Short-Range Entangled Phases of Fermions

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ABSTRACT

This thesis is an attempt to understand the physics of short-range entangled phases of fermions through several related approaches. The first angle is topological quantum field theory. We discuss the classification of interacting fermionic short-range entangled phases by spin cobordism and give an algebraic characterization of unoriented equivariant bosonic topological quantum field theories in one spatial dimension. A second tool is tensor network representation. We develop the formalism of fermionic matrix product states and use it to derive the stacking group law for one dimensional symmetry-enriched fermionic short-range entangled phases. We also study its relationship with state sum constructions of topological quantum field theories and develop a state sum construction for pin-minus theories in one spatial dimension. The third approach is topological band theory. We classify free fermionic phases enriched by a unitary symmetry in any dimension and determine the map into the interacting classification.

PUBLISHED CONTENT AND CONTRIBUTIONS

Much of this thesis is adapted from articles that originally appeared in other forms. All are available as arXiv preprints; where appropriate, the DOI for the published version is included as a hyperlink from the journal citation. Chapter 1 is based on

Kapustin, A., R. Thorngren, A. Turzillo, and Z. Wang (2015). "Fermionic symmetry protected topological phases and cobordisms". In: *JHEP* 2015.12. DOI: 10.1007/JHEP12(2015)052. arXiv: 1406.7329.

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INTRODUCTION

Understanding the highly entangled low energy states of many-body systems is a core problem of modern condensed matter physics. Strong correlations and massive superpositions are defining characteristics of the zoo of physical systems beyond the Landau paradigm, from topological insulators and symmetry protected phases, to topological orders, to quantum spin liquids, and beyond. With experimental realization of these exotic phases of matter has come an interest in theoretical models that capture their universal behaviors as well as proposals for new phases based on arguments in group theory and topology. The projects of classification and characterization require novel techniques to handle the large number of degrees of freedom and lack of local order parameters.

In studies of many-body systems, one is typically interested in families (called "phases") of systems, defined by shared qualitative features. These features are taken to be those of the long distance theories obtained by renormalization group flow or, equivalently, those invariant under under appropriate deformations of the microscopic systems. This thesis will focus on many-body systems that are gapped in the thermodynamic limit; in this case, the appropriate deformations are those that preserve the gap. While many gapped phases – the topological orders – exhibit exotic behavior like topology-dependent ground state degeneracy, anyonic excitations, and long-range entanglement, there are others – the short-range entangled (SRE) phases – that do not yet are nonetheless distinct from the "trivial" phase containing the product state ground state.

A pair of gapped systems may be stacked to obtain another gapped system. This operation is commutative, is compatible with the phase equivalence relation, and has the trivial phase as its unit. According to a standard definition, SRE phases (also called "invertible phases") are those that are invertible with respect to this operation; hence, stacking defines the structure of an abelian group on the classification of SRE phases.

It is interesting to study gapped systems that have been enriched by symmetry. In this context, the phase equivalence is modified so that the gap as well as the symmetry is preserved along the deformation. A related concept is that of symmetry protected trivial (SPT) phases: those that become trivial upon forgetting symmetry. In contexts (i.e. dimensions, bosons/fermions) where there exist nontrivial SRE phases without symmetry enrichment (sometimes called invertible topological orders (iTO)), not all symmetry-enriched SRE phases are SPT phases.

A 2013 proposal (X. Chen, Gu, Z.-X. Liu, et al., 2013) argued that bosonic SPT phases are classified by group cohomology. Later, bosonic SRE phases *beyond group cohomology* were classified (Kapustin, 2014b). Fermionic SPT phases have also been studied with group supercohomology (Gu and Wen, 2014), and fermionic SRE phases are the subject of Chapter 1. In the simplest case where the enriching symmetry is unitary or a product of a unitary symmetry and time reversal symmetry and does not mix the fermion parity symmetry, the classification of symmetry-enriched SRE phases splits as a product of SPT phases and iTO and is completely understood. Much of the recent literature on symmetry-enriched SRE phases focuses on more exotic symmetry groups. We discuss the classification in one spatial dimension with generic symmetry group in Chapter 3.

This thesis discusses three approaches to the study of short-range entangled phases of fermions: effective field theory, tensor network representations, and topological band theory, each of which constitutes a deep, rapidly evolving field of study in its own right. The powerful concepts have been key factors in our understanding of many-body systems by elucidating the roles of topology and many highly entangled degrees of freedom in generating their exotic behaviors. Now we briefly introduce each of these concepts and outline the rest of the thesis.

Effective field theories for many-body systems. It is commonly believed that the long-distance physics of a many-body system is characterized by an effective quantum field theory. When the system is gapped, its universal behavior is thought to be encoded in a topological quantum field theory (TQFT). The topological partition function encodes the response of the physical system to gravitational probes and also the anomaly of the boundary theory.

This relationship has been fruitful, especially in the case of short-range entangled (SRE) systems. The conjectures of (Kapustin, Thorngren, et al., 2015), discussed in chapter 1, that SRE phases are classified by certain cobordism groups was supported by the observation that these groups also classify invertible TQFTs (Freed and Hopkins, 2016). In a similar vein, chapter 2 discusses an axiomatic characterization of two (spacetime) dimensional invertible equivariant TQFTs and finds that their classification agrees with the (twisted) group cohomology classification of bosonic symmetry protected topological phases (in particular, when there are anti-unitary symmetries) (Kapustin and Turzillo, 2017). The precise connection between non-invertible TQFTs and topological orders in two and three dimensions is an evolving story with conjectures still being made (Zhu, Lan, and Wen, 2018).

In one dimension, the conjectured relationship between gapped phases and TQFTs has been made into a theorem. It was found in (Kapustin, Turzillo, and You, 2017), discussed in 3, (see also (Shiozaki and Ryu, 2016)) that the canonical forms of renormalization group fixed point matrix product states (MPS), which realize all gapped bosonic phases, naturally appear in state sum formulations of unitary TQFTs; moreover, both MPS and state sums are equivalent to basic algebraic data: a separable algebra and its modules. The dictionary extends to symmetry protected phases and equivariant TQFTs. In the sequel (Kapustin, Turzillo, and You, 2018), it was shown that fixed point fMPS, which realize all gapped fermionic phases, are related to state sum formulations of spin-TQFTs, which are QFTs that are sensitive to a spin structure as well as the spacetime topology. Chapter 4, based on (Turzillo, 2018), develops a diagrammatic state sum construction for pin-TQFTs and discusses their relation to time-reversal-invariant gapped fermionic phases; the algebraic characterization discovered in (Turzillo and You, 2019) is recovered.

Tensor network representations of many-body systems. The language of tensor networks has proven indispensable in the study – both numerical and analytical – of many-body systems and their low energy states. In one dimension, the MPS ansatz provides an efficient representation of ground states of gapped, local Hamiltonians (Hastings, 2007), and certain generalizations are suspected to represent a large class of higher dimensional systems whose ground states satisfy an entanglement entropy area law.

An important application of MPS is the classification of one dimensional gapped phases of bosonic matter protected by a symmetry G (X. Chen, Gu, and Wen, 2011a; Schuch, Perez-García, and I. Cirac, 2010). When $G = \mathbb{Z}/2$ such phases are related to their fermionic counterparts by the Jordan-Wigner transformation, a fact that extends the usefulness of MPS to classifying short-range entangled (SRE) phases of fermions as well (Fidkowski and Kitaev, 2010; X. Chen, Gu, and Wen, 2011b). In a series of papers (Kapustin, Turzillo, and You, 2018; Turzillo and You, 2019), discussed in chapter 3, an intrinsically fermionic formalism, dubbed fMPS, is developed. It not only recovers and extends the known classification of fermionic SRE phases but also allows one to derive a group law for their stacking. This fermionic group law differs subtly from that of their bosonized duals. It is computationally tractable, and well-known results – like the $\mathbb{Z}/8$ group law for time-reversal-invariant Majorana chains subject to arbitrarily strong interactions – are recovered as examples. The application of intrinsically fermionic tensor networks to symmetry enriched topological phases in higher dimensions has since been explored (Bultinck et al., 2017b).

The complement to the classification problem is the characterization problem, and the formalism of tensor networks is also applicable here. In the paper (Turzillo and You, 2019), discussed in chapter 3 the invariants (α, β, γ) of one dimensional fermionic SRE phases are interpreted in terms of bulk physical properties like the charges of twisted sectors and amplitudes for fusing domain walls; the three invariants were previously understood to describe the symmetry action on boundary modes. More generally, it is expected that the ground states of SRE phases are characterized by the values of certain non-local order parameters. In many cases, these were recently described (Shiozaki, Shapourian, et al., 2017).

Topological band theory. Some of the most familiar exotic quantum phases, such as topological insulators, may be realized in systems of free (quadratic) fermions, described by band Hamiltonians. In the presence of discrete translation symmetry, the ground state physics of these systems is encoded in the bundle of Bloch wavefunctions over the Brillouin zone. Deformation classes of these systems were shown by Kitaev and others to have a K-theoretic classification (Kitaev, 2009a; Schnyder et al., n.d.). This approach was extended to a classification of free fermionic phases with symmetries, both on-site and crystallographic, by equivariant K-theory.

It is natural to ask how this free classification is related to the (typically more complicated) classification of interacting phases. Some free phases are known to be destabilized by interactions: as a famous example, the $\mathbb{Z}$ classification of free class BDI phases in one dimension collapses to a $\mathbb{Z}/8$ when one allows deformations through a strongly interaction region of parameter space (Fidkowski and Kitaev, 2009). On the other hand, the cobordism classification predicts the existence of intrinsically interacting phases, with no free description, in high dimensions (Kapustin, Thorngren, et al., 2015). In chapter 5, based on (Y.-A. Chen et al., 2018), we describe the map from free to interacting phases with a unitary on-site symmetry and discuss both phenomena. The interacting invariants – like the aforementioned (α, β, γ) in one dimension – appear as characteristic classes of the Bloch bundle of the free theory. A surprising finding is that intrinsically interacting symmetric phases exist in dimensions as low as zero.

Chapter 1

FERMIONIC SYMMETRY PROTECTED TOPOLOGICAL PHASES AND COBORDISMS

Kapustin, A., R. Thorngren, A. Turzillo, and Z. Wang (2015). "Fermionic symmetry protected topological phases and cobordisms". In: *JHEP* 2015.12. DOI: `10.1007/JHEP12(2015)052`. arXiv: `1406.7329`.

Forward

It has been proposed that interacting Symmetry Protected Topological Phases can be classified using cobordism theory. This chapter tests this proposal in the case of Fermionic SPT phases with $\mathbb{Z}_2$ symmetry, where $\mathbb{Z}_2$ is either time-reversal or an internal symmetry. We find that cobordism classification correctly describes all known Fermionic SPT phases in space dimension $D \leq 3$ and also predicts that all such phases can be realized by free fermions. In higher dimensions we predict the existence of inherently interacting fermionic SPT phases.

Background and Overview

Classification of Symmetry Protected Topological Phases has been a subject of intensive activity over the last few years. In the case of free fermions, a complete classification has been achieved in (Ryu et al., 2010; Kitaev, 2009b) using such ideas as Anderson localization and K-theory. In the case of bosonic systems, all SPT phases are intrinsically interacting, so one has to use entirely different methods. Interactions are also known to affect fermionic SPT phases (Fidkowski and Kitaev, 2010; Fidkowski, X. Chen, and Vishwanath, 2013; C. Wang and Senthil, 2014; Gu and Levin, 2014). Recently it has been proposed that cobordism theory can provide a complete classification of both bosonic and fermionic interacting SPT phases in all dimensions. This improves on the previous proposal that group cohomology classifies interacting bosonic SPT phases (X. Chen, Gu, Z.-X. Liu, et al., 2013), while "group supercohomology" (Gu and Wen, 2014) classifies interacting fermionic SPT phases. For bosonic systems with time-reversal and $U(1)$ symmetries the cobordism proposal has been tested in (Kapustin, 2014b) and (Kapustin, 2014a) respectively. Cobordism theory has been found to describe all known bosonic SPT phases with such symmetries in $D \leq 3$. In this paper we test the proposal further by studying fermionic SPT phases with $\mathbb{Z}_2$ symmetry.

The $\mathbb{Z}_2$ symmetry in question can be either unitary or anti-unitary. In the former case we will assume that the symmetry is internal (does not act on space-time). In the latter case it must reverse the direction of time, so we will call it time-reversal symmetry. In either case, the generator can square either to 1 or to $(-1)^F$ (fermion parity). Fermionic SPT phases with time-reversal symmetry are also known as topological superconductors, so in particular we describe a classification scheme for interacting topological superconductors.

Compared to the bosonic case, fermionic SPT phases present several related difficulties. First of all, one needs to decide what one means by a fermionic system. In a continuum Lorentz-invariant field theory, anti-commuting fields are also spinors with respect to the Lorentz group, but condensed matter systems are usually defined on a lattice and lack Lorentz invariance on the microscopic level. Thus the connection between spin and statistics need not hold. A related issue is that all fermionic systems have $\mathbb{Z}_2$ symmetry called fermionic parity, usually denoted $(-1)^F$. But all observables, including the Hamiltonian and the action, are bosonic, i.e. invariant under $(-1)^F$. In a sense, every fermionic system has a $\mathbb{Z}_2$ gauge symmetry, which means that the partition function must depend on a choice of a background $\mathbb{Z}_2$ gauge field. It is tempting to identify this gauge field with the spin structure. However, it is not clear how a spin structure should be defined for a lattice system, except in the case of toroidal geometry.[1]

Instead of dealing with all these difficult questions, in this paper we take a more "phenomenological" approach: we make a few assumptions about the long-distance behavior of SPT phases which parallel those for bosonic SPT phases, and then test these assumptions by comparing the results in space-time dimensions $d \leq 4$ with those available in the condensed matter literature. For various reasons, we limit our selves to the cases of no symmetry, time-reversal symmetry, and unitary $\mathbb{Z}_2$ symmetry. Having found agreement with the known results, we make a conjecture about the classification of fermionic SPT phases with any symmetry group G.

1.1 Spin and Pin structures

A smooth oriented d-manifold M equipped with a Riemannian metric is said to have a spin structure if the transition functions for the tangent bundle, which take values

[1]In 2d, there is a good combinatorial description of spin structures via so called Kasteleyn orientations (Cimasoni and Reshetikhin, 2007). But a generalization of this construction to higher dimensions is unknown.

in $SO(d)$, can be lifted to $Spin(d)$ while preserving the cocycle condition on triple overlaps of coordinate charts. Let us unpack this definition. On a general manifold one cannot choose a global coordinate system, so one covers M with coordinate charts U_i, $i \in I$. If over every coordinate chart U_i one picks an orthonormal basis of vector fields with the correct orientation, then on double overlaps $U_{ij} = U_i \bigcap U_j$ they are related by transition functions g_{ij} which take values in the group $SO(d)$ and satisfy on $U_{ijk} = U_i \bigcap U_j \bigcap U_k$ the cocycle condition:

$$g_{ij} g_{jk} = g_{ik}. \tag{1.1}$$

The group $SO(d)$ has a double cover $Spin(d)$, i.e. one has $SO(d) = Spin(d)/\mathbb{Z}_2$. One can lift every smooth function $g_{ij} : U_{ij} \to SO(d)$ to a smooth function $h_{ij} : U_{ij} \to Spin(d)$, with a sign ambiguity. Thus on every U_{ijk} one has

$$h_{ij} h_{jk} = \pm h_{ik}. \tag{1.2}$$

M has a spin structure if and only if one can choose the functions h_{ij} so that the sign on the right-hand side is $+1$ for all U_{ijk}. We also identify spin structures which are related by $Spin(d)$ gauge transformations:

$$h_{ij} \mapsto h'_{ij} = h_i h_{ij} h_j^{-1}, \quad h_i : U_i \to Spin(d).$$

A spin structure allows one to define Weyl spinors on M.

For $d < 4$ every oriented d-manifold admits a spin structure, but it is not unique, in general. Namely, given any spin structure, one can modify it by multiplying every h_{ij} by constants $\zeta_{ij} = \pm 1$ satisfying

$$\zeta_{ij} \zeta_{jk} = \zeta_{ik}.$$

Such constants define a Cech 1-cochain on M with values in $\mathbb{Z}_2$. The same data also parameterize $\mathbb{Z}_2$ gauge fields on M, thus any two spin structures differ by a $\mathbb{Z}_2$ gauge field. It is easy to see that gauge fields differing by $\mathbb{Z}_2$ gauge transformations lead to equivalent transformations of spin structures, so the number of inequivalent spin structures is equal to the order of the Cech cohomology group $H^1(M, \mathbb{Z}_2)$, whose elements label gauge-equivalence classes of $\mathbb{Z}_2$ gauge fields.

In dimension $d > 3$ not every oriented manifold admits a spin structure. For example, the complex projective plane $\mathbb{CP}^2$ does not admit a spin structure. Nevertheless, if a spin structure on M exists, the above argument still shows that the number of inequivalent spin structures is given by $|H^1(M, \mathbb{Z}_2)|$. The necessary and sufficient condition for the existence of a spin structure is the vanishing of the 2nd Stiefel-Whitney class $w_2(M) \in H^2(X, \mathbb{Z}_2)$. This condition is purely topological and thus does not depend on the choice of Riemannian metric on M.

If M is not oriented, the transition functions g_{ij} take values in $O(d)$ rather than $SO(d)$. They still satisfy (1.1). An analog of *Spin* group in this case is called a *Pin* group. In the absence of orientation, fermions transform in a representation of the *Pin* group. In fact, for all $d > 0$ there exist two versions of the *Pin* group called $Pin^+(d)$ and $Pin^-(d)$. They both have the property $Pin^\pm(d)/\mathbb{Z}_2 = O(d)$. The difference between Pin^+ and Pin^- is the way a reflection of any one of coordinate axis is realized on fermions. Let $r \in O(d)$ be such a reflection. It satisfies $r^2 = 1$. If $\tilde{r} \in Pin^\pm(d)$ is a pre-image of r, it can satisfy either $\tilde{r}^2 = 1$ or $\tilde{r}^2 = -1$. The first possibility corresponds to Pin^+, while the second one corresponds to Pin^-.

If we are given an unoriented d-manifold M, we can ask whether it admits Pin^+ or Pin^- structures (that is, lifts of transition functions to either $Pin^+(d)$ or $Pin^-(d)$ so that the condition (1.2) on triple overlaps is satisfied). The conditions for this are again topological: in the case of Pin^+ it is the vanishing of $w_2(M)$, while in the case of Pin^- it is the vanishing of $w_2(M) + w_1(M)^2$. Note that if M happens to be orientable, then $w_1(M) = 0$, so the two conditions coincide and reduce to the condition that M admit a *Spin* structure.

Note that these topological conditions are nontrivial already for $d = 2$. More precisely, for $d = 2$ one has a relation between Stiefel-Whitney classes $w_1^2 + w_2 = 0$, so every 2d manifold admits a Pin^- structure, but not necessarily a Pin^+ structure. For example the real projective plane $\mathbb{RP}^2$ admits only Pin^- structures, while the Klein bottle admits both Pin^+ and Pin^- structures. Similarly, not every 3-manifold admits a Pin^+ structure, but all 3-manifolds admit a Pin^- structure.

1.2 Working assumptions

We assume that fermionic SPTs in d space-time dimensions without time-reversal symmetry can be defined on any oriented smooth d-manifold M equipped with a spin structure. Similarly, we assume that fermionic SPTs with time-reversal symmetry can be defined on any smooth manifold M equipped with a Pin^+ or Pin^- structure (we will see below that Pin^+ corresponds to $T^2 = (-1)^F$ while Pin^- corresponds to $T^2 = 1$). If there are additional symmetries beyond $(-1)^F$ and time-reversal, M can carry a background gauge field for this symmetry.

We also assume that given such M, a long-distance effective action is defined. The action is related to the partition function by $Z = \exp(2\pi i S_{eff})$, thus S_{eff} is defined modulo integers. The trivial SPT phase corresponds to the trivial (zero) action. The effective action is additive under the disjoint union of manifolds. It also changes sign under orientation-reversal. In the case of SPT phases with time-reversal symmetry, this implies $2S_{eff} \in \mathbb{Z}$.

The effective action, in general, is not completely topological: it may depend on the Levi-Civita connection on M. Such actions are gravitational Chern-Simons terms and can exist if $d = 4k - 1$. Since we will be interested only in low-dimensional SPT phases, the only case of interest is $d = 3$. The correspond gravitational Chern-Simons term has the form

$$S_{CS} = \frac{k}{192\pi} \int \mathrm{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right),$$

where the trace is in the adjoint representation of $SO(3)$. Note that such a term makes sense only on an orientable 3-manifold and therefore can appear only if the symmetry group of the SPT phase does not involve time reversal.

