Multilevel Accelerated Quadrature for PDEs With Log-Normal Distributed Random Coefficient*

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MULTILEVEL ACCELERATED QUADRATURE FOR PDES WITH LOG-NORMAL DISTRIBUTED RANDOM COEFFICIENT*

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Abstract. This article is dedicated to multilevel quadrature methods for the rapid solution of stochastic partial differential equations with a log-normal distributed diffusion coefficient. The key idea of these approaches is a sparse grid approximation of the occurring product space between the stochastic and the spatial variable. We develop the mathematical theory and present error estimates for the computation of the solution's statistical moments with focus on the mean and variance. Especially, the present framework covers the multilevel Monte Carlo method and the multilevel quasi Monte Carlo method as special cases. The theoretical findings are supplemented by numerical experiments.

Key words. multilevel quadrature, PDEs with stochastic data, log-normal diffusion, Karhunen-Loève expansion, finite element method

AMS subject classifications. 65N30, 65D32, 60H15, 60H35

1. Introduction. This article is dedicated to multilevel quadrature methods for the fast solution of stochastic, elliptic partial differential equations with log-normal distributed diffusion coefficient. The basic idea of the multilevel quadrature is a sparse-grid-like discretization of the underlying Bochner space $L^2_{\mathbb{P}}\big(\Omega, H^1_0(D)\big)$. The spatial variable is discretized by a classical finite element method and the stochastic variable is treated by an appropriately chosen quadrature rule. Since the problem's solution provides the necessary mixed Sobolev regularity, the approximation errors on the different levels of resolution can be equilibrated in a sparse-grid-like fashion, cf. [5, 14, 31]. This idea has already been proposed in case of uniformly elliptic diffusion coefficients in [17] for different quadrature strategies. On the one hand, the *Multilevel Monte Carlo Method* (MLMC), as introduced in [3, 11, 12, 18, 19], yields only stochastic error estimates in the mean-square sense. On the other hand, to avoid this drawback, two completely deterministic methods have been introduced in [17], namely the *Multilevel Quasi Monte Carlo Method* (MLQMC) and the *Multilevel Polynomial Chaos Method* (MLPC).

The treatment of the log-normal case is much more involved for the deterministic methods due to the unboundedness of the domain of integration, i.e. $\mathbb{R}^m$ for some $m \in \mathbb{N}$. This makes the analysis of the quadrature error difficult. In particular, special regularity results are required which extend those of [2, 7, 21]. Furthermore, a log-normal distributed diffusion coefficient depends non-linearly on the stochastics. Thus, MLPC is no longer feasible since a polynomial chaos expansion, cf. [9, 10], would yield a fully coupled system of partial differential equations. Instead, one has to apply stochastic collocation to overcome this obstruction, cf. [2, 25]. If output functionals of the solution, like the mean or the variance, are desired rather than the solution itself, the stochastic collocation coincides with a quadrature rule based on polynomial interpolation. Especially, for the log-normal case, quadrature formulae based on the Hermite polynomials are convenient. This yields the *Multilevel Gaussian Quadrature Method* (MLGQ) which we also analyze in this article.

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Although it seems quite encouraging that the techniques for stochastic, elliptic partial differential equations with uniformly elliptic diffusion coefficients can mostly be transferred to the log-normal situation in theory, it is not quite clear, how the transfer is performed numerically. The reason for the difficulties is that, in the log-normal situation, the diffusion coefficient is no more affine in the stochastic variable. As it turns out, the numerical treatment of the diffusion coefficient is thus much more involved and yields, by use of an h-refined quadrature, an unfavourable complexity which is caused by the non-linearity. The implementation and the connected problems will also be addressed in this article.

The rest of this article is organized as follows. Section 2 specifies the diffusion problem and the corresponding framework. In particular, the parametric reformulation as a high dimensional deterministic problem is performed here. In Section 3, we derive the crucial regularity estimates of the solution to the stochastic diffusion problem under consideration. Section 4 provides the theoretical background for the Gaussian quadrature in the stochastic variable. Here, the regularity estimates from the preceding section are employed to determine the degree of quadrature for each stochastic dimension which is required to guarantee the convergence of the quadrature. Section 5 is concerned with the Monte Carlo and the quasi Monte Carlo quadrature. In case of the quasi Monte Carlo quadrature, we restrict ourselves to a finite dimensional domain of integration and introduce an auxiliary density to get the error of quadrature bounded. Section 6 gives a brief outline of the multilevel finite element method which we will employ for the spatial discretization later on. Especially, the important regularity and convergence results are stated. In Section 7, the previously specified spatial and stochastic discretizations are combined by menas of the multilevel quadrature idea. Furthermore, convergence results for the solution's moments, especially its mean and variance, are presented. Finally, in Section 8, the theoretical findings are validated by numerical examples. Here, also the difficulties in the treatment of the diffusion coefficient are addressed.

In the following, in order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Problem setting. In the following, let $D \subset \mathbb{R}^n$ for $n = 2, 3$ be a polygonal or polyhedral domain and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with σ-field $\mathcal{F} \subset 2^\Omega$ and a complete probability measure $\mathbb{P}$, i.e. for all $A \subset B$ and $B \in \mathcal{F}$ with $\mathbb{P}[B] = 0$ it follows $A \in \mathcal{F}$. We intend to compute the random function $u(\omega) \in H_0^1(D)$ which solves for almost every $\omega \in \Omega$ the stochastic diffusion problem

$$-\operatorname{div}\big(a(\omega)\nabla u(\omega)\big) = f \text{ in } D. \tag{2.1}$$

Throughout this paper, we shall assume that the load f is purely deterministic and belongs to $L^2(D)$. Furthermore, we assume that the logarithm of the diffusion coefficient is a centered Gaussian field which can be represented by a Karhunen-Loève expansion, cf. [22],

$$b(\mathbf{x}, \omega) := \log\big(a(\mathbf{x}, \omega)\big) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\varphi_k(\mathbf{x})\psi_k(\omega). \tag{2.2}$$

Here, $\{\varphi_k\}_k \subset L^\infty(D)$ are pairwise orthonormal functions and $\{\psi_k\}_k$ are independent, standard normally distributed random variables, i.e. $\psi_k(\omega) \sim \mathcal{N}(0, 1)$. For the convergence of the series in (2.2) we assume that the sequence

$$\gamma_k := \sqrt{\lambda_k}\|\varphi_k\|_{L^\infty(D)} \tag{2.3}$$

satisfies $\{\gamma_k\}_k \in \ell^1(\mathbb{N})$.

In practice, one has of course to compute the expansion (2.2) from the given covariance kernel

$$\operatorname{Cov}_b(\mathbf{x},\mathbf{y}) := \int_\Omega b(\mathbf{x},\omega)b(\mathbf{y},\omega)\,\mathrm{d}\mathbb{P}(\omega).$$

Thus, the Karhunen-Loève expansion is either finite of length m or needs to be appropriately truncated after m terms. We will assume this in the following. Note that the truncation which arises in the case of the truncation has been discussed in [7].

The assumption that the random variables $\{\psi_k(\omega)\}_k$ are stochastically independent implies that the pushforward measure $\mathbb{P}_{\boldsymbol{\psi}} := \mathbb{P}\circ\boldsymbol{\psi}$ with respect to the measurable mapping

$$\boldsymbol{\psi}\colon \Omega \to \mathbb{R}^m, \quad \omega \mapsto \boldsymbol{\psi}(\omega) := \big(\psi_1(\omega),\ldots,\psi_m(\omega)\big).$$

is given by the joint density function with respect to the Lebesgue measure

$$\boldsymbol{\rho}(\mathbf{y}) := \prod_{k=1}^m \rho(y_k), \quad \text{where} \quad \rho(y) := \frac{1}{\sqrt{2\pi}}\exp\bigg(-\frac{y^2}{2}\bigg). \tag{2.4}$$

With this representation at hand, we can reformulate the stochastic problem (2.1) as a parametric deterministic problem. To that end, we replace the space $L^2_{\mathbb{P}}(\Omega)$ by $L^2_{\boldsymbol{\rho}}(\mathbb{R}^m)$ and substitute the random variables ψ_k by the coordinates $y_k \in \mathbb{R}$. Now, we define the parameterized and truncated diffusion coefficient $b\colon D\times\mathbb{R}^m \to \mathbb{R}$ via

$$b(\mathbf{x},\mathbf{y}) := \sum_{k=1}^m \sqrt{\lambda_k}\varphi_k(\mathbf{x})y_k \quad \text{and} \quad a(\mathbf{x},\mathbf{y}) := \exp\big(b(\mathbf{x},\mathbf{y})\big) \tag{2.5}$$

for all $\mathbf{x} \in D$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Thus, we arrive at a variational formulation for the parametric diffusion problem:

$$\begin{aligned} &\text{find } u \in L^2_{\boldsymbol{\rho}}(\mathbb{R}^m; H^1_0(D)) \text{ such that} \\ &\qquad -\operatorname{div}\big(a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})\big) = f(\mathbf{x}) \text{ in } D \text{ for all } \mathbf{y}\in\mathbb{R}^m. \end{aligned} \tag{2.6}$$

Here and in the sequel, for a given Banach space X, the Bochner space $L^p_{\boldsymbol{\rho}}(\mathbb{R}^m; X)$, $1 \le p \le \infty$, consists of all functions $v:\mathbb{R}^m \to X$ whose norm

$$\|v\|_{L^p_{\boldsymbol{\rho}}(\mathbb{R}^m;X)} := \begin{cases} \bigg(\displaystyle\int_{\mathbb{R}^m} \|v(\cdot,\mathbf{y})\|_X^p \boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y}\bigg)^{1/p}, & p<\infty \\ \operatorname*{ess\,sup}_{\mathbf{y}\in\mathbb{R}^m} \|v(\cdot,\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\|_X, & p=\infty \end{cases}$$

is finite. If $p = 2$ and X is a Hilbert space, then the Bochner space is isomorphic to the tensor product space $L^2_{\boldsymbol{\rho}}(\mathbb{R}^m)\otimes X$. Note that, for notational convenience, we will always write $v(\mathbf{x},\mathbf{y})$ instead of $\big(v(\mathbf{y})\big)(\mathbf{x})$ if $v \in L^p_{\boldsymbol{\rho}}(\mathbb{R}^m; X)$.

The stochastic diffusion coefficient $a(\mathbf{x},\mathbf{y})$ is neither uniformly bounded away from zero nor uniformly bounded from above for all $\mathbf{y}\in\mathbb{R}^m$. Consequently, it is impossible to show unique solvability in the classical way for elliptic boundary value problems. Especially the Lax-Milgram theorem does not directly apply to the problem (2.1). Nevertheless, in [28], it is shown that the set

$$\Gamma := \left\{ \mathbf{y} \in \mathbb{R}^m : \sum_{k=1}^{m} \gamma_k |y_k| < \infty \right\}$$

is of measure $\mathbb{P}_{\boldsymbol{\psi}}(\Gamma) = 1$ for all $m \leq \infty$. Moreover, for all $\mathbf{y} \in \Gamma$, the diffusion coefficient satisfies

$$0 < a_{\min}(\mathbf{y}) := \operatorname*{ess\,inf}_{\mathbf{x} \in D} a(\mathbf{x}, \mathbf{y}) \leq \operatorname*{ess\,sup}_{\mathbf{x} \in D} a(\mathbf{x}, \mathbf{y}) =: a_{\max}(\mathbf{y}) < \infty. \tag{2.7}$$

Remark 2.1. *In the framework of [28], the general case of $m \to \infty$ is considered. Then, the restriction* (2.3) *of the parameter domain ensures for all $\mathbf{y} \in \Gamma$ that $|b(\mathbf{x}, \mathbf{y})| < \infty$ holds uniformly in $\mathbf{x} \in D$. Obviously, we have $\Gamma = \mathbb{R}^m$ for all $m < \infty$. In the following, we may tacitly assume that condition* (2.7) *is satisfied for all $\mathbf{y} \in \mathbb{R}^m$. The case of an infinite dimensional stochastics, i.e. $m = \infty$, is easily obtained by straightforward modifications of the presented arguments.*

Due to (2.7), for every fixed $\mathbf{y} \in \mathbb{R}^m$, the problem

$$-\operatorname{div}\big(a(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y})\big) = f(\mathbf{x}) \text{ in } D \tag{2.8}$$

is elliptic and admits a unique solution $u(\cdot, \mathbf{y}) \in H_0^1(D)$ which satisfies

$$\|u(\cdot, \mathbf{y})\|_{H^1(D)} \lesssim \frac{1}{a_{\min}(\mathbf{y})} \|f\|_{L^2(D)}. \tag{2.9}$$

We refer the reader to e.g. [28] for a proof of this result.

Throughout this article, we intend to the compute the moments $\mathcal{M}^p u := \mathbb{E}[u(\cdot, \mathbf{y})^p]$ of the solution to (2.8). Especially, the solution's expectation

$$\mathbb{E}_u(\mathbf{x}) = \int_{\mathbb{R}^m} u(\mathbf{x}, \mathbf{y}) \boldsymbol{\rho}(\mathbf{y}) \,\mathrm{d}\mathbf{y} \in H_0^1(D) \tag{2.10}$$

and its variance

$$\mathbb{V}_u(\mathbf{x}) = \mathbb{E}_{u^2}(\mathbf{x}) - \mathbb{E}_u^2(\mathbf{x}) = \int_{\mathbb{R}^m} u^2(\mathbf{x}, \mathbf{y}) \boldsymbol{\rho}(\mathbf{y}) \,\mathrm{d}\mathbf{y} - \mathbb{E}_u^2(\mathbf{x}) \in W_0^{1,1}(D) \tag{2.11}$$

are of interest to us. They correspond to the first and the second (centered) moment of the solution u. Notice that the knowledge of all moments is sufficient to determine the solution's distribution.

