Limitations of the background field method applied to Rayleigh-Bénard convection

C. Nobili, F. Otto

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Camilla Nobili and Felix Otto

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Abstract

We consider Rayleigh-Bénard convection as modeled by the Boussinesq equations, in case of infinite Prandtl number and with no-slip boundary condition. There is a broad interest in bounds of the upwards heat flux, as given by the Nusselt number Nu, in terms of the forcing via the imposed temperature difference, as given by the Rayleigh number in the turbulent regime $\mathrm{Ra} \gg 1$. In several works, the background field method applied to the temperature field has been used to provide upper bounds on Nu in terms of Ra. In these applications, the background field method comes in form of a variational problem where one optimizes a stratified temperature profile subject to a certain stability condition; the method is believed to capture marginal stability of the boundary layer. The best available upper bound via this method is $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}} (\ln \mathrm{Ra})^{\frac{1}{15}}$; it proceeds via the construction of a stable temperature background profile that increases logarithmically in the bulk. In this paper, we show that the background temperature field method cannot provide a tighter upper bound in terms of the power of the logarithm. However, by another method one does obtain the tighter upper bound $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}} (\ln \ln \mathrm{Ra})^{\frac{1}{3}}$, so that the result of this paper implies that the background temperature field method is unphysical in the sense that it cannot provide the optimal bound.

Keywords. Rayleigh-Bénard convection, Stokes equations, no-slip boundary condition, infinite Prandtl number, Nusselt number, background field method, variational methods.

Contents

1 Introduction

In a $d-$dimensional container of height normalized to unity we consider Rayleigh-Bénard convection as modeled by the Boussinesq equations, which we consider in their infinite-Prandtl-number limit:

$$\partial_t T + u \cdot \nabla T = \Delta T \qquad \text{for} \quad 0 < z < 1 \,, \tag{1a}$$

$$-\Delta u + \nabla p = \mathrm{Ra} T e_z \qquad \text{for} \quad 0 < z < 1 \,, \tag{1b}$$

$$\nabla \cdot u = 0 \qquad \text{for} \quad 0 < z < 1 \,, \tag{1c}$$

$$u = 0 \qquad \text{for} \quad z \in \{0, 1\} \,, \tag{1d}$$

$$T = 1 \qquad \text{for} \quad z = 0 \,, \tag{1e}$$

$$T = 0 \qquad \text{for} \quad z = 1 \,. \tag{1f}$$

Here $u \in \mathbb{R}^d$ denotes the fluid velocity, $T \in \mathbb{R}$ its temperature and $p \in \mathbb{R}$ its pressure. We denote with z the vertical component of the $d-$dimensional position vector $x = (y, z)$ and with e_z the upward unit normal in the vertical direction. As a convenient proxy of the side-wall effect, the functions u, T and p, which depend on the spatial variable x and the time variable t, are supposed to be periodic in the $(d-1)-$horizontal directions y with period L, where L is the horizontal period. In our treatment, the dimension d is arbitrary and we think of L as being large. The first equation encodes the diffusion of the temperature, driven by the Dirichlet boundary conditions (1e)&(1f), and its advection by the fluid velocity. The second equation, the Stokes equation, encodes the fact that the fluid parcels move as a reaction to the buoyancy force $\mathrm{Ra} T e_z$ (hotter parcels expand and thus experience an upwards force under gravity) and are slowed down by viscosity $(-\Delta u)$ in conjunction with the no-slip boundary condition (1d). The last equation expresses the incompressibility of the fluid and is balanced by the pressure term ∇p acting as a Lagrangian multiplier in the previous equation. The parameter appearing in the Stokes equation, the Rayleigh number Ra, expresses the relative strength of the buoyancy force and is given by

$$\mathrm{Ra} = \frac{\alpha g (T_b - T_t) h^3}{\nu \kappa} \,, \tag{2}$$

where α is the thermal expansion coefficient, ν the kinematic viscosity, κ the thermal conductivity, $(T_b - T_t)$ the temperature gap between the bottom and the top plate and h the height of the container before the non-dimensionalization.

In (1), the inertia of the fluid has been neglected, which amounts to sending the Prandtl number $\mathrm{Pr} = \frac{\nu}{\kappa}$ to infinity. Therefore Ra, next to L, is the only non-dimensional parameter. The linear stability analysis identifies a critical value Ra_c, the Rayleigh number at which the solution of (1) bifurcates from the linear conduction profile $T = 1 - z$, $u = 0$, $p = z - \frac{z^2}{2}$, see for instance [1]. When $\mathrm{Ra} > \mathrm{Ra}_c$, the buoyancy forces trigger the formation of convection rolls. Eventually, when $\mathrm{Ra} \gg \mathrm{Ra}_c$, these convection rolls break down. This regime features boundary layers at the top and bottom plates, with a high vertical temperature gradient, from which small fluid parcels of different temperature than the ambient fluid detach and deform, the so called plumes. In this paper we are interested in this turbulent regime of

$$\mathrm{Ra} \gg 1 \,,$$

and in the experimentally observed enhancement of the heat transport over the pure conduction state. An appropriate measure to quantify the vertical heat flux is the Nusselt

number,

$$\mathrm{Nu} = \int_0^1 \langle (uT - \nabla T) \cdot e_z \rangle \, dz \,,$$

which represents the average heat flux $uT - \nabla T$ passing through an area element. The bracket $\langle \cdot \rangle$ denotes the time and horizontal space average

$$\langle f \rangle := \limsup_{t_0 \uparrow \infty} \frac{1}{t_0} \int_0^{t_0} \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} f(t,y) dy dt \,, \qquad (3)$$

and might be thought as a statistical average.

In the fifties Malkus [2], considering fluids with very high viscosity, performed experiments in which he noticed sharp transitions in the slope of the $\mathrm{Nu} - \mathrm{Ra}$ relation and suggested the scaling $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$ for very high Ra numbers based on the *marginal stability argument*, which we reproduce now. Since the main temperature drop happens near the boundary, we can assume that in a bottom boundary layer of thickness δ (to be determined) the temperature drops from 1 to its average $\frac{1}{2}$. Thanks to the average $\langle \cdot \rangle$, we may extract the Nusselt number from the boundary layer, where by the no-slip boundary condition we have $(uT - \nabla T)e_z \approx \partial_z T$ so that $\mathrm{Nu} \sim \frac{1}{\delta}$. The boundary layer is assimilated to the pure conduction state of height δ. Marginal stability refers to the assumption that this state is borderline stable, meaning that its Rayleigh number is critical, which in view of (2) means

$$\mathrm{Ra}_c = \frac{g\alpha(T_b - T_t)(\delta h)^3}{\nu \kappa} \,,$$

from which, because of $\mathrm{Ra}_c \sim 1$, we infer $\delta \sim \mathrm{Ra}^{-\frac{1}{3}}$. Inserting this in the scaling of Nusselt number above one finds

$$\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}} \,.$$

The same conclusion can be achieved by rescaling equation (1) according to

$$x = \mathrm{Ra}^{-\frac{1}{3}} \hat{x}, \ t = \mathrm{Ra}^{-\frac{2}{3}} \hat{t}, \ u = \mathrm{Ra}^{\frac{1}{3}} \hat{u}, \ p = \mathrm{Ra}^{\frac{2}{3}} \hat{p} \ \text{ and thus } \ \mathrm{Nu} = \mathrm{Ra}^{\frac{1}{3}} \widehat{\mathrm{Nu}} \,. \qquad (4)$$

In this way we end up with the parameter-free system

$$\begin{aligned}
\partial_{\hat{t}} T + \hat{u} \cdot \hat{\nabla} T &= \hat{\Delta} T \,, \\
-\hat{\Delta} \hat{u} + \hat{\nabla} \hat{p} &= T e_{\hat{z}} \,, \\
\hat{\nabla} \cdot \hat{u} &= 0 \,,
\end{aligned}$$

which naturally lives in the half space. Since for the latter system, it is natural to expect that the heat flux is universal, i. e. $\widehat{\mathrm{Nu}} \sim 1$, we also obtain $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$.

The scaling $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$ has been been confirmed by experiments at (relatively) high Prandtl numbers (cf. [3], p.30, for a list of experimental results). Rigorous analyses have produced upper bounds that capture this scaling up to logarithms which we report now. In the sixties Howard [4] obtained an upper bound that scales like $\mathrm{Ra}^{\frac{1}{2}}$, optimizing over a field of test functions satisfying physical constraints coming from the Navier-Stokes equation (cf. [4], Sec.3), while neglecting the incompressibility constraint. Later Busse [5] developed the theoretical tool of *multiple boundary layer solution* (multi-α solution) in order to solve Howard's variational problem when the incompressibility constraint is taken into account (cf. [5], Sec.2). The multi-α solution theory inspired Chan [6] in the seventies. He elegantly applied it, deriving an upper bound on the Nusselt number that scales like $\mathrm{Ra}^{\frac{1}{3}}$ when additional conditions in the asymptotic analysis are assumed. In the nineties, Constantin and

Doering, inspired by the works of Malkus, Howard and Busse, introduced the *background field method* in order to bound the average dissipation rate in plane Couette flow. This method was already implicitly used by Hopf [7] in the forties in the construction of solutions to the Navier-Stokes equations (in the sense of Leray) with inhomogeneous boundary data. Later, Constantin and Doering applied it to the Rayleigh-Bénard convection in order to derive rigorous upper bounds for the Nusselt number Nu. Although Howard's problem and the background field method constitute dual variational problems [8], the second method has the advantage to use simple test functions and functional estimates. Indeed this method turned out to be very fruitful: It has been extensively used and it has produced meaningful bounds in the theory of turbulence. In the context of Rayleigh-Bénard convection, this method consists of decomposing the temperature field T into a steady background temperature field profile $\tau = \tau(z)$ with driving boundary conditions, $\tau = 1$ for $z = 0$ and $\tau = 0$ for $z = 1$, and into temperature fluctuations θ. As we will see in detail in the next subsection, the advantage of the background field method is to transform the problem of finding upper bounds for the Nusselt number into a purely variational problem: Find profile (test) functions τ which satisfies a certain stability condition and then select the one with minimal Dirichlet energy. The solution of this variational method produces a new number $\mathrm{Nu}_{\mathrm{BF}}$ such that

$$\mathrm{Nu} \leq \mathrm{Nu}_{\mathrm{BF}}. \tag{5}$$

Experiments suggest to try a profile τ that displays a drop by $\frac{1}{2}$ in a boundary layer and it is constant in the bulk. Such a profile satisfies the stability condition only if the boundary layer size δ is chosen artificially small and gives only suboptimal bounds (see [9]). Replacing the constant bulk by a linearly increasing profile (at the expense of making the drop in the boundary layers deeper) does not improve the situation. The idea that the "bad" boundary layers can be more efficiently compensated by a profile that increases fast near the boundary and slowly (almost constant) away from them, brought Doering, Reznikoff and the second author in 2006 [10] to investigate the stability of a background profile that grows logarithmically in the bulk. This Ansatz indeed proved to be successful, yielding the bound $\mathrm{Nu}_{\mathrm{BF}} \lesssim \mathrm{Ra}^{\frac{1}{3}}(\ln \mathrm{Ra})^{\frac{1}{3}}$ and therefore reproducing the scaling proposed by Malkus up to a logarithmic correction. Seis and the second author [11] in 2011 improved the last bound by reducing the logarithmic correction

$$\mathrm{Nu}_{\mathrm{BF}} \lesssim \mathrm{Ra}^{\frac{1}{3}}(\ln \mathrm{Ra})^{\frac{1}{15}}. \tag{6}$$

They used the same logarithmic construction as in [10] with a (logarithmically) larger boundary layer thickness, which they could afford using an additional estimate on the vertical velocity component $w = u \cdot e_z$ in terms of θ.

In the context of the Rayleigh-Bénard convection, the background field method has also been used to study the case of free-slip boundary condition for the velocity field [12], of an imposed heat flux at the boundary [13] and in the bulk [14], of mixed thermal boundary conditions [15], and of rough boundaries [16]. This method has been fruitfully applied to a variety of other problems in fluid mechanics, namely plane Couette flow [17], pipe flow, and arbitrary Prandtl number convection. Nicolaenko, Scheuer and Temam [18] applied the background field method to derive an upper bound for the long-time limit of the L^2-norm of the solution of the Kuramoto-Sivashinsky equation.

In this paper, we address the question of the optimality of the background field method in two ways:

- What is the optimal bound (in terms of the two scaling exponents in $\mathrm{Ra}^\mu(\ln \mathrm{Ra})^\nu$) in the background field method? The answer given in our main result is that the construction in [10] and [11] leading to (6) is indeed optimal.

- Does the background field method catch the optimal bound on Nu, in other words, is (5) optimal (at least in terms of both scaling exponents)? The answer given by our main result in conjunction with the bound in (9) obtained in [11] is no.

The main result of this paper is stated in the following

Theorem. *For* $\mathrm{Ra} \gg 1$ *we have*

$$\mathrm{Nu}_{\mathrm{BF}} \gtrsim \mathrm{Ra}^{\frac{1}{3}} (\ln \mathrm{Ra})^{\frac{1}{15}} . \tag{7}$$

We refer to Theorem 1 for the full formulation and the explanation of the notation $\gtrsim$ and $\gg$. This lower bound on $\mathrm{Nu}_{\mathrm{BF}}$, together with the upper bound (6), implies

$$\mathrm{Nu}_{\mathrm{BF}} \sim \mathrm{Ra}^{\frac{1}{3}} (\ln \mathrm{Ra})^{\frac{1}{15}} ,$$

and in particular shows that the background field method cannot produce any smaller logarithmic correction than $(\ln \mathrm{Ra})^{\frac{1}{15}}$. However, a combination of the background field with another method improves the logarithmic correction in (6), as we shall explain now. This other method, which we refer to as maximal principle method, was also introduced by Constantin and Doering [19]. Indeed it is easy to verify that the temperature equation satisfies the maximum principle, leading to

$$0 \le T \le 1 , \tag{8}$$

possibly neglecting an initial layer. This L^∞ bound together with a maximal regularity estimate for Stokes equation in L^∞ yields the bound

$$\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}} (\ln \mathrm{Ra})^{\frac{2}{3}} ,$$

where, this time, the logarithm is an expression of the failure of the $L^\infty-$norm to be a Calderón-Zygmund norm. Recently, Seis and the second author [11] combined the maximum principle method with the background field method developed in [10], obtaining

$$\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}} (\ln \ln \mathrm{Ra})^{\frac{1}{3}} , \tag{9}$$

which, to our knowledge, is the best rigorous upper bound. We observe that the combination of all the previous results yields

$$\mathrm{Nu} \le \mathrm{Nu}_{\mathrm{BF}} \overset{(9)}{\lesssim} \mathrm{Ra}^{\frac{1}{3}} (\ln \ln \mathrm{Ra})^{\frac{1}{3}} \ll \mathrm{Ra}^{\frac{1}{3}} (\ln \mathrm{Ra})^{\frac{1}{15}} \overset{(7)}{\lesssim} \mathrm{Nu}_{\mathrm{BF}} .$$

So indeed, the background field method is not able to capture the behavior of the Nusselt number even in terms of the two scaling exponents. Therefore, the optimal background temperature profile cannot carry much of a physical meaning.

In 2005 Plasting and Ierley [20] considerer piecewise linear profiles with $\tau' \ge 0$ in the bulk and solved the variational problem numerically, finding $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{7}{20}}$. In 2006 inspired by Chan's multi$-\alpha$ solution treatment, Ierley, Kerswell and Plasting [21], with help of a mixture of numerical and analytical methods, improved the previous result finding

$$\mathrm{Nu} \sim c_1 \mathrm{Ra}^\mu (\ln \mathrm{Ra})^\nu ,$$

where $\mu = 0.33175$ and $\nu = 0.0325$. Clearly, since $0.0325 < 0.0\bar{6} = \frac{1}{15}$, our result (although slightly underestimated) has been anticipated ten years ago.

Incidentally, the background field method suffers a similar fate in the context of the Kuramoto-Sivashinsky equation: Bronski and Gambill [22] identified the optimal scaling of the upper bound that can be obtained by this method, and soon later, Giacomelli and the second author [23] showed that a tighter upper bound can be obtained by an alternative method, which has subsequently been further improved in [24] and [25].

1.1 Temperature background field method and main result

We start by the rescaling (4) suggested by Malkus' marginal stability argument, that is

$$x \rightsquigarrow \mathrm{Ra}^{\frac{1}{3}} x, \ t \rightsquigarrow \mathrm{Ra}^{\frac{2}{3}} t, \ u \rightsquigarrow \mathrm{Ra}^{-\frac{1}{3}} u \text{ and } p \rightsquigarrow \mathrm{Ra}^{-\frac{2}{3}} p \,,$$

and setting $H := \mathrm{Ra}^{\frac{1}{3}}$ we rewrite (1) as

$$\partial_t T + u \cdot \nabla T = \Delta T \qquad \text{for} \quad 0 < z < H \,, \tag{10a}$$

$$-\Delta u + \nabla p = T e_z \qquad \text{for} \quad 0 < z < H \,, \tag{10b}$$

$$\nabla \cdot u = 0 \qquad \text{for} \quad 0 < z < H \,, \tag{10c}$$

$$u = 0 \qquad \text{for} \quad z \in \{0, H\} \,, \tag{10d}$$

$$T = 1 \qquad \text{for} \quad z = 0 \,, \tag{10e}$$

$$T = 0 \qquad \text{for} \quad z = H. \tag{10f}$$

Notice that in this non-dimensionalization of the equation, the only parameter appearing is the height H of the container and Malkus' scaling $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$ corresponds to $\mathrm{Nu} \sim 1$.

We recall from the previous section that the Nusselt number is defined as

$$\mathrm{Nu} = \frac{1}{H} \int_0^H \langle (uT - \nabla T) \cdot e_z \rangle \, dz \,, \tag{11}$$

and now derive some useful representations starting from the equation for the temperature in (10a): Applying $\langle \cdot \rangle$ to the equation (10a) and qualitatively using the bound (8) on T given by the maximum principle it is easy to show that the upward heat flux is constant in the vertical direction,

$$\mathrm{Nu} = \langle Tw - \partial_z T \rangle \quad \text{for } z \in (0, H) \,. \tag{12}$$

Testing the equation with T, appealing to incompressibility (10c) and using (12) for $z = 0$, we obtain (see [9]) the alternative representation

$$\mathrm{Nu} = \int_0^H \langle |\nabla T|^2 \rangle dz \,. \tag{13}$$

For the convenience of the reader we sketch the derivation of the background field method, see [26] for more details. The background field method consists of decomposing the temperature field T into a steady background temperature profile τ which depends only on the vertical variable z and satisfies the inhomogeneous (driving) boundary conditions, $\tau = 1$ for $z = 0$ and $\tau = 0$ for $z = H$, and into temperature fluctuations θ, with homogeneous boundary conditions $\theta = 0$ for $z \in \{0, H\}$. Inserting this decomposition

$$T = \tau + \theta \tag{14}$$

in the equation for the temperature (10a) we find that the fluctuations θ evolve according to

$$\partial_t \theta + u \cdot \nabla \theta - \Delta \theta = \frac{d^2 \tau}{dz^2} - w \frac{d\tau}{dz} \,.$$

From the incompressibility condition (10c) we obtain by testing with θ

$$\int_0^H \langle |\nabla \theta|^2 \rangle \, dz = -\int_0^H \frac{d\tau}{dz} \langle \partial_z \theta \rangle \, dz - \int_0^H \frac{d\tau}{dz} \langle \theta w \rangle \, dz \,.$$

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Together with (13) this yields the final representation

$$\mathrm{Nu} = \int_0^H \left(\frac{d\tau}{dz}\right)^2 dz - \int_0^H \left\langle 2\frac{d\tau}{dz}\theta w + |\nabla\theta|^2 \right\rangle dz\,. \tag{15}$$

Applying the divergence to the Stokes equation (10b) we find that the pressure satisfies $\Delta p = \partial_z T$. Inserting Δp into the equation Δ (10b)$\cdot e_z$, we find the direct relationship between θ and the vertical velocity component $w := u \cdot e_z$:

$$\begin{aligned} \Delta^2 w &= -\Delta_y \theta & \text{for} \quad 0 < z < H\,, \\ w &= \partial_z w = 0 & \text{for} \quad z \in \{0, H\}\,. \end{aligned} \tag{16}$$

The representation (15) shows: Any $\tau = \tau(z)$ that satisfies the driving boundary conditions and is stable in the sense that the following quadratic form is non-negative

$$\int_0^H \left\langle 2\frac{d\tau}{dz}\theta w + |\nabla\theta|^2 \right\rangle dz \geq 0\,, \tag{17}$$

for every θ satisfying homogeneous boundary conditions (and w defined through (16)), yields an upper bound for the Nusselt number:

$$\mathrm{Nu} \leq \int_0^H \left(\frac{d\tau}{dz}\right)^2 dz\,.$$

Note that in (17), we may disregard the time variable, which is only a parameter in (16), so that $\langle\cdot\rangle$ in (17) reduces to the horizontal average. This motivates to define the Nusselt number associated to the background field method as

$$\mathrm{Nu}_{\mathrm{BF}} := \inf_{\substack{\tau:(0,H)\to\mathbb{R},\\ \tau(0)=1,\tau(H)=0}} \left\{ \int_0^H \left(\frac{d\tau}{dz}\right)^2 dz \,|\, \tau \text{ satisfies } (17) \right\}\,, \tag{18}$$

which in view of (15) satisfies

$$\mathrm{Nu} \leq \mathrm{Nu}_{\mathrm{BF}}\,.$$

Our objective is to derive an Ansatz-free lower bound for $\mathrm{Nu}_{\mathrm{BF}}$, trying to extract local information on τ from the completely non-local stability condition (17). The full formulation of the main result, already stated in the previous section, is contained in the following

Theorem 1. *Suppose that*

$$\tau(0) = 1\,, \quad \tau(H) = 0\,, \tag{19}$$

and τ satisfies (17) for all (θ, w) related by (16) in its Fourier transformed version (26). Then for $H \gg 1$

$$\int_0^H \left(\frac{d\tau}{dz}\right)^2 dz \gtrsim (\ln H)^{\frac{1}{15}}\,.$$

In particular for the Nusselt number associated to the background field method we have the lower bound

$$\mathrm{Nu}_{\mathrm{BF}} \gtrsim (\ln H)^{\frac{1}{15}}\,. \tag{20}$$

Here and in the sequel, $\lesssim$ stands for $\leq C$ for some generic universal constant $C < \infty$. Likewise, $H \gg 1$ means that there exists a universal constant $C < \infty$ such that the statement holds for $H \geq C$.

Besides implying the non-optimality of the background field method, this theorem offers some insights. Indeed the proof is based on a characterization of profiles that satisfy the stability condition (17) (see Section 2.2). This characterization is motivated by the analysis of a reduced form of the stability condition (28) which indicates that long-wave length stability implies (approximate) logarithmic growth of τ in z, while short wave-length stability implies (approximate) monotonicity (see Proposition 1 in the next section).

It is convenient to introduce the slope $\xi := \frac{d\tau}{dz}$ of the background temperature profile. With this convention the stability condition (17) can be rewritten explicitly as follows

$$2 \int_0^H \xi \langle w\theta \rangle \, dz + \int_0^H \langle |\nabla_y \theta|^2 \rangle \, dz + \int_0^H \langle |\partial_z \theta|^2 \rangle \, dz \geq 0, \tag{21}$$

for all functions θ (and w related to θ via the fourth-order boundary value problem (16)) that vanish at $z \in \{0, H\}$. A major advantage of the background field method is that it is amenable to (horizontal) Fourier transform: Indeed, denoting by $k \in \frac{2\pi}{L}\mathbb{Z}^{d-1}, k \neq 0$, the horizontal wavenumber, (16) turns into

$$\begin{aligned}
\left(|k|^2 - \tfrac{d^2}{dz^2} \right)^2 \mathcal{F}w &= |k|^2 \mathcal{F}\theta && \text{for} \quad 0 < z < H, \\
\mathcal{F}w &= \tfrac{d}{dz}\mathcal{F}w = 0 && \text{for} \quad z \in \{0, H\},
\end{aligned} \tag{22}$$

whereas (21) assumes the form

$$2 \int_0^H \xi \mathcal{F}w \overline{\mathcal{F}\theta} \, dz + \int_0^H |k|^2 |\mathcal{F}\theta|^2 dz + \int_0^H \left| \frac{d}{dz}\mathcal{F}\theta \right|^2 dz \geq 0, \tag{23}$$

where the bar denotes complex conjugation. Using equation (22) we can eliminate θ from the stability condition (23), obtaining

$$2 \int_0^H \xi \mathcal{F}w \left(-\frac{d^2}{dz^2} + |k|^2 \right)^2 \overline{\mathcal{F}w} \, dz \tag{24}$$

$$+ \int_0^H |k|^{-2} \left| \frac{d}{dz} \left(-\frac{d^2}{dz^2} + |k|^2 \right)^2 \mathcal{F}w \right|^2 dz + \int_0^H \left| \left(-\frac{d^2}{dz^2} + |k|^2 \right)^2 \mathcal{F}w \right|^2 dz \geq 0,$$

which has to be satisfied for all $k \in \frac{2\pi}{L}\mathbb{Z}^{d-1} \setminus \{0\}$ and all (complex valued) functions $\mathcal{F}w(z)$ satisfying the three boundary conditions

$$\mathcal{F}w = \frac{d}{dz}\mathcal{F}w = \left(-\frac{d^2}{dz^2} + |k|^2 \right)^2 \mathcal{F}w = 0 \quad \text{for } z \in \{0, H\}. \tag{25}$$

We now introduce a further simplification by letting $L \uparrow \infty$ so that (24) has to hold for all $k \in \mathbb{R}^d$. This strengthening of the stability condition has the additional advantage that it becomes independent of the dimension d: We will henceforth say that ξ satisfies the stability condition if

$$2 \int_0^H \xi w \left(-\frac{d^2}{dz^2} + k^2 \right)^2 \overline{w} \, dz$$

$$+ \int_0^H k^{-2} \left| \frac{d}{dz} \left(-\frac{d^2}{dz^2} + k^2 \right)^2 w \right|^2 dz + \int_0^H \left| \left(-\frac{d^2}{dz^2} + k^2 \right)^2 w \right|^2 dz \geq 0, \tag{26}$$

holds true for all $k \in \mathbb{R}$ and all (complex valued) functions $w(z)$ satisfying the three boundary conditions (25) with $\mathcal{F}w$ replaced by w. The analysis of the stability condition (26) imposed for all values of $k \in \mathbb{R}$ (which corresponds to assume $L \uparrow \infty$) amounts to consider profiles τ that are stable even under perturbation that have horizontal wavelength much larger than H. In [27] it is shown that at least Proposition 1 still holds true if the lateral size L is of order H .