In the bosonic case, one can show that k must be an integral multiple of 16. In the fermionic case, k can be an arbitrary integer. The quantization of k is explained in the appendix.

The physical meaning of S_{CS} is that it controls the thermal Hall response of the SPT phases (Read and Green, 2000). The thermal Hall conductivity is proportional to k (Read and Green, 2000):

$$\kappa_{xy} = \frac{k\pi k_B^2 T}{12\hbar},$$

where T is the temperature and k_B is the Boltzmann constant. Thus for both bosonic and fermionic SPT phases the quantity κ_{xy}/T is quantized, but in the fermionic case the quantum is smaller than in the bosonic case by a factor 16. This is derived in the appendix.

SPT phases with a particular symmetry form an abelian group, where the group operation amounts to forming the composite system. The effective action is additive under this operation. Taking the inverse corresponds to applying time-reversal to the SPT phase. The effective action changes sign under this operation. Thus the effective action can be regarded as a homomorphisms from the set of SPT phases to $\mathbb{R}/\mathbb{Z} \simeq U(1)$.

The difference of two SPT phases with the same thermal Hall conductivity is an SPT phase with zero thermal Hall conductivity. Thus it is sufficient to classify

Table 1.1: *Spin* and $Pin^{\pm}$ Bordism Groups

$d = D + 1$	$\Omega_d^{Spin}(pt)$	$\Omega_d^{Pin^-}(pt)$	$\Omega_d^{Pin^+}(pt)$	$\Omega_d^{Spin}(B\mathbb{Z}_2)$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_8$
4	$\mathbb{Z}$	0	$\mathbb{Z}_{16}$	$\mathbb{Z}$
5	0	0	0	0
6	0	$\mathbb{Z}_{16}$	0	0
7	0	0	0	$\mathbb{Z}_{16}$
8	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$	$\mathbb{Z}^2$
9	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^4$
10	$\mathbb{Z}_2^2 \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_{128}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4 \times \mathbb{Z}$

Table 1.2: Interacting Fermionic SPT Phases

$d = D + 1$	no symmetry	$T^2 = 1$	$T^2 = (-1)^F$	unitary $\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}$
4	0	0	$\mathbb{Z}_{16}$	0
5	0	0	0	0
6	0	$\mathbb{Z}_{16}$	0	0
7	$\mathbb{Z}^2$	0	0	$\mathbb{Z}_{16} \times \mathbb{Z}^2$
8	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$	0
9	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^4$
10	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_{128}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$

SPT phases with zero thermal Hall conductivity. In such a case the action is purely topological. Our final assumption is that this topological action depends only on the bordism class of M. Equivalently, we assume that if M is a boundary of some $d+1$-manifold with the same structure (*Spin* or $Pin^{\pm}$, as the case may be), then S_{eff} vanishes. This assumption is supposed to encode locality.

1.3 Fermionic SPT phases without any symmetry

In the case when the only symmetry is $(-1)^F$, the manifold M can be assumed to be a compact oriented manifold with a spin structure. As explained above, without loss of generality we may assume that the action is purely topological (depends only on the spin bordism class of M). Thus possible effective actions in space-time

Table 1.3: Free Fermionic SPT Phases

$d = D + 1$ mod 8	no symmetry	$T^2 = 1$	$T^2 = (-1)^F$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
2	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
3	$\mathbb{Z}$	0	$\mathbb{Z}_2$
4	0	0	$\mathbb{Z}$
5	0	0	0
6	0	$\mathbb{Z}$	0
7	$\mathbb{Z}$	0	0
8	0	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 1.4: Classification of free fermionic SPT phases according to (Ryu et al., 2010) and (Kitaev, 2009b). The "no symmetry" case corresponds to class D, the case $T^2 = 1$ corresponds to class BDI, the case $T^2 = (-1)^F$ corresponds to class DIII.

dimension d are classified by elements of the group $\mathrm{H}om(\Omega_d^{Spin}(pt), U(1))$, where $\Omega_d^{Spin}(pt)$ is the group of bordism classes of spin manifold of dimension d.

The spin bordism groups $\Omega_d^{Spin}(pt)$ have been computed by Anderson, Brown, and Peterson (Anderson, H. Brown, and Peterson, 1967). In low dimensions, one gets

$$\Omega_1^{Spin}(pt) = \mathbb{Z}_2, \quad \Omega_2^{Spin}(pt) = \mathbb{Z}_2, \quad \Omega_3^{Spin}(pt) = 0, \quad \Omega_4^{Spin}(pt) = \mathbb{Z},$$

If a bordism group contains a free part, its Pontryagin dual has a $U(1)$ factor. This means that the corresponding effective action can depend on a continuous parameter. If we want to classify SPT phases up to homotopy, we can ignore such parameters. This is equivalent to only considering the torsion subgroup of $\Omega_d^{Spin}(pt)$. Thus we propose that SPT phases in dimension d are classified by elements of the Pontryagin dual of the torsion subgroup of $\Omega_d^{Spin}(pt)$. We will denote this group $\Omega_{Spin}^{d,tors}(pt)$.

The groups Ω_d^{Spin} are displayed in Table 1. The classification of interacting fermionic SPT phases can be deduced from it in the manner just described and is displayed in Table 2. For comparison, the classification of free fermionic SPT phases described in (Ryu et al., 2010) and (Kitaev, 2009b) is shown in Table 3. We see that there are nontrivial interacting fermionic SPT phases with zero thermal Hall response in $D = 0$ and 1 but not in $D = 2$ and 3. However, for $D = 2$ there is a phase with a nontrivial thermal Hall response; it is also present in the table of free fermionic SPT phases. In higher dimensions the number of phases grows rapidly. For instance, the effective action can be any combination of the Stiefel-Whitney numbers modulo w_1 and w_2 (such effective actions correspond to fermionic phases which are independent of the spin structure on M and thus can also be regarded as bosonic phases).

Let us consider the cases $d = 1$ and $d = 2$ in slightly more detail. For $d = 1$, there is only one connected closed manifold, namely, the circle. There are two spin structures on a circle: the periodic one and the anti-periodic one. The nontrivial effective action assigns a different sign to each spin structure and is multiplicative over disjoint unions. From the point of view of quantum mechanics, such an effective action corresponds to the $d = 1$ SPT phase whose unique ground state is fermionic.

In two space-time dimensions, the situation is more complicated. Spin structures on an oriented 2d manifold X can be thought of as $\mathbb{Z}_2$ valued quadratic forms on $H_1(X, \mathbb{Z}_2)$ satisfying $q(x + y) = q(x) + x \cap y + q(y) \mod 2$, where $x \cap y$ denotes the $\mathbb{Z}_2$ intersection pairing. The bordism invariant is the Arf invariant, which is the obstruction to finding a Lagrangian subspace for this quadratic form. The effective action for the nontrivial SPT phase in $D = 1$ is given by the Arf invariant (R. C. Kirby and L. R. Taylor, 1989)

$$S(q) = \frac{1}{\sqrt{|H^1(X, \mathbb{Z}_2)|}} \sum_{A \in H^1(X,\mathbb{Z}_2)} \exp(2\pi i q(A)/2). \tag{1.3}$$

Another way to describe the Arf invariant is to consider zero modes for the chiral Dirac operator. Their number modulo 2 is an invariant of the spin structure and coincides with the Arf invariant (Blumenhagen, Lüst, and Theisen, 2013). In string theory, spin structures for which the Arf invariant is even (resp. odd) are called even (resp. odd).

The spin cobordism classification is consistent with existing results in condensed matter literature. Fidkowski and Kitaev (Fidkowski and Kitaev, 2010) have considered the Majorana chain with just fermion parity. There are two distinct phases: one where all sites are decoupled and unoccupied in the unique ground state and one with dangling Majorana operators which can be paired into a gapless Dirac mode representing a two-fold ground state degeneracy. In the absense of any symmetry beyond $(-1)^F$, a four-fermion interaction can gap out the dangling modes in pairs, so these are the only two phases.

1.4 Fermionic SPT phases with time-reversal symmetry

General considerations

In the presence of time-reversal symmetry, the manifold M can be unorientable. As discussed in section 2, there are two distinct unoriented analogs of a spin structure, called Pin^+ and Pin^- structures. They should correspond to the two possibilities for the action of time-reversal: $T^2 = 1$ and $T^2 = (-1)^F$.

Naively, it seems that $T^2 = 1$ should correspond to Pin^+ and $T^2 = (-1)^F$ should correspond to Pin^-. Indeed, for Pin^+ the reflection of a coordinate axis acts on a fermion by an element $\tilde{r}$ satisfying $\tilde{r}^2 = 1$, while for Pin^- it acts by $\tilde{r}$ satisfying $\tilde{r}^2 = -1$. However, one should take into account that the groups $Pin^\pm$ are suitable for space-time of Euclidean signature. A reflection of a coordinate axis in Euclidean space is related to time-reversal by a Wick rotation. Let r be a reflection of the coordinate axis which is to be Wick-rotated. The corresponding element of $Pin^\pm$ acts on the fermions by a Dirac matrix γ_d which satisfies $\gamma_d^2 = \pm 1$. Wick rotation amounts to $\gamma_d \mapsto i\gamma_d$, hence Pin^+ corresponds to $T^2 = (-1)^F$, while Pin^- corresponds to $T^2 = 1$. This identification will be confirmed by the comparison with the results from the condensed matter literature.

$T^2 = (-1)^F$

We propose that interacting fermionic SPT phases protected by time-reversal symmetry T with $T^2 = (-1)^F$ are classified by elements of

$$\Omega^d_{Pin^+}(pt) = \mathrm{H}om(\Omega_d^{Pin^+}(pt), U(1)).$$

We will call this group the Pin^+ cobordism group with $U(1)$ coefficients.

The Pin^+ bordism groups have been computed by Kirby and Taylor (R. Kirby and L. Taylor, 1990)

$$\Omega_1^{Pin^+}(pt) = 0, \quad \Omega_2^{Pin^+}(pt) = \mathbb{Z}_2, \quad \Omega_3^{Pin^+}(pt) = \mathbb{Z}_2, \quad \Omega_4^{Pin^+}(pt) = \mathbb{Z}_{16},$$

Pin^+ bordism groups grow quickly with dimension, soon having multiple cyclic factors.

In one space-time dimension, the Pin^+ cobordism group vanishes. This is easily interpreted in physical terms. Recall that without time-reversal symmetry, the ground state can be bosonic or fermionic, and the latter possibility corresponds to a nontrivial fermionic $d = 1$ SPT phases. However, if time-reversal symmetry T with $T^2 = (-1)^F$ is present, fermionic states are doubly-degenerate, and since by definition the ground state of an SPT phase are non-degenerate, the ground state cannot be fermionic.

In two space-time dimensions, there is an isomorphism

$$\Omega_2^{Pin^+}(pt) \to \Omega_2^{Spin}(pt),$$

see (R. C. Kirby and L. R. Taylor, 1989). The isomorphism arises from the fact that a Pin^+ structure on an unoriented manifold induces a spin structure on its orientation double cover. Thus there is a unique nontrivial fermionic SPT phase in $d = 2$, and the corresponding effective action is simply the action (1.3) on the orientation double cover:

$$S(q) = \frac{1}{\sqrt{|H^1(\tilde{X}, \mathbb{Z}_2)|}} \sum_{A \in H^1(\tilde{X}, \mathbb{Z}_2)} e^{2\pi i q(A)/2}.$$

The classification of the free fermionic SPTs in $d = 2$ also predicts a unique nontrivial phase with time-reversal symmetry $T^2 = (-1)^F$ (Ryu et al., 2010; Kitaev, 2009b). It can be realized by a time-reversal-invariant version of the Majorana chain and is characterized by the presence of a pair of dangling Majorana zero modes on the edge.

In three space-time dimensions, a similar map is not an isomorphism, as $\Omega_3^{Spin} = 0$. However, there is a map

$$[\cap w_1] : \Omega_3^{Pin^+} \to \Omega_2^{Spin} \tag{1.4}$$

taking a Pin^+ manifold to a codimension 1 submanifold Poincaré dual to the orientation class w_1. This submanifold is defined to be minimal for the property that the complement can be consistently oriented. With this choice of partial orientation, crossing this submanifold reverses the orientation, so it can be thought of as a time-reversal domain wall. For Pin^+ 3-manifolds, we have $w_1^2 = 0$, so this domain wall is oriented and inherits a Spin structure from the ambient spacetime.

The map (1.4) is an isomorphism (R. C. Kirby and L. R. Taylor, 1989). From the physical viewpoint this means that away from the time-reversal domain walls the SPT is trivial and the boundary can be gapped, but on the domain walls there is a $d = 2$ fermionic SPT, the Majorana chain, so at locations where the domain walls meet the boundary there are Majorana zero modes. This is a special case of a construction of SPT phases discussed in the bosonic case in (X. Chen, Y.-M. Liu, and Vishwanath, 2013). One starts with a system with symmetry G in a trivial phase, breaks the G symmetry, decorates the resulting domain walls with an SPT in 1 dimension lower, and finally proliferates the domain walls to restore the symmetry G. One can also do this with defects of higher codimension. A mathematical counterpart of this general construction is the Smith homomorphism discussed below.

The classification of free fermionic SPT phases also predicts a unique nontrivial $d = 3$ SPT phase. It can be realized by a spin-polarized $p \pm ip$ superconductor (Ryu et al., 2010; Kitaev, 2009b). It is characterized by the presence of a pair of counter-propagating massless Majorana fermions on the edge of the SPT phase.

In four space-time dimensions, the cobordism classification says that fermionic SPT phases are labeled by elements of $\mathbb{Z}_{16}$. Free fermionic SPTs in $d = 4$ are classified by $\mathbb{Z}$ (Ryu et al., 2010; Kitaev, 2009b), but with interactions turned on $\mathbb{Z}$ collapses to $\mathbb{Z}_{16}$ (Fidkowski, X. Chen, and Vishwanath, 2013). The generator of $\Omega_4^{Pin^+} = \mathbb{Z}_{16}$ is the eta invariant of a Dirac operator (Stolz, 1988). The corresponding free fermionic SPT phase can be realized by a spin-triplet superconductor (Ryu et al., 2010; Kitaev, 2009b). It is characterized by the property that on its boundary there is a single massless Majorana fermion.

Two layers of the basic phase can be constructed from the $d = 2$ phase with time-reversal symmetry $T^2 = 1$, via the map

$$[\cap w_1^2]\colon \Omega_4^{\nu Pin^+} \to \Omega_2^{\nu Pin^-}.$$

The map sends a the bordism class of a manifold X on the left hand side to the bordism class of a codimension-2 submanifold of X representing $w_1^2(TX)$. From the physical viewpoint, the order 8 phase with $T^2 = (-1)^F$ can be obtained from the trivial SPT phase by decorating certain codimension 2 defects (self-intersections of time-reversal domain walls, see the 3d case above) with the order 8 $D = 1$ phase with $T^2 = 1$, i.e. the Kitaev chain.

Eight copies of this fermionic SPT phase are equivalent to a bosonic SPT phase with time-reversal symmetry and the effective action $\int w_1^4$ (the bosonic SPT phase predicted by group cohomology, see (Kapustin, 2014b)). To show this, we need to show $8\eta = w_1^4$ for every Pin^+ 4-manifold. The space $\mathbb{RP}^4$ generates the Pin^+ bordism group in 4 dimensions, so every such manifold X is Pin^+ bordant to a disjoint union of k $\mathbb{RP}^4$s. Since η is a Pin^+ bordism invariant, it follows $8\eta(X) = 8k\eta(\mathbb{RP}^4)$. Now w_1^4 is also a bordism invariant, so $w_1^4(X) = kw_1^4(\mathbb{RP}^4)$. Thus, we just need to show $8\eta(\mathbb{RP}^4) = w_1^4(\mathbb{RP}^4)$. We know the left hand side is -1 since the bordism group is $\mathbb{Z}/16$ and η generates the dual group, and it is simple to show $w_1^4(\mathbb{RP}^4) = -1$ as well. The equivalence of these two phases was also argued in (C. Wang and Senthil, 2014).

Note that the eta-invariant cannot be written as an integral over a Lagrangian density $\mathcal{L}$ naturally associated to a lattice configuration on the underlying manifold M. In particular, if we have a covering map, we can pullback configurations to the cover. If the Lagrangian density were to simply pull back, then the action would just be multiplied by the number of sheets of the cover. However, for $M = \mathbb{RP}^4$ the eta-invariant associated to the standard Dirac operator is order 16 but trivial for its orientation double cover, S^4.

This signals that the effective field theory requires a certain amount of non-locality. It cannot have a description where each Pin^+ structure corresponds to a lattice configuration which respects covering maps of spacetimes up to gauge transformations.

It is interesting to note that the topological Pin^+ bordism group in 4d is $\mathbb{Z}_8$ rather than $\mathbb{Z}_{16}$. There is a manifold homeomorphic to the smooth generator $\mathbb{RP}^4$ but not smoothly Pin^+ cobordant to it which has a $\mathbb{Z}_{16}$ invariant equal to 9 as opposed to $\mathbb{RP}^4$'s 1 (these numbers are equal mod 8). The eta-invariant distinguishes these two manifolds. Since the classification of topological insulators in 3+1d is known to be at least $\mathbb{Z}_{16}$, this example shows that the spacetimes relevant to these systems always carry smooth structure.

$T^2 = 1$

We propose that interacting fermionic SPT phases protected by time-reversal symmetry with $T^2 = 1$ are classified by the Pin^- cobordism groups with $U(1)$ coefficients. In low dimensions the Pin^- bordism groups are (R. C. Kirby and L. R. Taylor, 1989)

$$\Omega_1^{Pin^-}(pt) = \mathbb{Z}_2, \quad \Omega_2^{Pin^-}(pt) = \mathbb{Z}_8, \quad \Omega_3^{Pin^-}(pt) = 0, \quad \Omega_4^{Pin^-}(pt) = 0,$$

and the cobordism groups are their Pontryagin duals.

In one space-time dimension, fermionic SPT phases are classified by $\mathbb{Z}_2$. This is easily interpreted in physical terms: the non-degenerate ground state can be either bosonic or fermionic, without breaking T.

In two space-time dimensions, a Pin^- structure can be thought of as a $\mathbb{Z}_4$-valued quadratic enhancement of the intersection form which in the oriented (Spin) case is even and reduces to our description above(R. C. Kirby and L. R. Taylor, 1989). Such a form q satisfies $q(x + y) = q(x) + 2x \cap y + q(y) \mod 4$, where $2x \cap y$ represents the mod 2 intersection of x and y mapped to $\mathbb{Z}_4$. The bordism group $\Omega_2^{Pin^-} = \mathbb{Z}_8$ is generated by $\mathbb{RP}^2$. The effective action is a generalization of the Arf invariant, the

Arf-Brown-Kervaire invariant:

$$S(q) = \frac{1}{\sqrt{|H^1(X,\mathbb{Z}_2)|}} \sum_{A \in H^1(X,\mathbb{Z}_2)} \exp(2\pi i q(A)/4). \tag{1.5}$$

It takes values in $\mathbb{Z}_8 \in U(1)$. If $q(x)$ is even for all x (that is, if q is $\mathbb{Z}_2$-valued), it reduces to the Arf invariant. This situation occurs when the space-time is orientable.

From the physical viewpoint, the generator of $\mathbb{Z}_8$ is the Majorana chain, which can be regarded as a time-reversal invariant system with $T^2 = 1$. Time-reversal protects the dangling Majorana zero modes from being gapped out in pairs. Instead, interactions can only gap out octets, yielding a $\mathbb{Z}_8$ classification of phases labeled by the number of dangling modes (Fidkowski and Kitaev, 2010). Moreover, four copies of the Majorana chain with $T^2 = 1$ have states on the boundary on which T acts projectively, $T^2 = -1$ (Fidkowski and Kitaev, 2010); hence, four copies of the basic fermionic SPT phases with time-reversal $T^2 = 1$ are equivalent to the basic bosonic SPT phase in $d = 2$ with time-reversal symmetry. We can easily see this from the cobordism viewpoint. The generator of the Pin^- bordism group in $d = 2$ is $\mathbb{RP}^2$, so the fourth power of the generator of the cobordism group is -1 for this spacetime (here we are thinking about $\mathbb{Z}_8$ as a subgroup of $U(1)$). Meanwhile, w_1^2 is also -1 on $\mathbb{RP}^2$. Since both of these are Pin^--bordism invariants, they are equal on all $d = 2$ spacetimes.

As with the eta-invariant discussed above, the Arf-Brown-Kervaire invariant does not admit a local expression. There is a νPin^+ structure on $\mathbb{RP}^2$ for which the Arf-Brown-Kervaire invariant is a primitive 8th root of unity. However, the corresponding $Spin$ structure on the orientation double cover S^2 has Arf-Brown-Kervaire invariant 1 (the unique $Spin$ structure on the 2-sphere extends to a 3-ball).

