Repeated Implementation*

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Abstract

We prove that a social choice function is repeatedly implementable if and only if it is dynamically monotonic when the number of agents is at least three. We show how to test dynamic monotonicity by building an associated repeated game. It follows that a weaker version of Maskin monotonicity is necessary and sufficient among the social choice functions that are efficient. As an application, we show that utilitarian social choice functions, which can only be one-shot implemented with side-payments, are repeatedly implementable, as continuation payoffs can play the role of transfers. Under some additional assumptions, our results also apply when the number of agents is two.

Keywords: mechanism design, dynamic monotonicity, efficiency, repeated implementation, repeated games, approximation of the equilibrium set, sufficient and necessary condition

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1 Introduction

We consider an infinite horizon problem when a new state of the world is realised in each period (i.i.d.). A social designer wants to select an outcome in each period, which depends on that period's state. However, the realised state is only observed by agents, and never by the designer. Therefore, the designer must construct a sequence of mechanisms, referred to as a regime, that would allow to elicit the state of the world from the agents in each period and, at the same time, implement the desired outcome.

Our objective is to characterise the social choice functions that are repeatedly implementable in Nash equilibrium. A social choice function – a mapping from states of the world into the set of alternatives – is repeatedly implementable if there exists a regime such that the set of Nash equilibria of the game, which is induced by this regime, is non-empty and, for any sequence of realized states, the sequence of outcomes in any Nash equilibrium is such that in each period, the outcome coincides with the socially desired one (i.e., we consider full outcome implementation). With the help of our characterization, we provide practical ways to check if a social choice function is repeatedly implementable. In particular, as an application, we prove that utilitarian social choice functions are repeatedly implementable by showing that continuation payoff promises can effectively play the role of side-payments, which are needed for implementation in static setups.

The same problem has recently been studied by Lee and Sabourian (2011) and Mezzetti and Renou (2014) with n agents.[1] Lee and Sabourian (2011) show in their Theorem 1 that if a social choice function is not weakly efficient in its range, then the function is not repeatedly implementable for sufficiently high discount factors. They also show in their Theorem 2 that for $n > 2$, strict efficiency in the range together with certain domain restrictions are sufficient for outcome implementation from period 2 onwards. In turn, Mezzetti and Renou (2014) introduce a condition, called dynamic monotonicity (DM), and show in their Theorem 1 that it is necessary, while in their Theorem 2, they show that for $n > 2$, dynamic monotonicity plus no-veto power are sufficient for repeated implementation both in finite and infinite horizon problems, irrespective of the magnitude of the discount factor. Both Lee and Sabourian (2011) and Mezzetti and Renou (2014) offer additional sufficient conditions to treat the case of $n = 2$ in their Theorem 3.

We contribute to this literature by closing the gap between necessary and sufficient conditions when $n > 2$. We also strengthen the results of both Theorem 3 in Lee and Sabourian (2011) and that in Mezzetti and Renou (2014) in the case of $n = 2$. In the following, we discuss our results in more detail starting with the

[1] Repeated implementation has also been studied by Kalai and Ledyard (1998) and Chambers (2004), but their setup is different: the socially desired outcome is allowed to change over time and according to the state the world but the latter is drawn only once and kept fixed for all periods.

case of $n > 2$.

The Case of $n > 2$:

First, we provide necessary and sufficient conditions for a function to be repeatedly implementable. Our Theorem 1 states that if the designer can only use regimes with deterministic stage mechanisms but possibly stochastic transitions, then a function is repeatedly implementable if and only if there exists a collection of sets C with respect to which this function is dynamically monotonic (in the sense of Mezzetti and Renou (2014)) and relative to the same collection C, the function satisfies two further conditions, which we call $\lambda 0$ and $\lambda 1$. Consequently, we improve on Theorem 2 of Mezzetti and Renou (2014) by replacing their non-necessary assumption of no-veto power with necessary Conditions $\lambda 0$ and $\lambda 1$.

To understand the role of the necessary and sufficient conditions, consider Condition μ of Moore and Repullo (1990), which is necessary and sufficient for implementation in a static setup. Dynamic monotonicity w.r.t. C can be thought as a dynamic version of μ(i), i.e., Maskin monotonicity, while Conditions $\lambda 0$ and $\lambda 1$ can be thought as dynamic versions of μ(ii) combined with μ(iii). Further, similar to Condition μ, the collection of sets C plays the role of lower contour sets and describes the outcomes that each agent can demand when deviating unilaterally from equilibrium strategies. These sets are chosen so as to give the incentives to agents to upset undesirable equilibria without introducing other undesirable equilibria. Specifically, dynamic monotonicity w.r.t. C ensures that there exists an agent who has incentives to deviate if a joint lie by agents, which we call a deception, about the true state of the world leads to an undesirable outcome in some period. On the other hand, Conditions $\lambda 0$ and $\lambda 1$ ensure that such a deviation by a single agent does not result in another undesirable equilibrium.

Second, we show that even simpler characterization of necessary and sufficient conditions is possible if the designer has more flexibility in the choice of mechanisms. Thus, Corollary 1, which we consider to be our main result, states that if the designer, in addition to stochastic transitions, can also use stochastic stage mechanisms, then a function is repeatedly implementable if and only if it is dynamically monotonic with respect to a collection of sets C.[2] To show this result, we build an auxiliary repeated game with random states and connect the properties of its equilibrium payoff set to DM. It turns out that a function is DM w.r.t. some C if and only if it is Maskin monotonic* (a weaker version of Maskin monotonicity, which we explain later) and the payoffs, when the function is implemented, are the unique efficient equilibrium payoffs of the repeated game (see Lemma 2). Then, by using recent results on the upper-semicontinuity of

[2]Mezzetti and Renou (2014) only allow for stochastic transitions and deterministic stage mechanisms. In a static setup, random mechanisms have been employed in Abreu and Sen (1991); Bochet (2007); Benoît and Ok (2008); Lombardi (2012).

the equilibrium payoff correspondence of repeated games (Plan, 2014), we show that C can always be modified so as Conditions $\lambda 0$ and $\lambda 1$ hold vacuously while preserving DM at the same time. The repeated game also turns out to be useful for another reason as we discuss next.

Third, checking dynamic monotonicity can be challenging in practice. However, we argue that it can be numerically verified or disproved by using the repeated game that we have constructed for the proof of Corollary 1. The numerical methods offered by Judd, Yeltekin, and Conklin (2003) can approximate the equilibrium payoff set of this repeated game, once we allow for public randomization. As mentioned before, the properties of the equilibrium payoff set then tell us about DM of the corresponding social choice function. Of course, Theorem 1 and Corollary 1 still hold, when agents have access to a public randomization device with the qualification that DM must now be satisfied for all, possibly random, deceptions. Additionally, by applying a recent result of Abreu and Sannikov (2014), we argue that when the number of agents is two and they have access to a public randomization device, only Markovian deceptions must be checked, i.e., those deceptions that, besides the current state, only depend on the previous period's state and on the realization of the previous period's public randomization. However, this result does not extend to the case with more than two agents.

Fourth, we also improve on Theorem 2 of Lee and Sabourian (2011) and show that Maskin monotonicity* makes the domain restrictions of Lee and Sabourian (2011) unnecessary, as long as the social choice function is efficient in the range. We state in Proposition 2 that if a function is efficient in its range and Maskin monotonic*, then it is DM. Hence, by Corollary 1, it is repeatedly implementable.[3] Maskin monotonicity* is a simple necessary condition, which is implied by both Maskin monotonicity and DM. Said simply, it rules out deceptions in which agents jointly lie about the state of the world only in a certain period and from the next period on they report honestly. Interestingly, it is also connected to one-shot implementation with side-payments as we discuss next.

Fifth, as an application of Proposition 2 and Corollary 1, we show that (generalized) utilitarian social choice functions are repeatedly implementable in an environment that is adapted from Laffont and Maskin (1982). We say that a function is a utilitarian social choice function if in each state it maximizes some weighted sum of agents' utilities, assuming that these weights are the same across the states. A utilitarian social choice function is obviously efficient in its range. We also show that it is Maskin monotonic* in Proposition 3. On the other hand, it follows from Laffont and Maskin (1982) that a social choice rule consisting of a utilitarian social choice function and a private transfer function is one-shot implementable, when agents have quasi-linear preferences. Therefore, one can say

[3]Note, however, that Lee and Sabourian (2011) work with fully deterministic regimes, while we allow for stochastic transitions and stochastic stage mechanisms.

that side-payments, which often facilitate implementation in static setups, can be redundant in repeated setups as their role is effectively played by continuation payoff promises.

The Case of $n = 2$:

Our results also extend to the case with two agents under some additional assumptions. Any necessary and sufficient conditions for the two agent case must include the necessary and sufficient conditions for the case of more than two agents. However, when $n = 2$, the designer must additionally face the problem that the agents are sending different messages about the state of the world and the "deviator" cannot be identified. To overcome this problem, we need to introduce additional conditions.

First, in Proposition 4, we establish a result similar to the one in Proposition 2 for $n = 2$. Besides Maskin monotonicity* and efficiency (not just efficiency in the range) of social choice function, we also assume that the function satisfies a version of (static) self-selection condition. The self-selection condition is necessary for the static implementation as shown by Moore and Repullo (1990) and Dutta and Sen (1991). It is also used by Lee and Sabourian (2011) in their Theorem 3, which states that efficiency in the range, self-selection, and their usual domain restrictions are sufficient for repeated implementation of social choice function when $n = 2$. However, we argue in Section 7 that either self-selection or efficiency in the range must be strengthened to obtain repeated implementation. Therefore, we assume efficiency in Proposition 4. Alternatively, we could assume efficiency in the range but then self-selection must be replaced with *self-selection in the range.*

Second, we show in Proposition 5 that the result of Corollary 1 carries over to the $n = 2$ case if we assume that there exists a bad outcome (for its use, see, e.g., Moore and Repullo (1990) or Mezzetti and Renou (2014)). Now, whenever the agents send different messages and the deviator cannot be identified, the designer can simply implement the bad outcome forever. This is sufficient to rule out contradictory messages in the equilibrium.

The rest of the paper is organized as follows. In Section 2, we introduce the model and basic notation. In Section 3, we provide the definitions of Maskin monotonicity* and dynamic monotonicity. We also derive Conditions $\lambda 0$ and $\lambda 1$ and state Theorem 1 in this section. In Section 4, we prove Corollary 1. To do it, in Section 4.1, we introduce an auxiliary repeated game with random states, which is derived from the repeated implementation environment, and in Section 4.2, we prove several lemmas that connect DM with the equilibrium payoff set of the associated repeated game (for any $n \geq 2$). In Section 5, we discuss how one can numerically test DM using the results of Section 4.2. In Section 6, we first relate our findings to Lee and Sabourian (2011) and show that efficiency in the range and the necessary condition of Maskin monotonicity* are sufficient for implementation when $n > 2$. As an application of this result, we next prove that the utilitarian social choice functions are repeatedly implementable in the environment of Laffont and Maskin (1982). In Section 7, we present and discuss our results for the $n = 2$ case. The proof of the sufficiency part of Theorem 1 can be found in Section A in Appendix. The proofs of the propositions for the $n = 2$ case are also relegated to Appendix and can be found in Section B. Finally, we illustrate and compare our results to those in the literature through several examples in Section C, which hopefully also make the reading of the paper easier.

2 The Model

2.1 Preferences

There is a finite set of agents, $I = \{1, 2, \ldots, n\}$, a finite set of alternatives, A, a finite set of states of the world, Θ, and an infinity of periods, $T = \{0, 1, 2, \ldots\}$.[4] Each period $t \in T$, a state of the world $\theta \in \Theta$ is independently and identically realized with probability $p(\theta)$. We assume that $p(\theta) > 0$ for all $\theta \in \Theta$. Let $\tilde{a} \in \Delta A$ denote a random alternative and let $\tilde{a}(a)$ denote the probability that the deterministic alternative $a \in A$ is selected. When we want to emphasize that the selected alternative depends on the state, we will write accordingly $\tilde{a}(\theta)$ and $\tilde{a}(\theta)(a)$. Throughout, we will use superscripts for variables to indicate a time period and subscripts to indicate an agent.

The preferences of the agents are represented by the discounting criterion. Given a sequence of random alternatives, $(\tilde{a}^\tau(\theta))_{\tau \in T, \theta \in \Theta}$, the period t (continuation) payoff of agent i before he has learnt the state of the world of that period, is given by

$$v_i^t = (1-\delta) \sum_{\tau=t}^{\infty} \sum_{\theta \in \Theta} \sum_{a \in A} \delta^{\tau - t} p(\theta) \tilde{a}^\tau(\theta)(a) u_i(a, \theta),$$

Let $v^t = (v_1^t, \ldots, v_n^t)$ be a payoff profile in period t, and let V denote the set of feasible payoff profiles. Note that the set V is the same for all t. We will write $u_i(\tilde{a}, \theta)$ for $\sum_{a \in A} \tilde{a}(a) u_i(a, \theta)$. Once agent i learns that the state of the world in period t is θ, his period t payoff is $(1-\delta) u_i(\tilde{a}^t, \theta) + \delta v_i^{t+1}$ if random alternative $\tilde{a}^t$ is selected in that period and the continuation payoff is v_i^{t+1}.

Agent i's preferences over $A \times V$ in any state θ are completely described by $U_i(a, v, \theta) = (1-\delta) u_i(a, \theta) + \delta v_i$. We assume that $U_i(\cdot, \cdot, \theta)$ is a Bernoulli utility function for all i, determining the preferences over $\Delta(A \times V)$ as expected utilities. Throughout, we assume that in each state, agents have strict preferences

[4] We keep finiteness of A for simplicity. All proofs can be modified to accommodate infinite A as well. In fact, in Section 6.1, we provide an application of our results, where $A = [0, 1]$. The finiteness of Θ can be relaxed for Theorem 1, but it is necessary for Corollary 1 and Propositions 4 and 5.

over the set of non-random alternatives, i.e., $u_i(a,\theta) \neq u_i(b,\theta)$ for all i, θ, $a \in A$, and $b \in A$. We will discuss in the text how our results can be extended to the case of weak preferences.

2.1.1 "Preferences" of the Designer

A social choice function[5] maps states of the world into alternatives, $f : \Theta \to A$. The objective of the designer is to select alternative $f(\theta^t)$ in period t if the state of that period is θ^t. However, the designer never observes the realized state of the world, while all agents observe θ^t at the beginning of period t. Note that we are only interested in the implementation of deterministic, time independent choice functions.

2.2 Repeated Implementation

2.2.1 Stage Mechanisms

Let Γ be a set of mechanisms or game forms. A deterministic mechanism $\gamma \in \Gamma$ is a pair $((\mathcal{M}_i)_{i\in I}, g)$ where $\mathcal{M}_i$ denotes a message space of agent i, and $g : \times_{i\in I}\mathcal{M}_i \to A$ is a deterministic allocation rule. Let $\mathcal{M} = \times_{i\in I}\mathcal{M}_i$ be the space of message profiles. Let m_i and $m = (m_1, \ldots, m_n)$ be generic elements of $\mathcal{M}_i$ and $\mathcal{M}$. If $g : \times_{i\in I}\mathcal{M}_i \to \Delta A$, we say that the mechanism is stochastic and $g(m)(a)$ denotes the probability of selecting alternative a.