The rest of the paper is organized as follow: In Section 2.1 we study a reduced stability condition (obtained by retaining only the indefinite term in the stability condition) and show that in this case a stable profile must be increasing and logarithmically growing. This result is obtained by exploring the limit for small and large wavelengths in the reduced stability condition, respectively. In Section 2.2, when working with the original stability condition we can no longer pass to the limit for small/large wavenumbers k to infer the positivity for $\xi = \frac{d\tau}{dz}$ and the logarithmic growth for τ. Nevertheless, by subtle averaging of the stability condition we construct a non-negative convolution kernel ϕ_0 with help of which we can express the positivity on average approximately in the bulk (see Lemma 1). Likewise we recover logarithmic growth, at least approximately in the bulk, on the level of the construction of ξ_0, see Lemma 2. Finally in Lemma 3 we connect the bulk with the boundary layers. The main result (Theorem 1) is proved in Section 3 and it consists of combining all the results contained in the lemmas together with an estimate that connects ξ_0 to ξ, and in particular to $\int_0^H \xi \, dz = -1$ (see Lemma 4, estimate (36)).

In the rest of the paper we omit the constant factor 2 in front of the indefinite term in (26), which is legitimate since we are interested in the scaling of the Nusselt number.

2 Characterization of stable profiles

2.1 Reduced stability condition

We note that the stability condition (26)&(25) is invariant under the following transformation

$$z = L\hat{z} \text{ and thus } k = \frac{1}{L}\hat{k}, \ H = L\hat{H} \text{ and } \xi = L^{-4}\hat{\xi}. \tag{27}$$

Hence in the bulk ($z \gg 1$ and $H - z \gg 1$) we expect that the first term in (26) dominates. This motivates to consider the *reduced stability condition*

$$\int_0^H \xi w \left(-\frac{d^2}{dz^2} + k^2 \right)^2 \overline{w} \, dz \geq 0, \tag{28}$$

for all $k \in \mathbb{R}$ and all (complex valued) functions $w(z)$ satisfying the three boundary conditions (25).

The following proposition is independent of the main result (Theorem 1) but it serves as a preparation: The ideas developed in the proof (cf. Section 2.3) will be adapted to the more challenging full stability condition (26).

Proposition 1. *Let $\xi = \xi(z)$ be such that for all $k \in \mathbb{R}$ and for all $w(z)$ satisfying (25), it satisfies the reduced stability condition (28) . Then*

$$\xi \geq 0, \qquad and \tag{29}$$

$$\int_{1/e}^1 \xi dz \lesssim \frac{1}{\ln H} \int_1^H \xi \, dz \,. \tag{30}$$

We notice that while (29) means that τ is an increasing functions, the second statement (30) corresponds to a logarithmic growth of τ (see Figure 1). Hence somewhat surprisingly, monotonicity is not sufficient for stability.
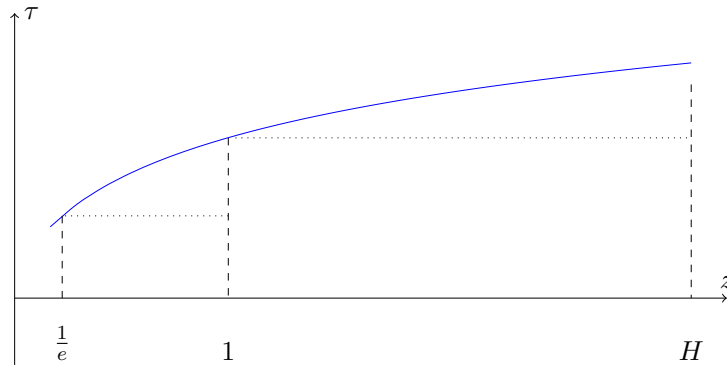

Figure 1: Logarithmic growth of the background profile τ as expressed in Proposition 1.

2.2 Original stability condition: statement of lemmas

In the following lemmas, we derive properties of those profiles τ that, in terms of their slope $\xi = \frac{d\tau}{dz}$, satisfy the original stability condition (26) and (for the last lemma) the driving boundary conditions (19). These four lemmas are the (only) ingredients of the main theorem. They all are formulated on the level of the logarithmic variables $s = \ln z$ and $\hat{\xi} = z\xi = \frac{d\tau}{ds}$, cf. (40). Lemma 1 establishes approximate positivity of the slope $\hat{\xi}$ in the bulk, and thus is the generalization of (29) in Proposition 1, replacing the stricter reduced stability condition (28) there by the original stability condition (26) here. It does so in terms of a suitable convolution $\hat{\xi}_0$ of $\hat{\xi}$ in the logarithmic variable s. Lemma 2 establishes approximate logarithmic growth of the profile in the bulk, again on the level of $\hat{\xi}_0$, and amounts to the generalization of (30) in Proposition 1. Lemma 3 is the most subtle and shows that the convolved slope $\hat{\xi}_0$ cannot be too negative in the boundary layer $-s \gg 1$ provided it is sufficiently small in the transition layer $|s| \lesssim 1$. Lemma 4 translates the driving boundary conditions (19) on τ in form of $\int_0^H \xi \, dz = -1$ from the slope ξ to its logarithmic-variable convolution $\hat{\xi}_0$.

Lemma 1.
There exists a ϕ_0, which will play the role of a convolution kernel, with the properties

$$\phi_0(s) \geq 0, \quad \int_{-\infty}^{\infty} \phi_0(s) \, ds = 1, \quad \operatorname{supp} \phi_0(z) \subset \left(\frac{1}{4}, \frac{3}{4}\right), \quad \phi_0\left(\frac{1}{2} - z\right) = \phi_0\left(\frac{1}{2} + z\right),$$
(31)

such that, for all $s' \leq \ln H$

$$\hat{\xi}_0(s') \gtrsim -\exp(-3s'),$$
(32)

where

$$\hat{\xi}_0(s') := \int_{-\infty}^{\infty} \hat{\xi}(s + s')\phi_0(s) \, ds.$$
(33)

Lemma 2.
For $S_1 \gg 1$ we have

$$\int_{-1}^{0} \hat{\xi}_0 ds \lesssim \frac{1}{S_1} \int_0^{S_1} \hat{\xi}_0 \, ds + 1.$$
(34)

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Lemma 3.
For all $S_2 \gg 1$ and $\varepsilon \leq 1$ we have

$$\int_{-S_2}^{-1} \hat{\xi}_0 ds \gtrsim - \left(\frac{1}{\varepsilon} \int_{-1}^{0} \hat{\xi}_0 \, ds + \frac{1}{\varepsilon} + \int_{-S_2}^{-S_2+1} |\hat{\xi}_0| ds + \varepsilon \exp(5S_2) \right). \tag{35}$$

Lemma 4.
Suppose that the slope ξ of the profile τ satisfies $\int_0^H \xi \, dz = -1$ and $\int_0^H \xi^2 \, dz \lesssim (\ln H)^{\frac{1}{15}}$. Then

$$\int_{-\infty}^{\ln H} \hat{\xi}_0 \, ds \lesssim -1. \tag{36}$$

2.3   Proof of Proposition 1

Argument for (29):
Letting $k \uparrow \infty$, (28) reduces to

$$\int_0^H \xi |w|^2 \, dz \geq 0$$

for all compactly supported w, from which we infer (29).
Argument for (30), heuristic version:
Letting $k \downarrow 0$, (28) reduces to

$$\int_0^H \xi w \frac{d^4}{dz^4} \overline{w} dz \geq 0 \tag{37}$$

for all functions $w(z)$ satisfying the three boundary conditions

$$w = \frac{dw}{dz} = \frac{d^4 w}{dz^4} = 0 \qquad \text{for } z \in \{0, H\}. \tag{38}$$

In fact, besides Subsection 4.3, we will work with w compactly supported in $z \in (0, H)$, so that the boundary condition (38) are trivially satisfied. Focusing on the lower half of the container we make the following Ansatz

$$w = z^2 \hat{w},$$

where $\hat{w}(z)$ is a real function with compact support in $(0, H)$. The merit of this Ansatz is that in the new variable $\hat{w}$, the multiplier in (37) can be written in the scale-invariant form

$$\phi = w \frac{d^4}{dz^4} \bar{w} = \hat{w} z^2 \frac{d^4}{dz^4} z^2 \hat{w} = \hat{w} \left(z \frac{d}{dz} + 2 \right) \left(z \frac{d}{dz} + 1 \right) z \frac{d}{dz} \left(z \frac{d}{dz} - 1 \right) \hat{w}. \tag{39}$$

Note that the fourth-order polynomial in $z \frac{d}{dz}$ appearing on the r. h. s. of (39) may be inferred, without lengthy calculations, from the fact that $z^2 \frac{d^4}{dz^4} z^2$ annihilates $\{\frac{1}{z^2}, \frac{1}{z}, 1, z\}$. This suggests to introduce the new variables

$$s = \ln z \text{ and } \xi = z^{-1} \hat{\xi}, \tag{40}$$

for which the stability condition turns into

$$\int_{-\infty}^{\ln H} \hat{\xi} \phi \, ds \geq 0, \tag{41}$$

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where

$$\phi = \hat{w}\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left(\frac{d}{ds}-1\right)\hat{w}\,,$$

for all functions $\hat{w}$ with compact support in $z \in (0, H)$. Here comes the heuristic argument for (30): For $H \gg 1$, we may think of test functions $\hat{w}$ that vary slowly in the logarithmic variable s. For these $\hat{w}$ we have

$$\phi = \hat{w}\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left(\frac{d}{ds}-1\right)\hat{w} \approx -2\hat{w}\frac{d}{ds}\hat{w} = -\frac{d}{ds}\hat{w}^2\,, \qquad (42)$$

which in particular implies

$$0 \le \int_{-\infty}^{\ln H} \hat{\xi}\,\phi\,ds \approx -\int_{-\infty}^{\ln H} \hat{\xi}\frac{d}{ds}\hat{w}^2\,ds = \int_{-\infty}^{\ln H} \frac{d\hat{\xi}}{ds}\hat{w}^2\,ds\,.$$

Since $\hat{w}$ was arbitrary besides varying slowly in s, it follows

$$\frac{d\hat{\xi}}{ds} \ge 0\,,$$

approximately on large $s-$scales. We expect that this implies that for any $1 \ll S_1 \le \ln H$:

$$\int_{-1}^{0} \hat{\xi}\,ds \lesssim \frac{1}{S_1}\int_{0}^{S_1} \hat{\xi}\,ds\,, \qquad (43)$$

which in the original variables (40), for S_1 turns into (30). We now establish rigorously that (37) and (29) imply (43).

Argument for (43), rigorous version:

We start by noticing that because of translation invariance in s, (41) can be reformulated as follows: For any function $\hat{w}(s)$ supported in $s \le 0$, and any $s' \le \ln H$ we have

$$\int_{-\infty}^{\infty} \hat{\xi}(s'')\phi(s''-s')\,ds'' \;=\; \int_{-\infty}^{\infty} \hat{\xi}(s+s')\phi(s)\,ds \;\ge\; 0, \qquad (44)$$

where the multiplier ϕ is defined as in (41). We note that (43) follows from (44) once for given S_1 we construct

- a family $\mathfrak{F} = \{w_{s'}\}_{s'}$ of smooth functions $w_{s'}$ parameterized by $s' \in \mathbb{R}$ and compactly supported in $z \in (0, 1)$ (i. e. $s \in (-\infty, 0]$) and

- a measure $\rho(ds') = \rho(s')\,ds'$ supported in $s' \in (-\infty, \ln H]$,

such that the corresponding convex combination of multipliers $\{\phi_{s'}\}_{s'}$ shifted by s', i. e.

$$\phi_1(s'') \;:=\; \int_{-\infty}^{\infty} \phi_{s'}(s''-s')\,\rho(s')ds'\,, \qquad (45)$$

satisfies

$$\phi_1(s'') \le \left\{\begin{array}{ll} -1 & \text{for } -1 \le s'' \le 0 \\ \frac{C}{S_1} & \text{for } 0 \le s'' \le S_1 \\ 0 & \text{else} \end{array}\right\}\,, \qquad (46)$$

for a (possibly large) universal constant C. Indeed, using (44), (45), and (46) in conjunction with the positivity (29) of the profile $\hat{\xi}$ we have

$$0 \le \int_{-\infty}^{\infty} \hat{\xi}\phi_1\,ds'' \le -\int_{-1}^{0} \hat{\xi}\,ds'' + \frac{C}{S_1}\int_{0}^{S_1} \hat{\xi}\,ds''\,,$$

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which implies (43).

We first address the form of the family $\mathfrak{F}$. The heuristic observation (42) motivates the change of variables

$$s = \lambda \hat{s} \quad \text{with} \quad \lambda \geq 1, \tag{47}$$

our "(logarithmic) length-scale", to be chosen sufficiently large. We fix a smooth, real-valued and compactly supported "mask" $\hat{w}_0(\hat{s})$; it will be convenient to restrict its support to $\hat{s} \in (-1, 0]$, say

$$\hat{w}_0^2 > 0 \quad \text{in} \quad \left(-\frac{1}{2}, 0 \right) \quad \text{and} \quad \hat{w}_0 = 0 \qquad \text{else}, \tag{48}$$

and, in order to justify the language of "mollification by convolution" we impose the normalization $\int \hat{w}_0^2 d\hat{s} = 1$. By mask we mean that in (41) we choose

$$\hat{w}(\lambda \hat{s}) = \lambda^{-1/2} \hat{w}_0(\hat{s}) \tag{49}$$

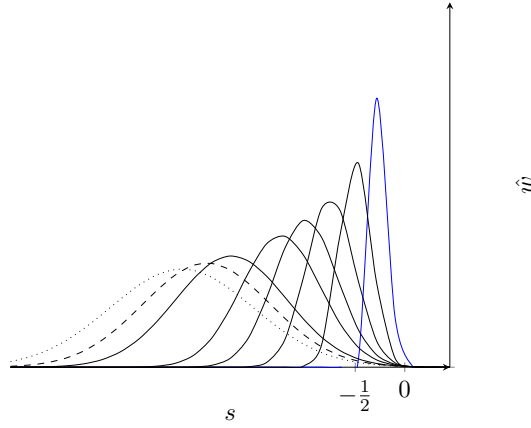(see Figure 2). With this change of variables, the multiplier can be rewritten as follows

Figure 2: Construction of the family of test functions starting from the mask $\hat{w}_0$ (blue line).

$$\phi_\lambda \overset{(42)}{=} \hat{w} \left(\frac{d}{ds} + 2 \right) \left(\frac{d}{ds} + 1 \right) \frac{d}{ds} \left(\frac{d}{ds} - 1 \right) \hat{w}$$

$$\overset{(47)\&(49)}{=} \frac{1}{\lambda} \hat{w}_0 \left(\frac{1}{\lambda} \frac{d}{d\hat{s}} + 2 \right) \left(\frac{1}{\lambda} \frac{d}{d\hat{s}} + 1 \right) \frac{1}{\lambda} \frac{d}{d\hat{s}} \left(\frac{1}{\lambda} \frac{d}{d\hat{s}} - 1 \right) \hat{w}_0$$

$$= \hat{w}_0 \left(\frac{1}{\lambda^5} \frac{d^4}{d\hat{s}^4} + \frac{2}{\lambda^4} \frac{d^3}{d\hat{s}^3} - \frac{1}{\lambda^3} \frac{d^2}{d\hat{s}^2} - \frac{2}{\lambda^2} \frac{d}{d\hat{s}} \right) \hat{w}_0,$$

and reordering the terms we have

$$\phi_\lambda = -\frac{2}{\lambda^2} \hat{w}_0 \frac{d}{d\hat{s}} \hat{w}_0 - \frac{1}{\lambda^3} \hat{w}_0 \frac{d^2}{d\hat{s}^2} \hat{w}_0 + \frac{2}{\lambda^4} \hat{w}_0 \frac{d^3}{d\hat{s}^3} \hat{w}_0 + \frac{1}{\lambda^5} \hat{w}_0 \frac{d^4}{d\hat{s}^4} \hat{w}_0. \tag{50}$$

Heuristically, for $\lambda \gg 1$ the multiplier ϕ_λ can be approximated by the first term on the r. h. s.

$$\phi_\lambda(s) \approx -\frac{d}{ds} \left(\frac{1}{\lambda} \hat{w}_0^2 \left(\frac{s}{\lambda} \right) \right).$$

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Inserting this approximation into the definition (45) of ϕ_1 we have

$$
\begin{aligned}
\phi_1(s'') &= \int_{-\infty}^{\infty} \phi(s'' - s')\rho(s')ds' = \int_{-\infty}^{\infty} \phi(s)\rho(s'' - s)ds \\
&= -\int_{-\infty}^{\infty} \frac{d}{ds}\left(\frac{1}{\lambda}\hat{w}_0^2\left(\frac{s}{\lambda}\right)\right)\rho(s'' - s)ds \approx -\int_{-\infty}^{\infty} \frac{1}{\lambda}\hat{w}_0^2\left(\frac{s}{\lambda}\right)\frac{d\rho}{ds'}(s'' - s)ds \,.\text{(51)}
\end{aligned}
$$

For λ smaller than the characteristic scale on which ρ varies, we may think of $\frac{1}{\lambda}\hat{w}_0^2\left(\frac{s}{\lambda}\right) \approx \delta_0(s)$, in view of our normalization. This yields

$$
\phi_1 \approx -\frac{d\rho}{ds'} \,, \tag{52}
$$

which in view of (46) suggests that ρ should have the form

$$
\rho(s') = \left\{ \begin{array}{ll} s' + 1 & \text{for } -1 \le s' \le 0 \\ 1 - \frac{s'}{S_1} & \text{for } 0 \le s' \le S_1 \end{array} \right\} \,, \tag{53}
$$

(see Figure 3). We will now argue that we have to modify the Ansatz for both ϕ_λ and ρ. To this purpose, we go through the above heuristic argument (heuristically) assessing the
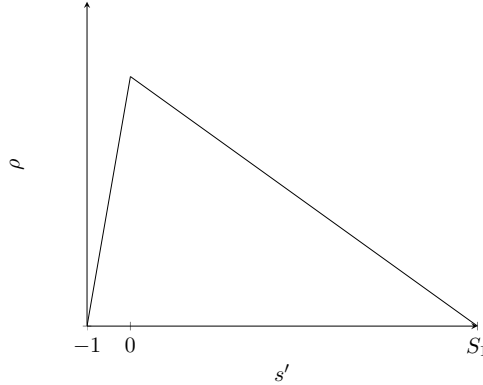

Figure 3: The measure ρ suggested by the heuristic argument.

error terms. Expanding ρ in a Taylor series around s''

$$
\rho(s'' - s) \approx \rho(s'') - \frac{d\rho}{ds'}(s'')s + \frac{1}{2}\frac{d^2\rho}{ds'^2}(s'')s^2 \,,
$$

we may write

$$
\begin{aligned}
\phi_1(s'') &\overset{(45)}{=} \int_{-\infty}^{\infty} \phi_\lambda(s)\rho(s'' - s)ds \\
&\approx \rho(s'')\int_{-\infty}^{\infty} \phi_\lambda \, ds - \frac{d\rho}{ds'}(s'')\int_{-\infty}^{\infty} s\phi_\lambda \, ds + \frac{1}{2}\frac{d^2\rho}{ds'^2}(s'')\int_{-\infty}^{\infty} s^2\phi_\lambda \, ds \,.
\end{aligned}
$$

We note that the first term in (50), i. e. $-\frac{2}{\lambda^2}\hat{w}_0\frac{d\hat{w}_0}{d\hat{s}} = -\frac{1}{\lambda^2}\frac{d\hat{w}_0^2}{d\hat{s}}$, gives the leading-order contribution to the first and the second moment

$$
\int_{-\infty}^{\infty} s\phi_\lambda \, ds \approx \int_{-\infty}^{\infty} s\left(-\frac{1}{\lambda^2}\frac{d\hat{w}_0^2}{d\hat{s}}\right)ds \overset{(47)}{=} \int_{-\infty}^{\infty} \hat{s}\left(-\frac{d\hat{w}_0^2}{d\hat{s}}\right)d\hat{s} = \int_{-\infty}^{\infty} \hat{w}_0^2 \, d\hat{s} = 1
$$

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and

$$\int_{-\infty}^{\infty} s^2 \phi_\lambda \, ds \approx \int_{-\infty}^{\infty} s^2 \left(-\frac{1}{\lambda^2} \frac{d\hat{w}_0^2}{d\hat{s}} \right) ds \overset{(47)}{=} -\lambda \int_{-\infty}^{\infty} \hat{s}^2 \left(\frac{d\hat{w}_0^2}{d\hat{s}} \right) d\hat{s} = \lambda \int_{-\infty}^{\infty} 2\hat{s}\hat{w}_0^2 \, d\hat{s} \,,$$

while the second term in (50) gives the leading-order contribution to the zeroth moment of the multiplier ϕ:

$$\int_{-\infty}^{\infty} \phi_\lambda \, ds \approx \int_{-\infty}^{\infty} \left(-\frac{1}{\lambda^3} \hat{w}_0 \frac{d^2 \hat{w}_0}{d\hat{s}^2} \right) ds \overset{(47)}{=} \frac{1}{\lambda^2} \int_{-\infty}^{\infty} \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 d\hat{s} \,.$$

Hence we obtain the following specification of (51)

$$
\begin{aligned}
\phi_1(s'') &= \int_{-\infty}^{\infty} \phi_\lambda(s)\rho(s'' - s)ds \\
&\approx \frac{1}{\lambda^2} \rho(s'') \int_{-\infty}^{\infty} \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 d\hat{s} - \frac{d\rho}{ds'}(s'') + \lambda \frac{d^2\rho}{ds'^2}(s'') \int_{-\infty}^{\infty} \hat{s}\hat{w}_0^2 \, d\hat{s} \,. \quad (54)
\end{aligned}
$$

Our goal is to specify the choice (53) of ρ such that (46) is satisfied. This shows a dilemma: On the one hand, in the "plateau region" $s'' \sim S_1$, we would need $\lambda^2 \gg S_1$ so that the first r. h. s. term in (54) does not destroy the desired $\frac{1}{S_1}$−behavior. On the other hand in the "foot region" $s'' \in [0,1]$, we would need $\lambda \lesssim 1$ so that the last term does not destroy the effect of the middle term. This suggests that λ should be chosen to be small in the foot regions and large on the plateau region. Therefore it is natural to choose

$$\lambda = s' \,, \quad (55)$$

so that $\phi_{s'}$ in (45) indeed acquires a dependency on s' besides the translation. For our choice of (55), (50) assumes the form

$$\phi_{s'} = -\frac{2}{(s')^2} \hat{w}_0 \frac{d}{d\hat{s}} \hat{w}_0 - \frac{1}{(s')^3} \hat{w}_0 \frac{d^2}{d\hat{s}^2} \hat{w}_0 + \frac{2}{(s')^4} \hat{w}_0 \frac{d^3}{d\hat{s}^3} \hat{w}_0 + \frac{1}{(s')^5} \hat{w}_0 \frac{d^4}{d\hat{s}^4} \hat{w}_0 \,. \quad (56)$$