1.5 Fermionic SPT phases with a unitary $\mathbb{Z}_2$ symmetry

Let g denote the generator of a unitary $\mathbb{Z}_2$ symmetry. There are two possibilities: either $g^2 = 1$ or $g^2 = (-1)^F$. In this section we discuss the former possibility only; the other one is discussed in the next section.

We propose that interacting fermionic SPT phases with unitary $\mathbb{Z}_2$ symmetry g, $g^2 = 1$, are classified by

$$\Omega^d_{Spin,tors}(B\mathbb{Z}_2) = \mathrm{H}om(\Omega_d^{Spin,tors}(B\mathbb{Z}_2), U(1))$$

The analogous group in the bosonic case is $\Omega^d_{SO,tors}(B\mathbb{Z}_2)$. In all dimensions there is an isomorphism called the Smith isomorphism

$$\tilde{\Omega}_d^{Spin}(B\mathbb{Z}_2) \to \Omega_{d-1}^{Pin^-}(pt),$$

where on the left hand side we use the tilde to denote reduced bordism: the kernel of the forgetful map to $\Omega_d^{Spin}(pt)$. The torsion part of reduced bordism is dual to SPT phases which can be made trivial after breaking the symmetry. Not all SPT phases are of this sort. One could imagine that after breaking the symmetry the system is reduced to some non-trivial SRE like the Kitaev chain. In general,

$$\Omega_d^{Spin}(BG) = \tilde{\Omega}_d^{Spin}(BG) \oplus \Omega_d^{Spin}(pt),$$

so these effects can be separated consistently and the Smith isomorphism is enough to classify the $G = \mathbb{Z}_2$ phases. This splitting fails if any elements of G are orientation reversing or if G acts projectively on fermions.

The Smith isomorphism is defined as follows. Starting with a Spin manifold X and some $A \in H^1(X, \mathbb{Z}_2)$ representing a class on the left hand side, we produce a submanifold Y Poincaré dual to A. (That we can do this is a special fact about codimension 1 classes with $\mathbb{Z}_2$ coefficients. Not all homology classes are represented by submanifolds.) The manifold Y is not necessarily orientable. The Spin structure on TX restricts to a Spin structure on $TY \oplus NY$, where NY is the normal bundle of Y in X. In fact, NY is classified by the restriction of A to Y. We compute

$$0 = w_1(TX)|_Y = w_1(TY \oplus NY) = w_1(TY) + A,$$

so on Y the gauge field A restricts to the orientation class, ie. the $\mathbb{Z}_2$ symmetry is orientation-reversing for Y. We also have

$$w_2(TY \oplus NY) = w_2(TY) + w_1(TY)^2,$$

so the Spin structure on X becomes a Pin^- structure on Y.

Physically, the submanifold Y Poincaré dual to A represents $\mathbb{Z}_2$ domain walls. The dual map from the Pin^- cobordism of a point in $d-1$ dimensions to the $Spin$ cobordism of $B\mathbb{Z}_2$ in d dimensions has the following physical meaning. Picking an element of the Pin^- cobordism group gives us a $d-1$-dimensional fermionic SPT with time-reversal symmetry $T^2 = 1$. To obtain a d-dimensional SPT, we decorate $\mathbb{Z}_2$ domain walls with this $d-1$-dimensional SPT and then proliferate the walls.

The inverse map can be described via compactification. One takes the d-dimensional SPT on a spacetime which is a circle bundle over the $d-1$-dimensional (perhaps unorientable) spacetime. This circle bundle is the unit circle bundle of the orientation line plus a trivial line, and is therefore oriented. We give the gauge field nontrivial holonomy around this circle and compactify. The effective field theory in $d-1$ dimensions is the $d-1$-dimensional SPT phase with time-reversal symmetry.

Fermionic SPT phases with a unitary $\mathbb{Z}_2$ symmetry have not been much studied in the physics literature. In one space-time dimension, they are classified by $\mathbb{Z}_2 \times \mathbb{Z}_2$, since the ground state can be either bosonic or fermionic, as well as g-even or g-odd. In three space-time dimensions, Levin and Gu (Gu and Levin, 2014) argued that fermionic SPT phases with $\mathbb{Z}_2$ symmetry and zero thermal Hall conductance are classified by $\mathbb{Z}_8$. Both of these results agree with the cobordism approach.

1.6 Fermionic SPT phases with a general symmetry

A choice of spin structure gives a lift of the oriented frame bundle $P_{SO(d)}$ to a spin frame bundle $P_{Spin(d)}$. Neutral Dirac spinors are sections of the bundle S associated to this one by the complex spin representation. For Dirac spinors charged under some G representation ρ, they are sections of the tensor bundle

$$\psi \in \Gamma(S \otimes_{\mathbb{C}} A^*\rho),$$

where $A^*\rho$ denotes the vector bundle associated to the gauge bundle by ρ. Bosonic observables are composed of fermion bilinears which are sections of the tensor square of this bundle or the tensor product of this bundle with its dual. These are composed of integral spin representations of $SO(d)$ and exterior powers of ρ^2.

However, the situations where the spacetime is not a spin manifold are still physically important if ρ is a projective representation. That is, while the spin frame bundle $P_{Spin(d)}$ or charge bundle $A^*\rho$ may not exist, the tensor product above does. For example, when ρ is a half-charge representation of $G = U(1)$ the choice of a tensor product bundle is the same as a $Spin^c$ structure with determinant line ρ^2. One also knows that such a $Spin^c$ structure is the same as a spin structure on $TX \oplus A^*\rho^2$.

One way to deal with this situation is to regard the fermions in d dimensions as dimensional reduction of fermions in $d+n$ dimensions. Under such a reduction, the rotation group $SO(n+d)$ decomposes into $SO(d)\times SO(n)$ (for the moment we assume that the d-dimensional theory does not have orientation-reversing symmetries, and accordingly the d-dimensional space-time is orientable). We imagine that the symmetry group G is embedded into $SO(d)$, and denote by ξ the G-representation in which the n-vector of $SO(n)$ transforms. We can think of ξ as a particular G-

bundle over BG. Spinors in $d+n$ dimensions are elements of an irreducible module over the Clifford algebra built from $\mathbb{R}^n \oplus \xi$.

Consider now the theory on a curved space-time X equipped with a G-bundle A. As usual, we can think of A as a map from X to BG, defined up to homotopy. To define the theory on such a space-time we must specify the bundle in which the fermions take value. This bundle must have the same rank as the spinor of $SO(d+n)$ and be a module over a bundle of Clifford algebras $T^*X \oplus A^*\xi$. Such a bundle is called a spin structure on the $SO(d+n)$-bundle $T^*X \oplus A^*\xi$.

If some of the symmetries are orientation-reversing, we need to allow X to be unorientable, so that the structure group of the tangent bundle is $O(d)$ rather than $SO(d)$. But we can compensate for this by embedding G into $O(n)$ so that the generators of the Clifford algebra transform as a vector of $SO(d+n)$. Then fermions must take values in the irreducible Clifford module over the corresponding bundle of Clifford algebras, as before.

This discussion leads us to the following proposal Given a bosonic symmetry group G, and its representation ξ, fermionic SPT phases in d space-time dimensions with this symmetry structure are classified by

$$\Omega^d_{Spin}(\flat BG, \xi),$$

a cobordism theory dual to the torsion part of the bordism theory of d-manifolds X with a map $A : X \to \flat BG$ (the gauge field) and a spin structure on $TX \oplus A^*\xi$. It is important for continuous groups to use $\flat BG$ rather than BG since gauging the G symmetry means coupling to a flat G gauge field. Turning on curvature for the gauge field requires a kinetic term which is non-canonical. One model for $\flat BG$ is to take the classifying space of G as a discrete group. For finite G this is of course automatic.

The data (G, ξ) may seem to depend on some uphysical details, like the embedding of G into $SO(n)$, but one can show that cobordism groups thus defined depend only on $w_1(\xi) : G \to \mathbb{Z}_2$, which picks out the orientation reversing elements, and $w_2(\xi) \in H^2(G, \mathbb{Z}_2)$ (Barrera-Yanez, n.d.), which determines how G is extended by fermion parity.

Let us illustrate this with some examples. For $G = \mathbb{Z}_2$, first there is the trivial representation, for which this twisted cobordism group is the ordinary ones classifying fermionic SPTs with an internal $\mathbb{Z}_2$ symmetry acting honestly on the fermions, so the total symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2^F$.

The other irreducible is the 1d sign representation. For this representation we have w_1 equal to the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, this being the determinant of the representation, and $w_2 = 0$ since ths representation is 1 dimensional. We compute

$$w_1(TX \oplus A^*\xi) = w_1(TX) + A^*w_1(\xi) = w_1(TX) + A,$$

so an orientation of $TX \oplus A^*\xi$ identifies A with the orientation class of X. We also have

$$w_2(TX \oplus A^*\xi) = w_2(TX) + w_1(TX)A^*w_1(\xi) = w_2(TX) + w_1(TX)^2,$$

a trivialization of which is a Pin^- structure on TX. Thus,

$$\Omega^d_{Spin}(B\mathbb{Z}_2, sign) = \Omega^d_{Pin^-}.$$

Since $w_1(\xi) \neq 0$ and $w_2(\xi) = 0$ we interpret this group as classifying fermionic SPTs with an orientation-reversing symmetry such as time reversal which satisfies $T^2 = 1$. Note that the same group classifies SPT phases with a reflection symmetry squaring to 1.

We can also consider a sum of two sign representations, for which we have $w_1(\xi) = 0$ and $w_2(\xi) \neq 0$. This gives a bordism theory of oriented manifolds with $A^2 = w_2(TX)$. This symmetry structure is that associated to an orientation preserving symmetry such as particle-hole symmetry which squares to the fermion parity.

The sum of three sign representations has both $w_1(\xi)$ and $w_2(\xi)$ nonzero. The cohomology of $B\mathbb{Z}_2$ implies also $w_2(\xi) = w_1(\xi)^2$. With this we compute

$$w_1(TX \oplus A^*\xi) = w_1(TX) + A$$

and

$$w_2(TX \oplus A^*\xi) = w_2(TX) + A^2 + A^2 = w_2(TX).$$

The first implies that A equals the orientation class of X. The second says that a spin structure on $TX \oplus A^*\xi$ is the same as a Pin^+ structure on TX. Thus

$$\Omega^d_{Spin}(B\mathbb{Z}_2, 3 \times sign) = \Omega^d_{Pin^+}.$$

Therefore fermionic SPT phases with an orientation reversing $\mathbb{Z}_2$ symmetry squaring to the fermion parity are classified by Pin^+ cobordism.

For $G = U(1)$ there are no continuous representations with $w_1 \neq 0$ and $w_2 \neq 0$ for a continuous representation precisely when the sum of charges is odd. In this case $A^*w_2(\xi)$ is the mod 2 reduction of the gauge curvature F_A. A spin structure on $w_2(TX \oplus A^*\xi)$ is therefore the same thing as a $Spin^c$ structure with determinant line F_A. Note that these are not the $Spin^c$ cobordism groups studied in most of the mathematical literature since we require the determinant line to be flat.

For $G = U(1) \times \mathbb{Z}_2$ we now have representations where the $\mathbb{Z}_2$ is orientation reversing. For example, consider $\xi = charge\ 1 \otimes trivial \oplus trivial \otimes sign$. For this representation, $w_1(\xi)$ is the map to $\mathbb{Z}_2$ which is trivial on $U(1)$ and the identity on $\mathbb{Z}_2$. We also find

$$w_2(TX \oplus A^*\xi) = w_2(TX) + w_1(TX)^2 + F_A.$$

If we instead used three copies of the sign representation, we would have

$$w_2(TX \oplus A^*\tilde{\xi}) = w_2(TX) + F_A.$$

It may first appear that these give different cobordism theories, but note that $w_1(TX)^2$ lifts to an integral class, so a redefinition of the $U(1)$ field produces an equivalence between the two bordism groups. This is the same redefinition used in (C. Wang and Senthil, 2014) to show that the $T^2 = 1$ and $T^2 = (-1)^F$ classifications agree, a result verified here in cobordism. This is also reflected in the uniqueness of the $Pin^c(d)$ group and we find that both types of phase are classified by Pin^c bordism with flat determinant line.

Now consider $G = U(1) \rtimes \mathbb{Z}_2$ with $\mathbb{Z}_2$ acting by conjugation. This group can be thought of as $SO(2) \rtimes \mathbb{Z}_2 = O(2)$. Consider first the standard 2d representation ξ. For this, $w_1(\xi)$ is the determinant $O(2) \to \mathbb{Z}_2$ and $w_2(\xi)$ is the obstruction to finding a section of

$$Pin^+(2) \to O(2),$$

ie. it is the class in group cohomology $H^2(BO(2), \mathbb{Z}_2)$ classifying $Pin^+(2)$. The ring $H^*(BO(2), \mathbb{Z}_2)$ is generated by the universal Stiefel-Whitney classes w_1 and w_2, and $w_2(\xi)$ is the universal w_2. This representation corresponds to $T^2 = 1$ since $T^2 = 1$ in $Pin^+(2)$.

One can also consider $T^2 = (-1)^F$ by using the representation $\tilde{\xi} = \xi + 2 \times sign$. For this, $w_1(\tilde{\xi}) = w_1(\xi)$, but $w_2(\tilde{\xi})$ is the universal $w_2 + w_1^2$, which differs from the other representation, demonstrating that these two classifications differ when time reversal does not commute with $U(1)$.

1.7 Decorated Domain Walls

The formulation above in terms of the global symmetry representation ξ carried by fermion bilinears highlights some interesting features of the so-called decorated domain wall construction described in (X. Chen, Y.-M. Liu, and Vishwanath, 2013).

Let us start with a concrete example with a unitary $\mathbb{Z}_2$ symmetry which squares to fermion parity. We consider in 1+1d a massless Dirac fermion ψ coupled to a massless real scalar ϕ by the Yukawa coupling $\phi\bar{\psi}\psi$. The $\mathbb{Z}_2$ symmetry we consider is $\phi \mapsto -\phi$, $\psi \mapsto \gamma^5\psi$, where $\gamma^5 = i\gamma^0\gamma^1$. We condense ϕ, making the domain wall infinitely heavy, and we consider the system on a line with boundary conditions $\phi \to \infty$ on the right and $\phi \to -\infty$ on the left. Then there is a domain wall at some fixed position and a ψ zero mode bound to it. The point is to define the quantum mechanical theory of this zero mode, we need to pick a time direction. The ambient 2 dimensional space-time is oriented, so we can orient the domain wall if we can orient its normal direction. This orientation has to come from which side has the boundary condition $\phi \to \infty$ and which side has the boundary condition $\phi \to -\infty$. We choose some convention such as the $\phi \to \infty$ side is the positive side and thus orient the domain wall. However, if we now perform a global $\mathbb{Z}/2$ symmetry transformation, it swaps the boundary conditions but not the ambient orientation, so it reverses the time direction on the domain wall.

We can understand what happened in terms of the representation theory of $\mathbb{Z}_2$. We have to find the representation of $\mathbb{Z}_2$ on the fermion bilinears. There are three of them: $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$, and $\bar{\psi}\gamma^5\psi$. The first and the last transform as the sign representation, while the vector is invariant. Thus, ξ is two copies of the sign representation. As calculated in the previous section, we have $w_1(\xi) = 0$, $w_2(\xi) \neq 0$, meaning that we have a unitary symmetry squaring to the fermion parity. Recall now that to define fermions in a background G gauge field A we used a spin structure on $TX \oplus A^*\xi$. If Y is a curve in X, then $TX = TY \oplus NY$. If Y is Poincaré dual to A, then NY is A^*sign. Altogether then, our fermions restricted to Y are defined using a spin structure on $TY \oplus A^*sign \oplus A^*\xi = TY \oplus A^*(\xi \oplus sign)$. That is, for the fermions on the domain wall, ξ is effectively shifted by a copy of the sign representation. To understand how the domain wall operators have different transformation properties, consider the operator $\bar{\psi}\gamma^x\psi$, where γ^x is the Clifford operator in the oriented normal to the domain wall. Because we need to use the oriented normal to define this operator in the 0+1d theory, we have $\gamma^x \mapsto -\gamma^x$ under the $\mathbb{Z}_2$ symmetry, so $\bar{\psi}\gamma^x\psi \mapsto -\bar{\psi}\gamma^x\psi$, contributing another copy of the sign representation. So for the example just described we now have $\xi' = 3 \times sign$. Accordingly, as computed in the previous section, $w_1(\xi') \neq 0$ and $w_2(\xi') \neq 0$, so on the domain wall, $\mathbb{Z}_2$ has been transmuted into an orientation-reversing symmetry squaring to the fermion parity.

We pause before considering the general case to note that this feature is independent of dimension and for $\mathbb{Z}_2$ has an interesting order 4 periodicity as we cycle through each type of $\mathbb{Z}_2$ symmetry:

$$... \to \text{u}nitary,\ squaring\ to\ (-1)^F \to \text{a}ntiunitary, squaring\ to\ (-1)^F$$

$$\to \text{u}nitary,\ squaring\ to\ 1 \to \text{a}ntiunitary,\ squaring\ to\ 1 \to ...$$

where the arrow denotes restriction to the $\mathbb{Z}_2$ domain wall.

Now let's consider the general case of G symmetry with fermion bilinear representation ξ. In order to study a domain wall as we did above, we need a real scalar ϕ transforming in some 1 dimensional representation of G. This is the same as a group homomorphism $\sigma : G \to \mathbb{Z}_2$. In any decorated domain wall picture, the degrees of freedom bound to the wall are defined in a symmetry broken regime where the domain wall is infinitely tense. This choice of regime corresponds to the choice of σ. After the coupling is made in this regime, domain walls are again proliferated, restoring the G symmetry. For such a σ, in the phase where ϕ is condensed, the domain wall Y is Poincaré dual to the $\mathbb{Z}/2$ gauge field $\sigma(A)$ induced from the G gauge field A. In particular, the normal bundle to the domain wall is $A^*\sigma$. Thus, the ambient spin structure restricts to a spin structure on $TY \oplus A^*(\xi \oplus \sigma)$. That is, the G symmetry properties on the domain wall correspond to the fermion bilinear representation $\xi \oplus \sigma$.

In terms of cobordism groups, every map $\sigma : G \to \mathbb{Z}_2$ induces a map

$$\Omega_d^{Spin}(BG, \xi) \to \Omega_{d-1}^{Spin}(BG, \xi \oplus \sigma)$$

and thus a map

$$\Omega^{d-1}_{Spin}(BG, \xi \oplus \sigma) \to \Omega^d_{Spin}(BG, \xi).$$

Note that domain walls may be coupled to different degrees of freedom in different symmetry breaking sectors, corresponding to adding the images of maps from different σs.

It is also possible to couple domain defects of higher codimension through higher dimensional representations σ. These representations may be irreducible over $\mathbb{R}$, so this procedure is not always equivalent to merely iterating the above construction.

For example, if $G = \mathbb{Z}_4$, then we can take σ to be the 2-dimensional representation rotating the plane by $\pi/2$. This representation is irreducible over $\mathbb{R}$ since the eigenvectors of this rotation are imaginary. This representation defines a map

$$\Omega^{d-2}_{Spin}(B\mathbb{Z}_4, \xi \oplus \sigma) \to \Omega^{d}_{Spin}(B\mathbb{Z}_4, \xi).$$

One must be careful in defining these maps in general, however, since not every homology class Poincaré dual to $A^*\sigma$ is representable by a manifold if the dimension of σ is too large. Happily this does not occur until the ambient dimension is at least 6.

1.8 Concluding remarks

We have seen that cobordism correctly predicts the known classification of interacting fermionic SPT phases in $D \leq 3$ with $\mathbb{Z}_2$ symmetry, either unitary or anti-unitary. We find that for $0 \leq D \leq 3$, all phases are realized by free fermions. However, in higher dimensions new phenomena occur. First of all, while the classification of free fermionic SPT phases with a fixed symmetry exhibits mod 8 periodicity in dimension (Kitaev, 2009b), in the interacting case there is no periodicity. Second, the deviations from the free fermionic classification occur for high enough D, but the precise point depends on the symmetry group. For example, for SPT phases with time-reversal symmetry T, $T^2 = (-1)^F$, deviations start at $D = 3$. For SPT phases with no symmetry beyond $(-1)^F$ deviations start at $D = 6$. (In $D = 6$ the free fermionic classification predicts $\mathbb{Z}$, but in the interacting case it is $\mathbb{Z} \times \mathbb{Z}$ because there are two different gravitational Chern-Simons terms possible based on the Pontryagin numbers p_1^2 and p_2, respectively.)