2.2.2 Histories

The designer chooses, possibly randomly, the current period's mechanism which he commits to. A state of the world is realized. All agents are informed about the state and the selected mechanism. That is, even if the mechanism was chosen randomly, the agents are informed about the realized mechanism. The agents send public, simultaneous messages to the designer. The designer implements the (possibly random) alternative, which he has committed to. Then the process is repeated in the next period and so on.

Let period 0 history be $h^0 = \emptyset$. The history that is observed by all agents in the beginning of period $t > 0$ is $h^t = (\theta^0, \gamma^0, m^0, a^0, \ldots, \theta^{t-1}, \gamma^{t-1}, m^{t-1}, a^{t-1})$ where $\theta^\tau \in \Theta$, $\gamma^\tau \in \Gamma$, $m^\tau \in \mathcal{M}^t$, where $\mathcal{M}^\tau$ is the space of message profiles corresponding to mechanism γ^τ, and $a^\tau \in A$ is the realization of $g^\tau(m^\tau)$, where g^τ is the possibly stochastic allocation rule corresponding to γ^τ. Hence, the period t history does not contain the period t state of the world and the mechanism, which will be used by the designer in that period. Let H^t be the space of all possible period t histories that are observed by the agents, with $H^0 = \{\emptyset\}$. The space

[5]The case of social choice correspondences can be tackled as in Lee and Sabourian (2011) and Mezzetti and Renou (2014).

of all possible agents' histories is $H = \cup_{t=0}^{\infty} H^t$. The designer cannot distinguish between any two period t histories that only differ in the realized states of the world.

2.2.3 Regimes

A *dynamic mechanism regime* or regime for short is a transition rule $r : H \to \Delta(\Gamma)$, where $r(\gamma|h^t)$ denotes the probability that mechanism γ is selected after history h^t. We say that a regime is *simply-stochastic* if the stage mechanisms must be deterministic, i.e., for any $\gamma \in \Gamma$, the corresponding g maps $\mathcal{M}$ to A. If γ can be stochastic, i.e., when g maps $\mathcal{M}$ to ΔA, we say that the regime is *doubly-stochastic*. Note that $r(\gamma|h^t) = r(\gamma|\tilde{h}^t)$ if the designer cannot distinguish between histories h^t and $\tilde{h}^t$.

2.2.4 Strategies and Payoffs

Fix a regime r. In period t, after the state θ^t is realized, the agents learn the state θ^t and the mechanism γ^t, which will be used in period t by the designer. The randomness of $r(\gamma|h^t)$ is resolved before agents send their messages. Hence, a pure[6] strategy, s_i of agent i selects a message $s_i(h^t, \theta^t, \gamma^t) \in \mathcal{M}_i^t$ for each $t \in T$ and each $(h^t, \theta^t, \gamma^t) \in H^t \times \Theta \times \Gamma$. Let $s = (s_1, \ldots, s_n)$ be a profile of messages.

The strategy profile s and the regime r together with the distribution of states of the world p induce a distribution over histories. Let $q(h^t|s, r)$ denote the probability that history h^t is realized given s and r. Define $q(h^0|s, r) = 1$. Given $q(h^t|s, r)$, $q((h^t, \theta^t, \gamma^t, m^t, a^t)|s, r) = q(h^t|s, r)p(\theta^t)r(\gamma^t|h^t)g^t(m^t)(a^t)$ if $s(h^t, \theta^t, \gamma^t) = m^t$ and $q((h^t, \theta^t, \gamma^t, m^t)|s, r) = 0$ otherwise. Given s and r, the payoff of agent i is

$$v_i(s|r) = (1 - \delta) \sum_{t \in T} \sum_{h^t \in H^t} \sum_{\theta^t \in \Theta} \sum_{\gamma^t \in \Gamma} \delta^t q(h^t|s, r)p(\theta^t)r(\gamma^t|h^t)u_i(g^t(s(h^t, \theta^t, \gamma^t)), \theta^t).$$

2.2.5 Repeated Implementation in Nash Equilibrium

A profile of strategies s is a Nash equilibrium if $v_i(s|r) \geq v_i((s_i', s_{-i})|r)$ for all i and s_i'. A regime r *repeatedly implements* a social choice function f if the set of Nash equilibria is non-empty and for each Nash equilibrium s, we have that $g^t(s(h^t, \theta^t, \gamma^t)) = f(\theta^t)$ in case of simply-stochastic regimes or $g^t(s(h^t, \theta^t, \gamma^t))(f(\theta^t)) = 1$ in case of doubly-stochastic regimes for all $t \in T$, all $\theta^t \in \Theta$, all h^t such that $q(h^t|s, r) > 0$, and all γ^t such that $r(\gamma^t|h^t) > 0$. A social choice function f is *repeatedly implementable* in Nash equilibrium if there exists a regime r that repeatedly implements f. The payoff of agent i if f is repeatedly implemented is $v_i^f := \sum_{\theta \in \Theta} p(\theta)u_i(f(\theta), \theta)$. Let $v^f = (v_1^f, \ldots, v_n^f)$.

[6]We only study Nash implementation in pure strategies.

3 Necessary Conditions and Theorem 1

In this section, we derive conditions that a social choice functions must satisfy for it to be repeatedly implementable. These conditions apply for any number of agents, $n \geq 2$. For an arbitrary set X, let $L_i(x, \theta)$ denote the lower contour set of agent i at outcome $x \in X$ in state θ. The set X can be a subset of $\Delta A \times V$, in which case the agent's preferences are described by $U_i(\cdot, \cdot, \theta)$, or X can be a subset of ΔA, in which case the agent's preferences are described by $u_i(\cdot, \theta)$. If the designer is only allowed to use deterministic mechanisms, then clearly X belongs to $A \times V$ or A.

A necessary condition for a function to be one-shot implementable is Maskin monotonicity due to Maskin (1999). We present it in a slightly modified form, which allows to compare it conveniently with other notions of monotonicity that will be defined later.

Definition 1 (Maskin monotonicity)**.** *f satisfies* Maskin monotonicity *with respect to $C = (C_i(\theta))_{i,\theta}$ if for each i and θ, $C_i(\theta) \subseteq L_i(f(\theta), \theta)$ and for all pairs (θ, θ^*), we have that (a) implies (b):*

a. $C_i(\theta) \subseteq L_i(f(\theta), \theta^*)$ holds for all i,

b. $f(\theta) = f(\theta^*)$.

Remark 1. *f is Maskin monotonic w.r.t. some C (i.e., there exists such C) if and only if f is Maskin monotonic w.r.t. $C = (L_i(f(\theta), \theta))_{i,\theta}$, which gives Maskin's original definition (Maskin, 1999).*[7]

The reason why Maskin monotonicity is necessary for one-shot implementation is that if all agents pretend in state θ^* that the state is θ and no agent has incentives to upset such a deception, then there is no hope for one-shot implementation unless $f(\theta) = f(\theta^*)$. Lee and Sabourian (2011) show that Maskin monotonicity is neither necessary nor sufficient for repeated implementation. One of the reasons is that the notion of deception becomes more complicated as it can take place over many periods. Mezzetti and Renou (2014) provide the right definition of monotonicity, which is necessary for repeated implementation. In the next subsections, we introduce the notion of (dynamic) deception and a slightly modified definition of dynamic monotonicity, which was originally proposed by Mezzetti and Renou (2014).

3.1 Deceptions

Suppose that $t \geq 1$. Let $\theta^{\to t} = (\theta^0, \ldots, \theta^{t-1})$ and $\Theta^{\to t}$ be the set of all such sequences. Let $\pi^t : \Theta \times \Theta^{\to t} \to \Theta$ be a deception in period t. That is, π^t specifies

[7] See, for example, the discussion in Moore and Repullo (1990) on page 1089.

a state $\theta' = \pi^t(\theta, \theta^{\to t})$ after any $\theta^{\to t}$ given that the period t state is θ. One can think of it as if after $\theta^{\to t}$ all the agents pretend that the period t state of the world is θ' while in fact it is θ. Let $\pi = (\pi^t)_{t\geq 0}$ be a *deception*, where $\pi^0 : \Theta \to \Theta$. We will refer to π^0 as a static deception. Also, let $\theta^{\to 0} = \emptyset$ and $\Theta^{\to 0} = \{\emptyset\}$. The continuation payoff of agent i, when the agents deceive according to π and the designer selects an alternative according to f, is

$$v_i^f(\pi) = (1-\delta)\sum_{t\in T}\sum_{\theta^{\to t}\in\Theta^{\to t}}\sum_{\theta^t\in\Theta}\delta^t p(\theta^{\to t})p(\theta^t)u_i(f(\pi^t(\theta^t,\theta^{\to t}),\theta^t),$$

where $p(\theta^{\to t}) = p(\theta^0)\cdot\ldots\cdot p(\theta^{t-1})$ and $p(\theta^{\to 0}) = 1$. Finally, given a deception π and some $\theta^{\to t}$ and θ, we denote by $\pi(\theta, \theta^{\to t})$ the continuation deception derived in the obvious way from π.[8]

3.2 Dynamic Monotonicity

To gain better intuition for dynamic monotonicity, we start by introducing a weaker version of Maskin monotonicity, which plays a useful role on its own in the sequel.

Definition 2 (Maskin monotonicity*)**.** *f satisfies* Maskin monotonicity* *with respect to $C = (C_i(\theta))_{i,\theta}$ if for each i and θ, $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta)$ and for all pairs (θ, θ^*), we have that (a) implies (b):*

a. $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta^*)$ holds for all i,

b. $f(\theta) = f(\theta^*)$.

Remark 2. *f is Maskin monotonic* w.r.t. some C if and only if f is Maskin monotonic* w.r.t. $C = (L_i((f(\theta), v^f), \theta))_{i,\theta}$. Sometimes we will suppress the sets with respect to which f is Maskin monotonic*. Also, f is Maskin monotonic* if and only if the function $(f(\cdot), v^f)$ is Maskin monotonic over the set $A \times V$.*

Remark 3. *Maskin monotonicity* is implied by Maskin monotonicity of f, but the converse is not true. Also, Maskin monotonicity* is necessary for repeated implementation (as argued below).*

One can think of the necessity of Maskin monotonicity* as follows. Suppose that there is a regime, which repeatedly implements f. Fix a Nash equilibrium of this regime. Let $C_i(\theta)$ represent the set of alternative and continuation payoff pairs that agent i can obtain by deviating from the equilibrium after some history when the state is θ, given that all other agents play according to the fixed Nash equilibrium. Now, as a thought experiment, suppose that f does not satisfy

[8] Note the distinction between a deception in period t, $\pi^t(\theta, \theta^{\to t})$ and a deception from period t *onwards*, $\pi(\theta, \theta^{\to t})$.

Definition 2 for C, that is, part (a) of Definition 2 holds for some pair (θ, θ^*) but we have that $f(\theta^*) \neq f(\theta)$. Consider now the following simple deception: in the initial period, and only in this period, all agents pretend that the state is θ when the true state is actually θ^* and they continue to play the Nash equilibrium strategies in the following periods as if they had not pretended in period 0. This is another Nash equilibrium, in which the alternative $f(\theta)$ is implemented in period 0 if the state is θ^* and from the next period on, the agents expect v^f since they play the original Nash equilibrium. It is an equilibrium because no agent has incentives to upset it in period 0 since, according to part (a) of Definition 2, they all prefer what they get from the deception compared to what they can get by demanding something from $C_i(\theta)$ for all i. No agent will also upset this Nash equilibrium from period 1 onwards because v^f is generated by a Nash equilibrium from period 1 on.

Notice, however, that Maskin monotonicity* is far from being a sufficient condition. The period 0 deception described above might be maintained by promising something better than v^f from period 1 on, which in turn can be obtained through deceptions, which support themselves. Hence the following definition:

Definition 3 (Dynamic Monotonicity (DM))**.** f *is* dynamically monotonic *with respect to* $C = (C_i(\theta))_{i,\theta}$ *if for each* i *and* θ*, we have* $C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta)$ *and for any deception* π*, we have that (1) implies (2):*

1. $C_i(\theta) \subseteq L_i((f(\theta), v^f(\pi(\theta^*, \theta^{\to t}))), \theta^*)$ *holds for all* $i \in I$*, all* $t \in T$*, all* $\theta^{\to t} \in \Theta^{\to t}$*, and all pairs* $(\theta, \theta^*) \in \Theta \times \Theta$ *for which* $\pi^t(\theta^*; \theta^{\to t}) = \theta$*,*

2. $f(\pi^t(\cdot; \theta^{\to t})) = f$ *holds for all* $t \in T$ *and all* $\theta^{\to t} \in \Theta^{\to t}$*.*

Remark 4. *f is dynamically monotonic w.r.t. some C if and only if f is dynamically monotonic w.r.t. $C = (L_i((f(\theta), v^f), \theta))_{i,\theta}$, which is the definition of dynamic monotonicity in Mezzetti and Renou (2014). Obviously, if f is dynamically monotonic, then it is Maskin monotonic*.*

Whether a function is dynamically monotonic or not depends on the regime considered. Namely, any function, which is DM for simply-stochastic regimes, is also DM for doubly-stochastic regimes. The converse, however, is not true. The reason is that for simply-stochastic regimes, $C_i(\theta)$ is a subset of $A \times V$, while for doubly-stochastic regimes, it is a subset of $\Delta A \times V$, and similarly for the lower contour sets. In the definition, we do not emphasize whether the sets $(C_i(\theta))_{i,\theta}$ belong to $A \times V$ or $\Delta A \times V$, but it should be understood from the regime, which one we are considering.

The proof of the following proposition follows directly from Theorem 1 in Mezzetti and Renou (2014) and from Remark 4 above.

Proposition 1 (Necessity of Dynamic Monotonicity)**.** *If f is repeatedly implementable, then f satisfies dynamic monotonicity with respect to some C.*

It turns out that dynamic monotonicity is also sufficient if the designer can use doubly-stochastic regimes as we prove in Corollary 1 in Section 4. However, if the designer is restricted to use only deterministic stage mechanisms (but possibly stochastic transitions), then we need additional conditions, which together with dynamic monotonicity are sufficient for repeated implementation. These conditions also turn out to be necessary and are derived in the spirit of Moore and Repullo (1990) in the next section.

3.3 Further Necessary Conditions for Simply-Stochastic Regimes

Here we use the assumption of strict preferences, but similar necessary conditions can be designed for weak preferences as well.

We start by introducing additional notation. For a set $X \subseteq A \times V$, let $M_i(X, \theta) = \arg\max_{x \in X} U_i(x, \theta)$. Note that $M_i(X, \theta)$ can be empty if X is not a closed set. Also, let $\overline{a}_i(\theta) = \arg\max_{a \in A} u_i(a, \theta)$, $\underline{a}_i(\theta) = \arg\min_{a \in A} u_i(a, \theta)$, $\overline{v}_i := \sum_{\theta \in \Theta} p(\theta) u_i(\overline{a}_i(\theta), \theta)$, and $\underline{v}_i := \sum_{\theta \in \Theta} p(\theta) u_i(\underline{a}_i(\theta), \theta)$. Finally, let $\underline{A}_i = \{\underline{a}_i(\theta) | \theta \in \Theta\}$.