Note that with the choice (55) and $s = s'' - s'$, (47) turns into the *nonlinear* change of variables

$$\hat{s} = \frac{s'' - s'}{s'} = \frac{s''}{s'} - 1 \Rightarrow s' = \frac{s''}{1 + \hat{s}} \,. \quad (57)$$

We consider this as a change of variables between s' and $\hat{s}$ (with s'' as a parameter); thanks to the support restriction (48) on $\hat{w}_0$, it is invertible in the relevant range $\hat{s} \in [-\frac{1}{2}, 0]$: $\frac{d}{d\hat{s}} = -\frac{s''}{(1+\hat{s})^2} \frac{d}{ds'} = -\frac{(s')^2}{s''} \frac{d}{ds'}$ and $ds' = \frac{s''}{(1+\hat{s})^2} d\hat{s}$. From (45) and (56) we thus get the first representation

$$
\begin{aligned}
\phi_1(s'') &= -\frac{1}{s''} \int_{-\infty}^{\infty} \frac{d\hat{w}_0^2}{d\hat{s}} \rho \, d\hat{s} - \frac{1}{(s'')^2} \int_{-\infty}^{\infty} (1 + \hat{s}) \, \hat{w}_0 \frac{d^2\hat{w}_0}{d\hat{s}^2} \rho \, d\hat{s} \\
&\quad + \frac{2}{(s'')^3} \int_{-\infty}^{\infty} (1 + \hat{s})^2 \, \hat{w}_0 \frac{d^3\hat{w}_0}{d\hat{s}^3} \rho \, d\hat{s} + \frac{1}{(s'')^4} \int_{-\infty}^{\infty} (1 + \hat{s})^3 \, \hat{w}_0 \frac{d^4\hat{w}_0}{d\hat{s}^4} \rho \, d\hat{s} \,.
\end{aligned}
$$

An approximation argument in $\hat{w}_0$ below necessitates a second representation that involves $\hat{w}_0$ only up to second derivatives. For this purpose, we rewrite (56) in terms of the three

quadratic quantities $\hat{w}_0^2$, $(\frac{d\hat{w}_0}{d\hat{s}})^2$, and $(\frac{d^2\hat{w}_0}{d\hat{s}^2})^2$:

$$
\begin{aligned}
\phi_{s'} &= -\frac{1}{(s')^2}\frac{d\hat{w}_0^2}{d\hat{s}} + \frac{1}{(s')^3}\left[\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 - \frac{1}{2}\frac{d^2\hat{w}_0^2}{d\hat{s}^2}\right] + \frac{1}{(s')^4}\left[-3\frac{d}{d\hat{s}}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{d^3\hat{w}_0^2}{d\hat{s}^3}\right] \\
&+ \frac{1}{(s')^5}\left[\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 - 2\frac{d^2}{d\hat{s}^2}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{1}{2}\frac{d^4\hat{w}_0^2}{d\hat{s}^4}\right] \\
&= \left(-\frac{1}{(s')^2}\frac{d}{d\hat{s}} - \frac{1}{2}\frac{1}{(s')^3}\frac{d^2}{d\hat{s}^2} + \frac{1}{(s')^4}\frac{d^3}{d\hat{s}^3} + \frac{1}{2}\frac{1}{(s')^5}\frac{d^4}{d\hat{s}^4}\right)\hat{w}_0^2 \\
&+ \left(\frac{1}{(s')^3} - 3\frac{1}{(s')^4}\frac{d}{d\hat{s}} - 2\frac{1}{(s')^5}\frac{d^2}{d\hat{s}^2}\right)\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{1}{(s')^5}\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2.
\end{aligned}
\tag{58}
$$

Now in this formula, using the change of variables (57), we want to substitute the derivations $\frac{1}{(s')^m}\frac{d^n}{d\hat{s}^n}$ by linear combinations of derivations of the form $\frac{1}{(s'')^{m-k}}\frac{d^k}{ds'^k}(1+\hat{s})^{m-n-k}$ for $k = 0, \cdots, n$. The reason why this can be done is explained in Appendix 5.1, where also the linear combinations are explicitly computed. The formulas (200), (201) & (202) allow to rewrite (58) as follows

$$
\begin{aligned}
\phi_{s'} &= \frac{1}{s''}\left(\frac{d}{ds'} - \frac{1}{2}\frac{d^2}{ds'^2}\frac{1}{(1+\hat{s})} - \frac{d^3}{ds'^3}\frac{1}{(1+\hat{s})^2} + \frac{1}{2}\frac{d^4}{ds'^4}\frac{1}{(1+\hat{s})^3}\right)\hat{w}_0^2 \\
&+ \left[\left(\frac{1}{(s'')^3} + \frac{6}{(s'')^4} - \frac{12}{(s'')^5}\right)(1+\hat{s})^3 + \left(\frac{3}{(s'')^3} - \frac{8}{(s'')^4}\right)\frac{d}{ds'}(1+\hat{s})^2\right. \\
&\left. - \frac{2}{(s'')^3}\frac{d^2}{ds'^2}(1+\hat{s})\right]\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{1}{(s'')^5}(1+\hat{s})^5\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2.
\end{aligned}
\tag{59}
$$

The advantage of this form is that integrations by part in s' become easy, so that we obtain

$$
\begin{aligned}
\phi_1 &= \frac{1}{s''}\int_{-\infty}^{\infty}\hat{w}_0^2\left(-\frac{d\rho}{ds'} - \frac{1}{2}\frac{1}{1+\hat{s}}\frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s})^2}\frac{d^3\rho}{ds'^3} + \frac{1}{2}\frac{1}{(1+\hat{s})^3}\frac{d^4\rho}{ds'^4}\right)ds' \\
&+ \left(\frac{1}{(s'')^3} + \frac{6}{(s'')^4} - \frac{12}{(s'')^5}\right)\int_{-\infty}^{\infty}(1+\hat{s})^3\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\rho\,ds' \\
&- \left(\frac{3}{(s'')^3} - \frac{8}{(s'')^4}\right)\int_{-\infty}^{\infty}(1+\hat{s})^2\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d\rho}{ds'}\,ds' \\
&- \frac{2}{(s'')^3}\int_{-\infty}^{\infty}(1+\hat{s})\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d^2\rho}{ds'^2}\,ds' \\
&+ \frac{1}{(s'')^5}\int_{-\infty}^{\infty}(1+\hat{s})^5\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2\rho\,ds'.
\end{aligned}
$$

Finally, using the substitution $\frac{ds'}{s''} = \frac{d\hat{s}}{(1+\hat{s})^2}$, the last formula turns into the desired second

representation

$$
\begin{aligned}
\phi_1 &= \int_{-\infty}^{\infty} \hat{w}_0^2 \left(-\frac{1}{(1+\hat{s})^2} \frac{d\rho}{ds'} - \frac{1}{2} \frac{1}{(1+\hat{s})^3} \frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s})^4} \frac{d^3\rho}{ds'^3} + \frac{1}{2} \frac{1}{(1+\hat{s})^5} \frac{d^4\rho}{ds'^4} \right) d\hat{s} \\
&+ \left(\frac{1}{(s'')^2} + \frac{6}{(s'')^3} - \frac{12}{(s'')^4} \right) \int_{-\infty}^{\infty} (1+\hat{s}) \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \rho \, d\hat{s} \\
&- \left(\frac{3}{(s'')^2} - \frac{8}{(s'')^3} \right) \int_{-\infty}^{\infty} \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \frac{d\rho}{ds'} \, d\hat{s} \\
&- \frac{2}{(s'')^2} \int_{-\infty}^{\infty} \frac{1}{1+\hat{s}} \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \frac{d^2\rho}{ds'^2} \, d\hat{s} \\
&+ \frac{1}{(s'')^4} \int_{-\infty}^{\infty} (1+\hat{s})^3 \left(\frac{d^2\hat{w}_0}{d\hat{s}^2} \right)^2 \rho \, d\hat{s} \,.
\end{aligned} \tag{60}
$$

From this representation we learn the following: If we assume that $\rho(s')$ varies on large length-scales only, so that $\frac{d\rho}{ds'}, \frac{d^2\rho}{ds'^2}, \cdots \ll \rho$ and $\frac{d^2\rho}{ds'^2}, \frac{d^3\rho}{ds'^3}, \cdots \ll \frac{d\rho}{ds'}$ then for $s'' \gg 1$, we obtain to leading order from the above

$$
\phi_1 \approx -\int_{-\infty}^{\infty} \frac{1}{(1+\hat{s})^2} \hat{w}_0^2 \frac{d\rho}{ds'} \, d\hat{s} + \frac{1}{(s'')^2} \int_{-\infty}^{\infty} (1+\hat{s}) \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \rho \, d\hat{s} \,.
$$

If $\rho(s')$ varies slowly even on a logarithmic scale (so that e. g. $s' \frac{d\rho}{ds'}$ is negligible with respect to ρ), the above further reduces to

$$
\phi_1 \approx -\frac{d\rho(s'')}{ds'} \int_{-\infty}^{\infty} \frac{1}{(1+\hat{s})^2} \hat{w}_0^2 \, d\hat{s} + \frac{\rho(s'')}{(s'')^2} \int_{-\infty}^{\infty} (1+\hat{s}) \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \, d\hat{s} \,, \tag{61}
$$

which should be compared with (54). We see that the first, negative, right-hand-side term of (61) dominates the second positive term provided

$$
\frac{d\rho}{ds'} \gg \frac{1}{(s')^2} \,.
$$

This is satisfied if ρ is of the form $\rho(s') = 1 - \frac{S_0}{s' - S_0}$ for some $S_0 \gg 1$ to be chosen later; indeed $\frac{d\rho}{ds'} = \frac{S_0}{(s' - S_0)^2} \approx \frac{S_0}{(s')^2} \gg \frac{1}{(s')^2}$ for $s' \gg S_0 \gg 1$.

Disregarding for a couple of pages the fact that ρ needs to be supported in $s' \in (-\infty, \ln H)$, which will be achieved by cutting off at scales $s' \sim S_1$, we define as our intermediate goal to construct a measure $0 \le \widetilde{\rho}(s') \le 1$ (with infinite support) such that

$$
\widetilde{\phi}_1(s'') := \int \phi_{s'}(s'' - s') \widetilde{\rho}(s') \, ds' \left\{ \begin{array}{ll} = 0 & s'' \le \frac{S_0}{2} \\ < 0 & s'' > \frac{S_0}{2} \end{array} \right\} , \tag{62}
$$

for some universal S_0, to be chosen later. The above considerations motivate the following Ansatz for $\widetilde{\rho}$: We fix a smooth mask $\widetilde{\rho}_0(\hat{s}')$ such that

$$
\widetilde{\rho}_0 = 0 \text{ for } \hat{s}' \le 0, \quad \frac{d\widetilde{\rho}_0}{d\hat{s}'} > 0 \text{ for } 0 < \hat{s}' \le 2, \quad \widetilde{\rho}_0 = 1 - \frac{1}{\hat{s}'} \text{ for } 2 \le \hat{s}', \tag{63}
$$

and consider the rescaled version

$$
\widetilde{\rho}(S_0(\hat{s}' + 1)) = \widetilde{\rho}_0(\hat{s}'), \quad \text{i. e. the change of variables } s' = S_0(\hat{s}' + 1) \tag{64}
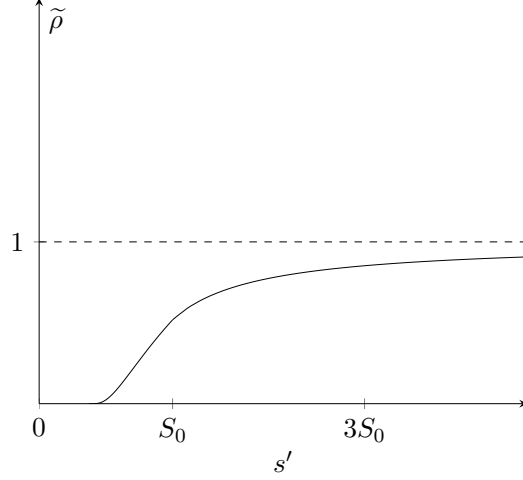$$

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Figure 4: The function $\tilde{\rho}$ of the variable $s' = S_0(\hat{s}' + 1)$.

with $S_0 \gg 1$ to be fixed later (see Figure 4). It is convenient to rescale s'' accordingly:

$$s'' = S_0 \hat{s}''. \tag{65}$$

With this new rescaling and the choice of the measure $\widetilde{\rho}$ (see (63) and (64)), (60) turns into

$$
\begin{aligned}
\widetilde{\phi}_1 &= -\frac{1}{S_0} \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^2} \frac{d\widetilde{\rho}_0}{d\hat{s}'} \, d\hat{s} - \frac{1}{2S_0^2} \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^3} \frac{d^2\widetilde{\rho}_0}{d\hat{s}'^2} \, d\hat{s} \\
&\quad + \frac{1}{S_0^3} \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^4} \frac{d^3\widetilde{\rho}_0}{d\hat{s}'^3} \, d\hat{s} + \frac{1}{2S_0^4} \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^5} \frac{d^4\widetilde{\rho}_0}{d\hat{s}'^4} \, d\hat{s} \\
&\quad + \left(\frac{1}{S_0^2} \frac{1}{(\hat{s}'')^2} + \frac{1}{S_0^3} \frac{6}{(\hat{s}'')^3} - \frac{1}{S_0^4} \frac{12}{(\hat{s}'')^4} \right) \int_{-\infty}^{\infty} (1+\hat{s}) \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \widetilde{\rho}_0 \, d\hat{s} \\
&\quad - \left(\frac{1}{S_0^4} \frac{3}{(\hat{s}'')^3} - \frac{1}{S_0^5} \frac{8}{(\hat{s}'')^4} \right) \int_{-\infty}^{\infty} \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \frac{d\widetilde{\rho}_0}{d\hat{s}'} \, d\hat{s} \\
&\quad - \frac{1}{S_0^5} \frac{2}{(\hat{s}'')^3} \int_{-\infty}^{\infty} \frac{1}{1+\hat{s}} \left(\frac{d\hat{w}_0}{d\hat{s}} \right)^2 \frac{d^2\widetilde{\rho}_0}{d\hat{s}'^2} \, d\hat{s} \\
&\quad + \frac{1}{S_0^4} \frac{1}{(\hat{s}'')^4} \int_{-\infty}^{\infty} (1+\hat{s})^3 \left(\frac{d^2\hat{w}_0}{d\hat{s}^2} \right)^2 \widetilde{\rho}_0 \, d\hat{s} \,. \tag{66}
\end{aligned}
$$

Since in the integrals in formula (60), the argument of $\widetilde{\rho}$ was given by $s' = \frac{s''}{1+\hat{s}}$, cf. (57), it follows from (64) and (65) that the argument of $\widetilde{\rho}_0$ is given by

$$\hat{s}' = \frac{\hat{s}''}{1+\hat{s}} - 1 \,. \tag{67}$$

Thus all the integrals in (66) just depend on $\hat{s}''$, not on S_0. Hence (66) makes the dependence of $\widetilde{\phi}_1$ on S_0 explicit. We are now ready to show that the construction of the family $w_{s'}$ (cf. (49) and (55)) and of the measure $\widetilde{\rho}$ (cf. (63) and (64)) yield the intermediate goal (62) when $S_0 \gg 1$. In order to establish (62) it is convenient to distinguish three regions (note that if $s'' \in (\infty, \frac{S_0}{2}]$ all the integrals in (66) vanish because the supports of $\hat{w}_0$ and $\widetilde{\rho}_0$ do not intersect, see below):

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The range of large s'':

$$s'' \geq 3S_0 \quad \text{or equivalently} \quad \hat{s}'' \geq 3. \tag{68}$$

Note that because of our support condition (48) on $\hat{w}_0$, all integrals in (66) are supported in $\hat{s} \in [-\frac{1}{2}, 0]$. Together with our range (68), this yields for the argument $\hat{s}' \overset{(67)}{=} \frac{\hat{s}''}{1+\hat{s}} - 1$ of $\widetilde{\rho}_0$ and its derivatives that $\hat{s}' \geq 2$. Because of $\frac{d\widetilde{\rho}_0}{d\hat{s}'} = \frac{1}{(\hat{s}')^2}$ for $\hat{s}' \geq 2$, cf. our Ansatz (63), the first integral in (66) reduces to

$$\int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^2} \frac{d\widetilde{\rho}_0}{d\hat{s}'} \, d\hat{s} = \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^2} \frac{1}{(\frac{\hat{s}''}{1+\hat{s}} - 1)^2} \, d\hat{s}$$

$$= \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(\hat{s}'' - (1+\hat{s}))^2} \, d\hat{s} \approx \frac{1}{(\hat{s}'')^2} \int_{-\infty}^{\infty} \hat{w}_0^2 \, d\hat{s}, \tag{69}$$

for $\hat{s}'' \gg 1$, whereas all the other integrals in (66) are $O(\frac{1}{(\hat{s}'')^2})$ or smaller in $\hat{s}'' \gg 1$ (because at least one derivation falls on $\widetilde{\rho}_0$) or have pre-factors $\frac{1}{(\hat{s}'')^2}$ or smaller. Since only the term in (66) coming from integral (69) has pre-factor $\frac{1}{S_0}$ while all the other terms have pre-factors $\frac{1}{S_0^2}$ or smaller (for $S_0 \gg 1$), the first term in (66) *uniformly* dominates all other terms for $S_0 \gg 1$:

$$\widetilde{\phi}_1 \approx -\frac{1}{S_0} \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(\hat{s}'' - (1+\hat{s}))^2} \, d\hat{s} \quad \text{uniformly in } \hat{s}'' \geq 3 \quad \text{for } S_0 \gg 1. \tag{70}$$

In conclusion we have

$$\widetilde{\phi}_1 \sim -\frac{1}{S_0} \frac{1}{(\hat{s}'')^2} < 0 \quad \text{in the range } \hat{s}'' \geq 3 \quad \text{for } S_0 \gg 1. \tag{71}$$

The range of intermediate s'':

$$s'' \in \left[\frac{3}{4} S_0, 3S_0\right] \quad \text{or equivalently} \quad \hat{s}'' \in \left[\frac{3}{4}, 3\right]. \tag{72}$$

Again, we consider the first integral in (66). Now we use that $\frac{\hat{w}_0^2}{(1+\hat{s})^2} \geq 0$ is *strictly positive* in $\hat{s} \in \left(-\frac{1}{2}, 0\right)$, cf. (48), and that $\frac{d\widetilde{\rho}_0}{d\hat{s}'} \geq 0$ is strictly positive in $\hat{s}' > 0$, cf. (63), that is, in $\hat{s} < \hat{s}'' - 1$, cf. (67). We note that the two $\hat{s}-$intervals $(-\frac{1}{2}, 0)$ and $(-\infty, \hat{s}'' - 1)$ intersect for $\hat{s}'' > \frac{1}{2}$. Hence by continuity of the first integral in (66) in its parameter $\hat{s}''$, there exists a universal constant C such that

$$\int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^2} \frac{d\widetilde{\rho}_0}{d\hat{s}'} \, d\hat{s} \geq \frac{1}{C} \quad \text{for } \hat{s}'' \in \left[\frac{3}{4}, 3\right].$$

Hence also in this range the first term in (66) dominates all other terms:

$$\widetilde{\phi}_1 \approx -\frac{1}{S_0} \int_{-\infty}^{\infty} \frac{\hat{w}_0^2}{(1+\hat{s})^2} \frac{d\widetilde{\rho}_0}{d\hat{s}'} \, d\hat{s} \quad \text{uniformly in } \hat{s}'' \in \left[\frac{3}{4}, 3\right] \quad \text{for } S_0 \gg 1, \tag{73}$$

and we may conclude that

$$\widetilde{\phi}_1 \sim -\frac{1}{S_0} < 0 \quad \text{in the range } s'' \in \left[\frac{3}{4} S_0, 3S_0\right] \quad \text{for } S_0 \gg 1. \tag{74}$$

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Note that the above discussion on supports also yields that $\widetilde{\phi}_1$ is supported in $\hat{s}'' \in \left[\frac{1}{2}, \infty\right)$. The range of small s'':

$$s'' \in \left(\frac{1}{2}S_0, \frac{3}{4}S_0\right) \quad \text{or equivalently} \quad \hat{s}'' \in \left(\frac{1}{2}, \frac{3}{4}\right). \tag{75}$$

We would like $\widetilde{\phi}_1$ to be strictly negative in this range for $S_0 \gg 1$. Here, we encounter the second difficulty: No matter how large $\lambda = s'$ in (50) is, the behavior of $\phi_{s'}$ near the left edge $-\frac{1}{2}$ of its support $\left[-\frac{1}{2}, 0\right]$ (and also at its right edge 0, but there we don't care), is dominated by the $\frac{1}{\lambda^5}\hat{w}_0 \frac{d^4 \hat{w}_0}{d\hat{s}^4}$-term and thus automatically is *strictly positive*. Taking the $\widetilde{\rho}(s')\,ds'$-average of the shifted $\phi_{s'}(s'' - s')$ does not alter this behavior as long as $\widetilde{\rho}$ is non-negative in $[S_0, \infty)$, cf. (63): $\widetilde{\phi}_1$ is strictly positive near the left edge $\frac{S_0}{2}$ of its support. The way out of this problem is to give *give up smoothness* of $\hat{w}_0$ near the left edge $-\frac{1}{2}$ of its support $\left[-\frac{1}{2}, 0\right]$. In fact, we shall first assume that $\hat{w}_0$ satisfies in addition

$$\hat{w}_0 = \frac{1}{2}\left(\hat{s} + \frac{1}{2}\right)^2 \quad \text{for } \hat{s} \in \left[-\frac{1}{2}, -\frac{1}{4}\right]. \tag{76}$$

This means that $\hat{w}_0$ has a bounded but discontinuous second derivative (i. e. $\hat{w}_0 \in H^{2,\infty}$). This is the main reason why in (66) we expressed $\widetilde{\phi}_1$ only in terms of up to second derivatives of $\hat{w}_0$. We argue that the so defined $\widetilde{\phi}_1$ is, as desired, strictly negative on $s'' \in (\frac{1}{2}S_0, \frac{3}{4}S_0]$ for all S_0. Indeed, in view of (56), the form (76) implies, in terms of $s = s'' - s'$,

$$\phi_{s'} = -\frac{1}{(s')^2}\left(\frac{s}{s'} + \frac{1}{2}\right)^3 - \frac{1}{2}\frac{1}{(s')^3}\left(\frac{s}{s'} + \frac{1}{2}\right)^2 < 0 \quad \text{for } s \in \left(-\frac{s'}{2}, -\frac{s'}{4}\right]. \tag{77}$$

In view of (63) & (64), $\widetilde{\rho} \geq 0$ is strictly positive for $s' \in (S_0, \infty)$. On the other hand, it follows from (77) that $s' \mapsto \phi_{s'}(s'' - s')$ is strictly negative for $s'' - s' \in (-\frac{s'}{2}, -\frac{s'}{4}]$, that is, for $s' \in [\frac{4}{3}s'', 2s'')$ (and supported in $s' \in [s'', 2s'']$). Hence, by (45), $\widetilde{\phi}_1$ is strictly negative for $S_0 \in [\frac{4}{3}s'', 2s'')$, that is, for $s'' \in (\frac{1}{2}S_0, \frac{3}{4}S_0]$, for any value of $S_0 > 0$.

Now we approximate $\hat{w}_0$ with a sequence of smooth functions $\hat{w}_0^\nu$ in $H^{2,2}$ and we call $\widetilde{\phi}_1^\nu$ the associated multiplier. Since $\widetilde{\phi}_1$ involves $\hat{w}_0$ only up to second derivatives (cf. (66)) then ϕ_1^ν converge uniformly in $\hat{s}''$ to $\widetilde{\phi}_1$. We conclude that

$$\widetilde{\phi}_1 < 0 \quad \text{for } s'' \in \left(\frac{1}{2}S_0, \frac{3}{4}S_0\right) \quad \text{and} \quad S_0 > 0. \tag{78}$$

Finally we fix a sufficiently large but universal S_0, so that (78) together with (71) and (74) imply our intermediate goal (62).

The choice (63)&(64) of $\widetilde{\rho}$ is not admissible, since ρ should be supported in $(-\infty, \ln H)$, which will be achieved by cutting off at scales $s' \sim S_1$. Only this cut-off will ensure (46) in the range $s'' \geq S_1$. More precisely, in (45) we choose ρ to be $\widetilde{\rho}(s')\eta(\frac{s'}{S_1})$ where $\widetilde{\rho}$ is defined in (64), η is a mask for a smooth cut-off function with

$$\eta(\hat{\hat{s}}') = \left\{ \begin{array}{ll} 1 & \hat{\hat{s}}' \leq \frac{1}{2} \\ 0 & \hat{\hat{s}}' \geq 1 \end{array} \right\}$$

(see Figure 5 for an illustration of ρ). We argue that $\phi_1(s'')$ satisfies (46) for $S_1 \gg 1$. This will follow immediately from the three claims
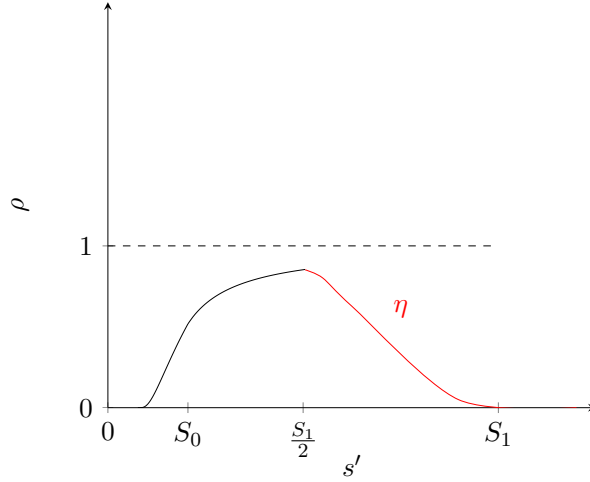
Figure 5: The measure ρ is constructed from $\widetilde{\rho}$ (see Figure 4) by cutting off at scales $s' \sim S_1$.

