta(z) \otimes (1\,\mathrm{d}\,z)) = h_F(\alpha(z), \beta(z)) h_{\Lambda_X^{1,0} \otimes \Lambda_X^{0,1}}(\mathrm{d}\,z, \mathrm{d}\,z)$ for any space F where α, β take values in. In our case $F = \mathbb{H}_n^{\mathbb{C}}$.[48].
Usually we will work on an open disc with the euclidean metric. For the euclidean inner product we get $h_{\Lambda_X^{1,0} \otimes \Lambda_X^{0,1}}(\omega, \eta) := \omega \wedge *\overline{\eta}$.[49] This is a positive differential form for $\omega = \eta$ since for $\omega = f\,\mathrm{d}\,z \Rightarrow \omega \wedge *\overline{\omega} = f\,\mathrm{d}\,z \wedge *\overline{f}\,\mathrm{d}\,\overline{z} = |f|^2\,\mathrm{d}\,z \wedge i\,\mathrm{d}\,\overline{z} = i|f|^2\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z} = 2|f|^2\,\mathrm{d}\,x \wedge \mathrm{d}\,y$. Further $\omega \wedge *\overline{\omega} = 0 \Leftrightarrow \omega = 0$ and linearity is obvious.

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[46] Rename $i \leftrightarrow j$ and interchange the sums to get $\sum_{i=1}^n \sum_{j=1}^n \delta_{i<j}\, {}^QS_{ji}\, {}^PS_{ji} = \sum_{i=1}^n \sum_{j=1}^n \delta_{i>j}\, {}^QS_{ij}\, {}^PS_{ij}$ and by symmetry ${}^QS_{ij}\, {}^PS_{ij} = {}^QS_{ji}\, {}^PS_{ji}$. Hence $2\sum_{i=1}^n \sum_{j=1}^n \delta_{i<j}\, {}^QS_{ji}\, {}^PS_{ji} = \sum_{i=1}^n \sum_{j=1}^n \delta_{i\neq j}\, {}^QS_{ji}\, {}^PS_{ji}$. Analogous for T.

[47] $\operatorname{tr}(H) = \operatorname{tr}(H^T) \Rightarrow \operatorname{tr}(ST) = \operatorname{tr}((ST)^T) = \operatorname{tr}(T^T S^T) = -\operatorname{tr}(TS) = -\operatorname{tr}(ST) \Rightarrow \operatorname{tr}(ST) = 0, \forall S \in \mathbb{S}_n, T \in \mathbb{S}_n^-$ by elementary properties of the trace.

[48] Maps $dH : TX \to T\mathbb{P}_n$ can be identified with an element of $(\Lambda_X^{1,0} \otimes \Lambda_X^{0,1}) \otimes \mathbb{H}_n$.

[49] cf. A.1.3.

Since our $\mathrm{Gl}_n(\mathbb{C})-$action is smooth and invertible

$$\rho^G_{\mathrm{Gl},\mathbb{P}_n} : \mathbb{P}_n \to \mathbb{P}_n, \rho^G_{\mathrm{Gl},\mathbb{P}_n}(Q) := \rho^G_{\mathrm{Gl},\mathbb{P}_n}(G,Q)$$

is a diffeomorphism. In particular $(d\rho^G_{\mathrm{Gl},\mathbb{P}_n})_E$ is an isomorphism. Thus $T_E\mathbb{P}_n \simeq T_{GG^*}\mathbb{P}_n \simeq T_Q\mathbb{P}_n, \forall Q \in \mathbb{P}_n$ by transitivity of the action. Note that $(d\rho^G_{\mathrm{Gl},\mathbb{P}_n})_E(Q) = GQG^* \in T_{GG^*}(\mathbb{P}^n)$ by a basic calculation.[50]
Finally we get the metric (corresponding to our topology) by

$$\langle Q, P\rangle_{T_H\mathbb{P}_n} := \mathrm{tr}(H^{-1}Q^*(H)^{-1}H^{-1}P), \quad \forall Q, P \in T_H\mathbb{P}_n$$

since this inner product is invariant under our action: For $Q = GQ_0G^*, P = GP_0G^* \in T_{GG^*}\mathbb{P}_n, Q_0, P_0 \in T_E\mathbb{P}_n$ we get

$$\begin{aligned}
\langle Q, P\rangle_{T_{G^*G}\mathbb{P}_n} &= \mathrm{tr}((GG^*)^{-1}(GQ_0G^*)^*(GG^*)^{-1}(GP_0G^*)) \\
&= \mathrm{tr}((G^*)^{-1}G^{-1}GQ_0^*G^*(G^*)^{-1}G^{-1}GP_0G^*) \\
&= \mathrm{tr}((G^*)^{-1}Q_0^*P_0G^*) = \mathrm{tr}(Q_0^*P_0G^*(G^*)^{-1}) = \mathrm{tr}(Q_0^*P_0) \\
&= \langle Q_0, P_0\rangle_{HS}.
\end{aligned}$$

At the end of this section we want to describe the Levi-Civita Connection on $\mathbb{P}_n$.

Lemma A.2.21. The Levi-Civita Connection on $\mathbb{P}_n$ is given by

$$\nabla_X Y = XY - \frac{1}{2}(XH^{-1}Y + YH^{-1}X),$$

with $H \in \mathbb{P}_n$, $X, Y \in \Gamma(\mathbb{P}_n, T\mathbb{P}_n)$ vector fields, i.e. $X_H, Y_H \in T_H\mathbb{P}_n = \{H\} \times \mathbb{H}_n$. Here XY shall be understood as the "product" of vector bundles, i.e. the derivative of Y in $X-$direction.

Proof. First note that the connection is torsion free, since

$$\nabla_X Y - \nabla_Y X = XY - XY = [X, Y].$$

In order to show "metric" choose a path $H(t) : [0,1] \to \mathbb{P}_n$. By the chain rule we get $\frac{\mathrm{d}}{\mathrm{d}t}H^{-1}(t) = -H^{-1}(t)\dot{H}(t)H^{-1}(t)$. Then
$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(H^{-1}(t)X_{H(t)}H^{-1}(t)Y_{H(t)})$
$= \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}H^{-1}(t)X_{H(t)}H^{-1}(t)Y_{H(t)}\right)$
$= -\mathrm{tr}\left(H^{-1}(t)\dot{H}(t)H^{-1}(t)X_{H(t)}H^{-1}(t)Y_{H(t)}\right) + \mathrm{tr}\left(H^{-1}(t)\dot{X}_{H(t)}H^{-1}(t)Y_{H(t)}\right)$
$-\mathrm{tr}\left(H^{-1}(t)X_{H(t)}H^{-1}(t)\dot{H}(t)H^{-1}(t)Y_{H(t)}\right) + \mathrm{tr}\left(H^{-1}(t)X_{H(t)}H^{-1}(t)\dot{Y}_{H(t)}\right).$

[50] γ :] $-1,1[\to \mathbb{P}_n, \gamma(0) = E, X = \frac{d}{dt}\gamma(t)\big|_{t=0}, (d\rho^G_{\mathrm{Gl},\mathbb{P}_n})_E(X) = \frac{d}{dt}G\gamma(t)G^*\big|_{t=0} = G\,\frac{d}{dt}\gamma(t)\big|_{t=0}G^* = GXG^*$. In particular $T_H\mathbb{P} = \mathbb{H}_n$.