Third, while in low dimensions the effect of interactions is to truncate the free fermionic classification, in high enough dimension inherently interacting fermionic SPT phases appear. For example, in $D = 7$ free fermionic SPT phases with time-reversal symmetry $T, T^2 = (-1)^F$, are classified by $\mathbb{Z}$, while the cobordism approach predicts $\mathbb{Z}_2 \times \mathbb{Z}_{32}$. The latter group is not a quotient of the former, so truncation alone cannot explain the discrepancy. The most likely interpretation is that $\mathbb{Z}_{32}$ is a truncation of $\mathbb{Z}$, while the $\mathbb{Z}_2$ factor corresponds to an inherently interacting fermionic SPT phase. Similarly, in $D = 6$ there should exist inherently interacting fermionic SPT phases with only fermion parity as a symmetry.

We have found that the correct classification requires the use of smooth manifolds rather than topological manifolds. It would be interesting to determine whether there is some physical difference between the smooth and piecewise linear categories.

We find also that the fermionic SPT effective action has a degree of non-locality that was not present in the case of bosonic SPTs. For $D = 1$, the effective action can be written in terms of a sum over an auxiliary $\mathbb{Z}_2$ gauge field. It is tempting to interpret it as a gauge field which couples to the fermion parity, but this needs to be tested. We leave this and the determination of possible boundary behaviors of fermionic SPT phases to further work.

Chapter 2

EQUIVARIANT TOPOLOGICAL QUANTUM FIELD THEORY AND SYMMETRY PROTECTED TOPOLOGICAL PHASES

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Forward

Short-Range Entangled topological phases of matter are closely related to Topological Quantum Field Theory. This chapter uses this connection to classify Symmetry Protected Topological phases in low dimensions, including the case when the symmetry involves time-reversal. To accomplish this, we generalize Turaev's description of equivariant TQFT to the unoriented case. We show that invertible unoriented equivariant TQFTs in one or fewer spatial dimensions are classified by twisted group cohomology, in agreement with the proposal of Chen, Gu, Liu and Wen. We also show that invertible oriented equivariant TQFTs in spatial dimension two or fewer are classified by ordinary group cohomology.

Background and Overview

Recently the problem of classifying gapped phases of matter whose ground state is short-range entangled[1] (SRE phases) has received a lot of attention. Two gapped local Hamiltonians (or *gapped systems*) are said to lie in the same *gapped phase* if there is a continuous family of gapped systems that interpolates between them. In the context of systems with a global symmetry G, a *phase with symmetry G* is defined by requiring the family of Hamiltonians to be symmetric. Gapped phases can be divided into two broad classes, bosonic and fermionic, depending on whether the fundamental degrees of freedom are bosons or fermions. The bosonic SRE phases are in many ways simpler, and there has been a substantial progress in their classification. In particular, it has been proposed in (X. Chen, Gu, Z.-X. Liu, et al., 2013) that D-dimensional bosonic SRE phases with a finite internal symmetry G are classified by the abelian group $H^{D+1}(BG, U(1))$. Here BG is the classifying

[1]There are at least two common definitions of short-range entanglement. Here, we follow (Kitaev, 2015) in defining SRE as *invertibility*. The notion of an SPT (defined and discussed below) captures the other definition (X. Chen, Gu, and Wen, 2010).

space of G, and D is the dimension of space (thus the dimension of spacetime is $D+1$). Later it was noticed that some SRE phases in spatial dimension 3 are not captured by the group cohomology classification (Vishwanath and Senthil, 2013), and it was proposed by one of the authors that the classification can be improved by replacing ordinary cohomology of BG with a particular generalized cohomology theory (the stable cobordism) (Kapustin, 2014b) (see also (Freed, 2014; Kitaev, 2015)). For $D \leq 2$ all classification schemes agree. In fact one can use the matrix product representation of SRE states to prove that $D=1$ bosonic SRE phases are classified by $H^2(BG, U(1))$ (X. Chen, Gu, and Wen, 2011a). The $D=1$ fermionic SRE phases have also been classified (Fidkowski and Kitaev, 2010). The $D=0$ case is even simpler.

One promising avenue for extending these results to higher dimensions is via equivariant Topological Quantum Field Theory (TQFT). It is an attractive conjecture that a large class of gapped phases is described at large scales by a TQFT. It is widely believed that if the large-volume limit of a quantum system exists, then its long distance behavior is described by an effective field theory. For a gapped system, which has no long wavelength propagating degrees of freedom, the effective theory is a TQFT. Gapped systems in the same phase are expected to share a single long distance effective description, or at least their effective descriptions can be continuously connected. If this is the case, then there is a one-to-one correspondence between gapped phases and deformation classes of TQFTs.[2]

Both gapped phases and TQFTs can be tensored, and each set has a neutral element $\mathbf{1}$ corresponding to the trivial phase or TQFT. This operation makes each set into a commutative monoid (a set with an associative and commutative binary operation and a unit). An element Φ of a monoid is said to be invertible if there exists an element $\bar{\Phi}$ such that $\Phi \circ \bar{\Phi} = \bar{\Phi} \circ \Phi = \mathbf{1}$. Thus it makes sense to talk about invertible gapped phases and invertible TQFTs. The set of invertible elements forms an abelian group. According to one definition of SRE phases (Kitaev, 2015), an invertible gapped phase is the same as an SRE phase; that is, a gapped system ϕ in a phase Φ is an SRE if there exists another gapped system $\bar{\phi}$ in $\bar{\Phi}$ such that $\phi \otimes \bar{\phi}$ can be deformed to the trivial (product state) system without closing the gap. If one believes into the correspondence between gapped phases and TQFTs, the classification of SRE phases is reduced to the classification of invertible TQFTs up

[2] There are exceptions to this rule, however, due to the existence of phases with non-vanishing thermal Hall conductivity. These only occur in $D = 2\, mod\, 4$, where there exist gravitational Chern-Simons terms.

to a continuous deformation.

Consider now phases with a symmetry G. These also form a commutative monoid, and forgetting the symmetry gives us a map to the monoid of all phases. Phases with a symmetry G are mapped to the neutral element under this map are usually called SPT phases. Note that it is not clear from this definition whether SPT phases with a symmetry G are invertible as G-symmetric phases, but it is believed to be true. SRE phases with a symmetry G are conjectured to correspond to invertible G-equivariant TQFTs.

While classifying TQFTs in $D > 1$ is unrealistic, classifying invertible ones is much simpler. In fact, using the known algebraic description of equivariant TQFTs in $D = 0$, 1 and 2, it is easy to check that in these dimensions invertible G-equivariant TQFTs are classified by $H^{D+1}(BG, U(1))$, provided the group G does not act on spacetime. But if some elements of G involve time-reversal, the problem is more complicated. From the TQFT viewpoint, time-reversal symmetry means that the theory can be defined on unorientable spacetimes. The difficulty is that an algebraic description of unoriented equivariant TQFTs is not known even in low dimensions. The main goal of this paper is to provide such an algebraic description in $D = 0$ and 1 and to show that invertible equivariant TQFTs are classified by twisted group cohomology $H^{D+1}(BG, U(1)_\rho)$, where $\rho : G \to \mathbb{Z}_2$ is a homomorphism which tells us which elements of G are time-reversing and which are not. This agrees with the proposal of (X. Chen, Gu, Z.-X. Liu, et al., 2013). It is likely that this method can be extended to $D = 2$. In higher dimensions an algebraic description of general TQFTs is prohibitively complicated, and this approach to classifying SRE phases becomes impractical. Note that equivariant TQFTs which are not necessarily invertible are interesting in their own right, as they describe Symmetry Enhanced Topological (SET) phases.

In Section 2.1 we deal with the case of a finite symmetry G which acts trivially on spacetime. We recall algebraic descriptions of oriented equivariant TQFTs in $D \leq 2$ and show that invertible equivariant TQFTs are classified by elements of $H^{D+1}(BG, U(1))$. All of this is either trivial ($D = 0$) or well-known to experts ($D = 1$ and 2).

In Section 2.2 we consider unoriented equivariant TQFT in $D = 0$ and the corresponding SRE phases with time-reversing symmetries.

In Section 2.3 we formulate axioms of unoriented equivariant TQFT in $D = 1$ by extending Turaev's axioms in the oriented case (Turaev, 1999). We show how these axioms lead to a generalization of Turaev's G-crossed algebra, which we call ρ-twisted G-crossed algebra. We prove that every ρ-twisted G-crossed algebra gives rise to an unoriented equivariant TQFT. Finally we show that invertible TQFTs in $D = 1$ give rise to ρ-twisted 2-cocycles on BG, and that conversely to every element of $H^2(BG, U(1)_\rho)$ one can associate a ρ-twisted G-crossed algebra which is unique up to isotopy.

It would be interesting to give an algebraic description of $D = 2$ unoriented equivariant TQFTs and show that in the invertible case they are classified by $H^3(BG, U(1)_\rho)$. The first step is to categorify our algebraic description of $D = 1$ unoriented equivariant TQFT by replacing vector spaces with categories, linear maps with functors, and equalities with isomorphisms. The nontrivial part is to find a complete set of coherence conditions between isomorphisms analogous to the pentagon and hexagon conditions in the oriented case which ensure consistency under gluing.

Since this paper was submitted to the arXiv, there have been several developments. Freed and Hopkins (Freed and Hopkins, 2016) proved a theorem relating invertible unitary TQFTs and stable cobordisms. For $D = 1$ it reduces to the statement that invertible unitary equivariant TQFTs are classified by elements of $H^2(BG, U(1)_\rho)$. More recently Bhardwaj (Bhardwaj, 2011) generalized the Turaev-Viro construction of equivariant $D = 2$ TQFTs to the unoriented case.

2.1 Oriented equivariant TQFT

$D = 0$

A $D = 0$ TQFT is ordinary quantum mechanics with zero Hamiltonian and is completely determined by its space of states (a finite-dimensional complex vector space V). Equivariant TQFT is merely a vector space V with an action of G. Since G is finite, this representation is unitarizable (unitary for a suitable choice of inner product on V). The trivial equivariant TQFT corresponds to $V = \mathbb{C}$ with a trivial action of G. Equivariant TQFTs which are invertible with respect to the tensor product are one-dimensional representations of G, i.e., elements of $H^1(BG, \mathbb{C}^*) \simeq H^1(BG, U(1))$.

$D = 1$

$D = 1$ TQFTs are in one-one correspondence with commutative Frobenius algebras (Atiyah, 1989) (see (G. W. Moore and Segal, 2006) for a nice exposition, including various generalizations). The vector space $\mathcal{A}$ underlying the algebra is the space of states of the TQFT on a circle. The state-operator correspondence identifies $\mathcal{A}$ with the space of local operators, which is clearly a commutative algebra. The Frobenius structure is a non-degenerate bilinear inner product

$$\eta(a, b) \in \mathbb{C}, \quad a, b \in \mathcal{A}, \tag{2.1}$$

satisfying $\eta(ab, c) = \eta(a, bc)$. It is a combination of the usual sesquilinear Hilbert space inner product and the anti-linear CPT transformation:

$$\eta(a, b) = (\mathrm{C}PT a, b). \tag{2.2}$$

The trivial $D = 1$ TQFT has $\mathcal{A} \simeq \mathbb{C}$ and $\eta(1, 1) = 1$. An invertible TQFT has $\mathcal{A} \simeq \mathbb{C}$, and thus is completely determined by $\eta(1, 1) \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. If we are interested only in classifying TQFTs up to isotopy (i.e. up to continuous deformations), then all these TQFTs can be identified (since $\pi_0(\mathbb{C}^*)$ is trivial). If we identify invertible TQFTs and SRE phases, this means that in the absence of symmetry there are no nontrivial $D = 1$ SRE phases.

To incorporate a symmetry G, we need to consider G-equivariant $D = 1$ TQFTs. G-equivariance means that we can couple the theory to an arbitrary G-bundle. The precise definition of equivariant TQFT will be recalled in Section 2.2. For now, we only need the algebraic description of such TQFTs due to Turaev (Turaev, 1999). He defines a G-crossed algebra as a Frobenius algebra ($\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g, \eta$) together with a homomorphism $\alpha : G \to \mathrm{A}ut\, \mathcal{A}$ such that

$$\mathcal{A}_g \cdot \mathcal{A}_h \subset \mathcal{A}_{gh} \text{ } \mathrm{a}nd \text{ } 1 \in \mathcal{A}_1. \tag{2.3}$$

$$\eta(\mathcal{A}_g, \mathcal{A}_h) = 0 \text{ } \mathrm{i}f \text{ } gh \neq 1. \tag{2.4}$$

$$\alpha_h(\mathcal{A}_g) \subset \mathcal{A}_{hgh^{-1}}. \tag{2.5}$$

$$\alpha \text{ } \mathrm{p}reserves \text{ } \eta \text{ } \mathrm{a}nd \text{ } \alpha_h|_{\mathcal{A}_h} = \mathrm{i}d. \tag{2.6}$$

$$\forall \psi_g \in \mathcal{A}_g, \psi_h \in \mathcal{A}_h \text{ } \mathrm{w}e \text{ } have \text{ } \psi_g \cdot \psi_h = \alpha_g(\psi_h) \cdot \psi_g. \tag{2.7}$$

$$\begin{gathered} \forall g \in G \text{ } \mathrm{l}et \text{ } \xi_i^g \text{ } \mathrm{a}nd \text{ } \xi_g^i \text{ } \mathrm{b}e \text{ } dual \text{ } bases \text{ } in \text{ } \mathcal{A}_g \text{ } \mathrm{a}nd \text{ } \mathcal{A}_{g^{-1}}. \text{ } \mathrm{T}hen \\ \sum_i \alpha_h(\xi_i^g)\xi_g^i = \sum_j \xi_j^h \alpha_g(\xi_h^j), \ \forall g, h \in G. \end{gathered} \tag{2.8}$$

Let us make a few remarks about this definition. $\mathcal{A}_g$ is the g-twisted sector of the space of states on a circle, and α_h describes the action of G on the space of states. If G is abelian, it acts on each twisted sector separately, but in general it mixes different twisted sectors. The penultimate axiom shows that $\mathcal{A}$ is not commutative, but is twisted-commutative. The last axiom arises from considering a punctured torus with twists by g and h along the two generators of its fundamental group and computing the corresponding state in two different ways. This axiom, together with the Frobenius condition $\eta(a, bc) = \eta(ab, c)$, implies

$$\dim \mathcal{A}_g = \mathrm{Tr}\ \alpha_g|_{\mathcal{A}_1}. \tag{2.9}$$

Both sides of this equality compute the partition function of a torus twisted by g along one direction and by 1 along the other direction. On the left-hand side, the direction twisted by g is regarded as space and the direction twisted by 1 is regarded as time. On the right-hand side, it is the other way around. Since the right-hand side is a character of a finite group G, we get an inequality $0 < \dim \mathcal{A}_g \leq \dim \mathcal{A}_1$. That is, twisting by g cannot increase the number of states.

In particular, let us consider an invertible G-equivariant TQFT. Then $\mathcal{A}_1 \simeq \mathbb{C}$, and therefore $\mathcal{A}_g \simeq \mathbb{C}$ for all $g \in G$. If we choose a basis vector ℓ_h in each $\mathcal{A}_h$, we see that the algebra structure is given by a collection of complex numbers $b(g, h)$ such that

$$\ell_g \cdot \ell_h = b(g, h) \cdot \ell_{gh}. \tag{2.10}$$

Twisted commutativity of $\mathcal{A}$ implies that $b(g, h)$ is nonzero for all g, h and fixes α_h in terms of b. Associativity of multiplication implies that b is a 2-cocycle, and changing a basis in $\mathcal{A}_g$ changes it by a coboundary. The rest of the axioms are easily checked. With b fixed, the only freedom left is the choice of the inner product η; all such choices lead to isotopic TQFTs, which means that isotopy classes of invertible oriented equivariant $D = 1$ TQFTs are classified by $[b] \in H^2(BG, \mathbb{C}^*) \simeq H^2(BG, U(1))$. This result has been proved in (Turaev, 1999).

$D = 2$

When studying oriented $D = 2$ TQFTs, one usually assumes that the space of local operators (i.e. the vector space attached to S^2) is one-dimensional, and thus the algebra of local operators is isomorphic to $\mathbb{C}$. If one is interested only in unitarizable TQFTs, one does not lose much by focusing on this special case. Indeed, it is easy to show that if the TQFT is unitarizable (i.e. the bilinear inner product arises from a Hermitian inner product and an antilinear CPT symmetry), the algebra of local operators is semisimple. It is also commutative, and therefore isomorphic to a sum of several copies of $\mathbb{C}$. The generators of this algebra label different superselection sectors, and one might as well focus on a single sector where all but one generator act trivially. The argument applies equally well for all $D > 0$, but in $D = 1$ it is traditional to allow the algebra of local operators to be non-semisimple, in view of string theory applications which require one to consider non-unitary TQFTs.

We are interested in unitarizable oriented $D = 2$ TQFTs, and therefore in this section we assume that the space of local operators is $\mathbb{C}$. Such theories are described by modular tensor categories with vanishing central charge $c \in \mathbb{Z}/8$ (G. W. Moore and Seiberg, 1989; Bakalov and (Jr)., 2000). (If the central charge is nonzero, one gets a framed $D = 2$ TQFT). The data of a modular tensor category attaches a vector space to every closed oriented 2-manifold, and a map of vector spaces to every oriented bordism between such 2-manifolds. Similarly, oriented equivariant $D = 2$ TQFT is described by a G-modular category (Turaev, 2010; Turaev and Virelizier, 2014). Its definition is a categorification of the notion of G-crossed algebra. In particular, for every $g \in G$ one has a category C_g, and a bi-functor $C_g \times C_h \to C_{gh}$ satisfying the associativity constraint. The data of a G-modular category attaches a vector space to every closed oriented 2-manifold with a G-bundle and a trivialization at a base point, and a map of vector spaces for every oriented G-bordism between such 2-manifolds (i.e. to every oriented 3-manifold with a G-bundle which "interpolates" between the two oriented 2-manifolds with G-bundles). Objects of the category C_g represent quasi-particles in the g-twisted sector.

An invertible oriented equivariant $D = 2$ TQFT is described by a G-modular category with $C_1 \simeq \mathrm{V}ect$, where $\mathrm{V}ect$ is the category of finite-dimensional vector spaces. This condition ensures that for the trivial G-bundle the vector space attached to any oriented 2-manifold is one-dimensional. If the TQFT describes a gapped phase, its space of ground states is non-degenerate for any topology. This is a hallmark of an SRE phase.

From $C_1 \simeq \mathrm{V}ect$ one can deduce that $C_g \simeq \mathrm{V}ect$ for all $g \in G$. Indeed, by the definition of a G-modular category (Turaev and Virelizier, 2014), C_g is nonempty for all $g \in G$. Then Prop. 4.58 in (Drinfeld et al., 2010) implies that $C_g \simeq \mathrm{V}ect$. As a consequence, the vector space attached to any 2-manifold with any G-bundle is one-dimensional. That is, there is no ground-state degeneracy even after twisting by an arbitrary G-bundle.

Finally, Prop. 4.61 in (Drinfeld et al., 2010) tells us that in the invertible case C is entirely determined by an element of $H^3(BG, \mathbb{C}^*) \simeq H^3(BG, U(1))$. This agrees with the proposal of (X. Chen, Gu, Z.-X. Liu, et al., 2013) that $D = 2$ bosonic SRE phases with symmetry G are classified by elements of $H^3(BG, U(1))$.

2.2 Unoriented equivariant $D = 0$ TQFT

A homomorphism $\rho : G \to \mathbb{Z}_2$ encodes whether a particular symmetry g preserves or reverses the direction of time. We identify $\mathbb{Z}_2$ with $\{1, -1\}$ and let $\rho(g) = -1$ if g is time-reversing and $\rho(g) = 1$ otherwise. Recall that a ρ-twisted 1-cochain on BG is the same as a function $\phi : G \to U(1)$ satisfying

$$\phi(gh) = \phi(g)\phi(h)^{\rho(g)}. \tag{2.11}$$

Two twisted cochains $\phi(g)$ and $\psi(g)$ are regarded as equivalent (i.e. cohomologous) if there exists $\mu \in U(1)$ such that for all $g \in G$ we have

$$\psi(g) = \mu^{\rho(g)-1}\phi(g) = \begin{cases} \phi(g), & \rho(g) = 1 \\ \mu^{-2}\phi(g), & \rho(g) = -1. \end{cases} \tag{2.12}$$

To each $g \in G$ an equivariant D=0 TQFT associates an operator on the vector space V assigned to the point:

$$\Lambda(g) : V \to V \tag{2.13}$$

where $\Lambda(g)$ is linear if $\rho(g) = 1$ and anti-linear if $\rho(g) = -1$. After choosing a basis in V, we can attach to every $\Lambda(g)$ a complex non-degenerate matrix $M(g)$, by letting

$$\Lambda(g) = \begin{cases} M(g), & \rho(g) = 1 \\ M(g)K, & \rho(g) = -1. \end{cases} \tag{2.14}$$

Here $K : V \to V$ is an operator which complex-conjugates the coordinates of a vector in the chosen basis. The matrices $M(g)$ do not form a complex representations of G, rather (Weyl, 1937):

$$M(gh) = \begin{cases} M(g)M(h), & \rho(g) = 1 \\ M(g)M(h)^*, & \rho(g) = -1. \end{cases} \tag{2.15}$$

In the invertible case $V \simeq \mathbb{C}$ the matrices $M(g)$ become elements of $\mathbb{C}^*$, and (2.15) becomes precisely the twisted cocycle condition for the $\mathbb{C}^*$-valued 1-cochain $M(g)$, where $\mathbb{Z}_2$ acts on $\mathbb{C}^*$ by complex conjugation.

