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In $\mathbb{R}^n$, $n \geqslant 2$, we study the constructive and numerical solution of minimizing the energy relative to the Riesz kernel $|\mathbf{x}-\mathbf{y}|^{\alpha-n}$, where $1<\alpha<n$, for the Gauss variational problem, considered for finitely many compact, mutually disjoint, boundaryless $(n-1)$-dimensional $C^{k-1,1}$-manifolds Γ_ℓ, $\ell \in L$, where $k > (\alpha-1)/2$, each Γ_ℓ being charged with Borel measures with the sign $\alpha_\ell := \pm 1$ prescribed. We show that the Gauss variational problem over a cone of Borel measures can alternatively be formulated as a minimum problem over the corresponding cone of surface distributions belonging to the Sobolev–Slobodetski space $H^{-\varepsilon/2}(\Gamma)$, where $\varepsilon := \alpha-1$ and $\Gamma := \bigcup_{\ell\in L}\Gamma_\ell$. An equivalent formulation leads in the case of two manifolds to a nonlinear system of boundary integral equations involving simple layer potential operators on Γ. A corresponding numerical method is based on the Galerkin–Bubnov discretization with piecewise constant boundary elements. Wavelet matrix compression is applied to sparsify the system matrix. Numerical results are presented to illustrate the approach.

1 Introduction

Carl Friedrich Gauss investigated in [12] the variational problem of minimizing the Newtonian energy evaluated in the presence of an external field, nowadays called the Gauss functional (or, in constructive function theory, the weighted energy), over nonnegative charges $\varphi\, ds$ on the boundary surface of a given domain. For this problem, later on the sign condition was given up in connection with boundary integral equation methods where distributional boundary charges had been introduced for solving boundary value problems. (For the history, see Costabel's article [8].) A different generalization of the original Gauss variational problem, maintaining the sign restriction but employing Borel or Radon measures μ as charges and replacing the Newtonian kernel by a much more general one (e.g., by the Riesz or Green kernel) has independently grown into an eminent branch of modern potential theory (see, e.g., [24] and the extensive works [28]–[32] and [34]; for two dimensions, see [25]).

In this paper, we consider the Gauss variational problem with the Riesz kernel $|\mathbf{x}-\mathbf{y}|^{\alpha-n}$, $1<\alpha<n$, on $\Gamma := \bigcup_{\ell\in L}\Gamma_\ell$, where Γ_ℓ, $\ell\in L$, are finitely many compact, connected, mutually disjoint, boundaryless $(n-1)$-dimensional orientable manifolds, immersed into $\mathbb{R}^n$, $n \geqslant 2$, which are assumed to be at least Lipschitz, and Γ is loaded by charges $\mu = \sum_{\ell\in L}\alpha_\ell\mu^\ell$, where α_ℓ is a function of ℓ taking the value $+1$ or -1 and μ^ℓ is a nonnegative Borel measure supported by Γ_ℓ. We first show that, if each Γ_ℓ is a $C^{k-1,1}$-manifold (see, e.g., [13, 21]), where $k\in\mathbb{N}$ and $k>(\alpha-1)/2$, then every Borel measure ν on Γ with finite Riesz energy can be identified with an element of the Sobolev–Slobodetski space $H^{-\varepsilon/2}(\Gamma)$, where $\varepsilon := \alpha-1$, in the sense that the functional ν on $C^\infty(\Gamma)$ can be extended by continuity to the whole space $H^{\varepsilon/2}(\Gamma)$ and, moreover, the Riesz energy norm of ν

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and the corresponding one in $H^{-\varepsilon/2}(\Gamma)$ are equivalent.[1] Therefore, under proper assumptions on the external field, for these Γ_ℓ, the Gauss problem over Borel measures is equivalent to the problem of minimizing the Gauss functional over the corresponding affine cone in $H^{-\varepsilon/2}(\Gamma)$, and then the Gauss functional can be expressed in terms of a simple layer boundary integral operator on Γ. This allows us to approximate the Gauss problem by employing the boundary element method. The latter corresponds to a nonlinear variational problem on the convex cone of all $\varphi = \sum_{\ell\in L} \alpha_\ell \varphi^\ell$ where $\varphi^\ell \in H^{-\varepsilon/2}(\Gamma_\ell)$ and $\varphi^\ell \geqslant 0$.

In [15, 23], under the assumptions admitted therein, we used a penalty formulation of the above-mentioned nonlinear variational problem, whose discrete version allowed us the application of the gradient projection method; corresponding convergence and error analysis has also been provided. The convergence of the gradient projection method depends on the degrees of freedom and the penalty parameter, and it becomes extremely slow for higher accuracy; whereas with an active set strategy the solution can be obtained significantly faster. As to the (much more general) case investigated in the present paper, corresponding work applying an active set strategy is in progress.

In this paper, numerical experiments are given in the case of two oppositely signed manifolds Γ_1 and Γ_2, immersed into $\mathbb{R}^3$, and they are based on an alternative approach to the Gauss problem, provided in [32]. This refers to distributions $\varphi = \sum_{\ell\in L} \alpha_\ell \varphi^\ell$ whose weighted potentials satisfy certain boundary conditions, involving the minimum weighted energy, but now with $\varphi^\ell \in H^{-\varepsilon/2}(\Gamma_\ell)$ not necessarily positive. In the special case where the equilibrium weighted potential takes constant values on each of Γ_i, $i = 1, 2$, we are led to a system of nonlinear boundary integral equations on Γ. The corresponding numerical solution is found with a few steps of Newton's iteration employing wavelet matrix compression [10, 14].

In applications, the numerical solution of the Gauss variational problem is of great interest if for practical reasons in electrical engineering on some of the Γ_ℓ only nonnegative while on the others only nonpositive charges are allowed (see "capacitors" in [18]). It also has applications in approximation theory and the development of efficient numerical integration (see [16]).

2 Gauss variational problem

We consider the problem of minimizing the energy relative to the Riesz kernel $|\mathbf{x} - \mathbf{y}|^{\alpha-n}$ of order $\alpha \in (1, n)$ for signed Borel measures on a given $(n-1)$-dimensional (in general, non-connected) manifold Γ in $\mathbb{R}^n$, $n \geqslant 2$, in the presence of an external field. The corresponding admissible measures (or charges) are associated with a (generalized) condenser, which is meant here as an ordered collection $\mathbf{A} = (A_i)_{i\in I}$ of finitely many mutually disjoint plates A_i, $i \in I$, and each A_i is the finite union of compact, nonintersecting, boundaryless, connected Lipschitz $(n-1)$-dimensional orientable manifolds Γ_ℓ, $\ell \in L_i$, immersed into $\mathbb{R}^n$. That is, $\Gamma = \bigcup_{i\in I} A_i$, where $A_i = \bigcup_{\ell\in L_i} \Gamma_\ell$. Each plate A_i, $i \in I$, is treated with the sign α_i prescribed, where α_i takes the value $+1$ for $i \in I^+$ and -1 for $i \in I^-$. Here, $I = I^+ \cup I^-$, $I^+ \cap I^- = \varnothing$, and I^- is allowed to be empty.

Changing notations if necessary, we assume the index sets L_i, $i \in I$, to be mutually disjoint. Write $L := \bigcup_{i\in I} L_i$, $L^+ := \bigcup_{i\in I^+} L_i$, $L^- := \bigcup_{i\in I^-} L_i$ and define $\alpha_\ell := +1$ for $\ell \in L^+$ and $\alpha_\ell := -1$ for $\ell \in L^-$.

To introduce notations and preliminary results, we consider the Riesz kernel of arbitrary order $0 < \alpha < n$. Let $\mathfrak{M} = \mathfrak{M}(\mathbb{R}^n)$ stand for the σ-algebra of all Borel measures ν on $\mathbb{R}^n$, equipped with the *vague* topology, i.e., that of pointwise convergence on the class $C_0(\mathbb{R}^n)$ of all real-valued continuous functions on $\mathbb{R}^n$ with compact support (see, e.g., [3]). For $\nu, \nu_1 \in \mathfrak{M}$, the mutual Riesz energy and the Riesz potential are given by

$$I_\alpha(\nu,\nu_1) := \int_{\mathbb{R}^n\times\mathbb{R}^n} |\mathbf{x}-\mathbf{y}|^{\alpha-n}\, d(\nu\otimes\nu_1)(\mathbf{x},\mathbf{y}) \quad \text{and} \quad U_\alpha^\nu(\mathbf{x}) := \int_{\mathbb{R}^n} |\mathbf{x}-\mathbf{y}|^{\alpha-n}\, d\nu(\mathbf{y}),$$

respectively, provided the corresponding integral on the right is well defined (as a finite number or $\pm\infty$). For $\nu = \nu_1$, we get the Riesz energy $I_\alpha(\nu) := I_\alpha(\nu,\nu)$ of ν.

Let $\mathcal{E}_\alpha = \mathcal{E}_\alpha(\mathbb{R}^n)$ consist of all $\nu \in \mathfrak{M}$ with finite energy. Since the Riesz kernel is strictly positive definite (see, e.g., [19]), the bilinear form $I_\alpha(\nu_1,\nu_2)$ defines on $\mathcal{E}_\alpha$ a scalar product and, hence, the norm

$$\|\nu\|_{\mathcal{E}_\alpha} := \sqrt{I_\alpha(\nu)}.$$

[1] In the case where two elements of two different topological spaces, respectively, can be identified in some sense, evident from the context, they are denoted by the same symbol.

The topology on $\mathcal{E}_\alpha$ defined by the norm $\|\cdot\|_{\mathcal{E}_\alpha}$ is called *strong*.

As has been shown by H. Cartan [6], $\mathcal{E}_\alpha$ is, in general, strongly incomplete[2] (and, hence, it is a pre-Hilbert space), while, by J. Deny [11] (see also [19]), $\mathcal{E}_\alpha$ can be isometrically imbedded into its completion, the space S^*_α of slowly increasing distributions $T \in S^*$ with finite energy

$$\|T\|^2_{S^*_\alpha} = C(n,\alpha) \int_{\mathbb{R}^n} \frac{|\hat{T}(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^\alpha}\, d\boldsymbol{\xi}. \tag{2.1}$$

Here,

$$C(n,\alpha) := 2^\alpha \pi^{\,n/2} \frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \tag{2.2}$$

$\Gamma(\cdot)$ being the Gamma function, and $\hat{T}(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathbb{R}^n$, is the Fourier transform of $T \in S^*$, i.e.

$$\hat{T}(\boldsymbol{\xi}) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, dT(\mathbf{x}).$$

Observe that the constant $C(n,\alpha)$, appeared in (2.1), differs from that in [19, (6.1.3)] because of the different normalizing factors used in the definitions of the Riesz kernel and the Fourier transform.

Given a Borel set $B \subset \mathbb{R}^n$, let $\mathfrak{M}(B)$ consist of all $\nu \in \mathfrak{M}$ concentrated in B, and let $\mathfrak{M}^+(B)$ be the convex cone of all nonnegative $\nu \in \mathfrak{M}(B)$. Also write $\mathcal{E}_\alpha(B) := \mathfrak{M}(B) \cap \mathcal{E}_\alpha$, $\mathcal{E}^+_\alpha(B) := \mathfrak{M}^+(B) \cap \mathcal{E}_\alpha$ and equip $\mathfrak{M}(B)$ and $\mathcal{E}_\alpha(B)$ with the vague and strong topologies inherited from $\mathfrak{M}$ and $\mathcal{E}_\alpha$, respectively. Then $\mathcal{E}_\alpha(B)$ is a pre-Hilbert (in general, strongly incomplete) space as well.

The condenser $\mathbf{A} = (A_i)_{i\in I}$, defined above, is supposed to be loaded by charges

$$\mu = \sum_{i\in I} \alpha_i \mu^i, \quad \text{where } \mu^i \in \mathcal{E}^+_\alpha(A_i).$$

The set of all those μ will be denoted by $\mathcal{E}_\alpha(\mathbf{A})$; it is a convex cone in the pre-Hilbert space $\mathcal{E}_\alpha(\Gamma) = \mathcal{E}_\alpha\big(\bigcup_{i\in I} A_i\big)$.

Further, let g be a given continuous, positive function on Γ and let $\mathbf{a} = (a_i)_{i\in I}$ be a given vector with $a_i > 0$, $i \in I$. Then the set of admissible charges for the Gauss problem is defined by

$$\mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g) := \Big\{\mu \in \mathcal{E}_\alpha(\mathbf{A}) : \int_{A_i} g\, d\mu^i = a_i \quad \text{for all } i \in I\Big\}.$$

Note that $\mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$ is an affine, convex cone in $\mathcal{E}_\alpha(\Gamma)$.

In addition, let f be a given continuous function on Γ, characterizing an exterior source of energy. Then

$$\mathbb{G}_f(\mu) := I_\alpha(\mu) + 2\int_\Gamma f\, d\mu$$

defines the value of the *Gauss functional* at $\mu \in \mathcal{E}_\alpha(\mathbf{A})$. The Gauss problem now reads as follows:

Problem 2.1 Let $\alpha \in (1,n)$. Find λ that minimizes $\mathbb{G}_f(\mu)$ in $\mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$, i.e., $\lambda \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$ with

$$\mathbb{G}_f(\lambda) = \inf_{\mu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)} \mathbb{G}_f(\mu) =: \mathbb{G}_f(\mathbf{A},\mathbf{a},g). \tag{2.3}$$

A minimizer λ is unique (if exists). This follows from the strict positive definiteness of the Riesz kernel and the convexity of the class of admissible measures; see [29]. But what about the existence of λ?

Assume for a moment that at least one of the A_i is noncompact. Then it is not clear at all whether the equilibrium state in the Gauss variational problem can be attained. Moreover, it has been shown by the third author that, in this case, a minimizing measure λ in general does not exist; necessary and sufficient conditions for λ to exist were given in [28, 30, 31]. See also Section 10 below for some related numerical experiments.

[2] At least, this is the case for $\alpha > 1$ (see [19, Theorem 1.19]).