3. Regularity of the solution. We are mainly interested in computing the mean and the variance of the solution of (2.6). To establish error bounds for the application of Gaussian quadrature rules, we consider in this section the regularity of the solution u and its square u^2. Under certain regularity assumptions it is also possible to obtain error bounds for arbitrary powers of the solution, i.e., u^p for $p \in \mathbb{N}$. Although this topic has already been addressed in a rather abstract way in [2, 7, 21, 28], the available results are not satisfactory for us. Thus, we will compile and augment here some of the results which originate from those articles for our framework.

Remark 3.1. *In the following, the Sobolev space $H_0^1(D)$ is considered to be equipped with the norm $\|\cdot\|_{H^1(D)} := \|\nabla \cdot\|_{L^2(D)}$. Likewise, we use corresponding* *norms for the Sobolev spaces $W_0^{1,p}(D)$, i.e., $\|\cdot\|_{W^{1,p}(D)} := \|\nabla\cdot\|_{L^p(D)}$. Since we only consider homogenous Dirichlet problems, by Sobolev's norm equivalence theorem, cf. [1], they all induce equivalent norms for these spaces. Of course, all results are straightforwardly extendable to the case of non-homogenous Dirichlet problems.*

Since the diffusion coefficient $a(\mathbf{x},\mathbf{y})$ is not uniformly elliptic with respect to $\mathbf{y}$, we cannot expect the solution to be uniformly bounded in $\mathbf{y}$. Thus, the solution u to (2.6) may not be contained in the Bochner space $C^k\big(\mathbb{R}^m; H_0^1(D)\big)$, which consists of the functions providing k-fold differentiability with respect to $\mathbf{y}$. Nonetheless, we can multiply u by an auxiliary weight and end up with a bounded product.

Definition 3.2. *For $\boldsymbol{\gamma} := (\gamma_1,\ldots,\gamma_m)$, cf. (2.3), let $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_m) \in \mathbb{R}^m$ be such that $\boldsymbol{\beta} \geq \boldsymbol{\gamma}$, that is $\beta_k \geq \gamma_k$ for all $k = 1,\ldots,m$. Then, we define the auxiliary weight*

$$\boldsymbol{\sigma}(\mathbf{y}) := \prod_{k=1}^{m} \sigma_k(y_k), \quad \textit{where} \quad \sigma_k(y_k) := \exp(-\beta_k|y_k|).$$

For the special case $\boldsymbol{\beta} = \boldsymbol{\gamma}$, the auxiliary weight is denoted by $\boldsymbol{\sigma}_{\min}(\mathbf{y})$.

Now let X denote some Banach space of functions which are defined on the domain D, for example $X = H_0^1(D)$ or $X = W_0^{1,1}(D)$. For X and a weight $w : \mathbb{R}^m \to \mathbb{R}$, we define the weighted space, cf. [2],

$$C_w^0(\mathbb{R}^m; X) := \Big\{ v \colon \mathbb{R}^m \to X : v \text{ is continuous and } \max_{\mathbf{y}\in\mathbb{R}^m} \|w(\mathbf{y})v(\cdot,\mathbf{y})\|_X < \infty \Big\}$$

equipped with the norm

$$\|v\|_{C_w^0(\mathbb{R}^m;X)} := \max_{\mathbf{y}\in\mathbb{R}^m} \|w(\mathbf{y})v(\cdot,\mathbf{y})\|_X.$$

In the case $w = \boldsymbol{\sigma}$, this space satisfies $C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m; X) \subset L_{\boldsymbol{\rho}}^p(\mathbb{R}^m; X)$ for all $p \in \mathbb{N}$. This comes from

$$\|v\|_{L_{\boldsymbol{\rho}}^p(\mathbb{R}^m;X)} \leq \|v\|_{C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m;X)} \bigg(\int_{\mathbb{R}^m} \big(\boldsymbol{\sigma}(\mathbf{y})\big)^{-p} \boldsymbol{\rho}(\mathbf{y}) \,\mathrm{d}\mathbf{y} \bigg)^{\frac{1}{p}} < \infty$$

because of $p\beta_k y_k = o(y_k^2)$ for $y_k \to \infty$ and the integrability of the normal distribution's tails. Typically, the value of the integral depends exponentially on m, since the domain of integration is $\mathbb{R}^m$. In particular, if we integrate $\boldsymbol{\sigma}_{\min}$ or even the p-th power of $\boldsymbol{\sigma}_{\min}$ with respect to the Gaussian measure, we get, cf. [13],

$$\bigg(\int_{\mathbb{R}^m} \big(\boldsymbol{\sigma}_{\min}(\mathbf{y})\big)^{-p} \rho(\mathbf{y}) \,\mathrm{d}\mathbf{y} \bigg)^{\frac{1}{p}} \leq \exp\bigg(p^2 \sum_{k=1}^{m} \gamma_k^2 + p\sqrt{\frac{2}{\pi}} \sum_{k=1}^{m} \gamma_k \bigg).$$

This expression depends only on the decay of the γ_k and is bounded independently of the dimension m due to (2.3).

Proposition 3.3. *The solution u of (2.6) is contained in $C_{\boldsymbol{\sigma}}^0\big(\mathbb{R}^m; H_0^1(D)\big)$ and it holds*

$$\|u\|_{C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m;H^1(D))} \lesssim \|f\|_{L^2(D)}.$$

Furthermore, we get $u^2 \in C_{\boldsymbol{\sigma}^2}^0\big(\mathbb{R}^m; W_0^{1,1}(D)\big)$ and

$$\|u^2\|_{C_{\boldsymbol{\sigma}^2}^0(\mathbb{R}^m;W^{1,1}(D))} \lesssim \|f\|_{L^2(D)}^2.$$

Proof. In view of inequality (2.9) and since $1/a_{\min}(\mathbf{y}) \leq \exp\big(\sum_{k=1}^m \gamma_k|y_k|\big)$, we conclude the first estimate

$$\boldsymbol{\sigma}(\mathbf{y})\|u(\cdot,\mathbf{y})\|_{H^1(D)} \lesssim \exp\Bigg(\sum_{k=1}^m(\gamma_k-\beta_k)|y_k|\Bigg)\|f\|_{L^2(D)} \leq \|f\|_{L^2(D)}.$$

The second estimate follows from

$$\begin{aligned}\boldsymbol{\sigma}^2(\mathbf{y})\|u^2(\cdot,\mathbf{y})\|_{W^{1,1}(D)} &\leq \boldsymbol{\sigma}^2(\mathbf{y})\|2u(\cdot,\mathbf{y})\nabla u(\cdot,\mathbf{y})\|_{L^1(D)}\\ &\lesssim \big(\boldsymbol{\sigma}(\mathbf{y})\|u(\cdot,\mathbf{y})\|_{H^1(D)}\big)^2 \lesssim \|f\|^2_{L^2(D)}.\end{aligned}$$

□

The differentiability of u follows straightforwardly from the differentiability of the diffusion coefficient α, cf. [21]. Therefore, we use a lemma from [21] which is adjusted for our purposes.

LEMMA 3.4. *For every $\mathbf{y}\in\mathbb{R}^m$, the following estimate holds*

$$\big\|\partial_{y_k}^j u(\cdot,\mathbf{y})\big\|_{H^1(D)} \leq j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}}\|u(\cdot,\mathbf{y})\|_{H^1(D)}.$$

With this lemma at hand, we are able to show the following result.

PROPOSITION 3.5. *For all weights $\boldsymbol{\sigma}$ with $\boldsymbol{\beta}\geq 2\boldsymbol{\gamma}$, the partial derivatives of the solution u to* (2.6) *satisfy*

$$\big\|\partial_{y_k}^j u\big\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;H^1(D))} \lesssim j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\|f\|_{L^2(D)}. \tag{3.1}$$

Especially, it holds $\partial_{y_k}^j u\in C^0_{\boldsymbol{\sigma}}\big(\mathbb{R}^m;H^1_0(D)\big)$.

Proof. From Lemma 3.4, we obtain

$$\begin{aligned}\big\|\partial_{y_k}^j u\big\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;H^1(D))} &= \max_{\mathbf{y}\in\mathbb{R}^m}\big\|\boldsymbol{\sigma}(\mathbf{y})\partial_{y_k}^j u(\cdot,\mathbf{y})\big\|_{H^1(D)}\\ &\leq j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\max_{\mathbf{y}\in\mathbb{R}^m}\sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}}\boldsymbol{\sigma}(\mathbf{y})\|u(\cdot,\mathbf{y})\|_{H^1(D)}.\end{aligned}$$

According to [7, 28], in view of (2.5) and (2.3), we can bound the ellipticity constant by

$$\sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}} \leq \exp\Bigg(\sum_{k=1}^m\gamma_k|y_k|\Bigg),$$

which yields the desired estimate according to

$$\begin{aligned}\big\|\partial_{y_k}^j u\big\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m,H^1(D))} &\leq j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\max_{\mathbf{y}\in\mathbb{R}^m}\exp\Bigg(\sum_{k=1}^m(\gamma_k-\beta_k)|y_k|\Bigg)\|u(\cdot,\mathbf{y})\|_{H^1(D)}\\ &\leq j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\max_{\mathbf{y}\in\mathbb{R}^m}\exp\Bigg(\sum_{k=1}^m-\gamma_k|y_k|\Bigg)\|u(\cdot,\mathbf{y})\|_{H^1(D)}\\ &\lesssim j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\|f\|_{L^2(D)}.\end{aligned}$$

□

The previous result shows the regularity of the solution u. In the following proposition, we consider the regularity of u^2.

PROPOSITION 3.6. *The partial derivatives of u^2, where u is the solution of (2.6), satisfy $\partial_{y_k}^j u^2 \in C^0_{\boldsymbol{\sigma}^2}\big(\mathbb{R}^m; W_0^{1,1}(D)\big)$ for all $\boldsymbol{\sigma}$ with $\boldsymbol{\beta} \geq 2\boldsymbol{\gamma}$. Especially, it holds*

$$\big\|\partial_{y_k}^j u^2\big\|_{C^0_{\boldsymbol{\sigma}^2}(\mathbb{R}^m;W^{1,1}(D))} \lesssim (j+1)!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j \|f\|^2_{L^2(D)}. \tag{3.2}$$

Proof. By the Leibniz rule, we obtain

$$\big\|\partial_{y_k}^j u^2(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} \leq \sum_{\ell=0}^{j}\binom{j}{\ell}\big\|\partial_{y_k}^\ell u(\cdot,\mathbf{y})\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)}. \tag{3.3}$$

Each summand can be estimated as follows

$$\begin{aligned}
&\big\|\partial_{y_k}^\ell u(\cdot,\mathbf{y})\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)}\\
&\quad= \big\|\nabla\partial_{y_k}^\ell u(\cdot,\mathbf{y})\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y}) + \partial_{y_k}^\ell u(\cdot,\mathbf{y})\nabla\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{L^1(D)}\\
&\quad\leq \big\|\nabla\partial_{y_k}^\ell u(\cdot,\mathbf{y})\big\|_{L^2(D)}\big\|\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{L^2(D)} + \big\|\partial_{y_k}^\ell u(\cdot,\mathbf{y})\big\|_{L^2(D)}\big\|\nabla\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{L^2(D)}\\
&\quad\lesssim \big\|\partial_{y_k}^\ell u(\cdot,\mathbf{y})\big\|_{H^1(D)}\big\|\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{H^1(D)} + \big\|\partial^\ell u(\cdot,\mathbf{y})\big\|_{H^1(D)}\big\|\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{H^1(D)}.
\end{aligned}$$

Application of Lemma 3.4 yields

$$\big\|\partial_{y_k}^\ell u(\cdot,\mathbf{y})\partial_{y_k}^{j-\ell}u(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} \lesssim 2\ell!(j-\ell)!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|^2_{H^1(D)}.$$

Inserting this inequality into (3.3) results in

$$\begin{aligned}
\big\|\partial_{y_k}^j u^2(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} &\lesssim 2\sum_{l=0}^{j} j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|^2_{H^1(D)}\\
&\leq 2(j+1)!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|^2_{H^1(D)}.
\end{aligned}$$

The estimate (3.2) is now obtained analogously to estimate (3.1) in Lemma 3.5. □

REMARK 3.7. *Given that the load satisfies $f \in L^p(D)$ and that the solution u satisfies the stronger regularity condition*

$$\big\|\partial_{y_k}^j u\big\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;W^{1,p}(D))} \lesssim j!\bigg(\frac{\gamma_k}{\log 2}\bigg)^j\|f\|_{L^p(D)}$$

for $p \in \mathbb{N}$, we obtain a similar regularity result for u^p, i.e.

$$\big\|\partial_{y_k}^j u^p\big\|_{C^0_{\boldsymbol{\sigma}^p}(\mathbb{R}^m;W^{1,1}(D))} \lesssim j!\bigg(\frac{c(p)\gamma_k}{\log 2}\bigg)^j\|f\|^p_{L^p(D)},$$

with a constant $1 < c(p) \leq p$ which depends on p. This constant results from the application of Faà di Bruno's formula, cf. [8], for the higher order derivatives.