1. $\phi_1(s'') = 0$ for $s'' \geq S_1$,

2. $\phi_1(s'') = \widetilde{\phi}_1(s'')$ for $s'' \leq \frac{S_1}{4}$,

3. $|\phi_1 - \widetilde{\phi}_1| \lesssim \frac{1}{S_1}$ for $\frac{S_1}{4} \leq s'' \leq S_1$.

Claims 1, 2 and 3 together with the bound (62) on $\widetilde{\phi}_1$ imply

$$\phi_1(s'') \begin{cases} = 0 & \text{for} & s'' \leq \frac{S_0}{2} \\ \lesssim -1 & \text{for} & \frac{S_0}{2} + 1 \leq s'' \leq \frac{S_0}{2} + 2 \\ \leq 0 & \text{for} & \frac{S_0}{2} \leq s'' \leq \frac{S_1}{4} \\ \lesssim \frac{1}{S_1} & \text{for} & \frac{S_1}{4} \leq s'' \leq S_1 \\ = 0 & \text{for} & s'' \geq S_1 \end{cases} . \tag{79}$$

Note that with help of the scale invariance (27), which turns into a shift invariance in the logarithmic variables (40), we may shift ϕ_1 and its estimate (79) by $\frac{S_0}{2} + 1$ to the left (and redefine $S_1 \gg S_0$ noting that the universal constant S_0 was fixed in the previous step) recovering the desired form (46). Let us now establish the claims 1, 2, and 3. We start by noting that by the change of variables (57) in conjunction with the support condition (48) on $\hat{s}$ we have the following relation between the argument s' of ρ and the variable s'' of ϕ_1 in the representation (60)

$$\left\{ \begin{matrix} & s'' & \leq & \frac{S_1}{4} \\ \frac{S_1}{4} & \leq s'' & \leq & S_1 \\ S_1 & \leq s'' & \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} & s' & \leq & \frac{S_1}{2} \\ \frac{S_1}{4} & \leq s' & \leq & 2S_1 \\ S_1 & \leq s' & \end{matrix} \right\} . \tag{80}$$

Hence Claims 1 and 2 are immediate consequences of the defining properties of the cut-off function η. We now turn to Claim 3 and appeal to the representation (60) for ϕ_1, which we also use for $\widetilde{\phi}_1$ and thus obtain a representation of $\widetilde{\phi}_1 - \phi_1$ with $\widetilde{\rho}$ replaced by $(1-\eta)\widetilde{\rho}$. Clearly, all the terms bearing a pre-factor of $\frac{1}{s''^2}$ or smaller are of higher order in the range $s'' \geq \frac{S_1}{4}$. Likewise, all the terms where at least one of the derivative on the product

$(1-\eta)\widetilde{\rho}$ falls on the cut-off $1-\eta$, thereby producing a factor $\frac{1}{S_1}$, are of the desired order or smaller. We are left with the terms

$$\int_{-\infty}^{\infty} \hat{w}_0^2 (1-\eta) \left(-\frac{1}{(1+\hat{s})^2} \frac{d\widetilde{\rho}}{ds'} - \frac{1}{2} \frac{1}{(1+\hat{s})^3} \frac{d^2\widetilde{\rho}}{ds'^2} + \frac{1}{(1+\hat{s})^4} \frac{d^3\widetilde{\rho}}{ds'^3} + \frac{1}{2} \frac{1}{(1+\hat{s})^5} \frac{d^2\widetilde{\rho}}{ds'^4} \right) d\hat{s}.$$
(81)

In our range $\frac{S_1}{4} \leq s'' \leq S_1$ the integrand is supported in $\frac{S_1}{4} \leq s' \leq 2S_1$, cf. (80). For these arguments we have by choice (63) & (64) of the averaging function $\widetilde{\rho}(s') = 1 - \frac{1}{\frac{s'}{S_0}-1} \approx 1 - \frac{S_0}{s'}$. Hence also the terms (81) are at least of order $\frac{1}{S_1^2}$ and thus of higher order. This establishes Claim 3 and thus (79).

3   Proof of the main theorem

Proof of Theorem 1.
We start by combining Lemmas 2 and 3. We first note that by rearranging (34) in Lemma 2 and then adding $\int_{-1}^{0} \hat{\xi}_0 \, ds$ to both sides, while using $S_1 \gg 1$, we obtain by rearranging

$$-\int_{-1}^{S_1} \hat{\xi}_0 \, ds \leq S_1 \left(-\frac{1}{C_1} \int_{-1}^{0} \hat{\xi}_0 \, ds + 1 \right),$$

where we momentarily retain the value C_1 of the universal constant. We now add this to (35) in Lemma 3 in form of

$$-\int_{-S_2}^{-1} \hat{\xi}_0 \, ds \leq C_2 \left(\frac{1}{\epsilon} \int_{-1}^{0} \hat{\xi}_0 \, ds + \frac{1}{\epsilon} + \int_{-S_2}^{-S_2+1} |\hat{\xi}_0| \, ds + \epsilon \exp(5S_2) \right)$$

and adjust $\epsilon = \frac{C_1 C_2}{S_1}$ so that the pre-factor of the transition term $\int_{-1}^{0} \hat{\xi}_0 \, ds$ vanishes; this choice of ϵ is consistent with $\epsilon \leq 1$ because of $S_1 \gg 1$. We end up with

$$-\int_{-S_2}^{S_1} \hat{\xi}_0 \, ds \lesssim S_1 + \int_{-S_2}^{-S_2+1} |\hat{\xi}_0| \, ds + \frac{1}{S_1} \exp(5S_2),$$

to which we add $-\int_{-\infty}^{-S_2} \hat{\xi}_0 \, ds \leq \int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds$, obtaining

$$-\int_{-\infty}^{S_1} \hat{\xi}_0 \, ds \lesssim \int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds + S_1 + \frac{1}{S_1} \exp(5S_2),$$
(82)

where we replaced S_2 by $S_2 + 1$ (at the expense of changing the multiplicative constant hidden in $\lesssim$).

In order to provide ourselves with an additional parameter S_0 next to $S_1, S_2 \gg 1$ to optimize in, we now appeal to a scaling argument: The change of variables (27) that leaves the stability condition (26) invariant assumes the form of a shift

$$s = S_0 + \hat{s}, \quad \hat{\xi} = \exp(-3S_0)\hat{\hat{\xi}} \quad \text{and thus} \quad \hat{\xi}_0 = \exp(-3S_0)\hat{\hat{\xi}}_0.$$

on the level of the logarithmic variables (40). We apply (82), which only relied on the stability condition, to the rescaled variables $\hat{s}$, $\hat{\hat{\xi}}_0$ (with the upper integral boundaries $\hat{S}_1$ and $-\hat{S}_2$), and then revert to the original variables:

$$-\int_{-\infty}^{\hat{S}_1+S_0} \hat{\xi}_0 \, ds \lesssim \int_{-\infty}^{-\hat{S}_2+S_0} |\hat{\xi}_0| \, ds + \exp(-3S_0) \left(\hat{S}_1 + \frac{1}{\hat{S}_1} \exp(5\hat{S}_2) \right).$$

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This estimate holds provided $\hat{S}_1, \hat{S}_2 \gg 1$. In terms of the original integral boundaries $S_1 = \hat{S}_1 + S_0$ and $-S_2 = -\hat{S}_2 + S_0$ this reads

$$-\int_{-\infty}^{S_1} \hat{\xi}_0 \, ds \lesssim \int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds + \exp(-3S_0) \left(S_1 - S_0 + \frac{1}{S_1 - S_0} \exp(5(S_2 + S_0)) \right), \quad (83)$$

which is valid provided $S_1 - S_0 = \hat{S}_1 \gg 1$, $S_2 + S_0 = \hat{S}_2 \gg 1$, and $S_1 \leq \ln H$. Now is the moment to optimize in S_0 by choosing it such that the two last terms are balanced, which is achieved by $S_1 - S_0 = \exp(\frac{5}{2}(S_2 + S_0))$. In our regime $S_2 + S_0 \gg 1$ we have $\exp(\frac{5}{2}(S_2 + S_0)) \approx \exp(\frac{5}{2}(S_2 + S_0)) - (S_2 + S_0)$ so that the above choice means $S_1 + S_2 \approx \exp(\frac{5}{2}(S_2 + S_0))$, which implies $\exp(-3S_0) \approx \exp(3S_2)(S_1 + S_2)^{-\frac{6}{5}}$. Hence with our choice, the entire second term in (83) is $\approx \exp(3S_2)(S_1 + S_2)^{-\frac{1}{5}}$:

$$-\int_{-\infty}^{S_1} \hat{\xi}_0 \, ds \lesssim \int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds + \exp(3S_2)(S_1 + S_2)^{-\frac{1}{5}}. \quad (84)$$

This estimate is valid provided $S_1 \leq \ln H$ and $S_1 + S_2 \gg 1$, since the latter by $S_1 + S_2 \approx \exp(\frac{5}{2}(S_2 + S_0))$ ensures $S_2 + S_0 \gg 1$ and by $S_1 - S_0 = \exp(\frac{5}{2}(S_2 + S_0))$ then also $S_1 - S_0 \gg 1$.

We now make use of Lemma 1 and Lemma 4. Note that w. l. o. g. we may assume that our background profile τ satisfies on the level of its slope $\int_0^H \xi^2 \, dz \lesssim (\ln H)^{\frac{1}{15}}$ next to $\int_0^H \xi \, dz = -1$, since if such a profile would not exist, the statement of Theorem 1 would be trivially true. Hence we are in the position to apply Lemma 4. By Lemma 1 we have $-\int_{S_1}^{\ln H} \hat{\xi}_0 \, ds \lesssim \exp(-3S_1)$. Adding this to (84), we obtain

$$-\int_{-\infty}^{\ln H} \hat{\xi}_0 \, ds \lesssim \int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds + \exp(3S_2)(S_1 + S_2)^{-\frac{1}{5}} + \exp(-3S_1),$$

so that by (36) in Lemma 4 we get

$$1 \lesssim \int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds + \exp(3S_2)(S_1 + S_2)^{-\frac{1}{5}} + \exp(-3S_1).$$

Clearly, the optimal choice of S_1 is given by saturating the constraint in form of $S_1 = \ln H$, which by $H \gg 1$ turns into

$$1 \lesssim \int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds + \exp(3S_2)(\ln H)^{-\frac{1}{5}} \quad (85)$$

and holds provided $S_2 \geq 0$.

We finally connect this to the Nusselt number $\mathrm{Nu}_{\mathrm{BF}}$, which on the level of the slope ξ and the logarithmic variables turns into

$$\mathrm{Nu}_{\mathrm{BF}} \geq \int_0^H (\frac{d\tau}{dz})^2 \, dz = \int_0^H \xi^2 \, dz = \int_{-\infty}^{\ln H} \exp(-s) \hat{\xi}^2 \, ds. \quad (86)$$

This allows us to estimate the first r. h. s. term in (85): By definition (33) of the convolution $\hat{\xi}_0$ and the support property (31) of the kernel ϕ_0 we have

$$\int_{-\infty}^{-S_2} |\hat{\xi}_0| \, ds \lesssim \int_{-\infty}^{-S_2} |\hat{\xi}| \, ds \lesssim \exp(-\frac{1}{2}S_2) \left(\int_{-\infty}^{-S_2} \exp(-s) \hat{\xi}^2 \, ds \right)^{\frac{1}{2}}, \quad (87)$$

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where we used the Cauchy-Schwarz inequality in the last step. Inserting (86) into (87) and then into (85), we obtain

$$1 \lesssim \exp(-\frac{1}{2}S_2)\mathrm{Nu}_{\mathrm{BF}}^{\frac{1}{2}} + \exp(3S_2)(\ln H)^{-\frac{1}{5}}.$$

Finally choosing $S_2 \geq 0$ such that $\exp(3S_2)$ is a small multiple of $(\ln H)^{\frac{1}{5}}$ so that the last term can be absorbed into the l. h. s. and to the effect of $\exp(-\frac{1}{2}S_2) \sim (\ln H)^{-\frac{1}{30}}$, the above turns into the desired $1 \lesssim (\ln H)^{-\frac{1}{30}}\mathrm{Nu}_{\mathrm{BF}}^{\frac{1}{2}}$.

4 Proofs of lemmas

In this section we will give the proofs of the four lemmas stated in Subsection 2.2.

4.1 Approximate positivity in the bulk: proof of Lemma 1

Much of the effort of this construction will consist in designing the kernel ϕ_0 in such a way that it is both non-negative and compactly supported. Non-negativity of $\phi_0(s)$ and its fast decay for $s \downarrow -\infty$ and support in $s \leq 0$ will be crucial in Subsections 4.2 and 4.3, where we will work with the convolution (33). In order to infer non-negativity, we can no longer let $k \uparrow \infty$ in (26) (as in the proof of Proposition 1), since the two last terms would blow up. To quantify this qualitative observation, we restrict to $k > 0$ and introduce the change of variable

$$z = \frac{\hat{z}}{k}, \tag{88}$$

so that the stability condition (26) turns into

$$\int_0^{kH} \hat{\xi}\left(\frac{\hat{z}}{k}\right) w \left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 \bar{w}\, \frac{d\hat{z}}{\hat{z}}$$
$$+ k^3 \int_0^{kH} \left| \frac{d}{d\hat{z}}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w \right|^2 d\hat{z} + k^3 \int_0^{kH} \left| \left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w \right|^2 d\hat{z} \geq 0. \tag{89}$$

We shall restrict ourselves to k with $kH \geq 1$ and real functions $w(\hat{z})$ compactly supported in $\hat{z} \in (0, 1]$ so that the boundary conditions (25) are automatically satisfied. In particular, an integration by parts (based on $(-\frac{d^2}{dz^2}+1)^4 = \frac{d^8}{dz^8} - 4\frac{d^6}{dz^6} + 6\frac{d^4}{dz^4} - 4\frac{d^2}{dz^2} + 1$) in the two last terms of (89), yielding

$$\int_0^{kH} \left| \frac{d}{d\hat{z}}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w \right|^2 d\hat{z} + \int_0^{kH} \left| \left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w \right|^2 d\hat{z}$$
$$= \int_0^\infty \left[\left(\frac{d^5 w}{d\hat{z}^5}\right)^2 + 5\left(\frac{d^4 w}{d\hat{z}^4}\right)^2 + 10\left(\frac{d^3 w}{d\hat{z}^3}\right)^2 + 10\left(\frac{d^2 w}{d\hat{z}^2}\right)^2 + 5\left(\frac{dw}{d\hat{z}}\right)^2 + w^2 \right] d\hat{z}, \tag{90}$$

shows that there are no fortuitous cancellations: Provided the multiplier $\phi := w(-\frac{d^2}{d\hat{z}^2}+1)^2 w$ of $\hat{\xi}$ is non-negligible in the sense of $\int_0^\infty \phi\,\frac{d\hat{z}}{\hat{z}} = O(1)$, the two last terms of (89) are at least of $O(k^3)$. Hence we are forced to work with $k \ll 1$ and thus, as expected, the conclusion is only effective for $z = \frac{\hat{z}}{k} \gg 1$.

In anticipation of (33) we introduce the logarithmic variables

$$s = \ln \hat{z} \quad \text{and} \quad s' = -\ln k, \tag{91}$$

so that the first term in the stability condition (89) can be rewritten as follows

$$\int_0^{kH} \hat{\xi}\left(\frac{\hat{z}}{k}\right) w \left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 \bar{w} \, \frac{d\hat{z}}{\hat{z}} = \int_{-\infty}^{\infty} \hat{\xi}(s+s') w \left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 w \, ds \,.$$

In view of this and (90), the stability condition (89) turns into: For all $s' \leq \ln H$ we have

$$\int_{-\infty}^{\infty} \hat{\xi}(s+s') w \left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 w \, ds \geq -\exp(-3s') \int_0^1 \left[\left(\frac{d^5 w}{d\hat{z}^5}\right)^2 + \cdots + w^2\right] d\hat{z} \,. \quad (92)$$

Let us consider the l. h. s. of (92) in more detail. In order to derive a result of the type of (32), it would be convenient to have a smooth *compactly supported* w such that the multiplier $\phi = w(-\frac{d^2}{d\hat{z}^2} + 1)^2 w$ is *non-negative*. Although we don't have an argument, we believe that such a w does not exist. Instead, we will construct

- a family $\mathfrak{F}$ of smooth functions w supported in $\hat{z} \in (0, 1]$

- and a probability measure $\rho(dw)$ on $\mathfrak{F}$ which is invariant under the symmetry transformation $w \to \hat{w}$ defined through the change of variables $\hat{w}(\frac{1}{2} + z) = w(\frac{1}{2} - z)$

such that the convex combination of the multipliers

$$\phi_0(\hat{z}) := \int_{\mathfrak{F}} \phi(\hat{z}) \, \rho(dw) \,, \quad (93)$$

where

$$\phi := w \left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 w \,,$$

is non-negative, supported in $[\frac{1}{4}, \frac{3}{4}]$ (and non-trivial) — and thus satisfies (31) after normalization. Roughly speaking, the reason why this can be achieved is the following: For any (non-trivial) smooth, compactly supported w we have

- $\phi = w \left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 w$ is positive on average:

$$\int_0^1 \phi \, d\hat{z} = \int_0^1 \left[\left(\frac{d^2 w}{d\hat{z}^2}\right)^2 + 2 \left(\frac{dw}{d\hat{z}}\right)^2 + w^2\right] d\hat{z} \,.$$

- $\phi = w \frac{d^4 w}{dz^4} + \cdots + w^2$ is positive near the edge of the support of w (incidentally this would *not* be true for the positive *second* order operator $-\frac{d^2}{d\hat{z}^2} + 1$).

Before becoming much more specific let us address the error term stemming from the r. h. s. of (92) for our construction, that is

$$\int_{\mathcal{F}} \int_0^1 \left[\left(\frac{d^5 w}{d\hat{z}^5}\right)^2 + \cdots + w^2\right] d\hat{z} \, \rho(dw) \,. \quad (94)$$

The functions in our family $\mathfrak{F}$ will be of the form

$$w_{\ell, \hat{z}'}(\hat{z}) := \left(\sqrt{\ell}\right)^3 w_0 \left(\frac{\hat{z} - \hat{z}'}{\ell}\right) \,, \quad (95)$$

that is, translations and rescalings of a "mask" w_0. The mask w_0 is some compactly supported smooth function that we fix now, say

$$w_0(\hat{\hat{z}}) := \left\{ \begin{array}{cc} \frac{1}{\sqrt{C_0}} \exp\left(-\frac{1}{(1-\hat{\hat{z}}^2)^2}\right) & \text{for} \quad \hat{\hat{z}} \in (-1,1) \\ 0 & \text{else} \end{array} \right\}, \tag{96}$$

and the normalization constant C_0 chosen such that

$$\int \left(\frac{dw_0}{d\hat{\hat{z}}^2}\right)^2 d\hat{\hat{z}} = 1. \tag{97}$$

Provided

$$\ell \leq \frac{1}{8} \quad \text{and} \quad \hat{z}' \in \left(\frac{3}{8}, \frac{5}{8}\right), \tag{98}$$

then $w_{\ell,\hat{z}'}$ is, as desired, supported in $\hat{z} \in [\frac{1}{4}, \frac{3}{4}]$. If we choose the length-scale to be bounded away from zero, i. e.

$$\ell \geq \frac{1}{C}, \tag{99}$$

then the error term (94) is clearly finite, so that (32) follows from (92) via integration w. r. t. $\rho(dw)$.

It thus remains to construct a probability measure ρ in ℓ and $\hat{z}'$ with (98) & (99) such that (93) is non-negative (and non-trivial). Note that $w_{\ell,\hat{z}'}$ in (95) is scaled such that the corresponding multipliers satisfy

$$\phi_{\ell,\hat{z}'}(\hat{z}) = \left(\frac{1}{\ell} w_0 \frac{d^4}{d\hat{\hat{z}}^4} w_0 - 2\ell w_0 \frac{d^2}{d\hat{\hat{z}}^2} w_0 + \ell^3 w_0^2\right)\left(\frac{\hat{z} - \hat{z}'}{\ell}\right), \tag{100}$$

and w_0 is normalized in (97) in such a way that $\int w_0 \frac{d^4}{d\hat{\hat{z}}^4} w_0 \, d\hat{\hat{z}} = 1$. Hence for all $\hat{z}'$ we have convergence as $\ell \downarrow 0$

$$\phi_{\ell,\hat{z}'}(\hat{z}) \rightharpoonup \delta(\hat{z}' - \hat{z}) \quad \text{when tested against smooth functions of } \hat{z}'. \tag{101}$$

On the other hand we note the following: Writing $w_0 = \frac{1}{\sqrt{C_0}} \exp(I)$ with $I = -\frac{1}{(1-\hat{\hat{z}}^2)^2}$, we have

$$\frac{d^2 w_0}{d\hat{\hat{z}}^2} = \frac{1}{\sqrt{C_0}}\left[\left(\frac{dI}{d\hat{\hat{z}}}\right)^2 + \frac{d^2 I}{d\hat{\hat{z}}^2}\right]\exp(I),$$

$$\frac{d^4 w_0}{d\hat{\hat{z}}^4} = \frac{1}{\sqrt{C_0}}\left[\left(\frac{dI}{d\hat{\hat{z}}}\right)^4 + 6\left(\frac{dI}{d\hat{\hat{z}}}\right)^2 \frac{d^2 I}{d\hat{\hat{z}}^2} + 3\left(\frac{d^2 I}{d\hat{\hat{z}}^2}\right)^2 + 4\frac{dI}{d\hat{\hat{z}}}\frac{d^3 I}{d\hat{\hat{z}}^3} + \frac{d^4 I}{d\hat{\hat{z}}^4}\right]\exp(I).$$

Since near the edges $\{-1,1\}$ of the support $[-1,1]$ of w_0, i. e. for $1 - |\hat{\hat{z}}| \ll 1$, the term $\left(\frac{dI}{d\hat{\hat{z}}}\right)^4 \gg 1$ dominates the other terms thanks to the *quadratic* blow up of I near the edges, we have, according to (100)

$$\phi_{\ell,\hat{z}'}(\hat{z}) = \frac{1}{\ell} w_0 \frac{d^4}{d\hat{\hat{z}}^4} w_0 - 2\ell w_0 \frac{d^2}{d\hat{\hat{z}}^2} w_0 + \ell^3 w_0^2 \approx \frac{1}{C_0}\frac{1}{\ell}\left[\left(\frac{dI}{d\hat{\hat{z}}}\right)^4 \exp(2I)\right]\left(\frac{z - \hat{z}'}{l}\right).$$

Hence in particular for $\ell = \frac{1}{8}$ and $\hat{z}' = \frac{1}{2}$, $w_{\frac{1}{8},\frac{1}{2}}$ and thus $\phi_{\frac{1}{8},\frac{1}{2}}$ are supported in $[\frac{3}{8}, \frac{5}{8}]$, $\phi_{\frac{1}{8},\frac{1}{2}}$ is positive near the edges of the support (and thus bounded away from zero at some

small distance of the edges of the support), and trivially bounded away from $-\infty$ in the support. The universal constants $\delta_0 > 0$, $\delta_1 > 0$, and $0 < C_1 < \infty$ are to quantify this:

$$
\phi_{\frac{1}{8},\frac{1}{2}}
\left\{
\begin{array}{lll}
= 0 & \text{for} & \hat{z} \notin (\frac{3}{8}, \frac{5}{8}) \\
> 0 & \text{for} & \hat{z} \in (\frac{3}{8}, \frac{3}{8} + \delta_0] \cup [\frac{5}{8} - \delta_0, \frac{5}{8}) \\
> \delta_1 & \text{for} & \hat{z} \in [\frac{3}{8} + \delta_0, \frac{3}{8} + 3\delta_0] \cup [\frac{5}{8} - 3\delta_0, \frac{5}{8} - \delta_0] \\
> -C_1 & \text{for} & \hat{z} \in [\frac{3}{8} + 3\delta_0, \frac{5}{8} - 3\delta_0]
\end{array}
\right\} .
\tag{102}
$$

We now choose a universal smooth $\rho_0(\hat{z}')$ such that

$$
\rho_0
\left\{
\begin{array}{lll}
= 0 & \text{for} & \hat{z}' \notin (\frac{3}{8} + 2\delta_0, \frac{5}{8} - 2\delta_0) \\
\geq 0 & \text{for} & \hat{z}' \in [\frac{3}{8} + 2\delta_0, \frac{5}{8} - 2\delta_0] \\
= 2C_1 & \text{for} & \hat{z}' \in [\frac{3}{8} + 3\delta_0, \frac{5}{8} - 3\delta_0]
\end{array}
\right\} ,
\tag{103}
$$

and that is even w. r. t. $\hat{z}' = \frac{1}{2}$. Since ρ_0 is smooth in $\hat{z}'$ we have according to (101)

$$
\int_{-\infty}^{\infty} \phi_{\ell,\hat{z}'}(\hat{z}) \rho_0(\hat{z}') \, d\hat{z}' \;\to\; \rho_0(\hat{z}) \quad \text{uniformly in } \hat{z} \text{ as } \ell \downarrow 0 .
$$

In view of the properties (103), there thus exists (a possibly small) $\ell_0 > 0$ such that

$$
\int_{-\infty}^{\infty} \phi_{\ell_0,\hat{z}'}(\hat{z}) \rho_0(\hat{z}') \, d\hat{z}'
\left\{
\begin{array}{lll}
= 0 & \text{for} & \hat{z} \notin (\frac{3}{8} + \delta_0, \frac{5}{8} - \delta_0) \\
\geq -\delta_1 & \text{for} & \hat{z} \in [\frac{3}{8} + \delta_0, \frac{5}{8} - \delta_0] \\
\geq C_1 & \text{for} & \hat{z} \in [\frac{3}{8} + 3\delta_0, \frac{5}{8} - 3\delta_0]
\end{array}
\right\} .
\tag{104}
$$

In view of (102), the properties (104) are made to ensure that

$$
\phi_0(\hat{z}) \;:=\; \phi_{\frac{1}{8},\frac{1}{2}}(\hat{z}) + \int_{-\infty}^{\infty} \phi_{\ell_0,\hat{z}'}(\hat{z}) \rho_0(\hat{z}') \, d\hat{z}'
\left\{
\begin{array}{lll}
= 0 & \text{for} & \hat{z} \notin (\frac{3}{8}, \frac{5}{8}) \\
> 0 & \text{for} & \hat{z} \in (\frac{3}{8}, \frac{5}{8})
\end{array}
\right\}
$$

defines a ϕ_0 that is strictly positive in its support and that is of the form (93) (after a gratuitous normalization to obtain a probability measure).