We should also investigate the effect of a change of basis in V. In the invertible case, if we replace the basis element $\ell \in V$ by $\lambda^{-1}\ell$, $\lambda \in \mathbb{C}^*$, the function $M(g)$ transforms as follows:

$$M(g) \mapsto \begin{cases} M(g), & \rho(g) = 0 \\ \lambda^{-1}\lambda^* M(g), & \rho(g) = 1. \end{cases} \tag{2.16}$$

This is precisely the shift of the twisted 1-cocycle $M(g)$ by a twisted coboundary. Thus equivalence classes of invertible unoriented equivariant $D = 0$ TQFTs are classified by elements of the twisted cohomology group $H^1(BG, \mathbb{C}^*_\rho) \simeq H^1(BG, U(1)_\rho)$.

2.3 Unoriented equivariant $D = 1$ TQFT

Definition of unoriented equivariant TQFT

For $D > 0$ we can avoid anti-linear operators by interpreting the orientation-reversing symmetry as a parity symmetry (P or CP). Since CPT is a symmetry of any local unitary QFT, we do not loose generality by doing this. Thus $\rho(g) = -1$ if g reverses spatial orientation and $\rho(g) = 1$ otherwise.

At first we will try to be as general as possible and do not fix the spatial dimension D. Consider a finite group G together with a homomorphism $\rho : G \to \mathbb{Z}_2$, and let G_0 denote the kernel of ρ. For any manifold X we will denote by $o(X)$ its orientation bundle. Any TQFT is defined as a functor from a geometric source category with a symmetric monoidal structure to the category of finite-dimensional vector spaces Vect (or more generally, to a symmetric monoidal category).

In the case of equivariant TQFT based on the pair (G, ρ), the source category C is defined as follows. An object of C is a closed D-manifold M, a base point for every connected component of M, a G-bundle E over M, a trivialization of G at every base point, and a trivialization of $o(M) \otimes \rho(E)$ everywhere on M. Here, $\rho(E)$ denotes the $\mathbb{Z}_2$-bundle given by the quotient of $E \times \mathbb{Z}_2$ by $(e, x) \sim (eg^{-1}, \rho(g)x)$, and the last datum expresses the fact that $\rho(E)$ is isomorphic to the orientation bundle of X. A morphism of C is an isomorphism class of a $D + 1$-dimensional bordism N equipped with a G-bundle E and a trivialization of $o(N) \otimes \rho(E)$, with every connected component of the boundary given a base point and a trivialization of E there. Two such bundles are said to be isomorphic if they are related by a bundle map that is an homeomorphism of the total space, covers a homeomorphism of the base space, and preserves the trivialization and boundary data. There is also a decomposition of the boundary into two disjoint parts, corresponding to the source and target of the morphism. Composition of morphisms is obvious. The symmetric monoidal structure arises from the operation of disjoint union.

Let us now specialize to the case $D = 1$. In this case the definition can be simplified, because all 1d manifolds are orientable. Since we are given trivializations of E at all base points, as well a trivialization of $o(M) \otimes \rho(E)$, we also have a trivialization of $o(M)$ at all base points. But since M is orientable, this means that we are given a trivialization of $o(M)$ everywhere, i.e. an orientation. Then $\rho(E)$ is also trivialized everywhere, and the G-bundle reduces to a G_0-bundle. Thus the objects for C are exactly the same as in the oriented equivariant TQFT with symmetry group G_0. Morphisms are different however, for example because unorientable bordisms are now allowed. Moreover, even when bordisms are orientable, they are not given an orientation. More precisely, if the boundary of a bordism is connected, there is a base point with an orientation on it, and one can use this to extend orientation to the whole N. But if more than one base point is present, there is no guarantee that orientations so obtained agree between each other. This will be discussed in more detail below.

Algebraic description for $D = 1$

From the above definition we extract the following algebraic data. First of all, let $M = S^1$. As remarked above, S^1 is actually oriented, and the structure group G is reduced to G_0. Thus unoriented equivariant TQFT assigns a vector space $\mathcal{A}_g$ to every $g \in G_0$.

Now consider a cylinder regarded as a bordism from S^1 to S^1. It has two marked points on the boundaries which we call p_- and p_+ (source and target). A G-bundle over a cylinder trivialized over p_- is determined by the holonomy around the source S^1 and thus is labeled by an element $g \in G$. We are also given a trivialization at p_+, and the holonomy along a path from p_- to p_+ gives a well-defined element $h \in G$. We know that $g \in G_0$, but h can be an arbitrary element of G. If $\rho(h) = 1$, the two trivializations of $\rho(E)$ obtained from the trivializations of E at p_- and p_+ agree. Then, since $o(N) \otimes \rho(E)$ is trivialized everywhere, the orientations at p_- and p_+ also agree, and the source and target circles have the same orientation. Thus the source is labeled by g, and the target by hgh^{-1}, and the cylinder is assigned a map $\alpha_h : \mathcal{A}_g \to \mathcal{A}_{hgh^{-1}}$. Similarly, if $\rho(h) = -1$, the two orientations disagree, and the target is labeled by $hg^{-1}h^{-1}$, while the source is still labeled by g. Such a cylinder is assigned a map $\alpha_h : \mathcal{A}_g \to \mathcal{A}_{hg^{-1}h^{-1}}$. We can summarize both cases by saying that α_h maps $\mathcal{A}_g$ to $\mathcal{A}_{hg^{\rho(g)}h^{-1}}$. Since gluing two cylinders labeled by (g, h) and $(hg^{\rho(g)}h^{-1}, h')$ using the trivial identification of target and source circles gives a cylinder labeled by $(g, h'h)$, we must have $\alpha_{h'} \circ \alpha_h = \alpha_{h'h}$. In particular, each α_h is invertible.

In general, we note that if N is an orientable bordism, and the paths between base points on different boundary components all lie in G_0, the morphism becomes a morphism in the oriented equivariant theory with symmetry group G_0. Thus we get all the same algebraic data as in the oriented G_0-equivariant theory. That is, a G_0-crossed algebra

$$\mathcal{A} = \oplus_{g \in G_0} \mathcal{A}_g, \quad \eta : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}, \quad \alpha : G_0 \to \mathrm{A}ut\, \mathcal{A}, \tag{2.17}$$

satisfying (2.3)-(2.8). In particular, for $h \in G_0$ the map α_h is an automorphism of $\mathcal{A}$. On the other hand, for $h \notin G_0$ the map α_h is an anti-automorphism:

$$\alpha_h(ab) = \alpha_h(b)\alpha_h(a), \quad \forall h \notin G_0, \forall a, b \in \mathcal{A}. \tag{2.18}$$

To see this, we compare the pants diagrams with cylinders attached either to the torso or to the pant legs and note that for $h \notin G_0$ they are related by a reflection rather than the identity homeomorphism.

Finally, in the unoriented case we have cross-cap states $\theta_g \in \mathcal{A}_{g^2}$, $g \notin G_0$. The state θ_g, $g \notin G_0$, arises from a Möbius strip with an oriented boundary and a base point on the boundary. The fundamental group of the Möbius strip is isomorphic to $\mathbb{Z}$, where an orientation-reversing generator is fixed once the orientation of the boundary has been fixed. θ_g corresponds to a G-bundle whose holonomy along this generator is g.

The cross-cap states have the following properties:

$$\alpha_{h\in G_0}(\theta_g) = \theta_{hgh^{-1}} \text{ and } \alpha_{h\notin G_0}(\theta_g) = \theta_{hg^{-1}h^{-1}} \tag{2.19}$$

$$\theta_g \cdot \psi_k = \alpha_g(\psi_k) \cdot \theta_{gk} \text{ for all } \psi_k \in \mathcal{A}_k. \tag{2.20}$$

$$\sum_i \alpha_g(\xi^i_{gh})\xi^{gh}_i = \theta_g \cdot \theta_h. \tag{2.21}$$

The first of these properties is illustrated in Figure 2.1. The vectors $\theta_{hg^{(-1)}h^{-1}}$ and $\alpha_h(\theta_g)$ are defined by the two pictures which happen to be related by an isotopy. The second property arises from an isotopy of the punctured Möbius strip shown in Figure 2.2. The third property arises from the fact that a Klein bottle with two holes can be represented in two apparently different ways: as a cylinder with an orientation-reversing twist, or as a cylinder with an insertion of two cross-caps, see Figure 2.3.

We will call the data $(\mathcal{A}, \eta, \alpha, \theta_g, g \notin G_0)$ a ρ-twisted G-crossed algebra.

Proposition 1. *Unoriented equivariant $D = 1$ TQFTs with symmetry $(G, \rho : G \to \mathbb{Z}_2)$ are in bijective correspondence with ρ-twisted G-crossed algebras $(\mathcal{A}, \eta, \alpha, \theta_g, g \notin G_0)$.*

We have already explained how to assign this algebraic data to any unoriented equivariant $D = 1$ TQFT. The converse procedure is described in Appendix B.1.

Invertible unoriented equivariant $D = 1$ TQFT

Let us now specialize to the invertible case. For an invertible unoriented equivariant $D = 1$ TQFT, the vector spaces $\mathcal{A}_{g \in G_0}$ are one-dimensional. After fixing a basis $\{\ell_g\}_{g \in G_0}$ of $\mathcal{A}$ so that $\eta(\ell_g, \ell_{g^{-1}}) = 1$, the ρ-twisted G-crossed algebra is determined by nonzero complex numbers $\theta(g)$, $g \notin G_0$, $b(h,k), z(h,k)$, $h, k \in G_0$, $w(h,k)$, $h \notin G_0$, $k \in G_0$ defined as follows:

$$\begin{aligned} m_{k,l}(\ell_k, \ell_l) &= b(k,l)\ell_{kl}, & \theta_g &= \theta(g)\ell_{g^2}, \\ \alpha_{h \in G_0}(\ell_k) &= z(h,k)\ell_{hkh^{-1}}, & \alpha_{h \notin G_0}(\ell_k) &= w(h,k)\ell_{hk^{-1}h^{-1}}. \end{aligned} \tag{2.22}$$

These numbers satisfy a number of identities due to the properties of $\mathcal{A}$.

Proposition 2. *Invertible unoriented equivariant $D = 1$ TQFTs with symmetry (G, ρ) are in bijective correspondence with elements of the ρ-twisted group cohomology $H^2(BG, U(1)_\rho)$.*

Twisted cohomology is the cohomology of the usual group cochain complex with respect to the ρ-twisted coboundary maps

$$\delta^n_\rho : C^n(G, U(1)) \to C^{n+1}(G, U(1)). \tag{2.23}$$

In degree 2, the ρ-twisted cocycle condition reads

$$a(g,h)a(gh,k) = a(h,k)^{\rho(g)}a(g,hk) \tag{2.24}$$

A proof of Proposition 2 is rather lengthy, see Appendix B.2. But the map in one direction, from twisted group cohomology to the set of algebraic data (2.22), is easy to describe:

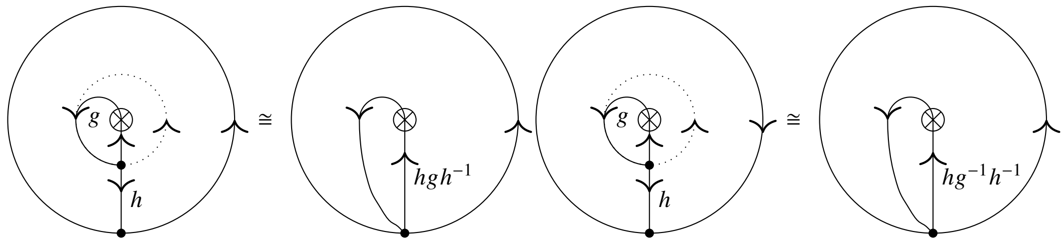
$$b(k,l) = a(k,l) \tag{2.25}$$

$$\theta(g) = a(g,g) \tag{2.26}$$

$$z(h,k) = \frac{a(h,k)a(hk,h^{-1})}{a(h,h^{-1})} \tag{2.27}$$

$$w(h,k) = \frac{a(h,k^{-1})a(hk^{-1},h^{-1})a(k,k^{-1})}{a(h,h^{-1})} \tag{2.28}$$

To prove Proposition 2, we must show that these numbers satisfy the TQFT axioms (2.18)-(2.21) and that the map is injective and surjective.



(a) $h \in G_0$.

(b) $h \notin G_0$.

Figure 2.1: Axiom (2.19).

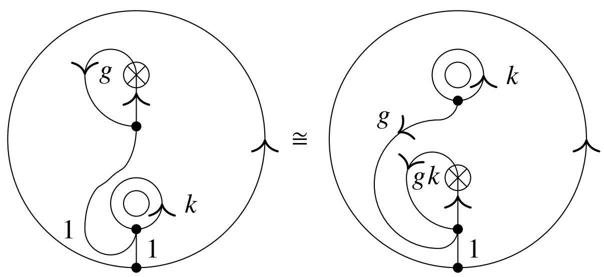


Figure 2.2: Axiom (2.20). To obtain the right figure from the left, the puncture with holonomy k is pulled through the crosscap along the path with holonomy g.

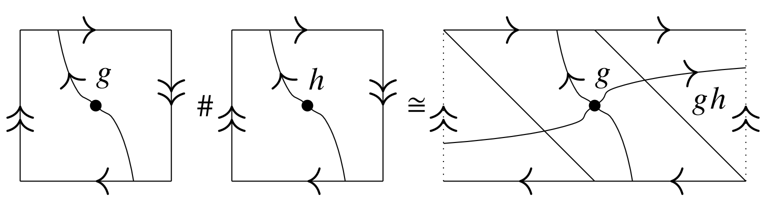


Figure 2.3: Axiom (2.21). Two projective planes are punctured and sewed along their boundaries - the diagonal lines - to obtain their connected sum, the Klein bottle.

Chapter 3

TOPOLOGICAL FIELD THEORY, MATRIX PRODUCT STATES, AND THE STACKING LAW IN ONE DIMENSION

Kapustin, A., A. Turzillo, and M. You (2017). "Topological field theory and matrix product states". In: *Phys. Rev. B* 96 (7), p. 075125. DOI: 10.1103/PhysRevB.96.075125. arXiv: 1607.06766.

– (2018). "Spin topological field theory and fermionic matrix product states". In: *Phys. Rev. B* 98 (12), p. 125101. DOI: 10.1103/PhysRevB.98.125101. arXiv: 1610.10075.

Turzillo, A. and M. You (2019). "Fermionic matrix product states and one-dimensional short-range entangled phases with antiunitary symmetries". In: *Phys. Rev. B* 99 (3), p. 035103. DOI: 10.1103/PhysRevB.99.035103. arXiv: 1710.00140.

Forward

It is believed that most (perhaps all) gapped phases of matter can be described at long distances by Topological Quantum Field Theory (TQFT). On the other hand, it has been rigorously established that in 1+1d ground states of gapped Hamiltonians can be approximated by Matrix Product States (MPS). This chapter shows that the state-sum construction of 2d TQFT naturally leads to MPS in their standard form. In the case of systems with a global symmetry G, this leads to a classification of gapped phases in 1+1d in terms of Morita-equivalence classes of G-equivariant algebras. Non-uniqueness of the MPS representation is traced to the freedom of choosing an algebra in a particular Morita class. In the case of Short-Range Entangled phases, we recover the group cohomology classification of SPT phases. We also study state-sum constructions of G-equivariant spin-TQFTs and their relationship to Matrix Product States. In the Neveu-Schwarz, Ramond, and twisted sectors, states of the TQFT are generalized MPS. Our results are applied to the classification of fermionic Short-Range-Entangled phases with a unitary symmetry G to determine the group law on the set of such phases. Interesting subtleties appear when the total symmetry group is a nontrivial extension of G by fermion parity. Later, we extend the formalism of MPS to describe one-dimensional gapped systems of fermions with both unitary and anti-unitary symmetries. Additionally, systems with orientation-reversing spatial symmetries are considered. The short-ranged entangled phases of such systems are classified by three invariants, which characterize the projective action of the symmetry on edge states. We give interpretations of these invariants as properties of states on the closed chain. The relationship between fermionic MPS systems at an RG fixed point and equivariant algebras is exploited to derive a group law for the stacking of fermionic phases. The result generalizes known classifications to symmetry groups that are non-trivial extensions of fermion parity and time-reversal.

Background and Overview

Chapter 1 discussed a classification of SRE phases in all dimensions (Kapustin, 2014b; Kapustin, Thorngren, et al., 2015). In the case of bosonic (resp. fermionic) SRE phases phases with an internal finite symmetry G in d spatial dimensions, the claim is that they are classified by the torsion part of the $(d + 1)$-dimensional oriented cobordism (resp. spin-cobordism) of BG with $U(1)$ coefficients. Here BG is a certain infinite-dimensional topological space known as the classifying space of G. This conjecture is partially explained by the recently proved mathematical theorem (Freed and Hopkins, 2016) which states that oriented (resp. spin) $(d + 1)$-dimensional cobordism groups classify unitary invertible oriented (resp. spin) Topological Quantum Field Theories in $d + 1$ space-time dimensions. This is only a partial explanation, because the relation between SRE phases and TQFTs remains conjectural. It is a widely held belief that the universal long-distance behavior of a quantum phase of matter at zero temperature can be encoded into an effective field theory.[1] In the case of gapped phases of matter, the extreme infrared should be described by a Topological Quantum Field Theory.

Matrix Product States (MPS) have proven useful at describing the ground states of gapped local Hamiltonians in one spatial dimension (Hastings, 2007; Verstraete, J. I. Cirac, and Murg, 2008). This representation leads to a classification of interacting short-range entangled (SRE) bosonic phases with a symmetry G in terms of the group cohomology of G (X. Chen, Gu, and Wen, 2011a; Fidkowski and Kitaev, 2010). One-dimensional systems of fermions are related to these bosonic systems by the Jordan-Wigner transformation, and this fact has been exploited to classify fermionic SRE phases (Fidkowski and Kitaev, 2010; X. Chen, Gu, and Wen, 2011b).

In 1d, one could hope for a more direct connection between the cobordism/TQFT data and the MPS data. This chapter explores the connection between these two approaches to gapped phases of matter. This approach has the benefits of straight-

[1] It is hard to make this rigorous since neither the notion of a phase of matter nor that of an effective field theory has been formalized.

forwardly describing systems on a closed chain with twisted boundary conditions and allowing one to derive a group law for the stacking of fermionic SRE phases. Let us now describe the structure of this chapter and the main results.

We begin by studying bosonic phases. We show that a standard-form MPS is naturally associated with a module M over a finite-dimensional semisimple algebra A. The universality class of the MPS depends only on the center $Z(A)$. On the other hand, every unitary 2d TQFT has a state-sum construction which uses a semisimple algebra as an input. Further, given a module M over this algebra, one naturally gets a particular state in the TQFT space of states. We show that this state is precisely the MPS associated to the pair (A, M). Since the TQFT depends only on $Z(A)$, we reproduce the fact that the universality class of the MPS depends only on $Z(A)$. In the case of an MPS with a symmetry G, a similar story holds. A G-equivariant MPS is encoded in a G-equivariant module M over a G-equivariant semisimple algebra A. Such an algebra can be used to give a state-sum construction of a G-equivariant TQFT, while every G-equivariant module M gives rise to a particular state. This state is an equivariant MPS state. Again, different A can give rise to the same TQFT. This leads to an equivalence relation on G-equivariant algebras which is a special case of Morita equivalence. An indecomposable phase with symmetry G is therefore associated with a Morita-equivalence class of indecomposable G-equivariant algebras. The classification of such algebras is well known(Ostrik, 2003) and leads to an (also well-known (X. Chen, Gu, and Wen, 2011a; X. Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011)) classification of bosonic 1+1d gapped phases of matter with symmetry G. In the special case of Short-Range Entangled gapped phases, we recover the group cohomology classification of SPT phases.