Lemma 3.4 provides only a bound on the derivatives of $\partial_{y_k}^j u$ when the spatial regularity is measured in $H_0^1(D)$. We shall thus complete this section by a result from [21] which establishes estimates on $\partial_{y_k}^j u$ when the spatial regularity is measured in the space $\mathcal{W} := H^2(D) \cap H_0^1(D)$. This result guarantees us the mixed regularity which is necessary for the sparse grid construction between the spatial and the stochastic variable. To establish this result we shall assume that the corresponding eigenfunctions of the Karhunen-Loève decomposition belong to $W^{1,\infty}(D)$, which is for example fulfilled in case of a Gaussian covariance. If we then replace γ_k by

$$\tilde{\gamma}_k := \sqrt{\lambda_k}\big(\|\varphi_k\|_{L^\infty(D)} + \|\nabla\varphi_k\|_{L^\infty(D)}\big).$$

in the definition of the set Γ, cf. (2.3), we still have the parameter domain $\mathbb{R}^m$ for each finite $m \in \mathbb{N}$. Hence, this sharpened condition induces no further restriction to the parameter domain. For convex or sufficiently smooth curved domains, a norm on $\mathcal{W}$ is given by

$$\|u\|_{\mathcal{W}} := \|\nabla u\|_{L^2(D)} + \|\Delta u\|_{L^2(D)},$$

cf. [21]. Along the lines of [21], we have the following

PROPOSITION 3.8. *For all $\mathbf{y} \in \mathbb{R}^m$, the solution to problem (2.8) satisfies*

$$\begin{aligned}\big\|\sqrt{a(\cdot,\mathbf{y})}\Delta\partial_{y_k}^j u(\cdot,\mathbf{y})\big\|_{L^2(D)} \\ \lesssim j!\bigg(\frac{2\tilde{\gamma}_k}{\log 2}\bigg)^j\Big(\big\|\sqrt{a(\cdot,\mathbf{y})^{-1}}f\big\|_{L^2(D)} + 2g(\mathbf{y})\big\|\sqrt{a(\cdot,\mathbf{y})}u(\cdot,\mathbf{y})\big\|_{H_0^1(D)}\Big)\end{aligned}$$

with $g(\mathbf{y}) = 1 + 2\sum_{k=1}^m |y_k|\|\nabla\varphi_k\|_{L^\infty(D)} < \infty$.

This proposition implies the estimate

$$\big\|\Delta\partial_{y_k}^j u(\cdot,\mathbf{y})\big\|_{L^2(D)} \lesssim \sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}}j!\bigg(\frac{2\tilde{\gamma}_k}{\log 2}\bigg)^j\Big(\|f\|_{L^2(D)} + 2g(\mathbf{y})\|u(\cdot,\mathbf{y})\|_{H_0^1(D)}\Big).$$

It follows together with Lemma 3.4 and with $g(\mathbf{y}) \lesssim \big(a_{\max}(\mathbf{y})/a_{\min}(\mathbf{y})\big)^\delta$ for all $\delta > 0$ that

$$\big\|\partial_{y_k}^j u(\cdot,\mathbf{y})\big\|_{H^2(D)} \lesssim \big\|\partial_{y_k}^j u(\cdot,\mathbf{y})\big\|_{\mathcal{W}} \lesssim \bigg(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\bigg)^{1+\delta} j!\bigg(\frac{2\tilde{\gamma}_k}{\log 2}\bigg)^j\|f\|_{L^2(D)}. \tag{3.4}$$

This establishes $u(\cdot,\mathbf{y}) \in \mathcal{W}$ for all $\mathbf{y} \in \mathbb{R}^m$. Moreover, we get also $u \in C_{\boldsymbol{\sigma}}^0\big(\mathbb{R}^m, H^2(D)\big)$ since there holds

$$\big\|\partial_{y_k}^j u\big\|_{C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m,H^2(D))} \lesssim j!\bigg(\frac{2\tilde{\gamma}_k}{\log 2}\bigg)^j\|f\|_{L^2(D)} \tag{3.5}$$

for all $\boldsymbol{\sigma}$ with $\boldsymbol{\beta} > 2\boldsymbol{\gamma}$.

REMARK 3.9. *All estimates in this section are given for one-dimensional derivatives of the solution u to (2.6). Nevertheless, they can easily be extended for multidimensional derivatives.*

4. Gauss-Hermite quadrature for the stochastic variable. For given sets of points

$$\big\{\eta_0^{(k)},\ldots,\eta_{N_k}^{(k)}\big\}\subset\mathbb{R},\quad N_k\in\mathbb{N},\quad k=1,\ldots,m, \tag{4.1}$$

we define the Lagrangian basis polynomials $L_0^{(k)},\ldots,L_{N_k}^{(k)}$ of degree N_k by the property $L_i^{(k)}(\eta_j^{(k)})=\delta_{i,j}$. Then, for a multiindex $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)\in\mathcal{J}$ with $\mathcal{J}:=\bigtimes_{k=1}^m\{0,\ldots,N_k\}$, we define the corresponding tensor product Lagrangian basis polynomial

$$\mathbf{L}_{\boldsymbol{\alpha}}(\mathbf{y}):=\prod_{k=1}^m L_{\alpha_k}^{(k)}(y_k)\quad\text{with}\quad \mathbf{L}_{\boldsymbol{\alpha}}(\boldsymbol{\eta}_{\boldsymbol{\alpha}'})=\delta_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\quad\text{for}\quad \boldsymbol{\eta}_{\boldsymbol{\alpha}}:=\big(\eta_{\alpha_1}^{(1)},\ldots,\eta_{\alpha_m}^{(m)}\big).$$

For continuous functions $v\colon\mathbb{R}^m\to\mathbb{R}$, we can thus introduce the associated interpolation operator by

$$\big(\boldsymbol{\Pi}_{\mathcal{J}}v\big)(\mathbf{y}):=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}v(\boldsymbol{\eta}_{\boldsymbol{\alpha}})\mathbf{L}_{\boldsymbol{\alpha}}(\mathbf{y}).$$

If we choose for each k the sets (4.1) to be the roots of the Hermite polynomials of respective degree N_k+1 which are known to be orthogonal with respect to the inner product

$$(q,r)_{L^2_\rho(\mathbb{R})}=\int_{\mathbb{R}}q(y)r(y)\rho(y)\,\mathrm{d}y,\quad q,r\in L^2_\rho(\mathbb{R}),$$

we can straightforwardly derive Gaussian quadrature from the interpolation operator, which are taylored to our problem setting. The resulting quadrature rules are known to be exact of degree $2N_k+1$. The associated quadrature weights are given by

$$\omega_i^{(k)}=\int_{\mathbb{R}}L_i^{(k)}(y)\rho(y)\,\mathrm{d}y.$$

Further, one easily verifies

$$\int_{\mathbb{R}}L_i^{(k)}(y)L_j^{(k)}(y)\rho(y)\,\mathrm{d}y=\omega_i^{(k)}\delta_{i,j}.$$

By tensor product construction we get the multivariate weights

$$\boldsymbol{\omega}_{\boldsymbol{\alpha}}:=\prod_{k=1}^m\omega_{\alpha_k}^{(k)}\quad\text{with}\quad\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\boldsymbol{\omega}_{\boldsymbol{\alpha}}=1.$$

Then, for the $\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\mathcal{J}$, we have the corresponding multivariate relations

$$\boldsymbol{\omega}_{\boldsymbol{\alpha}}=\int_{\mathbb{R}^m}\mathbf{L}_{\boldsymbol{\alpha}}(\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y}\quad\text{and}\quad\int_{\mathbb{R}^m}\mathbf{L}_{\boldsymbol{\alpha}}(\mathbf{y})\mathbf{L}_{\boldsymbol{\alpha}'}(\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y}=\boldsymbol{\omega}_{\boldsymbol{\alpha}}\delta_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}.$$

Now, we can interpolate the solution $u\in L^2_{\boldsymbol{\rho}}\big(\mathbb{R}^m;H^1_0(D)\big)$ of (2.6) in the stochastic variable

$$u(\mathbf{x},\mathbf{y})\approx(\mathrm{Id}\otimes\boldsymbol{\Pi}_{\mathcal{J}}u)(\mathbf{x},\mathbf{y})=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}u(\mathbf{x},\boldsymbol{\eta}_{\boldsymbol{\alpha}})\mathbf{L}_{\boldsymbol{\alpha}}(\mathbf{y}). \tag{4.2}$$

The occurring interpolation error is estimated in e.g. [2].

We intend to compute output functionals of the solution, like the solution's moments. Especially, with (4.2) at hand, we can approximate the solution's expectation (2.10) and its variance (2.11) by

$$\mathbb{E}_u(\mathbf{x}) \approx \sum_{\boldsymbol{\alpha}\in\mathcal{J}} u(\mathbf{x},\boldsymbol{\eta}_{\boldsymbol{\alpha}})\boldsymbol{\omega}_{\boldsymbol{\alpha}} \quad \text{and} \quad \mathbb{V}_u(\mathbf{x}) \approx \sum_{\boldsymbol{\alpha}\in\mathcal{J}} u(\mathbf{x},\boldsymbol{\eta}_{\boldsymbol{\alpha}})^2\boldsymbol{\omega}_{\boldsymbol{\alpha}} - \bigg(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} u(\mathbf{x},\boldsymbol{\eta}_{\boldsymbol{\alpha}})\boldsymbol{\omega}_{\boldsymbol{\alpha}}\bigg)^2.$$

Therefore, we rather have to consider the quadrature error than the interpolation error. Hence, the remainder of this section is dedicated to the analysis of the occurring quadrature error for the Gauss-Hermite quadrature. To that end, we define the quadrature operator

$$\mathbf{Q}_{\mathcal{J}}\colon C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)\to X, \quad (\mathbf{Q}_{\mathcal{J}}v)(\mathbf{x}) := \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \boldsymbol{\omega}_{\boldsymbol{\alpha}} v(\mathbf{x},\boldsymbol{\eta}_{\boldsymbol{\alpha}}).$$

Of course, it is possible to consider quadrature operators with respect to other weighted spaces $C^0_\nu(\mathbb{R}^m;X)$ analogously.

In the following, we adopt the analysis and the notation presented in [2], where the polynomial approximation error in case of stochastic collocation for uniformly elliptic equations is analyzed. In [7], this analysis has been extended to the case of log-normal and thus no more uniformly elliptic diffusion coefficients. According to [2], we shall introduce the one-dimensional Gaussian auxiliary measure $\sqrt{\rho(y)} \eqsim \exp(-y^2/4)$ and the corresponding space $C^0_{\sqrt{\rho}}(\mathbb{R};X)$. This norm is weaker than the norm of $C^0_\sigma(\mathbb{R};X)$ from the previous section, that is $C^0_\sigma(\mathbb{R};X) \subset C^0_{\sqrt{\rho}}(\mathbb{R};X)$. On $C^0_{\sqrt{\rho}}(\mathbb{R};X)$, we can now define the one-dimensional quadrature operator of degree $N\in\mathbb{N}$ by

$$Q_N\colon C^0_{\sqrt{\rho}}(\mathbb{R};X)\to X, \quad (Q_N v)(\mathbf{x}) := \sum_{i=0}^N \omega_i v(\mathbf{x},\eta_i),$$

where $\eta_0,\ldots,\eta_N$ are again the $N+1$ roots of the Hermite polynomial of degree $N+1$.

The following two lemmata imply that the quadrature error in one dimension can be bounded by the polynomial approximation error. They are modifications of the corresponding lemmata in [2] for the polynomial interpolation.

Lemma 4.1. *The quadrature operator $Q_N\colon C^0_{\sqrt{\rho}}(\mathbb{R};X)\to X$ is continuous.*

Proof. Consider $v\in C^0_{\sqrt{\rho}}(\mathbb{R};X)$. By using the triangle inequality and exploiting the positivity of the weights w_k of the Gauss-Hermite quadrature we have

$$\begin{aligned}\|Q_N v\|_X &= \bigg\|\sum_{i=0}^N \omega_i v(\cdot,\eta_i)\bigg\|_X \le \sum_{i=0}^N \omega_i \|v(\cdot,\eta_i)\|_X = \sum_{i=0}^N \frac{\omega_i}{\sqrt{\rho(\eta_i)}}\big\|\sqrt{\rho(\eta_i)}v(\cdot,\eta_k)\big\|_X\\ &\le \max_{i=0,\ldots,N}\big\|\sqrt{\rho(\eta_i)}v(\cdot,\eta_i)\big\|_X \sum_{i=0}^N \frac{\omega_i}{\sqrt{\rho(\eta_i)}} \lesssim \|v\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}.\end{aligned}$$

The last inequality follows from [30], where the convergence

$$\sum_{i=0}^N \frac{\omega_i}{\sqrt{\rho(\eta_i)}} \stackrel{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} \frac{\rho(y)}{\sqrt{\rho(y)}}\,\mathrm{d}y < \infty$$

is shown. ◻

The next lemma relates the quadrature error to the best approximation error in $C^0_\sigma(\mathbb{R};X)$. We denote by $\mathcal{P}_d(\mathbb{R})$ the space of polynomials of degree at most d.

LEMMA 4.2. *For every $v \in C^0_{\sqrt{\rho}}(\mathbb{R};X)$, the quadrature error of the $(N+1)$-point Gauss-Hermite quadrature is bounded by*

$$\|\mathbb{E}_v - Q_N v\|_X \lesssim \inf_{w \in \mathcal{P}_{2N+1}(\mathbb{R})\otimes X} \|v - w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}.$$

Proof. Since the $(N+1)$-point Gauss-Hermite quadrature has degree of precision $2N+1$, it holds $Q_N w = \mathbb{E}_w$ for all $w \in \mathcal{P}_{2N+1}(\mathbb{R}) \otimes X$. Thus, for arbitrary $w \in \mathcal{P}_{2N+1}(\mathbb{R}) \otimes X$, we have

$$\begin{aligned}\|\mathbb{E}_v - Q_N v\|_X &\le \|\mathbb{E}_{v-w}\|_X + \|Q_N(v-w)\|_X \lesssim \|v-w\|_{L^1_\rho(\mathbb{R};X)} + \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)} \\ &\lesssim \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}.\end{aligned}$$

□

In the next step, we show that functions $v \in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)$ admit an analytic extension under certain decay properties of their derivatives. This is crucial to bound error of the best approximation in the polynomial space. Following the notation in [2], we introduce

$$\boldsymbol{\rho}^\star_k(\mathbf{y}^\star_k) := \prod_{\substack{i=1\\ i\neq k}}^m \rho(y_i) \quad \text{and} \quad \mathbf{y}^\star_k := (y_1,\dots,y_{k-1},y_{k+1},\dots,y_m) \in \mathbb{R}^{m-1}$$

to continue with the one-dimensional analysis.