However, in the case under consideration, where all the A_i are assumed to be compact, the Gauss variational problem has a (unique) solution λ. Indeed, this follows from the vague compactness of $\mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$ when combined with the fact that the Gauss functional $\mathbb{G}_f$ is vaguely lower semicontinuous on $\mathcal{E}_\alpha(\mathbf{A})$; cf. [24].

If each Γ_ℓ is a $C^{k-1,1}$-manifold with $k>(\alpha-1)/2$ then, under proper additional restrictions on g and f, in Section 6 we give an equivalent formulation of the Gauss variational problem (2.3), now based on distributions concentrated on Γ with densities from the Sobolev–Slobodetski space $H^{-\varepsilon/2}(\Gamma)$, where $\varepsilon:=\alpha-1$.[3] This becomes possible due to the fact that, for these Γ_ℓ, every $\nu\in\mathcal{E}_\alpha(\Gamma)$ can be interpreted as an element of $H^{-\varepsilon/2}(\Gamma)$ in the sense that the functional ν on $C^\infty(\Gamma)$ can be extended by continuity to the whole space $H^{\varepsilon/2}(\Gamma)$ and, moreover, its norm in $\mathcal{E}_\alpha(\Gamma)$ and the one in $H^{-\varepsilon/2}(\Gamma)$ are equivalent; see Theorem 5.1.

3 Riesz potentials in $\mathbb{R}^n$

Let $D\subset\mathbb{R}^n$ be a given bounded domain. For any $s>0$, let $\widetilde{H}^{-s}(D)$ denote the Sobolev space of order $-s$ in D. Recall that $\widetilde{H}^{-s}(D)$ consists of all $\varphi\in H^{-s}(\mathbb{R}^n)$ supported by $\overline{D}$ (see, e.g., [17, (4.1.17)]), where $H^{-s}(\mathbb{R}^n)$ is the Sobolev space of order $-s$ in $\mathbb{R}^n$ (see, e.g., [1]). It also can be obtained as the closure of $C_0^\infty(D)$ with respect to the Sobolev norm $\|\cdot\|_{H^{-s}(\mathbb{R}^n)}$. Below, we shall also use the fact (see, e.g., [17, (4.1.28)]) that the Sobolev space $H^{-s}(\mathbb{R}^n)$ consists of all slowly increasing distributions $\varphi\in S^*$ with

$$\|\varphi\|_{H^{-s}(\mathbb{R}^n)}:=\Big\{\int_{\mathbb{R}^n}\big(1+|\boldsymbol{\xi}|^2\big)^{-s}|\hat\varphi(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\Big\}^{1/2}<\infty. \tag{3.1}$$

For the Riesz potentials of order $\alpha\in(0,n)$ in $\mathbb{R}^n$, $n\geqslant 2$, we have the following

Lemma 3.1 *The operator $\mathbf{V}_{-\alpha}$, given by the formula*

$$\mathbf{V}_{-\alpha}\varphi(\mathbf{x}):=\int_{\mathbb{R}^n}|\mathbf{x}-\mathbf{y}|^{\alpha-n}\varphi(\mathbf{y})\,d\mathbf{y},\quad \textit{where }\ \varphi\in C_0^\infty(D)\ \textit{ and }\ \mathbf{x}\in\mathbb{R}^n,$$

is a strongly elliptic classical pseudodifferential operator of order $-\alpha$. Moreover, there exist positive constants c_1 and c_2 depending on D only such that

$$c_1\|\varphi\|^2_{\widetilde{H}^{-\alpha/2}(D)}\leqslant\big(\mathbf{V}_{-\alpha}\varphi,\varphi\big)_{L_2(D)}\leqslant c_2\|\varphi\|^2_{\widetilde{H}^{-\alpha/2}(D)}\quad \textit{for all }\ \varphi\in\widetilde{H}^{-\alpha/2}(D). \tag{3.2}$$

P r o o f. Observe that the Schwartz kernel of the integral operator $\mathbf{V}_{-\alpha}$ is homogeneous of degree $\alpha-n<0$ and, by Seeley [26], the homogeneous symbol of $\mathbf{V}_{-\alpha}$ can be given by

$$a_{-\alpha}(\mathbf{x},\boldsymbol{\xi})=C(n,\alpha)|\boldsymbol{\xi}|^{-\alpha},\quad \boldsymbol{\xi}\in\mathbb{R}^n,$$

where $C(n,\alpha)$ is defined by (2.2). Since for $|\boldsymbol{\xi}|=1$, $a_{-\alpha}(\mathbf{x},\boldsymbol{\xi})$ is a positive constant, $\mathbf{V}_{-\alpha}$ is strongly elliptic and, as a pseudodifferential operator on the (bounded) domain D, it is continuous. This yields the inequality on the right in (3.2) with a constant c_2 depending on D only. The one on the left follows with the Fourier transform and Parseval's equality (see [17, Section 7.1.1]); actually, c_1 does not depend on D. □

Let $S^*_\alpha(D)$ be the topological subspace of S^*_α consisting of all $T\in S^*_\alpha$ with $\operatorname{supp}T\subset\overline{D}$. We next establish relationships between the pre-Hilbert space $\mathcal{E}_\alpha(D)$, the Sobolev space $\widetilde{H}^{-\alpha/2}(D)$, and the space $S^*_\alpha(D)$.

Lemma 3.2 *The spaces $\widetilde{H}^{-\alpha/2}(D)$ and $S^*_\alpha(D)$ are topologically equivalent.*

P r o o f. For any $T\in S^*_\alpha(D)$ we get, by (2.1) and (3.1),

$$\|T\|^2_{H^{-\alpha/2}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\big(1+|\boldsymbol{\xi}|^2\big)^{-\alpha/2}|\hat T(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\leqslant C(n,\alpha)^{-1}\|T\|^2_{S^*_\alpha},$$

[3] These distributions define bounded linear functionals on $H^{\varepsilon/2}(\Gamma)$, whereas Borel measures $\mu\in\mathfrak{M}(\Gamma)$ define bounded linear functionals on $C(\Gamma)$; however, $C(\Gamma)\not\subset H^{\varepsilon/2}(\Gamma)\not\subset C(\Gamma)$ (for more details, see Section 3 below).

so that $T \in \widetilde{H}^{-\alpha/2}(D)$. Conversely, for any $\varphi \in \widetilde{H}^{-\alpha/2}(D)$, we have $\varphi \in S^*_\alpha(D)$ since, by the Parseval–Plancherel formula and relation (2.1),

$$\big(\mathbf{V}_{-\alpha}\varphi, \varphi\big)_{L_2(\mathbb{R}^n)} = C(n,\alpha)\int_{\mathbb{R}^n} \frac{|\widehat{\varphi}(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^\alpha}\, d\boldsymbol{\xi} = \|\varphi\|^2_{S^*_\alpha}.$$

When combined with (3.2), the last relation also shows that the norms $\|\cdot\|_{\widetilde{H}^{-\alpha/2}(D)}$ and $\|\cdot\|_{S^*_\alpha(D)}$ are equivalent as claimed. $\square$

Corollary 3.3 *The pre-Hilbert space $\mathcal{E}_\alpha(D)$ is topologically equivalent to a certain subspace of $\widetilde{H}^{-\alpha/2}(D)$. That is, each $\nu \in \mathcal{E}_\alpha(D)$ can be interpreted as an element of $\widetilde{H}^{-\alpha/2}(D)$ (we denote it by ν as well) and*

$$c_1\|\nu\|_{\widetilde{H}^{-\alpha/2}(D)} \leqslant \|\nu\|_{\mathcal{E}_\alpha} \leqslant c_2\|\nu\|_{\widetilde{H}^{-\alpha/2}(D)}, \tag{3.3}$$

where c_1 and c_2 are positive and independent of ν. Moreover, $\widetilde{H}^{-\alpha/2}(D)$ is the completion of $\mathcal{E}_\alpha(D)$ with respect to the norm $\|\cdot\|_{\widetilde{H}^{-\alpha/2}(D)}$. The same holds true for $S^*_\alpha(D)$ instead of $\widetilde{H}^{-\alpha/2}(D)$.

Proof. Indeed, this is an immediate consequence of Deny's theorem (cf. Section 2 above), Lemma 3.2 and the fact that $C_0^\infty(D)$ is dense in $\widetilde{H}^{-\alpha/2}(D)$. $\square$

4 Riesz potentials in $\mathbb{R}^n$ and on $C^{k-1,1}$-manifolds

From now on, we shall always assume α, the order of the Riesz kernel, to satisfy the requirement $1 < \alpha < n$, and we write $\varepsilon := \varepsilon(\alpha) := \alpha - 1$. Then $0 < \varepsilon < n-1$.

Also, we shall always tacitly assume that Γ_ℓ, $\ell \in L$, are compact, connected, mutually disjoint, boundaryless, $(n-1)$-dimensional orientable $C^{k-1,1}$-manifolds with $k > (\alpha-1)/2$, immersed into $\mathbb{R}^n$, and $\Gamma = \bigcup_{\ell\in L}\Gamma_\ell$.

Let $\Omega \subset \mathbb{R}^n$ be the domain (bounded or unbounded) with the boundary $\partial_{\mathbb{R}^n}\Omega = \Gamma$ and let $H^{\varepsilon/2}(\Gamma)$ be the space of traces of elements from the Sobolev space $H^{\alpha/2}(\Omega)$ on Γ (see [1, 13]). Let $C^\infty(\Gamma)$ be the trace space of $C_0^\infty(\mathbb{R}^n)$ on Γ, and define for $\varphi \in C^\infty(\Gamma)$

$$\|\varphi\|_{H^{\varepsilon/2}(\Gamma)} := \inf\Big\{\|\tilde\varphi\|_{H^{\alpha/2}(\Omega)}, \quad \text{where } \tilde\varphi \in C_0^\infty(\mathbb{R}^n) \text{ and } \tilde\varphi|_\Gamma = \varphi\Big\}. \tag{4.1}$$

Since Γ is Lipschitz, $C^\infty(\Gamma)$ is dense in the trace space $H^{\varepsilon/2}(\Gamma)$, its closure with respect to the norm given by (4.1) (see [1]).

Moreover, the surface measure ds on Γ is well defined and generates on $C^\infty(\Gamma)$ the L_2-scalar product,

$$(\varphi,\psi) := (\varphi,\psi)_{L_2(\Gamma)} := \int_\Gamma \varphi\psi\, ds, \quad \text{where } \varphi,\psi \in C^\infty(\Gamma). \tag{4.2}$$

In fact, $H^{\varepsilon/2}(\Gamma)$ is a Hilbert space equipped with the scalar product

$$(\!(\varphi,\psi)\!)_{H^{\varepsilon/2}(\Gamma)} := (\varphi,\psi)_{L_2(\Gamma)} + \int_\Gamma\int_\Gamma \frac{\big(\varphi(\mathbf{x})-\varphi(\mathbf{y})\big)\big(\psi(\mathbf{x})-\psi(\mathbf{y})\big)}{|\mathbf{x}-\mathbf{y}|^{n-1+\varepsilon}}\, ds(\mathbf{x})\, ds(\mathbf{y})$$

and the norms given by (4.1) and by $\sqrt{(\!(\varphi,\varphi)\!)_{H^{\varepsilon/2}(\Gamma)}}$ are equivalent (see [1, Th. 7.48]).

The L_2-scalar product (4.2) continuously extends to the duality between $H^{\varepsilon/2}(\Gamma)$ and its dual space $H^{-\varepsilon/2}(\Gamma)$, which is equipped with the norm

$$\|\varphi\|_{H^{-\varepsilon/2}(\Gamma)} := \sup\Big\{|(\varphi,\psi)|, \text{ where } \psi \in H^{\varepsilon/2}(\Gamma) \text{ and } \|\psi\|_{H^{\varepsilon/2}(\Gamma)} \leqslant 1\Big\}.$$

We denote that extension by the same symbol $(\cdot,\cdot) = (\cdot,\cdot)_{L_2(\Gamma)}$. Note that the function space $C^\infty(\Gamma)$ is also dense in each of the spaces $L_2(\Gamma)$ and $H^{-\varepsilon/2}(\Gamma)$.

We shall show below that, under proper additional restrictions on g and f, the solution to the Gauss problem (2.3) can be obtained with the help of the simple layer potential

$$V_{-\alpha}\psi(\mathbf{x}) := \int_\Gamma |\mathbf{x}-\mathbf{y}|^{\alpha-n}\psi(\mathbf{y})\,ds(\mathbf{y}), \quad \text{where } \mathbf{x}\in\mathbb{R}^n \text{ and } \psi\in H^{-\varepsilon/2}(\Gamma).$$

In our analysis, the operator V defined by

$$V := \gamma_0 V_{-\alpha}, \tag{4.3}$$

where γ_0 is the Gagliardo trace operator onto Γ (see [13]), will play a decisive role. The operator γ_0 is characterized by the following slightly extended version of the trace theorem (compare with [9, 21, 22]).

Theorem 4.1 *Given Γ of the class $C^{k-1,1}$, let $1/2 < s < k+1/2$. Then, for the Gagliardo trace operator γ_0 and its adjoint γ_0^*,*

$$\gamma_0 : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\Gamma) \quad \textit{and} \quad \gamma_0^* : H^{\frac{1}{2}-s}(\Gamma) \to H^{-s}(\mathbb{R}^n),$$

there exist positive constants c, c' and c'' depending on s, n, and Γ only such that

$$\|\gamma_0\Phi\|_{H^{s-\frac{1}{2}}(\Gamma)} \leqslant c\,\|\Phi\|_{H^s(\mathbb{R}^n)} \quad \textit{for all } \Phi\in C_0^\infty(\mathbb{R}^n),$$
$$c'\|\psi\|_{H^{\frac{1}{2}-s}(\Gamma)} \leqslant \|\gamma_0^*\psi\|_{H^{-s}(\mathbb{R}^n)} \leqslant c''\|\psi\|_{H^{\frac{1}{2}-s}(\Gamma)} \quad \textit{for all } \psi\in H^{\frac{1}{2}-s}(\Gamma), \tag{4.4}$$

and so γ_0 and γ_0^* are continuous.