We then move on to fermionic phases. We review the state-sum construction of spin-TQFTs in two space-time dimensions from $\mathbb{Z}_2$-graded algebras following (Novak and Runkel, 2015; Gaiotto and Kapustin, 2016). We also show that stacking fermionic systems together corresponds to taking the supertensor product of the corresponding algebras. This gives a very clean and simple derivation of the spin-statistics relation in the topological case. Next we evaluate the annulus diagram and show that it gives rise to a generalized MPS both in the Neveu-Schwarz and the Ramond sector. We then work out the commuting projector Hamiltonian starting from the TQFT data describing an invertible spin-TQFT. We show that for a nontrivial spin-TQFT the resulting Hamiltonian describes the Majorana chain (Fidkowski and Kitaev, 2010).

We discuss G-equivariant spin-TQFT and G-equivariant fermionic MPS. We show that fermionic SRE phases with a symmetry G times the fermion parity are in 1-1 correspondence with invertible G-equivariant spin-TQFTs, and that the TQFT data give rise to fermionic G-equivariant MPS. We also discuss the case when the symmetry is a nontrivial extension $\mathcal{G}$ of G by fermion parity, which is related to $\mathcal{G}$-Spin TQFTs. In all cases we determine the group law on the set of fermionic SRE phases. As an example, we discuss in some detail fermionic SRE phases with symmetry $\mathbb{Z}_2$. Before continuing to a discussion of anti-unitary symmetries, we review the formalism of $\mathcal{G}$-equivariant fMPS for a unitary on-site symmetry $\mathcal{G}$. We recall how fermionic SRE phases are classified by Morita classes of equivariant algebras and how the invariants α, β, and γ that characterize these algebras appear in the action of $\mathcal{G}$ on edge degrees of freedom. We then derive interpretations of the invariants on the closed chain that extend the results of Ref. (Kapustin and Thorngren, 2017), which were discovered in the context of spin-TQFT. Next, time-reversing symmetries and their relation to spatial parity are discussed. The generalizations of the three invariants to phases with such symmetries are derived and interpreted. A general stacking law (3.185) is derived for fermionic SRE phases with a symmetry $\mathcal{G}$ that is a central extension by fermion parity of a bosonic symmetry group that may contain anti-unitary symmetries. We contrast this result with the bosonic group structure and emphasize the origin of the difference. Finally, we demonstrate our result with several examples, recovering the $\mathbb{Z}/8$ classification of fermionic SRE phases in the symmetry class BDI ($T^2 = 1$) and the $\mathbb{Z}/2$ classification in the class DIII ($T^2 = P$).

3.1 Matrix Product States at RG Fixed Points

Matrix Product States

In this section, we review Matrix Product States (MPS) and extract the algebraic data that characterizes them at fixed points of the Renormalization Group (RG). We find that a fixed point MPS is described by a module over a finite-dimensional semisimple algebra. We discuss the notion of a gapped phase and argue that they are classified by finite-dimensional semisimple commutative algebras. Given a fixed point MPS and the corresponding semisimple algebra A, the commutative algebra characterizing the gapped phase is the center of A, denoted $\mathcal{A} = Z(A)$.

The models we consider are defined on Hilbert spaces that are tensor products of finite-dimensional state spaces A on the sites of a 1D chain. We are interested in Hamiltonians with an energy gap that persists in the thermodynamic limit of an infinite chain. A large class of examples of gapped systems come from local commuting projector (LCP) Hamiltonians; that is, $H = \sum h_{s,s+1}$, where the $h_{s,s+1}$ are projectors that act on sites s, $s+1$ and commute with each other. Since the local projectors commute, an eigenstate of H is an eigenstate of each projector. It follows that the gap of H is at least 1. Thus LCP Hamiltonians are gapped in the thermodynamic limit. In one spatial dimension, ground states of gapped Hamiltonians are efficiently approximated by an ansatz called a matrix product state (MPS),(Hastings, 2007) which we recall below.[2] From each MPS, one can construct a gapped *parent Hamiltonian* that has the MPS as a ground state.(Fannes, Nachtergaele, and Werner, 1992) At RG fixed points, which we consider below, the parent Hamiltonian is an LCP Hamiltonian. To discuss and classify 1D gapped Hamiltonians, it suffices to consider the parent Hamiltonians of the MPS that approximate their ground states.

Consider a closed chain of N sites, each with a copy of a *physical* Hilbert space $A \simeq \mathbb{C}^d$ and two copies V^L, V^R of a *virtual* space $\mathbb{C}^D$. We identify $V^L = V$ and $V^R = V^*$ and choose a Hilbert space structure on V. Between each adjacent pair $(s, s+1)$ of sites, place the maximally entangled state

$$|\omega\rangle_{s,s+1} = \sum_{i=1}^{D} |i\rangle \otimes |i\rangle \in V_s^R \otimes V_{s+1}^L \tag{3.1}$$

An MPS tensor[3] is a linear map $\mathcal{P} : V^L \otimes V^R \to A$. The MPS associated to $\mathcal{P}$ is the state

$$\begin{aligned} |\psi_{\mathcal{P}}\rangle = &(\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_N) \\ &(|\omega\rangle_{12} \otimes |\omega\rangle_{23} \otimes \cdots \otimes |\omega\rangle_{N1}) \end{aligned} \tag{3.2}$$

in $A^{\otimes N}$. Since $|\psi_{\mathcal{P}}\rangle$ lies in the image of $\mathcal{P}^{\otimes N}$, we do not lose generality by truncating A to i$m\,P$. We will assume we have done so in the following. Equivalently, we assume that the adjoint MPS tensor $T = \mathcal{P}^\dagger$ is injective[4]. The MPS wavefunction can be expressed as a trace of a product of matrices, hence its name. In the basis $\{e_i\}_{i=1,\ldots,d}$ of A, the conjugate state takes the form

$$\langle\psi_T| = \sum_{i_1 \cdots i_N = 1}^{d} \mathrm{Tr}\left[T(e_{i_1}) \cdots T(e_{i_N})\right] \langle i_1 \cdots i_N| \tag{3.3}$$

[2] We only consider translationally-invariant MPS.

[3] More generally, the tensors $\mathcal{P}_s$ may depend on the site index s. But any translationally-invariant state has an MPS representation with a site-independent tensor.(Perez-García et al., 2006)

[4] To avoid confusion, we stress that injectivity of T is unrelated to the notion of an injective MPS in the sense of (Schuch, Perez-García, and I. Cirac, 2010). In particular, while we will always assume that T is injective, we will not assume that the the ground state of the parent Hamiltonian is unique.

There may be many different ways to represent a given state in $A^{\otimes N}$ in an MPS form. Even the dimension of the virtual space V is not uniquely defined. In general, it is not immediate to read off the properties of the state ψ_T from the tensor T.

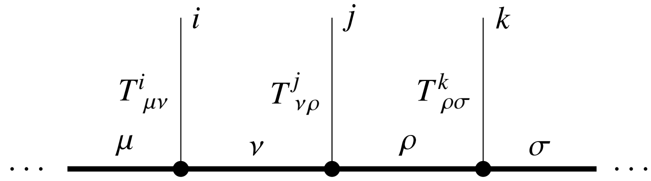


Figure 3.1: An MPS represented as a tensor network

For the tensor T, one can construct a LCP Hamiltonian H_T, called the *parent Hamiltonian*[5] of $|\psi_T\rangle$, which has $|\psi_T\rangle$ as a ground state. It is given as a sum of 2-site terms $h_{s,s+1}$ that project onto the orthogonal complement of $\ker h = (\mathcal{P} \otimes \mathcal{P})(V \otimes |\omega\rangle \otimes V^*)$. Explicitly,

$$
\begin{aligned}
H_T &= \sum_s h_{s,s+1} \qquad \text{where} \\
h_{s,s+1} &= \mathbb{1} - (\mathcal{P}_s \otimes \mathcal{P}_{s+1})\delta(\mathcal{P}_s^+ \oplus \mathcal{P}_{s+1}^+)
\end{aligned}
\tag{3.4}
$$

where δ is the projector onto $(V_s \otimes |\omega\rangle \otimes V_{s+1}^*)$ and $\mathcal{P}_s^+ := (T_s\mathcal{P}_s)^{-1}T_s$ is a left inverse of $\mathcal{P}_s$. The local projectors $h_{s,s+1}$ commute, so H_T is gapped. $|\psi_T\rangle$ is annihilated by $h_{s,s+1}$, $\forall s$ and therefore also by H_T.

In general, H_T has other ground states. Consider a state of the form

$$
\begin{aligned}
\left|\psi_T^X\right\rangle = &(\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_N) \\
&\Big(|\omega\rangle_{12} \otimes |\omega\rangle_{23} \otimes \cdots \otimes \left|\omega^X\right\rangle_{N1}\Big)
\end{aligned}
\tag{3.5}
$$

for some virtual state

$$
\left|\omega^X\right\rangle = \sum_{i=1}^{D} X_{ij}\, |i\rangle \otimes |j\rangle \in V^* \otimes V \tag{3.6}
$$

where X is a matrix that commutes with $T(a)$ for all $a \in A$. Note that $\left|\omega^{\mathbb{1}}\right\rangle = |\omega\rangle$ and so $\left|\psi_T^{\mathbb{1}}\right\rangle = |\psi_T\rangle$. The states (3.5) are clearly annihilated by $h_{s,s+1}$ for $s \neq N$. To see that they are annihilated by h_{N1}, note that tensor $T(e_i)XT(e_j)$ is expressible as a linear combination of tensors $T(e_i)T(e_j)$ if and only if X commutes with every $T(e_i)$. The conjugate states have wavefunctions

$$
\langle\psi_T^X| = \sum \mathrm{Tr}\big[X^\dagger T(e_{i_1}) \cdots T(e_{i_n})\big] \langle i_1 \cdots i_N| \tag{3.7}
$$

[5] There is a more general notion of a parent Hamiltonian where h is any operator with this kernel; however, we will always take h to be the projector.

We will refer to these states as *generalized MPS*.

It turns out that all ground states of H_T can be written as generalized MPS. One can always take T to be an isometry with respect to some inner product on A and the standard inner product

$$\langle M|N\rangle = \mathrm{Tr}\left[M^\dagger N\right] \quad M, N \in \mathrm{E}nd(V) \tag{3.8}$$

on $\mathrm{E}nd(V)$. For an orthogonal basis $\{e_i\}$ of A, $\mathrm{Tr}\left[T(e_i)^\dagger T(e_j)\right] = \delta_{ij}$. Consider the case $N = 1$. An arbitrary state

$$\langle\psi| = \sum_i a_i \langle i| \tag{3.9}$$

can be written in generalized MPS form (3.7) if one takes

$$X = \sum_j a_j T(e_j)^\dagger \tag{3.10}$$

Thus generalized MPS with commuting X are the only ground states. Neither the number of generalized MPS nor the number of ground states depends on N; thus, the argument extends to all N.

Suppose the data (A_1, V_1, T_1) and (A_2, V_2, T_2) define two MPS systems with parent Hamiltonians H_1 and H_2. Consider the composite system $(A_1 \otimes A_2, V_1 \otimes V_2, T_1 \otimes T_2)$. It has $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$ and $\delta = \delta_1 \otimes \delta_2$. Then

$$\begin{aligned} h_{A\otimes B} &= \mathbb{1}_{A_1\otimes A_2} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_1\otimes A_2} \\ &= \mathbb{1}_{A_1} \otimes \mathbb{1}_{A_2} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_1} \otimes \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_2} \\ &= (\mathbb{1}_{A_1} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_1}) \otimes \mathbb{1}_{A_2} + \mathbb{1}_{A_1} \otimes (\mathbb{1}_{A_2} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_2}) \\ &= h_{A_1} \otimes \mathbb{1}_{A_2} + \mathbb{1}_{A_1} \otimes h_{A_2} \end{aligned} \tag{3.11}$$

where the penultimate line follows from the fact that $\mathcal{P}^2\delta\mathcal{P}^{+2}$ is a projector. Therefore, the composite parent Hamiltonian is

$$H_{A\otimes B} = H_{A_1} \otimes \mathbb{1}_{A_2} + \mathbb{1}_{A_1} \otimes H_{A_2}. \tag{3.12}$$

RG-fixed MPS and gapped phases

Under real-space renormalization group (RG) flow,(Verstraete, J. I. Cirac, Latorre, et al., 2005) adjacent pairs of sites are combined into blocks with physical space $A \otimes A$. The MPS form of the state is preserved, with the new MPS tensor being

$$T'(a \otimes b) = T(a)T(b), \tag{3.13}$$

where on the r.h.s. the multiplication is matrix multiplication. We also define $\mathcal{P}' = T'^{\dagger}$. Though an RG step squares the dimension of the codomain of the MPS tensor, the rank is bounded above by D^2, and so the truncated physical space $im(\mathcal{P}')$ never grows beyond dimension D^2.

An *RG fixed MPS tensor* is an MPS tensor such that $\mathcal{P}$ and $\mathcal{P}'$ have isomorphic images and are identical (up to this isomorphism) as maps. That is, there exists an injective map $\mu : A \longrightarrow A \otimes A$ such that

$$\mu \circ \mathcal{P} = \mathcal{P}'. \tag{3.14}$$

If we denote $m = \mu^{\dagger}$, this is equivalent to

$$T(m(a \otimes b)) = T(a)T(b). \tag{3.15}$$

Since T was assumed to be injective, this equation completely determines m. Similarly, the fact that matrix multiplication is associative implies that $m : A \otimes A \longrightarrow A$ is an associative multiplication on A. The map $T : A \longrightarrow End(V)$ then gives V the structure of a module over A. Since T is injective, this module is faithful (all nonzero elements of A act nontrivially). The statement that X commutes with T in the ground state of the parent Hamiltonian is the statement that X is a module endomorphism of V.

As previously stated, a state in $A^{\otimes N}$ may have multiple distinct MPS descriptions. One can always choose T to have a certain *standard form*(Schuch, Perez-García, and I. Cirac, 2010) – regardless of whether it is RG fixed. When this is done, the matrices $T(a)$ are simultaneously block-diagonalized, for all $a \in A$. Moreover, if we denote by $T^{(\alpha)}$ the α^{th} block, say of size $L_\alpha \times L_\alpha$, then the matrices $T^{(\alpha)}(e_i)$ span the space of $L_\alpha \times L_\alpha$ matrices. That is, $T^{(\alpha)}$ defines a surjective map from A to the space of $L_\alpha \times L_\alpha$ matrices.

For an RG-fixed MPS tensor in its standard form, one can easily see that A is a direct sum of matrix algebras. Indeed, each block A^α defines a surjective homomorphism T^α from A to the algebra of $L_\alpha \times L_\alpha$ matrices, and if an element of A is annihilated by all these homomorphisms, then it must vanish. Thus we get a decomposition

$$A = \oplus_\alpha A^\alpha, \tag{3.16}$$

where each $A^\alpha = (\ker T^\alpha)^\perp$ is isomorphic to a matrix algebra. We stress that some of these homomorphisms might be linearly dependent, so the number of summands may be smaller than the number of blocks in the standard form of T. An algebra of such a form is semisimple, that is, any module is a direct sum of irreducible modules. More specifically, any module over a matrix algebra of $L \times L$ matrices is a direct sum of several copies of the obvious L-dimensional module. This basic module is irreducible. If, for a particular A^{α}, T contains more than one copy of the irreducible module, the corresponding blocks in the standard form of T are not independent.

The ground-state degeneracy is simply related to the properties of the algebra A. Namely, the number of ground states is equal to the number of independent blocks in a standard-form MPS, or equivalently the number of summands in the decomposition (3.16). Since the center of a matrix algebra consists of scalar matrices and thus is isomorphic to $\mathbb{C}$, one can also say that the number of ground states is equal to the dimension of $\mathcal{A} = Z(A)$.

Two gapped systems are said to be in the same phase if their Hamiltonians can be connected by a Local Unitary (LU) evolution, i.e. if they are related by conjugation with a finite-time evolution operator for a local time-dependent Hamiltonian.(X. Chen, Gu, and Wen, 2010) Clearly, the ground-state degeneracy is the same for all systems in a particular phase. In fact, for 1+1d gapped bosonic systems, it completely determines the phase.(Schuch, Perez-García, and I. Cirac, 2010; X. Chen, Gu, and Wen, 2011a)

It is convenient to introduce an addition operation $\oplus$ on systems and phases. Given two 1+1d systems with local Hilbert spaces A_1 and A_2, we can form a new 1+1d system with the local Hilbert space $A_1 \oplus A_2$. The Hamiltonian is taken to be the sum of the Hamiltonians of the two systems plus projectors which enforce the condition that neighboring "spins" are either both in the A_1 subspace or in the A_2 subspace. The ground state degeneracy is additive under this operation. A phase is called decomposable if it is a sum of two phases, otherwise it is called indecomposable. Clearly, it is sufficient to classify indecomposable phases.

It is easy to see that if A decomposes as a sum of subalgebras, the corresponding phase is decomposable. Further, an indecomposable semisimple algebra A is isomorphic to a matrix algebra. The corresponding ground state is unique. Moreover, while the parent Hamiltonians for different matrix algebras are different, they all correspond to the same phase,(X. Chen, Gu, and Wen, 2011a) i.e. are related by a Local Unitary evolution. Hence the phase is determined by the number of components in the decomposition (3.16), or in other words, by $Z(A)$.

3.2 Topological Quantum Field Theory

We have seen above that an RG-fixed MPS state is associated with a finite-dimensional semisimple algebra A, and that the universality class of the corresponding phase depends only on the center of A. On the other hand, it is known since the work of Fukuma, Hosono, and Kawai (Fukuma, Hosono, and Kawai, 1994) that for any finite-dimensional semisimple algebra A with an invariant scalar product one can construct a unitary 2D TQFT, and that the isomorphism class of the resulting TQFT depends only on the center of A. In this section we show that this is not a mere coincidence, and that the ground states of this TQFT can be naturally written in an MPS form, with an RG-fixed MPS tensor.

State-sum construction of 2d TQFTs

A (closed) 2D TQFT associates a space of states $\mathcal{A}$ to an oriented circle, and a vector space $\mathcal{A}^{\otimes n}$ to n disjoint oriented circles. Further, suppose we are given an oriented bordism from n circles to l circles, i.e. a compact oriented 2d manifold Σ whose boundary consists of l circles oriented in the same way as Σ and n circles oriented in the opposite way. A 2d TQFT associates to Σ a linear map from $\mathcal{A}^{\otimes n}$ to $\mathcal{A}^{\otimes l}$. This map is invariant under diffeomorphisms. Also, gluing bordisms taking care that orientations agree corresponds to composing linear maps.

Let us mention some special cases. If Σ is closed (i.e. has an empty boundary), then the 2D TQFT associates to it a linear map $\mathbb{C} \to \mathbb{C}$, i.e. a complex number Z_Σ, called the partition function. If Σ is a pair-of-pants bordism from two circles to one circle, the corresponding map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defines an associative, commutative product on $\mathcal{A}$. The cap bordism defines a symmetric trace function $\mathrm{T}r : \mathcal{A} \to \mathbb{C}$ such that the scalar product $\eta(a, b) = \mathrm{T}r(ab)$ is symmetric and non-degenerate. These data make $\mathcal{A}$ into a commutative Frobenius algebra. It is known that a two-dimensional TQFT is completely determined by the commutative Frobenius algebra structure on $\mathcal{A}$.(Atiyah, 1989; G. W. Moore and Segal, 2006; Abrams, 1996) The state-operator correspondence identifies $\mathcal{A}$ with the algebra of local operators. This Frobenius algebra encodes the 2- and 3-point functions on the sphere, from which all other correlators, including the partition function, can be reconstructed.

In 2d there is an essentially trivial family of unitary oriented TQFTs parameterized by a positive real number λ. The partition function of such a TQFT on a closed oriented 2d manifold Σ is $\lambda^{\chi}(\Sigma)$, while the Hilbert space attached to a circle is one-dimensional. Such 2d TQFTS are called invertible, since the partition function is a nonzero number for any Σ. Since, by the Gauss-Bonnet theorem, $\chi(\Sigma)$ can be expressed as an integral of scalar curvature, tensoring a 2d TQFT by an invertible 2d TQFT is equivalent to redefining the TQFT action by a local counterterm which depends only on the background curvature. One usually disregards such counterterms. In what follows we will follow this practice and regard TQFTs related by tensoring with an invertible TQFT as equivalent.

Every unitary oriented 2d TQFT[6] has an alternative construction called the state-sum construction,(Fukuma, Hosono, and Kawai, 1994) which is combinatorial and manifestly local. The input for this construction is a finite-dimensional semisimple algebra A, which is not necessarily commutative. To compute the linear maps associated to a particular bordism Σ, one needs to choose a triangulation of Σ. Nevertheless, the result is independent of the choice of the triangulation. The connection between the not-necessarily commutative algebra A and the commutative algebra $\mathcal{A}$ is that $\mathcal{A}$ is $Z(A)$, the center of A. From the perspective of open-closed TQFTs, A is the algebra of states on the interval for a particular boundary condition. The scalar product on $\mathcal{A}$ is also fixed by the structure of A.