LEMMA 4.3. *Let $(y_k, \mathbf{y}^\star_k) \in \mathbb{R}^m$, $v \in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)$, and assume that there holds*

$$\left\|\partial^j_{y_k} v\right\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)} \lesssim j!\mu_k^j$$

with some constant $\mu_k \in (0,\infty)$. Then, for $\tau_k \in (0, 1/\mu_k)$, the function

$$v\colon \mathbb{R} \to C^0_{\boldsymbol{\sigma}^\star_k}(\mathbb{R}^{m-1};X), \quad y_k \mapsto v(\mathbf{x}, y_k, \mathbf{y}^\star_k)$$

admits an analytic extension $v(\mathbf{x}, z, \mathbf{y}^\star_k)$ for $z \in \Sigma(\tau_k) := \{z \in \mathbb{C},\ \operatorname{dist}(z,\mathbb{R}) \le \tau_k\}$.

Proof. For given $y_k \in \mathbb{R}$, we define a formal Taylor expansion in $z \in \mathbb{C}$:

$$v(\mathbf{x}, z, \mathbf{y}^\star_k) = \sum_{j=0}^\infty \frac{(z-y_k)^j}{j!} \partial^j_{y_k} v(\mathbf{x}, y_k, \mathbf{y}^\star_k).$$

Thus, given an arbitrary $y_k \in \mathbb{R}$, we can estimate

$$\begin{aligned}\sigma_k(y_k)\|v(\cdot,z,\cdot)\|_{C^0_{\boldsymbol{\sigma}^\star_k}(\mathbb{R}^{m-1};X)} &\le \sum_{j=0}^\infty \frac{|z-y_k|^j}{j!} \sigma_k(y_k) \left\|\partial^j_{y_k} v(\cdot, y_k, \cdot)\right\|_{C^0_{\boldsymbol{\sigma}^\star_k}(\mathbb{R}^{m-1};X)} \\ &\le \sum_{j=0}^\infty \frac{|z-y_k|^j}{j!} \left\|\partial^j_{y_k} v\right\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)} \\ &\lesssim \sum_{j=0}^\infty \big(|z-y_k|\mu_k\big)^j.\end{aligned}$$

The last expression converges for all $|z-y_k|\le\tau_k<1/\mu_k$. Hence, since we can cover $\Sigma(\tau_k)$ by the union of balls $|z-y_k|\le\tau_k$, the function v can be extended analytically to the whole region $\Sigma(\tau_k)$. $\square$

Finally, we have to bound the approximation error of a function v which admits an analytic extension. This was done in [2], where the next lemma was proven.

Lemma 4.4. *Suppose that $v\in C^0_\sigma(\mathbb{R};X)$ admits an analytic extension in $\Sigma(\tau)$ for some $\tau>0$. Then, there exists a function $\Theta(N)=\mathcal{O}(\sqrt{N})$ such that*

$$\min_{w\in\mathcal{P}_N\otimes X}\max_{y\in\mathbb{R}}\left|\sqrt{\rho(y)}\|v(y)-w(y)\|_X\right|\lesssim\Theta(N)\exp(-\tau\sqrt{N})\max_{z=y+iw\in\Sigma(\tau)}\sigma(y)\|v(z)\|_X.$$

We are now able to estimate the error of the tensor product Gaussian quadrature for functions $v\in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)$ which fulfill the condition of Lemma 4.3. Therefore, we define the following tensor product integral operator:

$$\mathbf{I}:=\bigotimes_{k=1}^m I_k\quad\text{with}\quad I_k v=\int_{\mathbb{R}}v(y_k)\rho(y_k)\,\mathrm{d}y_k.$$

Note that $\mathbf{I}v$ coincides with $\mathbb{E}_v$ due to the product structure of the measure $\boldsymbol{\rho}$.

Theorem 4.5. *Let $v\in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)$ satisfy the conditions of Lemma 4.3 in every direction, i.e. for all $k=1,\ldots,m$ there exists a $\tau_k\in(0,\infty)$ such that $v(\mathbf{x},y_k,\mathbf{y}_k^\star)$ as a function of y_k admits an analytic extension in $\Sigma(\tau_k)$. Furthermore, let $\varepsilon>0$ and choose $\varepsilon_1,\ldots,\varepsilon_m>0$ such that $\sum_{k=1}^m\varepsilon_k=\varepsilon$. If we determine the number of quadrature points N_k such that*

$$N_k\ge\frac{|\log\varepsilon_k|^2}{2\tau_k^2(1-\delta)^2}-\frac{1}{2}$$

for some $\delta>0$, we get the quadrature error bounded by

$$\|(\mathbf{I}-\mathbf{Q}_{\mathcal{J}})u\|_{H^1(D)}\lesssim\sum_{k=1}^m\varepsilon_k\left\{\max_{\mathbf{y}_k^\star\in\mathbb{R}^{m-1}}\boldsymbol{\sigma}_k^\star(\mathbf{y}_k^\star)\right\}\left\{\max_{z\in\Sigma(\tau_k)}\boldsymbol{\sigma}(\mathrm{Re}(z))\|v(\cdot,z,\mathbf{y}_k^\star)\|_X\right\}.$$

Hence, the quadrature error is then bounded by $\mathcal{O}(\varepsilon)$ due to the choice of ε_k.

Proof. Let $v\in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;X)$ be a function which fulfills the conditions of Lemma 4.3 for all directions y_k, $k=1,\ldots,m$. We estimate the tensor product quadrature error by

$$\begin{aligned}\|(\mathbf{I}-\mathbf{Q}_{\mathcal{J}})v\|_X&=\|(I_1\otimes\ldots\otimes I_m-Q_{N_1}\otimes\ldots\otimes Q_{N_m})v\|_X\\&\le\big\|\big((I_1-Q_{N_1})\otimes I_2\otimes\ldots\otimes I_m\big)v\big\|_X\\&\qquad+\big\|\big(Q_{N_1}\otimes(I_2\otimes\ldots\otimes I_m-Q_{N_2}\otimes\ldots\otimes Q_{N_m})\big)v\big\|_X.\end{aligned}$$

Repeating this procedure m-times yields

$$\|(\mathbf{I}-\mathbf{Q}_{\mathcal{J}})v\|_X\le\sum_{k=1}^m\big\|\big(Q_{N_1}\otimes\ldots\otimes Q_{N_{k-1}}\otimes(I_k-Q_{N_k})\otimes I_{k+1}\otimes\ldots\otimes I_m\big)v\big\|_X.$$

To achieve an over-all error of order $\mathcal{O}(\varepsilon)$ we have to bound the k-th summand by ε_k. Since the one-dimensional Gaussian quadrature operator Q_{N_ℓ} and the integral operator I_ℓ are continuous mappings from $C^0_{\sigma_\ell}(\mathbb{R};X)$ to X for all $\ell = 1,\dots,m$, we get

$$\begin{aligned}
&\big\|\big(Q_{N_1}\otimes\ldots\otimes Q_{N_{k-1}}\otimes(I_k-Q_{N_k})\otimes I_{k+1}\otimes\ldots\otimes I_m\big)v\big\|_X\\
&\quad\le \max_{y_1\in\mathbb{R}}\sigma_1(y_1)\big\|\big(Q_{N_2}\otimes\ldots\otimes Q_{N_{k-1}}\otimes(I_k-Q_{N_k})\otimes I_{k+1}\otimes\ldots\otimes I_m\big)v(\cdot,y_1)\big\|_X\\
&\quad\le \max_{\mathbf{y}_k^\star\in\mathbb{R}^{m-1}}\boldsymbol{\sigma}_k^\star(\mathbf{y}_k^\star)\|(I_k-Q_{N_k})v(\cdot,\mathbf{y}_k^\star)\|_X.
\end{aligned}$$

By the Lemmata 4.2, 4.3 and 4.4, we conclude

$$\begin{aligned}
&\big\|\big(Q_{N_1}\otimes\ldots\otimes Q_{N_{k-1}}\otimes(I_k-Q_{N_k})\otimes I_{k+1}\otimes\ldots\otimes I_m\big)v\big\|_X\\
&\quad\lesssim \max_{\mathbf{y}_k^\star\in\mathbb{R}^{m-1}}\boldsymbol{\sigma}_k^\star(\mathbf{y}_k^\star)\min_{w\in\mathcal{P}_{2N_k+1}\otimes X}\|v(\cdot,\cdot,\mathbf{y}_k^\star)-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}\\
&\quad\lesssim \sqrt{2N_k+1}\exp\big(-\tau_k\sqrt{2N_k+1}\big)\Big\{\max_{\mathbf{y}_k^\star\in\mathbb{R}^{m-1}}\boldsymbol{\sigma}_k^\star(\mathbf{y}_k^\star)\Big\}\Big\{\max_{z\in\Sigma(\tau_k)}\boldsymbol{\sigma}(\mathrm{Re}(z))\|v(\cdot,z,\mathbf{y}_k^\star)\|_X\Big\}\\
&\quad\lesssim \exp\big(-\tau_k\sqrt{2N_k+1}(1-\delta)\big)\Big\{\max_{\mathbf{y}_k^\star\in\mathbb{R}^{m-1}}\boldsymbol{\sigma}_k^\star(\mathbf{y}_k^\star)\Big\}\Big\{\max_{z\in\Sigma(\tau_k)}\boldsymbol{\sigma}(\mathrm{Re}(z))\|v(\cdot,z,\mathbf{y}_k^\star)\|_X\Big\}.
\end{aligned}$$

Here, the last step holds for arbitrary $\delta > 0$. Thus, we can determine the polynomial degree to ensure an exactness of ε_k by

$$\exp\big(-\tau_k\sqrt{2N_k+1}(1-\delta)\big)\le\varepsilon_k \quad\Longrightarrow\quad N_k\ge 0.5\bigg(\frac{|\log\varepsilon_k|^2}{\tau_k^2(1-\delta)^2}-1\bigg).$$

□

Remark 4.6. *The constants hidden in the error estimate obviously depend on the choice of τ_k and tend to infinity if τ_k comes close to the boundary of the analyticity region by means of $\tau_k\to\log 2/\gamma_k$. The solution u of (2.6), the second moment u^2, and, under the conditions of Remark 3.7, even the higher order moments u^p satisfy the conditions of Lemma 4.3 and, therefore ,of Theorem 4.5.*

5. (Quasi) Monte Carlo quadrature for the stochastic variable. In this section, we discuss the use of Monte Carlo and quasi Monte Carlo quadrature rules. These quadrature rules are classically of the form

$$\mathbf{Q}_{\mathrm{(Q)MC}}v=\frac{1}{N}\sum_{i=1}^{N}v(\cdot,\boldsymbol{\xi}_i),$$

where N denotes the number of *samples* and $\boldsymbol{\xi}_i\in\mathbb{R}^m$ is one *sample point*. In case of the Monte Carlo quadrature the sample points are chosen randomly. Therefore, we need a (pseudo-) random number generator, which produces m-dimensional normal distributed random vectors. There are two main advantages of the Monte Carlo quadrature. The method does not suffer from the curse of dimensionality and requires very weak regularity assumptions on the integrand. The drawback of this method is that it produces only stochastical error estimates, also known as *root mean square error*, cf. [6], and converges only with a rate of $\mathcal{O}(N^{-1/2})$, i.e. we have to increase the number of samples by a factor 100 to get a further digit of exactness. More precisely, one has

$$\|(\mathbf{I}-\mathbf{Q}_{MC})v\|_{L^2_\rho(\mathbb{R}^m;X)}\lesssim N^{-\frac{1}{2}}\|v\|_{L^2_\rho(\mathbb{R}^m;X)}.$$

For the error estimation of the quasi Monte Carlo method, it is required that the integrand has integrable, mixed first order derivatives. Then, the error of the standard quasi Monte Carlo method over the unit cube $[0,1]^m$ is bounded by means of the L^∞-*star discrepancy*

$$\mathcal{D}_\infty^\star(\Xi) := \sup_{\mathbf{t}\in[0,1]^m} \left| \operatorname{Vol}\big([\mathbf{0},\mathbf{t})\big) - \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{[\mathbf{0},\mathbf{t})}(\boldsymbol{\xi}_i)\right|$$

of the set of sample points $\Xi = \{\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_N\} \subset [0,1]^m$, where $\operatorname{Vol}\big([\mathbf{0},\mathbf{t})\big)$ denotes the Lebesgue measure of the cuboid $[\mathbf{0},\mathbf{t})$, cf. [24]. In case of certain point sequences, this discrepancy is typically estimated to be of the order $\mathcal{O}\big(N^{-1}(\log N)^m\big)$.