Suppose that a social choice function f is repeatedly implementable using a simply-stochastic regime r. Take a strategy profile s that is a Nash equilibrium of the game induced by regime r. Consider a history h^t and a mechanism γ^t for some t such that $q(h^t|s, r) > 0$ and $r(\gamma^t|h^t) > 0$. Let $C_i(h^t, \theta, \gamma^t) \subseteq A \times V$ be the set of alternative and continuation payoff pairs that agent i can attain by deviating in period t given that the period t state is θ and the other agents follow s_{-i}. Let $C_i(\theta) = \cup_t \cup_{\{h^t | q(h^t|s,r)>0\}} \cup_{\{\gamma^t | r(\gamma^t|h^t)>0\}} C_i(h^t, \theta, \gamma^t)$.

Consider the following situation. Suppose that, on the one hand, there exist an agent i, a pair of states (θ, θ^*), and an alternative and continuation payoff pair (b, v) such that $(b, v) \in M_i(C_i(\theta), \theta^*)$, $v_i = \underline{v}_i$ and $b = \overline{a}_j(\theta^*)$ for all $j \in I \backslash \{i\}$. And, on the other hand, there exists a static deception $\pi^0 : \Theta \to \Theta$ such that $\underline{a}_i(\pi^0(\theta')) = \overline{a}_j(\theta')$ for all θ' and $j \in I \backslash \{i\}$. Then, it is easy to see that one can construct a new Nash equilibrium in the following way. There must be a history (h^t, θ, γ^t) for some t such that $q(h^t|s, r) > 0$, $r(\gamma^t|h^t) > 0$, and $(b, v) \in C_i(h^t, \theta, \gamma^t)$. Consider now history $(h^t, \theta^*, \gamma^t)$. Let agents $-i$ play after that history as if the state was θ instead of θ^* and let agent i "demand" (b, v). In the continuation, agents $-i$ pretend to play according to s_{-i}, using π^0 in each period. Let us denote the continuation payoff that agent i gets under such a strategy profile by $v_i(\pi^0)$, which can only be larger or equal to v_i since $v_i = \underline{v}_i$. Agent i cannot have a continuation strategy, which would increase his continuation payoff above $v_i(\pi^0)$, as otherwise he could already obtain a higher continuation payoff than v_i, when agents $-i$ do not pretend according π^0 but truly follow s_{-i}.[9] This joint

[9] In case of weak preferences, the argument is somewhat more involved, which leads to a slightly more complicated necessary condition.

deviation from s can be clearly maintained as a Nash equilibrium since agent i will be (weakly) worse off if he does not demand (b, v) and agents $-i$ play truly s_{-i} (i.e., because $(b, v) \in M_i(C_i(\theta), \theta^*)$). On the other hand, agents $-i$ get their best possible payoffs. Therefore, if f is repeatedly implementable, the following condition must necessarily hold:[10]

Condition $\lambda 0$. f satisfies $\lambda 0$ relative to $C = (C_i(\theta))_{i,\theta}$:

If for some agent i and some pair of states (θ, θ^*), there exists an alternative and continuation payoff pair $(b, v) \in M_i(C_i(\theta), \theta^*)$ such that $v_i = \underline{v}_i$ and $b = \overline{a}_j(\theta^*)$ for all $j \in I\backslash\{i\}$, and there exists a static deception $\pi^0 : \Theta \to \Theta$ such that $\underline{a}_i(\pi^0(\theta')) = \overline{a}_j(\theta')$ for all θ' and $j \in I\backslash\{i\}$, then $f(\theta^*) = b$ and $f(\theta') = \underline{a}_i(\pi^0(\theta')) = \overline{a}_j(\theta')$ for all θ' and $j \in I\backslash\{i\}$.

Similarly, consider the following situation. Suppose that there exist an agent i, a pair of states (θ, θ^*), and an alternative and continuation payoff pair (b, v) such that $(b, v) \in M_i(C_i(\theta), \theta^*)$, $v_i = \overline{v}_i$ and $b = \overline{a}_j(\theta^*)$ for all $j \in I\backslash\{i\}$ and it is true that $\overline{a}_i(\theta') = \overline{a}_j(\theta')$ for all θ' and $j \in I\backslash\{i\}$. Then again, given the Nash equilibrium s, we can construct another Nash equilibrium in the following way. There must be a history (h^t, θ, γ^t) for some t such that $q(h^t|s, r) > 0$, $r(\gamma^t|h^t) > 0$, and $(b, v) \in C_i(h^t, \theta, \gamma^t)$. Consider now history $(h^t, \theta^*, \gamma^t)$. Let agents $-i$ play after that history as if the state was θ and let agent i "demand" (b, v). In period t and in the continuation, all agents simply get their best possible alternatives. This is clearly a Nash equilibrium. Thus, if f is repeatedly implementable, then the following condition must also necessarily hold:

Condition $\lambda 1$. f satisfies $\lambda 1$ relative to $C = (C_i(\theta))_{i,\theta}$:

If for some agent i and some pair of states (θ, θ^*), there exists an alternative and continuation payoff pair $(b, v) \in M_i(C_i(\theta), \theta^*)$ such that $v_i = \overline{v}_i$ and $b = \overline{a}_j(\theta^*)$ for all $j \in I\backslash\{i\}$, and if it is true that $\overline{a}_i(\theta') = \overline{a}_j(\theta')$ for all θ' and $j \in I\backslash\{i\}$, then $f(\theta^*) = b$ and $f(\theta') = \overline{a}_j(\theta')$ for all θ' and $j \in I$.

The interested reader can compare these conditions (see also Example 1 in Section C) to the non-necessary conditions imposed by Lee and Sabourian (2011) (Assumption A and Condition ω) and by Mezzetti and Renou (2014) (no veto power or Assumption A). The main idea here is that we do not have to restrict attention to the lower contour sets but we look at the sets $(C_i(\theta))_{i,\theta}$, which are generated by some equilibrium of some regime. That is, it can be that the lower contour sets are not appropriate for implementation and one has to restrict what a "whistle blower" can demand when upsetting undesired equilibria without introducing new ones.

[10]See Example 2 for f that does not satisfy Condition $\lambda 0$.

3.4 Theorem 1

Conditions $\lambda 0$ and $\lambda 1$ are not only necessary, but together with dynamic monotonicity, are also sufficient for repeated implementation when $n > 2$. It is formally stated in the following theorem:

Theorem 1. *When $n > 2$, f is repeatedly implementable with a simply-stochastic regime if and only if there is a collection C with respect to which f is dynamically monotonic and relative to which f satisfies Conditions $\lambda 0$ and $\lambda 1$.*

Proof. To prove the only-if-part, consider C as constructed in Section 3.3. It immediately follows from Proposition 1 that f is dynamically monotonic with respect C. As argued in Section 3.3, f must also satisfy Conditions $\lambda 0$ and $\lambda 1$ relative to C. The proof of the if-part can be found in Section A in the Appendix. □

Remark 5. *The only-if-part of Theorem 1 also holds for $n = 2$.*

Now, we informally describe the regime used to prove the sufficiency part of Theorem 1. It is a modification of the regime in Mezzetti and Renou (2014). To highlight our modifications, we start by briefly outlining their regime. It starts in period 0 with a mechanism that is similar to the canonical mechanism used in one-shot implementation. The regime continues to employ this mechanism as long as there has been no disagreement in agents' messages in the past. If, in any period, there is a single agent whose message differs from the messages of the other agents and who demands an alternative and continuation payoff pair from his lower contour set, then the demanded alternative is implemented in that period. Further, a scalar $\lambda \in [0, 1]$ is calculated, which determines the probabilities of using two different mechanisms from the next period onwards. The scalar is calculated so that the deviator expects his demanded continuation payoff provided that with probability λ, he becomes a dictator forever and receives his highest continuation payoff, and with probability $1-\lambda$, he is punished forever and gets his lowest continuation payoff. For all other message profiles of the agents, the agent announcing the highest integer becomes a dictator forever with probability 1.

Unwanted equilibria, in which agents' reports differ, are taken care in Mezzetti and Renou (2014) by no-veto power assumption (or Assumption A). The main idea behind our modifications is that such equilibria can almost always be eliminated by creating competition between the agents if their messages differ. First, whenever $0 < \lambda < 1$, we additionally allow the designer to choose one of the alternatives forever with strictly positive, uniform probabilities. Because of strict preferences, the agents do not get their highest continuation payoffs with certainty, giving them incentives to trigger the integer game. Second, in the integer game, an agent becomes a dictator always with a probability strictly less than 1, although by announcing higher and higher integers, he can increase this probability. With the remaining probability, again a constant alternative is uniformly chosen forever. This eliminates any equilibria in the integer game.

The power of competition cannot be exploited only when a deviator demands his lowest or highest continuation payoff, i.e., when $\lambda = 0$ or $\lambda = 1$. In that case, Conditions $\lambda 0$ and $\lambda 1$ ensure that there are no undesirable equilibria. However, these conditions might be violated if the deviator is allowed to choose anything in his lower contour set. Therefore, our third modification is to replace the lower contour sets with smaller sets given by collection C that still preserve the dynamic monotonicity of the social choice function but, at the same time, satisfy Conditions $\lambda 0$ and $\lambda 1$. Taken together, these modifications allow us to dispense of no-veto power. Finally, any unwanted equilibria, in which agents' reports are unanimous, are taken care by dynamic monotonicity just as in Mezzetti and Renou (2014).

4 Repeated Implementation with Doubly Stochastic Regimes

In this section, we are going to show that if the designer can additionally use random stage mechanisms, then the premises of the cumbersome necessary Conditions $\lambda 0$ and $\lambda 1$, introduced in Section 3.3, become empty for some appropriately chosen C and, hence, these conditions are dispensable. Then, it immediately follows that dynamic monotonicity in itself is a necessary and sufficient condition for repeated implementation. Moreover, we can implement strictly more functions than with deterministic stage mechanisms. There are two independent reasons for this.

First, if we have to pick the sets $C = (C_i(\theta))_{i \in I, \theta \in \Theta}$ as subsets of $A \times V$, it can be that we do not find such a collection C with respect to which f is dynamically monotonic. However, if these sets can be chosen from $\Delta A \times V$, we can find such a collection for the same function.[11]

Second, it still can be that we find a collection C, consisting of subsets of $A \times V$, with respect to which f is dynamically monotonic, but relative to any such collection C, f does not satisfy either Condition $\lambda 0$ or $\lambda 1$. However, if the collection can be built from subsets of $\Delta A \times V$, then satisfying dynamic monotonicity and Conditions $\lambda 0$ and $\lambda 1$ might become possible. In fact, we show that whenever a function is dynamically monotonic with respect to some C, then there always exists another collection D consisting of sets from $\Delta A \times V$ such that our function is dynamically monotonic with respect to D and the premises of Conditions $\lambda 0$ and $\lambda 1$ become empty, that is, these conditions are satisfied

[11]Now, of course, in the definition of dynamic monotonicity, one should consider the lower contour sets corresponding to the preferences over the set $\Delta A \times V$.

automatically. Thus, we have the following result.

Corollary 1. *When $n > 2$, f is repeatedly implementable with a doubly-stochastic regime if and only if there is a collection $C = (C_i(\theta))_{i\in I,\theta\in\Theta}$ of subsets of $\Delta A \times V$ with respect to which f is dynamically monotonic.*

To prove this result, we connect the repeated implementation problem to the theory of repeated games and make use of some recent results. Then, we have our result as a simple corollary of the following lemmas and Theorem 1. We should stress that it is not obvious that one can innocuously replace a "whistle", i.e., a pair (a, v), which violates Conditions $\lambda 0$ or $\lambda 1$, with a nearby whistle without violating dynamic monotonicity. The reason is that a single whistle might be responsible for breaking infinitely many deceptions.[12]

4.1 A Repeated Game Associated to an Implementation Environment

Given the repeated implementation environment as described in Section 2, we now construct an associated repeated game with discounting, perfect monitoring, and random states. This game will be useful to show that we can always replace elements of the sets $(C_i(\theta))_{i,\theta}$ with nearby elements while preserving the dynamic monotonicity of f. Further, in Section 5, we will discuss how the dynamic monotonicity of f can be checked numerically with the help of this game.

Let us fix $C_i(\theta) \subseteq \cup_{\theta^*\in\Theta} M_i(L_i((f(\theta), v^f), \theta^*)$ for all i and θ such that there exists $(a, v) \in C_i(\theta) \cap M_i(L_i((f(\theta), v^f), \theta^*))$ for all θ^*, and $C_i(\theta)$ is minimal with respect to set inclusion; hence, $C_i(\theta)$ is finite. Let $C = (C_i(\theta))_{i,\theta}$. It is easy to see that f is dynamically monotonic with respect to lower contour sets if and only if it is dynamically monotonic with respect to C.

For any pair of states (θ, θ^*), with slight abuse of notation, let $u_i(\theta, \theta^*) = U_i(b, v, \theta^*)/(1 - \delta) = u_i(b, \theta^*) + \delta v_i/(1 - \delta)$ for some $(b, v) \in M_i(C_i(\theta), \theta^*)$. Note that $u_i(\theta, \theta^*)$ is the same for all $(b, v) \in M_i(C_i(\theta), \theta^*)$. Also, $u_i(\theta, \theta) = U_i(f(\theta), v^f, \theta)/(1 - \delta)$.

The repeated game Γ^C, given C, is as follows. A state $\theta \in \Theta$ is drawn each period i.i.d. according to p, each player i corresponds to agent i, and the action sets of the stage game are $\mathcal{A}_i = \mathcal{A} = \Theta \cup \{\omega, o\}$. Without loss of generality, we can assume that $u_i(a, \theta) > 0$ for all i, a, and θ in the implementation problem. Then, for an action profile $x \in \mathcal{A}^n$ in state θ^*, the stage game payoffs of the players are defined as follows:

1. $u_i(x, \theta^*) = u_i(f(\theta), \theta^*)$ if $x_j = \theta$ for all j. (In the implementation problem,

[12] An alternative proof is also available that does not require building an associated repeated game. However, in that proof, the collection C that vacuously satisfies Conditions $\lambda 0$ and $\lambda 1$ consists of open sets. Besides, the current proof is also useful for discussion in Section 5.

it corresponds to the situation when everyone claims that the state is θ while the true state is θ^*.)

2. $u_i(x, \theta^*) = u_i(\theta, \theta^*)$ if $x_j = \theta$ for all $j \in I \backslash \{i\}$ and $x_i = \omega$. (It corresponds to the situation when all but agent i claim that the state is θ but the true state is θ^*, while agent i "blows the whistle" and demands an element in $M_i(C_i(\theta), \theta^*)$.)

3. $u_i(x, \theta^*) = 0$ if $x_i = o$.

4. $u_i(x, \theta^*) \ll 0$ for any other $x \in \mathcal{A}^n$, which does not fall under any of the above points. (When we write that a payoff is $\ll 0$, we mean that it is so negative that the corresponding action profile can never be played on a Nash equilibrium path of the repeated game.)