On the other hand

$$\operatorname{tr}\left(H^{-1}(t)\nabla_{\dot H(t)}X_{H(t)}H^{-1}(t)Y_{H(t)}\right)$$
$$= \operatorname{tr}\left(H^{-1}(t)\dot H(t)X_{H(t)}H^{-1}(t)Y_{H(t)}\right) - \frac{1}{2}\operatorname{tr}\left(H^{-1}(t)\dot H(t)H^{-1}(t)X_{H(t)}\cdot\right.$$
$$\left.\cdot H^{-1}(t)Y_{H(t)}\right) - \frac{1}{2}\operatorname{tr}\left(H^{-1}(t)X_{H(t)}H^{-1}(t)\dot H(t)H^{-1}(t)Y_{H(t)}\right)$$
$$\operatorname{tr}\left(H^{-1}(t)X_{H(t)}H^{-1}(t)\nabla_{\dot H(t)}Y_{H(t)}\right)$$
$$= \operatorname{tr}\left(H^{-1}(t)X_{H(t)}H^{-1}(t)\dot H(t)Y_{H(t)}\right) - \frac{1}{2}\operatorname{tr}\left(H^{-1}(t)X_{H(t)}H^{-1}(t)\dot H(t)\right.$$
$$\left.\cdot H^{-1}(t)Y_{H(t)}\right) - \frac{1}{2}\operatorname{tr}\left(H^{-1}(t)X_{H(t)}H^{-1}(t)Y_{H(t)}H^{-1}(t)\dot H(t)\right)$$

Now using $\dot X_{H(t)} = \dot H(t)\frac{\mathrm{d}}{\mathrm{d}\,H(t)}X_{H(t)}$ we get the claim. □

Lemma A.2.22. The space $\mathbb{P}_n$ is non-positively curved.

Proof. The curvature tensor of the Levi-Civita connection follows by an elementary calculation using torsion freeness (cf. [CPR93].)

$$\mathcal{R}_H(X,Y)Z \;=\; -\frac{1}{4}H[[H^{-1}X, H^{-1}Y], H^{-1}Z]$$

Next compute

$$\langle \mathcal{R}_H(X,Y)Y, X\rangle_{\mathbb{H}_n,H} = -\frac{1}{4}\langle [[H^{-1}X, H^{-1}Y], H^{-1}Y], H^{-1}X\rangle_{\mathbb{H}_n,E}$$
$$= -\frac{1}{4}\left(\langle H^{-1}X(H^{-1}Y)^2, H^{-1}X\rangle_{\mathbb{H}_n,E} - 2\langle H^{-1}YH^{-1}XH^{-1}Y, H^{-1}X\rangle_{\mathbb{H}_n,E}\right.$$
$$\left.+\langle (H^{-1}Y)^2H^{-1}X, H^{-1}X\rangle_{\mathbb{H}_n,E}\right)$$

Every positive hermitian matrix H has a square root, even more a matrix is positive-definite if and only if it has a square root.[51] Using the properties of the trace we get for $X_{-1/2} := H^{-1/2}XH^{-1/2}, Y_{-1/2} := H^{-1/2}YH^{-1/2}$

$$\langle \mathcal{R}_H(X,Y)Y, X\rangle_{\mathbb{H}_n,H}$$
$$= -\frac{1}{4}\left(\langle X_{-1/2}(Y_{-1/2})^2, X_{-1/2}\rangle_{\mathbb{H}_n,E} - 2\langle Y_{-1/2}X_{-1/2}Y_{-1/2}, X_{-1/2}\rangle_{\mathbb{H}_n,E}\right.$$
$$\left.+\langle (Y_{-1/2})^2X_{-1/2}, X_{-1/2}\rangle_{\mathbb{H}_n,E}\right)$$

[51] Any book on linear algebra or by $H = UDU^*$, $D = \operatorname{diag}(d_1,\dots,d_n)$ diagonal, U unitary, $D^{1/2} = \operatorname{diag}(d_1^{1/2},\dots,d_n^{1/2}) \Rightarrow H = UDU^* = U(D^{1/2})^2U^* = UD^{1/2}U^*UD^{1/2}U^* = (H_0)^2$, i.e. H_0 square root of H.

$$= -\frac{1}{4}\left(\mathrm{tr}((X_{-1/2})^2(Y_{-1/2})^2) - 2\mathrm{tr}(Y_{-1/2}X_{-1/2}Y_{-1/2}X_{-1/2}) \right.$$
$$\left. +\mathrm{tr}((Y_{-1/2})^2(X_{-1/2})^2)\right)$$

By Cauchy-Schwarz

$$\begin{aligned}
&\mathrm{tr}(Y_{-1/2}X_{-1/2}Y_{-1/2}X_{-1/2})\\
&= \mathrm{tr}((X_{-1/2}Y_{-1/2})^*Y_{-1/2}X_{-1/2})\\
&\leq \mathrm{tr}((X_{-1/2}Y_{-1/2})^*(X_{-1/2}Y_{-1/2}))^{1/2}\mathrm{tr}((Y_{-1/2}X_{-1/2})^*(Y_{-1/2}X_{-1/2}))^{1/2}\\
&= \mathrm{tr}(Y_{-1/2}X_{-1/2}X_{-1/2}Y_{-1/2})^{1/2}\mathrm{tr}(X_{-1/2}Y_{-1/2}Y_{-1/2}X_{-1/2})^{1/2}\\
&= \mathrm{tr}((X_{-1/2})^2(Y_{-1/2})^2)
\end{aligned}$$

Hence

$$\langle \mathcal{R}_H(X,Y)Y,X\rangle_{\mathbb{H}_n,H} \leq 0.$$

□

Definition A.2.23. Define a metric on $\mathbb{P}_n$ as $\delta_{\mathbb{P}_n}(A,B) = \inf\{L(\gamma)|\gamma : I \to \mathbb{P}_n$ path from A to $B.\}$ where $L(\gamma) = \int_0^1 \left\|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right\|_{\mathbb{H}_n,\gamma(t)} \mathrm{d}t$ is the length of γ.

Define $De^H(K) = \lim_{t\to 0}\frac{e^{H+tB}-e^H}{t} = \frac{\mathrm{d}}{\mathrm{d}t}e^{H+tB}\Big|_{t=0}$ the differential of exp at the point $H \in \mathbb{H}_n$ in $B-$direction, $B \in \mathbb{H}_n$.

Lemma A.2.24. $\|De^H(B)\|_{\mathbb{H}_n,e^{-H}} = \|e^{-H/2}De^H(B)e^{-H/2}\|_{\mathbb{H}_n,E} \geq \|B\|_{\mathbb{H}_n,E}$

Proof. In order to show this inequality and afterwards that $\mathbb{P}_n$ is complete, we follow the approach of Bhatia [Bha06], p. 203ff.
Remember first:

Theorem A.2.25 (Continuous functional calculus). $\forall A \in \mathbb{H}_n$ $\exists$ a unique $\Phi : C(\mathrm{spec}(A)) \to (BL)(\mathbb{C}^n)$ bounded linear operator on $\mathbb{C}^n$ with

(i) $\Phi(\mathrm{id}) = A$ and $\Phi(1) = \mathrm{id}_{\mathbb{C}^n}$ $(1 : \lambda \to 1)$,

(ii) $\forall \alpha, \beta \in \mathbb{C}, f, g \in C(\mathrm{spec}(A))$: $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g), \Phi(fg) = \Phi(f)\Phi(g), \Phi(\overline{f}) = \Phi(f)^*$,

(iii) Φ continuous.