Let us describe the state-sum construction for the partition function Z_Σ of a closed oriented 2D manifold Σ, following FHK.(Fukuma, Hosono, and Kawai, 1994) Fix a basis e_i, $i \in S$, of A. We define the following tensors:

$$\eta_{ij} = \eta(e_i, e_j) = \mathrm{Tr}_A\, P_i P_j, \qquad C_{ijk} = \mathrm{Tr}_A\, P_i P_j P_k \tag{3.17}$$

Here $P_i : A \to A$ is the operator of multiplication by e_i. The tensor η_{ij} is symmetric and non-degenerate (if the algebra A is semi-simple); the tensor C_{ijk} is cyclically symmetric. We also denote by η^{ij} the inverse to the tensor η_{ij}. Note also that C_{ijk} is related to the structure constants $C^i{}_{jk}$ in this basis by

$$C^i{}_{jk} = \sum_l \eta^{il} C_{ljk}. \tag{3.18}$$

Let $T(\Sigma)$ be a triangulation of Σ. A coloring of a 2-simplex F of $T(\Sigma)$ is a choice of a basis vector e_i for each 1-simplex $E \in \partial F$. A coloring of $T(\Sigma)$ is a coloring of all 2-simplices of $T(\Sigma)$. Note that each 1-simplex of $T(\Sigma)$ has two basis vectors attached to it, one from each 2-simplex that it bounds. The weight of a coloring is the product of C_{ijk} over 2-simplices and η^{ij} over 1-simplices, where the cyclic

[6]More precisely, every equivalence class of unitary oriented 2d TQFTs, in the sense explained in the previous paragraph.

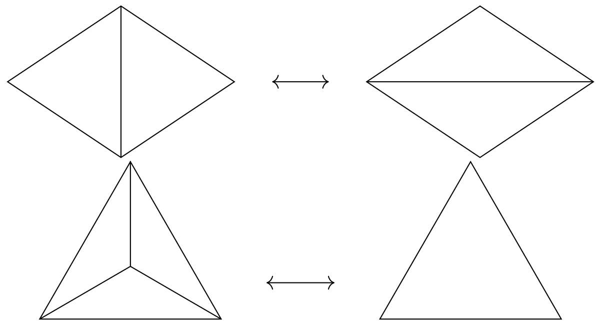


Figure 3.2: The 2-2 and the 3-1 Pachner moves

ordering of indices for each 2-simplex is determined by the orientation of Σ. The partition function is the sum of these weights over all colorings.

Topological invariance of Z_Σ can be shown as follows. It is known that any two triangulations of a smooth manifold are related by a finite sequence of local moves.(Pachner, 1991) In two dimensions, there are two moves - the 2-2 move and the 3-1 move, depicted in Figure 3.2 - which swap two or three faces of a tetrahedron with their complement. Invariance of the state-sum under the 2-2 "fusion" move reads

$$C_{ij}{}^{p} C_{pk}{}^{l} = C_{jk}{}^{p} C_{ip}{}^{l} \tag{3.19}$$

Similarly the 3-1 move reads

$$C_i^{mn} C_{nl}{}^{k} C^{l}{}_{mj} = C_{ij}{}^{k} \tag{3.20}$$

These axioms are satisfied by any finite-dimensional semisimple algebra A;(Fukuma, Hosono, and Kawai, 1994) therefore, the partition sum is a topological invariant[7].

Open-closed 2d TQFT

So far we have discussed what is known as closed 2D TQFTs. That is, the boundary circles were interpreted as spacelike hypersurfaces, and thus each spatial slice had an empty boundary. The notion of a TQFT can be extended to incorporate spatial boundaries; such theories are called open-closed TQFTs. In such a theory a spatial slice is a compact oriented manifold, possibly with an nonempty boundary. That is, it is a finite collection of oriented intervals and circles. A bordism between such spatial slices is a smooth oriented *surface with corners*: paracompact Hausdorff spaces for which each point has a neighborhood homeomorphic to an open subset of a half-plane. Surfaces with corners are homeomorphic, but typically not diffeomorphic, to smooth surfaces with a boundary.

[7] In two dimensions, there is no difference between topological and smooth manifolds.

The corner points subdivide the boundary of the bordism into two parts: the initial and final spatial slices, and the rest. We will refer to the initial and final spatial slices as the cut boundary, while the rest will be referred to as the brane boundary. The cut boundary can be thought of as spacelike, while the brane boundary is timelike. Bordisms are composed along their cut boundary (hence the name), while on the brane boundary one needs to impose boundary conditions (known as D-branes in the string theory context, hence the name). More precisely, if C is the set of boundary conditions, one needs to label each connected component of the brane boundary with an element of C.

An open-closed 2d TQFT associates a vector space $V_{MM'}$ to every oriented interval with the endpoints labeled by $M, M' \in C$, and a vector space $\mathcal{A}$ to every oriented circle. To a collection of thus labeled compact oriented 1D manifolds it attaches the tensor product of spaces $V_{MM'}$ and $\mathcal{A}$. To every bordism with corners labeled in the way explained above, it attaches a linear map from a vector space of the 'incoming" cut boundary to the vector space of the "outgoing" cut boundary. Gluing bordisms along their cut boundaries corresponds to composing the linear maps.

Just like in the case of a closed 2d TQFT, one can describe algebraically the data which are needed to construct a 2d open-closed TQFT. We refer to Moore and Segal(G. W. Moore and Segal, 2006) for details. Suffice it to say that each space V_{MM} is a (possibly noncommutative) Frobenius algebra, and each space $V_{MM'}$ is a left module over V_{MM} and a right module over $V_{M'M'}$. That is, to every element $x \in V_{MM}$ one associates a linear operator $T^M(x) : V_{MM'} \to V_{MM'}$ so that composition of elements of V_{MM} corresponds to the composition of linear operators: $T^M(x)T^M(x') = T^M(xx')$ (and similarly for $V_{M'M'}$). Also, for every $M \in C$ there is a map $\iota^M : \mathcal{A} \to V_{MM}$ which is a homomorphism of Frobenius algebras. The dual map $\iota_M : V_{MM} \to \mathcal{A}$ is known as the generalized boundary-bulk map. In particular, if we act with ι_M on the identity element of the algebra V_{MM}, we get a distinguished element $\psi_M \in \mathcal{A}$ called the boundary state corresponding to the boundary condition M. Geometrically, ψ_M is the element of $\mathcal{A}$ which the open-closed TQFT associates to an annulus whose interior circle is a brane boundary labeled by M, while the exterior circle is an outgoing cut boundary.

One may wonder if it is possible to reconstruct the open-closed TQFT from the closed TQFT. The answer turns out to be yes if $\mathcal{A}$ is a semisimple, i.e. if every module over $\mathcal{A}$ is a sum of irreducible modules.(G. W. Moore and Segal, 2006)

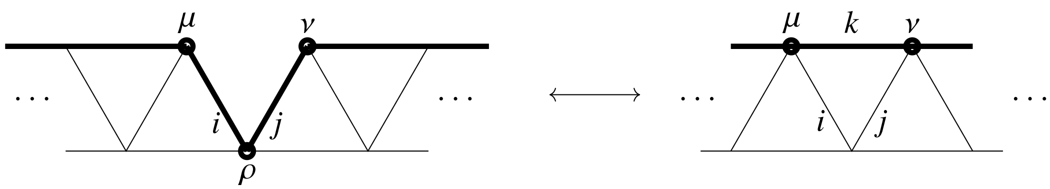


Figure 3.3: An elementary shelling representing $T^{\mu}_{\rho i}T^{\rho}_{\nu j} = C^{k}_{ij}T^{\mu}_{\nu k}$ (3.21). The thick line is a physical boundary.

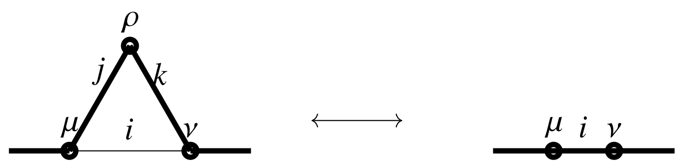


Figure 3.4: An elementary shelling representing $T^{\mu}_{\rho j}T^{\rho}_{\nu k}C^{jk}_{i} = T^{\mu}_{\nu i}$ (3.22).

[8] Then C is the set of finite-dimensional modules over $\mathcal{A}$, and $V_{MM'}$ is the space of linear maps from the module M to the module M' commuting with the action of $\mathcal{A}$ (i.e. $V_{MM'}$ is the space of module homomorphisms). Conversely, one can reconstruct the algebra $\mathcal{A}$ from any "sufficiently large" brane $M \in C$: if we assume that the module M is faithful (i.e. all nonzero elements of $\mathcal{A}$ act nontrivially), then $\mathcal{A} = Z(V_{MM})$.

The state-sum construction generalizes to the open-closed case.(Lauda and Pfeiffer, 2006) Let us describe it for a semisimple $\mathcal{A}$, and assuming that the bordism Σ only has a brane boundary. Each connected component of $\partial\Sigma$ is then labeled by a brane $M \in C$. We pick a sufficiently large brane M_0 such that $\mathcal{A} = Z(V_{M_0M_0})$. Let $A = V_{M_0M_0}$. We also choose a basis f^M_μ, $\mu \in S_M$ in each module M. Denote the matrix elements of the action of A on M by $T^{\mu}_{M\,\nu i}$. We choose a triangulation of Σ, which also gives us a triangulation of each connected component of the boundary. 2-simplices of Σ are labeled as before. Label boundary 0-simplices on any M-labeled boundary component by the basis vectors f^M_μ. Thus each boundary 1-simplex is labeled by a basis vector of A and a pair of basis vectors of a module. We assign a weight to each 2-simplex and each interior 1-simplex before. We also assign a weight to each boundary 1-simplex as follows. Suppose the boundary 1-simplex is labeled by $e_i \in A$ and $f^M_\mu, f^M_\nu \in M$. Then the weight of the boundary 1-simplex is $T^{\mu}_{M\,\nu i}$. The total weight is the product of weights of all 2-simplices and all 1-simplices

[8]This might seem like a rather uninteresting case, since by the Wedderburn theorem every commutative semisimple algebra is isomorphic to a sum of several copies of $\mathbb{C}$. But as explained below unitarity forces $\mathcal{A}$ to be semisimple. Also, in the case of TQFTs with symmetries and fermionic TQFTs the classification of semisimple algebras is more interesting.

(both interior and exterior).

Due to the introduction of brane boundaries, there are two more moves, called the 2-2 and 3-1 *elementary shellings* and depicted in Figures 3.3 and 3.4, that must be considered when demonstrating topological invariance. (Lauda and Pfeiffer, 2006) They yield conditions

$$T^{\mu}_{M\rho i}T^{\rho}_{M\nu j} = C^{k}_{ij}T^{\mu}_{M\nu k} \tag{3.21}$$

and

$$T^{\mu}_{M\rho j}T^{\rho}_{M\nu k}C^{jk}_{i} = T^{\mu}_{M\nu i} \tag{3.22}$$

respectively. The first one is the definition of a module, and the second one follows from the semisimplicity of A. Therefore the state-sum is a well-defined open-closed TQFT. Moreover, such structures are precisely those required to define a topologically invariant state-sum.

Unitary TQFTs and semisimplicity

The state-sum construction defines a perfectly good topological invariant for any finite-dimensional semisimple algebra A; however, if it is to model an actual physical system, its space of states must carry a Hilbert space structure, and linear maps corresponding to bordisms must be compatible in some sense with this structure. To be precise, for any oriented bordism Σ whose source is a disjoint union of n circles and whose target is a disjoint union of l circles, let $-\Sigma$ denote its orientation-reversal. $-\Sigma$ has l circles in its source and n circles in its target. A 2d TQFT attaches to Σ a linear map $\mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes l}$, and to $-\Sigma$ a linear map $\mathcal{A}^{\otimes l} \to \mathcal{A}^{\otimes n}$. A unitary structure on a 2d TQFT is a Hilbert space structure on $\mathcal{A}$ such that the maps corresponding to Σ and $-\Sigma$ are adjoint to each other. For an open-closed 2D TQFT, we require that the state-space assigned to each boundary-colored interval has a non-degenerate Hermitian metric, and that cobordisms with nonempty brane boundary also satisfy the Hermiticity condition. In particular, the product m and coproduct μ are adjoints. It then follows from the Pachner moves that μ is an isometry. Likewise, the module structure T is an isometry.

Let $\langle a, b\rangle$ denote the Hilbert space inner product of $a, b \in \mathcal{A}$. Since $\mathcal{A}$ also has a bilinear scalar product η, we can define an antilinear map

$$* : \mathcal{A} \to \mathcal{A}, \quad a \mapsto a^*, \tag{3.23}$$

such that $\langle a, b\rangle = \eta(a^*, b)$. It can be shown that this map is an involution (i.e. $a^{**} = a$) and an anti-automorphism (i.e. $(ab)^* = b^* a^*$).(Turaev, 2010) This can also be expressed by saying that $\mathcal{A}$ is a $*$-algebra. Conversely, one can show that any commutative Frobenius $*$-algebra such that the sesquilinear product $\eta(a^*, b)$ is positive-definite gives rise to a unitary 2d TQFT.(Turaev, 2010)

A corollary of this result is that for a unitary 2d TQFT the algebra $\mathcal{A}$ is semisimple. To see this, note first that any nonzero self-adjoint element a, $a = a^*$, cannot be nilpotent. Indeed, if n is the smallest n such that $a^n = 0$, then $a^{2m} = 0$, where $m = \lfloor (n+1)/2 \rfloor$. Then $\langle a^m | a^m \rangle = \left\langle 1 | a^{2m} | 1 \right\rangle = 0$, and therefore $a^m = 0$. Since $n \leq m$, repeat with $n' = m$ until $n = 1$, i.e. $a = 0$. Now we can use a result(Kapustin, 2013) which says that a $*$-algebra with no nilpotent self-adjoint elements (apart from zero) is semisimple.

By the Artin-Wedderburn theorem, a finite-dimensional semisimple algebra over complex numbers is isomorphic to a sum of matrix algebras. Since $\mathcal{A}$ is also commutative, this means that it is isomorphic to a sum of several copies of $\mathbb{C}$. Frobenius and $*$-algebra structures exist and are unique up to isomorphism. This means that the only invariant of the 2d TQFT is the dimension of $\mathcal{A}$, i.e. the ground-state degeneracy of the corresponding phase.

As discussed above, for a semisimple algebra $\mathcal{A}$ boundary conditions correspond to finite-dimensional modules over $\mathcal{A}$. It is easy to see that for the open-closed TQFT to be unitary, the algebra V_{MM} must also have a Hilbert space structure such that

$$T(a)^\dagger = T(a^*). \tag{3.24}$$

Such a structure always exists and is unique. Thus a boundary condition for a unitary 2d TQFT can be simply identified with a module over $\mathcal{A}$. One can use any faithful module over $\mathcal{A}$ as an input for the state-sum construction.

State-sum construction of the space of states

We have discussed above the state-sum construction of the partition function $Z(\Sigma)$ for an oriented 2d manifold Σ without boundary (or more generally, with only brane

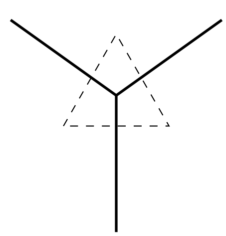


Figure 3.5: The Poincare dual of a triangle

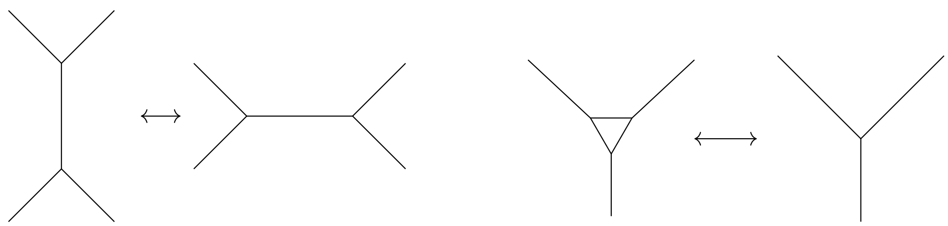


Figure 3.6: The dual 2-2 and 3-1 Pachner moves

boundary). More generally, one also needs to describe in similar terms the state space $\mathcal{A}$ and a linear map $\mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes l}$ for every bordism Σ whose source is a disjoint union of n circles and target is a disjoint union of l circles. That is, one needs to describe $Z(\Sigma)$ for the case when Σ has nonempty cut boundary.

Consider a bordism Σ with a nonempty cut boundary. For simplicity let us assume that there is no brane boundary; the general case is a trivial generalization, but requires a more cumbersome notation. We choose a triangulation $\mathcal{T}$ of Σ. It induces a triangulation of each boundary circle. We label the edges of 2-simplices with basis elements of A, as before. The only difference is that boundary 1-simplices have only one label rather than two. If we assign the weights to every 2-simplex and every internal 1-simplex as before and sum over the labelings of internal 1-simplices, we get a number $Z_{\mathcal{T}}(\Sigma)$ which depends on the labelings of the boundary 1-simplices. Suppose some boundary circle is divided into N intervals. Then a labeling by $e_{i_1}, \ldots, e_{i_N}$ corresponds to a vector

$$e_{i_1} \otimes \ldots \otimes e_{i_N} \in A^{\otimes N}. \tag{3.25}$$

We can think of the number $Z_{\mathcal{T}}(\Sigma)$ computed by the state-sum as a matrix element of a linear map

$$A^{\otimes N_1} \otimes \ldots \otimes A^{\otimes N_n} \longrightarrow A^{\otimes M_1} \otimes \ldots \otimes A^{\otimes M_l}, \tag{3.26}$$

where $N_1, \ldots, N_n$ denote the number of 1-simplices in the source circles, and $M_1, \ldots, M_l$ denote the number of 1-simplices in the target circles of Σ. It can be shown (Fukuma, Hosono, and Kawai, 1994) that the map $Z_{\mathcal{T}}(\Sigma)$ does not depend on the triangulation of Σ, provided we fix the triangulation of the boundary circles.

$Z_{\mathcal{T}}(\Sigma)$ is not yet the desired $Z(\Sigma)$ because it depends on the way the boundary circles are triangulated. To get rid of this dependence, we need to restrict this map to a certain subspace in each source factor $A^{\otimes N_i}$ and project to a certain subspace in each target factor $A^{\otimes M_j}$. Both tasks are accomplished by means of projectors

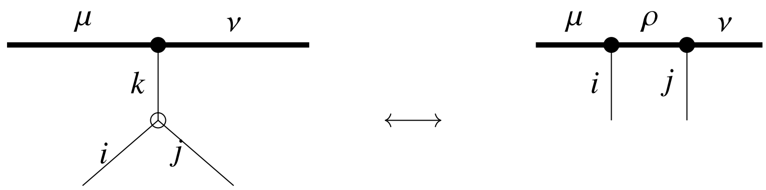


Figure 3.7: The dual shelling of (3.21). A filled dot represents T, while an empty dot represents C.

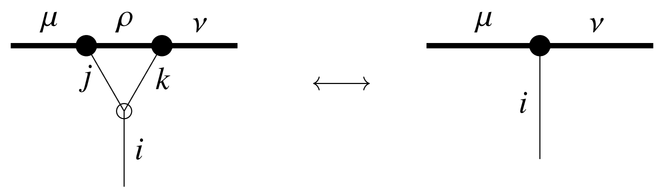


Figure 3.8: The dual shelling of (3.22), representing $T^j_{\mu\rho}T^k_{\rho\nu}C_{ijk} = T^i_{\mu\nu}$

$C_N : A^{\otimes N} \to A^{\otimes N}$. The projector C_N is simply $Z_{\mathcal{T}_N}(C)$, where C is a cylinder and $\mathcal{T}_N$ is any triangulation of C such that both boundary circle are subdivided into N intervals. The image of each C_N is a certain subspace of $A^{\otimes N}$ isomorphic to $Z(A)$.(Fukuma, Hosono, and Kawai, 1994) Restricting $Z_{\mathcal{T}}(\Sigma)$ to these subspaces and then projecting to the image of each C_{M_j} gives us the desired map

$$Z(\Sigma) : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes l}, \tag{3.27}$$

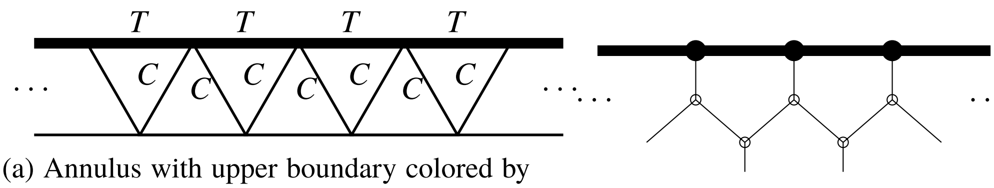
where $\mathcal{A} = Z(A)$.