To obtain a quasi Monte Carlo method for the domain of integration $\mathbb{R}^m$, the sample points have to be mapped to $\mathbb{R}^m$ by the inverse distribution function. Numerically, this can be done very efficiently by employing a rational interpolant of the inverse distribution function, cf. [23]. It has been shown in e.g. [20] that the error can again be bounded by $\mathcal{D}_\infty^\star(\Xi)$ for a certain set of functions. To specify this, we define the Bochner space $W_{\text{mix}}^{1,1}(\mathbb{R}^m;X)$ which consists of all functions $v:\mathbb{R}^m\to X$ with finite norm

$$\|v\|_{W_{\text{mix}}^{1,1}(\mathbb{R}^m;X)} := \sum_{\|\mathbf{q}\|_\infty\le 1} \int_{\mathbb{R}^m} \big\|\partial_{\mathbf{y}}^{\mathbf{q}} v(\mathbf{y})\big\|_X \,\mathrm{d}\mathbf{y} < \infty, \tag{5.1}$$

where

$$\partial_{\mathbf{y}}^{\mathbf{q}} v(\mathbf{y}) := \frac{\partial^{|\mathbf{q}|}}{\partial y_1^{q_1}\partial y_2^{q_2}\cdots\partial y_m^{q_m}} v(\mathbf{y}).$$

Then, the error of the quasi Monte Carlo method is typically estimated by

$$\|(\mathbf{I}-\mathbf{Q}_{QMC})v\|_X \lesssim \mathcal{D}_\infty^\star(\Xi)\|v\|_{W_{\text{mix}}^{1,1}(\mathbb{R}^m;X)},$$

cf. [20], which is an extension of the *Koksma-Hlawka* inequality, cf. [24], to unbounded domains. The condition that the norm $\|v\|_{W_{\text{mix}}^{1,1}(\mathbb{R}^m;X)}$ is bounded, is in general very restrictive and is not necessarily fulfilled in our application. Hence, we follow a suggestion of [20] and rewrite the integral under consideration as

$$\int_{\mathbb{R}^m} v(\mathbf{x},\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \overline{\rho}\int_{\mathbb{R}^m} v(\mathbf{x},\mathbf{y})\sqrt{\boldsymbol{\rho}(\mathbf{y})}\,\frac{\sqrt{\boldsymbol{\rho}(\mathbf{y})}}{\overline{\rho}}\,\mathrm{d}\mathbf{y},$$

with the scaling factor $\overline{\rho}$ being defined by

$$\overline{\rho} := \int_{\mathbb{R}^m} \sqrt{\boldsymbol{\rho}(\mathbf{y})}\,\mathrm{d}\mathbf{y}.$$

We now employ a quasi Monte Carlo method with respect to the auxiliary density function $\sqrt{\boldsymbol{\rho}(\mathbf{y})}/\overline{\rho}$ and obtain the error estimate

$$\|(\mathbf{I}-\mathbf{Q}_{QMC})v\|_X \lesssim \mathcal{D}_\infty^\star(\Xi)\|v\sqrt{\boldsymbol{\rho}}\|_{W_{\text{mix}}^{1,1}(\mathbb{R}^m;X)}.$$

Herein, the last norm is bounded (with a constant which depends on m but not on N) in case of the moment computation as it is proven in the next theorem.

Theorem 5.1. *For the solution u to* (2.6)*, the following bound is valid*

$$\|v\sqrt{\boldsymbol{\rho}}\|_{W_{\text{mix}}^{1,1}(\mathbb{R}^m;H^1(D))} \lesssim \Bigg(\sum_{\|\mathbf{q}\|_\infty\le 1}\frac{1}{2^{|\mathbf{q}|}}\sum_{\boldsymbol{\alpha}\le\mathbf{q}}\left(\frac{2\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}|\boldsymbol{\alpha}|!\Bigg)\|f\|_{L^2(D)} < \infty, \quad p = 1.$$

Furthermore, under the condition of 3.7, it holds for the p-th power $v = u^p$ of u

$$\|v\sqrt{\boldsymbol{\rho}}\|_{W^{1,1}_{\text{mix}}(\mathbb{R}^m;W^{1,1}(D))} \lesssim \Bigg(\sum_{\|\mathbf{q}\|_\infty \le 1} \frac{1}{2^{|\mathbf{q}|}} \sum_{\boldsymbol{\alpha}\le\mathbf{q}} \left(\frac{2c(p)\gamma}{\log 2}\right)^{\boldsymbol{\alpha}} |\boldsymbol{\alpha}|! \Bigg) \|f\|^p_{L^p(D)} < \infty, \quad p \ge 2.$$

Proof. Each summand in the expression

$$\|v\sqrt{\boldsymbol{\rho}}\|_{W^{1,1}_{\text{mix}}(\mathbb{R}^m;X)} = \sum_{\|\mathbf{q}\|_\infty \le 1} \int_{\mathbb{R}^m} \big\|\partial^{\mathbf{q}}_{\mathbf{y}}\big(v(\mathbf{y})\sqrt{\boldsymbol{\rho}(\mathbf{y})}\big)\big\|_X \,\mathrm{d}\mathbf{y}$$

can be estimated by

$$\int_{\mathbb{R}^m} \big\|\partial^{\mathbf{q}}_{\mathbf{y}}\big(v(\mathbf{y})\sqrt{\boldsymbol{\rho}(\mathbf{y})}\big)\big\|_X \,\mathrm{d}\mathbf{y} = \sum_{\boldsymbol{\alpha}\le\mathbf{q}} \frac{\mathbf{q}!}{\boldsymbol{\alpha}!(\mathbf{q}-\boldsymbol{\alpha})!} \int_{\mathbb{R}^m} \big\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}} v(\mathbf{y}) \partial^{\mathbf{q}-\boldsymbol{\alpha}}_{\mathbf{y}} \sqrt{\boldsymbol{\rho}(\mathbf{y})}\big\|_X \,\mathrm{d}\mathbf{y}.$$

We find

$$\partial^{\mathbf{q}-\boldsymbol{\alpha}}_{\mathbf{y}} \sqrt{\boldsymbol{\rho}(\mathbf{y})} = \frac{\mathbf{y}^{\mathbf{q}-\boldsymbol{\alpha}}}{2^{|\mathbf{q}-\boldsymbol{\alpha}|}} \sqrt{\boldsymbol{\rho}(\mathbf{y})}$$

and

$$\mathbf{q}! = 1, \quad \boldsymbol{\alpha}! = 1, \quad (\mathbf{q}-\boldsymbol{\alpha})! = 1,$$

since the entries of these vectors are always either 1 or 0. Hence, we arrive at

$$\int_{\mathbb{R}^m} \big\|\partial^{\mathbf{q}}_{\mathbf{y}}\big(v(\mathbf{y})\sqrt{\boldsymbol{\rho}(\mathbf{y})}\big)\big\|_X \,\mathrm{d}\mathbf{y} = \sum_{\boldsymbol{\alpha}\le\mathbf{q}} \frac{1}{2^{|\mathbf{q}-\boldsymbol{\alpha}|}} \int_{\mathbb{R}^m} \big\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}} v(\mathbf{y})\big\|_X \mathbf{y}^{\mathbf{q}-\boldsymbol{\alpha}} \sqrt{\boldsymbol{\rho}(\mathbf{y})} \,\mathrm{d}\mathbf{y}.$$

Thus, for all functions v whose mixed first order derivatives grow at most exponentially in $\|\mathbf{y}\|$, the norm $\big\|v\sqrt{\boldsymbol{\rho}}\big\|_{W^{1,1}_{\text{mix}}(\mathbb{R}^m;X)}$ is bounded.

For the solution u, the multivariate version of (3.1) is

$$\big\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}} u\big\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m;H^1(D))} \lesssim |\boldsymbol{\alpha}|! \left(\frac{\gamma}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)}.$$

Therefore, we obtain

$$\begin{aligned}\int_{\mathbb{R}^m} \big\|\partial^{\mathbf{q}}_{\mathbf{y}}\big(u(\mathbf{y})\sqrt{\boldsymbol{\rho}(\mathbf{y})}\big)\big\|_{H^1(D)} \,\mathrm{d}\mathbf{y} &\lesssim \sum_{\boldsymbol{\alpha}\le\mathbf{q}} \frac{|\boldsymbol{\alpha}|!}{2^{|\mathbf{q}-\boldsymbol{\alpha}|}} \left(\frac{\gamma}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)} \int_{\mathbb{R}^m} \boldsymbol{\sigma}^{-1}(\mathbf{y})\mathbf{y}^{\mathbf{q}-\boldsymbol{\alpha}} \sqrt{\boldsymbol{\rho}(\mathbf{y})} \,\mathrm{d}\mathbf{y} \\ &\lesssim \sum_{\boldsymbol{\alpha}\le\mathbf{q}} \frac{|\boldsymbol{\alpha}|!}{2^{|\mathbf{q}-\boldsymbol{\alpha}|}} \left(\frac{\gamma}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)}.\end{aligned}$$

Note that the last step holds since $\int_{\mathbb{R}^m} \boldsymbol{\sigma}^{-\ell}(\mathbf{y})\mathbf{y}^{\mathbf{q}-\boldsymbol{\alpha}} \sqrt{\boldsymbol{\rho}(\mathbf{y})} \,\mathrm{d}\mathbf{y} < \infty$ for all $\ell \in \mathbb{N}$.

For the p-th power $v = u^p$ of the solution, the multivariate version of 3.7 reads

$$\big\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}} u^p\big\|_{C^0_{\boldsymbol{\sigma}^p}(\mathbb{R}^m;W^{1,1}(D))} \lesssim j! \left(\frac{c(p)\gamma}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|^p_{L^p(D)}$$

and we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big\|\partial_{\mathbf{y}}^{\mathbf{q}}\big(u^p(\mathbf{y})\sqrt{\boldsymbol{\rho}(\mathbf{y})}\big)\big\|_{W^{1,1}(D)}\,\mathrm{d}\mathbf{y}\\
&\qquad\lesssim \sum_{\boldsymbol{\alpha}\le\mathbf{q}} \frac{|\boldsymbol{\alpha}|!}{2^{|\mathbf{q}-\boldsymbol{\alpha}|}}\bigg(\frac{c(p)\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\|f\|_{L^p(D)}^p\int_{\mathbb{R}^m}\boldsymbol{\sigma}^{-p}(\mathbf{y})\mathbf{y}^{\mathbf{q}-\boldsymbol{\alpha}}\sqrt{\boldsymbol{\rho}(\mathbf{y})}\,\mathrm{d}\mathbf{y}\\
&\qquad\lesssim \sum_{\boldsymbol{\alpha}\le\mathbf{q}} \frac{|\boldsymbol{\alpha}|!}{2^{|\mathbf{q}-\boldsymbol{\alpha}|}}\bigg(\frac{c(p)\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\|f\|_{L^p(D)}^p.
\end{aligned}$$

Putting all together, yields the desired result. □

REMARK 5.2. *The estimation of the discrepancy of a set* $\Xi\subset[0,1]^m$*, especially for high dimensions* m*, has been the topic of many publications in the past fifteen years. The aim is to avoid the factor* $(\log N)^m$ *in the estimation of the discrepancy which grows exponentially in the dimension* m*. This exponential dependence is called* intractability *in the literature, cf. [26, 29]. To avoid intractability, further regularity assumptions on the integrand are necessary. These assumptions are allowed to be violated in the analysis presented above. Furthermore, if we do not take into account the decay of the sequence* $\{\gamma_k\}_k$*, the constant occurring in the previous theorem can only be bounded by*

$$\sum_{\|\mathbf{q}\|_\infty\le 1}\frac{1}{2^{|\mathbf{q}|}}\sum_{\boldsymbol{\alpha}\le\mathbf{q}}\bigg(\frac{2c(p)\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}}|\boldsymbol{\alpha}|!\lesssim\exp(cm\log m)$$

with some constant $c>0$*. Thus, for larger values of* m*, one has to consider another approach for the analysis of QMC. This can be done by extending the ideas in [27] and additionally taking into account the decay of the sequence* $\{\gamma_k\}_k$*, see [16].*

6. Finite element approximation in the spatial variable. For the spatial discretization of the diffusion problem under consideration, we will employ multilevel finite elements. This constitutes the key ingredient for the multilevel quadrature idea. To this end, we consider a coarse grid triangulation $\mathcal{T}_0=\{\tau_{0,k}\}$ of the domain D. Then, for $\ell\ge 1$, a uniform and shape regular triangulation $\mathcal{T}_\ell=\{\tau_{\ell,k}\}$ is recursively obtained by uniformly refining each simplex $\tau_{\ell-1,k}$ into 2^n simplices with diameter $h_\ell\sim 2^{-\ell}$. For some $d\ge 1$, we define the finite element spaces on level ℓ by

$$\mathcal{S}_\ell^d(D):=\{v\in C(D): v|_{\partial D}=0 \text{ and } v|_\tau\in\mathcal{P}_d \text{ for all } \tau\in\mathcal{T}_\ell\}\subset H_0^1(D),$$

where $\mathcal{P}_d$ denotes the space of all polynomials of total degree d. Then, for given $\mathbf{y}\in\mathbb{R}^m$, the Galerkin solution $u_\ell(\cdot,\mathbf{y})\in\mathcal{S}_\ell^d(D)$ to the solution $u(\cdot,\mathbf{y})\in H_0^1(D)$ of the diffusion problem (2.6) with respect to the finite element space $\mathcal{S}_\ell^d(D)$ is known to fulfill the following error estimate.*

LEMMA 6.1. *Let the domain* D *be convex,* $f\in L^2(D)$ *and* $a(\cdot,\mathbf{y})\in C^1(\overline{D})$*. Then, the finite element solution* $u_\ell(\mathbf{y})\in\mathcal{S}_\ell^d(D)$ *of the diffusion problem* (2.6) *satisfies the error estimates*

$$\|u(\cdot,\mathbf{y})-u_\ell(\cdot,\mathbf{y})\|_{H^1(D)}\lesssim 2^{-\ell}\sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}}\|u(\cdot,\mathbf{y})\|_{H^2(D)}. \tag{6.1}$$

*Error estimates in respectively $L^2(D)$ and $L^1(D)$ are derived by straightforward modifications, yielding the convergence rate $4^{-\ell}$. Then, the error analysis of the multilevel quadrature can be performed with respect to these norms, provided that the precision of the underlying quadrature rule is also chosen as $\varepsilon_\ell=4^{-\ell}$.

Moreover if $f \in L^p(D)$, for given $p > 1$, it holds $u_\ell^p(\cdot,\mathbf{y}) \in \mathcal{S}_\ell^{pd}(D)$ with

$$\|u^p(\cdot,\mathbf{y}) - u_\ell^p(\cdot,\mathbf{y})\|_{W^{1,1}(D)} \lesssim 2^{-\ell} p \left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{\frac{p}{2}} \|u(\cdot,\mathbf{y})\|_{W^{2,p}(D)}^p. \tag{6.2}$$

Here, the constants hidden in (6.1) and (6.2) do not depend on $\mathbf{y} \in \mathbb{R}^m$.