An inspection of our construction shows that ϕ_0 satisfies the symmetry property in (31).

4.2 Approximate logarithmic growth: proof of Lemma 2

In this subsection, we return to the approximate logarithmic growth of τ worked out in case of the reduced stability condition in Section 2.1. Compared to Section 2.1, we have to work with the mollified version $\hat{\xi}_0$ of $\hat{\xi}$, cf. (33) in Lemma 1, since only for the former we have approximate positivity in the bulk according to Lemma 1. As stated in Lemma 2, we shall show that for $S_1 \gg 1$ we have

$$
\int_{-1}^{0} \hat{\xi}_0 \, ds \;\lesssim\; \frac{1}{S_1} \int_{0}^{S_1} \hat{\xi}_0 \, ds + 1 .
\tag{105}
$$

We start the proof recalling

- The starting point for Subsection 4.1, that is (92), which we rewrite as

$$\int_{-\infty}^{\infty} \hat{\xi}(s + s' + s'') w \left(-\frac{d2}{d\hat{z}^2} + 1 \right)^2 w \, ds$$

$$\geq - \exp(-3s' - 3s'') \int_0^1 \left(\left(\frac{d^5 w}{d\hat{z}^5} \right)^2 + \cdots + w^2 \right) d\hat{z}, \quad (106)$$

  for all $s' \leq \ln H$, $s'' \leq 0$ and all smooth w compactly supported in $\hat{z} \in (0, 1]$.

- The outcome of Subsection 4.1, that is (32):

$$\hat{\xi}_0(s') \ = \ \int_{-\infty}^{\infty} \hat{\xi}(s' + s'') \phi_0(s'') \, ds'' \ \gtrsim \ - \exp(-3s') \quad (107)$$

  for all $s' \leq \ln H$.

Since the kernel $\phi_0(s'')$ is non-negative and compactly supported in $s'' \in (-\infty, 0]$, we obtain by testing the inequality in (106) with $\phi_0(s'') \, ds''$:

$$\int_{-\infty}^{\infty} \hat{\xi}_0(s'') \phi(s'' - s') \, ds'' \ = \ \int_{-\infty}^{\infty} \hat{\xi}_0(s + s') \phi(s) \, ds \quad (108)$$

$$\gtrsim \ - \exp(-3s') \int_0^1 \left[\left(\frac{d^5 w}{d\hat{z}^5} \right)^2 + \cdots + w^2 \right] d\hat{z},$$

where we continue to use the abbreviation ϕ for the multiplier corresponding to the generic w: $\phi \ = \ w \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w$.

We recall that in terms of $w = \hat{z}^2 \hat{w}$ and $s = \ln \hat{z}$, the multiplier assumes the form $\phi = \hat{w} \left(\frac{d^4}{d\hat{s}^4} + 2\frac{d^3}{ds^3} - \frac{d^2}{ds^2} - 2\frac{d}{ds} \right) \hat{w}$, cf. (41). The structure of the argument is similar to the one for (30) in Section 2.1. We seek

- a family $\mathfrak{F} = \{w_{s'}\}_{s'}$ of smooth functions $w_{s'}$ parametrized by $s' \in \mathbb{R}$ and compactly supported in $\hat{z} \in (0, 1]$, that is $s = \ln z \in (-\infty, 0]$, and

- a probability measure $\rho(ds')$ supported in $s' \in (-\infty, \ln H]$,

such that the corresponding convex combination of multipliers shifted by s', i. e.

$$\phi_1(s'') \ := \ \int_{-\infty}^{\infty} \phi_{s'}(s'' - s') \, \rho(ds'), \quad \text{where} \quad \phi_{s'} := \hat{w}_{s'} \left(\frac{d^4}{ds^4} + 2\frac{d^3}{ds^3} - \frac{d^2}{ds^2} - 2\frac{d}{ds} \right) \hat{w}_{s'},$$

$$(109)$$

is estimated by above as follows

$$\phi_1(s'') \ \leq \ \left\{ \begin{array}{ll} -1 & \text{for} \quad -1 \leq s'' \leq 0 \\ \frac{C_1}{S_1} & \text{for} \quad 0 \leq s'' \leq S_1 \\ 0 & \text{else} \end{array} \right\}, \quad (110)$$

where $C_1(\leq S_1)$ is some universal constant, the value of which we want to remember momentarily, and estimated by below

$$\phi_1(s'') \ \gtrsim \ - \left\{ \begin{array}{ll} \exp(\frac{s''}{S_2}) & \text{for} \quad s'' \leq 0 \\ 1 & \text{for} \quad s'' \geq 0 \end{array} \right\}, \quad (111)$$

where the exponential rate $\frac{1}{S_2} \gg 1$ could be replaced by any rate larger than 3. Furthermore, we need that

$$\int_{-\infty}^{\infty} \exp(-3s') \int_0^1 \left[\left(\frac{d^5 w_{s'}}{d\hat{z}^5} \right)^2 + \cdots + w_{s'}^2 \right] d\hat{z} \, \rho(ds') \lesssim 1 . \tag{112}$$

For later use we note that in terms of $w_{s'} = \hat{z}^2 \hat{w}_{s'}$, (112) turns into

$$\int_{-\infty}^{\infty} \exp(-3s') \int_0^1 \left[\hat{z}^4 \left(\frac{d^5 \hat{w}_{s'}}{d\hat{z}^5} \right)^2 + \hat{z}^2 \left(\frac{d^4 \hat{w}_{s'}}{d\hat{z}^4} \right)^2 + \left(\frac{d^3 \hat{w}_{s'}}{d\hat{z}^3} \right)^2 + \cdots + \hat{w}_{s'}^2 \right] d\hat{z} \rho(ds') \lesssim 1 .$$

In terms of $s = \ln \hat{z}$, this means

$$\int_{-\infty}^{\infty} \exp(-3s') \int_0^1 \exp(-5s) \left[\left(\frac{d^5 \hat{w}_{s'}}{d\hat{z}^5} \right)^2 + \cdots + \hat{w}_{s'}^2 \right] ds \rho(ds') \lesssim 1 . \tag{113}$$

It is almost obvious how (110), (111) & (112) allow to pass from (108) to (105) by substituting w with $w_{s'}$ and integrating in $\rho(ds')$. We just need to show how (110) & (111) yield

$$\int_{-\infty}^{\infty} \hat{\xi}_0 \phi_1 \, ds'' \leq -\int_{-1}^0 \hat{\xi}_0 \, ds'' + \frac{C_1}{S_1} \int_0^{S_1} \hat{\xi}_0 \, ds'' + C .$$

Indeed, we write

$$\int_{-\infty}^{\infty} \hat{\xi}_0 \phi_1 \, ds'' + \int_{-1}^0 \hat{\xi}_0 \, ds'' - \frac{C_1}{S_1} \int_0^{S_1} \hat{\xi}_0 \, ds''$$

$$= \int_{-\infty}^{\infty} (-\hat{\xi}_0) \left(\left\{ \begin{array}{lll} -1 & \text{for} & -1 \leq s'' \leq 0 \\ \frac{C_1}{S_1} & \text{for} & 0 \leq s'' \leq S_1 \\ 0 & \text{else} \end{array} \right\} - \phi_1 \right) ds''$$

$$\overset{(110),(107)}{\lesssim} \int_{-\infty}^{\infty} \exp(-3s'') \left(\left\{ \begin{array}{lll} -1 & \text{for} & -1 \leq s'' \leq 0 \\ \frac{C_1}{S_1} & \text{for} & 0 \leq s'' \leq S_1 \\ 0 & \text{else} \end{array} \right\} - \phi_1 \right) ds''$$

$$\overset{(111)}{\lesssim} \int_{-\infty}^{\infty} \exp(-3s'') \left\{ \begin{array}{ll} \exp(\frac{s''}{S_2}) & \text{for} \quad s'' \leq 0 \\ 1 & \text{for} \quad s'' \geq 0 \end{array} \right\} ds'' \overset{S_2 < \frac{1}{3}}{\lesssim} 1 .$$

The proof of (110) is the same as the one for (46) in Section 2.1 until the point where we used an approximation argument to allow for the non-smooth choice (76) of $\hat{w}_{s'}(s)$. This approximation argument in $H^{2,\infty}$ was sufficient for ϕ_1, which in view of the representation (60) is continuous with respect to $H^{2,2}$. In our situation this approximation argument is not sufficient because we need to control the error term (113) which requires boundedness in $H^{5,2}$. We now summarize the main steps of the proof of (110), (111) and (113) leaving out the details that can be found in Section 2.1.

Construction of the family $\{w_{s'}\}_{s'}$:

- As in Section 2.1, we fix a smooth mask $\hat{w}_0$ such that

$$\hat{w}_0 \quad \text{is supported in } \hat{s} \in \left[-\frac{1}{2}, 0 \right] \quad \text{and nonvanishing in} \left(-\frac{1}{2}, 0 \right) , \tag{114}$$

and normalize it by $\int \hat{w}_0^2 \, d\hat{s} = 1$.

- As in Section 2.1, we introduce the change of variables

$$s = \lambda \hat{s} \quad \text{and} \quad \hat{w}_\lambda = \lambda^{-\frac{1}{2}} \hat{w}_0 \,, \tag{115}$$

and rewrite the corresponding multiplier as

$$\phi_\lambda = -\frac{2}{\lambda^2} \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}} - \frac{1}{\lambda^3} \hat{w}_0 \frac{d^2 \hat{w}_0}{d\hat{s}^2} + \frac{2}{\lambda^4} \hat{w}_0 \frac{d^3 \hat{w}_0}{d\hat{s}^3} + \frac{1}{\lambda^5} \hat{w}_0 \frac{d^4 \hat{w}_0}{d\hat{s}^4} \,, \tag{116}$$

a form that highlights the desired dominance of the term $-\frac{2}{\lambda^2} \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}} = -\frac{1}{\lambda^2} \frac{d\hat{w}_0^2}{d\hat{s}}$ for $\lambda \gg 1$ and (heuristically) suggests the shape of the probability measure $\rho(s')$ as increasing over scales of order 1 and eventually decreasing over scales of order S_1 (see (53)). We note for later reference that in these variables, (113) assumes the form

$$\int_{-\infty}^{\infty} \exp(-3s') \int_{-\infty}^{0} \exp(-5\lambda \hat{s}) \left[\frac{1}{\lambda^{10}} \left(\frac{d^5 \hat{w}_0}{d\hat{s}^5} \right)^2 + \cdots + \hat{w}_0^2 \right] d\hat{s} \rho(ds') \lesssim 1 \,,$$

and because of (114) for $\lambda \geq 1$ follows from

$$\int_{-\infty}^{\infty} \exp\left(-3s' + \frac{5}{2} \lambda \right) \int_{-\infty}^{\infty} \left(\left(\frac{d^5 \hat{w}_0}{d\hat{s}^5} \right)^2 + \cdots + \hat{w}_0^2 \right) d\hat{s} \rho(ds') \lesssim 1 \,. \tag{117}$$

- In oder to obtain $\phi_1(s'') \sim -1$ over an s''-interval of length of the order 1 followed by $\phi_1(s'') \lesssim \frac{1}{S_1}$, we choose as in Section 2.1

$$\lambda = s' \,, \tag{118}$$

meaning that λ is small in the foot regions and large in the plateau region (see argument after (53), leading to this choice). Eventually we will need to modify (118) for moderate and small s', cf. (131). With the choice of (118), (116) turns into

$$\begin{aligned} \phi_{s'}(s) &= -\frac{1}{(s')^2} \left(\frac{d\hat{w}_0^2}{d\hat{s}} \right) \left(\frac{s}{s'} \right) - \frac{1}{(s')^3} \left(\hat{w}_0 \frac{d^2 \hat{w}_0}{d\hat{s}^2} \right) \left(\frac{s}{s'} \right) \\ &+ \frac{2}{(s')^4} \left(\hat{w}_0 \frac{d^3 \hat{w}_0}{d\hat{s}^3} \right) \left(\frac{s}{s'} \right) + \frac{1}{(s')^5} \left(\hat{w}_0 \frac{d^4 \hat{w}_0}{d\hat{s}^4} \right) \left(\frac{s}{s'} \right) \,. \end{aligned} \tag{119}$$

- As in Section 2.1, replacing in (109) the integration over s' by the integration over the argument $\hat{s} = \frac{s}{\lambda}$ of $\hat{w}_0$ according to the nonlinear change of variable

$$\hat{s} \overset{(115)}{=} \frac{s'' - s'}{\lambda} = \frac{s'' - s'}{s'} = \frac{s''}{s'} - 1 \quad \Longleftrightarrow \quad s' = \frac{s''}{1 + \hat{s}} \,, \tag{120}$$

ϕ_1 can be written as

$$\begin{aligned} \phi_1(s'') &= -\int_{-\infty}^{\infty} \frac{1}{(1 + \hat{s})^2} \hat{w}_0^2 \frac{d\rho}{ds'} \, d\hat{s} - \frac{1}{(s'')^2} \int_{-\infty}^{\infty} (1 + \hat{s}) \hat{w}_0 \frac{d^2 \hat{w}_0}{d\hat{s}^2} \rho \, d\hat{s} \\ &+ \frac{2}{(s'')^3} \int_{-\infty}^{\infty} (1 + \hat{s})^2 \hat{w}_0 \frac{d^3 \hat{w}_0}{d\hat{s}^3} \rho \, d\hat{s} + \frac{1}{(s'')^4} \int_{-\infty}^{\infty} (1 + \hat{s})^3 \hat{w}_0 \frac{d^4 \hat{w}_0}{d\hat{s}^4} \rho \, d\hat{s} \,. \end{aligned} \tag{121}$$

Construction of the probability measure $\rho(s') \, ds'$ supported in $s' \in (-\infty, \ln H]$:

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- As in Section 2.1, we formulate an intermediate goal: Find a measure $0 \leq \widetilde{\rho}(s') \leq 1$ but not supported in $(-\infty, \ln H)$ such that $\widetilde{\phi}_1(s'') = \int \phi_{s'}(s'' - s')\widetilde{\rho}(s') \, ds'$ satisfies

$$
-\widetilde{\phi}_1(s'') \in \left\{ \begin{array}{lrcccc} [\frac{1}{C}\frac{1}{(s'')^2}, C\frac{1}{(s'')^2}] & \text{for} & S_0 & \leq & s'' & \\ (0, C] & \text{for} & \frac{1}{2}S_0 & < & s'' & \leq & S_0 \\ \{0\} & \text{for} & & & s'' & \leq & \frac{1}{2}S_0 \end{array} \right\}. \tag{122}
$$

- From the representation (121) and the assumption that ρ varies slowly, we learn that $\widetilde{\phi}_1$ is negative if $\frac{d\widetilde{\rho}}{ds'} \gg \frac{1}{(s')^2}$. In Section 2.1 this motivated the following Ansatz for $\widetilde{\rho}$ in the range $1 \ll s' \ll S_1$: we fix a smooth mask $\widetilde{\rho}_0(\hat{s}')$ such

$$
\widetilde{\rho}_0 = 0 \ \text{ for } \hat{s}' \leq 0, \quad \frac{d\widetilde{\rho}_0}{d\hat{s}'} > 0 \ \text{ for } 0 < \hat{s}' \leq 2, \quad \widetilde{\rho}_0 = 1 - \frac{1}{\hat{s}'} \ \text{ for } 2 \leq \hat{s}'. \tag{123}
$$

For $S_0 \gg 1$, consider the rescaled version

$$
\widetilde{\rho}(S_0(\hat{s}' + 1)) = \widetilde{\rho}_0(\hat{s}'). \tag{124}
$$

Eventually, for (117) and departing from the argument in Section 2.1, we will have to modify $\widetilde{\rho}$ for moderate and small s', cf. (129).

- Finally, this $\widetilde{\rho}$ does not decrease to zero on the large scales $s' \sim S_1$, which has to be done by cutting it off as in Section 2.1, cf. (62). This allows to pass from the intermediate goal (122) to its final version (110).

Exactly as in the proof of (62), we distinguish the regions of small, intermediate and large s'' (note that for $s'' \in (-\infty, \frac{S_0}{2}]$ all the integrals in (121) vanish because the supports of $\hat{w}_0$ and ρ do not intersect). In Section 2.1 we established

$$
\widetilde{\phi}_1 \sim -\frac{1}{S_0}\frac{1}{(s'')^2} \quad \text{uniformly in } s'' \geq 3S_0 \quad \text{for } S_0 \gg 1 \tag{125}
$$

and

$$
\widetilde{\phi}_1 \sim -\frac{1}{S_0} \quad \text{uniformly in } s'' \in \left[\frac{3}{4}S_0, 3S_0 \right] \quad \text{for } S_0 \gg 1. \tag{126}
$$

As we have seen in Section 2.1, in the range of small s'', i. e. $s'' \in \left(\frac{1}{2}S_0, \frac{3}{4}S_0 \right)$, the behavior of $\phi_{s'}$ near the left edge $-\frac{1}{2}$ of is dominated by the $\frac{1}{\lambda^5}\hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}$-term and thus automatically is *strictly positive*. In Section 2.1, we solved this problem by *giving up smoothness* of $\hat{w}_0$ near the left edge $-\frac{1}{2}$ of its support $[-\frac{1}{2}, 0]$ and eventually using an approximation argument in $H^{2,2}$. As discussed earlier, we cannot use this approximation in the present situation, since we need to keep the error term (117) under control. The remainder of this section is devoted to the way out to this dilemma and it consists of three steps (the first one is the same as in (75) and we report it just for the sake of clarity).

- In the first stage, we *give up smoothness* of $\hat{w}_0$ near the left edge $-\frac{1}{2}$ of its support $[-\frac{1}{2}, 0]$. In fact, as in Section 2.1, we shall first assume that $\hat{w}_0$ is of the specific form

$$
\hat{w}_0 = \frac{1}{2}\left(\hat{s} + \frac{1}{2} \right)^2 \quad \text{for } \hat{s} \in \left[-\frac{1}{2}, -\frac{1}{4} \right]. \tag{127}
$$

This means that $\hat{w}_0$ has a bounded but discontinuous second derivative. Our non-smooth Ansatz together with (116) implies

$$
\phi_{s'} = -\frac{1}{(s')^2}\left(\frac{s}{s'} + \frac{1}{2} \right)^3 - \frac{1}{2}\frac{1}{(s')^3}\left(\frac{s}{s'} + \frac{1}{2} \right)^2 < 0 \quad \text{for } s \in \left(-\frac{s'}{2}, -\frac{s'}{4} \right]. \tag{128}
$$

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As shown in Section 2.1, the corresponding $\widetilde{\phi}_1$ is, as desired, strictly negative on $s'' \in (\frac{S_0}{2}, \frac{3}{4}S_0]$ for all S_0 sufficiently large. We fix a sufficiently large but universal S_0 such that together with (125)& (126) we obtain

$$-\widetilde{\phi}_1(s'') \in \left\{ \begin{array}{lllll} [\frac{1}{C}\frac{1}{(s'')^2}, C\frac{1}{(s'')^2}] & \text{for} & S_0 & \leq & s'' \\ (0, C] & \text{for} & \frac{1}{2}S_0 & < & s'' & \leq & S_0 \\ \{0\} & \text{for} & & & s'' & \leq & \frac{1}{2}S_0 \end{array} \right\},$$

for some generic universal constant C.

- In the second stage, *we modify the definition (124)* of $\widetilde{\rho}(s')$ by adding a small-amplitude and fast-decaying (exponential) tail for $s' \downarrow -\infty$. More precisely, we make the Ansatz

$$\widetilde{\widetilde{\rho}} = \widetilde{\rho} + \varepsilon \delta\widetilde{\rho} \quad \text{with} \quad \delta\widetilde{\rho} := \exp\left(\frac{s'}{S_2}\right)\eta_0\left(\frac{s'}{S_0}\right), \tag{129}$$

where $\eta_0(\hat{s}')$ is the mask of a smooth cut-off function with

$$\eta_0 = 1 \quad \text{for } \hat{s}' \leq 2 \quad \text{and} \quad \eta_0 = 0 \quad \text{for } \hat{s}' \geq 3. \tag{130}$$

Here $0 < S_2 \ll 1$ is some small length-scale and $\varepsilon \ll 1$ is some small amplitude to be chosen below. Recall that S_0 is the universal constant fixed in the first stage. Since $\widetilde{\widetilde{\rho}}$ is no longer supported on $s' \in [S_0, \infty)$ but is positive on the entire line, we need to extend our definition of the function $\hat{w}_{s'}$ from $s' \geq S_0$ to all s'. In view of (115) we just have to extend the definition (118) of the rescaling parameter $\lambda(s')$ to

$$\lambda = \left\{ \begin{array}{lll} s' & \text{for} & s' \geq S_0 \\ S_0 & \text{for} & s' \leq S_0 \end{array} \right\}. \tag{131}$$

We will show that we can first choose a universal $0 < S_2 \ll 1$ and then a universal $0 < \varepsilon \ll 1$ such that we obtain for $\widetilde{\widetilde{\phi}}_1(s'') := \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s'' - s')\widetilde{\widetilde{\rho}}(s')\, ds'$ the following estimates

$$-\widetilde{\widetilde{\phi}}_1(s'') \in \left\{ \begin{array}{lllll} [\frac{1}{C}\frac{1}{(s'')^2}, C\frac{1}{(s'')^2}] & \text{for} & S_0 & \leq & s'' \\ [\frac{1}{C}, C] & \text{for} & \frac{1}{2}S_0 & < & s'' & \leq & S_0 \\ [\frac{1}{C}\exp(\frac{s''}{S_2}), C\exp(\frac{s''}{S_2})] & \text{for} & & & s'' & \leq & \frac{1}{2}S_0 \end{array} \right\}, \tag{132}$$

for some generic universal constant C. The gain with respect to $\widetilde{\phi}_1$ is that $\widetilde{\widetilde{\phi}}_1$ is strictly negative also for $s'' \leq \frac{1}{2}S_0$ which will allow us to pass to the third stage.