Proof. Müller Functional Analysis Course II, [FA09], Chapter 1.2, page 5. □

As usual write $\Phi(f) = f(H)$. Restrict to $f \in C^1(\operatorname{spec}(A))$. Further denote f^Δ the first divided sum of f defined on $\operatorname{spec}(A) \times \operatorname{spec}(A)$:

$$\begin{aligned} f^\Delta(\lambda,\mu) &= \frac{f(\lambda)-f(\mu)}{\lambda-\mu}, \qquad \text{if } \lambda \neq \mu, \\ f^\Delta(\lambda,\lambda) &= f'(\lambda). \end{aligned}$$

Back to $f = \exp$. Choose an orthonormal frame such that $H = \operatorname{diag}(h_1, \dots, h_n)$. Then $e^H = \operatorname{diag}(e^{h_1}, \dots, e^{h_n})$. We have already seen, that exp induces $\exp : \mathbb{H}_n \to \mathbb{P}_n \subset \mathbb{H}_n$. We want to show that $D\exp(H)(B) = \exp^\Delta(H) \bullet B$[52], where $\exp^\Delta(H)$ is the matrix with entries $\exp^\Delta(\lambda_i, \lambda_j)$ for the eigenvalues of H in the fixed order corresponding to our basis, i.e. how they occur in the diagonal form.

$$\begin{aligned} \Rightarrow \qquad (\exp^\Delta(H))_{ij} &= \frac{e^{\lambda_i}-e^{\lambda_j}}{\lambda_i-\lambda_j}, \qquad \text{if } i \neq j, \\ (\exp^\Delta(H))_{ii} &= e^{\lambda_i}. \end{aligned}$$

Moreover the commutator of a hermitian matrix H and a skew-skew hermitian matrix A is again hermitian. So we may apply D. By Leibniz rule

$$\begin{aligned} &\exp(H)A - A\exp(H) \\ =\ & \frac{\mathrm{d}}{\mathrm{d}\,t} e^{-tA}e^{H}e^{tA}\Big|_{t=0} \\ =\ & \frac{\mathrm{d}}{\mathrm{d}\,t}\sum_{n=0}^{\infty} e^{-tA}\frac{H^n}{n!}e^{tA}\Big|_{t=0} \\ =\ & \frac{\mathrm{d}}{\mathrm{d}\,t}\sum_{n=0}^{\infty} \frac{e^{-tA}He^{tA}e^{-tA}H\cdots He^{tA}e^{-tA}He^{tA}}{n!}\Big|_{t=0} \\ =\ & \frac{\mathrm{d}}{\mathrm{d}\,t}\sum_{n=0}^{\infty} \frac{(e^{-tA}He^{tA})^n}{n!}\Big|_{t=0} \\ =^{53}\ & \frac{\mathrm{d}}{\mathrm{d}\,t}\exp\left(e^{-tA}He^{tA}\right)\Big|_{t=0} \\ =\ & \frac{\mathrm{d}}{\mathrm{d}\,t}\exp\left(\left(\sum_{n=0}^{\infty}\frac{(-tA)^n}{n!}\right)H\left(\sum_{j=0}^{\infty}\frac{(tA)^j}{j!}\right)\right)\Big|_{t=0} \\ =\ & \frac{\mathrm{d}}{\mathrm{d}\,t}\exp\left(H + HtA - tAH + O(t^2)\right)\Big|_{t=0} \end{aligned}$$

[52] $\bullet$ is the entry-wise multiplication. If it is clear that we are working with divided sum, we might omit $\bullet$ in future.

$$= \frac{\mathrm{d}}{\mathrm{d}\,t} \exp\left(H + t[H,A] + O(t^2)\right)\Big|_{t=0}$$

Next the differentiability of exp allows us to move the limes from the differential quotient under the exponential function and hence $O(t^2)/t$ drops out. Hence

$$[\exp(H), A]\ \frac{\mathrm{d}}{\mathrm{d}\,t} \exp\left(H + t[H,A]\right)\Big|_{t=0} = \lim_{t\to 0} \frac{e^{H+t[H,A]} - e^{H}}{t} = De^{H}([H,A]).$$

Further define the subspace $\mathbb{S}\mathcal{C}_H := \{[H,A] : A^* = -A\}$ of the commutators of H with skew-hermitian matrices. This is an additive subspace of $\mathbb{H}_n$, as the commutator is linear. Therefore $\mathbb{H}_n$ induces an inner product $\mathrm{tr}(AB)$ on $\mathbb{S}\mathcal{C}_H$. But

$$\mathrm{tr}([H,A],B) = \mathrm{tr}(HAB - AHB) = \mathrm{tr}(ABH - AHB) = \mathrm{tr}(A[B,H]) = 0,$$

$\forall A \in \mathbb{S}H, \quad B \in \mathbb{H}_n$ if and only if $[B,H] = 0$. This follows from $[B,H] =: M$ skew hermitian: $\mathrm{tr}(AM) = \sum_{i,j=1}^{n} a_{ij} m_{ji} = 0$ and for $a_{ij} = a_{ij}\delta_{ki}\delta_{lj} - \overline{a_{ji}}\delta_{kj}\delta_{li}$ $\Rightarrow a_{kl}m_{lk} = 0$ for $k \neq l$ implies $m_{lk} = 0$ and for $l = k$: $m_{ll} = 0$ skew-hermitian. Hence $M = 0$. So

$$\mathcal{Z}_H := \{A \in \mathbb{H}_n : [A,H] = 0\} \Rightarrow \mathcal{Z}_H \oplus \mathcal{C}_H = \mathbb{H}_n$$

$\mathcal{Z}_H$ orthogonal complement.
Let $B \in \mathbb{S}\mathcal{C}_H \Rightarrow B = [H,A]$ for some skew-hermitian matrix A. Then $De^{H}(B) = [\exp(H), A]$ and in the basis with $H = \mathrm{diag}(h_1, \ldots, h_n) \Rightarrow e^{H} = \mathrm{diag}(e^{h_1}, \ldots, e^{h_n})$. For $\lambda_i \neq \lambda_j$

$$\begin{aligned}
(De^{H}(B))_{ij} &= \left([\exp(H), A]\right)_{ij} = \left(e^{h_i}a_{ij} - a_{ij}e^{h_j}\right)_{ij} \\
&= \left(\frac{e^{h_i} - e^{h_j}}{h_i - h_j}(h_i - h_j)a_{ij}\right)_{ij} \\
&= \left(\frac{e^{h_i} - e^{h_j}}{h_i - h_j}[H,A]_{ij}\right)_{ij} \\
&= \left(\frac{e^{h_i} - e^{h_j}}{h_i - h_j}B_{ij}\right)_{ij}
\end{aligned}$$