Proof. The parametric diffusion problem (2.6) is H^2-regular since D is convex and $f \in L^2(D)$. Hence, the first error estimate immediately follows from the standard finite element theory. We further find

$$\begin{aligned}
&\|u^p(\cdot,\mathbf{y}) - u_\ell^p(\cdot,\mathbf{y})\|_{W^{1,1}(D)} \\
&\quad = \left\| \big(u(\cdot,\mathbf{y}) - u_\ell(\cdot,\mathbf{y})\big) \sum_{i=0}^{p-1} u^i(\cdot,\mathbf{y}) u_\ell^{p-1-i}(\cdot,\mathbf{y}) \right\|_{W^{1,1}(D)} \\
&\quad \le \left\| |u(\cdot,\mathbf{y}) - u_\ell(\cdot,\mathbf{y})| \sum_{i=0}^{p-1} \binom{p-1}{i} \big|u(\cdot,\mathbf{y})\big|^i \big|u_\ell(\cdot,\mathbf{y})\big|^{p-1-i} \right\|_{W^{1,1}(D)} \\
&\quad = \big\| |u(\cdot,\mathbf{y}) - u_\ell(\cdot,\mathbf{y})| \big(|u(\cdot,\mathbf{y})| + |u_\ell(\cdot,\mathbf{y})|\big)^{p-1} \big\|_{W^{1,1}(D)}.
\end{aligned}$$

Then, by the generalized Hölder inequality and the Leibniz rule for derivatives, we obtain

$$\begin{aligned}
&\big\| |u(\cdot,\mathbf{y}) - u_\ell(\cdot,\mathbf{y})| \big(|u(\cdot,\mathbf{y})| + |u_\ell(\cdot,\mathbf{y})|\big)^{p-1} \big\|_{W^{1,1}(D)} \\
&\qquad \lesssim p \big\| u(\cdot,\mathbf{y}) - u_\ell(\cdot,\mathbf{y}) \big\|_{W^{1,p}(D)} \big\| |u(\cdot,\mathbf{y})| + |u_\ell(\cdot,\mathbf{y})| \big\|_{W^{1,p}(D)}^{p-1}.
\end{aligned}$$

By using the estimate

$$\big\| u(\cdot,\mathbf{y}) - u_\ell(\cdot,\mathbf{y}) \big\|_{W^{1,p}(D)} \lesssim 2^{-\ell} \sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}} \|u(\cdot,\mathbf{y})\|_{W^{2,p}(D)},$$

cf. [4], it follows

$$\begin{aligned}
\|u_\ell(\cdot,\mathbf{y})\|_{W^{1,p}(D)} &\le \|u(\cdot,\mathbf{y})\|_{W^{1,p}(D)} + \|u(\cdot,\mathbf{y}) - u_\ell(\cdot,\mathbf{y})\|_{W^{1,p}(D)} \\
&\lesssim \sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}} (1 + 2^{-\ell}) \|u(\cdot,\mathbf{y})\|_{W^{2,p}(D)},
\end{aligned}$$

and we finally arrive at

$$\|u^p(\cdot,\mathbf{y}) - u_\ell^p(\cdot,\mathbf{y})\|_{W^{1,1}(D)} \lesssim p \left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{\frac{p}{2}} 2^{-\ell} \|u(\cdot,\mathbf{y})\|_{W^{2,p}(D)}^p.$$

□

7. Multilevel quadrature. We now want to use the previous results to approximate the expectation and the moments of the solution by a multilevel quadrature method. The crucial idea of the multilevel quadrature is a finite dimensional approximation of the mapping

$$\mathbb{E}\colon C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m; X) \to X, \quad v \mapsto \mathbb{E}_v.$$

Therefore, we have to combine an appropriate quadrature rule for the stochastic variable with the multilevel finite element discretization in the spatial variable. More precisely, for a function $v \in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m; X)$, we perform a multilevel splitting of $\mathbb{E}_v$ in X and approximate each level with a level dependent quadrature accuracy. This accuracy is chosen antipodal to the approximation power of the finite element spaces in the spatial domain.

For given discretization level $j \in \mathbb{N}$ and $\mathbf{y} \in \mathbb{R}^m$, we shall introduce the Galerkin projection

$$G_j(\mathbf{y}) : H^1_0(D) \to \mathcal{S}^d_j(D), \quad v \mapsto v_j$$

to discretize in the spatial variable. It is defined via the Galerkin orthogonality

$$\int_D a(\mathbf{x},\mathbf{y}) \nabla\big(v(\mathbf{x}) - v_j(\mathbf{x})\big) \nabla w_j(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0 \text{ for all } w_j \in \mathcal{S}^d_j(D).$$

Moreover, we set $G_{-1}(\mathbf{y}) := 0$ for all $\mathbf{y} \in \mathbb{R}^m$.

For the approximation in the stochastic variable $\mathbf{y}$, we shall provide a sequence of quadrature formulae $\{\mathbf{Q}_\ell\}$ for the Bochner integral

$$\mathbf{I}v = \int_{\mathbb{R}^m} v(\cdot,\mathbf{y}) \boldsymbol{\rho}(\mathbf{y}) \,\mathrm{d}\mathbf{y}$$

of the form

$$\mathbf{Q}_\ell : C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m; X) \to X, \quad \mathbf{Q}_\ell v = \sum_{i=1}^{N_\ell} \boldsymbol{\omega}_{\ell,i} v(\cdot, \boldsymbol{\xi}_{\ell,i}).$$

For our purpose, we assume that the number of points N_ℓ of the quadrature formula $\mathbf{Q}_\ell$ is chosen such that the corresponding accuracy is

$$\varepsilon_\ell = 2^{-\ell}. \tag{7.1}$$

The multilevel quadrature of the expectation of a function $v \in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m; X)$ is defined by

$$\begin{aligned} \mathbb{E}_v(\mathbf{x}) &\approx \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}\big(G_\ell(\mathbf{y})v(\mathbf{x},\mathbf{y}) - G_{\ell-1}(\mathbf{y})v(\mathbf{x},\mathbf{y})\big) \\ &= \sum_{\ell=0}^{j} \sum_{i=0}^{N_{j-\ell}} \boldsymbol{\omega}_{j-\ell,i}\big(G_\ell(\boldsymbol{\xi}_{j-\ell,i})v(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i}) - G_{\ell-1}(\boldsymbol{\xi}_{j-\ell,i})v(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i})\big). \end{aligned} \tag{7.2}$$

The higher order moments are approximated in complete analogy by

(7.3)

$$\begin{aligned} \mathcal{M}^p v(\mathbf{x}) &\approx \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}\Big(\big(G_\ell(\mathbf{y})v(\mathbf{x},\mathbf{y})\big)^p - \big(G_{\ell-1}(\mathbf{y})v(\mathbf{x},\mathbf{y})\big)^p\Big) \\ &= \sum_{\ell=0}^{j} \sum_{i=0}^{N_{j-\ell}} \boldsymbol{\omega}_{j-\ell,i}\Big(\big(G_\ell(\boldsymbol{\xi}_{j-\ell,i})v(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i})\big)^p - \big(G_{\ell-1}(\boldsymbol{\xi}_{j-\ell,i})v(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i})\big)^p\Big). \end{aligned}$$

To analyze the errors of these multilevel quadratures for each type of quadrature formulae in case of the solution to (2.6), we will need in Theorem 7.4 an estimate of the form

$$\Big\|(\mathbf{I}-\mathbf{Q}_{j-\ell})\Big(\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^p-\big(G_{\ell-1}(\mathbf{y})u(\cdot,\mathbf{y})\big)^p\Big)\Big\|_X \lesssim p\varepsilon_{j-\ell}2^{-\ell}\|f\|_{L^{p+e}(D)}^p, \tag{7.4}$$

where $X=H^1(D)$ and $e=1$ if $p=1$ and $X=W^{1,1}(D)$ and $e=0$ if $p\geq 2$, respectively.

For the Monte Carlo quadrature, we get the error (7.4) simply bounded in the mean square sense:

$$\begin{aligned}
&\Big\|(\mathbf{I}-\mathbf{Q}_{j-\ell})\Big(\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^p-\big(G_{\ell-1}(\mathbf{y})u(\cdot,\mathbf{y})\big)^p\Big)\Big\|_{L^2_{\boldsymbol\rho}(\mathbb{R}^m;X)}\\
&\quad\lesssim \varepsilon_{j-\ell}\big\|\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^p-\big(G_{\ell-1}(\mathbf{y})u(\cdot,\mathbf{y})\big)^p\big\|_{L^2_{\boldsymbol\rho}(\mathbb{R}^m;X)}\\
&\quad\lesssim p\varepsilon_{j-\ell}2^{-\ell}\|f\|_{L^{p+e}(D)}^p.
\end{aligned}$$

The last inequality is obtained by using the estimate (6.1) if $p=1$ and by using the estimate (6.2) if $p\geq 2$.

As we have seen, the error analysis in case of the Gaussian or the quasi Monte Carlo quadrature is based on the derivatives of the integrand. Hence, in the following, we shall show that the derivatives of the term $\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^p-\big(G_{\ell-1}(\mathbf{y})u(\cdot,\mathbf{y})\big)^p$ exhibit a behaviour similar to the derivatives of $u(\cdot,\mathbf{y})^p$ up to an additional factor $2^{-\ell}$. This will then lead to the estimate (7.4) for the Gaussian and the quasi Monte Carlo quadrature.

Lemma 7.1. *For the error $\delta_\ell(\cdot,\mathbf{y}):=G_\ell(\mathbf{y})u(\cdot,\mathbf{y})-u(\cdot,\mathbf{y})$ of the Galerkin projection, there holds the estimate*

$$\big\|\partial^{\boldsymbol\alpha}\delta_\ell(\cdot,\mathbf{y})\big\|_{H^1(D)}\lesssim 2^{-\ell}|\boldsymbol\alpha|!\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{2+\delta}\left(\frac{2\tilde\gamma}{\log 2}\right)^{\boldsymbol\alpha}\|f\|_{L^2(D)}\quad \textit{for all } |\boldsymbol\alpha|\geq 0. \tag{7.5}$$

Proof. Since the Galerkin projection satisfies

$$\int_D a(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\delta_\ell(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}v_\ell(\mathbf{x})\,\mathrm{d}\mathbf{x}=0 \text{ for all } v_\ell\in\mathcal{S}_\ell^d(D),$$

it follows by differentiation that

$$-\int_D a(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{y_k}^i\delta_\ell(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}v_\ell(\mathbf{x})\,\mathrm{d}\mathbf{x}=\sum_{s=1}^{i}\binom{i}{s}\int_D\partial_{y_k}^s a(\mathbf{x},\mathbf{y})\partial_{y_k}^{i-s}\delta_\ell(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}v_\ell(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

for all $v_\ell\in\mathcal{S}_\ell^d(D)$. For an arbitrary function $v_\ell\in\mathcal{S}_\ell^d(D)$, we therefore obtain:

$$\begin{aligned}
\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}^2 &= \int_D a(\mathbf{x},\mathbf{y})\big|\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y})\big|^2\,\mathrm{d}\mathbf{x}\\
&= \int_D a(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y})\big[\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{x}}v_\ell(\mathbf{x})\big]\,\mathrm{d}\mathbf{x}\\
&\qquad+\int_D \partial_{y_k}a(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\delta_\ell(\mathbf{x},\mathbf{y})\big[\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{x}}v_\ell(\mathbf{x})\big]\,\mathrm{d}\mathbf{x}\\
&\qquad-\int_D \partial_{y_k}a(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\delta_\ell(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

From (2.5), we derive $\|\partial_{y_k} a(\cdot,\mathbf{y})/a(\cdot,\mathbf{y})\|_{L^\infty(D)} \le \gamma_k$. Hence, we can further estimate

$$\begin{aligned}\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}^2 \le \int_D & a(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y})\big[\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}} v_\ell(\mathbf{x})\big]\,\mathrm{d}\mathbf{x}\\ &+ \gamma_k\int_D a(\mathbf{x},\mathbf{y})\big|\nabla_{\mathbf{x}}\delta_\ell(\mathbf{x},\mathbf{y})\big[\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}} v_\ell(\mathbf{x})\big]\big|\,\mathrm{d}\mathbf{x}\\ &+ \gamma_k\int_D a(\mathbf{x},\mathbf{y})\big|\nabla_{\mathbf{x}}\delta_\ell(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{y_k}\delta_\ell(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}\mathbf{x}.\end{aligned}$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}&\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}^2\\ &\quad\le \big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}\big\|\sqrt{a(\cdot,\mathbf{y})}\big[\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y}) - \nabla v_\ell\big]\big\|_{L^2(D)}\\ &\quad+ \gamma_k\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}\big\|\sqrt{a(\cdot,\mathbf{y})}\big[\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y}) - \nabla v_\ell\big]\big\|_{L^2(D)}\\ &\quad+ \gamma_k\|\sqrt{a(\cdot,\mathbf{y})}\nabla\delta_\ell(\cdot,\mathbf{y})\|_{L^2(D)}\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}.\end{aligned}$$

Since $v_\ell \in \mathcal{S}_\ell^d(D)$ can be chosen arbitrarily, we get on the one hand

$$\inf_{v_\ell\in\mathcal{S}_\ell^d(D)}\big\|\sqrt{a(\cdot,\mathbf{y})}\big[\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y}) - \nabla v_\ell\big]\big\|_{L^2(D)} \lesssim 2^{-\ell}\sqrt{a_{\max}(\mathbf{y})}\|\partial_{y_k}u(\cdot,\mathbf{y})\|_{H^2(D)}.$$

On the other hand, due to $0 \in \mathcal{S}_\ell^d(D)$, we find

$$\inf_{v_\ell\in\mathcal{S}_\ell^d(D)}\big\|\sqrt{a(\cdot,\mathbf{y})}\big[\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y}) - \nabla v_\ell\big]\big\|_{L^2(D)} \le \big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}.$$

By combining these two estimates with

$$\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)} \lesssim 2^{-\ell}\sqrt{a_{\max}(\mathbf{y})}\sqrt{\frac{a_{\min}(\mathbf{y})}{a_{\max}(\mathbf{y})}}\|u(\cdot,\mathbf{y})\|_{H^2(D)},$$

we arrive at

$$\begin{aligned}\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}^2 &\lesssim 2^{-\ell}\sqrt{a_{\max}(\mathbf{y})}\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}\\ &\cdot\left[\|\partial_{y_k}u(\cdot,\mathbf{y})\|_{H^2(D)} + 2\gamma_k\sqrt{\frac{a_{\min}(\mathbf{y})}{a_{\max}(\mathbf{y})}}\|u(\cdot,\mathbf{y})\|_{H^2(D)}\right].\end{aligned}$$

Division by $\big\|\sqrt{a(\cdot,\mathbf{y})}\nabla\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{L^2(D)}$ results in

$$\big\|\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{H^1(D)} \lesssim 2^{-\ell}\sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}}\left(\|\partial_{y_k}u(\cdot,\mathbf{y})\|_{H^2(D)} + \sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}}\frac{\gamma_k}{\log 2}\|u(\cdot,\mathbf{y})\|_{H^2(D)}\right).$$

In view of (3.4), this yields

$$\big\|\partial_{y_k}\delta_\ell(\cdot,\mathbf{y})\big\|_{H^1(D)} \lesssim 2^{-\ell}\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{2+\delta}\left(\frac{2\tilde{\gamma}_k}{\log 2}\right)\|f\|_{L^2(D)}$$

which establishes the estimate (7.5) for $|\boldsymbol{\alpha}| = 1$. From this, the desired estimate (7.5) for $|\boldsymbol{\alpha}| > 1$ follows finally by induction. □

To prove the convergence of the multilevel quadrature in case of the higher order moments, we need the following result.