- In a third stage, *we smoothen out* $\hat{w}_0$: We define a sequence of smooth functions $\{\tilde{w}_0^\alpha\}_{\alpha\downarrow 0}$ which approximate $\hat{w}_0$ in such a way that the corresponding $\widetilde{\widetilde{\phi}}_1^\alpha$ still satisfies (132). This takes care of (117): Since for fixed $\alpha > 0$ to be chosen later, (117) with $\hat{w}_0$ replaced by $\hat{w}_0^\alpha$ reduces to

$$\int_{-\infty}^{\infty} \exp\left(-3s' + \frac{5}{2}\lambda(s')\right)\widetilde{\widetilde{\rho}}(s')\, ds' \lesssim 1.$$

For $s' \leq S_0$ this follows from $\widetilde{\widetilde{\rho}}(s') \overset{(124)\&(129)}{=} \varepsilon\exp\left(\frac{s'}{S_2}\right)$ and $\lambda(s') \overset{(131)}{=} S_0$ because of $S_2 \ll 1$. For $s' \geq S_0$, this follows from $\widetilde{\widetilde{\rho}} \lesssim 1$ and $\lambda(s') \overset{(131)}{=} s'$.

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We turn to the details for the second stage, i. e. the effect of the modification $\widetilde{\widetilde{\rho}}(s')$ of $\widetilde{\rho}(s')$. Consider the perturbation $\delta\widetilde{\phi}_1$ of the multiplier $\widetilde{\phi}_1$:

$$\delta\widetilde{\phi}_1(s'') := \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s'' - s')\delta\widetilde{\rho}(s')\, ds' \overset{(131)}{=} \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s'' - s') \exp\left(\frac{s'}{S_2}\right) \eta_0\left(\frac{s'}{S_0}\right) ds'. \tag{133}$$

In order to show that the unperturbed (122) upgrades to (132), it is enough to establish

$$-\delta\widetilde{\phi}_1(s'') \in \left\{ \begin{array}{llll} \{0\} & \text{for} & 3S_0 \leq s'' \\ [-C, C] & \text{for} & S_0 \leq s'' \leq 3S_0 \\ [\frac{1}{C}, C] & \text{for} & \frac{1}{2}S_0 \leq s'' \leq S_0 \\ [\frac{1}{C}\exp(\frac{s''}{S_2}), C\exp(\frac{s''}{S_2})] & \text{for} & s'' \leq \frac{1}{2}S_0 \end{array} \right\}, \tag{134}$$

for some sufficiently small *but fixed* S_2, where C denotes a universal constant. Indeed, choosing $\varepsilon \ll 1$, we see from $\widetilde{\widetilde{\phi}}_1 = \widetilde{\phi}_1 + \varepsilon\delta\widetilde{\phi}_1$ that (134) upgrades (122) to (132).

We start the argument of (134) with the range of large s'', i. e. $s'' \geq 3S_0$, and consider the integral $\delta\widetilde{\phi}_1(s'') = \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s'' - s') \exp(-\frac{s'}{S_2})\eta_0(\frac{s'}{S_0})\, ds'$. Because of our choice (130) of the cut-off η_0, the second factor $\exp(-\frac{s'}{S_2})\eta_0(\frac{s'}{S_0})$ is supported in $s' \in (-\infty, 3S_0]$. We note that in view of our choice (131) of the scaling factor λ, $\hat{w}_{s'}(s)$ and thus $\phi_{\lambda(s')}(s)$ are supported in $s \in [-\frac{1}{2}S_0, 0]$ for $s' \leq S_0$ and in $s \in [-\frac{1}{2}s', 0]$ for $s' \geq S_0$. Hence $(s', s'') \mapsto \phi_{\lambda(s')}(s'' - s')$ is supported in $s'' \in [s' - \frac{1}{2}S_0, s']$ for $s' \leq S_0$ and in $s'' \in [\frac{1}{2}s', s']$ for $s' \geq S_0$, or — equivalently — in $s' \in [s'', s'' + \frac{1}{2}S_0]$ for $s'' \leq \frac{S_0}{2}$ and in $s' \in [s'', 2s'']$ for $s'' \geq \frac{S_0}{2}$. Since $s'' \geq 3S_0$, we are in the latter case and $\phi_{\lambda(s')}(s'' - s')$ is supported in $s' \in [s'', 2s''] \subset [3S_0, \infty)$. Hence both factors $\exp(-\frac{s'}{S_2})\eta_0(\frac{s'}{S_0})$ and $\phi_{\lambda(s')}(s'' - s') = \phi_{s'}(s'' - s')$ have disjoint support in s' and thus the integral (133) in s' vanishes. This establishes the first line in (134).

We now turn to the very small s'', i. e. $s'' \leq \frac{S_0}{2}$ in (134). By the above, $s' \mapsto \phi_{\lambda(s')}(s'' - s')$ is supported in $s' \in [s'', s'' + \frac{S_0}{2}] \subset (-\infty, S_0]$; in this s'-range we have for the cut-off function $\eta_0(\frac{s'}{S_0}) = 1$, and $\phi_{\lambda(s')} = \phi_{S_0}$. Hence the definition (133) simplifies to

$$\frac{1}{\varepsilon}\delta\widetilde{\phi}_1(s'') = \int_{-\infty}^{\infty} \phi_{S_0}(s'' - s') \exp\left(\frac{s'}{S_2}\right) ds' = \exp\left(\frac{s''}{S_2}\right) \int_{-\infty}^{\infty} \phi_{S_0}(s) \exp\left(-\frac{s}{S_2}\right) ds. \tag{135}$$

We note that by (128) we have

$$\phi_{S_0} < 0 \text{ for } s \in \left(-\frac{S_0}{2}, -\frac{S_0}{4}\right) \quad \text{and supported in } s \in \left[-\frac{S_0}{2}, 0\right].$$

This allows us to use Laplace's method for $S_2 \ll 1$ in the integral in (135),

$$
\begin{aligned}
&-\int_{-\infty}^{\infty} \phi_{S_0}(s) \exp\left(-\frac{s}{S_2}\right) ds \\
\approx\ & -\int_{-\infty}^{-\frac{S_0}{4}} \phi_{S_0}(s) \exp\left(-\frac{s}{S_2}\right) ds \\
\overset{(128)}{=}\ & \int_{-\frac{S_0}{2}}^{-\frac{S_0}{4}} \left(\frac{1}{S_0^2} \left(\frac{s}{S_0} + \frac{1}{2} \right)^3 + \frac{1}{2S_0^3} \left(\frac{s}{S_0} + \frac{1}{2} \right)^2 \right) \exp\left(-\frac{s}{S_2}\right) ds \\
\approx\ & \int_{-\frac{S_0}{2}}^{\infty} \frac{1}{2S_0^3} \left(\frac{s}{S_0} + \frac{1}{2} \right)^2 \exp\left(-\frac{s}{S_2}\right) ds \\
=\ & \frac{1}{S_0^2} \int_{-\frac{1}{2}}^{\infty} \frac{1}{2} \left(\hat{s} + \frac{1}{2} \right)^2 \exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
=\ & \frac{S_2^3}{S_0^5} \exp\left(\frac{1}{2}\frac{S_0}{S_2}\right) .
\end{aligned}
$$

Plugging this into (135) yields as claimed in (134)

$$
-\delta\widetilde{\phi}_1(s'') \approx \frac{S_2^3}{S_0^5} \exp\left(\frac{1}{2}\frac{S_0}{S_2}\right) \exp\left(\frac{s''}{S_2}\right) \quad \text{uniformly in } s'' \leq \frac{S_0}{2} \text{ for } S_2 \ll 1 . \tag{136}
$$

We now treat the intermediary small values $\frac{S_0}{2} \leq s'' \leq S_0$ in (134). This time, the function $s' \mapsto \phi_{\lambda(s')}(s'' - s')$ is supported in $s' \in [s'', 2s''] \subset (-\infty, 2S_0]$, so that also in this s'-range we have for the cut-off function $\eta_0(\frac{s'}{S_0}) = 1$. Hence the representation simplifies to

$$
\delta\widetilde{\phi}_1(s'') = \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s'' - s') \exp\left(\frac{s'}{S_2}\right) ds' .
$$

On this integral, we can again use the Laplace's method for $S_2 \ll 1$: By (128) we have for the continuous function $(s', s'') \mapsto \phi_{\lambda(s')}(s'' - s')$

$$
\phi_{\lambda(s')}(s'' - s') \begin{cases} < & 0 \quad \text{for } s' \in (\frac{3}{2}s'', 2s'') \\ = & 0 \quad \text{for } s' \notin (s'', 2s'') \end{cases} .
$$

Hence we obtain as claimed in (134)

$$
\delta\widetilde{\phi}_1(s'') < 0 \quad \text{uniformly in } s'' \in \left[\frac{S_0}{2}, S_0\right] \text{ for } S_2 \ll 1 . \tag{137}
$$

We finally address the remaining intermediary range, that is, $S_0 \leq s'' \leq 3S_0$. We clearly have by continuity of $(s', s'') \mapsto \phi_{\lambda(s')}(s'' - s')$ and $\eta_0(\hat{s}')$ that

$$
\delta\widetilde{\phi}_1(s'') = \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s'' - s) \exp\left(\frac{s'}{S_2}\right) \eta_0\left(\frac{s'}{S_0}\right) ds' \tag{138}
$$

is uniformly bounded for $s'' \in [S_0, 3S_0]$. Estimate (134) now follows from (136), (137) & (138) for a choice of sufficiently small S_2.

We now turn to the details for the third stage. We approximate $\hat{w}_0$, which is non-smooth at the left edge of its support, cf. (127), by a sequence of smooth $\hat{w}_0^\alpha$ in such a

way that the corresponding $\phi_{\lambda(s')}$ and $\phi^\alpha_{\lambda(s')}$ are close in L^1. More precisely, we select a smooth function $F(w)$ with

$$F(w) \;=\; 0 \quad \text{for } w \le 0 \quad \text{and} \quad F(w) \;=\; w \quad \text{for } w \ge 1\,.$$

For a small parameter $0 < \alpha \ll 1$ we now define $\hat{w}_0^\alpha(\hat{s})$ via

$$\hat{w}_0^\alpha \;:=\; \alpha^2 F\left(\frac{\hat{w}_0}{\alpha^2}\right) \;\stackrel{(127)}{=}\; \alpha^2 F\left(\frac{(\hat{s}+\frac{1}{2})^2}{2\alpha^2}\right) \quad \text{for } \hat{s} \in \left[-\frac{1}{2}, -\frac{1}{4}\right]\,;$$

for $\hat{s} \notin [-\frac{1}{2}, -\frac{1}{4}]$, $\hat{w}_0^\alpha$ is set equal to $\hat{w}_0$. Clearly, the so defined $\hat{w}_0^\alpha$ is smooth on the whole line. We want to show that the convex combination of multipliers

$$\widetilde{\widetilde{\phi}}_1^\alpha(s'') = \int_{-\infty}^{\infty} \phi^\alpha_{\lambda(s')}(s'' - s')\widetilde{\widetilde{\rho}}(s')\,ds'\,,$$

still satisfies (132), that is

$$-\widetilde{\widetilde{\phi}}_1^\alpha(s'') \;\in\; \left\{ \begin{array}{lll} [\frac{1}{C}\frac{1}{(s'')^2}, C\frac{1}{(s'')^2}] & \text{for} & S_0 \le s'' \\ [\frac{1}{C}, C] & \text{for} & \frac{1}{2}S_0 < s'' \le S_0 \\ [\frac{1}{C}\exp(\frac{s''}{S_2}), C\exp(\frac{s''}{S_2})] & \text{for} & s'' \le \frac{1}{2}S_0 \end{array} \right\}\,, \qquad (139)$$

for some choice of $0 < \alpha \ll 1$ and a generic universal constant C. For this purpose we consider the difference of the combination of multipliers, that is, $\delta\widetilde{\widetilde{\phi}}_1^\alpha := \widetilde{\widetilde{\phi}}_1^\alpha - \widetilde{\widetilde{\phi}}_1$. In order to pass from (132) to (139), it is sufficient to establish

$$|\delta\widetilde{\widetilde{\phi}}_1^\alpha(s'')| \;\lesssim\; \alpha \left\{ \begin{array}{lll} \frac{1}{(s'')^2} & \text{for} & 3S_0 \le s''\,, \\ 1 & \text{for} & \frac{1}{2}S_0 \le s'' \le 3S_0\,, \\ \exp\left(\frac{s''}{S_2}\right) & \text{for} & s'' \le \frac{1}{2}S_0 \end{array} \right\}\,. \qquad (140)$$

To this aim, we first observe that

$$\left|\hat{w}_0^\alpha \frac{d^k\hat{w}_0^\alpha}{d\hat{s}^k} - \hat{w}_0\frac{d^k\hat{w}_0}{d\hat{s}^k}\right| \;\lesssim\; \alpha^{4-k} \qquad \text{with } k = 0, \cdots, 4\,. \qquad (141)$$

which follows from the fact that

all these differences are supported on the interval $\qquad \hat{s} \in \left[-\frac{1}{2}, -\frac{1}{2} + \sqrt{2}\alpha\right]\,, \quad (142)$

and that on this interval, the two terms forming the difference are by themselves of the claimed size.

We first treat the case of large s''-values in (139), that is, of $s'' \ge 3S_0$. In this case, $s' \mapsto \phi_{\lambda(s')}(s'' - s')$ and $s' \mapsto \phi^\alpha_{\lambda(s')}(s'' - s')$ are supported in $s' \in [s'', 2s'']$. In particular, $s' \ge S_0$ so that $\lambda \stackrel{(131)}{=} s'$. Hence (121) takes the form

$$\begin{aligned} \delta\widetilde{\widetilde{\phi}}_1^\alpha(s'') \;=\; & -\int_{-\infty}^{\infty} \frac{1}{(1+\hat{s})^2}((\hat{w}_0^\alpha)^2 - \hat{w}_0^2)\frac{d\widetilde{\widetilde{\rho}}}{ds'}\,d\hat{s} \\ & -\frac{1}{(s'')^2}\int_{-\infty}^{\infty} (1+\hat{s})\left(\hat{w}_0^\alpha\frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\ & +\frac{2}{(s'')^3}\int_{-\infty}^{\infty} (1+\hat{s})^2\left(\hat{w}_0^\alpha\frac{d^2\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\ & +\frac{1}{(s'')^4}\int_{-\infty}^{\infty} (1+\hat{s})^3\left(\hat{w}_0^\alpha\frac{d^2\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\widetilde{\widetilde{\rho}}\,d\hat{s}\,. \end{aligned}$$

In particular, we also have $s' \geq 3S_0$ so that $\overset{\approx}{\rho}(s') \overset{(129)}{=} \widetilde{\rho}(s') \overset{(123)}{=} 1 - \frac{1}{\frac{s'}{S_0}-1} = 1 - \frac{S_0}{s'-S_0} \leq 1$, and thus $\frac{d\rho}{ds'} = \frac{S_0}{(s'-S_0)^2} \leq \left(\frac{3}{2}\right)^2 S_0 \frac{1}{(s'')^2}$ since $s' \geq s'' \geq 3S_0$. Hence the above representation yields

$$
\begin{aligned}
|\delta\overset{\approx}{\phi}_1^\alpha(s'')| \;\lesssim\; & \frac{S_0}{(s'')^2} \int_{-\infty}^\infty \left|(\hat{w}_0^\alpha)^2 - \hat{w}_0^2\right| d\hat{s} \\
& + \frac{1}{(s'')^2} \int_{-\infty}^\infty (1+\hat{s}) \left|\hat{w}_0^\alpha \frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0 \frac{d^2\hat{w}_0}{d\hat{s}^2}\right| d\hat{s} \\
& + \frac{1}{(s'')^3} \int_{-\infty}^\infty (1+\hat{s})^2 \left|\hat{w}_0^\alpha \frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0 \frac{d^3\hat{w}_0}{d\hat{s}^3}\right| d\hat{s} \\
& + \frac{1}{(s'')^4} \int_{-\infty}^\infty (1+\hat{s})^3 \left|\hat{w}_0^\alpha \frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0 \frac{d^4\hat{w}_0}{d\hat{s}^4}\right| d\hat{s}\,.
\end{aligned}
$$

Using (142) and inserting the estimate (141) for $k = 0, 2, 3, 4$ we obtain, as claimed in (140),

$$
|\delta\overset{\approx}{\phi}_1^\alpha(s'')| \;\leq\; C\alpha \left(\frac{\alpha^4}{(s'')^2} + \frac{\alpha^2}{(s'')^2} + \frac{\alpha}{(s'')^3} + \frac{1}{(s'')^4}\right) \;\lesssim\; C\frac{\alpha}{(s'')^2} \quad \text{for } s'' \geq 3S_0. \tag{143}
$$

We now address the small s''-values, that is, $s'' \leq \frac{S_0}{2}$. In this case, $s' \mapsto \phi_{\lambda(s')}(s''-s')$ and $s' \mapsto \phi_{\lambda(s')}^\alpha(s''-s')$ are supported in $s' \in [s'', s'' + \frac{S_0}{2}]$. In particular, $s' \leq S_0$ so that $\lambda \overset{(131)}{=} S_0$. Hence by (116) and (109) we obtain the representation

$$
\begin{aligned}
\delta\overset{\approx}{\phi}_1^\alpha(s'') \;=\; & -\frac{2}{S_0^2} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0^\alpha \frac{d\hat{w}_0^\alpha}{d\hat{s}}\right) \overset{\approx}{\rho}\, d\hat{s} \\
& - \frac{1}{S_0^3} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0 \frac{d^2\hat{w}_0}{d\hat{s}^2}\right) \overset{\approx}{\rho}\, d\hat{s} \\
& + \frac{2}{S_0^4} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0 \frac{d^3\hat{w}_0}{d\hat{s}^3}\right) \overset{\approx}{\rho}\, d\hat{s} \\
& + \frac{1}{S_0^5} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0 \frac{d^4\hat{w}_0}{d\hat{s}^4}\right) \overset{\approx}{\rho}\, d\hat{s}\,.
\end{aligned}
$$

Moreover, $s' \leq S_0$ implies $\rho(s') = 0$ (cf. (123)&(124)), $\eta_0(\frac{s'}{S_0}) = 1$ and (cf. (130)) thus $\widetilde{\rho}(s') = \varepsilon\exp(\frac{s'}{S_2})$. In terms of $\hat{s}$ given by $s' = s'' - S_0\hat{s}$, this translates into $\widetilde{\rho}(s') = \varepsilon\exp(\frac{s''}{S_2})\exp(-\frac{S_0}{S_2}\hat{s})$. Hence the above representation specifies to

$$
\begin{aligned}
\delta\overset{\approx}{\phi}_1^\alpha(s'') \;=\; & -\varepsilon\frac{2\exp(\frac{s''}{S_2})}{S_0^2} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}}\right) \exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
& - \varepsilon\frac{\exp(\frac{s''}{S_2})}{S_0^3} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0 \frac{d^2\hat{w}_0}{d\hat{s}^2}\right) \exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
& + \varepsilon\frac{2\exp(\frac{s''}{S_2})}{S_0^4} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0 \frac{d^3\hat{w}_0}{d\hat{s}^3}\right) \exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
& + \varepsilon\frac{\exp(\frac{s''}{S_2})}{S_0^4} \int_{-\infty}^\infty \left(\hat{w}_0^\alpha \frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0 \frac{d^4\hat{w}_0}{d\hat{s}^4}\right) \exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s}\,.
\end{aligned}
$$

Inserting the estimate (141) for $k = 1, 2, 3, 4$ and using (142) we obtain as claimed in (140)

$$|\delta\widetilde{\widetilde{\phi}}_1^\alpha(s'')| \;\lesssim\; \alpha\varepsilon\exp\left(\frac{s''}{S_2}\right)(\alpha^3 + \alpha^2 + \alpha^1 + 1) \;\lesssim\; \alpha\exp\left(\frac{s''}{S_2}\right) \quad \text{for } s'' \le \frac{S_0}{2}. \quad (144)$$

We finally address the intermediate values of s'', that is, $\frac{S_0}{2} \le s'' \le 3S_0$. Splitting the ds'-integrals into $s' \in [S_0, \infty)$ and $s' \in (-\infty, S_0]$ in order to treat $\lambda \overset{(131)}{=} \max\{s', S_0\}$, we obtain

$$
\begin{aligned}
\delta\widetilde{\widetilde{\phi}}_1^\alpha(s'') \;=\; & -\frac{2}{s''}\int_{-\infty}^{\frac{s''}{S_0}-1}\left(\hat{w}_0^\alpha\frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0\frac{d\hat{w}_0}{d\hat{s}}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\
& -\frac{1}{(s'')^2}\int_{-\infty}^{\frac{s''}{S_0}-1}(1+\hat{s})\left(\hat{w}_0^\alpha\frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\
& +\frac{2}{(s'')^3}\int_{-\infty}^{\frac{s''}{S_0}-1}(1+\hat{s})^2\left(\hat{w}_0^\alpha\frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\
& +\frac{1}{(s'')^4}\int_{-\infty}^{\frac{s''}{S_0}-1}(1+\hat{s})^3\left(\hat{w}_0^\alpha\frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\
& -\frac{2}{S_0^2}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0\frac{d\hat{w}_0}{d\hat{s}}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\
& -\frac{1}{S_0^3}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\
& +\frac{2}{S_0^4}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\widetilde{\widetilde{\rho}}\,d\hat{s} \\
& +\frac{1}{S_0^5}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\widetilde{\widetilde{\rho}}\,d\hat{s}\,.
\end{aligned}
$$

Since $|\widetilde{\widetilde{\rho}}| \le 1$ and since $|\frac{1}{s''}| \le \frac{2}{S_0}$, we obtain from using (142) and inserting the estimate (141) for $k = 1, 2, 3, 4$:

$$|\delta\widetilde{\widetilde{\phi}}_1^\alpha(s'')| \;\lesssim\; \alpha(\alpha^3 + \alpha^2 + \alpha^1 + 1) \;\lesssim\; \alpha \quad \text{for } s'' \in \left[\frac{S_0}{2}, 2S_0\right]. \quad (145)$$

Now (143), (144) and (145) establish (140).

As in the proof of (110) in Section 2.1, in order to obtain (110) in the range $s'' \ge S_1$, we need to cut-off the measure $\widetilde{\rho}$ (defined in (123)&(124) and modified in (129)) in the range $\frac{S_1}{2} \le s' \le S_1$ so that

$$\phi_1(s'') = \int_{-\infty}^{\infty}\phi_{\lambda(s')}^\alpha(s'' - s')\widetilde{\rho}(s')\eta\left(\frac{s'}{S_1}\right)ds'$$

satisfies (110). In this region ($s' \ge \frac{S_1}{2}$ or $s'' \ge \frac{S_1}{4}$) the modification (129) of ρ and (131) of λ, are not effective. So we may directly quote the argument of Section 2.1 for the modification of $\widetilde{\rho}$ through η. Note that this argument is unaffected by having replaced $\hat{w}_0$ by $\hat{w}_0^\alpha$. This concludes the proof of (110) and (111).