For $h_i = h_j \Rightarrow B_{ij} = 0$ and $e^{h_i}a_{ij} - a_{ij}e^{h_j} = 0$ so the equality holds anyway.
The other case is $B \in \mathcal{Z}_H \Rightarrow [H,B] = 0$: If H, B commute we may diagonalize simultaneously - $B = \mathrm{diag}(b_1, \ldots, b_n)$ - and so

$$(De^{H}(B))_{ij} = \lim_{t\to 0}\left(\frac{e^{H+tB} - e^{H}}{t}\right)_{ij}$$

[53] $e^{-tA}He^{tA}$ is again hermitian, since e^{tA} is unitary, i.e. exp well-defined.

$$= \lim_{t\to 0}\left(\frac{e^{h_i+tb_i}-e^{h_i}}{t}\right)_{ij}$$
$$= \left(e^{tb_i}b_i\delta_{ij}\right)_{ij}$$

Putting all together we get $De^H(B) = \exp^{\Delta}(H) \bullet B$.
Our original problem becomes (B is no longer diagonal, since it must not commute with H - but hermitian)

$$\begin{aligned}
\|e^{-H/2}De^H(B)e^{-H/2}\|^2_{\mathbb{H}_n,E} &= \sum_{i=1}^{n} e^{-h_i}\frac{e^{h_i}-e^{h_j}}{h_i-h_j}B_{ij}e^{-h_j}\frac{e^{h_j}-e^{h_i}}{h_j-h_i}B_{ji} \\
&= \sum_{i,j=1}^{n} e^{-h_i-h_j}\frac{\left(e^{h_i}-e^{h_j}\right)^2}{(h_i-h_j)^2}|B_{ij}|^2 \\
&= \sum_{i,j=1}^{n} \frac{\left(e^{-h_i/2-h_j/2}e^{h_i}-e^{-h_i/2-h_j/2}e^{h_j}\right)^2}{(h_i-h_j)^2}|B_{ij}|^2 \\
&= \sum_{i,j=1}^{n} \frac{\left(e^{h_i/2-h_j/2}-e^{-h_i/2+h_j/2}\right)^2}{(h_i-h_j)^2}|B_{ij}|^2 \\
&= \sum_{i,j=1}^{n} \frac{\left(e^{h_i/2-h_j/2}-e^{-h_i/2+h_j/2}\right)^2}{(h_i-h_j)^2}|B_{ij}|^2 \\
&= \sum_{i,j=1}^{n} \frac{\left(2\sinh(h_i/2-h_j/2)\right)^2}{(h_i-h_j)^2}|B_{ij}|^2 \\
&= \sum_{i,j=1}^{n} \frac{\sinh^2(h_i/2-h_j/2)}{((h_i-h_j)/2)^2}|B_{ij}|^2 \\
&\geq \sum_{i,j=1}^{n} |B_{ij}|^2 = \operatorname{tr}(B^*B) = \|B\|^2_{\mathbb{H}_n,E}
\end{aligned}$$

where we used in the last step that $\frac{\sinh(x)}{x} \geq 1, \forall x \in \mathbb{R}$. □

Lemma A.2.26. For any path $H(t)$ in $\mathbb{H}_n$ and the induces path $\gamma(t) = e^{H(t)}$:

$$L(\gamma) \geq \int_0^1 \|H'(t)\|_{\mathbb{H}_n,E}\,\mathrm{d}\,t.^{54}$$

Moreover

$$L(\gamma) \geq \|(\log(A) - \log(B))\|_{\mathbb{H}_n,E},$$

which implies $\delta_{\mathbb{P}_n}(A,B) \geq \|(\log(A) - \log(B))\|_{\mathbb{H}_n,E}$.

[54] The tangent space to $\mathbb{H}_n$ is $T\mathbb{H}_n \simeq T\mathbb{R}^n \simeq \mathbb{R}^n \simeq \mathbb{H}_n$.

Proof. By the chain rule $\gamma'(t) = De^{H(t)}(H'(t))$. By the previous proposition for $B = H'(t)$

$$L(\gamma) = \int_0^1 \|e^{H(t)} De^{H(t)}(H'(t))\|_{\mathbb{H}_n,E} \,\mathrm{d}\, t \geq \int_0^1 \|H'(t)\|_{\mathbb{H}_n,E} \,\mathrm{d}\, t.$$

Remember that we had the bijection $\exp : \mathbb{H}_n \to \mathbb{P}_n$. Therefore we may define log the unique function with $\log(\exp) = \mathrm{id}_{\mathbb{H}_n}$. Now take a path γ from A to B in $\mathbb{P}_n$, then $H(t) := \log\gamma(t)$ is path joining $\log(A)$ and $\log(B)$ in $\mathbb{H}_n$. Therefore $\int_0^1 \|H'(t)\|_{\mathbb{H}_n,E} \,\mathrm{d}\, t$ is the length of $H(t)$. Recall that our metric on $\mathbb{H}_n$ is induced by the euclidean metric on $\mathbb{R}^n$. Hence a geodesic in $\mathbb{H}_n$ is a straight line and therefore

$$\begin{aligned} L(\gamma) &\geq \int_0^1 \|H'(t)\|_{\mathbb{H}_n,E} \,\mathrm{d}\, t \geq \int_0^1 \|(\log(A)(1-t) + t\log(B))\|_{\mathbb{H}_n,E} \,\mathrm{d}\, t \\ &= \int_0^1 \|(\log(B) - \log(A))\|_{\mathbb{H}_n,E} \,\mathrm{d}\, t = \|(\log(A) - \log(B))\|_{\mathbb{H}_n,E}. \end{aligned}$$

□

Thus we get

Proposition A.2.27. $(\mathbb{P}_n, \delta_{\mathbb{P}_n})$ is complete.

Proof. Take a Cauchy-sequence F_m in $(\mathbb{P}_n, \delta_{\mathbb{P}_n})$ and let $H_m = \log F_m$ be the corresponding sequence in $\mathbb{H}_n$. By A.2.26: $\forall l \in \mathbb{N}, \exists N \in \mathbb{N}\ \forall k, m \geq N$:

$$\|H_m - H_k\|_{\mathbb{H}_n,E} = \|\log(F_m) - \log(F_k)\|_{\mathbb{H}_n,E} \leq \delta_{\mathbb{P}_n}(F_m, F_k) \leq 2^{-l}.$$

Then H_m converges to $H \in \mathbb{H}_n$. As mentioned before ([Che46], p. 14) exp is even a homeomorphism, i.e. continuous, and we may interchange exp and the limit to receive

$$\lim_{m\to\infty} F_m = \lim_{m\to\infty} \log(H_m) = \log(H) \in \mathbb{P}_n.$$

□

This result is of particular importance, because it enables us to use some results for complete geodesic spaces.

A.3. Local Systems and $\mathscr{D}_X$-modules

After defining local systems we will recall the Riemann-Hilbert correspondence. Then we will define the pushforward bundle, which will appear again, when we define our functors later on; and recall two equivalent notions of regularity due to Deligne [Del70].