LEMMA 7.2. *Let $f \in L^p(D)$ and assume that*

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\cdot,\mathbf{y})\|_{W^{2,p}} \lesssim \left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{1+\delta} |\boldsymbol{\alpha}|! \left(\frac{2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^p(D)}$$

for all $\delta > 0$.[†] *Then, with the constant $c(p)$ from 3.7, it holds*

$$\left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^p - u(\cdot,\mathbf{y})^p\Big)\right\|_{W^{1,1}(D)} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{c(p)2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{p+1+\delta}\|f\|_{L^p(D)}^p. \tag{7.6}$$

Proof. For sake of convenience, we demonstrate the proof only in case of $p = 2$. The case of a general p can be treated in a similar way. It holds

$$\begin{aligned}
&\left\|\partial_{y_k}^{i}\Big(\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^2 - u(\cdot,\mathbf{y})^2\Big)\right\|_{W^{1,1}(D)} \\
&\quad\lesssim \sum_{s=0}^{i}\binom{i}{s}\big\|\partial_{y_k}^{s}\delta_\ell(\cdot,\mathbf{y})\partial_{y_k}^{i-s}\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y}) + u(\cdot,\mathbf{y})\big)\big\|_{W^{1,1}(D)} \\
&\quad\lesssim \sum_{s=0}^{i}\binom{i}{s}\big\|\partial_{y_k}^{s}\delta_\ell(\cdot,\mathbf{y})\big\|_{H^1(D)}\big\|\partial_{y_k}^{i-s}\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y}) + u(\cdot,\mathbf{y})\big)\big\|_{H^1(D)}.
\end{aligned}$$

Using the estimate (7.5), the fact that the Galerkin projection $G_\ell(\mathbf{y})u(\cdot,\mathbf{y})$ has the same regularity with respect to the parametric variable as the solution itself, and the Lemma 3.4, we obtain

$$\begin{aligned}
&\big\|\partial_{y_k}^{i}\big((G_\ell(\mathbf{y})u(\cdot,\mathbf{y}))^2 - u(\cdot,\mathbf{y})^2\big)\big\|_{W^{1,1}(D)} \\
&\quad\lesssim \sum_{s=0}^{i}\binom{i}{s}2^{-\ell}s!\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{2+\delta}\left(\frac{2\tilde{\gamma}_k}{\log 2}\right)^{s}\|f\|_{L^2(D)}(i-s)!\left(\frac{2\gamma_k}{\log 2}\right)^{i-s}\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|f\|_{L^2(D)} \\
&\quad\lesssim 2^{-\ell}(i+1)i!\left(\frac{2\tilde{\gamma}_k}{\log 2}\right)^{i}\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{3+\delta}\|f\|_{L^2(D)}^2 \\
&\quad\lesssim 2^{-\ell}i!\left(\frac{2c(2)\tilde{\gamma}_k}{\log 2}\right)^{i}\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{3+\delta}\|f\|_{L^2(D)}^2.
\end{aligned}$$

A similar result for multidimensional derivatives is obtained by induction. □

Employing Lemma 7.1 and Lemma 7.2, we immediately get (7.4):

LEMMA 7.3. *In case of MLQMC and MLGQ, the estimate* (7.4) *holds for all $p \geq 1$, where $X = H^1(D)$ if $p = 1$ and $X = W^{1,1}(D)$ if $p \geq 2$.*

Proof. The results of Lemma 7.1 and Lemma 7.2 imply

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_\ell\|_{C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m;H^1(D))} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}$$

[†]In case $p = 2$, this is estimate (3.4).

for all $\boldsymbol{\alpha} \geq \mathbf{0}$ provided that the weight $\boldsymbol{\sigma}$ from Definition 3.2 satisfies $\boldsymbol{\beta} > 4\boldsymbol{\gamma}$ and

$$\Big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^p - u(\cdot,\mathbf{y})^p\Big)\Big\|_{C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m, W^{1,1}(D))} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{2\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)}$$

for all $\boldsymbol{\alpha} \geq \mathbf{0}$ provided that the weight satisfies $\boldsymbol{\beta} > (2p+2)\boldsymbol{\gamma}$. We thus obtain the estimate (7.4) by replacing $\boldsymbol{\gamma}$ by $2\tilde{\boldsymbol{\gamma}}$ in the analysis of Sections 4 and 5, respectively. □

THEOREM 7.4. *Let $\{\mathbf{Q}_\ell\}$ be a sequence of quadrature rules which satisfy* (7.1) *and let $u \in C_{\boldsymbol{\sigma}}^0\big(\mathbb{R}^m, H_0^1(D)\big)$ be the solution to* (2.6) *which satisfies* (6.1) *and* (6.2)*. Assume that for $p \geq 2$ the weight $\boldsymbol{\sigma}$ is chosen such that*

$$\max_{\mathbf{y}\in\mathbb{R}^m} \boldsymbol{\sigma}(\mathbf{y})\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{\frac{p}{2}} \|u(\cdot,\mathbf{y})\|_{W^{2,p}(D)}^p \lesssim \|f\|_{L^p(D)}^p. \tag{7.7}$$

Then, there holds the error estimate

$$\left\|\mathbb{E}_{u^p}(\mathbf{x}) - \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}\Big(\big(G_\ell(\mathbf{y})u(\cdot,\mathbf{y})\big)^p - \big(G_{\ell-1}(\mathbf{y})u(\cdot,\mathbf{y})\big)^p\Big)\right\|_X \lesssim 2^{-j} jp \|f\|_{L^{p+e}(D)}^p$$

where $e = 1$ if $p = 1$ and $e = 0$ if $p \geq 2$. Here, in case of MLMC, the error is measured in the Bochner space $X = L_{\boldsymbol{\rho}}^2\big(\mathbb{R}^m, H^1(D)\big)$ if $p = 1$ and in the Bochner space $X = L_{\boldsymbol{\rho}}^2\big(\mathbb{R}^m, W^{1,1}(D)\big)$ if $p \geq 2$. In case of MLGQ or MLQMC, the error is measured in the Sobolev space $X = H^1(D)$ if $p = 1$ and $X = W^{1,1}(D)$ if $p \geq 2$.

Proof. We shall apply the following multilevel splitting of the error

$$\begin{aligned}
&\left\|\mathbb{E}_{u^p} - \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}\big(G_\ell(\mathbf{y})u^p(\cdot,\mathbf{y}) - G_{\ell-1}(\mathbf{y})u^p(\cdot,\mathbf{y})\big)\right\|_X \\
&\quad \leq \big\|\mathbb{E}_{u^p} - \mathbf{I}\big(G_j(\mathbf{y})u^p(\cdot,\mathbf{y})\big)\big\|_X + \sum_{\ell=0}^{j} \big\|(\mathbf{I} - \mathbf{Q}_{j-\ell})\big(G_\ell(\mathbf{y})u^p(\cdot,\mathbf{y}) - G_{\ell-1}(\mathbf{y})u^p(\cdot,\mathbf{y})\big)\big\|_X.
\end{aligned}$$

The second term is estimated by (7.4). The first term does not depend on the particular quadrature method since $\mathbb{E}_{u^p} - \mathbf{I}\big(G_j(\mathbf{y})u(\cdot,\mathbf{y})\big)^p$ is a function of $\mathbf{x}$ which is independent of $\mathbf{y}$. Thus, it holds

$$\big\|\mathbb{E}_{u^p} - \mathbf{I}\big(G_j(\mathbf{y})u(\cdot,\mathbf{y})\big)^p\big\|_{L_{\boldsymbol{\rho}}^2(\mathbb{R}^m;X)} = \big\|\mathbb{E}_{u^p} - \mathbf{I}\big(G_j(\mathbf{y})u(\cdot,\mathbf{y})\big)^p\big\|_X$$

with $X = H^1(D)$ or $X = W^{1,1}(D)$. For $p = 1$, due to (6.1) and the continuity of $\mathbf{I}$, there holds

$$\begin{aligned}
\big\|\mathbb{E}_u - \mathbf{I}\big(G_j(\mathbf{y})u(\cdot,\mathbf{y})\big)\big\|_{H^1(D)} &\lesssim 2^{-\ell} \max_{\mathbf{y}\in\mathbb{R}^m} \boldsymbol{\sigma}(\mathbf{y})\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{1/2} \|u(\cdot,\mathbf{y})\|_{H^2(D)} \\
&\lesssim 2^{-\ell}\|f\|_{L^2(D)},
\end{aligned}$$

where the weight $\boldsymbol{\sigma}$ is chosen such that $\boldsymbol{\beta} > 3\boldsymbol{\gamma}$. For $p \geq 2$, we use (6.1), the assumption (7.7), and again the continuity of $\mathbf{I}$ to obtain

$$\begin{aligned}
\Big\|\mathbb{E}_{u^p} - \mathbf{I}\Big(\big(G_j(\mathbf{y})u(\cdot,\mathbf{y})\big)^p\Big)\Big\|_{H^1(D)} &\lesssim 2^{-\ell} \max_{\mathbf{y}\in\mathbb{R}^m} \boldsymbol{\sigma}(\mathbf{y})\left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{p/2} \|u(\cdot,\mathbf{y})\|_{W^{2,p}(D)}^p \\
&\lesssim 2^{-\ell}\|f\|_{L^p(D)}^p.
\end{aligned}$$

This completes the proof. □

REMARK 7.5. *The solution u of (2.6) belongs to $C^0_{\boldsymbol\sigma}\big(\mathbb{R}^m, H^1_0(D)\big)$ for all weight $\boldsymbol\sigma$ which satisfy $\boldsymbol\beta \geq \boldsymbol\gamma$. Therefore, the assumption in Theorem 7.4 is not very restrictive provided that the diffusion coefficient $a(\cdot,\mathbf{y})$ belongs at least to $C(\overline{D})$ for all $\mathbf{y}\in\mathbb{R}^m$.*

8. Numerical results. In this section, we present numerical examples to quantify and verify the presented methods by studying the convergence for the mean and the variance of the solution to (2.1). To that end, we consider two different settings. On the one hand, we consider a diffusion coefficient that can be represented exactly by a Karhunen-Loève expansion of finite rank. On the other hand, we consider a diffusion coefficient given by a Gaussian correlation function. In this case, we have to truncate the Karhunen-Loève expansion appropriately, where the truncation rank m has to tend to ∞ as the over-all accuracy increases. The domain of the spatial variable is always given by unit square. For the approximation of the Karhunen-Loève expansion, we employ the pivoted Cholesky decomposition as described in [15, 17] together with a piecewise constant finite element discretization of the two-point correlation.

The finite element method we use is based on piecewise linear ansatz and test functions. Therefore, we have to provide a quadratic prolongation for the solution's second moment to perform the inter-grid transfer. Since no reference solution is analytically known, we have to compute it numerically. Although it is tempting to employ a multilevel quadrature solution on a finer grid for this purpose, we employ here a single-level method, which is of much more computational effort. By this choice, we rule out occurring convergence effects, caused by the decay of the hierarchical surpluses, i.e.

$$\|G_\ell u - G_{\ell-1}u\|_{C^0_{\boldsymbol\sigma}(\mathbb{R}^m;H^1_0(D))} \lesssim 2^{-\ell}\left\|\sqrt{\frac{a_{\max}}{a_{\min}}}u\right\|_{C^0_{\boldsymbol\sigma}(\mathbb{R}^m;H^2(D))}.$$

The reference solution is obtained by a quasi Monte Carlo method on a finer grid with a fairly large number of samples ($\approx 10^6$) based on a Halton sequence, as described in Section 5.

8.1. Treatment of the diffusion coefficient. According to the multigrid idea, the diffusion coeffcient has to be approximated with the full precision 2^{-j} on each grid. In the case of a diffusion coefficient which is affine in the stochastic variable, it is possible to compute the Karhunen-Loève expansion on the finest level j and restrict it by means of L^2-projections to the coarser grids, cf. [17]. This coincides with an h-refined quadrature, which provides the desired precision of 2^{-j} on each level.