4.3 Approximate positivity in the boundary layers: proof of Lemma 3

The approximate non-negativity of $\hat{\xi}_0$, cf. (33), is lost in the boundary layer $s \ll -1$, cf. (32). However, in this subsection we show that $\hat{\xi}_0$ cannot be too negative in the boundary layer *provided $\hat{\xi}_0$ is sufficiently small in the transition region $|s| \lesssim 1$*. We recall the statement of Lemma 3: For all $S_2 \gg 1$ and $\varepsilon \leq 1$ we have

$$-\int_{-S_2}^{-1} \hat{\xi}_0 ds \lesssim \frac{1}{\varepsilon} \int_{-1}^{0} \hat{\xi}_0 ds + \frac{1}{\varepsilon} + \int_{-S_2}^{-S_2+1} |\hat{\xi}_0| ds + \varepsilon \exp(5S_2) \,.$$

With the standard rescaling (cf. proof of Theorem 1)

$$s \rightsquigarrow s + S_0, \quad \hat{\xi} \rightsquigarrow \exp(-3S_0)\hat{\xi} \quad \text{and thus also} \quad \hat{\xi}_0 \rightsquigarrow \exp(-3S_0)\hat{\xi}_0 \,,$$

the above turns into

$$-\int_{-S_2-S_0}^{-S_0-1} \hat{\xi}_0 \, ds \lesssim \frac{1}{\varepsilon} \int_{-S_0-1}^{-S_0} \hat{\xi}_0 \, ds + \frac{1}{\varepsilon} \exp(3S_0) + \int_{-S_2-S_0}^{-S_2-S_0+1} |\hat{\xi}_0| \, ds + \varepsilon \exp(5S_2 + 3S_0) \,. \tag{146}$$

Hence it is enough to establish the latter for *some* S_0. In fact, we shall show that for *all* $S_0 \gg 1$ and $S_1 \gg S_0$

$$-\int_{-S_1}^{-S_0-1} \hat{\xi}_0 \, ds \lesssim \frac{1}{\varepsilon} \int_{-S_0-1}^{-S_0} \hat{\xi}_0 \, ds + \frac{1}{\varepsilon} \exp(5S_0) + \int_{-S_1}^{-S_1+1} |\hat{\xi}_0| \, ds + \varepsilon \exp(5S_1) \,, \tag{147}$$

where we write $S_1 = S_2 + S_0$. Indeed, fixing an order-one S_0 which is sufficiently large so that (147) is valid, we obtain (146). Multiplying both sides of (92) by $\phi_0(s')$ (see definition (31)) and integrating in $(-\infty, \infty)$, we deduce

$$\int_{-\infty}^{\infty} \hat{\xi}_0 \phi \, ds \geq -C \left\{ \int_0^1 \left[\frac{d}{d\hat{z}} \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w \right]^2 + \left[\left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w \right]^2 d\hat{z} \right\}, \tag{148}$$

for any smooth w, supported in $\hat{z} \in [0, 1]$ and satisfying the boundary conditions $w = \frac{dw}{d\hat{z}} = \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w = 0$ at $\hat{z} = 0$, where as before we use the abbreviation

$$\phi := w \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w$$

for the multiplier. This time, w will *not* be compactly supported in $\hat{z} \in (0, 1]$ (only in $[0, 1]$) so that the boundary conditions matters. Using the fact that the function $\hat{z} \sinh \hat{z}$ satisfies these boundary conditions, we enforce them for w by the Ansatz

$$w = (\hat{z} \sinh \hat{z})\hat{w} \quad \text{with} \quad \hat{w} = const \text{ for } \hat{z} \ll 1 \,. \tag{149}$$

As in the previous subsections, it is more telling to express (148) in terms of the s-variable $s = \ln \hat{z}$. Appealing to the representations

$$(-\partial_{\hat{z}}^2 + 1)^2 \hat{z} \sinh \hat{z}$$
$$= \hat{z}^{-2} \left(\hat{z}^{-1} \sinh \hat{z} \, (\partial_s - 2)(\partial_s - 1) + 4 \cosh \hat{z}(\partial_s - 1) + 4\hat{z} \sinh \hat{z} \right)(\partial_s + 1)\partial_s \tag{150}$$

and

$$\partial_{\hat{z}}(-\partial_{\hat{z}}^2 + 1)^2 \hat{z} \sinh \hat{z}$$
$$= \hat{z}^{-3}\Big(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 3)(\partial_s - 2)(\partial_s - 1) + 5\cosh\hat{z}(\partial_s - 2)(\partial_s - 1)$$
$$+ 8\hat{z}\sinh\hat{z}(\partial_s - 1) + 4\hat{z}^2\cosh\hat{z}\Big)(\partial_s + 1)\partial_s \tag{151}$$

(their proofs are reported in the Appendix to Section 4.3) we obtain

$$\int_{-\infty}^{\infty} \hat{\xi}_0 \phi \, ds \geq -C \int_{-\infty}^{\infty} \exp(-5s)\left[\left(\frac{d^5\hat{w}}{ds^5}\right)^2 + \cdots + \left(\frac{d\hat{w}}{ds}\right)^2\right] ds\,, \tag{152}$$

where according to the formula

$$\hat{z}\sinh\hat{z}\left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 \hat{z}\sinh\hat{z}$$
$$= \left(\frac{d}{ds} + 1\right)\frac{d}{ds}\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds} + 1\right)\frac{d}{ds} - 2\left(\frac{d}{ds} + 1\right)\frac{d}{ds}\,,$$

(the argument for the formula above is given at the end of this section, see (173)), the multiplier is given by

$$\phi = \hat{w}\left(\frac{d}{ds} + 1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds} + 1\right)\frac{d}{ds} - 2\right]\hat{w}\,. \tag{153}$$

We now make the following Ansatz for $\hat{w}$:

$$\hat{w} = \frac{1}{\sqrt{\varepsilon}}\hat{w}_0 + \sqrt{\varepsilon}\hat{w}_1\,, \tag{154}$$

with the constraints

$$\hat{w}_0 = \left\{\begin{array}{ll} 1 & \text{for} \quad s \leq -S_0 - 1 \\ 0 & \text{for} \quad s \geq -S_0 \end{array}\right\}, \quad \hat{w}_1 = \left\{\begin{array}{ll} const & \text{for} \quad s \leq -S_1 \\ 0 & \text{for} \quad s \geq -S_0 - 1 \end{array}\right\}, \tag{155}$$

so that (149) is satisfied. We don't want to specify the value of the constant appearing in the definition of w since it will not appear in the future estimates. The merit of the Ansatz (154) is that, because $\frac{d\hat{w}_0}{ds}$ and $\frac{d\hat{w}_1}{ds}$ have disjoint support, the multiplier ϕ, cf. (153), splits into three parts

$$\phi = \frac{1}{\varepsilon}\phi_0 + \phi_{01} + \varepsilon\phi_1\,, \tag{156}$$

where

$$\phi_0 := \hat{w}_0\left(\frac{d}{ds} + 1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds} + 1\right)\frac{d}{ds} - 2\right]\hat{w}_0,$$

$$\phi_{01} := \hat{w}_0\left(\frac{d}{ds} + 1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds} + 1\right)\frac{d}{ds} - 2\right]\hat{w}_1, \tag{157}$$

$$\phi_1 := \hat{w}_1\left(\frac{d}{ds} + 1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds} + 1\right)\frac{d}{ds} - 2\right]\hat{w}_1\,.$$

As a related side effect of the disjoint support of the functions $\frac{d\hat{w}_0}{ds}$ and $\frac{d\hat{w}_1}{ds}$, the error term in (152) splits into two parts:

$$\int_{-\infty}^{\infty} \exp(-5s) \left[\left(\frac{d^5\hat{w}}{ds^5} \right)^2 + \cdots + \left(\frac{d\hat{w}}{ds} \right)^2 \right] ds$$

$$= \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \exp(-5s) \left[\left(\frac{d^5\hat{w}_0}{ds^5} \right)^2 + \cdots + \left(\frac{d\hat{w}_0}{ds} \right)^2 \right] ds \tag{158}$$

$$+ \varepsilon \int_{-\infty}^{\infty} \exp(-5s) \left[\left(\frac{d^5\hat{w}_1}{ds^5} \right)^2 + \cdots + \left(\frac{d\hat{w}_1}{ds} \right)^2 \right] ds . \tag{159}$$

Hence in the sequel, we will have to consider five terms:

- three multiplier terms: $\frac{1}{\varepsilon} \int_{-\infty}^{\infty} \hat{\xi}_0 \phi_0 \, ds$, $\int_{-\infty}^{\infty} \hat{\xi}_0 \phi_{01} \, ds$, and $\varepsilon \int_{-\infty}^{\infty} \hat{\xi}_0 \phi_1 \, ds$,

- two error terms: the $\hat{w}_0$-error term (158) and the $\hat{w}_1$-error term (159).

Below, we will construct $\hat{w}_1$ such that the mixed expression ϕ_{01}, cf. (157), in the multiplier ϕ gives rise to the left-hand side of (147).

Before, we address the multiplier and the error term that only involve $\hat{w}_0$. Clearly, $\hat{w}_0$ can be chosen to satisfy S_0-independent bounds: $\sup_{s\in\mathbb{R}} |\hat{w}_0|, \; \cdots , \sup_{s\in\mathbb{R}} \left| \frac{d^5\hat{w}_0}{ds^5} \right| \lesssim 1$. Hence in view of (155), we obtain for the $\hat{w}_0$-error term (158)

$$\frac{1}{\varepsilon} \int_{-\infty}^{\infty} \exp(-5s) \left[\left(\frac{d^5\hat{w}_0}{ds^5} \right)^2 + \cdots + \left(\frac{d\hat{w}_0}{ds} \right)^2 \right] ds \lesssim \frac{1}{\varepsilon} \exp(5S_0). \tag{160}$$

Moreover, in view of (155), we obtain

$$|\phi_0| \leq \left\{ \begin{array}{llll} 0 & \text{for} & s & \leq -S_0 - 1 \\ C_0 & \text{for} & -S_0 - 1 \leq s & \leq -S_0 \\ 0 & \text{for} & -S_0 \leq s & \end{array} \right\}, \tag{161}$$

where we momentarily want to remember the value of the universal constant C_0. Since, by (32) in Lemma 1, there exists a specific constant C_1 such that $\int_{-S_0-1}^{-S_0} \hat{\xi}_0 \, ds + C_1 \exp(3S_0) \geq 0$, we obtain from (161)

$$\begin{aligned} \int_{-S_0-1}^{-S_0} \hat{\xi}_0(\phi_0 - C_0) \, ds &= \int_{-S_0-1}^{-S_0} (-\hat{\xi}_0)(-\phi_0 + C_0) \, ds \\ &\overset{(32)}{\leq} C_1 \int_{-S_0-1}^{-S_0} \exp(-3s)(-\phi_0 + C_0) \, ds \\ &\overset{(161)}{\leq} 2C_1 C_0 \int_{-S_0-1}^{-S_0} \exp(-3s) \, ds \\ &\leq C_1 C_0 \exp(3S_0) , \end{aligned} \tag{162}$$

so that for the ϕ_0-multiplier term we obtain

$$\begin{aligned} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \hat{\xi}_0 \phi_0 \, ds &\overset{(161)}{=} \frac{1}{\varepsilon} \int_{-S_0-1}^{-S_0} \hat{\xi}_0 \phi_0 \, ds \\ &\leq \frac{1}{\varepsilon} \left(C_0 \int_{-S_0-1}^{-S_0} \hat{\xi}_0 \, ds + C_1 C_0 \exp(3S_0) \right) \\ &\lesssim \frac{1}{\varepsilon} \left(\int_{-S_0-1}^{-S_0} \hat{\xi}_0 \, ds + C_1 \exp(3S_0) \right) . \end{aligned} \tag{163}$$

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We now specify $\hat{w}_1$ with the goal that ϕ_{01}, cf. (157), gives rise to the l. h. s. of (147). This motivates the construction of a universal function $\hat{w}_2$ with the property that

$$\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left[\left(\frac{\sinh \hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds}-2\right]\hat{w}_2 \;=\; 1 \quad \text{for } s \ll -1\,, \qquad (164)$$

which will be carried out below in such a way that

$$\frac{|\hat{w}_2|}{|s|+1},\; \left|\frac{d\hat{w}_2}{ds}\right|,\; \cdots \; \left|\frac{d^5\hat{w}_2}{ds^5}\right| \;\lesssim\; 1\,. \qquad (165)$$

Equipped with $\hat{w}_2$, we now make the Ansatz of blending $\hat{w}_2$ to $\hat{w}_2(-S_1)$ for $s < -S_1$ and to zero for $s < -S_0 - 1$:

$$\hat{w}_1(s) \;=\; \eta(s + S_1)\eta(-(s + S_0 + 1))\hat{w}_2(s) + (1 - \eta(s + S_1))\hat{w}_2(-S_1)\,, \qquad (166)$$

where η is a universal cut-off function with

$$\eta(s) = \left\{ \begin{array}{ll} 0 & \text{for } s \leq 0 \\ 1 & \text{for } s \geq 1 \end{array} \right\}\,, \qquad (167)$$

so that (155) is satisfied. The main merit of Ansatz (166) & (167) is that it makes use of (164) which by definition (157) yields

$$\phi_{01} = \left\{ \begin{array}{lll} 0 & \text{for} & s \leq -S_1 \\ 1 & \text{for} & -S_1 + 1 \leq s \leq -S_0 - 2 \\ 0 & \text{for} & -S_0 - 1 \leq s \end{array} \right\}\,. \qquad (168)$$

Furthermore, the estimates (165) turn into

$$|\phi_{01}|,\; \frac{|\hat{w}_1|}{S_1},\; \left|\frac{d\hat{w}_1}{ds}\right|,\; \cdots,\; \left|\frac{d^5\hat{w}_1}{ds^5}\right| \;\leq\; C_0\,. \qquad (169)$$

In particular, we obtain for the ϕ_{01}-multiplier term

$$C_0 \int_{-\infty}^{\infty} \hat{\xi}_0 \phi_{01}\, ds$$

$$\overset{(168)}{=} \quad C_0 \int_{-S_1}^{-S_0-1} \hat{\xi}_0\, ds + \int_{-S_1}^{-S_1+1} \hat{\xi}_0(\phi_{01} - C_0)\, ds + \int_{-S_0-2}^{-S_0-1} \hat{\xi}_0(\phi_{01} - C_0)\, ds$$

$$\overset{(169)}{\leq} \quad C_0 \int_{-S_1}^{-S_0-1} \hat{\xi}_0\, ds + 2C_0 \int_{-S_1}^{-S_1+1} |\hat{\xi}_0|\, ds + C_1 C_0 \exp(3(S_0 + 1))\,, \qquad (170)$$

where for $\int_{-S_0-2}^{-S_0-1} \hat{\xi}_0(\phi_{01} - 1)\, ds$, we have used the same argument as in (162).

Because of $\phi_1 = \hat{w}_1 \phi_{01}$ another consequence of (169) and (168) is

$$|\phi_1| \;\lesssim\; \left\{ \begin{array}{lll} 0 & \text{for} & s \leq -S_1 \\ S_1 & \text{for} & -S_1 \leq s \leq -S_0 - 1 \\ 0 & \text{for} & -S_0 - 1 \leq s \end{array} \right\}\,.$$

By the same argument that leads to (163), this implies for the ϕ_1-multiplier term

$$\varepsilon \int_{-\infty}^{\infty} \hat{\xi}_0 \phi_1\, ds \;\lesssim\; \varepsilon S_1 \left(\int_{-S_1}^{-S_0-1} \hat{\xi}_0\, ds + C_1 \exp(3(S_1 - 1)) \right)\,. \qquad (171)$$

We finally address the $\hat{w}_1$-error term (159): It follows from (155) and (169) that

$$\varepsilon \int_{-\infty}^{\infty} \exp(-5s) \left[\left(\frac{d^5 \hat{w}_1}{ds^5} \right)^2 + \cdots + \left(\frac{d\hat{w}_1}{ds} \right)^2 \right] ds \ \leq \ C\varepsilon \exp(5S_1). \qquad (172)$$

We now collect the five estimates (160), (163), (170), (171), and (172). Via (156) and (159) we obtain from (152) that

$$-C_0 \int_{-S_1}^{-S_0-1} \hat{\xi}_0 \, ds$$

$$\lesssim \ \frac{1}{\varepsilon} \exp(5S_0) + \frac{1}{\varepsilon} \left(\int_{-S_0-1}^{-S_0} \hat{\xi}_0 \, ds + C_1 \exp(3S_0) \right)$$

$$+ \ C_0 \int_{-S_1}^{-S_1+1} |\hat{\xi}_0| \, ds + C_0 C_1 \exp(3S_0)$$

$$+ \ \varepsilon S_1 \left(\int_{-S_1}^{-S_0-1} \hat{\xi}_0 \, ds + C_1 \exp(3S_1) \right) + \varepsilon \exp(5S_1) \, ,$$

where we recall that C_1 was chosen such that the terms in the parentheses are non-negative. Hence we may discard the term $\int_{-S_1}^{-S_0-1} \hat{\xi}_0 \, ds$ on the r. h. s. : If it is negative we may omit it; if it is positive, then the estimate comes for free. Dividing by C_0 we thus obtain

$$-\int_{-S_1}^{-S_0-1} \hat{\xi}_0 \, ds \ \lesssim \ \frac{1}{\varepsilon} \exp(5S_0) + \frac{1}{\varepsilon} \left(\int_{-S_0-1}^{-S_0} \hat{\xi}_0 \, ds + C_1 \exp(3S_0) \right)$$

$$+ \int_{-S_1}^{-S_1+1} |\hat{\xi}_0| \, ds + \exp(3S_0) + \varepsilon S_1 \exp(3S_1) + \varepsilon \exp(5S_1) \, .$$

which implies (147) because $\varepsilon \leq 1$ and $S_1 \gg S_0 \gg 1$.

We derive now the operator-valued formula

$$\hat{z} \sinh \hat{z} \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 \hat{z} \sinh \hat{z}$$

$$= \left(\frac{d}{ds} + 1 \right) \frac{d}{ds} \left(\frac{\sinh \hat{z}}{\hat{z}} \right)^2 \left(\frac{d}{ds} + 1 \right) \frac{d}{ds} - 2 \left(\frac{d}{ds} + 1 \right) \frac{d}{ds} \, , \qquad (173)$$

that is a non-homogeneous generalization of $\hat{z}^2 \frac{d^4}{d\hat{z}^4} \hat{z}^2 = (\frac{d}{ds}+2)(\frac{d}{ds}+1)\frac{d}{ds}(\frac{d}{ds}-1)$ (cf. (39)). The fairly simple structure of this formula is not a surprise: Since the functions $\sinh \hat{z}$ and $\hat{z} \sinh \hat{z}$ are in the kernel of $(-\frac{d^2}{d\hat{z}^2} + 1)^2$, the functions 1 and $\hat{z}^{-1}$ are in the kernel of $(-\frac{d^2}{d\hat{z}^2}+1)^2 \hat{z} \sinh \hat{z}$. In s coordinates, these functions are 1 and $\exp(-s)$, respectively. This explains the *right* factor $(\frac{d}{ds} + 1)\frac{d}{ds}$ on the r. h. s. of (173). On the other hand, the *adjoint* of the l. h. s. of (173) *w. r. t. to the measure* $\frac{d\hat{z}}{\hat{z}} = ds$ is given by $\hat{z}^2 \sinh \hat{z}(-\frac{d^2}{d\hat{z}^2}+1)^2 \sinh \hat{z}$ and thus has a kernel containing 1 and $\hat{z} = \exp(s)$. Hence the adjoint of the r. h. s. of (173) w. r. t. ds has to contain the right factor $(\frac{d}{ds} - 1)\frac{d}{ds}$, which means that the operator itself should contain the *left* factor $(\frac{d}{ds} + 1)\frac{d}{ds}$.

We claim that the formula (173) can be factorized into the two formulas

$$\left(\frac{d^2}{d\hat{z}^2} - 1 \right) \hat{z} \sinh \hat{z} \ = \ \left(\frac{\sinh \hat{z}}{\hat{z}} \frac{d}{ds} + 2 \cosh \hat{z} \right) \left(\frac{d}{ds} + 1 \right) , \qquad (174)$$

$$\hat{z} \sinh \hat{z} \left(\frac{d^2}{d\hat{z}^2} - 1 \right) \ = \ \frac{d}{ds} \left[\left(\frac{d}{ds} + 1 \right) \frac{\sinh \hat{z}}{\hat{z}} - 2 \cosh \hat{z} \right] . \qquad (175)$$

Indeed, the composition of (174) and (175) yields

$$
\hat{z}\sinh\hat{z}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2\hat{z}\sinh\hat{z}
$$

$$
=\ \frac{d}{ds}\left(\frac{d}{ds}+1\right)\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\frac{d}{ds}\left(\frac{d}{ds}+1\right)
$$
$$
-2\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\frac{d}{ds}\left(\frac{d}{ds}+1\right)+2\frac{d}{ds}\left(\frac{d}{ds}+1\right)\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\left(\frac{d}{ds}+1\right)
$$
$$
-4\frac{d}{ds}(\cosh\hat{z})^2\left(\frac{d}{ds}+1\right)
$$

$$
=\ \frac{d}{ds}\left(\frac{d}{ds}+1\right)\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\frac{d}{ds}\left(\frac{d}{ds}+1\right)
$$
$$
+2\frac{d}{ds}\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right)\left(\frac{d}{ds}+1\right)+2\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\left(\frac{d}{ds}+1\right)
$$
$$
-4\frac{d}{ds}(\cosh\hat{z})^2\left(\frac{d}{ds}+1\right)
$$

$$
=\ \frac{d}{ds}\left(\frac{d}{ds}+1\right)\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\frac{d}{ds}\left(\frac{d}{ds}+1\right)
$$
$$
+2\frac{d}{ds}\left[\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right)+\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}-2(\cosh\hat{z})^2\right]\left(\frac{d}{ds}+1\right),\qquad(176)
$$

where $\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right)$ denotes the multiplication with the s-derivative of the function $\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}$. This implies (173) since because of

$$
\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right)=\hat{z}\left(\frac{d}{d\hat{z}}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right)=(\sinh\hat{z})^2+(\cosh\hat{z})^2-\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}},
$$

the factor in the last term of (176) simplifies to

$$
\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right)+\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}-2(\cosh\hat{z})^2=(\sinh\hat{z})^2-(\cosh\hat{z})^2=-1.
$$

We now turn to the argument for (174) and (175). We first note that (174) and (175) reduce to

$$
\left(\frac{d^2}{d\hat{z}^2}-1\right)\hat{z}\exp\hat{z}\ =\ \left(\frac{\exp\hat{z}}{\hat{z}}\frac{d}{ds}+2\exp\hat{z}\right)\left(\frac{d}{ds}+1\right)\qquad(177)
$$
$$
=\ \left(\exp\hat{z}\frac{d}{d\hat{z}}+2\exp\hat{z}\right)\left(\hat{z}\frac{d}{d\hat{z}}+1\right)
$$
$$
=\ \exp\hat{z}\left(\frac{d}{d\hat{z}}+2\right)\frac{d}{d\hat{z}}\hat{z}\qquad\text{and}\qquad(178)
$$
$$
\hat{z}\exp\hat{z}\left(\frac{d^2}{d\hat{z}^2}-1\right)\ =\ \frac{d}{ds}\left[\left(\frac{d}{ds}+1\right)\frac{\exp\hat{z}}{\hat{z}}-2\exp\hat{z}\right]\qquad(179)
$$
$$
=\ \hat{z}\frac{d}{d\hat{z}}\left[\left(\hat{z}\frac{d}{d\hat{z}}+1\right)\frac{\exp\hat{z}}{\hat{z}}-2\exp\hat{z}\right]
$$
$$
=\ \hat{z}\frac{d}{d\hat{z}}\left(\frac{d}{d\hat{z}}-2\right)\exp\hat{z}.\qquad(180)
$$

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Indeed, replacing $\hat{z}$ by $-\hat{z}$ in (177), using the invariance of $\frac{d}{ds} = \hat{z}\frac{d}{d\hat{z}}$ under this change of variables, and adding both identities yields (174). Likewise, (179) yields (175). The identities (178) and (180) can easily be checked using the commutator relation $\frac{d}{d\hat{z}}\exp\hat{z} = \exp\hat{z}\left(\frac{d}{d\hat{z}}+1\right)$ on their left hand sides:

$$\left(\frac{d^2}{d\hat{z}^2}-1\right)\exp\hat{z} = \exp\hat{z}\left[\left(\frac{d}{d\hat{z}}+1\right)^2-1\right] = \exp\hat{z}\left(\frac{d}{d\hat{z}}+2\right)\frac{d}{d\hat{z}} \quad\text{and}$$

$$\exp\hat{z}\left(\frac{d^2}{d\hat{z}^2}-1\right) = \left[\left(\frac{d}{d\hat{z}}-1\right)^2-1\right]\exp\hat{z} = \frac{d}{d\hat{z}}\left(\frac{d}{d\hat{z}}-2\right)\exp\hat{z}\,.$$

This concludes the argument for (173).