Definition A.3.1. A Local System $\mathcal{L}$ is a locally constant sheaf over X.

Proposition A.3.2. There is an equivalence of categories between local systems (modulo sheaf isomorphism) and representations of the fundamental group $\pi_1(X, x) \to \mathrm{Gl}_n(\mathbb{C})$ (modulo $\mathrm{Gl}_n(\mathbb{C})-$conjugation) for a fixed base point $x \in X$ and flat vector bundles up to Gauge equivalence.

Proof. A proof can be found in [Atk08], p. 13 or Reisert [Rei10]. Essentially we associate to each flat vector bundle the sheaf of sections killed by the connection, i.e. the corresponding local system and on the other hand reconstruct V with the stalks of the local system as fibers. □

Carlos Simpson uses for vector bundles with a flat connection - in contrast to vector bundles with constant transition function - the name $\mathscr{D}_X-$module. However, in general:

Remark A.3.3. The space $\mathscr{D}_X$ is the sheaf of differential operators, i.e. the $\mathcal{O}_X$-(resp. $\mathcal{O}_X^{an}$-, $\mathcal{E}_X$-)module generated[55] by derivations given by the vector fields on X. A space V is an $\mathscr{D}_X-$module if there is an action of $\mathscr{D}_X$ on V.[56] If V has the structure of a locally free $\mathcal{O}_X$- (resp. $\mathcal{O}_X^{an}$-, $\mathcal{E}_X$-)module, it is the same as equipping V with a flat connection.

We will use the $\mathscr{D}_X-$module mainly in the part on filtered vector bundles, where we add some more restrictions on the vector bundles resp. coherent sheaves.

The following definitions and results can be found in Deligne [Del70].

Definition A.3.4. Assume that $X = \overline{X} \setminus \{s\}$. Let (V, ∇) be a $\mathscr{D}_X$-module (flat vector bundle), $\mathcal{F}$ the sheaf of holomorphic sections into V. Denote $j : X \to \overline{X}$ the inclusion and $j_*(\mathcal{F})$ the pushforward sheaf, i.e. $j_*(\mathcal{F})(U) := \mathcal{F}(j^{-1}(U))$ for all open sets $U \subset \overline{X}$.[57] For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-(resp. $\mathcal{E}_X-$, $\mathcal{O}_X^{an}-$)modules the pushforward is a sheaf of $j_*\mathcal{O}_X-$(resp. $j_*\mathcal{E}_X-$, $j_*\mathcal{O}_X^{an}-$)modules, i.e. in the case of isolated punctures $j_*\mathcal{O}_X$ is the set of functions holomorphic outside the punctures.

Definition A.3.5. $(V, [\overline{V}])$ with V vector bundle on X, $\overline{V}$ vector bundle on $\overline{X}$ is called meromorphic vector bundle iff

(i) $\overline{V}$ extends V, in terms of sheaves $\overline{V} \subset j_*V$.

---

[55] sum, composition.

[56] One might differentiate between left and right $\mathscr{D}_X$-module.

[57] j continuous $\Rightarrow$ $j^{-1}(U)$ open. Further for $W \subset U$ the restriction maps are $\rho^*_{U,W}$: $j_*(\mathcal{F})(U) \to j_*(\mathcal{F})(W)$ just $\rho^*_{U,W} := \rho_{j^{-1}(U),j^{-1}(W)}$, where $\rho_{j^{-1}(U),j^{-1}(W)}$ are the restrictions of $\mathcal{F}$.

(ii) $[\overline{V}]$ is an equivalence class, with $V_1 \sim V_2$ for V_1, V_2 with (i) iff $\exists n \in \mathbb{N}$ such that for the corresponding sheaves[58]

$$z^n \mathcal{V}_1 \subset \mathcal{V}_2 \subset z^{-n} \mathcal{V}_1 \subset j_* \mathcal{V}.$$

Definition A.3.6. A frame of $\overline{V}$ is called a meromorphic frame of $[V, [\overline{V}]]$ if it restricts to a holomorphic frame of V. A frame on V is called meromorphic if it is the restriction of such a frame.

Definition A.3.7. A holomorphic connection ∇ on V is called meromorphic if V extends to a meromorphic bundle with meromorphic frame (s_i) and $\nabla = d + A$ w.r.t. (s_i) with all entries of A meromorphic.
A connection ∇ is regular if it is meromorphic and one can choose (s_i) such that A has at most a pole of order 1.

Lemma A.3.8 (Deligne)**.** Let (V, h) be a metric bundle, $\mathcal{V}$ the corresponding locally free $\mathcal{O}_X-$sheaf and $\mathcal{V}_1$ an extension of $\mathcal{V}$. A section $s \in \mathcal{V}_1$ is meromorphic if and only if it has "moderate crossing", i.e. if it is bounded by $\|s\|_h \leq C|z|^N$ for some $C, N > 0$.

Proof. This is proposition 2.18. [Del70], p. 68. □

Remark A.3.9. (i) There are indeed meromorphic sections on an open neighbourhood of a puncture s with pole at the puncture, even global functions on all of $\overline{X}$ (cf. Forster [For81] 29.17, p. 225.).

(ii) Let (s_i) be a holomorphic frame of V. If m is our meromorphic function from (i), $\varphi_{U\setminus\{s\}}$ a local trivialization, then $m_i(x) := \varphi^{-1}_{U\setminus\{s\}}(x, m(x)e_i)$ forms a meromorphic frame, since the trivialization is a $\mathbb{C}-$isomorphism. Hence there always is a meromorphic extension defined by the $\mathcal{O}_{\overline{X}}-$span of the m_i.

(iii) The stalk of $j_*(\mathcal{F})$ at s consists of equivalence classes of local sections of $\mathcal{F}$. In particular $\forall v \in j_*(\mathcal{F})(U), s \in U$ $\exists$ a possibly smaller neighbourhood V of s such that $v|_{V\setminus\{s\}} \in j_*(\mathcal{F})(V \setminus \{s\}) = \mathcal{F}(V \setminus \{s\})$ can be uniquely identified with v, i.e. there is only one element in the equivalence class of v. Therefore every sheaf homomorphism θ on $\mathcal{F}$ rises to a sheaf homomorphism on $j_*\mathcal{F}$. An isomorphism induces an isomorphism. We denote the morphism on $j_*\mathcal{F}$ by θ_*. Analogously for connections D resp. ∇ holomorphic.

Theorem A.3.10. Let (V, ∇) be a flat vector bundle on X. The connection ∇ has a regular singularity at s if and only if each branch of a flat holomorphic section e ($\nabla e = 0$) grows at most polynomially along each ray ρ out of s (in terms of a meromorphic basis).

[58] z local coordinate vanishing at the puncture.