Here, the adjoint operator γ_0^* is defined by

$$(\gamma_0^*\psi,\Phi)_{L_2(\mathbb{R}^n)} = (\psi,\gamma_0\Phi)_{L_2(\Gamma)}, \quad \text{where } \Phi\in C_0^\infty(\mathbb{R}^n) \text{ and } \psi\in H^{\frac{1}{2}-s}(\Gamma). \tag{4.5}$$

Observe that then $\operatorname{supp}(\gamma_0^*\psi)\subset\Gamma$ for all $\psi\in H^{\frac{1}{2}-s}(\Gamma)$.

Remark 4.2 If in Theorem 4.1, Γ is replaced by $\mathbb{R}^{n-1}$, then its assertion holds true for all $s > 1/2$ (see [20]).

The proof of Theorem 4.1 will be given in the Appendix.

Theorem 4.3 *Under the stated assumptions on α and Γ, the operator V, defined by* (4.3)*, is a linear, continuous, invertible mapping*

$$V:\ H^{-\varepsilon/2}(\Gamma) \to H^{\varepsilon/2}(\Gamma).$$

Moreover, it is $H^{-\varepsilon/2}(\Gamma)$-elliptic; i.e., there exist positive constants c_c and c_V depending on n, Γ, and ε only such that

$$c_V\|\psi\|^2_{H^{-\varepsilon/2}(\Gamma)} \leqslant \|\psi\|^2_V \leqslant c_c\|\psi\|^2_{H^{-\varepsilon/2}(\Gamma)} \quad \textit{for all } \psi\in H^{-\varepsilon/2}(\Gamma), \tag{4.6}$$

where

$$\|\psi\|_V^2 := (\psi, V\psi)_{L_2(\Gamma)}.$$

P r o o f. Fix $\psi\in H^{-\varepsilon/2}(\Gamma)$ and choose r such that $\Gamma\subset B_r$, where B_r is an open ball of radius r. Having observed that $1/2<\alpha/2<k+1/2$, from Theorem 4.1 with $s=\alpha/2$ we get $\gamma_0^*\psi\in H^{-\frac{\varepsilon}{2}-\frac{1}{2}}(\mathbb{R}^n) = H^{-\frac{\alpha}{2}}(\mathbb{R}^n)$. Actually,

$$\gamma_0^*\psi \in \widetilde{H}^{-\frac{\alpha}{2}}(B_r) \tag{4.7}$$

because of $\operatorname{supp}(\gamma_0^*\psi)\subset\Gamma$. Therefore, in consequence of Lemma 3.1 with $D=B_r$,

$$\mathbf{V}_{-\alpha}\gamma_0^*\psi \in H^{\alpha/2}(B_r). \tag{4.8}$$

Repeated application of Theorem 4.1 with $s = \alpha/2$ then shows that the trace of $\mathbf{V}_{-\alpha}\gamma_0^*\psi$ on Γ exists and, due to (3.2), (4.4) and (4.5),

$$(V\psi,\psi)_{L_2(\Gamma)} = (\gamma_0\mathbf{V}_{-\alpha}\gamma_0^*\psi,\psi)_{L_2(\Gamma)} = (\mathbf{V}_{-\alpha}\gamma_0^*\psi,\gamma_0^*\psi)_{L_2(B_r)} \geqslant c_1\|\gamma_0^*\psi\|^2_{\widetilde{H}^{-\alpha/2}(B_r)} \geqslant c_1{c'}^2\|\psi\|^2_{H^{-\varepsilon/2}(\Gamma)}$$

and also

$$(V\psi,\psi)_{L_2(\Gamma)} \leqslant c_2\|\gamma_0^*\psi\|^2_{\widetilde{H}^{-\alpha/2}(B_r)} \leqslant c_2{c''}^2\|\psi\|^2_{H^{-\varepsilon/2}(\Gamma)},$$

which is (4.6). Here, c_1, c_2 and c', c'' are taken from (3.2) and (4.4), respectively.

The invertibility of V then follows with the Lax–Milgram lemma. This completes the proof. □

Remark 4.4 If $n = 2$ or $n = 3$, then Theorem 4.3 is valid for any $\alpha \in (1, n)$ provided Γ is just Lipschitz. See [9, Th. 3.6] and [21, pp. 98–102].

5 Relations between $\mathcal{E}_\alpha(\Gamma)$ and $H^{-\varepsilon/2}(\Gamma)$

The main purpose of this section is to characterize the Borel measures on Γ with finite Riesz energy, namely $\nu \in \mathcal{E}_\alpha(\Gamma)$ where $1 < \alpha < n$, via distributions in $H^{-\varepsilon/2}(\Gamma)$ with $\varepsilon = \alpha - 1$. Recall that Γ is a $C^{k-1,1}$-manifold with $k > (\alpha-1)/2$. The characterization obtained is given by the following principal result (cf. Corollary 3.3).

Theorem 5.1 *Under the stated assumptions on α and Γ, $\mathcal{E}_\alpha(\Gamma)$ is topologically equivalent to a certain subspace of $H^{-\varepsilon/2}(\Gamma)$. That is, each $\nu \in \mathcal{E}_\alpha(\Gamma)$ can be interpreted as an element of $H^{-\varepsilon/2}(\Gamma)$ in the sense that the functional ν on $C^\infty(\Gamma)$ can be extended by continuity to the whole space $H^{\varepsilon/2}(\Gamma)$ and*

$$c_1\|\nu\|_{H^{-\varepsilon/2}(\Gamma)} \leqslant \|\nu\|_{\mathcal{E}_\alpha} \leqslant c_2\|\nu\|_{H^{-\varepsilon/2}(\Gamma)}, \tag{5.1}$$

where the constants c_1 and c_2 are positive and independent of ν. Moreover, $H^{-\varepsilon/2}(\Gamma)$ is the completion of the pre-Hilbert space $\mathcal{E}_\alpha(\Gamma)$ with respect to the norm $\|\cdot\|_{H^{-\varepsilon/2}(\Gamma)}$.

P r o o f. The proof is based on Theorem 4.3 and Corollary 3.3.

Choose r so that $\Gamma \subset B_r$. Since V is invertible, for a given $\varphi \in C^\infty(\Gamma)$ there exists $\psi \in H^{-\varepsilon/2}(\Gamma)$ such that

$$V\psi = \varphi.$$

Hence, for any $\nu \in \mathcal{E}_\alpha(\Gamma)$,

$$\nu(\varphi) = \int_\Gamma (V\psi)\,d\nu = \int_\Gamma \gamma_0(\mathbf{V}_{-\alpha}\gamma_0^*\psi)\,d\nu = \big(\mathbf{V}_{-\alpha}\gamma_0^*\psi,\nu\big)_{L_2(B_r)}, \tag{5.2}$$

the last equality being obtained with exploiting the fact that ν can be treated as an element of $\widetilde{H}^{-\alpha/2}(B_r)$ (see Corollary 3.3). Taking (4.7) and (4.8) into account, with the help of Lemma 3.1, relations (3.3) and (4.4), and Theorem 4.3, from (5.2) we get

$$\begin{aligned}|\nu(\varphi)| &\leqslant \|\mathbf{V}_{-\alpha}\gamma_0^*\psi\|_{H^{\alpha/2}(B_r)}\|\nu\|_{\widetilde{H}^{-\alpha/2}(B_r)} \leqslant c\|\gamma_0^*\psi\|_{\widetilde{H}^{-\alpha/2}(B_r)}\|\nu\|_{\mathcal{E}\alpha}\\ &\leqslant c'\|\nu\|_{\mathcal{E}\alpha}\|\psi\|_{H^{-\varepsilon/2}(\Gamma)} \leqslant c''\|\nu\|_{\mathcal{E}\alpha}\|\varphi\|_{H^{\varepsilon/2}(\Gamma)},\end{aligned}$$

which proves that, actually, ν can be identified with a distribution in $H^{-\varepsilon/2}(\Gamma)$. Therefore, applying Theorem 4.3 to $\nu \in \mathcal{E}_\alpha(\Gamma)$, treated now as an element of $H^{-\varepsilon/2}(\Gamma)$, we have

$$\|\nu\|^2_{\mathcal{E}_\alpha} = (V\nu,\nu)_{L_2(\Gamma)} = \|\nu\|^2_V \cong \|\nu\|^2_{H^{-\varepsilon/2}(\Gamma)} \tag{5.3}$$

which proves (5.1). Finally, combining (5.1) with the fact that $C^\infty(\Gamma)$ is dense in $H^{-\varepsilon/2}(\Gamma)$, we see that, indeed, $H^{-\varepsilon/2}(\Gamma)$ is the completion of $\mathcal{E}_\alpha(\Gamma)$ with respect to the norm $\|\cdot\|_{H^{-\varepsilon/2}(\Gamma)}$ as claimed. □

Corollary 5.2 *Under the stated assumptions on α and Γ, for every $\nu \in \mathcal{E}_\alpha(\Gamma)$ there exist absolutely continuous measures $\nu_k \in \mathcal{E}_\alpha(\Gamma)$, $k \in \mathbb{N}$, with densities $\varphi_k \in C^\infty(\Gamma)$ $\big($i.e., $d\nu_k(\mathbf{x}) = \varphi_k(\mathbf{x})\, ds(\mathbf{x})\big)$ such that $\nu_k \to \nu$ vaguely and strongly, i.e.*[4]

$$\nu_k(\varphi) \to \nu(\varphi) \quad \textit{for all } \ \varphi \in C(\Gamma) \quad \textit{and} \quad \lim_{k\to\infty} \|\nu_k - \nu\|_{\mathcal{E}_\alpha} = 0.$$

P r o o f. Without loss of generality, we can assume $\nu \in \mathcal{E}_\alpha(\Gamma)$ to be nonnegative, i.e. $\nu \in \mathcal{E}^+_\alpha(\Gamma)$. We consider it to be an element of $H^{-\varepsilon/2}(\Gamma)$, which is possible due to Theorem 5.1. Since $C^\infty(\Gamma)$ is dense in $H^{-\varepsilon/2}(\Gamma)$, there exists a sequence $\varphi_k \in C^\infty(\Gamma)$, $k \in \mathbb{N}$, converging to ν in $H^{-\varepsilon/2}(\Gamma)$ and, because of (5.1), also in $\mathcal{E}_\alpha$. Since for the Riesz kernel the strong convergence of nonnegative measures implies the vague convergence to the same limit (see, e.g., Lemma 1.2 in [19]), the corollary follows. □

6 Variational formulation in the space $H^{-\varepsilon/2}(\Gamma)$

From now on, for the given functions g and f we require that $f, g \in C(\Gamma) \cap H^{\varepsilon/2}(\Gamma)$. Define

$$\mathbb{V}_f(\varphi) := \|\varphi\|^2_V + 2(f,\varphi)_{L_2(\Gamma)}, \quad \text{where } \ \varphi \in H^{-\varepsilon/2}(\Gamma).$$

The following theorem shows that the Gauss problem (2.3) on $\mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$ (for the Riesz kernel $|\mathbf{x}-\mathbf{y}|^{\alpha-n}$ of order $\alpha \in (1,n)$) can alternatively be formulated as the problem of minimizing the functional $\mathbb{V}_f$ over the affine cone $\mathcal{K}(\mathbf{A},\mathbf{a},g)$ in $H^{-\varepsilon/2}(\Gamma)$, where

$$\mathcal{K}(\mathbf{A},\mathbf{a},g) := \Big\{\varphi = \sum_{\ell\in L} \alpha_\ell \varphi^\ell : \ \varphi^\ell \in H^{-\varepsilon/2}(\Gamma_\ell), \ \varphi^\ell \geqslant 0 \ \text{ and } \sum_{\ell \in L_i} (g,\varphi^\ell)_{L_2(\Gamma_\ell)} = a_i \ \text{ for all } \ i \in I\Big\}.$$

Theorem 6.1 *Under the stated assumptions on α, g, f, and Γ, the solution $\lambda \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$ of the Gauss problem* (2.3)*, treated as an element of $H^{-\varepsilon/2}(\Gamma)$, belongs to $\mathcal{K}(\mathbf{A},\mathbf{a},g)$ and satisfies the relation*

$$\mathbb{V}_f(\lambda) = \mathbb{G}_f(\lambda) = \mathbb{G}_f(\mathbf{A},\mathbf{a},g). \tag{6.1}$$

This λ is the unique minimizer of the functional $\mathbb{V}_f$ over $\mathcal{K}(\mathbf{A},\mathbf{a},g)$, i.e.,

$$\mathbb{V}_f(\lambda) = \min_{\varphi\in\mathcal{K}(\mathbf{A},\mathbf{a},g)} \mathbb{V}_f(\varphi) =: \mathbb{V}_f(\mathbf{A},\mathbf{a},g). \tag{6.2}$$

P r o o f. By Theorem 5.1, any Borel measure $\mu = \sum_{\ell\in L} \alpha_\ell \mu^\ell \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$ can be treated as an element of $H^{-\varepsilon/2}(\Gamma)$, while all the μ^ℓ, $\ell \in L$, as elements of $H^{-\varepsilon/2}(\Gamma_\ell)$, correspondingly. The latter implies that, actually, $\mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g) \subset \mathcal{K}(\mathbf{A},\mathbf{a},g)$. Furthermore, applying (5.3), one also gets

$$\mathbb{V}_f(\mu) = \|\mu\|^2_V + 2(\mu,f)_{L_2(\Gamma)} = \|\mu\|^2_{\mathcal{E}_\alpha} + 2\mu(f) = \mathbb{G}_f(\mu) \quad \text{for all } \ \mu \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g), \tag{6.3}$$

which yields that the solution λ of the Gauss problem (2.3) satisfies (6.1). To establish (6.2), we observe that one can construct a sequence $\varphi_k \in C^\infty(\Gamma) \cap \mathcal{K}(\mathbf{A},\mathbf{a},g)$ converging to λ in $H^{-\varepsilon/2}(\Gamma)$. Hence, by (6.1),

$$\mathbb{V}_f(\varphi_k) \to \mathbb{V}_f(\lambda) = \mathbb{G}_f(\mathbf{A},\mathbf{a},g).$$

Moreover, $\varphi ds \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{a},g)$ for all $\varphi \in C^\infty(\Gamma) \cap \mathcal{K}(\mathbf{A},\mathbf{a},g)$ and so, by (6.3),

$$\mathbb{G}_f(\mathbf{A},\mathbf{a},g) \leqslant \inf_{\varphi\in\mathcal{K}(\mathbf{A},\mathbf{a},g)\cap C^\infty(\Gamma)} \mathbb{V}_f(\varphi) \leqslant \mathbb{V}_f(\varphi_k) \quad \text{for all } \ k \in \mathbb{N},$$

which implies with $k \to \infty$

$$\inf_{\varphi\in\mathcal{K}(\mathbf{A},\mathbf{a},g)\cap C^\infty(\Gamma)} \mathbb{V}_f(\varphi) = \mathbb{V}_f(\lambda) = \mathbb{G}_f(\mathbf{A},\mathbf{a},g).$$

Repeated application of the fact that $C^\infty(\Gamma)$ is a dense subspace of $H^{-\varepsilon/2}(\Gamma)$ yields (6.2) as required. □

[4] Compare with Lemma 1.2 and Corollary 2 in [19, Chapt. 1], where Cartan's approximating measures have, in fact, n-dimensional supports in $\mathbb{R}^n$.