We turn now to the construction of the function $\hat{w}_2$ with (164) and (165). We start by reducing (164) to a second-order problem with bounded right-hand side: It is enough to construct a universal smooth $\hat{v}_2$ with

$$\left[\frac{d}{ds}\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)-2\right]\hat{v}_2 = 1 \quad\text{for } s \leq -S_0 \tag{181}$$

and

$$|\hat{v}_2|,\ \left|\frac{d\hat{v}_2}{ds}\right|,\ \cdots,\ \left|\frac{d^4\hat{v}_2}{ds^4}\right| \lesssim 1 \quad\text{for all } s\,. \tag{182}$$

Indeed, consider the anti derivative $\hat{w}_2(s) := \int_0^s \hat{v}_2 ds'$. Since $\frac{d\hat{w}_2}{ds} = \hat{v}_2$, the estimates (182) turn into the estimates (165). Likewise (181) turns into (164) because of

$$\left(\frac{d}{ds}+1\right)\left[\frac{d}{ds}\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)-2\right]\frac{d}{ds} = \left(\frac{d}{ds}+1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds}-2\right].$$

We now extend (181) to a problem on the entire line with nearly constant coefficients. Note that the coefficient $\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2$ is an entire, even function in $\hat{z}$ with value 1 at $\hat{z} = 0$. Hence for every $S_0 \gg 1$, we may write

$$\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2 = 1-a \quad\text{for all } s \leq -S_0\,,$$

where

$$\sup_{s\in\mathbb{R}}|a|,\sup_{s\in\mathbb{R}}\left|\frac{da}{ds}\right|,\cdots,\sup_{s\in\mathbb{R}}\left|\frac{d^3a}{ds^3}\right| \lesssim \exp(-2S_0)\,. \tag{183}$$

Thus we construct a universal smooth $\hat{v}_2(s)$ with

$$\left[\frac{d}{ds}(1-a)\left(\frac{d}{ds}+1\right)-2\right]\hat{v}_2 = 1 \quad\text{for all } s \tag{184}$$

and

$$\sup_{s\in\mathbb{R}}|\hat{v}_2|,\sup_{s\in\mathbb{R}}\left|\frac{d\hat{v}_2}{ds}\right|,\cdots,\sup_{s\in\mathbb{R}}\left|\frac{d^4\hat{v}_2}{ds^4}\right| < \infty\,. \tag{185}$$

We finally reformulate (184) as a fixed point problem. Note that since $\frac{d}{ds}\left(\frac{d}{ds}+1\right)-2 = \left(\frac{d}{ds}-1\right)\left(\frac{d}{ds}+2\right)$, the bounded solution of $\left[\frac{d}{ds}\left(\frac{d}{ds}+1\right)-2\right]\hat{v} = \hat{f}$ for some bounded

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continuous $\hat{f}$ is given by

$$
\begin{aligned}
\hat{v}(s) &= -\int_{-\infty}^{s} \exp(2(s'-s)) \int_{s'}^{\infty} \exp(s'-s'') \hat{f}(s'') \, ds'' \, ds' \\
&= -\frac{1}{3} \int_{-\infty}^{\infty} \exp(3 \min\{s, s''\} - 2s - s'') \hat{f}(s'') \, ds'' \\
&= : (T\hat{f})(s),
\end{aligned} \tag{186}
$$

defining an operator T. From its above representation with the bounded and Lipschitz-continuous kernel $\exp(3 \min\{s, s''\} - 2s - s'')$ we read off that T is a bounded operator from C^0 (the space of bounded continuous functions endowed with the sup norm) into C^1 and by the solution property of T thus also into C^2. Note that (184) can be reformulated as

$$
\begin{aligned}
&\left[\frac{d}{ds} \left(\frac{d}{ds} + 1 \right) - 2 \right] \hat{v}_2 \\
&= 1 + \frac{d}{ds} a \left(\frac{d}{ds} + 1 \right) \hat{v}_2 \\
&= 1 + \left[\left(\frac{d^2}{ds^2} + \frac{d}{ds} \right) a - \frac{d}{ds} \frac{da}{ds} \right] \hat{v}_2 \\
&= 1 + \left[\left(\frac{d}{ds} \left(\frac{d}{ds} + 1 \right) - 2 \right) a - \frac{d}{ds} \frac{da}{ds} + 2a \right] \hat{v}_2.
\end{aligned} \tag{187}
$$

An application of the translation-invariant operator T (formally) yields

$$
\hat{v}_2 = T 1 + \left(a - \frac{d}{ds} T \frac{da}{ds} + 2T a \right) \hat{v}_2. \tag{188}
$$

We view this equation as a fixed-point equation for $\hat{v}_2$ in the Banach space C^0. As mentioned above, T and even the composition $\frac{d}{ds} T$ are bounded operators (in C^0). In view of (183), the multiplication with a and with $\frac{da}{ds}$ are operators with C^0-operator norm estimated by $C \exp(-2S_0)$. Hence for sufficiently large S_0, the operator $a - \frac{d}{ds} T \frac{da}{ds} + 2T a$ has norm strictly less than one. Thus the contraction mapping theorem ensures the existence of a solution of (188), that is, a C^2-solution $\hat{v}_2$ of (184) with $\sup_{s \in \mathbb{R}} |\hat{v}_2|$, $\sup_{s \in \mathbb{R}} |\frac{d\hat{v}_2}{ds}|$, $\sup_{s \in \mathbb{R}} |\frac{d^2 \hat{v}_2}{ds^2}| < \infty$. Finally, we obtain the rest of (185) from (183) by a booth-strap argument.

4.4 Proof of Lemma 4

Here we give the argument for (36). We note that by definition (33) of the convolution $\hat{\xi}_0$, the change of variables $s = \ln \hat{z}$ and $s' = -\ln k$ already used in (88) & (91), and by definition (40) of $\hat{\xi}$ we have

$$\int_{-\infty}^{\ln H} \hat{\xi}_0 \, ds' \overset{(33)}{=} \int_{-\infty}^{\ln H} \int_{-\infty}^{\infty} \hat{\xi}(s+s')\phi_0(s) \, ds \, ds'$$

$$\overset{(88)\&(91)}{=} \int_{\frac{1}{H}}^{\infty} \int_0^1 \hat{\xi}\left(\frac{\hat{z}}{k}\right) \phi_0(\hat{z}) \, \frac{d\hat{z}}{\hat{z}} \, \frac{dk}{k}$$

$$\overset{(40)}{=} \int_0^1 \int_{\frac{1}{H}}^{\infty} \xi\left(\frac{\hat{z}}{k}\right) \frac{dk}{k^2} \phi_0(\hat{z}) \, d\hat{z}$$

$$= \int_0^1 \int_0^{H\hat{z}} \xi(z) \frac{dz}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z}$$

$$= \int_0^H \xi(z) \int_{\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} \, dz \, .$$

In view of this identity and the up-down symmetry (i. e. the symmetry of the problem under $z \rightsquigarrow H - z$), (36) will follow if we show that $\int_0^H \xi \, dz = -1$ implies

$$\int_0^H \xi(z) \left(\int_{\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} + \int_{1-\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} \right) dz \lesssim -1 \, . \tag{189}$$

With the normalization (31) and our assumption $\int_0^H \xi \, dz = -1$, (189) will follow once we show

$$\int_0^H \xi(z) \left(1 - \int_{\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} - \int_{1-\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} \right) dz \gtrsim -\frac{(\ln H)^{\frac{1}{45}}}{H^{\frac{2}{3}}} \, , \tag{190}$$

for which we will use the second assumption on $\int_0^H \xi^2 \, dz$: Claim (190) clearly implies (189) in the regime of $H \gg 1$. Let us reformulate (190) as

$$\int_0^H \xi(z)\rho(z) \, dz \gtrsim -\frac{(\ln H)^{\frac{1}{45}}}{H^{\frac{2}{3}}} \, , \tag{191}$$

where we introduced

$$\rho(z) := \rho_0\left(\frac{z}{H}\right) \quad \text{with} \quad \rho_0(\hat{z}) := 1 - \int_{\hat{z}}^1 \frac{1}{\hat{z}'} \phi_0(\hat{z}') \, d\hat{z}' - \int_{1-\hat{z}}^1 \frac{1}{\hat{z}'} \phi_0(\hat{z}') \, d\hat{z}' \, . \tag{192}$$

The symmetry (31) of $\phi_0(\hat{z}) \geq 0$ implies that

$$\frac{d\rho_0}{d\hat{z}}(\hat{z}) = \left(\frac{1}{\hat{z}} - \frac{1}{1-\hat{z}} \right) \phi_0(\hat{z}) \begin{cases} \geq 0 & \text{for } \hat{z} \leq \frac{1}{2} \, , \\ \leq 0 & \text{for } \hat{z} \geq \frac{1}{2} \, , \end{cases}$$

so that using the normalization (31) we have

$$\rho_0 \geq 0 \, , \tag{193}$$

and

$$\rho_0 \leq 1 \, . \tag{194}$$

Hence (191) is yet another way of expressing approximate non-negativity of ξ, this time in and up-down symmetric way in the bulk.

The strategy to establish (190) is now to construct an even (but not necessary nonnegative) mollification kernel $\phi(z)$ of length-scale $\ell \ll H$ such that

$$(\xi * \phi)(z) \gtrsim -\frac{1}{\ell^4} \qquad \text{for } z \in (\ell, H - \ell), \tag{195}$$

$$\int_0^H (\phi * \rho - \rho)^2 \, dz \lesssim \frac{\ell^4}{H^3} \qquad \text{for } \ell \ll H. \tag{196}$$

We first argue how (195) and (196) imply (190). Indeed by the evenness of ϕ we have the representation

$$\int_{-\infty}^{\infty} \xi \rho \, dz = \int_{-\infty}^{\infty} (\xi * \phi) \rho \, dz - \int_{-\infty}^{\infty} \xi (\rho * \phi - \rho) \, dz \,,$$

from which, since $\rho \geq 0$ (cf. (193)), we get

$$\int_{-\infty}^{\infty} \xi \rho \, dz \geq \inf_{z \in \mathrm{supp}\rho} (\xi * \phi)(z) \int_{-\infty}^{\infty} \rho \, dz - \left(\int_{-\infty}^{\infty} \xi^2 dz \int_{-\infty}^{\infty} (\rho * \phi - \rho)^2 \, dz \right)^{\frac{1}{2}} \,.$$

We note that since $\phi_0(\hat{z})$ is supported in $[\frac{1}{4}, \frac{3}{4}]$, cf. Lemma 1, $\rho_0(\hat{z})$ is supported in the same interval. Hence ρ is supported in $[\frac{1}{4}H, \frac{3}{4}H]$. Hence we may apply (195) as soon as $\ell \leq \frac{H}{4}$. Using (195) and (196) together with our assumption that $\int \xi^2 \, dz \lesssim (\ln H)^{\frac{1}{15}}$ and $\int_0^H \rho \, dz \leq H$ (from (194)) we obtain the estimate

$$\int_{-\infty}^{\infty} \xi \rho \, dz \gtrsim -\frac{H}{\ell^4} - \left((\ln H)^{\frac{1}{15}} \frac{\ell^4}{H^3} \right)^{\frac{1}{2}} \,.$$

The balancing choice of $\ell = \left(\frac{H^5}{(\ln H)^{\frac{1}{15}}} \right)^{\frac{1}{12}}$ turns this estimate into (190).

We now turn to the construction of the mollification kernel ϕ. We select a (nonvanishing) smooth and even $w_0(\hat{z})$, compactly supported in $\hat{z} \in [-1, 1]$, and consider the corresponding multiplier

$$\phi_0 = w_0 \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w_0 \,.$$

Notice that $\int_0^H \phi_0 \, d\hat{z} = \int_{-\infty}^{\infty} \left(\left(\frac{d^2 w_0}{d\hat{z}^2} \right)^2 + \left(\frac{dw_0}{d\hat{z}} \right)^2 + w_0^2 \right) d\hat{z} > 0$, so that by changing w_0 by a multiplicative constant we may achieve

$$\int_{-\infty}^{\infty} \phi_0 \, d\hat{z} = 1 \,.$$

We change variables according to $z = \ell \hat{z}$ and rescale the mask ϕ_0 by ℓ so as to preserve its integral

$$\ell \phi(\ell \hat{z}) = \phi_0(\hat{z}) \,, \tag{197}$$

and note that also ϕ is a multiplier in the sense of

$$\phi = w \left(-\frac{d^2}{d\hat{z}^2} + \frac{1}{\ell^2} \right)^2 w \,, \tag{198}$$

provided w is the following rescaling of w_0:

$$\frac{1}{\ell^{\frac{3}{2}}} w(\ell \hat{z}) = w_0(\hat{z}) \,. \tag{199}$$

For any translation $z' \in (\ell, H - \ell)$, the translated test function $z \mapsto w(z - z')$ is compactly supported in $z \in (0, H)$ and we may thus apply the stability condition (26) with $k = \frac{1}{\ell}$. Because of (198), this yields (195):

$$\int_0^H \xi(z)\,\phi(z - z')\,dz$$

$$\geq \quad -\int_0^H \left[\ell^2 \left(\frac{d}{dz} \left(-\frac{d^2}{dz^2} + \frac{1}{\ell^2} \right)^2 w \right)^2 + \left(\left(-\frac{d^2}{dz^2} + \frac{1}{\ell^2} \right)^2 w \right)^2 \right] (z - z')\,dz$$

$$\stackrel{(199)}{=} \quad -\frac{1}{\ell^4} \int_{-1}^1 \left[\left(\frac{d}{d\hat{z}} \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w_0 \right)^2 + \left(\left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w_0 \right)^2 \right] d\hat{z} \sim -\frac{1}{\ell^4}\,.$$

We finally turn to (196). From the representation

$$(\rho * \phi - \rho)(z') \quad = \quad \int_{-\infty}^\infty (\rho(z' - z) - \rho(z'))\,\phi(z)\,dz$$

$$\stackrel{\phi \text{ is even}}{=} \quad \frac{1}{2} \int_{-\infty}^\infty (\rho(z' + z) + \rho(z' - z) - 2\rho(z'))\,\phi(z)\,dz\,,$$

we obtain the inequality

$$|(\rho * \phi - \rho)(z')| \quad \leq \quad \frac{1}{2} \sup \left| \frac{d^2\rho}{dz^2} \right| \int_{-\infty}^\infty z^2 |\phi(z)|\,dz$$

$$\stackrel{(192),(197)}{=} \quad \frac{1}{H^2} \sup \left| \frac{d^2\rho_0}{d\hat{z}^2} \right| \ell^2 \int_{-\infty}^\infty \hat{z}^2 |\phi_0(\hat{z})|\,d\hat{z}\,,$$

which yields (196) after integration in $z' \in [0, H]$.

5 Appendix

5.1 Appendix for Section 2.1

Here, we argue how to derive (59). Recall the change of variables (57) for $s' \longleftrightarrow \hat{s}$ with s'' as a fixed parameter. If $p, \tilde{p}$ denote generic polynomials of degree n, we have

$$\frac{1}{(s')^m} \frac{d^n}{d\hat{s}^n} \quad \stackrel{(57)}{=} \quad \frac{1}{(s'')^n} \frac{1}{(s')^{m-n}} (1 + \hat{s})^n \frac{d^n}{d\hat{s}^n}$$

$$= \quad \frac{1}{(s'')^n} \frac{1}{(s')^{m-n}} p\left((1 + \hat{s}) \frac{d}{d\hat{s}} \right)$$

$$\stackrel{(57)}{=} \quad \frac{1}{(s'')^n} \frac{1}{(s')^{m-n}} p\left(-s' \frac{d}{ds'} \right)$$

$$= \quad \frac{1}{(s'')^n} \tilde{p}\left(s' \frac{d}{ds'} \right) \frac{1}{(s')^{m-n}}$$

$$= \quad \frac{1}{(s'')^n} \sum_{k=0}^n a_n \frac{d^k}{ds'^k} \frac{1}{(s')^{m-n-k}}$$

$$\stackrel{(57)}{=} \quad \sum_{k=0}^n a_n \frac{1}{(s'')^{m-k}} \frac{d^k}{ds'^k} (1 + \hat{s})^{m-n-k}\,.$$

This shows that the desired relations exist in principle, it remains to determine the coefficients $a_0, \cdots, a_n$. We start with the case $m = n+1$ (which yields the shortest formula). To this purpose, we again use $(1+\hat{s})\frac{d}{d\hat{s}} = -s'\frac{d}{ds'}$, which we rewrite as $\frac{d}{d\hat{s}}(1+\hat{s}) = -(s')^2\frac{d}{ds'}\frac{1}{s'}$. The latter yields

$$\left(\frac{d}{d\hat{s}}(1+\hat{s})\right)^n = (-1)^n s'\left(s'\frac{d}{ds'}\right)^n \frac{1}{s'} \quad \text{for every } n \in \mathbb{N},$$

which implies inductively

$$\frac{d^n}{d\hat{s}^n}(1+\hat{s})^n = (-1)^n (s')^{1+n}\frac{d^n}{ds'^n}\frac{1}{s'} \quad \text{for every } n \in \mathbb{N},$$

and which we rewrite as (using again $s'' = s'(1+\hat{s})$)

$$\frac{1}{(s')^{n+1}}\frac{d^n}{d\hat{s}^n} = (-1)^n \frac{d^n}{ds'^n}\frac{1}{s'}\frac{1}{(1+\hat{s})^n} = (-1)^n \frac{1}{s''}\frac{d^n}{ds'^n}\frac{1}{(1+\hat{s})^{n-1}}. \tag{200}$$

In view of the first line on the r. h. s. of (58), we need the latter transformation formula for $n = 1, 2, 3, 4$. In view of the second line, we also need:

$$\frac{1}{(s')^4}\frac{d}{d\hat{s}} \overset{(200)}{=} -\frac{1}{s''}\frac{1}{(s')^2}\frac{d}{ds'}$$

$$= -\frac{1}{s''}\left(\frac{d}{ds'}+\frac{2}{s'}\right)\frac{1}{(s')^2}$$

$$= -\frac{1}{(s'')^3}\frac{d}{ds'}(1+\hat{s})^2 - \frac{2}{(s'')^4}(1+\hat{s})^3, \tag{201}$$

$$\frac{1}{(s')^5}\frac{d^2}{d\hat{s}^2} \overset{(200)}{=} \frac{1}{s''}\frac{1}{(s')^2}\frac{d^2}{ds'^2}\frac{1}{1+\hat{s}}$$

$$= \frac{1}{s''}\left(\frac{d^2}{ds'^2}+4\frac{d}{ds'}\frac{1}{s'}+6\frac{1}{(s')^2}\right)\frac{1}{(s')^2}\frac{1}{1+\hat{s}}$$

$$= \frac{1}{(s'')^3}\frac{d^2}{ds'^2}(1+\hat{s}) + \frac{4}{(s'')^4}\frac{d}{ds'}(1+\hat{s})^2 + \frac{6}{(s'')^5}(1+\hat{s})^3. \tag{202}$$

5.2 Appendix for Subsection 4.3

In this subsection we derive the formulas (150) and (151). The main step is to establish

$$\hat{z}^2(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z} \tag{203}$$

$$= \left(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 2)(\partial_s - 1) + 4\cosh\hat{z}(\partial_s - 1) + 4\hat{z}\sinh\hat{z}\right)(\partial_s + 1)\partial_s.$$

Let us give a motivation for formula (203): The factor $(\partial_s + 1)\partial_s$ has to be there since $\hat{z}^{-1} = e^{-s}$ and 1 are in the kernel of $(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z}$, which in turn follows from the fact that $\sinh\hat{z}$ and $\hat{z}\sinh\hat{z}$ are in the kernel of $\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1$. Note that for $\hat{z} \ll 1$,

$$\hat{z}^{-1}\sinh\hat{z} = 1 + O(\hat{z}^2), \quad \cosh\hat{z} = 1 + O(\hat{z}^2), \quad \hat{z}\sinh\hat{z} = O(\hat{z}^2),$$

so that for $\hat{z} \ll 1$, (203) collapses to the identity already used in (39)

$$\hat{z}^2\partial_{\hat{z}}^4\hat{z}^2 = (\partial_s + 2)(\partial_s + 1)\partial_s(\partial_s - 1). \tag{204}$$

This identity is easily seen to be true because both differential operators are of fourth order and are homogeneous of degree zero in $\hat{z}$, because the four functions $\hat{z}^{-2} = e^{-2s}$, $\hat{z}^{-1} = e^{-s}$, 1, and $\hat{z} = e^s$ are in the kernel of both differential operators, and because on $\hat{z}^2 = e^{2s}$, both operators give $4!\hat{z}^2 = 4!e^{2s}$.

Let us give the argument for (203). Because of the transformation properties under $\hat{z} \rightsquigarrow -\hat{z}$, it suffices to show

$$\hat{z}^2(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\exp(\hat{z})$$
$$= \left[\hat{z}^{-1}\exp(\hat{z})\,(\partial_s - 2)(\partial_s - 1) + 4\exp(\hat{z})(\partial_s - 1) + 4\hat{z}\exp(\hat{z})\right](\partial_s + 1)\partial_s\,,$$

which we rearrange as

$$\hat{z}^3(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\exp(\hat{z})$$
$$= \exp(\hat{z})\left[(\partial_s - 2)(\partial_s - 1) + 4\hat{z}(\partial_s - 1) + 4\hat{z}^2\right](\partial_s + 1)\partial_s\,. \tag{205}$$

We note that because of $\partial_{\hat{z}}\exp(\hat{z}) = \exp(\hat{z})(\partial_{\hat{z}} + 1)$, we have

$$(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\exp(\hat{z}) = \exp(\hat{z})\left[(\partial_{\hat{z}} + 1)^4 - 2(\partial_{\hat{z}} + 1)^2 + 1\right]$$
$$= \exp(\hat{z})(\partial_{\hat{z}}^4 + 4\partial_{\hat{z}}^3 + 4\partial_{\hat{z}}^2)\,,$$

so that

$$\hat{z}^2(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\exp(\hat{z}) = \exp(\hat{z})\left[\hat{z}^3\partial_{\hat{z}}^4\hat{z} + 4\hat{z}(\hat{z}^2\partial_{\hat{z}}^3\hat{z}) + 4\hat{z}^2(\hat{z}\partial_{\hat{z}}^2\hat{z})\right]\,.$$

Now (205) follows by inserting the formulas

$$\begin{aligned}
\hat{z}\partial_{\hat{z}}^2\hat{z} &= (\partial_s + 1)\partial_s, \\
\hat{z}^2\partial_{\hat{z}}^3\hat{z} &= (\partial_s + 1)\partial_s(\partial_s - 1), \\
\hat{z}^3\partial_{\hat{z}}^4\hat{z} &= (\partial_s + 1)\partial_s(\partial_s - 1)(\partial_s - 2)\,.
\end{aligned} \tag{206}$$

These formulas can easily seen to be true; let us address (206): Both sides are differential operators of order 4 that are homogeneous of degree 0 in $\hat{z}$; the kernel of both operators is spanned by the four functions $\hat{z}^{-1} = e^{-s}$, 1, $\hat{z} = e^s$, and $\hat{z}^2 = e^{2s}$; On $\hat{z}^3 = e^{3s}$, both operators yield $4!\hat{z}^3 = 4!e^{3s}$.

Formulas (150) and (151) easily follow from (203). Formula (150) is an immediate consequence of (203). Formula (151) follows from (150) using the identities $\partial_{\hat{z}} = \hat{z}^{-1}\partial_s$ and

$$\begin{aligned}
\partial_{\hat{z}}\left(\hat{z}^{-3}\sinh\hat{z}\right) &= \hat{z}^{-3}(\cosh\hat{z} - 3\hat{z}^{-1}\sinh\hat{z}), \\
\partial_{\hat{z}}\left(4\hat{z}^{-2}\cosh\hat{z}\right) &= \hat{z}^{-3}(4\hat{z}\sinh\hat{z} - 8\cosh\hat{z}), \\
\partial_{\hat{z}}\left(4\hat{z}^{-1}\sinh\hat{z}\right) &= \hat{z}^{-3}(4\hat{z}^2\cosh\hat{z} - 4\hat{z}\sinh\hat{z})\,,
\end{aligned}$$

which lead as desired to

$$\begin{aligned}
&\partial_{\hat{z}}(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z} \\
&= \hat{z}^{-3}\Big[(\hat{z}^{-1}\sinh\hat{z}\,((\partial_s - 2)(\partial_s - 1)\partial_s - 3(\partial_s - 2)(\partial_s - 1)) \\
&\quad + \cosh\hat{z}\,(4(\partial_s - 1)\partial_s + (\partial_s - 2)(\partial_s - 1) - 8(\partial_s - 1)) \\
&\quad + \hat{z}\sinh\hat{z}\,(4\partial_s + 4(\partial_s - 1) - 4) \quad + \hat{z}^2\cosh\hat{z}\,4\Big] \times (\partial_s + 1)\partial_s \\
&= \hat{z}^{-3}\Big(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 3)(\partial_s - 2)(\partial_s - 1) + 5\cosh\hat{z}(\partial_s - 2)(\partial_s - 1) \\
&\quad + 8\hat{z}\sinh\hat{z}(\partial_s - 1) + 4\hat{z}^2\cosh\hat{z}\Big)(\partial_s + 1)\partial_s\,. \tag{207}
\end{aligned}$$

5.3 Notations

The spatial vector:
$$x = (y, z) \in [0, L)^{d-1} \times [0, H],$$

where H denotes the height of the container and L is the lateral horizontal cell-size.

Vertical velocity component:
$$w := u \cdot e_z \qquad \text{where} \qquad u = u(y, z, t).$$

Background profile:
$$\tau : [0, H] \to \mathbb{R} \quad \text{such that} \quad \tau(0) = 1 \text{ and } \tau(H) = 1,$$

$$\tau = \tau(z), \qquad\qquad \xi := \frac{d\tau}{dz}.$$

Long-time and horizontal average:
$$\langle f \rangle := \limsup_{t_0 \uparrow \infty} \frac{1}{t_0} \int_0^{t_0} \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} f(t, y) dy dt.$$

Gradient:
$$\nabla f = \begin{pmatrix} \nabla_y \\ \partial_z \end{pmatrix} f.$$

Laplacian:
$$\Delta f = \Delta_y f + \partial_z^2 f.$$

Horizontal Fourier transform:
$$\mathcal{F}f(k, z) = \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} e^{-ik \cdot y} f(y, z) dy,$$

where $k \in \frac{2\pi}{L} \mathbb{Z}^{d-1}$ is the dual variable of y.

Real part of an imaginary number : Re stands for the real part of a complex number.

Complex conjugate : $\overline{\mathcal{F}w}$ and $\overline{\mathcal{F}\theta}$ are the complex conjugates of the (complex valued) functions $\mathcal{F}w$ and $\mathcal{F}\theta$.

Universal and specific constants: We call *universal constant* a constant C such that $0 < C < \infty$ and it only depends on d but not on H, on L and on the initial data. Throughout the paper $A \lesssim B$ means $A \leq CB$ with C a universal constant. Likewise a condition $A \ll B$ means that there exists a possibly large universal constant C such that $A \leq \frac{1}{C}B$. We indicate specific constants with $C_0, C_1, C_2, \cdots$.

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