For a numerical example on how to calculate the stage game payoffs, see Example 3 in Section C.

Given a sequence of actions, $(x^t(\theta))_{t \in T, \theta \in \Theta}$, the payoff of player i in the repeated game is given by $(1-\delta) \sum_{t \in T} \sum_{\theta \in \Theta} \delta^t p(\theta) u_i(x^t(\theta), \theta)$. We are going to look at the set of subgame perfect Nash equilibrium payoffs, $\mathcal{E}(\Gamma^C)$ of the repeated game Γ^C, which in our case coincides with the set of Nash equilibrium payoffs of Γ^C.[13]

4.2 The Set of Equilibrium Payoffs of the Repeated Game

Consider an arbitrary function f, the sets C, which are derived from the lower contour sets as described in the previous section, and the associated repeated game Γ^C. As C is fixed in the sequel, we simply write $\Gamma = \Gamma^C$. We say that $D = (D_i(\theta))_{i,\theta}$ is a *good collection* if $D_i(\theta) \subseteq L_i((f(\theta), v^f)), \theta) \subseteq \Delta A \times V$ is compact for all i and θ. In particular, C is a good collection. Then, for any good collection D, one can define an associated game Γ^D as before. In particular, we can pick any element $(b, v) \in M_i(D_i(\theta), \theta^*)$ for each i, θ, θ^* and calculate $u_i^D(\theta, \theta^*) = U_i(b, v, \theta^*)/(1-\delta) = u_i(b, \theta^*) + \delta v_i/(1-\delta)$. We also specify payoffs in Γ^D, which are $\ll 0$, to be the same as in Γ.

For any two vectors k, k' from some finite dimensional vector space $\mathbb{R}^N$, denote by $d(k, k') = \max_{i=1,\dots,N} |k_i - k'_i|$ the distance between k and k'. Then, for a compact set X in this vector space, the ε-fattening of X is $X^\varepsilon = \cup_{k \in X} \{k' \in \mathbb{R}^N | d(k, k') \leq \varepsilon\}$. We say that D is $\xi > 0$ close to C (which we have fixed before) if it is a good collection and $d(\Gamma^D, \Gamma) \leq \xi$. (By $d(\Gamma^D, \Gamma)$, we mean the distance between the stage game payoffs of the two games across all players, states, and

[13]We have to look at subgame perfect Nash equilibria because the operator, introduced by Abreu, Pearce, and Stacchetti (1990), does not work for the set of Nash equilibria in general, which was pointed out to us by George Mailath.

action profiles. This distance is well-defined because both games have the same dimensions.)

For all pairs of states, (θ, θ^*), denote by $Q_\Gamma(\theta, \theta^*) = \{v \in V | \forall i : u_i(\theta, \theta^*) \leq u_i(f(\theta), \theta^*) + \delta/(1-\delta) v_i\}$ the compact set of continuation payoffs for which it is incentive compatible for all players to play action θ in the stage game of Γ when the state is θ^*. Similarly, let $Q_{\Gamma^D}(\theta, \theta^*) = \{v \in V | \forall i : u_i^D(\theta, \theta^*) \leq u_i(f(\theta), \theta^*) + \delta/(1-\delta) v_i\}$.

It must be obvious that the players can play θ in state θ^* on an equilibrium path of Γ^D for some good collection D if and only if $Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma^D) \neq \emptyset$. Hence, the following lemma holds trivially:

Lemma 1. *f is dynamically monotonic with respect to a good collection D if and only if for all θ, θ^* with $\theta \neq \theta^*$, $Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma^D) = \emptyset$.*

We have the following result:

Lemma 2. *f is dynamically monotonic with respect to a good collection D if and only if:*

1. *f is Maskin monotonic* with respect to D,*
2. *v^f is the unique strictly efficient equilibrium payoff of Γ^D.*

Proof. Suppose f is dynamically monotonic with respect D. Then it is trivially Maskin monotonic* with respect to D. To establish that v^f is the unique strictly efficient equilibrium payoff, we first claim that $(0, \ldots, 0)$ and v^f are equilibrium payoffs of Γ^D. $(0, \ldots, 0)$ is obtained by all players playing o in each stage in every state. v^f is attained if in each stage, all players play the action that coincides with the state of the world of that stage. This can be supported as an equilibrium if after any deviation, all players play o forever. Therefore, the deviator gets the same payoff that he would obtain if he followed the original strategy. Now, by construction, in any equilibrium of Γ^D, only the action profiles on the diagonal of the stage game can be played. Even more, from Lemma 1, it follows that in equilibrium, at any stage, either all players must choose o or θ where θ is the state of the world of that stage. However, any equilibrium strategy that involves playing o by all players at some stage, will result in equilibrium payoffs $v < v^f$. This establishes point 2 in the lemma.

On the other hand, suppose that points 1 and 2 are satisfied but f is not dynamically monotonic with respect D. By Lemma 1, this means that there is a pair (θ, θ^*) such that $\theta \neq \theta^*$ and $Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma^D) \neq \emptyset$. But then because v^f is the unique efficient equilibrium payoff, it follows that $v^f \in Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma^D)$, which in turn contradicts Maskin monotonicity* of f with respect to D. $\square$

Lemma 3. *If f is dynamically monotonic with respect to C, then there exists $\varepsilon > 0$ such that for all θ, θ^* with $\theta \neq \theta^*$, it is true that $Q_\Gamma(\theta, \theta^*) \cap \mathcal{E}(\Gamma)^\varepsilon = \emptyset$. Moreover, there is $\xi > 0$ such that $Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma)^\varepsilon = \emptyset$ if D is $\xi-$close to C.*

Proof. Due to finiteness of the action and state spaces of the stage game, we can find $\varepsilon > 0$ such that for all θ, θ^* with $\theta \neq \theta^*$, we have that $\{v^f\}^\varepsilon \cap Q_\Gamma(\theta, \theta^*) = \emptyset$. We know from point 2 of Lemma 2 (by taking $D = C$) that $v < v^f$ for all $v \in \mathcal{E}(\Gamma) \backslash \{v^f\}$. This implies that $\{v\}^\varepsilon \cap Q_\Gamma(\theta, \theta^*) = \emptyset$ for all $v \in \mathcal{E}(\Gamma)$. This establishes the first claim of the lemma. Finally, by compactness, there is $\xi > 0$ such that $Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma)^\varepsilon = \emptyset$ if D is $\xi-$close to C. □

Lemma 4. *If f is dynamically monotonic with respect to C, then there exists $\xi > 0$ such that f is dynamically monotonic with respect to any good collection D, which is ξ-close to C.*

Proof. Suppose f is dynamically monotonic with respect to C. Due to finiteness of the state and action spaces, it follows from the upper-semicontinuity of $\mathcal{E}(\cdot)$ (see Propositions 19 and 20 in Plan (2014)[14]) that for any $\varepsilon > 0$, there is $\xi > 0$ such that $\mathcal{E}(\Gamma^D) \subset \mathcal{E}(\Gamma)^\varepsilon$ if $d(\Gamma^D, \Gamma) \leq \xi$. By Lemma 3, if $\varepsilon > 0$ and $\xi > 0$ are small enough, we have that for all θ, θ^* with $\theta \neq \theta^*$, $Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma)^\varepsilon = \emptyset$ and, hence, $Q_{\Gamma^D}(\theta, \theta^*) \cap \mathcal{E}(\Gamma^D) = \emptyset$. Lemma 1 completes the proof. □

4.3 Proof of Corollary 1

We only have to prove sufficiency. Take a social choice function f, which is dynamically monotonic. Consider the collection C as constructed in the very beginning of Section 4.1. If for all i and θ, we have that $C_i(\theta)$ does not contain the alternative and continuation payoff pairs, in which agent i gets his minimal or maximal continuation payoff, then Corollary 1 follows directly from Theorem 1 as f vacuously satisfies Conditions $\lambda 0$ and $\lambda 1$ relative to C. If this is not the case, then we are going to replace $C_i(\theta)$ with some $D_i(\theta)$ for all i and θ so as to preserve dynamic monotonicity of f with respect to $D = (D_i(\theta))_{i,\theta}$, while at the same time no player can ever demand his worst or best continuation payoff (i.e., Conditions $\lambda 0$ and $\lambda 1$ become empty). Then the proof of Corollary 1 again follows from Theorem 1 by setting $C = D$. We only have to show how to replace some elements of the sets of C to obtain D.

Suppose that for some $(b, v) \in C_i(\theta)$, we have that v_i is the maximal continuation payoff of agent i. But then v_i can be slightly decreased and, due to Lemma 4,[15] (b, v) can be replaced with (b, v'), which eliminates the case when the premise of Condition $\lambda 1$ applies.

The case of Condition $\lambda 0$ is slightly more involved and requires the use of stochastic stage mechanisms. Suppose that for some $(b, v) \in C_i(\theta)$, we have that v_i is the minimal continuation payoff of agent i. Now, if $(f(\theta), v^f)$ is strictly preferred by agent i to (b, v) at θ, then v_i can be slightly increased (while ensuring

[14] The upper-semicontinuity of $\mathcal{E}(\cdot)$ is established in Plan (2014) only for games without random states but the proof also works for games with random states.

[15] $C_i(\theta)$ is finite for all i and θ, and we obtain $D_i(\theta)$ by replacing a finite number of elements in $C_i(\theta)$. Therefore, D is a good collection and we can invoke Lemma 4.

that $(f(\theta), v^f)$ is still the best for agent i under θ) and (b, v) can be replaced with $(b, v') \in L_i((f(\theta), v^f), \theta)$ where $v'_i > v_i$. This maintains the dynamic monotonicity of f, again due to Lemma 4. The interesting case is when agent i is indifferent between $(f(\theta), v^f)$ and (b, v) in state θ. There are two cases to consider. If $v_i = v_i^f$, then by strict preferences,[16] we have that $b = f(\theta)$, in which case (b, v) is never used to destroy a deception. If $v_i \neq v_i^f$, then it cannot be that b is the worst alternative for i at θ. In this case, we replace (b, v) with a close enough alternative and continuation payoff pair $(\tilde{b}, v') \in \Delta A \times V$ by slightly increasing the continuation payoff and by slightly decreasing the expected payoff from the alternative, while ensuring $(\tilde{b}, v') \in L_i((f(\theta), v^f), \theta)$. Due to Lemma 4, this can be done without losing dynamic monotonicity. This eliminates the case when the premise of Condition $\lambda 0$ applies.

Finally, given the constructed good collection D, Corollary 1 simply follows from Theorem 1 since, by Lemma 4, f is dynamically monotonic with respect to D and f satisfies Conditions $\lambda 0$ and $\lambda 1$ relative to D vacuously.

5 Testing DM by Using Repeated Games

It is obvious from the definition of DM that to check in practice whether f is DM or not, one has to consider infinity of possible deceptions, which is impossible (unless one is lucky to find a deception, which does not satisfy the condition required for DM). In this section, we informally discuss existing results from the literature of repeated games, which can help in verifying DM, given the connection between DM and the equilibrium payoff set of the the associated repeated game that we have established in Section 4.2.

In what follows, we assume that the agents have access to a public randomization device, which draws a uniformly distributed random variable from the unit interval independently in each period. Now, of course, one should consider also random deceptions in the definition of DM. It should be clear that the set of functions, which are DM, decreases when the agents can play correlated pure strategies. However, Theorem 1 and Corollary 1 continue to hold with public randomization, as do Lemmas 1 to 4.

Lemma 2 tells us that DM of a function can be checked by finding the equilibrium payoff set of the associated repeated game with random states and a fixed discount factor, which is described in Section 4.1. To find the equilibrium payoff set, one can apply the methods developed in Abreu, Pearce, and Stacchetti

[16] Notice that this is the only place where we use the assumption of strict preferences. If preferences are not strict, then it can be that agent i is indifferent between $f(\theta)$ and b at θ, and (b, v) is used to destroy a deception at some θ^*. A simple, though, extra assumption would suffice to circumvent such situations, namely, requiring that there is a unique worst alternative for each agent at each θ.

(1990) and Judd, Yeltekin, and Conklin (2003).[17] While the method of Abreu, Pearce, and Stacchetti (1990) is an outer approximation of the equilibrium set, Judd, Yeltekin, and Conklin (2003) offer both outer and inner approximations of the equilibrium set. If a social choice function is DM, then according to Lemma 2, the outer and inner approximations should[18] converge to a set in which v^f is strictly efficient. On the other hand, if the social choice function is not DM, then again according to Lemma 2, the outer and inner approximations should converge to a set in which v^f is not strictly efficient.

Finally, we must mention a recent result in Abreu and Sannikov (2014), which states that there are finitely many extreme points of the equilibrium payoff set of the repeated game if the number of agents is two. Although their result is without random states, it easily extends to repeated games with random states. A closer look at their result (see their proof of Theorem 4) tells that if we are only interested in the efficient equilibria (since it is enough to consider efficient deceptions), any efficient equilibrium payoff can be achieved in strategies that use public randomization, in which the current period's randomization is solely determined by the realization of the previous period's randomization. In our case, the randomization should also depend on the realized states in the current and previous periods. That is, to check dynamic monotonicity of a function, there is no need to check all deceptions. It is enough if one checks all deceptions with public randomization, which are Markov in the current and previous period's states and in the realization of the previous period's public randomization. In case the number of players is more than 2, it is easy to construct examples of efficient equilibria that are not Markov (as specified above) but use longer memory.

6 Implementing Efficient Functions

Here we connect Lee and Sabourian (2011)'s notion of efficiency in the range to dynamic monotonicity.

Definition 4 (Efficiency in the range). *Let $V^f = \{v \in V | \exists \pi^0 : \Theta \to \Delta\Theta : v = \sum_{\theta \in \Theta} p(\theta) u(f(\pi^0(\theta)), \theta)\}$. f is* efficient *(resp.,* weakly efficient*) in the range if there is no $v \in V^f$ such that $v_i \geq v_i^f$ (resp., $v_i > v_i^f$) for all i and $v_j > v_j^f$ for some j. That is, f is efficient in the range when v^f is Pareto efficient within the*

[17] Abreu, Pearce, and Stacchetti (1990)'s result is established without public randomization but covering the case of random states, while Judd, Yeltekin, and Conklin (2003)'s result is established with public randomization but without covering the case of random states. Both results easily extend to our setup.

[18] Judd, Yeltekin, and Conklin (2003) present several examples in which their method converges to the true equilibrium set. However, as the speed of convergence is unknown, one has no guarantee that the almost convergence is reached after some given, finite number of iterations. Abreu and Sannikov (2014) offer a modification of Judd, Yeltekin, and Conklin (2003) algorithm, which improves its performance but one must still count with the same problem.

set V^f. f is strictly efficient in the range *if it is efficient in the range and there does not exist $\pi^0 : \Theta \to \Delta\Theta$, with π^0 being different from the identity map, such that $v^f = \sum_{\theta\in\Theta} p(\theta)u(f(\pi^0(\theta)), \theta)$.*

Lee and Sabourian (2011) in their Theorem 1 show that if f is not weakly efficient in the range, then f is not implementable for δ sufficiently large. To understand this result, one can think that the agents use a stationary (i.e., history independent) random deception, which strictly Pareto dominates v^f.[19] On the other hand, for their sufficiency result, Lee and Sabourian (2011) in their Theorem 2 require that f is strictly efficient in the range to obtain outcome implementation. Further, they invoke their Assumption A and Condition ω. In the following proposition, we improve on this result by requiring only efficiency in the range and Maskin monotonicity*, where the latter is a necessary conditions for implementation.