7 Alternative approach to the Gauss problem

For any $\varphi \in H^{-\varepsilon/2}(\Gamma)$, write

$$\varphi^i := \alpha_i \varphi|_{A_i}, \quad i \in I;$$

then

$$\varphi = \sum_{i\in I} \alpha_i \varphi^i.$$

Note that φ^i belongs to $H^{-\varepsilon/2}(A_i)$, but it is no longer necessarily positive — in contrast to what we have had for elements from $\mathcal{K}(\mathbf{A}, \mathbf{a}, g)$. Given $\varphi \in H^{-\varepsilon/2}(\Gamma)$ and $i \in I$, define

$$\Psi^i(\mathbf{x}, \varphi) := a_i \frac{V_{-\alpha}\varphi(\mathbf{x}) + f(\mathbf{x})}{g(\mathbf{x})} + (f, \varphi^i)_{L_2(\Gamma)}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Observe that, if $\varphi = \mu \in \mathcal{E}_\alpha(\Gamma)$, then $V_{-\alpha}\mu(\mathbf{x}) = U^\mu_\alpha(\mathbf{x})$ and, hence, $\Psi^i(\mathbf{x}, \mu)$ is well defined and finite nearly everywhere (n.e.) in $\mathbb{R}^n$ (see, e.g., [19]), i.e., excepting at most a subset of $\mathbb{R}^n$ with the Riesz capacity zero.

We denote by $\mathcal{G}_\alpha(\mathbf{A}, \mathbf{a}, g)$ the cone of all $\varphi \in \mathcal{E}_\alpha(\Gamma)$ for which there exist $\eta_i(\varphi) \in \mathbb{R}$, $i \in I$, such that

$$\alpha_i \Psi^i(\mathbf{x}, \varphi) \geqslant \alpha_i \eta_i(\varphi) \quad \text{n.e. in } A_i, \tag{7.1}$$

$$\sum_{i\in I} \alpha_i \eta_i(\varphi) = \mathbb{V}_f(\mathbf{A}, \mathbf{a}, g). \tag{7.2}$$

Then there holds the following assertion (cf. [32, Th. 2] and [33, Corollary 8.4]).

Theorem 7.1 *The solution λ to the Gauss problem is also the unique minimizer of $\mathbb{V}_f(\varphi)$ over $\mathcal{G}_\alpha(\mathbf{A}, \mathbf{a}, g)$, i.e.*

$$\lambda \in \mathcal{G}_\alpha(\mathbf{A}, \mathbf{a}, g), \tag{7.3}$$

$$\inf_{\varphi\in\mathcal{G}_\alpha(\mathbf{A},\mathbf{a},g)} \mathbb{V}_f(\varphi) = \mathbb{V}_f(\lambda) = \mathbb{V}_f(\mathbf{A}, \mathbf{a}, g). \tag{7.4}$$

P r o o f. For brevity, write $S_\nu := \operatorname{supp}\nu$. According to [29, Th. 1], for every $i \in I$,

$$\alpha_i \Psi^i(\mathbf{x}, \lambda) \geqslant \alpha_i \eta_i(\lambda) \quad \text{n.e. in } A_i, \tag{7.5}$$

$$\alpha_i \Psi^i(\mathbf{x}, \lambda) \leqslant \alpha_i \eta_i(\lambda) \quad \text{for all } \mathbf{x} \in S_{\lambda^i}, \tag{7.6}$$

where

$$\eta_i(\lambda) = I_\alpha(\lambda^i, \lambda) + 2\int f\, d\lambda^i = (V\lambda, \lambda^i)_{L_2(\Gamma)} + 2(f, \lambda^i)_{L_2(\Gamma)}, \tag{7.7}$$

the latter equality in (7.7) is obtained with the application of Theorems 4.3 and 5.1.

Hence, by (7.7) and Theorem 6.1,

$$\sum_{i\in I} \alpha_i \eta_i(\lambda) = \mathbb{V}_f(\lambda) = \mathbb{V}_f(\mathbf{A}, \mathbf{a}, g), \tag{7.8}$$

which together with (7.5) proves inclusion (7.3). In turn, this yields

$$\mathbb{V}_f(\lambda) \geqslant \inf_{\varphi\in\mathcal{G}_\alpha(\mathbf{A},\mathbf{a},g)} \mathbb{V}_f(\varphi). \tag{7.9}$$

To show that this inequality is, in fact, an equality, for any given $\varphi \in \mathcal{G}_\alpha(\mathbf{A}, \mathbf{a}, g)$ and $i \in I$ we multiply (7.1) by $g(\mathbf{x})$ and then we integrate the inequality obtained with respect to λ^i, having used the fact that a set of capacity zero is necessarily of exterior ν-measure zero provided ν has finite energy (see, e.g., [19]). This gives

$$\alpha_i \big[(V\varphi, \lambda^i)_{L_2(\Gamma)} + (f, \lambda^i)_{L_2(\Gamma)} + (f, \varphi^i)_{L_2(\Gamma)}\big] \geqslant \alpha_i \eta_i(\varphi), \quad i \in I.$$

Summing up these inequalities over all $i \in I$ and then substituting (7.2) into the result obtained, after simple transformations we get

$$\mathbb{V}_f(\varphi) \geqslant \|\varphi - \lambda\|_V^2 + \mathbb{V}_f(\lambda) \geqslant \mathbb{V}_f(\lambda) \quad \text{for all } \ \varphi \in \mathcal{G}_\alpha(\mathbf{A}, \mathbf{a}, g),$$

which together with (7.9) establishes (7.4). The proof is complete. □

Corollary 7.2 *Let $\mathcal{H}(\mathbf{A}, \mathbf{a}, g)$ consist of all $\varphi \in H^{-\varepsilon/2}(\Gamma)$ for which there exist $\eta_i(\varphi) \in \mathbb{R}$, $i \in I$, satisfying* (7.2) *and, as well,*

$$\Psi^i(\mathbf{x}, \varphi) = \eta_i(\varphi) \quad \textit{for all } \ \mathbf{x} \in A_i, \quad \textit{where } \ i \in I,$$

and let $\mathbf{A}$, $\mathbf{a}$, g, and f be such that, instead of (7.5), *λ satisfies this very last relation.*[5] *Then λ can also be obtained as the* (*unique*) *minimizer of $\mathbb{V}_f(\varphi)$ over the cone $\mathcal{H}(\mathbf{A}, \mathbf{a}, g)$.*

8 Two manifolds problem

If $L^+ = \{1\}$ and $L^- = \{2\}$, then $\mathcal{H}(\mathbf{A}, \mathbf{a}, g)$ consists of all $\varphi \in H^{-\varepsilon/2}(\Gamma)$ for which there exists $c(\varphi) \in \mathbb{R}$ such that

$$\Psi^1(\mathbf{x}, \varphi) = c(\varphi) + \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) \quad \text{on } \ \Gamma_1, \tag{8.1}$$

$$\Psi^2(\mathbf{x}, \varphi) = c(\varphi) - \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) \quad \text{on } \ \Gamma_2. \tag{8.2}$$

Due to Corollary 7.2, we are led to the following theorem.

Theorem 8.1 *Let $L^+ = \{1\}$, $L^- = \{2\}$, $g = 1$, and let Γ_1, Γ_2, a_1, a_2 and f be such that λ, the solution of the corresponding Gauss problem, satisfies relations* (8.1) *and* (8.2) *with $C := c(\lambda)$. Then, equivalently,*

$$V\lambda^1 - V\lambda^2 = \begin{cases} a_1^{-1}\big[C + \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) - (f, \lambda^1)_{L_2(\Gamma_1)}\big] - f & \textit{on} \quad \Gamma_1, \\ a_2^{-1}\big[C - \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) - (f, \lambda^2)_{L_2(\Gamma_2)}\big] - f & \textit{on} \quad \Gamma_2. \end{cases} \tag{8.3}$$

If, moreover,

$$d_0 := a_2(\dot{\lambda}^1, 1)_{L_2(\Gamma_1)} - a_1(\dot{\lambda}^2, 1)_{L_2(\Gamma_2)} \neq 0, \tag{8.4}$$

then the constant C can be written in the form

$$C = d_0^{-1}\Big\{a_2(\dot{\lambda}^1, 1)_{L_2(\Gamma_1)}\big[(f, \lambda^1)_{L_2(\Gamma_1)} - \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g)\big] - a_1(\dot{\lambda}^2, 1)_{L_2(\Gamma_2)}\big[(f, \lambda^2)_{L_2(\Gamma_2)} + \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g)\big]\Big\} \tag{8.5}$$

where $\dot{\lambda}^i \in H^{-\varepsilon/2}(\Gamma_i)$, $i = 1,\, 2$, solve the system of boundary integral equations

$$V\dot{\lambda}^1 - V\dot{\lambda}^2 = \begin{cases} a_1^{-1}\big[1 - (f, \dot{\lambda}^1)_{L_2(\Gamma_1)}\big] & \textit{on} \quad \Gamma_1, \\ a_2^{-1}\big[1 - (f, \dot{\lambda}^2)_{L_2(\Gamma_2)}\big] & \textit{on} \quad \Gamma_2. \end{cases} \tag{8.6}$$

P r o o f. Observe that for any $c \in \mathbb{R}$ there exist $\varphi_c^i \in H^{-\varepsilon/2}(\Gamma_i)$, $i = 1,\, 2$, satisfying (8.3) with λ^i and C replaced by φ_c^i and c, respectively, i.e.

$$V\varphi_c^1 - V\varphi_c^2 = \begin{cases} a_1^{-1}\big[c + \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) - (f, \varphi_c^1)_{L_2(\Gamma_1)}\big] - f & \text{on} \quad \Gamma_1, \\ a_2^{-1}\big[c - \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) - (f, \varphi_c^2)_{L_2(\Gamma_2)}\big] - f & \text{on} \quad \Gamma_2, \end{cases} \tag{8.7}$$

and these φ_c^i, $i = 1,\, 2$, are determined uniquely. Then $\varphi_c := \varphi_c^1 - \varphi_c^2 \in \mathcal{H}(\mathbf{A}, \mathbf{a}, g)$, and therefore the cone $\mathcal{H}(\mathbf{A}, \mathbf{a}, g)$ can be considered as a one-dimensional family with the parameter $c \in \mathbb{R}$.

[5] In the general case of arbitrary $\mathbf{A}$, $\mathbf{a}$, g and f, this is not so.

Since λ is the minimizer of $\mathbb{V}_f(\varphi_c) = (\varphi_c, V\varphi_c)_{L_2(\Gamma)} + 2(f, \varphi_c)_{L_2(\Gamma)}$ over $c \in \mathbb{R}$ and both φ_c and $\mathbb{V}_f(\varphi_c)$ are continuously differentiable with respect to c, we conclude that

$$\frac{d}{dc}\mathbb{V}_f(\varphi_c)\Big|_{c=C} = 0 = \big\{(\dot\varphi_c, V\varphi_c)_{L_2(\Gamma)} + (\varphi_c, V\dot\varphi_c)_{L_2(\Gamma)} + 2(f, \dot\varphi_c)_{L_2(\Gamma)}\big\}\Big|_{c=C},$$

where $\dot\varphi_c := d\varphi_c/dc$. Having denoted

$$\dot\lambda^i := \dot\varphi_c^i\big|_{c=C} \quad \text{for } i = 1,\, 2, \tag{8.8}$$

we therefore get

$$\begin{aligned} 0 = &+ (\dot\lambda^1, V\lambda^1 - V\lambda^2)_{L_2(\Gamma_1)} - (\dot\lambda^2, V\lambda^1 - V\lambda^2)_{L_2(\Gamma_2)} \\ &+ (\lambda^1, V\dot\lambda^1 - V\dot\lambda^2)_{L_2(\Gamma_1)} - (\lambda^2, V\dot\lambda^1 - V\dot\lambda^2)_{L_2(\Gamma_2)} + 2(f, \dot\lambda^1)_{L_2(\Gamma_1)} - 2(f, \dot\lambda^2)_{L_2(\Gamma_2)}. \end{aligned} \tag{8.9}$$

Differentiating (8.7) with respect to c, in view of (8.8) we find the system of equations (8.6). Now, inserting (8.3) and (8.6) into (8.9) results in

$$\begin{aligned} 0 = &+ \big(\dot\lambda^1, a_1^{-1}[C + \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) - (f, \lambda^1)_{L_2(\Gamma_1)}]\big)_{L_2(\Gamma_1)} - (\dot\lambda^1, f)_{L_2(\Gamma_1)} \\ &- \big(\dot\lambda^2, a_2^{-1}[C - \tfrac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g) - (f, \lambda^2)_{L_2(\Gamma_2)}]\big)_{L_2(\Gamma_2)} + (\dot\lambda^2, f)_{L_2(\Gamma_2)} \\ &+ \big(\lambda^1, a_1^{-1}[1 - (f, \dot\lambda^1)_{L_2(\Gamma_1)}]\big)_{L_2(\Gamma_1)} - \big(\lambda^2, a_2^{-1}[1 - (f, \dot\lambda^2)_{L_2(\Gamma_2)}]\big)_{L_2(\Gamma_2)} \\ &+ 2(f, \dot\lambda^1)_{L_2(\Gamma_1)} - 2(f, \dot\lambda^2)_{L_2(\Gamma_2)}. \end{aligned}$$

Employing here the fact that $(\lambda^i, 1)_{L_2(\Gamma_i)} = a_i$ for $i = 1,\, 2$ and then multiplying the relation obtained by a_1a_2, one gets C in the form (8.5) as was to be proved. □

Remark 8.2 In the case $f = 0$, assumption (8.4) does hold automatically since then $d_0 = a_1a_2(V\dot\lambda, \dot\lambda) > 0$.