Proof. Deligne [Del70], p. 55, theorem 1.19. Parts of the proof involving monodromy will be given later. □

In signs: Let D be an open neighbourhood of p containing no other puncture and z a local coordinate around D. Set $\rho(z_0,t_0) = \{z \in D : z = t(t_0 z_0 - p) + p, 0 < t < 1, t_0 \in \mathbb{R}_+\}$ for $z_0 \neq p$. For t_0 small enough exists U_{z_0} in the trivializing cover of V such that $\rho(z_0,t_0) \subset U_{z_0}$. The connection ∇ can locally be written as $\nabla = d + A$[59], with A a matrix valued one-form (4.2.5 on page 174 in [Huy05]) w.r.t. a meromorphic frame of $\overline{V}$ on U, e.g. the meromorphic frame constructed before the last example. Than $\nabla s = 0 \Leftrightarrow ds = -As$, i.e. an ordinary differential equation which can be solved in some small neighbourhood of an initial value and by flatness of the connection be extended along some paths.[60] As long as we stay in some contractible neighbourhood the solution will be single-valued, in general multivalued (e.g. possible on $D\backslash\{p\}$). If necessary restrict U_{z_0} to a possibly smaller neighbourhood named U_{z_0} as well. Let $\varphi_{U_{z_0}}$ denote the trivialization on U_{z_0} and $\rho_{z_0}(t)$ be a parametrization of $\rho(z_0,t_0)$ with $\rho(z_0,t_0) \subset U_{z_0}$ and $\lim_{t\to 0}\rho_{z_0}(t) = p$. Now ∇ has a regular singularity at p iff $\forall z_0 \forall \rho_{z_0}(t) \forall s$ (multivalued) horizontal section s on U_{z_0}: $\exists k \in \mathbb{N} \exists c \in \mathbb{R}_+$

$$|\mathrm{pr}_m \varphi_{U_{z_0}} s(\rho_{z_0}(t))| \leq c|t|^{-k} \quad \forall 2 \leq m \leq n+1, t \text{ small enough.}$$ [61]

A.4. Endomorphism bundle

This section on the endomorphism bundle starts with a fundamental construction of a metric on $\mathrm{End}(E)$ as well as the construction of the induced differential operators. This subsection is of particular importance, since it will be often advantageous to consider the endomorphism bundle instead of E itself, in order to get the right result. So we will often use the description of this subsection.

Consider the bundle $\mathrm{End}(E)$ over X. If (E,h) is a hermitian bundle with orthonormal frame $(e_i)_{1\leq i\leq n}$, then $h^*(e_i^*, e_j^*) = \delta_{ij}$ defines a hermitian scalar product on E^*, where $(e_i^*)_{1\leq i\leq n}$ is the dual basis. Recall that $\mathrm{End}(E) \simeq E^* \otimes E$ via the isomorphism $f : \varphi \mapsto \sum_{i=1}^n (e_i^* \circ \varphi) \otimes (e_i)$. Now $h_{\mathrm{End}}(\varphi,\chi) := \sum_{i,j=1}^n h^*(e_i^* \circ \varphi, e_j^* \circ \chi) h(e_i,e_j)$. Obviously this is a positive-definite hermitian inner product.

[59] d is the trivial connection, i.e. $d\sum_{i=1}^n \alpha_i s_i = \sum_{i=1}^n (d\alpha_i)s_i$, s_i frame.
[60] Flatness guarantees the independence of the chosen path.
[61] In abuse of notation s stands for a local horizontal section on U or a branch of a multivalued horizontal section.

To introduce a connection on $\mathrm{End}(E) \simeq E^* \otimes E$ recall the induced connection on E^*: $D_*(f)(s) = d(f(s)) - f(D(s)), f \in E^*, \forall s \in E$. Here D is metric and h^* is the induced metric. If (E, h) is a hermitian bundle with orthonormal frame $(e_i)_{1\le i\le n}$, then $h^*(e_i^*, e_j^*) = \delta_{ij}$ defines a hermitian scalar product on E^*, where $(e_i^*)_{1\le i\le n}$ is the dual basis. Recall that there is an anti-linear bijection $E \to E^*, \xi \to h(\cdot, \xi) = \xi^*$. Our metric becomes

$$\begin{aligned} h^*(\xi^*, \eta^*) &= h^*(\sum_{i=1}^n \xi_i e_i^*, \sum_{j=1}^n \eta_i e_i^*) = \sum_{i=1}^n \xi_i \overline{\eta}_i \\ &= \sum_{i=1}^n \xi_i \overline{\eta}_i h(e_i, e_i) = h(\sum_{i=1}^n \xi_i e_i, \sum_{j=1}^n \eta_j e_j) \\ &= h(\xi, \eta), \end{aligned}$$

i.e. we could equally define $h^*(\xi^*, \eta^*) = \overline{h(\xi, \eta)}$.[62] Our induced connection satisfies $D_*(\xi^*)(\eta) = D_*(h(\eta, \xi)) = dh(\eta, \xi) - h(D\eta, \xi)$. If D is metric, than $D_*(\xi^*)(\eta) = (D\xi)^*(\eta)$. Further

$$\begin{aligned} dh^*(\xi^*, \eta^*) &= d\overline{h(\xi, \eta)} = \overline{h(D\xi, \eta) + h(\xi, D\eta)} \\ &= h^*((D\xi)^*, \eta^*) + h^*(\xi^*, (D\eta)^*) \\ &= h^*(D_*(\xi^*), \eta^*) + h^*(\xi^*, D_*(\eta^*)). \end{aligned}$$

Now we have seen, that on the tensor product $E \otimes E^* \simeq \mathrm{End}(E)$ a metric is defined by $h_{E\otimes E^*}((s, \xi^*), (r, \eta^*)) = h(s, r)h^*(\xi^*, \eta^*)$. D induces a connection $D_{E\otimes E^*} = D \otimes 1 + 1 \otimes D_*$. If D is metric so is $D_{E\otimes E^*}$:

$$\begin{aligned} & dh_{E\otimes E^*}((s, \xi^*), (r, \eta^*)) \\ & = (dh(s, r))h^*(\xi^*, \eta^*) + h(s, r)(dh^*(\xi^*, \eta^*)) \\ & = (h(Ds, r) + h(s, Dr))h(\xi, \eta) + h(s, r)(h^*(D_*\xi^*, \eta^*) + h^*(\xi^*, D_*\eta^*)) \\ & = h_{E\otimes E^*}((D \otimes 1 + 1 \otimes D_*)(s, \xi^*), (r, \eta^*)) \\ & \quad + h_{E\otimes E^*}((s, \xi^*), (D \otimes 1 + 1 \otimes D_*)(r, \eta^*)). \end{aligned}$$

Finally we want to find the induced metric on $\mathrm{End}(E)$. We have

$$\begin{aligned} E \otimes E^* & \to \mathrm{End}(E), \\ (\xi, \eta^*) & \mapsto \{s \to \eta^*(s)\xi = h(s, \eta)\xi\} \\ \mathrm{End}(E) & \to E \otimes E^*, \\ \varphi & \mapsto \sum_{i=1}^n e_i \otimes (e_i^* \circ \varphi) = \sum_{i=1}^n e_i \otimes h(\varphi\cdot, e_i) \end{aligned}$$

[62] Conjugation since $\alpha\xi^* = \alpha h(\cdot, \xi) = h(\cdot, \overline{\alpha}\xi) = (\overline{\alpha}\xi)^*$.