Proposition 2. *If f is efficient in the range and Maskin monotonic*, then f is dynamically monotonic and, hence, it is repeatedly implementable with a doubly-stochastic regime when $n > 2$.*

Proof. The proof follows exactly the argument in the proof of Remark 4 in Mezzetti and Renou (2014), but instead of Maskin monotonicity (which is not necessary), Maskin monotonicity* (which is necessary) is used in the last step. Corollary 1 then completes the proof. □

Since verifying efficiency in the range and Maskin monotonicity* is relatively easy, the result of Proposition 2 offers a simple way to confirm DM of f, especially, when δ is large.

6.1 An Application

As an application of Proposition 2, we study the repeated implementation of (generalized) utilitarian social choice functions.

Definition 5. *A social choice correspondence, f^u, is* utilitarian *if there exists $(\beta_i)_{i\in I}$ such that for each $\theta \in \Theta$,*

$$f^u(\theta) = \left\{ a : a \in \arg\max_{b\in A} \sum_{i\in I} \beta_i u_i(b, \theta) \right\},$$

and $\beta_i \geq 0$ for all i, and $\beta_j > 0$ for some j.

Remark 6. *f^u is ex ante weakly efficient and, hence, it is weakly efficient in the range. If $f^u(\theta)$ is single-valued for all θ, then f^u is (strictly) efficient in the range.*

[19] Lee and Sabourian (2011) dispense with public randomization by invoking the result of Fudenberg and Maskin (1991).

The one-shot implementation of the utilitarian social choice correspondences, especially in dominant strategies, has been extensively studied in the literature (see, for example, Groves (1973)). We show that they are repeatedly implementable when we restrict the domain of preferences as in Laffont and Maskin (1982). Specifically, we assume that $A = [0, 1]$, and for each i and θ, $u_i(a, \theta)$ is strictly concave and differentiable function in the first argument and it attains its maximum for some $a \in (0, 1)$. We also assume that in each state θ, $f^u(\theta)$ is single-valued. Thus, $f^u(\theta)$ is a solution to $\sum_{i \in I} \beta_i u_i'(f^u(\theta), \theta) = 0$ (where the derivative is taken with respect to the first argument). The assumptions also imply that $f^u(\theta) \in (0, 1)$ for all θ and $v_i^{f^u} > \underline{v}_i$ for all i. Unlike Laffont and Maskin (1982), we rule out monetary transfers, and we maintain the assumption that the state space Θ is finite.

The utilitarian social choice function does not need to be Maskin monotonic, but Laffont and Maskin (1982) have shown that a social choice *rule* consisting of the utilitarian social choice function f^u and, for example, a constant private transfer function, is Maskin monotonic (see Theorem 5 and its proof in Laffont and Maskin (1982)). The following proposition, in essence, establishes that in the repeated setup, the continuation payoffs can play the role of monetary transfers. All we have to make sure of is that the transfers can be chosen to be arbitrarily small.

Proposition 3. *f^u is Maskin monotonic*.*

Proof. Suppose, first, that $u_i'(f^u(\theta), \theta) = u_i'(f^u(\theta), \theta')$ for all i with $\beta_i > 0$. It follows that $\sum_{i \in I} \beta_i u_i'(f^u(\theta), \theta') = 0$ holds, implying that $f^u(\theta') = f^u(\theta)$. Therefore, suppose $u_i'(f^u(\theta), \theta) \neq u_i'(f^u(\theta), \theta')$ for some i with $\beta_i > 0$. We will argue that there exists (a, v_i) such that $u_i(f^u(\theta), \theta) + v_i^{f^u} \geq u_i(a, \theta) + v_i$ and $u_i(f^u(\theta), \theta') + v_i^{f^u} < u_i(a, \theta') + v_i$. Because of the strict concavity of $u_i(\cdot, \theta)$, we have that

$$u_i(f^u(\theta), \theta) > u_i(f^u(\theta) + \eta, \theta) - u_i'(f^u(\theta), \theta)\eta.$$

Likewise, by taking the Taylor expansion of $u_i(\cdot, \theta')$ around $f^u(\theta)$, we have

$$u_i(f^u(\theta), \theta') = u_i(f^u(\theta) + \eta, \theta') - u_i'(f^u(\theta), \theta')\eta + h(\eta)\eta$$

where $\lim_{\eta \to 0} h(\eta) = 0$. By choosing $\eta > 0$ when $u_i'(f^u(\theta), \theta') > u_i'(f^u(\theta), \theta)$ and $\eta < 0$ when $u'(f^u(\theta), \theta') < u'(f^u(\theta), \theta)$, we have that

$$u_i(f^u(\theta), \theta') < u_i(f^u(\theta) + \eta, \theta') - u_i'(f^u(\theta), \theta)\eta$$

since the second order effect of $h(\eta)\eta$ can be ignored if $|\eta|$ is small enough. To summarize, we can set $a = f^u(\theta) + \eta$ and $v_i = v_i^{f^u} - u_i'(f^u(\theta), \theta)\eta$ where the sign of η is determined as before. Note that v_i can be chosen sufficiently close to $v_i^{f^u}$ to ensure $\underline{v}_i < v_i < \overline{v}_i$ as long as $v_i^{f^u} < \overline{v}_i$ holds. If $v_i^{f^u} = \overline{v}_i$, then one can set $a = f^u(\theta')$ and $v_i = v_i^{f^u}$ since $f^u(\cdot) = \overline{a}_i(\cdot)$ and, consequently, $u_i(f^u(\theta), \theta) >$

$u_i(f^u(\theta'), \theta)$ and $u_i(f^u(\theta), \theta') < u_i(f^u(\theta'), \theta')$ due to strict concavity must hold. Hence, whenever part (b) of Definition 2 does not hold, part (a) also does not hold for some i. □

From Proposition 2 and Remark 6, it then follows that f^u is dynamically monotonic and is repeatedly implementable when $n > 2$. On the other hand, f^u does not need to satisfy no-veto power, as Example 4 in Section C illustrates. Therefore, one cannot apply the results of Mezzetti and Renou (2014) to show that f^u is repeatedly implementable. In fact, the example demonstrates that the regime of Mezzetti and Renou (2014) possesses an undesirable equilibrium. Also, while f^u is strictly efficient in its range, Condition ω of Lee and Sabourian (2011) does not need to hold. Therefore, their Theorem 2 cannot also be invoked to establish that f^u is implementable.

7 The Case of Two Agents

In this section, we assume that the number of agents is $n = 2$. While we have not identified sufficient and necessary conditions for implementation when $n = 2$, we offer sufficient conditions that improve on the existing results in literature.

Clearly, any necessary and sufficient conditions for the two agent case must include the necessary and sufficient conditions for the case of more than two agents. However, when $n = 2$, the designer must additionally and independently of other conditions face the problem that the agents are sending different messages but the "deviator" cannot be identified. In the static setup, this is known as the self-selection problem and the corresponding necessary condition has been identified by Moore and Repullo (1990) and Dutta and Sen (1991).

For our first result, we are going to use a version of static self-selection.

Definition 6 (Self-selection*). *f satisfies* self-selection* *relative to $C = (C_i(\theta))_{i,\theta}$ if for each i and θ, we have $(f(\theta), v^f) \in C_i(\theta) \subseteq L_i((f(\theta), v^f), \theta) \subseteq \Delta A \times V$ and for all pairs (θ_1, θ_2),*

1. *There exists $b(\theta_1, \theta_2) \in \Delta A$ such that $(b(\theta_1, \theta_2), v^f) \in C_1(\theta_2) \cap C_2(\theta_1)$;*
2. *If for some θ^*, it is true that $(b(\theta_1, \theta_2), v^f) \in M_1(C_1(\theta_2), \theta^*) \cap M_2(C_2(\theta_1), \theta^*)$, then $f(\theta^*) = b(\theta_1, \theta_2)$.*

Part 1 in Definition 6 is just a static self-selection condition that has also been assumed by Lee and Sabourian (2011).[20] It ensures that the designer can pick an alternative when observing contradictory messages in a way that does not upset the truth-telling equilibrium, i.e., if one agent tells the truth about the state, the

[20]Lee and Sabourian (2011) require that there exists $b(\theta_1, \theta_2) \in L_1(f(\theta_2), \theta_2) \cap L_2(f(\theta_1), \theta_1)$. However, if such $b(\theta_1, \theta_2)$ exists, then one can always include $(b(\theta_1, \theta_2), v^f)$ into both $C_1(\theta_2)$ and $C_2(\theta_1)$.

other agent cannot gain by lying about the state if $b(\theta_1, \theta_2)$ is chosen. Part 2 can be thought of as a variation of Maskin monotonicity*. It says that if neither agent has incentives to deviate in some state in some period when they receive $b(\theta_1, \theta_2)$ in that period and expect a payoff of v^f in the continuation, then $b(\theta_1, \theta_2)$ must be the desirable alternative according to f in that state. Neither part 1 nor part 2 of Definition 6 is, however, necessary for dynamic implementation. Part 1 is unnecessary for a similar reason that Maskin monotonicity of f is not necessary in the dynamic setup. Part 2 is not necessary because the agents might not be able to get v^f from the next period on since the designer can react to the contradictory messages by "changing" the regime.

Proposition 4. *If f is efficient and satisfies Maskin monotonicity* with respect to some $C = (C_i(\theta))_{i,\theta}$ and self-selection* relative to that C, then f is repeatedly implementable with a doubly-stochastic regime.*

Proof. The detailed proof appears in Section B.1 in the Appendix. Here, we just give a sketch of the proof. First, we show that for doubly-stochastic regimes, the collection C can be modified in a way that f also satisfies Conditions $\lambda 0$ and $\lambda 1$, while preserving self-selection* and Maskin monotonicity* with respect to the modified sets. This is done similarly to Corollary 1, except that we do not need to invoke Lemma 4 because to preserve Maskin monotonicity*, we only need to satisfy finitely many inequalities when modifying the sets. Besides, now we also need to ensure that self-selection* continues to hold. Second, we slightly modify our regime, which was designed for the case of at least 3 agents. The modification takes care of the case when the designer observes contradictory messages in some period, but the deviator cannot be identified. We specify that in such case, the corresponding $b(\theta_1, \theta_2)$ is implemented in that period (where θ_i is the state announced by agent i), while in the next period, the regime continues as if there has been no contradiction in messages. Third, we show that in any equilibrium in period 0, both agents should expect a continuation payoff of v^f due to efficiency of f. Finally, Maskin monotonicity* and part 2 in the definition of self-selection* ensure that the right alternative is implemented in period 0. The same argument can then be applied for any period. $\square$

Couple observations are in order. First, we cannot weaken the assumption of efficiency to efficiency in the range in Proposition 4 without strengthening the definition of self-selection*. If alternative $b(\theta_1, \theta_2)$ does not belong to the range of f, then it can be that both agents prefer this alternative to what f would give them in some state. Consequently, nothing guarantees that the equilibrium continuation payoffs do not weakly Pareto dominate v^f if f is only efficient in the range. However, we can assume that f is efficient in the range in Proposition 4 if we additionally require in part 1 of Definition 6 that $b(\theta_1, \theta_2) \in f(\Theta)$ where $f(\Theta) = \{a \in A | a = f(\theta) \text{ for some } \theta \in \Theta\}$ denotes the range of f.

Second, Lee and Sabourian (2011) in their Theorem 3 state that f is payoff implementable from the second period on if f satisfies self-selection (part 1 in Definition 6), efficiency in the range and their usual domain restrictions.[21] The discussion in the previous paragraph tells that their assumptions about f are not sufficient to guarantee the payoff implementation.[22]

Third, we cannot replace efficiency and Maskin monotonicity* with dynamic monotonicity in Proposition 4 without strengthening the definition of self-selection* even further. It is because efficiency (in the range) pins down the continuation payoffs while dynamic monotonicity does not. Nevertheless, we show that Corollary 1 carries over to the two agent case if additionally we assume the existence of a bad outcome (for its use, see, e.g., Moore and Repullo (1990) or Mezzetti and Renou (2014)).

Definition 7 (Bad Outcome)**.** *$\tilde{b} \in \Delta A$ is a* bad outcome *relative to f if $u_i(\tilde{b}, \theta') < u_i(f(\theta), \theta')$ for all i, θ, and θ'.*

To prove their Theorem 3, besides the bad outcome, Mezzetti and Renou (2014) also invoke Assumption A. We improve on their result by dispensing of Assumption A.[23]

Proposition 5. *If there exists a bad outcome $\tilde{b}$ relative to f and f is dynamically monotonic, then f is repeatedly implementable with a doubly-stochastic regime.*

Proof. The detailed proof appears in Section B.2. Here, we give a sketch of the proof. We modify the regime used in the proof of Proposition 4. First, obviously, instead of implementing $b(\theta_1, \theta_2)$ in the case of disagreement, we now implement the bad outcome. Second, while in the proof of Proposition 4, the designer proceeds in the next period as if there was no disagreement in the previous period, here we require that the bad outcome is implemented forever. Then, in any equilibrium, it must be that the agents send the same messages, in which case dynamic monotonicity ensures that the right alternative is implemented. $\square$

Similar to self-selection*, having a bad outcome is not necessary for repeated implementation. Mezzetti and Renou (2014) show that what they call dynamic self-selection is necessary for repeated implementation. Essentially, it means that function $f'(\cdot) := (f(\cdot), v^f)$ rather than $f(\cdot)$ must satisfy self-selection. However, even if one assumes that $f'(\cdot)$ satisfies self-selection, it is not immediate how the designer can implement the desired continuation payoffs in the case of disagreement.[24] Finding the necessary and sufficient conditions for repeated implementation in the two agent case is left for future work.

[21] f is payoff implementable from period t if in any Nash equilibrium, the equilibrium payoffs are v^f from period t on.

[22] In Part (iii) of Lemma 3 in Lee and Sabourian (2011), it is only implicitly assumed that $b(\theta_1, \theta_2) \in f(\Theta)$.

[23] Even if we only allow for simply-stochastic regimes, we can still improve by replacing unnecessary Assumption A with necessary Conditions $\lambda 0$ and $\lambda 1$.

[24] For large discount factors, one could possibly apply Blackwell's approachability result.

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A Proof of Theorem 1

We only need to prove the sufficiency part of Theorem 1. The proof is constructive – we define a regime and show that it implements f. Fix $C = (C_i(\theta))_{i,\theta}$ with respect to which f is dynamically monotonic and f satisfies Conditions $\lambda 0$ and $\lambda 1$ relative to C. In our regime, we are going to use the following deterministic stage mechanisms.