In the remainder of this section we shall tacitly require all the assumptions of Theorem 8.1 to be satisfied.

Lemma 8.3 *If $\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g)$ is given, then the systems of equations* (8.3) *and* (8.6) *are both uniquely solvable.*

Proof. Indeed, since (8.3) and (8.6) are the gradient equations to the minimization of a strictly convex, quadratic functional over $\mathcal{H}(\mathbf{A}, \mathbf{a}, g)$, which has a unique solution due to Corollary 7.2, the corresponding linear gradient equations are uniquely solvable. □

The solution of the linear equations (8.6) can be obtained with the Sherman–Morrison formula [27].

Lemma 8.4 *The following procedure provides us with the solution of* (8.3) *and* (8.6)*:*

i) *Determine $\sigma = \sigma^1 - \sigma^2$, where $\sigma^i \in H^{-\varepsilon/2}(\Gamma_i)$ for $i = 1, 2$, as the solution of*

$$V\sigma = \begin{cases} 1/a_1 & \textit{on} \quad \Gamma_1, \\ 1/a_2 & \textit{on} \quad \Gamma_2, \end{cases}$$

and let $\chi = \chi^1 - \chi^2$, where $\chi^i \in H^{-\varepsilon/2}(\Gamma_i)$ for $i = 1, 2$, be the solution of

$$V\chi = 1 \quad \textit{on } \Gamma.$$

ii) *Then the solution of* (8.6) *is given by*

$$\dot\lambda = \sigma + d_1k\chi \quad \textit{on } \Gamma, \tag{8.10}$$

where

$$\begin{aligned} d_1 &:= (\sigma^1, a_1^{-1}f)_{L_2(\Gamma_1)} + (\sigma^2, a_2^{-1}f)_{L_2(\Gamma_2)}, \\ -k^{-1} &:= 1 + (\chi^1, a_1^{-1})_{L_2(\Gamma_1)} + (\chi^2, a_2^{-1})_{L_2(\Gamma_2)}. \end{aligned}$$

iii) *For solving* (8.3) *determine* C *from* (8.5) *by the use of* $\dot{\lambda}$*, the solution of* (8.6)*, and also find* $\eta = \eta^1 - \eta^2$*, where* $\eta^i \in H^{-\varepsilon/2}(\Gamma_i)$ *for* $i = 1, 2$*, by solving*

$$V\eta = \begin{cases} a_1^{-1}\big[C + \frac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g)\big] - f & \textit{on} \quad \Gamma_1, \\ a_2^{-1}\big[C - \frac{1}{2}\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g)\big] - f & \textit{on} \quad \Gamma_2. \end{cases}$$

Then

$$\lambda = \eta + kd_2\chi$$

where $d_2 := (\eta^1, a_1^{-1}f)_{L_2(\Gamma_1)} + (\eta^2, a_2^{-1}f)_{L_2(\Gamma_2)}$.

Proof. With

$$\tilde{f} := \begin{cases} a_1^{-1}f & \text{on} \quad \Gamma_1, \\ -a_2^{-1}f & \text{on} \quad \Gamma_2, \end{cases}$$

the equation (8.6) can be written as

$$M\dot{\lambda} := V\dot{\lambda} + (\tilde{f}, \dot{\lambda})_{L_2(\Gamma)} = h := \begin{cases} 1/a_1 & \text{on} \quad \Gamma_1, \\ 1/a_2 & \text{on} \quad \Gamma_2. \end{cases}$$

Here $M : H^{-\varepsilon/2}(\Gamma) \to H^{\varepsilon/2}(\Gamma)$ is a linear Fredholm operator of index zero since V is invertible and $(\tilde{f}, \cdot)_{L_2(\Gamma)}$ is compact. Inserting $\dot{\lambda}$ as given by (8.10) and taking the definition of k into account, we obtain

$$M\dot{\lambda} = V\sigma + k(\tilde{f}, \sigma)_{L_2(\Gamma)}V\chi + \big(\tilde{f}, \sigma + k(\tilde{f}, \sigma)_{L_2(\Gamma)}\chi\big)_{L_2(\Gamma)} = h,$$

which justifies ii).

Since the proof of iii) can be given in exactly the same manner, we omit the details. □

In Theorem 8.1 and Lemmata 8.3 and 8.4, it is supposed that $\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g)$ is known. However, if the equation (8.5) for the constant C is inserted into (8.3) and $\mathbb{V}_f(\mathbf{A}, \mathbf{a}, g)$ is replaced by

$$\mathbb{V}_f(\lambda) = (\lambda, V\lambda)_{L_2(\Gamma)} + 2(f, \lambda)_{L_2(\Gamma)},$$

then we obtain the nonlinear system of boundary integral equations for λ.

For brevity, define the nonlinear operator

$$\mathfrak{C}(\lambda, \dot{\lambda}, f) := d_0^{-1}\Big\{a_2(\dot{\lambda}^1, 1)_{L_2(\Gamma_1)}\big[(f, \lambda^1)_{L_2(\Gamma_1)} - \tfrac{1}{2}\mathbb{V}_f(\lambda)\big] - a_1(\dot{\lambda}^2, 1)_{L_2(\Gamma_2)}\big[(f, \lambda^2)_{L_2(\Gamma_2)} + \tfrac{1}{2}\mathbb{V}_f(\lambda)\big]\Big\}.$$

The nonlinear system of boundary integral equations for λ now reads as follows:

$$V\lambda^1 - V\lambda^2 = \begin{cases} a_1^{-1}\big[\mathfrak{C}(\lambda, \dot{\lambda}, f) - (f, \lambda^1)_{L_2(\Gamma_1)} + \frac{1}{2}\mathbb{V}_f(\lambda)\big] - f & \text{on} \quad \Gamma_1, \\ a_2^{-1}\big[\mathfrak{C}(\lambda, \dot{\lambda}, f) - (f, \lambda^2)_{L_2(\Gamma_2)} - \frac{1}{2}\mathbb{V}_f(\lambda)\big] - f & \text{on} \quad \Gamma_2. \end{cases} \tag{8.11}$$

Note that $\dot{\lambda}$ in (8.11) is already determined by means of (8.6), and (8.11) can be solved via Newton's iteration for $\lambda \in H^{-\varepsilon/2}(\Gamma)$.

9 Example

The aim of this section is to provide an example where, in Theorem 8.1, both the requirements (8.1) and (8.2) for λ do hold. To this end, we restrict ourselves to the case where $\alpha \in (1, 2]$, $\alpha < n$; then the following concepts of Riesz equilibrium and balayage measures are well known (see, e.g., [19]).

Given a compact set $K \subset \mathbb{R}^n$, let $C_\alpha(K)$ denote the Riesz capacity of K and $\gamma_K \in \mathcal{E}^+_\alpha(K)$ its (Riesz) equilibrium measure, uniquely determined by the following relations:

$$\gamma_K(\mathbb{R}^n) = C_\alpha(K), \tag{9.1}$$

$$U_\alpha^{\gamma_K}(\mathbf{x}) = 1 \quad \text{n.e. in } K. \tag{9.2}$$

If $\nu \in \mathcal{E}_\alpha(\mathbb{R}^n)$ is also given, then there exists $\beta^\alpha_K\nu \in \mathcal{E}_\alpha(K)$, called the (Riesz) balayage, uniquely determined by

$$U_\alpha^{\beta^\alpha_K\nu}(\mathbf{x}) = U_\alpha^\nu(\mathbf{x}) \quad \text{n.e. in } K. \tag{9.3}$$

Furthermore, one can see from [19], Sections 3 and 5 in Chapters II and IV, respectively, that

$$S_{\gamma_K} = S_{\beta^\alpha_K\nu} = K \tag{9.4}$$

provided K is a connected $(n-1)$-dimensional orientable manifold. If, moreover, this manifold does not contain any α-irregular points (which is the case if it is Lipschitz; see [2, Lemma 10] or [7, Th. 2.2]), then, by [19], the equalities in (9.2) and (9.3) hold everywhere in K.

Example 9.1 Let $n \geqslant 2$, $L^+ = \{1\}$, $L^- = \{2\}$, $g(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{R}^n$, $\Gamma_1 = S_r := \big\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = r\big\}$,

$$f(\mathbf{x}) = \mathbb{V}_{-\alpha}\theta(\mathbf{x}) = U_\alpha^\theta(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

θ being a nonnegative measure of total mass $\theta(\mathbb{R}^n) = q \geqslant 0$ that coincides up to a constant factor with the $(n-1)$-dimensional Lebesgue surface measure of S_{r_1}, and let Γ_2 be a compact, connected $(n-1)$-dimensional orientable $C^{k-1,1}$-manifold in $\mathbb{R}^n \setminus B_R$, where $k > (\alpha-1)/2$ and $R > r_1 \geqslant r > 0$.

Under these requirements, there holds the following assertion (cf. [33, Corollary 10.1]).

Theorem 9.2 *If, moreover,* $1 < \alpha \leqslant 2$, $\alpha < n$, *and*

$$a_1\big(Rr^{-1} - 1\big)^{n-\alpha} \geqslant a_2 \geqslant a_1 + q, \tag{9.5}$$

then λ, *the solution of the corresponding Gauss problem, satisfies both* (8.1) *and* (8.2). *Furthermore, then*

$$S_{\lambda^i} = \Gamma_i, \quad i = 1,\, 2. \tag{9.6}$$

P r o o f. Let $\eta_i(\lambda)$, $i = 1,\, 2$, be determined by (7.7); then (7.5), (7.6), and (7.8) hold true.

Since, under the assumptions made, there exists $p \geqslant 0$ such that

$$f(\mathbf{x}) = p \quad \text{for all } \mathbf{x} \in \Gamma_1, \tag{9.7}$$

relations (7.5) and (7.6) yield

$$a_1 U_\alpha^\lambda(\mathbf{x}) \geqslant c_1^* \quad \text{n.e. in } \Gamma_1, \tag{9.8}$$

$$a_1 U_\alpha^\lambda(\mathbf{x}) = c_1^* \quad \text{n.e. in } S_{\lambda^1}, \tag{9.9}$$

where $c_1^* := \eta_1(\lambda) - 2pa_1$. The measure λ^1 is nonzero and has finite energy; therefore, $C_\alpha(S_{\lambda^1}) > 0$ and, by (9.1), $\gamma^1 := \gamma_{S_{\lambda^1}} \neq 0$. In view of (9.2), relation (9.9) can be rewritten in the form

$$a_1 U_\alpha^{\lambda^1}(\mathbf{x}) - c_1^* U_\alpha^{\gamma^1}(\mathbf{x}) = a_1 U_\alpha^{\lambda^2}(\mathbf{x}) \quad \text{n.e. in } S_{\lambda^1},$$

which means that, actually, $a_1\lambda^1 - c_1^*\gamma^1 = a_1\beta^\alpha_{S_{\lambda^1}}\lambda^2$. This implies

$$c_1^* = \frac{a_1\big[a_1 - \big(\beta^\alpha_{S_{\lambda^1}}\lambda^2\big)(\mathbb{R}^n)\big]}{\gamma^1(\mathbb{R}^n)}. \tag{9.10}$$

Since, due to (9.1)–(9.3),

$$\big(\beta^{\alpha}_{S_{\lambda^1}}\lambda^2\big)(\mathbb{R}^n) = \int 1\,d\beta^{\alpha}_{S_{\lambda^1}}\lambda^2 = I_\alpha(\gamma^1, \beta^{\alpha}_{S_{\lambda^1}}\lambda^2) = I_\alpha(\gamma^1, \lambda^2)$$
$$\leqslant \frac{\lambda^2(\mathbb{R}^n)\gamma^1(\mathbb{R}^n)}{\operatorname{dist}(S_{\gamma^1}, S_{\lambda^2})^{n-\alpha}} \leqslant \frac{a_2 C_\alpha(\Gamma_1)}{\operatorname{dist}(\Gamma_1, \Gamma_2)^{n-\alpha}} = \frac{a_2 r^{n-\alpha}}{(R-r)^{n-\alpha}},$$

we conclude from (9.10) and the left-hand side of (9.5) that $c_1^* \geqslant 0$.

Consequently, $c_1^* + a_1 U_\alpha^{\lambda^2}(\mathbf{x})$ is an α-superharmonic function (see, e.g., [19, Chapter I, Section 6]). Therefore, applying [19, Th. 1.29], we conclude from (9.9) that $a_1 U_\alpha^{\lambda}(\mathbf{x}) \leqslant c_1^*$ for all $\mathbf{x} \in \mathbb{R}^n$. Combined with (9.8), this gives

$$a_1 U_\alpha^{\lambda}(\mathbf{x}) = c_1^* \quad \text{n.e. in } \Gamma_1, \tag{9.11}$$

and so

$$a_1\lambda^1 - c_1^*\gamma_{\Gamma_1} = a_1\beta^{\alpha}_{\Gamma_1}\lambda^2. \tag{9.12}$$

Since Γ_1 contains no α-irregular points, (9.12) yields that (9.11) holds, in fact, everywhere in Γ_1, which together with (9.7) proves (8.1). Furthermore, due to (9.4), (9.12) implies $S_{\lambda^1} = \Gamma_1$, i.e. (9.6) for $i = 1$.