Hence we define the metric on End(E) as the pullback of the metric on $E \otimes E^*$:

$$
\begin{aligned}
h_{\mathrm{End}}(\varphi,\psi) &:= h_{E\otimes E^*}\left(\sum_{i=1}^n e_i \otimes h(\varphi\cdot, e_i), \sum_{j=1}^n e_j \otimes h(\psi\cdot, e_j)\right)\\
&= \sum_{i,j=1}^n h_{E\otimes E^*}\left(e_i \otimes h(\varphi\cdot, e_i), e_j \otimes h(\psi\cdot, e_j)\right)\\
&= \sum_{i,j=1}^n h_{ji} h^*((\varphi^* e_i)^*, (\psi^* e_j)^*)\\
&= \sum_{i,j=1}^n h_{ji} \overline{h(\sum_{j=1}^n \overline{\varphi_{il}} e_l, \sum_{j=1}^n \overline{\psi_{jk}} e_k)}\\
&= \sum_{i,j,l,k=1}^n h_{ji} \overline{\psi_{jk} h_{kl}} \varphi_{il}\\
&= \mathrm{tr}(H\psi^* H\varphi)\\
&= \mathrm{tr}(\psi^*\varphi) \quad \text{for } (e_i) \text{ orthonormal, i.e. } H = E\\
&= \|\varphi\|_F^2 \quad \text{for } \varphi = \psi.
\end{aligned}
$$

In general for $\varphi = \psi$ we still have a norm $\|H\varphi\|_F^2$ by $H^* = H$. Usually we define H on a basis (e_i) as $h_{ij} = h(e_j, e_i)$.

Remark A.4.1. (i) The inner product on End(E) is just the Hilbert-Schmidt inner product, i.e. $h_{\mathrm{End}}(A,B) = \sum_{i=1}^n h(Ae_i, Be_i) = \mathrm{tr}(B^*A) = \sum_{i,j=1}^n a_{ji}\overline{b_{ji}}$ for e_i orthonormal frame. The corresponding norm is the Frobenius norm. This justifies our choice of the Frobenius norm.
Further we see that $h_{\mathrm{End}}(A,B) = h_{\mathrm{End}}(E, A^*B)$. Even more $h_{\mathrm{End}}(A, BC) = h_{\mathrm{End}}(AC^*, B)$ for matrices A, B, C in orthonormal coordinates $H = E$. The last equality follows for example by

$$
\begin{aligned}
h_{\mathrm{End}}(A, BC) &= \mathrm{tr}(HAH(BC)^*) = \mathrm{tr}(HAHC^*B^*) \overset{H=E}{=} \mathrm{tr}(AC^*EB^*)\\
&= h_{\mathrm{End}}(AC^*, B).
\end{aligned}
$$

(ii) If we have to differ between different metrics we will sometimes denote $\|\varphi\|_H := \mathrm{tr}(\varphi^* H\varphi H)$.

(iii) The induced norm on the endomorphism bundle is consistent with the original norm H in the following sense:

$$
\|\varphi(e)\|_h^2 \le \frac{\mathrm{tr}(\varphi^* H\varphi H)}{\mathrm{tr}(H^2)}\|e\|_h^2 = \frac{\|\varphi\|_H^2}{\|E\|_H^2},
$$

for any section $e \in E$.

Remark A.4.2. Before we proceed with the construction of connections on the endomorphism bundle a short interlude on bundles with different metrics. Assume that we want to find an inner product for maps from (E,h) to (E,k) for two different metrics on the same bundle E. From the previous calculation we read off (using the general identification $\mathrm{Hom}(V,W) \to V^* \otimes W, f \mapsto \sum_{i=1}^n w_i^* \circ f \otimes w_i$ for w_i basis of W.)

$$h_{\mathrm{End},h\to k}(\varphi,\psi) \;\; := \;\; \sum_{i,j=1}^{n} k(e_i,e_k) h^* \left(e_i \otimes k(\varphi\cdot, e_i), e_j \otimes k(\psi\cdot, e_j)\right).$$

Let e_j be an $k-$orthonormal frame and denote all matrices in this frame. Let $K = E$ be the representation of k and H the representation of H. Since $h(e_i,e_j) = e_j^* H e_i$ we have

$$\begin{aligned} k(e,f) &= f^* K e = f^* K^{1/2} H^{-1/2} H H^{-1/2} K^{1/2} e \\ &= h(H^{-1/2} K^{1/2} e, H^{-1/2} K^{1/2} f) \\ &= h(H^{-1/2} e, H^{-1/2} f), \quad e,f \in E, K = E. \end{aligned}$$

Then

$$\begin{aligned} & h_{\mathrm{End},h\to k}(\varphi,\psi) \\ &= \sum_{i,j=1}^{n} k(e_i,e_j) h^* \big(k(H^{-1/2}K^{1/2}\cdot, \ldots \\ &\quad \ldots H^{-1/2}K^{1/2}\varphi^* e_i), k(H^{-1/2}K^{1/2}\cdot, H^{-1/2}K^{1/2}\psi^* e_j)\big) \\ &= \sum_{i,j=1}^{n} E_{ji} h^* \big(h(\cdot, K^{1/2}H^{-1}K^{1/2}\varphi^* e_i), h(\cdot, K^{1/2}H^{-1}K^{1/2}\psi^* e_j)\big) \\ &= \mathrm{tr}(K K^{1/2} H^{-1} K^{1/2} \psi^* H K^{1/2} H^{-1} K^{1/2} \varphi) \\ &= \mathrm{tr}(H^{-1}\psi^*\varphi), \qquad K = E \\ &= \langle \varphi H^{-1/2}, \psi H^{-1/2} \rangle_{HS}. \end{aligned}$$

Instead of a $k-$orthonormal frame we may choose a $K^{1/2}H^{-1}K^{1/2}-$orthonormal one.[63] Then norm becomes $\|\varphi\|^2_{\mathrm{End},h\to k} = \mathrm{tr}(K\varphi^* H\varphi) = \langle H\varphi, \varphi K\rangle_{HS} = |\langle H\varphi, \varphi K\rangle_{HS}| \leq \|H\varphi\|_F \|K\varphi\|_F$ by Cauchy-Schwarz. We will later on use that the last formula shows: If φ is L^p-integrable with respect to h and with respect to k then φ is L^p-integrable w.r.t. $\|\cdot\|_{\mathrm{End},h\to k}$.

Furthermore note that we didn't use any properties of the endomorphism bundle, that a general homomorphism bundle does not have. Indeed the identification

[63] $K^{1/2}H^{-1}K^{1/2}$ positive-definite since $z^* K^{1/2} H^{-1} K^{1/2} z = (K^{1/2}z)^* H^{-1} (K^{1/2} z) > 0$ by $K^{1/2}$ invertible, H positive-definite.

$\mathrm{Hom}(V,W) \to V^* \otimes W, f \mapsto \sum_{i=1}^n w_i^* \circ f \otimes w_i$ for w_i basis of W works perfectly with general homomorphism. As well the basis (w_i) only depends on the range; the usual identification with $X \times \mathbb{C}^n$ via trivializations leads to a compatible basis in V.