Mechanism $\hat{\gamma}$. For each agent $i \in I$, let the message space of agent i be $\mathcal{M}_i = \{(\theta, b, v, z) \in \Theta \times A \times V \times \mathbb{Z}_+\}$, where $\mathbb{Z}_+$ denotes the set of nonnegative integers. For a message profile m, let i_z denote an agent who sends the highest integer z, and let $m_{i_z} = (\theta_z, b_z, v_z, z_z)$. Given $C = (C_i(\theta))_{i,\theta}$, define the allocation rule g as follows:

I. If there exists $(\theta, b, v, z) \in \Theta \times A \times V \times \mathbb{Z}_+$ such that $m_i = (\theta, b, v, z)$ for all $i \in I$, then $g(m) = f(\theta)$.

II. If there exists $(\theta, b, v, z) \in \Theta \times A \times V \times \mathbb{Z}_+$ and $i^* \in I$ such that $m_i = (\theta, b, v, z)$ for all $i \neq i^*$ and $m_{i^*} = (\theta', b', v', z') \neq (\theta, b, v, z)$, then

   (a) $g(m) = b'$ if $(b', v') \in C_{i^*}(\theta)$.

   (b) $g(m) = f(\theta)$ otherwise.

III. If neither (I) nor (II) applies, then $g(m) = b_z$.

The second mechanism is a dictatorial mechanism for agent i:

Mechanism $\tilde{\gamma}_i(\mathcal{M}_i)$. Let $\mathcal{M}_i \subseteq A$, while $\mathcal{M}_j = \{\emptyset\}$ for each $j \in I \backslash \{i\}$. Let $g(m) = m_i$.

In case $\mathcal{M}_i = \{a\}$ for some $a \in A$ and $\mathcal{M}_j = \{\emptyset\}$, we refer to it as a constant mechanism.

Let $\hat{v}_i := \frac{\sum_{a\in A}\sum_{\theta\in\Theta} p(\theta)u_i(a,\theta)}{|A|}$ for all $i \in I$, where $|A|$ denotes the cardinality of the set A. Our regime is defined as follows:

Regime r.

1. $r(\hat{\gamma}|h^0) = 1$.
2. For $t \geq 1$ if $r(\hat{\gamma}|h^{t-1}) = 1$ and $m^{t-1} = (m_i)_{i\in I}$ is such that
   (a) Parts (I) or (IIb) of $\hat{\gamma}$ applies, then $r(\hat{\gamma}|h^t) = 1$.
   (b) Part (IIa) of $\hat{\gamma}$ applies with $m_{i^*} = (\theta', b', v', z')$, then
      i. If $\underline{v}_{i^*} < v'_{i^*} \leq \hat{v}_{i^*}$, then
      $$\begin{aligned} \check{r}(\tilde{\gamma}_{i^*}(\{a\})|h^t) &= \lambda/|A| \text{ for all } a \in A, \\ \check{r}(\tilde{\gamma}_{i^*+1}(\underline{A}_i)|h^t) &= (1-\lambda) \end{aligned}$$
      where
      $$\lambda = \frac{v'_{i^*} - \underline{v}_{i^*}}{\hat{v}_{i^*} - \underline{v}_{i^*}}.$$
      ii. If $\hat{v}_{i^*} < v'_{i^*} < \overline{v}_{i^*}$, then
      $$\begin{aligned} \check{r}(\tilde{\gamma}_{i^*}(A)|h^t) &= \lambda, \\ \check{r}(\tilde{\gamma}_{i^*}(\{a\})|h^t) &= (1-\lambda)/|A| \text{ for all } a \in A \end{aligned}$$
      where
      $$\lambda = \frac{v'_{i^*} - \hat{v}_{i^*}}{\overline{v}_{i^*} - \hat{v}_{i^*}}.$$
      iii. If $v'_{i^*} = \overline{v}_{i^*}$, then
      $$r(\tilde{\gamma}_{i^*}(A)|h^t) = 1.$$
      iv. If $v'_{i^*} = \underline{v}_{i^*}$, then
      $$r(\tilde{\gamma}_{i^*+1}(\underline{A}_{i^*})|h^t) = 1.$$
   (c) Part (III) of $\hat{\gamma}$ applies, then $r(\tilde{\gamma}_{i_z}(A)|h^t) = \frac{z_z}{1+z_z}$ and $r(\tilde{\gamma}_{i_z}(\{a\})|h^t) = \frac{1}{(1+z_z)|A|}$ for all $a \in A$.
3. If $r(\tilde{\gamma}_i(\mathcal{M}_i)|h^{t-1}) = 1$ for some i and $\mathcal{M}_i$, then $r(\tilde{\gamma}_i(\mathcal{M}_i)|h^t) = 1$.

In words, suppose that the agents face mechanism $\hat{\gamma}$ in period $t-1$. Depending on the agents' messages in that period, period t mechanism is determined as follows. If their reports are unanimous or there is a single agent i^* who sends a message $m_{i^*} = (\theta', b', v', z')$ different from (θ, b, v, z), which is sent by all the other agents, and $(b', v') \notin C_{i^*}(\theta)$, then the mechanism $\hat{\gamma}$ is again selected in period t. If instead the message of agent i^* is such that $(b', v') \in C_{i^*}(\theta)$, then the demanded alternative b' is implemented and from the next period on either agent i^* or i^*+1 is given the right to choose alternatives from set A (in case of i^*) or $\underline{A}_{i^*}$ (in case of i^*+1) forever, or one of the constant mechanisms is applied forever. The probabilities of these scenarios are chosen so as to ensure that agent i^* expects his announced continuation payoff v'_{i^*}, assuming that, on one hand, agent i^*+1 will always choose agent i^*'s worst alternative from $\underline{A}_{i^*}$ if agent i^*+1 becomes a dictator and, on the other hand, agent i^* plays optimally if he becomes the dictator. Finally, for all other message profiles, either the agent with the highest announced integer is given the right to choose alternatives from A (with probability $z_z/(1+z_z)$) or one of the constant mechanisms is applied forever (with probability $1/(|A|(1+z_z))$).[25]

We now prove that the defined regime implements f. Lemma 5 establishes that there exists an equilibrium that selects the desirable alternative in each period, while Lemma 6 establishes that in any equilibrium, only the desirable alternative is selected in each period.

Lemma 5. *There exists a Nash equilibrium s such that $g(s(h^t, \theta^t, \gamma^t)) = f(\theta^t)$ for all $t \in T$, $\theta^t \in \Theta$, $h^t \in H^t$, and $\gamma^t \in \Gamma$ such that $q(h^t|s, r)r(\gamma^t|h^t) > 0$.*

Proof. Let i^* be the agent that is defined in part (2b) of r. Also, let $j = i^*+1$. For all t, θ^t, and h^t, let s be defined as follows:

1. $s_i(h^t, \hat{\gamma}, \theta^t) = (\theta^t, f(\theta^t), v^f, 0)$ for all $i \in I$.
2. $s_{i^*}(h^t, \tilde{\gamma}_{i^*}(A), \theta^t) = \overline{a}_{i^*}(\theta^t)$ and $s_i(h^t, \tilde{\gamma}_{i^*}, \theta^t) = \emptyset$ for all $i \neq i^*$.
3. $s_j(h^t, \tilde{\gamma}_j(\underline{A}_{i^*}), \theta^t) = \underline{a}_{i^*}(\theta^t)$ and $s_i(h^t, \tilde{\gamma}_j, \theta^t) = \emptyset$ for all $i \neq j$.

Note that we have left s unspecified when multilateral deviations occur.

Given s, the payoff of agent i in period t when the state of the world is θ is $(1-\delta)u_i(f(\theta), \theta) + \delta v_i^f$. Suppose period t is the first period in which agent i deviates from s_i. The only period t deviations that matter are the ones that fall under Part (IIa) of $\hat{\gamma}$. However, for any such deviation, $(b, v) \in C_i(\theta)$ and i's payoff is $(1-\delta)u_i(b, \theta) + \delta v_i$, which is weakly smaller than $(1-\delta)u_i(f(\theta), \theta) + \delta v_i^f$. Note that v_i is, indeed, the highest continuation payoff that agent i can expect given the strategy of agent j if the mechanism $\tilde{\gamma}_j(\underline{A}_{i^*})$ is selected in period $t+1$. □

Lemma 6. *In any Nash equilibrium s of r, $g(s(h^t, \theta^t, \gamma^t)) = f(\theta^t)$ for all $t \in T$, $\theta^t \in \Theta$, $h^t \in H^t$, and $\gamma^t \in \Gamma$ such that $q(h^t|s, r)r(\gamma^t|h^t) > 0$.*

[25] We could also design a regime without the unbounded (or open) "integer game" in part (III) of $\hat{\gamma}$ by instead allowing the agents to announce numbers from the compact $[0, 1]$ interval. For example, the agent who announces the largest number becomes a dictator with the probability equal to his number if this number is strictly less than 1, and with the remaining probability the constant mechanisms are played. If anyone announces 1, then a constant mechanism is selected with equal probabilities. Hence, the regime (resp., mechanism) is discontinuous in strategies (resp., messages).

If $(C_i(\theta))_{i,\theta}$ can be chosen such that for all $i \in I$, $\theta \in \Theta$, and $(b, v) \in C_i(\theta)$, it is true that $v_i \notin \{\underline{v}_i, \overline{v}_i\}$, then the only possible Nash equilibria are the ones that fall under Claim 1 below.

Proof. Fix some Nash equilibrium s.

Claim 1. *[Full deception] If for all t, θ^t, h^t such that $q(h^t|s, r)r(\hat{\gamma}|h^t) > 0$, $s_i(h^t, \theta^t, \hat{\gamma}) = (\theta, b, v, z)$ for all i, then $g(s(h^t, \theta^t, \hat{\gamma})) = f(\theta) = f(\theta^t)$. Similarly, if there exist t, θ^t, h^t such that $q(h^t|s, r)r(\hat{\gamma}|h^t) > 0$, $s_i(h^t, \theta^t, \hat{\gamma}) = (\theta, b, v, z)$ for all $i \neq i^*$ and $s_{i^*}(h^t, \theta^t, \hat{\gamma}) = (\cdot, b', v', \cdot) \neq (\theta, b, v, z)$ such that $(b', v') \notin C_{i^*}(\theta)$ (that is, part (IIb) of $\hat{\gamma}$ and part (2a) of r apply), then $g(s(h^t, \theta^t, \hat{\gamma})) = f(\theta) = f(\theta^t)$.*

Proof of Claim 1. This follows directly from dynamic monotonicity, as in the proof of Claim 3 in Theorem 2 of Mezzetti and Renou (2014). □

Claim 2. *[An odd-man-out] If there exist t, θ^t, h^t such that $q(h^t|s, r)r(\hat{\gamma}|h^t) > 0$, $s_i(h^t, \theta^t, \hat{\gamma}) = (\theta, b, v, z)$ for all $i \neq i^*$ and $s_{i^*}(h^t, \theta^t, \hat{\gamma}) = (\cdot, b', v', \cdot) \neq (\theta, b, v, z)$ such that $(b', v') \in C_{i^*}(\theta)$ (that is, part (IIa) of $\hat{\gamma}$ and part (2b) of r apply), then $g(s(h^t, \theta^t, \hat{\gamma})) = b' = f(\theta^t)$.*

Proof of Claim 2. Cases 2(b)i and 2(b)ii of r: This cannot be an equilibrium because there is a positive probability that one of the constant mechanisms is played forever. Hence, due to strict preferences[26], there is an agent $j \neq i$ who prefers to play the integer game and to announce high enough integer to decrease the probability with which the constant mechanisms are played.

Case 2(b)iii of r: This can only be an equilibrium if the hypothesis (i.e., *if*-part) in Condition $\lambda 1$ applies. But then, by the conclusion (i.e., *then*-part) of Condition $\lambda 1$, $f(\theta^t) = b'$ as required. If the hypothesis of Condition $\lambda 1$ does not apply, there is an agent $j \neq i$, who wants to deviate and trigger the integer game with a sufficiently high integer, which allows him to get a payoff arbitrarily close to his best payoff.

Case 2(b)iv of r: This can only be an equilibrium if the hypothesis in Condition $\lambda 0$ applies. But then, by the conclusion of Condition $\lambda 0$, $f(\theta^t) = b'$ as required. If the hypothesis of Condition $\lambda 0$ does not apply, there is an agent $j \neq i$, who wants to deviate and trigger the integer game with a sufficiently high integer, which allows him to get a payoff arbitrarily close to his best payoff.

□

[26] In the case of weak preferences, we have to suppose that there are two agents who are not always indifferent between all alternatives. Otherwise, if there are $n - 1$ agents who are always indifferent between all alternatives, it is as if the designer only faced a single agent - the one who is not indifferent. Then, according to dynamic monotonicity, f must give this agent his unique best alternative in the range of f in each state. Such f can be implemented by simply making this agent the dictator over the range of f. If all agents are indifferent between all alternatives in all states, then dynamic monotonicity is not satisfied and the implementation of f is hopeless.

Claim 3. *[Integer game] There is no equilibrium s such that $q(h^t|s,r)r(\hat{\gamma}|h^t) > 0$ for some t, θ^t, h^t but neither the hypothesis of Claim 1 nor Claim 2 applies.*

Proof of Claim 3. It is trivially true because agents want to announce higher and higher integers to decrease the probability of the constant mechanisms. (Footnote 26 again applies for weak preferences.) $\square$

Claim 4. *$g(s(h^t, \theta^t, \tilde{\gamma}_i(\mathcal{M}_i))) = f(\theta^t)$ for all t, θ^t, i, $\mathcal{M}_i \in A \cup \{A\} \cup \{\underline{A}_{i-1}\}$, and h^t such that $q(h^t|s,r)r(\tilde{\gamma}_i(\mathcal{M}_i)|h^t) > 0$.*

Proof of Claim 4. From the proofs of Claims 2 and 3, we know that there can only be an equilibrium s in which a mechanism $\tilde{\gamma}_i(\mathcal{M}_i)$ is played, if either Condition $\lambda 0$ or $\lambda 1$ applies. If that happens, then it follows immediately from either condition that $g(s(h^t, \theta^t, \tilde{\gamma}_i(\mathcal{M}_i))) = f(\theta^t)$ for all t, θ^t, and h^t such that $q(h^t|s,r)r(\tilde{\gamma}_i(\mathcal{M}_i)|h^t) > 0$. $\square$

$\square$

B Proofs for the Two Agent Case

B.1 Proof of Proposition 4

Suppose that $v_i^f = \overline{v}_i$ for some $i \in \{1, 2\}$. Then, we can implement f by making agent i a dictator over A. (Note that we continue to assume that the preferences are strict.) Hence, we assume that $v_i^f < \overline{v}_i$ for $i = 1, 2$. Also, it cannot be that $v_i^f = \underline{v}_i$ for some $i \in \{1, 2\}$ because f is efficient and $v_j^f < \overline{v}_j$ for $j \neq i$. (If $v_i^f = \underline{v}_i$, then, for example, $\overline{a}_j(\cdot)$ would give higher payoffs to both agents than f.) Hence, we can assume that $\overline{v}_i > v_i^f > \underline{v}_i$ for $i = 1, 2$.

