Abcological anecdotes

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Preprint No. 2017-16
December 2017

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In June 1985 Joseph Oesterlé gave a lecture at the Max-Planck-Institut in Bonn (then the other side of the river). He discussed the conductor and discriminant of elliptic curves and a conjectural relationship between them due to Lucien Szpiro. He mentioned that for the particular curve defined by $y^2 = x(x-a)(x+b)$ with non-zero coprime rational integers $a \neq -b$ this amounted to an inequality $|abc| \leq C(\prod_{p|abc} p)^\kappa$, with c defined by $a+b+c=0$ and the product over primes p. Here C, κ are independent of a, b, c but I can no longer remember if this was just for some κ or for all $\kappa > 3$.

Anyway, one could clearly now forget about elliptic curves; and then if one is not interested in a precise value of κ one may as well estimate a, b, c separately using $\max\{|a|, |b|, |c|\}$. I recognized the subsequent inequality as a version of an analogue of a 1984 result of Richard Mason about polynomials (actually anticipated by Wilson Stothers). After the talk I rushed down the steps to the library and found his result, which (to highlight the analogy) can be stated in the exponential form

$$\max\{e^{\deg \mathcal{A}}, e^{\deg \mathcal{B}}, e^{\deg \mathcal{C}}\} \ \leq \ e^{-1} \prod_{\pi | \mathcal{ABC}} e$$

for all non-zero coprime $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in $\mathbf{C}[t]$, not all in $\mathbf{C}$, with $\mathcal{A} + \mathcal{B} + \mathcal{C} = 0$. Here the $\pi = t - \tau$ for τ in $\mathbf{C}$ are the normalized primes of $\mathbf{C}[t]$ and $e = \exp 1$ (by convention). Thus Mason has $\kappa = 1$, which was known to be best possible, and even a bit extra (also best possible). Converting back from $\mathbf{C}[t]$ to $\mathbf{Z}$, I followed standard practice by loosening up to any $\kappa > 1$ to accommodate archimedean valuations (and indeed it would be false with $\kappa = 1$, as is also believed for Klaus Roth's famous $|\alpha - r/s| \geq C^{-1} s^{-\kappa-1}$).

A couple of weeks later there occurred a Symposium on Analytic Number Theory in honour of Roth, and accordingly at a Problem Session I wrote on the blackboard the following:

Disprove (or prove) that for every $\epsilon > 0$ there exists $C(\epsilon)$ such that

$$\max\{|a|, |b|, |c|\} \ \leq \ C(\epsilon) \left(\prod_{p|abc} p \right)^{1+\epsilon}$$

for all coprime integers a, b, c with $a + b + c = 0$.

Of course I forgot then to say that a, b, c are all non-zero.

Since then, in connexion with the origin of *abc*, several authors have referred to the Symposium Proceedings. In fact these were available only to the participants and thus not generally accessible. By the publication of the present note I hope to regularize this situation (especially in view of the developments of the last few years).

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14th June 2017.

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