Further, by (7.5) and (7.6),

$$a_2 U_\alpha^{\lambda+\theta}(\mathbf{x}) \leqslant c_2^* \quad \text{n.e. in } \Gamma_2, \tag{9.13}$$

$$a_2 U_\alpha^{\lambda+\theta}(\mathbf{x}) = c_2^* \quad \text{n.e. in } S_{\lambda^2}, \tag{9.14}$$

where $c_2^* := \eta_2(\lambda) - I_\alpha(\theta, \lambda^2)$. Hence, by (9.14),

$$a_2 U_\alpha^{\lambda^1+\theta}(\mathbf{x}) = a_2 U_\alpha^{\lambda^2}(\mathbf{x}) + c_2^* U_\alpha^{\gamma^2}(\mathbf{x}) \quad \text{n.e. in } S_{\lambda^2},$$

where $\gamma^2 := \gamma_{S_{\lambda^2}}$, so that

$$a_2\lambda^2 + c_2^*\gamma^2 = a_2\beta^{\alpha}_{S_{\lambda^2}}(\lambda^1 + \theta),$$

and consequently

$$c_2^* = \frac{a_2\big[\beta^{\alpha}_{S_{\lambda^2}}(\lambda^1+\theta)(\mathbb{R}^n) - a_2\big]}{\gamma^2(\mathbb{R}^n)}.$$

In view of the right-hand side of (9.5) and the fact that $\beta^{\alpha}_K\nu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n)$ for any compact K and $\nu \in \mathcal{E}^+(\mathbb{R}^n)$ (see, e.g., [19]), we therefore get $c_2^* \leqslant 0$. Hence, $a_2 U_\alpha^{\lambda^1+\theta}(\mathbf{x}) - c_2^*$ is α-superharmonic, which due to [19, Th. 1.29] enables us to conclude from (9.14) that $a_2 U_\alpha^{\lambda+\theta}(\mathbf{x}) \geqslant c_2^*$ for all $\mathbf{x} \in \mathbb{R}^n$. When combined with (9.13), this gives

$$a_2 U_\alpha^{\lambda+\theta}(\mathbf{x}) = c_2^* \quad \text{n.e. in } \Gamma_2, \tag{9.15}$$

and so

$$a_2\lambda^2 + c_2^*\gamma_{\Gamma_2} = a_2\beta^{\alpha}_{\Gamma_2}(\lambda^1 + \theta). \tag{9.16}$$

Since Γ_2 is Lipschitz, (9.16) implies that (9.15) holds, in fact, everywhere in Γ_2, which proves (8.2). Furthermore, due to (9.4), (9.16) yields $S_{\lambda^2} = \Gamma_2$, which is (9.6) for $i = 2$. □

10 Numerical results

We consider Example 9.1 for $n = 3$ with Γ_1 being the unit sphere (i.e., $r = 1$) and Γ_2 being a rotational body of length X, namely

$$\Gamma_2 = \big\{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : \; y^2 + z^2 \leqslant 1 \quad \text{for } \; x = 3,$$
$$y^2 + z^2 = r^2(x) \quad \text{for } \; 3 \leqslant x \leqslant 3 + X, \quad y^2 + z^2 \leqslant r^2(3 + X) \quad \text{for } \; x = 3 + X\big\}.$$

In particular, for different lengths X, we focus on the rational function $r(x) = 1/(1 + x)$ and the exponential function $r(x) = \exp(-x)$. The distance of the bodies Γ_1 and Γ_2 is 2 (i.e., $R = 3$). Thus, choosing $a_1 = 1$, $a_2 = 2$, and $q = a_2 - a_1 = 1$ (in fact, we use the choice $r_1 = 1.5$), the inequality (9.5) is satisfied for all $\alpha \in (1, 2]$. Theorem 9.2 implies that both (8.1) and (8.2) hold, and therefore Theorem 8.1 applies. Let λ_X denote the solution of the corresponding Gauss problem.

We discretize the given manifolds Γ_1 and Γ_2 by a quadrangulation with maximal mesh width h. On the quadrangulation we use the characteristic functions as piecewise constant boundary elements and define a corresponding basis of vectors $\Phi_i \subset L^2(\Gamma_i)$, $i = 1, 2$. Set

$$\mathbf{f}_h^i := (f, \Phi_i)_{L^2(\Gamma_i)}, \quad \mathbf{g}_h^i := \frac{1}{a_i}(1, \Phi_i)_{L^2(\Gamma_i)}, \quad \mathbf{V}_h^{i,j} := (V\Phi_j, \Phi_i)_{L^2(\Gamma_i)}, \quad i, j = 1, 2.$$

Then, the Galerkin formulation of the nonlinear equation (8.11) reads as follows. Find $\lambda_{h,k} = \Phi_1 \boldsymbol{\lambda}_{h,k}^1 - \Phi_2 \boldsymbol{\lambda}_{h,k}^2 \in L_2(\Gamma) \subset H^{-\varepsilon/2}(\Gamma)$ such that

$$F(\lambda_h) := \begin{bmatrix} \mathbf{V}_h^{1,1} + \mathbf{g}_h^1(\mathbf{f}_h^1)^T & -\mathbf{V}_h^{1,2} \\ -\mathbf{V}_h^{2,1} & \mathbf{V}_h^{2,2} - \mathbf{g}_h^2(\mathbf{f}_h^2)^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_h^1 \\ \boldsymbol{\lambda}_h^2 \end{bmatrix} - \begin{bmatrix} \big\{\frac{1}{2}\mathbb{V}_f(\lambda_h) + \mathfrak{C}_h(\lambda_h, \dot{\lambda}_h, f)\big\}\mathbf{g}_h^1 \\ \big\{\frac{1}{2}\mathbb{V}_f(\lambda_h) - \mathfrak{C}_h(\lambda_h, \dot{\lambda}_h, f)\big\}\mathbf{g}_h^2 \end{bmatrix} + \begin{bmatrix} \mathbf{f}_h^1 \\ -\mathbf{f}_h^2 \end{bmatrix} = \mathbf{0}, \tag{10.1}$$

where $\mathbb{V}_f(\lambda_h)$, the discrete version of the Gauss functional, is expressed as

$$\mathbb{V}_f(\lambda_h) = \begin{bmatrix} \boldsymbol{\lambda}_h^1 \\ \boldsymbol{\lambda}_h^2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{V}_h^{1,1} & -\mathbf{V}_h^{1,2} \\ -\mathbf{V}_h^{2,1} & \mathbf{V}_h^{2,2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_h^1 \\ \boldsymbol{\lambda}_h^2 \end{bmatrix} + 2 \begin{bmatrix} \mathbf{f}_h^1 \\ -\mathbf{f}_h^2 \end{bmatrix} \right)$$

and

$$\mathfrak{C}_h(\lambda_h, \dot{\lambda}_h, f) := \Big\{(\mathbf{g}_h^1)^T \dot{\boldsymbol{\lambda}}_h^1 - (\mathbf{g}_h^2)^T \dot{\boldsymbol{\lambda}}_h^2\Big\}^{-1}$$
$$\times \Big\{(\mathbf{g}_h^1)^T \dot{\boldsymbol{\lambda}}_h^1 \big[(\mathbf{f}_h^1)^T \boldsymbol{\lambda}_h^1 - \tfrac{1}{2}\mathbb{V}_f(\lambda_h)\big] - (\mathbf{g}_h^2)^T \dot{\boldsymbol{\lambda}}_h^2 \big[(\mathbf{f}_h^2)^T \boldsymbol{\lambda}_h^2 + \tfrac{1}{2}\mathbb{V}_f(\lambda_h)\big]\Big\}.$$

In particular, the derivative $\dot{\lambda}_h = \Phi_1 \dot{\boldsymbol{\lambda}}_h^1 - \Phi_2 \dot{\boldsymbol{\lambda}}_h^2$ of the solution λ_h satisfies the linear system of equations

$$\begin{bmatrix} \mathbf{V}_h^{1,1} + \mathbf{g}_h^1(\mathbf{f}_h^1)^T & -\mathbf{V}_h^{1,2} \\ -\mathbf{V}_h^{2,1} & \mathbf{V}_h^{2,2} - \mathbf{g}_h^2(\mathbf{f}_h^2)^T \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\lambda}}_h^1 \\ \dot{\boldsymbol{\lambda}}_h^2 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_h^1 \\ -\mathbf{g}_h^2 \end{bmatrix}. \tag{10.2}$$

In order to solve the nonlinear system of equations (10.1) we use the Newton scheme. To this end, we note that the derivative $F'(\lambda_h)$ of $F(\lambda_h)$ in the direction $\psi_h = \Phi_1 \boldsymbol{\psi}_h^1 - \Phi_2 \boldsymbol{\psi}_h^2$ is given by

$$F'(\lambda_h) \cdot \psi_h = \begin{bmatrix} \mathbf{V}_h^{1,1} + \mathbf{g}_h^1(\mathbf{f}_h^1)^T & -\mathbf{V}_h^{1,2} \\ -\mathbf{V}_h^{2,1} & \mathbf{V}_h^{2,2} - \mathbf{g}_h^2(\mathbf{f}_h^2)^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_h^1 \\ \boldsymbol{\psi}_h^2 \end{bmatrix} - \begin{bmatrix} \big\{\frac{1}{2}\mathbb{V}'_f(\lambda_h) \cdot \psi_h + \mathfrak{C}'_h(\lambda_h, \dot{\lambda}_h, f) \cdot \psi_h\big\}\mathbf{g}_h^1 \\ \big\{\frac{1}{2}\mathbb{V}'_f(\lambda_h) \cdot \psi_h - \mathfrak{C}'_h(\lambda_h, \dot{\lambda}_h, f) \cdot \psi_h\big\}\mathbf{g}_h^2 \end{bmatrix}$$

where the real numbers involved in the last term are computed as

$$\mathbb{V}'_f(\lambda_h) \cdot \psi_h = 2 \begin{bmatrix} \boldsymbol{\psi}_h^1 \\ \boldsymbol{\psi}_h^2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{V}_h^{1,1} & -\mathbf{V}_h^{1,2} \\ -\mathbf{V}_h^{2,1} & \mathbf{V}_h^{2,2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_h^1 \\ \boldsymbol{\lambda}_h^2 \end{bmatrix} + \begin{bmatrix} \mathbf{f}_h^1 \\ -\mathbf{f}_h^2 \end{bmatrix} \right),$$

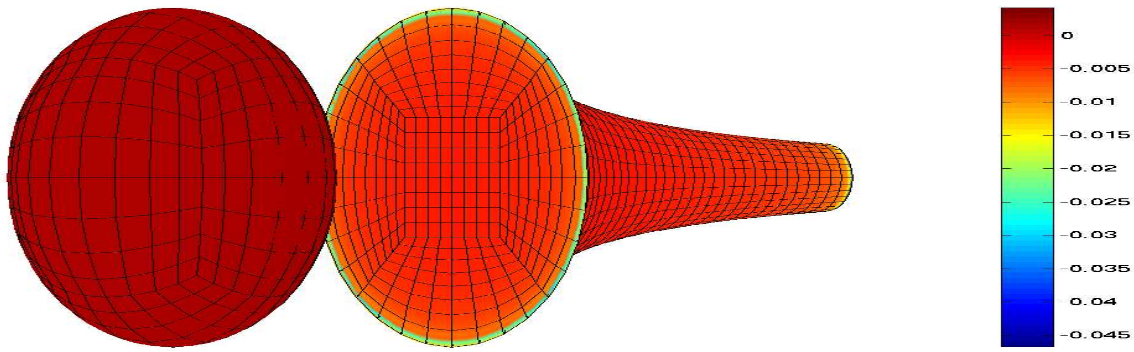

Fig. 1 Charge distribution in the case of $r(x) = 1/(1 + x)$ and $\alpha = 2$.

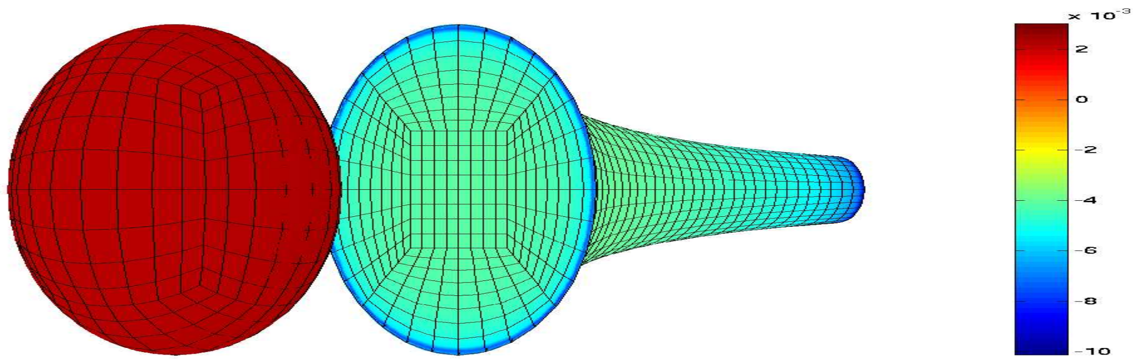

Fig. 2 Charge distribution in the case of $r(x) = 1/(1 + x)$ and $\alpha = 1.5$.

$$\mathfrak{E}'_h(\lambda_h, \dot{\lambda}_h, f) \cdot \psi_h = \Big\{ (\mathbf{g}_h^1)^T \dot{\boldsymbol{\lambda}}_h^1 - (\mathbf{g}_h^2)^T \dot{\boldsymbol{\lambda}}_h^2 \Big\}^{-1}$$
$$\times \Big\{ (\mathbf{g}_h^1)^T \dot{\boldsymbol{\lambda}}_h^1 \big[(\mathbf{f}_h^1)^T \boldsymbol{\psi}_h^1 - \tfrac{1}{2} \mathbb{V}'_f(\lambda_h) \cdot \psi_h \big] - (\mathbf{g}_h^2)^T \dot{\boldsymbol{\lambda}}_h^2 \big[(\mathbf{f}_h^2)^T \boldsymbol{\psi}_h^2 + \tfrac{1}{2} \mathbb{V}'_f(\lambda_h) \cdot \psi_h \big] \Big\}.$$

Then, the Newton scheme to solve (10.1) consists of the following steps:

1. Choose the initial approximation $\lambda_h^{(0)} := \frac{a_1}{|\Gamma_1|} \Phi_1 \mathbf{1} - \frac{a_2}{|\Gamma_2|} \Phi_2 \mathbf{1}$.
2. For $k = 0, 1, \ldots$, repeat
   (a) compute the derivative $\dot{\lambda}_h^{(k)}$ by solving (10.2) with the GMRES method with initial guess $\dot{\lambda}_h^{(k)} = 0$;
   (b) solve the equation $F'(\lambda_h^{(k)}) \cdot \psi_h = -F(\lambda_h^{(k)})$ by the GMRES method with initial guess $\psi_h = 0$;
   (c) update $\lambda_h^{(k+1)} = \lambda_h^{(k)} + \psi_h$.