Let us fix $C = (C_i(\theta))_{i,\theta}$ such that f is Maskin monotonic* with respect to C and f satisfies self-selection* relative to C. Also, let $b(\theta_1, \theta_2)$ for all $(\theta_1, \theta_2) \in \Theta^2$ be the alternatives that are defined in Definition 6. We argue that we can always pick another collection C' such that f is Maskin monotonic* w.r.t. C', f satisfies self-selection* relative to C', and Conditions $\lambda 0$ and $\lambda 1$ are vacuously satisfied as we have done when proving Corollary 1. There are two differences, however. First, we need additionally to ensure that part 2 in Definition 6 continues to hold. Second, unlike dynamic monotonicity, in the case of Maskin monotonicity*, we only have to take care of finitely many strict inequalities when modifying the collection C.

Thus, suppose we want to replace $(\tilde{a}, v) \in C_1(\theta_2)$ for some $\theta_2 \in \Theta$ with another alternative and continuation payoff pair $(\tilde{b}, w)$. To preserve Maskin monotonicity*, we need to ensure that for any θ, if the following inequalities hold

$$\begin{aligned} (1-\delta)u_1(f(\theta_2), \theta_2) + \delta v_1^f &\geq (1-\delta)u_1(\tilde{a}, \theta_2) + \delta v_1, && (1) \\ (1-\delta)u_1(f(\theta_2), \theta) + \delta v_1^f &< (1-\delta)u_1(\tilde{a}, \theta) + \delta v_1, && (2) \end{aligned}$$

then they continue to hold once we replace $(\tilde{a}, v)$ with $(\tilde{b}, w)$. Additionally, to satisfy part 2 in Definition 6, it is enough to ensure that for any pair (θ, θ'), if the following inequality holds

$$(1-\delta)u_1(b(\theta', \theta_2), \theta) + \delta v_1^f < (1-\delta)u_1(\tilde{a}, \theta) + \delta v_1 \tag{3}$$

then it continues to hold once we replace $(\tilde{a}, v)$ with $(\tilde{b}, w)$.

Suppose that $(\tilde{a}, v)$ is such that $v_1 = \overline{v}_1$. Then, because of strict inequalities in (2) and (3), we can always set $\tilde{b} = \tilde{a}$ and pick w such that w_1 is slightly below $\overline{v}_1$. Suppose now that $(\tilde{a}, v)$ is such that $v_1 = \underline{v}_1$. If the inequality in (1) is strict, we can again set $\tilde{b} = \tilde{a}$ and pick w such that w_1 is slightly above $\underline{v}_1$. If (1) holds with equality, it must be that $u_1(\tilde{a}, \theta_2) > u_1(f(\theta_2), \theta_2)$. Then, again because of strict inequalities in (2) and (3), we can choose $\tilde{b}$ to be a convex combination of $\tilde{a}$ and $f(\theta_2)$ (with sufficiently high weight on $\tilde{a}$) and pick w such that w_1 is slightly above $\underline{v}_1$. We can repeat this process for all θ_2 and all $(\tilde{a}, v) \in C_1(\theta_2)$ such that either $v_1 = \overline{v}_1$ or $v_1 = \underline{v}_1$ holds. The same can be done for agent 2. In this way, we have constructed a collection C' such that f is Maskin monotonic* w.r.t. C', f satisfies self-selection* relative to C', and Conditions $\lambda 0$ and $\lambda 1$ are vacuously satisfied. By abusing notation, let $C' = C$ in the continuation.

Consider the following mechanism and regime, which is well defined since agent $i \in \{1, 2\}$ cannot demand either $\overline{v}_i$ or $\underline{v}_i$.

Mechanism $\check{\gamma}$. Let $\mathcal{M}_i = \{\Theta \times A \times V \times \mathbb{Z}_+\}$. Let g be as follows:

I. If $m_1 = m_2 = (\theta, a, v, 0)$, then $g(m) = f(\theta)$.

II. If $m_1 = (\theta, a, v, 0)$ and $m_2 = (\theta', a', v', 0) \neq m_1$, then $g(m) = b(\theta, \theta')$.

III. If $m_i = (\theta, a, v, z)$ with $z > 0$ for some $i \in \{1, 2\}$ and $m_j = (\theta', a', v', 0)$ for $j \in \{1, 2\} \setminus \{i\}$, then
   (a) $g(m) = a$ if $(a, v) \in C_i(\theta')$.
   (b) $g(m) = f(\theta')$ otherwise.

IV. If $m_1 = (\theta, a, v, z)$ with $z > 0$ and $m_2 = (\theta', a', v', z') \neq m_1$ with $z' > 0$, then
   (a) $g(m) = a$ if $z \geq z'$.
   (b) $g(m) = a'$ otherwise.

Regime $\check{r}$.

1. $\check{r}(\check{\gamma}|h^0) = 1$.
2. For $t \geq 1$ if $\check{r}(\check{\gamma}|h^{t-1}) = 1$ and $m^{t-1} = (m_1, m_2)$ is such that
   (a) Part (I), (II) or (IIIb) of $\check{\gamma}$ applies, then $\check{r}(\check{\gamma}|h^t) = 1$.

(b) Part (IIIa) of $\check{\gamma}$ applies, then

i. If $\underline{v}_i < v_i \leq \hat{v}_i$, then

$$\begin{aligned}\check{r}(\tilde{\gamma}_i(\{a\})|h^t) &= \lambda/|A| \text{ for all } a \in A,\\ \check{r}(\tilde{\gamma}_{3-i}(\underline{A}_i)|h^t) &= (1-\lambda)\end{aligned}$$

where
$$\lambda = \frac{v_i - \underline{v}_i}{\hat{v}_i - \underline{v}_i}.$$

ii. If $\hat{v}_i < v_i < \overline{v}_i$, then

$$\begin{aligned}\check{r}(\tilde{\gamma}_i(A)|h^t) &= \lambda,\\ \check{r}(\tilde{\gamma}_i(\{a\})|h^t) &= (1-\lambda)/|A| \text{ for all } a \in A\end{aligned}$$

where
$$\lambda = \frac{v_i - \hat{v}_i}{\overline{v}_i - \hat{v}_i}.$$

(c) Part (IVa) of $\check{\gamma}$ applies, then $\check{r}(\tilde{\gamma}_1(A)|h^t) = \frac{z}{1+z}$ and $\check{r}(\tilde{\gamma}_1(\{a\})|h^t) = \frac{1}{(1+z)|A|}$ for all $a \in A$.

(d) Part (IVb) of $\check{\gamma}$ applies, then $\check{r}(\tilde{\gamma}_2(A)|h^t) = \frac{z'}{1+z'}$ and $\check{r}(\tilde{\gamma}_2(\{b\})|h^t) = \frac{1}{(1+z')|A|}$ for all $a \in A$.

3. If $\check{r}(\tilde{\gamma}_i(\mathcal{M}_i)|h^{t-1}) = 1$ for some i and $\mathcal{M}_i$, then $\check{r}(\tilde{\gamma}_i(\mathcal{M}_i)|h^t) = 1$.

There can be no equilibrium in which messages that fall under Parts (III) or (IV) of $\check{\gamma}$ are sent with a positive probability because the agents would compete to become the dictator. Consider now any equilibrium. Period 0 messages in that equilibrium must take the form of $m_1 = (\theta_1, \cdot, \cdot, 0)$ and $m_2 = (\theta_2, \cdot, \cdot, 0)$. Suppose in particular that the state in period 0 is θ^* and the agents send the messages $m_1 = (\theta_1, \cdot, \cdot, 0)$ and $m_2 = (\theta_2, \cdot, \cdot, 0) \neq m_1$. Then, alternative $b(\theta_1, \theta_2)$ is implemented in period 0. Further, each agent i can guarantee a continuation payoff of v_i^f by instead announcing $m_i' = (\theta_i, b(\theta_1, \theta_2), v^f, 1)$. Therefore, by efficiency of f, it must be that their equilibrium continuation payoffs at the start of period 1 are v^f. Now, suppose instead that in the equilibrium, the agents send $m_1 = m_2 = (\theta, \cdot, \cdot, 0)$ in period 0 when the state is θ^*. Then, alternative $f(\theta)$ is implemented in that period. Again, each agent i can guarantee a continuation payoff of v_i^f by instead announcing $m_i' = (\theta, f(\theta), v^f, 1)$. Therefore, again by efficiency of f, it must be that their equilibrium continuation payoffs at the start of period 1 are v^f.

Finally, if nobody has incentives to deviate in state θ^* in period 0, it must be that $(b(\theta_1, \theta_2), v^f) \in M_1(C_1(\theta_2), \theta^*) \cap M_2(C_2(\theta_1), \theta^*)$ and, by self-selection*, $b(\theta_1, \theta_2) = f(\theta^*)$ when the messages are $m_1 = (\theta_1, \cdot, \cdot, 0)$ and $m_2 = (\theta_2, \cdot, \cdot, 0) \neq m_1$. Or, $(f(\theta), v^f) \in M_1(C_1(\theta), \theta^*) \cap M_2(C_2(\theta), \theta^*)$ and, by Maskin monotonicity*, $f(\theta) = f(\theta^*)$ when the messages are $m_1 = m_2 = (\theta, \cdot, \cdot, 0)$.

Hence, we have established that in any equilibrium, $f(\theta^*)$ is implemented for any θ^* in period 0. The same argument can be repeated for all periods $t > 0$. This establishes that there are no undesirable equilibria. To prove that there exists a desirable equilibrium, we can use the result of Lemma 5 where we replace $\hat{\gamma}$ with $\check{\gamma}$.

B.2 Proof of Proposition 5

To prove Proposition 5, we have to modify slightly the regime described in the previous subsection. In particular, when both agents disagree by sending messages of the form $m_1 = (\theta_1, \cdot, \cdot, 0)$ and $m_2 = (\theta_2, \cdot, \cdot, 0) \neq m_1$ (i.e., Part (II) of $\check{\gamma}$ applies), we now assume that the bad outcome $\tilde{b}$ is implemented from that point on forever. Also, instead of modifying the collection C as it was done in the proof of Proposition 4, we can directly invoke Corollary 1 to conclude that there exists a collection C with respect to which f is dynamically monotonic and relative to which f satisfies Conditions $\lambda 0$ and $\lambda 1$ vacuously since no agent $i \in \{1, 2\}$ can demand either $\overline{v}_i$ or $\underline{v}_i$. Further, because of the bad outcome assumption, it is true that $v_i^f \neq \underline{v}_i$ and we can also assume that $v_i^f \neq \overline{v}_i$ because otherwise f can be implemented by making agent i the dictator. Therefore, we can also include $(f(\theta), v^f)$ in $C_i(\theta)$ for all i and θ without violating Conditions $\lambda 0$ and $\lambda 1$.

Given these changes, we can now prove that $\check{r}$ implements f. As before, there can be no equilibrium in which messages that fall under Parts (III) or (IV) of $\check{\gamma}$ are sent with a positive probability. Also, there cannot be an equilibrium in which messages that fall under Part (II) of $\check{\gamma}$ are sent with a positive probability: given that agent j reports θ, agent $i \neq j$ can opt for $(f(\theta), v^f)$, which is clearly better than getting the bad outcome forever. Hence, in any equilibrium, the messages must fall under Part (I) of $\check{\gamma}$ but then dynamic monotonicity ensures that the desired alternative is implemented in each period.

C Examples

Example 1

The purpose of this example is to illustrate how the introduction of threat in the regime to randomly implement a constant alternative forever in the case of disagreement can help to implement a social choice function. There are three agents, $1, 2, 3$, five alternatives a, b, c, d, e, and two states of the world, θ' and θ'', each occurring with equal probability. The discount factor is assumed to be $\delta = \frac{1}{3}$. The period payoffs are summarized in the table below (where Ai stands for agent i).

	θ'			θ''		
	$A1$	$A2$	$A3$	$A1$	$A2$	$A3$
a	2	3	3	1	1	1
b	0	3	3	1/2	3	3
c	3	3	3	4/3	3	3
d	1	1	1	1/2	0	0
e	1	3	3	0	3	3

Let the social choice function be $f(\theta') = d$ and $f(\theta'') = a$. Note that this function does not satisfy no-veto power, does not satisfy Assumptions A(i) (as we show later) and A(ii) of Mezzetti and Renou (2014), and it is not even weakly efficient in the range. It is also true that there is no θ and two agents i, j such that the intersection of the best alternatives of agents i and j at state θ is empty, i.e., Assumption A in Lee and Sabourian (2011) does not hold because c is the best alternative for all agents in both states. Therefore, we cannot apply Theorem 2 of Mezzetti and Renou (2014) even if f is dynamically monotonic or Theorem 2 of Lee and Sabourian (2011). Condition ω of Lee and Sabourian (2011), however, is satisfied: always implementing alternative d gives on average less to each player than v^f, that is, $\sum_\theta p(\theta)u_i(d,\theta) < v_i^f$ for all i. This assumption, however, is not necessary for our construction.

We verify that f is indeed dynamically monotonic. Note that $v^f = (1, 1, 1)$. The only deception that we need to take care of is a stationary one that is obtained by applying a static deception $\pi^0(\theta') = \theta''$ and $\pi^0(\theta'') = \theta''$ in each period since it results in the highest payoffs $v^f(\pi^0) = (\frac{3}{2}, 2, 2)$. If we can eliminate this deception, then we can also eliminate any non-stationary deception, where at state θ', the agents sometimes pretend that the state is θ'' and sometimes report truthfully that the state is θ', since such a deception would result in lower payoffs.[27] Suppose that $C_1(\theta'')$ contains an element (c, v) such that $v_1 \leq \frac{1}{3}$. We verify that it gives incentives for agent 1 to destroy the deception. For that we need to verify that the following inequalities are satisfied:

$$\begin{aligned}(1-\delta)u_1(c,\theta'') + \delta v_1 &\leq (1-\delta)u_1(a,\theta'') + \delta v_1^f,\\ (1-\delta)u_1(c,\theta') + \delta v_1 &> (1-\delta)u_1(a,\theta') + \delta v_1^f(\pi^0).\end{aligned}$$

That is, the first equation ensures that $(c, v) \in L_1((f(\theta''), v^f), \theta'')$ and the truth-telling equilibrium is not eliminated, while the second equation ensures that $(c, v) \notin L_1((f(\theta''), v^f), \theta')$ and the undesirable equilibrium is eliminated. After substituting the values, they become

$$\begin{aligned}\frac{8}{9} + \frac{1}{3}v_1 &\leq 1,\\ 2 + \frac{1}{3}v_1 &> \frac{11}{6}.\end{aligned}$$

[27] The agents have no incentives to ever pretend in state θ'' that the state is θ'.

Thus, they are indeed satisfied for $v_1 \leq \frac{1}{3}$.