Unfortunately, this approach is now longer valid in case of a log-normal diffusion coefficient since it is non-linear in the stochastic variable. Thus, in the general situation of a diffusion coefficient which cannot be resolved on the coarsest grid, we have to approximate it for each sample on the finest grid. This becomes unfortunately very costly. Nevertheless, we are of the opinion that this is a crucial point in the implementation of a multilevel quadrature method. Therefore, we provide here some example to validate this obstruction.

We consider the one-dimensional diffusion problem

$$-\partial_x\big(a(x,\mathbf{y})\partial_x u(x,\mathbf{y})\big) = 1 \text{ in } D=(0,1)$$

with homogenous boundary conditions, i.e. $u(0,\mathbf{y}) = u(1,\mathbf{y}) = 0$. The two-point correlation is given by

$$\operatorname{Cov}_b(x,y) = \exp\big(-100(x-y)^2\big).$$

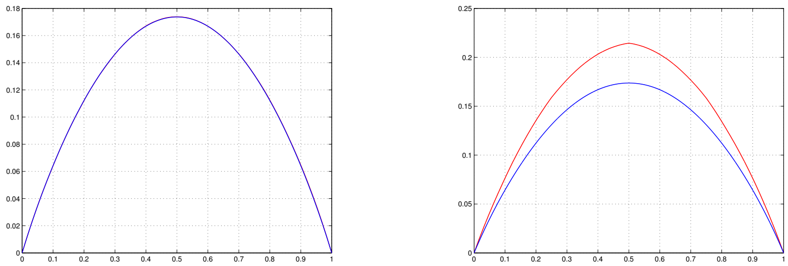

Fig. 8.1. *The left picture shows the correct solution's mean, the right picture the wrong solution's mean.*

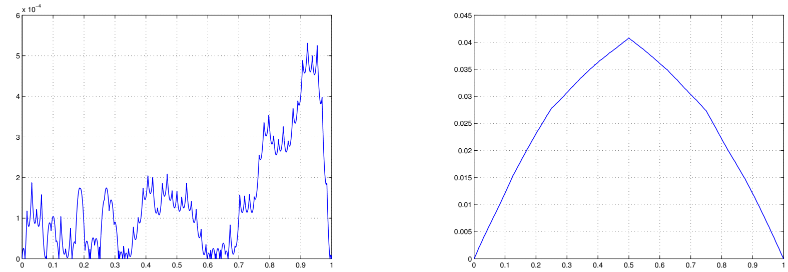

Fig. 8.2. *Corresponding error functions to the plots in Figure 8.1.*

The MLMC with $j = 6$ is chosen as quadrature method since it involves the lowest regularity assumptions to the solution. To obtain reasonable solutions, we averaged five MLMC solutions for each plot. A reference solution is computed by a single-level QMC method with about ($\approx 10^6$) samples on level 16. The Karhunen-Loève expansion is truncated after $m = 36$ terms.

An adequate approximation of the diffusion coefficient in the MLMC is computed by assembling it on the finest grid, that is on level 6, and restricting it to the actual grid by means of an L^2-projection. Alternatively, as a cheap but wrong approximation, the L^2-projection of the Karhunen-Loève expansion is restricted from the finest grid to the actual grid and then the exponential function is applied to get the diffusion coefficient. Although, the latter approach provides an approximation order of $2^{-\ell}$ on the grid of level ℓ, it is not sufficient to sustain the over-all approximation order of 2^{-j}.

In the left picture of Figure 8.1, we find the solution's mean obtained by assembling the diffusion coefficient on the finest grid and then restricting it to the actual grid. As can be seen, the multilevel approximation (red) perfectly coincides with the reference solution (blue). Furthermore, the corresponding error function on the left of Figure 8.2 shows only the highly oscillatory parts of the solution's mean, as one would expect of a multilevel approximation.

The right picture of Figure 8.1 contains the solution's mean (red) in case of the wrong approach. Again, the reference solution is indicated in blue. For this approach, the error function on the right of Figure 8.2 is dominated by the smooth contributions from the coarser grids which indicates that there the related approximation error is large.

Similar observations can be made for the approximation of the second moment.

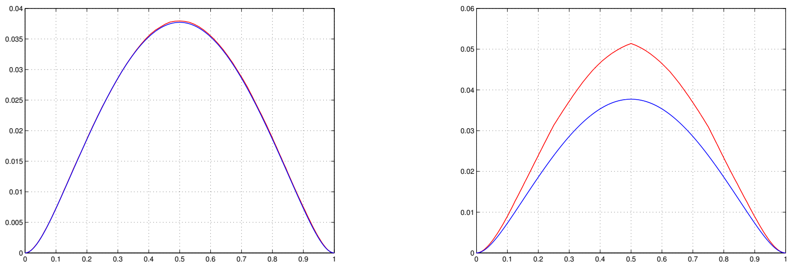

FIG. 8.3. *The left picture shows the correct solution's second moment, the right picture the wrong solution's second moment.*

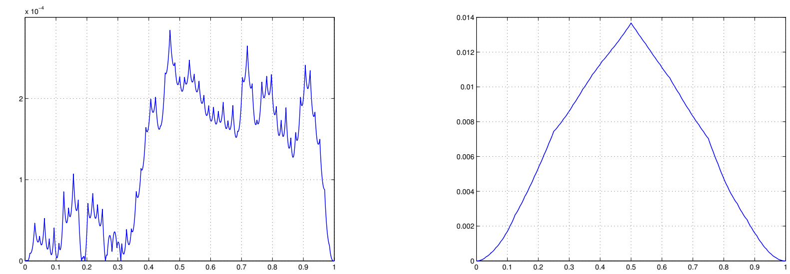

FIG. 8.4. *Corresponding error functions to the plots in Figure 8.1.*

The left picture in Figure 8.3 shows the solution's second moment obtained by the correct approach (red) and the reference solution (blue). Here, the wrong approach produces wrong results, too. This can be seen in the right picture of Figure 8.3 where the multilevel solution is again indicated in red and the reference solution is indicated in blue. The corresponding error functions in Figure 8.4 show the same behavior as in case of the mean.

Possibly, the obstruction of approximating the diffusion coefficient on each level with sufficient precision can be overcome by means of a p-refined quadrature, which would yield only a logarithmic increase of the computational complexity. Although, it is not quite clear yet how a p-refined quadrature performs with respect to the oscillations of the covariance's eigenfunctions. Therefore, we will simply employ the proposed h-approximation for all subsequent computations.

8.2. An example with finite dimensional stochastics. In our first numerical example, we focus on the covariance function

$$\mathrm{Cov}_b(\mathbf{x},\mathbf{y}) = 2\big(\mathbb{1}_{B_1}(\mathbf{x})\mathbb{1}_{B_1}(\mathbf{y}) + \mathbb{1}_{B_2}(\mathbf{x})\mathbb{1}_{B_2}(\mathbf{y}) + \mathbb{1}_{B_3}(\mathbf{x})\mathbb{1}_{B_3}(\mathbf{y}) + \mathbb{1}_{B_4}(\mathbf{x})\mathbb{1}_{B_4}(\mathbf{y})\big),$$

where the sets $B_1, \ldots, B_4$ are discs of diameter 0.3 equispaced in $D = (0,1)^2$. A visualization of the associated triangulation can be seen in the left picture of Figure 8.5. We consider $f \equiv 1$ as load vector. A reference solution is obtained by a single-level QMC computation with 6 mesh refinements which results in 262144 triangles. Figure 8.6 shows the solution's mean (on the left) and the solution's variance (on the right).

The error plots in Figure 8.7 indicate that the three methods, i.e. MLGC, MLMC, MLQMC, provide the desired order of convergence of 2^{-j} of the approximate mean

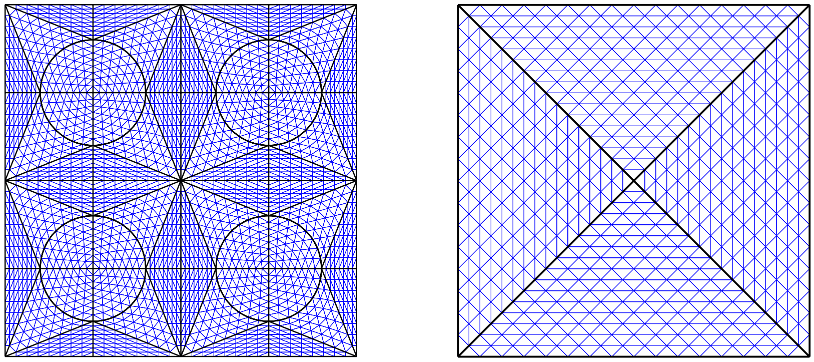

FIG. 8.5. *Computational domains with inscribed coarse grids.*

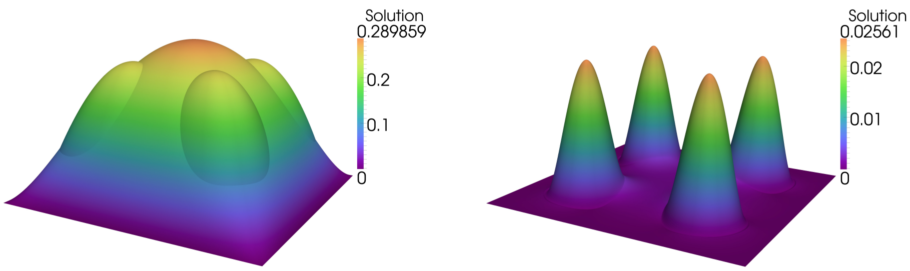

FIG. 8.6. *Solution's mean (left) and solution's variance in the case of finite dimensional stochastics.*

with respect to the H^1-norm (picture on the left). The approximation of the second moment (picture on the right) even seems to provide a better rate of convergence with respect to the $W^{1,1}$-norm. Note that, for the MLMC, we averaged five solutions in order to obtain smooth error plots.

8.3. An example with infinite dimensional stochastics. For our second example, we consider the covariance function

$$\operatorname{Cov}_b(\mathbf{x},\mathbf{y}) = \exp\left(-\frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2\right).$$

Here, the computational domain is given by $D = (0,1)^2$, as can be seen in the right picture of Figure 8.5. The load vector is again $f \equiv 1$. The reference solution is computed by a single-level QMC with eight mesh refinements which results in 262144 triangles. Figure 8.8 shows the solution's mean (on the left) and the solution's variance (on the right). Notice that the QMC approach with auxiliary density is no longer feasible as m tends to infinity, cf. Section 5. Therefore, we apply the QMC approach without auxiliary density. According to [16], convergence can then be shown if that the sequence $\{\gamma_k\}_k$ decays fast enough. Nevertheless, we emphasize that in case of a finite and relatively small dimensional stochastics like in the previous example, the QMC approach with auxiliary density performs better.

The approximation order of 2^{-j} suggests to truncate the Karhunen-Loève expansion for this example at $m = 22$ terms. The error plots in Figure 8.9 show the behavior of the three methods, i.e. MLGC, MLMC, MLQMC, for this example. As can be seen, the MLGC now exhibits an offset before it starts to converge with the

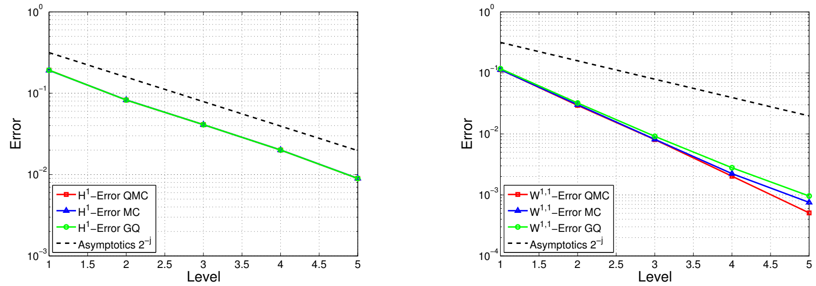

FIG. 8.7. *Error of the mean (left) and error of the second moment (right) in the finite dimensional stochastics case.*

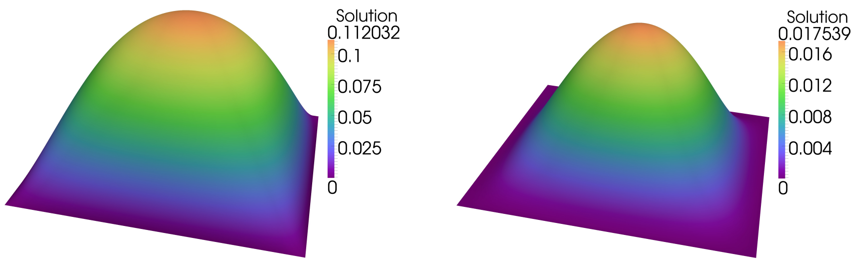

FIG. 8.8. *Solution's mean (left) and solution's variance in the case of infinite dimensional stochastics.*

desired rates. This yields approximation errors, which are up to a constant factor, as good as that of the sample methods, i.e. MLMC and MLQMC. Both sample methods perfectly produce the order of convergence 2^{-j} of the approximate mean with respect to the H^1-norm (picture on the left) and a slightly better rate of the approximate second moment with respect to the $W^{1,1}$-norm (picture on the right).

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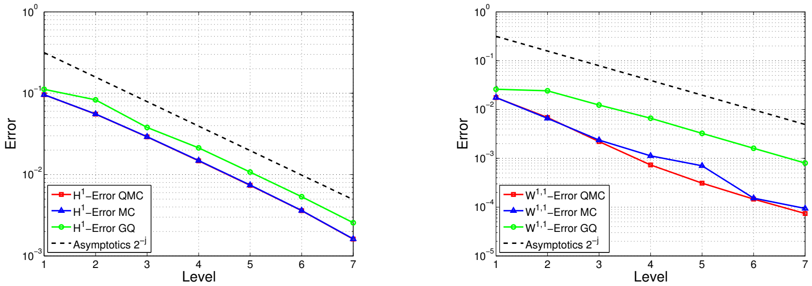

Fig. 8.9. *Error of the mean (left) and error of the second moment (right) in the infinite dimensional stochastics case.*

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