Note that we have used that density as initial approximation which is constant on both manifolds and satisfies there the constraints $(1, \lambda_{h,0})_{L^2(\Gamma_1)} = a_1$ and $(1, \lambda_{h,0})_{L^2(\Gamma_2)} = -a_2$. To our experience, with this initial approximation, the Newton scheme converges within a rather small number of iteration steps. For example, in all our numerical examples, we needed at most 5 iteration steps to solve (10.1) up to an accuracy of 10^{-6}, independently of α.

In Figures 1, 2, and 3, we have plotted the computed charge distributions for $\alpha = 2$, $\alpha = 1.5$, and $\alpha = 1.1$, respectively, where we consider $r(x) = 1/(1 + x)$ and $X = 4$. These computations have been carried out with piecewise constant boundary elements on a quadrangulation by about 50000 elements. It is observed that

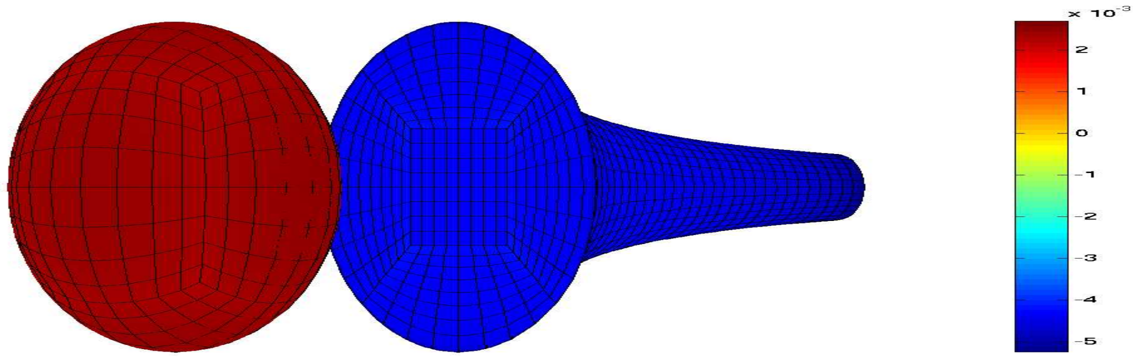

Fig. 3 Charge distribution in the case of $r(x) = 1/(1+x)$ and $\alpha = 1.1$.

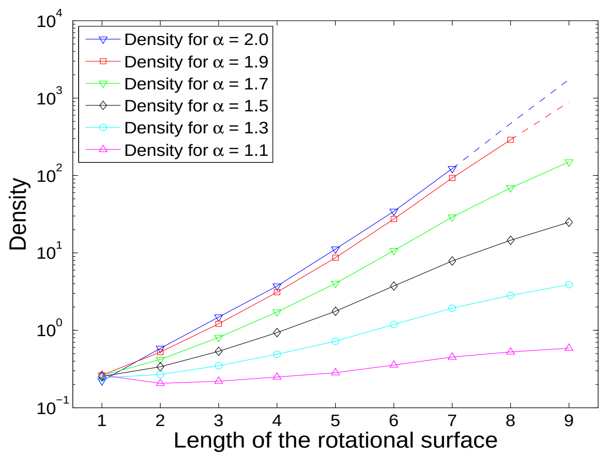

Fig. 4 Asymptotics for $X \to \infty$ in the case of $r(x) = \exp(-x)$.

X	area	$\alpha = 2.0$		$\alpha = 1.9$		$\alpha = 1.1$	
		charge	density	charge	density	charge	density
1	$4.2 \cdot 10^{-1}$	$9.4 \cdot 10^{-2}$	$2.2 \cdot 10^{-1}$	$1.1 \cdot 10^{-1}$	$2.6 \cdot 10^{-1}$	$1.1 \cdot 10^{-1}$	$2.7 \cdot 10^{-1}$
2	$5.8 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$	$5.8 \cdot 10^{-1}$	$3.0 \cdot 10^{-2}$	$5.2 \cdot 10^{-1}$	$1.2 \cdot 10^{-2}$	$2.1 \cdot 10^{-1}$
3	$7.8 \cdot 10^{-3}$	$1.2 \cdot 10^{2}$	1.5	$9.5 \cdot 10^{-3}$	1.2	$1.7 \cdot 10^{-3}$	$2.2 \cdot 10^{-1}$
4	$1.1 \cdot 10^{-3}$	$3.9 \cdot 10^{-3}$	3.7	$3.3 \cdot 10^{-3}$	3.1	$2.6 \cdot 10^{-4}$	$2.5 \cdot 10^{-1}$
5	$1.4 \cdot 10^{-4}$	$1.6 \cdot 10^{-3}$	$1.1 \cdot 10^{1}$	$1.2 \cdot 10^{-3}$	8.7	$4.0 \cdot 10^{-5}$	$2.8 \cdot 10^{-1}$
6	$1.9 \cdot 10^{-5}$	$6.6 \cdot 10^{-4}$	$3.4 \cdot 10^{1}$	$5.3 \cdot 10^{-4}$	$2.7 \cdot 10^{1}$	$6.9 \cdot 10^{-6}$	$3.6 \cdot 10^{-1}$
7	$2.6 \cdot 10^{-6}$	$3.2 \cdot 10^{-4}$	$1.2 \cdot 10^{2}$	$2.4 \cdot 10^{-4}$	$9.3 \cdot 10^{1}$	$1.2 \cdot 10^{-6}$	$4.5 \cdot 10^{-1}$
8	$3.5 \cdot 10^{-7}$	$(1.7 \cdot 10^{-4}$	$4.7 \cdot 10^{2})$	$1.0 \cdot 10^{-4}$	$2.9 \cdot 10^{2}$	$1.9 \cdot 10^{-7}$	$5.3 \cdot 10^{-1}$
9	$4.8 \cdot 10^{-8}$	$(8.3 \cdot 10^{-5}$	$1.7 \cdot 10^{3})$	$(4.3 \cdot 10^{-5}$	$8.9 \cdot 10^{2})$	$2.8 \cdot 10^{-8}$	$5.9 \cdot 10^{-1}$

Table 1 Asymptotics for $X \to \infty$ in the case of $r(x) = \exp(-x)$.

for $\alpha \to 1$ the charge distribution becomes constant on each sub-manifold. Vice versa, it becomes the more inhomogeneous the more α increases. It is also seen from Figures 1, 2, and 3 that the supports of the charges we have computed coincide with the whole surfaces Γ_1 and Γ_2, which is in agreement with the theoretical result (9.6).

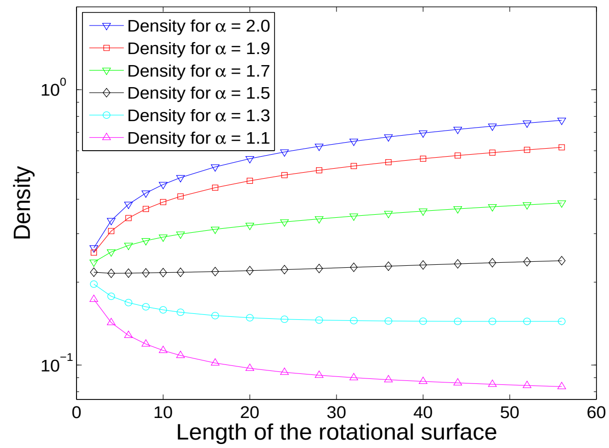

Fig. 5 Asymptotics for $X \to \infty$ in the case of $r(x) = 1/(1+x)$.

We next study the asymptotic behaviour of λ_X if the length X of the rotational body tends to infinity. We compute the module of the total charge at the tip of Γ_2, i.e.,

$$\Lambda_X := \int_{\Sigma_X} \lambda_X^2 \, ds, \quad \text{where} \quad \Sigma_X := \big\{\mathbf{x} \in \mathbb{R}^3 : \, y^2 + x^2 = r^2(X+3)\big\},$$

as well as the density $\Lambda_X/|\Sigma_X|$, where $|\Sigma_X| := \int_{\Sigma_X} 1 \, ds$. We are interested in their behaviours as $X \to \infty$ since, as has been shown in [28, 30, 31, 34], the Gauss variational problem for the noncompact condenser $\mathbf{A} = (\Gamma_1, \Gamma_2)$ can in general be nonsolvable, and then the infimum $\mathbb{G}_f(\mathbf{A}, \mathbf{a}, g)$ is attained at $\gamma \in \mathcal{E}_\alpha(\mathbf{A})$ with $\int_{\Gamma_2} g \, d\gamma^2 < a_2$, whereas $\lambda_X \to \gamma$ vaguely and strongly as $X \to \infty$.

According to [30, Theorems 4, 8], under our particular assumptions, such a phenomenon of nonsolvability occurs for $\mathbf{A} = (\Gamma_1, \Gamma_2)$ with Γ_2 being infinitely long, if and only if $C_\alpha(\Gamma_2) = \infty$ while Γ_2 is α-thin at $\infty_{\mathbb{R}^3}$, the latter by [4, 5] means that the inverse of Γ_2 relative to the unit sphere is α-irregular at the origin $\mathbf{x} = 0$. In the case $r(x) = \exp(-x)$, both these conditions hold true for $\alpha = 2$ (hence, also for α close to 2), so that then

$$\lim_{X \to \infty} \Lambda_X > 0, \tag{10.3}$$

while in the case $r(x) = 1/(1+x)$, Γ_2 is *not* α-thin at $\infty_{\mathbb{R}^3}$ for *any* $\alpha \in (1, 2]$, so that for this geometry

$$\lim_{X \to \infty} \Lambda_X = 0. \tag{10.4}$$

In Figure 4, in the case of $r(x) = \exp(-x)$, we have plotted the densities $\Lambda_X/|\Sigma_X|$ for $\alpha = 2.0$ (blue graph), $\alpha = 1.9$ (red graph), $\alpha = 1.7$ (green graph), $\alpha = 1.5$ (black graph), $\alpha = 1.3$ (cyan graph), and $\alpha = 1.1$ (magenta graph) in the range $1 \leqslant X \leqslant 9$. In the case of $\alpha = 2.0$, we were able to compute the charge distribution only for $X \leqslant 7$ and thus we have extrapolated the total charge for $X > 7$. Likewise, in the case of $\alpha = 1.9$, we had to extrapolate the total charge up to $X = 9$.

The area of the tip $|\Sigma_X|$, the module of the corresponding total charge Λ_X, and the density $\Lambda_X/|\Sigma_X|$ are also tabulated in Table 1 for $\alpha = 2.0$, $\alpha = 1.9$, and $\alpha = 1.1$. One can see that the density for $\alpha = 2$ is unbounded in X, as has been predicted by (10.3). The behaviour is quite similar for $\alpha = 1.9$, whereas for $\alpha = 1.1$ it seems to be bounded in X.

We have performed the same asymptotic study also in the case of $r(x) = 1/(1+x)$ (see Figure 5 and Table 2). Here, we were able to compute the total charges for a much larger range of X. The density $\Lambda_X/|\Sigma_X|$ is now always bounded, which is in agreement with the theoretical result (10.4). Also observe that the corresponding upper bound is the smaller the smaller the α is.

X	area	$\alpha = 2.0$ charge	$\alpha = 2.0$ density	$\alpha = 1.9$ charge	$\alpha = 1.9$ density	$\alpha = 1.1$ charge	$\alpha = 1.1$ density
8	$3.9 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$	$4.2 \cdot 10^{-1}$	$1.4 \cdot 10^{-2}$	$3.7 \cdot 10^{-1}$	$4.6 \cdot 10^{-3}$	$1.2 \cdot 10^{-1}$
16	$1.1 \cdot 10^{-2}$	$5.7 \cdot 10^{-3}$	$5.2 \cdot 10^{-1}$	$4.8 \cdot 10^{-3}$	$4.1 \cdot 10^{-1}$	$2.0 \cdot 10^{-3}$	$1.1 \cdot 10^{-1}$
24	$5.0 \cdot 10^{-3}$	$3.0 \cdot 10^{-3}$	$5.9 \cdot 10^{-1}$	$2.5 \cdot 10^{-3}$	$4.7 \cdot 10^{-1}$	$6.9 \cdot 10^{-4}$	$9.7 \cdot 10^{-2}$
32	$2.9 \cdot 10^{-3}$	$1.9 \cdot 10^{-3}$	$6.5 \cdot 10^{-1}$	$1.5 \cdot 10^{-3}$	$5.3 \cdot 10^{-1}$	$3.4 \cdot 10^{-4}$	$9.2 \cdot 10^{-2}$
40	$1.9 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	$7.0 \cdot 10^{-1}$	$1.0 \cdot 10^{-3}$	$5.6 \cdot 10^{-1}$	$2.0 \cdot 10^{-4}$	$8.8 \cdot 10^{-2}$
48	$1.3 \cdot 10^{-3}$	$9.6 \cdot 10^{-4}$	$7.4 \cdot 10^{-1}$	$7.7 \cdot 10^{-4}$	$5.9 \cdot 10^{-1}$	$1.1 \cdot 10^{-4}$	$8.5 \cdot 10^{-2}$
56	$9.7 \cdot 10^{-4}$	$7.5 \cdot 10^{-4}$	$7.7 \cdot 10^{-1}$	$6.0 \cdot 10^{-4}$	$6.2 \cdot 10^{-1}$	$8.1 \cdot 10^{-5}$	$8.4 \cdot 10^{-2}$

Table 2 Asymptotics for $X \to \infty$ in case of the rational function $r(x) = 1/(1+x)$.