If we apply the regime of Mezzetti and Renou (2014), then agent 1 becomes a dictator with the probability of $\lambda = \frac{v_1 - \underline{v}_1}{\overline{v}_1 - \underline{v}_1}$ (in the example, $\underline{v}_1 = 0$ and $\overline{v}_1 = \frac{13}{6}$) from the next period onwards after he has announced (c, v), while the other agents have announced that the state is θ''. With the remaining probability of $1 - \lambda$, agent 2 becomes the dictator.[28] However, this regime has an unwanted equilibrium, in which in state θ', agent 1 announces (c, v), while the other agents announce that the state is θ'', and then, in the continuation, the dictator chooses alternative c forever. Since c is the best alternative for all agents in both states, nobody has incentives to deviate.[29]

We can eliminate this unwanted equilibrium by modifying the regime as follows. After agent 1 announces (c, v), instead of allowing agent 1 to become the dictator, the regime now chooses each of the five alternatives forever with equal probability of $\lambda/5$. With the remaining probability of $1 - \lambda$, agent 2 still becomes the dictator. To give the continuation payoff of v_1 to agent 1, we now set $\lambda = \frac{v_1 - \underline{v}_1}{\hat{v}_1 - \underline{v}_1}$ where $\hat{v}_1$ is the continuation payoff of agent 1 when one of the alternatives is selected uniformly ($\hat{v}_1 = \frac{31}{30}$ in the example; hence, $\hat{v}_1 > v_1$ holds). With this modification in the regime, the undesirable equilibrium is eliminated because agents 2 and 3 have incentives to trigger the integer game. Further, to ensure that the integer game does not have undesirable equilibria, we require that while the agents can reduce the probability of having a constant alternative forever by announcing higher and higher integers, this probability cannot be made equal to 0.

Example 2

The purpose of this example is to demonstrate the necessity of Condition $\lambda 0$. It also illustrates how the use of doubly-stochastic regimes helps to satisfy this condition when simply-stochastic regimes fail. A similar example can be constructed for Condition $\lambda 1$.

Consider a modification of the previous example. The only change is in the payoffs of the agents when alternative c is selected in state θ'' (see the table below). Everything else remains the same as in the previous example.

[28]In fact, Mezzetti and Renou (2014) assume that with probability $1 - \lambda$, the outcome is decided by qualified majority, but this difference in the regime does not matter for the argument.

[29]This proves that Assumption A(i) of Mezzetti and Renou (2014) is not satisfied. It also means that there is an undesirable equilibrium, in which the agents trigger the integer game.

	θ'			θ''		
	$A1$	$A2$	$A3$	$A1$	$A2$	$A3$
a	2	3	3	1	1	1
b	0	3	3	1/2	3	3
c	3	3	3	3/2	2	2
d	1	1	1	1/2	0	0
e	1	3	3	0	3	3

We again verify that f is dynamically monotonic. However, we want to show that the only possible way to destroy the repeated use of static deception $\pi^0(\theta') = \theta''$ and $\pi^0(\theta'') = \theta''$ in each period is for agent 1 to announce $(c, (0, 3, 3))$. For that we consider each possible deviation by each agent. The deception will be destroyed if the following inequalities will hold for some agent i and some pair (x, v):

$$\begin{aligned}(1-\delta)u_i(x,\theta'') + \delta v_i &\leq (1-\delta)u_i(a,\theta'') + \delta v_i^f, \\ (1-\delta)u_i(x,\theta') + \delta v_i &> (1-\delta)u_1(a,\theta') + \delta v_i^f(\pi^0).\end{aligned}$$

In case of agent 1, these inequalities become

$$\begin{aligned}\frac{2}{3}u_1(x,\theta'') + \frac{1}{3}v_1 &\leq 1, \\ \frac{2}{3}u_1(x,\theta') + \frac{1}{3}v_1 &> \frac{11}{6}.\end{aligned}$$

The above inequalities require that

- If $x = a$, then $v_1 \leq 1$ and $v_1 > \frac{3}{2}$, which is impossible.
- If $x = b$, then $v_1 \leq 2$ and $v_1 > \frac{11}{2}$, which is impossible.
- If $x = c$, then $v_1 \leq 0$ and $v_1 > -\frac{1}{2}$. Since $\underline{v}_1 = 0$, it follows that $(c, (0, 3, 3))$ is the unique alternative-continuation payoff pair that agent 1 can announce to destroy the deception.
- If $x = d$, then $v_1 \leq 2$ and $v_1 > \frac{7}{2}$, which is impossible.
- If $x = e$, then $v_1 \leq 3$ and $v_1 > \frac{7}{2}$, which is impossible.

In case of agent 2 (and agent 3), these inequalities become

$$\begin{aligned}\frac{2}{3}u_2(x,\theta'') + \frac{1}{3}v_2 &\leq 1, \\ \frac{2}{3}u_2(x,\theta') + \frac{1}{3}v_2 &> \frac{8}{3}.\end{aligned}$$

The above inequalities require that

- If $x = a$, then $v_2 \leq 1$ and $v_2 > 2$, which is impossible.
- If $x = b$, then $v_2 \leq -3$ and $v_2 > 2$, which is impossible.
- If $x = c$, then $v_2 \leq -1$ and $v_2 > 2$, which is impossible.
- If $x = d$, then $v_2 \leq 3$ and $v_2 > 6$, which is impossible.
- If $x = e$, then $v_2 \leq -3$ and $v_2 > 2$, which is impossible.

We conclude that f is dynamically monotonic but the only way to destroy the deception, given by $\pi^0(\theta') = \theta''$ and $\pi^0(\theta'') = \theta''$ in each period, is for agent 1 to announce $(c, (0, 3, 3))$.

Even though f is dynamically monotonic, it is not repeatedly implementable using a simply-stochastic regime. Since $C_1(\theta'')$ necessarily contains $(c, (0, 3, 3))$, and the premiss of Condition $\lambda 0$ is satisfied while its implication not ($f(\theta') \neq c$), our Theorem 1 tells us that f is not repeatedly implementable with simply-stochastic regimes. Note that our simply-stochastic regime, which fails in this case, and the regime of Mezzetti and Renou (2014) are the same now since agent 1 becomes dictator with probability 0 and, therefore, we cannot introduce constant alternatives with positive probabilities. Therefore, f is also not repeatedly implementable by the regime of Mezzetti and Renou (2014).

To see why f is not implementable, consider a strategy profile in which in state θ', agent 1 announces $(c, (0, 3, 3))$, while agents 2 and 3 report that it is state θ''. In the continuation, agent 2 selects alternative b in state θ' and alternative e in state θ''. If agent 1 does not announce $(c, (0, 3, 3))$ in state θ', then agents 2 and 3 report the state honestly in all future periods. Then, if agent 1 announces $(c, (0, 3, 3))$ in state θ', he receives a payoff of

$$\frac{2}{3}3 + \frac{1}{3}\left(\frac{1}{2}0 + \frac{1}{2}0\right) = 2.$$

If agent 1 does not announce $(c, (0, 3, 3))$ in state θ', his payoff is

$$\frac{2}{3}2 + \frac{1}{3}1 = \frac{5}{3} < 2.$$

Therefore, agent 1 does not want to deviate from the specified strategy. Since agents 2 and 3 receive their best alternatives, they also do not want to deviate. Hence, we have a Nash equilibrium, in which an undesirable alternative is implemented. The same argument can be obtained for any regime, which does not use stochastic stage mechanisms.

However, we can repeatedly implement f using a doubly-stochastic regime. With the help of lottery, we can replace the alternative-continuation payoff pair $(c, (0, 3, 3))$ with another pair that gives lower current period payoff in period θ' to agent 1 while we increase his continuation payoff. For example, agent 1 can demand a lottery between a and c with probabilities of $\frac{1}{3}$ and $\frac{2}{3}$, respectively. We can verify that the relevant inequalities

$$\begin{aligned}\frac{2}{3}\left(\frac{1}{3}u_1(a,\theta'')+\frac{2}{3}u_1(c,\theta'')\right)+\frac{1}{3}v_1 &\leq 1,\\ \frac{2}{3}\left(\frac{1}{3}u_1(a,\theta')+\frac{2}{3}u_1(c,\theta')\right)+\frac{1}{3}v_1 &> \frac{11}{6}\end{aligned}$$

are satisfied for $\frac{1}{6} < v_1 \leq \frac{1}{3}$. Since $v_1 > 0$, we can again introduce the threat of having constant alternatives with a positive probability in the case of disagreement.

Example 3

We illustrate how to construct the stage game payoffs of the repeated game in Section 4.1. For that, we use the setup of Example 1. For simplicity, we only present the payoffs for the case when players 2 and 3 choose the same action.

One can choose $C_1(\theta') = \{(d,(1,1,1)),(b,(\frac{13}{6},3,3))\}$ and $C_1(\theta'') = \{(a,(1,1,1)),(c,(\frac{1}{3},\frac{7}{3},\frac{7}{3}))\}$. Therefore,

$$\begin{aligned}u_1(\theta',\theta') &= U_1(d,(1,1,1),\theta')/(1-\delta) = \left(\frac{2}{3}1+\frac{1}{3}1\right)\frac{3}{2} = \frac{3}{2},\\ u_1(\theta'',\theta') &= U_1\left(c,\left(\frac{1}{3},\frac{7}{3},\frac{7}{3}\right),\theta'\right)/(1-\delta) = \left(\frac{2}{3}3+\frac{1}{3}\frac{1}{3}\right)\frac{3}{2} = \frac{19}{6},\\ u_1(\theta'',\theta'') &= U_1(a,(1,1,1),\theta'')/(1-\delta) = \left(\frac{2}{3}1+\frac{1}{3}1\right)\frac{3}{2} = \frac{3}{2},\\ u_1(\theta',\theta'') &= U_1\left(b,\left(\frac{13}{6},3,3\right),\theta''\right)/(1-\delta) = \left(\frac{2}{3}\frac{1}{2}+\frac{1}{3}\frac{13}{6}\right)\frac{3}{2} = \frac{19}{12}.\end{aligned}$$

The stage game payoffs when players 2 and 3 choose the same action are given in the following matrices, where we have set $u_i(x,\theta^*) = -5$ for $\theta^* = \theta', \theta''$ if x falls under point 4 in Section 4.1.

	$u(x,\theta')$	Players 2 and 3: θ'	θ''	ω	o
Player 1	θ'	$1,\ 1,\ 1$	$-5,-5,-5$	$-5,-5,-5$	$-5,\ 0,\ 0$
	θ''	$-5,-5,-5$	$2,\ 3,\ 3$	$-5,-5,-5$	$-5,\ 0,\ 0$
	ω	$\frac{3}{2},-5,-5$	$\frac{19}{6},-5,-5$	$-5,-5,-5$	$-5,-5,-5$
	o	$0,-5,-5$	$0,-5,-5$	$0-5,-5$	$0,\ 0,\ 0$

	$u(x,\theta'')$	Players 2 and 3: θ'	θ''	ω	o
Player 1	θ'	$\frac{1}{2},\ 0,\ 0$	$-5,-5,-5$	$-5,-5,-5$	$-5,\ 0,\ 0$
	θ''	$-5,-5,-5$	$1,\ 1,\ 1$	$-5,-5,-5$	$-5,\ 0,\ 0$
	ω	$\frac{19}{12},-5,-5$	$\frac{3}{2},-5,-5$	$-5,-5,-5$	$-5,-5,-5$
	o	$0,-5,-5$	$0,-5,-5$	$0-5,-5$	$0,\ 0,\ 0$

Example 4

This example illustrates that the regime of Mezzetti and Renou (2014) can fail to implement a utilitarian social choice function.

There are two states of the world, θ and θ' that occur with equal probabilities. There are three agents. The preferences of agent 1 in the two states are given by $u_1(a,\theta) = 1-(a-\frac{1}{4})^2$ and $u_1(a,\theta') = 1-\frac{6}{5}(a-\frac{1}{4})^2$, respectively. The preferences of agents 2 and 3 are identical and are given by $u_2(a,\theta) = u_3(a,\theta) = 5-5(a-\alpha)^2$ and $u_2(a,\theta') = u_3(a,\theta') = 5-6(a-\frac{3}{10})^2$ where $\alpha = \frac{1}{4}+\sqrt{\frac{1}{16}+\frac{52\,517}{120\,000}} \approx 0.9572$. The discount factor is $\delta = \frac{1}{2}$.

The utilitarian social choice function when $\beta_i = 1$ for all i, is given by $f^u(\theta) = \frac{40\alpha+1}{44} \approx 0.8929 \neq \alpha$ and $f^u(\theta') = \frac{13}{44} \neq \frac{3}{10}$. Hence, f^u does not satisfy no-veto power. Further, $v_1^{f^u} = 1-\frac{1}{2}(\frac{40\alpha+1}{44}-\frac{1}{4})^2 - \frac{1}{2}\frac{6}{5}(\frac{13}{44}-\frac{1}{4})^2 \approx 0.7921$ and $\underline{v}_1 = 1-\frac{1}{2}(1-\frac{1}{4})^2 - \frac{1}{2}\frac{6}{5}(1-\frac{1}{4})^2 = \frac{61}{160}$. The indifference curve of agent 1 that passes through the point $(f^u(\theta), v_1^{f^u})$ in state θ is given by pairs (a, v_1) that satisfy the following equation:

$$u_1(f^u(\theta),\theta) + v_1^{f^u} = u_1(a,\theta) + v_1. \tag{4}$$

Additionally, $v_1 \geq \underline{v}_1$ must hold. One can verify that (4) is satisfied for $v_1 = \underline{v}_1$ when $a = \frac{1}{5}$ and $a = \frac{3}{10}$. Since $v_1 < \underline{v}_1$ for $a \in (\frac{1}{5}, \frac{3}{10})$, it follows that no (a, v) such that $a \in (\frac{1}{5}, \frac{3}{10})$ can belong to the lower contour set $L_1((f^u(\theta), v^{f^u}), \theta)$.

We argue that the regime of Mezzetti and Renou (2014) possesses an undesirable equilibrium because they allow an agent to demand anything in A once he becomes a dictator. Thus, let $C_i(\theta) = L_i((f^u(\theta), v^{f^u}), \theta)$ for all i and θ as it is the case in Mezzetti and Renou (2014). Suppose that in state θ', agents 2 and 3 send identical messages that claim that the state is θ, while agent 1 sends a different message. We look for $(a, v) \in M_1(L_1((f^u(\theta), v^{f^u}), \theta), \theta')$. It is equivalent to finding (a, v) that maximizes the payoff of agent 1, $(1-\delta)u_1(a,\theta') + \delta v_1$ subject to (4) and the restriction that $a \notin (\frac{1}{5}, \frac{3}{10})$. (For any (a, v) that does not satisfy (4), we can find another v' such that agent 1 receives a strictly higher continuation payoff and $(a, v') \in L_1((f^u(\theta), v^{f^u}), \theta)$.) Substituting for v_1 from (4) into the objective function and noting that $\delta = \frac{1}{2}$, we find that the payoff of agent 1,

$$\frac{1}{2}\left(-\frac{1}{5}\left(a-\frac{1}{4}\right)^2 + u_1(f^u(\theta),\theta) + v_1^{f^u}\right)$$

is maximized for $(a, v_1) = (\frac{3}{10}, \underline{v}_1)$. Since $v_1 = \underline{v}_1$, agent 2 becomes a dictator with probability 1 from the next period onwards. He will choose $a = \alpha$ whenever the state is θ and $a = \frac{1}{3}$ whenever the state is θ'. Clearly, no agent has incentives to deviate. Therefore, we have an equilibrium, which does not implement f^u. But, of course, f^u is implementable using our doubly-stochastic regime as noted after the proof of Proposition 3.