The connection D becomes

$$\begin{aligned}
D_{E\otimes E^*}(\varphi) &:= \sum_{i=1}^n De_i \otimes h(\varphi\cdot, e_i) + e_i \otimes D_* \left(h(\varphi\cdot, e_i)\right) \\
&= \sum_{i=1}^n De_i \otimes h(\varphi\cdot, e_i) + e_i \otimes D_*(\varphi^* e_i)^* \\
&= \sum_{i=1}^n De_i \otimes h(\varphi\cdot, e_i) + e_i \otimes dh(\varphi, e_i) - e_i \otimes h(\varphi D\cdot, e_i) \\
\Rightarrow D_{\mathrm{End}}(\varphi) &:= \sum_{i=1}^n De_i h(\varphi\cdot, e_i) + e_i dh(\varphi\cdot, e_i) - e_i h(\varphi D\cdot, e_i) \\
&= \sum_{i=1}^n De_i h(\varphi\cdot, e_i) + e_i dh(\varphi\cdot, e_i) - \varphi D \\
&= \sum_{i=1}^n D(e_i h(\varphi\cdot, e_i)) - \varphi D = D(\sum_{i=1}^n e_i h(\varphi\cdot, e_i)) - \varphi D \\
&= D\varphi - \varphi D.
\end{aligned}$$

By construction this connection is metric again.

Remark A.4.3. e_i induces a basis on $\mathrm{End}(E)$ by $e_j^* \otimes e_i$, which as a matrix looks like $(E_{ij})_{lk} = \delta_{il}\delta_{jk}$ - all entries 0 apart from the $(i,j)-$entry 1. If e_i is $D-$flat, then for $s = \sum_{k=1}^n s_k e_k$

$$\begin{aligned}
(DE_{ij} - E_{ij}D)(s) &= Ds_j e_i - E_{ij} \sum_{k=1}^n De_k \otimes s_k + e_k \otimes ds_k \\
&= De_i \otimes s_j + e_i \otimes ds_j - E_{ij}(\sum_{k=1}^n e_k \otimes ds_k) \\
&= e_i \otimes ds_j - e_i \otimes ds_j \\
&= 0.
\end{aligned}$$

Hence E_{ij} is a $D-$flat frame.

We showed:

Lemma A.4.4. A metric connection $D = \partial_E + \overline{\partial}_E$ on a hermitian bundle E induces a metric connection $D^0_{\mathrm{End}} : \Gamma(U, \mathrm{End}(E)) \to \Gamma(U, \mathrm{End}(E)) \otimes \Omega^1_X$ by

$$D^0_{\mathrm{End}}(\varphi) := [D_{\mathrm{End}}, \varphi] = D\varphi(s) - \varphi(Ds), \quad s \in \Gamma(U, E), \varphi \in \Gamma(U, \mathrm{End}(E)).^{64}$$

We see that it obeys the Leibniz rule

$$\begin{aligned} D^0_{\mathrm{End}}(\varphi \otimes \alpha) &= D(\varphi \otimes \alpha)(s) - (\varphi \otimes \alpha)(Ds) \\ &= D\varphi(s) \otimes \alpha + \varphi(s) \otimes d\alpha - \varphi D(s) \otimes \alpha \\ &= D^0_{\mathrm{End}}(\varphi)(s) \otimes \alpha + \varphi(s) \otimes d\alpha, \end{aligned}$$

for $s \in \Gamma(U, E), \varphi \in \Gamma(U, \mathrm{End}(E)), \alpha \in \Gamma(U, \mathbb{C})$.
By the requirement $D^1_{\mathrm{End}}(\varphi \otimes \omega) = D^1_{\mathrm{End}}(\varphi) \otimes \omega + \varphi \otimes d\omega$ we get further

$$\begin{aligned} D^1_{\mathrm{End}} &: \Gamma(U, \mathrm{End}(E)) \otimes \Omega^1_X \to \Gamma(U, \mathrm{End}(E)) \otimes \Omega^2_X, \\ D^1_{\mathrm{End}}(\varphi \otimes \omega)(s) &= D^1(\varphi \otimes \omega)(s) + (\varphi \otimes \omega) D^0(s) \\ &= D^0(\varphi)(s) \otimes \omega + \varphi(s) \otimes d\omega + (\varphi \otimes \omega)(s_{D^0} \otimes \omega_{D^0}) \\ &= D^0(\varphi)(s) \otimes \omega + \varphi(s) \otimes d\omega + (\varphi(s_{D^0}) \otimes \omega \wedge \omega_{D^0}) \\ &= D^0(\varphi)(s) \otimes \omega + \varphi(s) \otimes d\omega - (\varphi(s_{D^0}) \otimes \omega_{D^0} \wedge \omega) \\ &= D^0(\varphi)(s) \otimes \omega + \varphi(s) \otimes d\omega - (\varphi D^0)(s) \wedge \omega \\ &= ((D^0\varphi - \varphi D^0) \otimes \omega)(s) + \varphi(s) \otimes d\omega \\ &= D^0_{\mathrm{End}}(\varphi) \otimes \omega(s) + \varphi(s) \otimes d\omega. \end{aligned}$$

The curvature of D_{End} is

$$\begin{aligned} D^1_{\mathrm{End}} D^0_{\mathrm{End}}(\varphi) &= D^1_{\mathrm{End}}(D\varphi - \varphi D) = D^2\varphi - D\varphi D + D\varphi D - \varphi D^2 \\ &= [D^2, \varphi] = \mathrm{ad}(D^2)(\varphi). \end{aligned}$$

Remark A.4.5. The construction extends naturally to higher order differential operators. For two operators D_E, D_F of the same degree (p, q) on E resp. F the induced operator on $\mathrm{Hom}(E, F)$ is just $D_{\mathrm{Hom}(E,F)}(\varphi) = D_F\varphi + (-1)^{p+q}\varphi D_E$. This follows directly by adding some indices in the calculation above.

Let's take another look at the curvature in general. Note that for any hermitian connection D on any holomorphic bundle E':

$$\begin{aligned} &h_{E'}(D^2 s, s) + h_{E'}(s, D^2 s) \\ &= dh_{E'}(Ds, s) + h_{E'}(Ds, Ds) + dh_{E'}(s, Ds) - h_{E'}(Ds, Ds) \end{aligned}$$

[64] $\varphi = \varphi_t \otimes \eta_\varphi$ acts on $s \otimes \omega$ by $\varphi(s \otimes \omega) = \{x \mapsto (\varphi_t(x))(s(x)) \otimes (\eta_\varphi \wedge \omega)(x)$.

$$
\begin{aligned}
&= dh_{E'}(Ds, s) + dh_{E'}(s, Ds) \\
&= d^2 h_{E'}(s, s) = 0, \quad s \in \Gamma(X, E') \\
&\Rightarrow h_{E'}(D^2 s, s) + h_{E'}(s, D^2 s) = h_{E'}(D^2 s, s) + \overline{h_{E'}(D^2 s, s)} = 0 \\
&\Rightarrow h_{E'}(D^2 s, s) \text{ imaginary form, i.e. } f \,\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}, f \text{ real.}^{65}
\end{aligned}
$$

In our case $E' = \mathrm{End}(E)$.

[65] The name imaginary comes from $\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z} = -2i \,\mathrm{d}\, x \wedge \mathrm{d}\, y$, i.e. as a real differential form the function part is imaginary.

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Index

Declaration of Primary Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise.

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