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A Proof of Theorem 4.1

For $1/2 < s \leqslant k$, the proof can be found in [13, Theorem 1.5.2] and for $1/2 < s < 3/2$ see [9, Lemma 3.6]. Hence, it remains to consider the case $1 < s < k + 1/2$; then $k \geqslant 2$.

We follow closely the proof by Costabel in [9]. Since Γ is compact, by a partition of the unity the statement of Theorem 4.1 is in fact local. Therefore, without any loss of generality, one can assume Γ to be of the form

$$\Gamma = \big\{(\mathbf{x}', x_n) : \ \mathbf{x}' \in \mathbb{R}^{n-1}, \ x_n = \psi(\mathbf{x}')\big\},$$

where ψ is a function of $C^{k-1}(\mathbb{R}^{n-1})$ whose derivatives $\partial^{k-1}\psi$ are uniformly Lipschitz, i.e.,

$$\|\partial^k \psi\|_{L^\infty(\mathbb{R}^{n-1})} < \infty.$$

For any $f \in C_0^\infty(\mathbb{R}^n)$ define

$$f_\psi(\mathbf{x}', x_n) := f\big(\mathbf{x}', x_n + \psi(\mathbf{x}')\big).$$

Then the trace of f on Γ can be written as

$$(\gamma_0 f)(\mathbf{x}', x_n) = f_\psi(\mathbf{x}', 0) = f\big(\mathbf{x}', \psi(\mathbf{x}')\big).$$

Denote $\partial_p := \partial/\partial \mathbf{x}'_p$, $p = 1, \ldots, n-1$, and $\partial_n := \partial/\partial x_n$. Then with the chain and product rules we get

$$\partial_p f_\psi = (\partial_p f)_\psi + (\partial_n f)_\psi \, \partial_p \psi,$$
$$\partial_p \partial_j f_\psi = (\partial_p \partial_j f)_\psi + (\partial_p \partial_n f)_\psi \, \partial_j \psi + (\partial_j \partial_n f)_\psi \, \partial_p \psi + (\partial_n f)_\psi \, \partial_p \partial_j \psi + (\partial_n^2 f)_\psi \, \partial_j \psi \, \partial_p \psi$$

for all $p, j = 1, \ldots, n-1$, and, for the higher order derivatives,

$$\partial_{\mathbf{x}'}^{\boldsymbol{\alpha}} f_\psi = (\partial_{\mathbf{x}'}^{\boldsymbol{\alpha}} f)_\psi + (\partial_n f)_\psi \, \partial_{\mathbf{x}'}^{\boldsymbol{\alpha}} \psi + \sum_{\ell=1}^{|\boldsymbol{\alpha}|} \sum_{0 \leqslant |\boldsymbol{\beta}_\ell| \leqslant |\boldsymbol{\alpha}| - \ell} (\partial_n^\ell \partial_{\mathbf{x}'}^{\boldsymbol{\beta}_\ell} f)_\psi \, P_{\boldsymbol{\beta}_\ell}\big(\partial_{\mathbf{x}'}^{\boldsymbol{\gamma}} \psi\big), \tag{A.1}$$

where the multi-index $\boldsymbol{\beta}_\ell$ is obtained from $\boldsymbol{\alpha}$ by deleting some of its components, while $P_{\boldsymbol{\beta}_\ell}(\partial_{\mathbf{x}'}^{\boldsymbol{\gamma}} \psi)$ are certain products (depending on $\boldsymbol{\beta}_\ell$) of at most ℓ derivatives $\partial_{\mathbf{x}'}^{\boldsymbol{\gamma}} \psi$ with $|\boldsymbol{\gamma}| < |\boldsymbol{\alpha}|$, and $P_{\boldsymbol{\beta}_\ell}(\partial_{\mathbf{x}'}^{\boldsymbol{\gamma}} \psi)$ is to be zero if so is the number of all its factors.

By $\tilde{f}(\mathbf{x}', \xi_n)$ we denote the Fourier transform of f with respect to the last variable x_n, i.e.,

$$\tilde{f}(\mathbf{x}', \xi_n) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\mathbf{x}', x_n) e^{-i\xi_n x_n}\, dx_n.$$

Then

$$\widetilde{f_\psi}(\mathbf{x}', \xi_n) = e^{i\psi(\mathbf{x}')\xi_n} \tilde{f}(\mathbf{x}', \xi_n) \quad \text{and} \quad \|\widetilde{f_\psi}(\cdot, \xi_n)\|_{L^2(\mathbb{R}^{n-1})} = \|\tilde{f}(\cdot, \xi_n)\|_{L^2(\mathbb{R}^{n-1})}. \tag{A.2}$$

Hence, the Fourier transform of $\partial^{\boldsymbol{\alpha}}_{\mathbf{x}'} f_\psi$, where $|\boldsymbol{\alpha}| \geqslant 1$, with respect to the last variable has the form

$$\begin{aligned}\widetilde{\partial^{\boldsymbol{\alpha}}_{\mathbf{x}'} f_\psi}(\mathbf{x}', \xi_n) = e^{i\psi(\mathbf{x}')\xi_n} \Big\{ &\partial^{\boldsymbol{\alpha}}_{\mathbf{x}'} \tilde{f}(\mathbf{x}', \xi_n) + i\xi_n \tilde{f}(\mathbf{x}', \xi_n) \partial^{\boldsymbol{\alpha}} \psi(\mathbf{x}') \\ &+ \sum_{\ell=1}^{|\boldsymbol{\alpha}|} (i\xi_n)^\ell \sum_{0 \le |\boldsymbol{\beta}_\ell| \leqslant |\boldsymbol{\alpha}| - \ell} \partial^{\boldsymbol{\beta}_\ell}_{\mathbf{x}'} \tilde{f}(\mathbf{x}', \xi_n) P_{\boldsymbol{\beta}_\ell}(\partial^{\boldsymbol{\gamma}}_{\mathbf{x}'} \psi) \Big\}.\end{aligned}$$

Fix r, $1 \leqslant |\boldsymbol{\alpha}| \leqslant r \leqslant k$. Then for every $\xi_n \in \mathbb{R}$ we have the estimates

$$\begin{aligned}\|\widetilde{\partial^{\boldsymbol{\alpha}}_{\mathbf{x}'} f_\psi}(\cdot, \xi_n)\|^2_{L^2(\mathbb{R}^{n-1})} \leqslant c_1 \|\partial^{\boldsymbol{\alpha}}_{\mathbf{x}'} \tilde{f}(\cdot, \xi_n)\|^2_{L^2(\mathbb{R}^{n-1})} &+ c_2 \xi_n^2 \|\tilde{f}(\cdot, \xi_n)\|^2_{L^2(\mathbb{R}^{n-1})} \|\partial^{\boldsymbol{\alpha}} \psi\|^2_{L_\infty(\mathbb{R}^{n-1})} \\ &+ \sum_{\ell=1}^{r} \xi_n^{2\ell} c'_\ell \sum_{0 \leqslant |\boldsymbol{\beta}_\ell| \leqslant r - \ell} \|\partial^{\boldsymbol{\beta}_\ell}_{\mathbf{x}'} \tilde{f}(\cdot, \xi_n)\|^2_{L^2(\mathbb{R}^{n-1})}\end{aligned}$$

where the constants c_1, c_2, c'_ℓ depend only on k, ψ and do not depend on ξ_n. Multiplying the last inequality by $(1 + |\xi_n|)^{2t}$, where $t \in \mathbb{R}$, and then integrating the result obtained with respect to ξ_n, in view of (A.2) we get

$$\|f_\psi\|_{H^t(\mathbb{R}, H^r(\mathbb{R}^{n-1}))} \leqslant C \sum_{\ell=0}^{r} \|f\|_{H^{t+\ell}(\mathbb{R}, H^{r-\ell}(\mathbb{R}^{n-1}))} \quad \text{for all } \ t \in \mathbb{R} \ \text{ and } \ 0 \leqslant r \leqslant k, \tag{A.3}$$

where C depends only on t, r, k and ψ. Here, for any given $f \in C_0^\infty(\mathbb{R}^n)$ we use the notation (see [9])

$$\|f\|^2_{H^j(\mathbb{R}, H^d(\mathbb{R}^{n-1}))} := \int_{-\infty}^{\infty} (1 + |\xi_n|)^{2j}\, \|\tilde{f}(\cdot, \xi_n)\|^2_{H^d(\mathbb{R}^{n-1})}\, d\xi_n.$$

For a given s, $1 < s < k + 1/2$, define

$$m(\boldsymbol{\xi}', \xi_n) := \sum_{\ell=0}^{k} (1 + |\xi_n|)^{2(s-\ell)} (1 + |\boldsymbol{\xi}'|)^{2\ell}, \quad \text{where } \ \boldsymbol{\xi}' \in \mathbb{R}^{n-1}.$$

Then

$$\int_{-\infty}^{\infty} (1 + |\boldsymbol{\xi}'|)^{2s-1} m(\boldsymbol{\xi}', \xi_n)^{-1}\, d\xi_n \leqslant 2 \int_0^\infty \Big\{ \sum_{\ell=0}^{k} \tau^{2(s-\ell)} \Big\}^{-1} d\tau = c_{ks} < \infty.$$

In view of the definition of $H^{s-\frac{1}{2}}(\Gamma)$ (see [21, pp. 98–99]), we have

$$c_1 \|f_\psi(\cdot, 0)\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leqslant \|\gamma_0 f\|^2_{H^{s-\frac{1}{2}}(\Gamma)} \leqslant c_2 \|f_\psi(\cdot, 0)\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})}, \tag{A.4}$$

where the constants c_1 and c_2 are positive and independent of f. Having observed that

$$f_\psi(\cdot, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{f_\psi}(\cdot, \xi_n)\, d\xi_n,$$

with the help of the Cauchy–Schwarz inequality we therefore get

$$\|\gamma_0 f\|^2_{H^{s-1/2}(\Gamma)} \leqslant c' \int_{\mathbb{R}^{n-1}} (1+|\boldsymbol{\xi}'|)^{2s-1} \Big| \int_{-\infty}^{\infty} \widehat{f_\psi}(\boldsymbol{\xi}', \xi_n)\, d\xi_n \Big|^2 d\boldsymbol{\xi}'$$

$$\leqslant c' \int_{\mathbb{R}^{n-1}} \Big\{ \int_{-\infty}^{\infty} (1+|\boldsymbol{\xi}'|)^{2s-1} m(\boldsymbol{\xi}', \xi_n)^{-1}\, d\xi_n \int_{-\infty}^{\infty} m(\boldsymbol{\xi}', \xi_n)\, |\widehat{f_\psi}(\boldsymbol{\xi}', \xi_n)|^2\, d\xi_n \Big\}\, d\boldsymbol{\xi}'$$

$$\leq c' c_{ks} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \sum_{\ell=0}^{k} (1+|\xi_n|)^{2(s-\ell)} (1+|\boldsymbol{\xi}'|)^{2\ell}\, |\widehat{f_\psi}(\boldsymbol{\xi}', \xi_n)|^2\, d\xi_n\, d\boldsymbol{\xi}'$$

$$= c'_{ks} \sum_{\ell=0}^{k} \|f_\psi\|^2_{H^{s-\ell}(\mathbb{R}, H^\ell(\mathbb{R}^{n-1}))},$$

where $\widehat{f_\psi}$ is now the n-dimensional Fourier transform of f_ψ. Hence, with (A.3) we obtain the desired result

$$\begin{aligned}
\|\gamma_0 f\|^2_{H^{s-\frac{1}{2}}(\Gamma)} &\leqslant c \sum_{\ell=0}^{k} \|f\|^2_{H^{s-\ell}(\mathbb{R}, H^\ell(\mathbb{R}^{n-1}))} \\
&= c \sum_{\ell=0}^{k} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} (1+|\xi_n|)^{2(s-\ell)} (1+|\boldsymbol{\xi}'|)^{2\ell}\, |\hat{f}(\boldsymbol{\xi}', \xi_n)|^2\, d\boldsymbol{\xi}'\, d\xi_n \\
&\leqslant c' \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \big(1+|(\boldsymbol{\xi}', \xi_n)|\big)^{2s} |\hat{f}(\boldsymbol{\xi}', \xi_n)|^2\, d\boldsymbol{\xi}'\, d\xi_n = c' \|f\|^2_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Finally, using the definition of γ_0^* (see (4.5)), we obtain

$$\|\gamma_0^* \varphi\|_{H^{-s}(\mathbb{R}^n)} = \sup_{\|\Phi\|_{H^s(\mathbb{R}^n)} \leqslant 1} |(\gamma_0^* \varphi, \Phi)| \leqslant \|\varphi\|_{H^{\frac{1}{2}-s}(\Gamma)} \|\gamma_0 \Phi\|_{H^{s-\frac{1}{2}}(\Gamma)} \leqslant c'' \|\varphi\|_{H^{\frac{1}{2}-s}(\Gamma)},$$

which proves the right-hand side inequality in (4.4).

To establish its left-hand side, we observe that, according to [22, (2.7)],

$$\begin{aligned}
\|\gamma_0^* \varphi\|^2_{H^{-s}(\mathbb{R}^n)} &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{d\xi_n}{(1+|\boldsymbol{\xi}'|^2+\xi_n^2)^s} (2\pi)^{-1/2} |\hat{\varphi}(\boldsymbol{\xi}')|^2\, d\boldsymbol{\xi}' \\
&= c_s \int_{\mathbb{R}^{n-1}} (1+|\boldsymbol{\xi}'|^2)^{-s+\frac{1}{2}} |\hat{\varphi}(\boldsymbol{\xi}')|^2\, d\boldsymbol{\xi}' \\
&= c_s \|\varphi\|^2_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})} \geqslant {c'}^2 \|\varphi\|^2_{H^{\frac{1}{2}-s}(\Gamma)},
\end{aligned}$$

where in the very last inequality the equivalence (A.4) has been applied. Here, $c_s := 2(2s-1)^{-1}(2\pi)^{-1/2}$. □

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