MPS from TQFT

Let us consider the special case when Σ is an annulus such that one of the circles is a cut boundary, while the other one is a brane boundary corresponding to an A-module M. Let $T(a) \in \mathrm{H}om(M, M)$ represent an action of $a \in A$ in this module. For definiteness, we choose the cut boundary to be the source of Σ, while the target is empty. Thus $Z(\Sigma)$ is a linear map $\mathcal{A} \to \mathbb{C}$. It is the dual of the boundary state corresponding to the module M.

Let us now pick a triangulation of the annulus such that the cut boundary is divided into N intervals. Then $Z_{\mathcal{T}}(\Sigma)$ is a linear map $A^{\otimes N} \to \mathbb{C}$ which depends only on $\mathcal{T}$ and N. We claim that this map is the dual of the MPS state with the dual MPS tensor given by $T : A \to \mathrm{H}om(M, M)$.

To see this, it is convenient to reformulate the state-sum on the Poincare dual complex. This complex is built from the triangulation $\mathcal{T}(\Sigma)$ by replacing k-cells



(a) Annulus with upper boundary colored by a module T

(b) The Poincare dual of **(a)**

(c) The move (3.21) applied to **(b)**

(d) The move (3.22) applied to **(c)**

Figure 3.9: The equivalence of the annulus to the tensor network representation of an MPS

with $(2-k)$-cells, as in Figure 3.5. The dual of a triangulation is not a simplicial complex but a more general cell complex; since we will only be interested in the edges and vertices of this dual complex, we will refer to it as a *skeleton* for Σ. The Pachner moves are the same for skeleton as for triangulations, see Figure 3.6. Recall that for a unitary TQFT, one can choose $\eta_{ij} = \delta_{ij}$, so that indices may be freely raised and lowered; nonetheless, keeping track of index positions now will pay off later when we generalize to equivariant theories. Choose a direction for each edge; the state-sum does not depend on this choice. Choose these directions so that all edges on incoming boundaries are incoming and all edges on outgoing boundaries are outgoing. To define a state-sum on a skeleton, label its non-boundary edges with elements e_i and assign structure coefficients C to each non-boundary vertex according to orientation and using lower indices for incoming arrows and upper for outgoing. With these conventions, the Pachner moves algebrize to (3.19) and (3.20) as before. To incorporate brane boundaries, color brane boundary edges by elements v_μ and attach the module tensor T to each boundary vertex. The boundary moves recover (3.21) and (3.22). The dual state-sum is naturally a tensor network: it defines a circuit between the incoming and outgoing legs. Note that the "virtual" module indices are all contracted, so these legs are physical.

Consider the triangulation, shown in Figure 3.9a, of the annulus with boundary condition T on one of its boundary components. Its state-sum defines a state in the physical space $\mathcal{A}^N$. We claim that this state is the fixed point MPS $|\psi_T\rangle$. The proof of this fact is straightforward: by Pachner invariance, the annulus and MPS tensor networks are equivalent, see Figure 3.9.

More generally, one can insert a local observable on the brane boundary of the annulus. Such a local observable is parameterized by $X \in \mathrm{H}om(M, M)$ which commutes with $T(a)$ for all $a \in A$. The corresponding dual state is $\mathrm{Tr}\big[X^\dagger TT \cdots T\big]$, i.e. it is a generalized MPS state, with A being the physical space.

Since the linear operators $T(a)$ satisfy $T(a)T(b) = T(ab)$, all these MPS states are RG-fixed MPS states. The RG-step is described by the algebra structure on A, $m : A \otimes A \to A$. Moreover, the MPS is automatically in a standard form. The module $T : A \to \mathrm{E}nd(V)$ is semisimple, so it has a decomposition into simple modules $T^{(\alpha)} : A \to \mathrm{E}nd(V^{(\alpha)})$. The collection of spaces $\mathrm{E}nd(V^{(\alpha)})$ form a block-diagonal subspace of $\mathrm{E}nd(V)$. Since $V^{(\alpha)}$ is simple, $T^{(\alpha)}$ surjects onto the block $\mathrm{E}nd(V^{(\alpha)})$. Moreover, as we have seen, unitarity of the TQFT enforces that T is an isometry.

The parent Hamiltonian of the MPS on an N-site closed chain has a TQFT interpretation as well: it is the linear map $C_N = Z_{\mathcal{T}_N}(C) : A^{\otimes N} \to A^{\otimes N}$ assigned to a triangulated cylinder C whose boundary consists of two circles triangulated into N intervals. As previously stated, C_N projects onto a subspace $\mathcal{A} = Z(A) \subset A^{\otimes N}$, precisely the space of ground states of the parent Hamiltonian. In the continuum TQFT, topological invariance requires that the cylinder is the identity; this is consistent with our already having projected to $\mathcal{A}$ in defining the continuum state spaces.

We have seen that a unitary TQFT is completely determined by its space of states $\mathcal{A}$ on a circle and that each finite-dimensional commutative algebra $\mathcal{A}$ defines a unitary TQFT. Therefore, the classification of unitary TQFTs is quite simple: there is one for every positive integer n, in agreement with the MPS-based classification of gapped phases. (X. Chen, Gu, and Wen, 2011a; X. Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011)

3.3 Equivariant TQFT and Equivariant MPS

In this section, we generalize the relation between 2D TQFT and MPS states to systems with a global symmetry G. We show that both G-equivariant TQFTs and G-equivariant RG-fixed MPS states are described by semisimple G-equivariant algebras. In particular, we show that invertible G-equivariant TQFTs correspond to short-range entangled phases with symmetry G, and that both are classified by $H^2(G, U(1))$.

G-equivariant Matrix Product States

Let G be a finite symmetry group acting on the physical space A via a unitary representation R, $g \mapsto R(g) \in \mathrm{E}nd(A)$. A G-invariant MPS tensor is a map $\mathcal{P} : U \otimes U^* \to A$ equivariant in the following sense:

$$R(g)\mathcal{P}(X) = \mathcal{P}\left(Q(g)XQ(g^{-1})\right), \tag{3.28}$$

where the linear maps $Q(g) \in \mathrm{E}nd(U)$ form a projective representation of G. Let $T = \mathcal{P}^\dagger$. In terms of T, the equivariance condition looks as follows:

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1}, \tag{3.29}$$

for any $a \in A$ and any $g \in G$. The dual MPS state corresponding to T is

$$\langle\psi_T| = \sum_{i_1,\ldots,i_N} \mathrm{Tr}_U[T(e_{i_1})\ldots T(e_{i_N})]\langle i_1 \ldots i_N| \tag{3.30}$$

It is easy to see that the state ψ_T is G-invariant, thanks to the equivariance condition on P. More generally, let $X \in \mathrm{E}nd(U)$. Note that $\mathrm{E}nd(U)$ is a genuine (not projective) representation of G. Then the generalized MPS state $\mathrm{Tr}[XTT\ldots T]$ transforms in the same way as X.

G-equivariant TQFT

Roughly speaking, a definition of a G-equivariant TQFT is obtained from the definition of an ordinary TQFT by replacing oriented manifolds with oriented manifolds with principal G-bundles. This reflects the intuition that a model with a global non-anomalous symmetry G can be coupled to a background G gauge field. (For a finite group G, there is no difference between a G gauge field and a principal G-bundle.)

Some care is required regarding marked points and trivializations. Namely, each source and each target circle must be equipped with a marked point and a trivialization of the G-bundle at this point. This means that the holonomy of the gauge field around the circle is a well-defined element $g \in G$, rather than a conjugacy class. A G-equivariant TQFT associates a vector space $\mathcal{A}_g$ to a circle with holonomy g. A generic G-equivariant bordism has more than one marked point, and the holonomies between marked points along chosen paths are well-defined elements of G as well. Of course, these holonomies depend only on the homotopy classes of paths. For example, a G-equivariant cylinder bordism has two marked points (one for each boundary circle) and depends on two arbitrary elements of G. On the other hand, a G-equivariant torus, regarded as bordism with an empty source and empty target, has no marked points and depends on two commuting elements of G defined up to an overall conjugation.

One can describe a G-equivariant TQFT purely algebraically in terms of a G-crossed Frobenius algebra.(Turaev, 2010; G. W. Moore and Segal, 2006) This notion generalizes the commutative Frobenius algebra $\mathcal{A}$ and encodes the linear maps $Z(\Sigma, \mathcal{P})$ in a fairly complicated way.

We will use instead a state-sum construction of 2D equivariant TQFTs which is manifestly local. Its starting point is a finite-dimensional semisimple G-equivariant algebra A. This is an algebra with an action of G that preserves the multiplication $m : A \otimes A \to A$. That is, G acts on A via a linear representation $R(g)$, $g \in G$, such that

$$m(R(g)a \otimes R(g)b) = R(g)m(a \otimes b). \tag{3.31}$$

This condition implies that the group action also preserves the scalar product η defined in (3.17):

$$\eta(R(g)a, R(g)b) = \eta(a, b). \tag{3.32}$$

The condition (3.32) says that $R(g)$ is orthogonal with respect to η. As a consequence, if $R(g)$ commutes with the anti-linear map (3.23), it is unitary with respect to the Hilbert space inner product.

A large class of examples of G-equivariant algebras is obtained by taking $A = \mathrm{E}nd(U)$, where U is a vector space, and G acts on U via a projective representation $Q(g)$. It is clear that this gives rise to a genuine action of G on $\mathrm{E}nd(U)$ which preserves the usual matrix multiplication on $\mathrm{E}nd(U)$. Moreover, the standard Frobenius structure

$$\eta(a, b) = \mathrm{T}r(ab) \tag{3.33}$$

is clearly G-invariant.

A G-equivariant module over a G-equivariant algebra A is a vector space V with compatible actions of both A and G. That is, for every $a \in A$ we have a linear map $T(a) : V \to V$ such that $T(a)T(a') = T(aa')$, and for every $g \in G$ we have an invertible linear map $Q(g) : V \to V$ such that $Q(g)Q(g') = Q(gg')$. The compatibility condition that they satisfy reads

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1} \tag{3.34}$$

If we take $A = \mathrm{E}nd(U)$, where U is a projective representation of G with a 2-cocycle $\omega \in H^2(G, U(1))$, then U is not a G-equivariant module over A unless ω vanishes. However, if W is a projective representation of G with a 2-cocycle $-\omega$, then $U \otimes W$ is a G-equivariant module.[9]

Equivariant TQFTs admit a lattice description as well. It is simplest to describe a Poincare dual formulation in the sense of Section 3.5; spaces in this formulation also have direct interpretations as tensor networks. A trivialized background gauge field is represented on a skeleton as a decoration of each oriented edge with an element $g \in G$. Flipping the orientation of the edge replaces g with g^{-1}. We require that the field is flat: that the product of the group elements around the boundary of each face is the identity element.[10] In a basis e_i, $i \in S$ of A, the weight of a coloring of the skeleton is the product of the structure constants C^{ijk} over vertices (with the cyclic order given by the orientation) and a factor $\eta(R(g)e_i, e_j) = R(g)^k{}_i \eta_{jk}$ for each edge directed from i to j labeled by g. The partition sum is the sum of these weights over all colorings; we emphasize that the group labels represent a background gauge field and are not summed. To incorporate brane boundaries, choose a G-equivariant module V over A. Fix a basis f_μ of V. For each brane boundary vertex, label its adjacent boundary edges each with a basis element, so that each boundary edge has a total of two labels. The weight of a skeleton with a brane boundary is a product of C's and R's as well as a module tensor T for each brane boundary vertex and a matrix element $Q(g)^\mu_\nu$ for each brane boundary edge.

As before, topological invariance of the state-sum amounts to checking the conditions (3.19), (3.20), (3.21), and (3.22). These are satisfied by any finite-dimensional semisimple A. In order for the equivariant state-sum to constitute a well-defined equivariant TQFT, it must also be independent of the choice of trivialization of the background gauge field; in order words, it must be gauge invariant. A gauge transformation by $h \in G$ on a vertex acts by changing the decorations of the three edges whose boundary contains the vertex: incoming edges with g become hg, outgoing gh^{-1}, as in Figure 3.10. Invariance under a gauge transformation on a vertex in the interior is ensured by axioms (3.31) and (3.32) of a G-equivariant algebra. For vertices in the brane boundary, the analogous result follows from the G-equivariant

[9] In fact, the category of projective representations of G with a 2-cocycle $-\omega$ is equivalent to the category of G-equivariant modules over $\mathrm{E}nd(U)$, and the equivalence sends a projective representation W to $U \otimes W$.

[10] In the triangulation picture, we require the product of all group elements corresponding to edges entering a particular vertex to be the identity element.

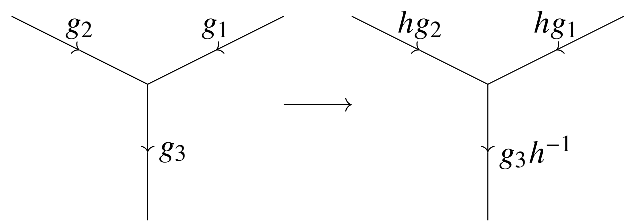


Figure 3.10: A gauge transformation at the vertex by h

module condition (3.34).[11] Finally, invariance under simultaneously reversing an edge direction and inverting its group label is enforced by the axiom (3.32).

G-equivariant semisimple algebras

The classic Wedderburn theorem implies that every finite-dimensional semisimple algebra is a sum of matrix algebras. Let us discuss a generalization of this result to the G-equivariant case following Ostrik(Ostrik, 2003) and Etingof(Etingof, 2015).

First, we can write every G-equivariant semisimple algebra as a sum of indecomposable ones, so it is sufficient to classify indecomposable G-equivariant semisimple algebras. A large class of examples is given by algebras of the form $\mathrm{E}nd(U)$, where U is a projective representation of G. Another set of examples is obtained as follows: let $H \subset G$ be a subgroup. Consider the space of complex-valued functions on G invariant with respect to left translations by H, i.e. $f(h^{-1}g) = f(g)$ for all $g \in G$ and all $h \in H$. The group G acts on this space by right translations:

$$(R(g)f)(g') = f(g'g) \tag{3.35}$$

Pointwise multiplication makes this space of functions into an associative algebra, and it is clear that the G-action commutes with the multiplication. This G-equivariant algebra is indecomposable for any H.

The most general indecomposable G-equivariant semisimple algebra is a combination of these two constructions called the induced representation $\mathrm{Ind}_H^G \mathrm{E}nd(U)$.(Ostrik, 2003; Etingof, 2015) One picks a subgroup $H \subset G$ and a projective representation (U, Q) of H. Here U is a vector space and Q is a map $H \to \mathrm{E}nd(U)$ defining a projective action with a 2-cocycle $\omega \in H^2(H, U(1))$. Then one considers the space of functions on G with values in $\mathrm{E}nd(U)$ which have the following transformation property under the left H action:

$$f(h^{-1}g) = Q(h)f(g)Q(h)^{-1} \tag{3.36}$$

[11] Here it is crucial that linear transformations $Q(g)$ form an ordinary (i.e. not projective) representation of G.

It is easy to check that the right G translations act on this space of functions. Pointwise multiplication makes this space into a G-equivariant algebra, and one can show that it is indecomposable. To summarize, indecomposable G-equivariant semisimple algebras are labeled by triples (H, U, Q), where $H \subset G$ is a subgroup, and (U, Q) is a projective representation of H. All these algebras are actually Frobenius algebras: the trace function $A \to \mathbb{C}$ is given by

$$\sum_{g \in G} \mathrm{T}r_U f(g) \tag{3.37}$$

A G-equivariant module over such an algebra A is obtained as follows. Start with an H-equivariant module $(M, \mathcal{Q})$ over $\mathrm{E}nd(U)$. Here M is a module over $\mathrm{E}nd(U)$ and $\mathcal{Q} : H \to \mathrm{E}nd(M)$ is a compatible action of H on M. As explained above, M must have the form $U \otimes W$, where W carries a projective action $S(h)$ of H with a 2-cocycle $-\omega$. Then consider functions on G with values in M which transform as follows under the left H-translations:

$$m(h^{-1}g) = (Q(h) \otimes S(h))m(g), \quad m : G \to U \otimes W. \tag{3.38}$$

The group G acts on this space by right translations, and it is easy to see that the pointwise action of $A = (H, U, Q)$ makes it into a G-equivariant module over A. One can show that any G-equivariant module over such an A is a direct sum of modules of this sort.

G-equivariant MPS from G-equivariant TQFT

It is sufficient to consider indecomposable TQFTs and G-equivariant algebras. Let us begin with the case $H = G$. Then the algebra $A = (G, U, Q)$ is isomorphic to the algebra $\mathrm{E}nd(U)$, and a G-equivariant module over it is simply a vector space M with a G-equivariant action of $\mathrm{E}nd(U)$. In other words, $M = U \otimes W$, where U carries a projective representation of G with the 2-cocycle ω, and W carries a projective representation of G with a 2-cocycle $-\omega$.

Consider an annulus whose outer boundary is labeled by a brane M and whose inner boundary is a cut boundary. Let us triangulate both boundary circles into N intervals. Let $g_{i,i+1}$ be the element of G labeling the interval from the $(i+1)^{th}$ to the i^{th} points on the boundary. We also assume that the holonomy of the gauge field between the points labeled by 1 on the two boundary circles is trivial. We get the

the following dual state:

$$\langle\psi_T| = \sum \mathrm{Tr}_{U\otimes W}[T(e_{i_1})Q(g_{1,2})\cdots \\ \cdots T(e_{i_N})Q(g_{N,1})]\langle i_1\cdots i_N|. \tag{3.39}$$

Note that although $T(e_i)$ is an operator on $U\otimes W$, it has the form $T(e_i)\otimes \mathbf{1}_W$. Therefore, if $g_{i,i+1} = 1$ for all i, the trace over W gives an overall factor $\dim W$, and up to this factor we get the equivariant MPS (3.30). Inserting an observable $X \in \mathrm{E}nd(U)$ on the brane boundary, we get a generalized equivariant MPS. The case when $X \in \mathrm{E}nd(U\otimes W)$ does not give anything new, since the trace over V factors out.

The generalized equivariant MPS (cf. eq 3.7)

$$\langle\psi_T^X| = \sum \mathrm{Tr}\big[X^\dagger T(e_{i_1})\cdots T(e_{i_n})\big]\langle i_1\cdots i_n| \tag{3.40}$$

may be charged under the action of $h\in G$:

$$\begin{aligned} R(h)^{\otimes N}\left\langle\psi_T^X\right| &= \sum \mathrm{Tr}\big[X^\dagger T(e_{i_1})\cdots T(e_{i_n})\big]\langle(h^{-1}\cdot i_1)\cdots(h^{-1}\cdot i_n)| \\ &= \sum \mathrm{Tr}\big[X^\dagger T(h\cdot e_{i_1})\cdots T(h\cdot e_{i_n})\big]\langle i_1\cdots i_n| \\ &= \sum \mathrm{Tr}\big[Q(h^{-1})X^\dagger Q(h)T(e_{i_1})\cdots T(e_{i_n})\big]\langle i_1\cdots i_n| \end{aligned} \tag{3.41}$$

Let us now consider the case when H is a proper subgroup of G and $A = \mathrm{Ind}_H^G\mathrm{E}nd(U)$, for some projective representation U of H. If we choose right H-coset representatives g_a, $a\in H\backslash G$, and a basis e_i in $\mathrm{E}nd(U)$, then a basis in A is given by e_i^a. Similarly, if f_μ is a basis in an H-equivariant module $U\otimes W$, then a basis in the corresponding G-equivariant module M is f_μ^a.

The action of A on M is diagonal as far as the a index is concerned. Therefore the dual state corresponding to a triangulated annulus with $g_{i,i+1} = 1$ for all i vanishes unless all a indices are the same. Then

$$\langle\psi_T| = \dim(W)\sum_{a,i_1,\ldots,i_N} \mathrm{Tr}_U[T(e_{i_1})\cdots \\ \cdots T(e_{i_N})]\langle i_1 a\, i_2 a\ \cdots i_N a|. \tag{3.42}$$

This state has equal components along all $|H\backslash G|$ directions. We can get a state concentrated at a particular value of a by inserting a suitable observable $X\in\mathrm{E}nd(M)$ on the brane boundary. Such an observable must commute with the action of A, so it must have the form $X^{\mu a}_{\nu\, b} = f(a)\delta^{\mu}_{\nu}\delta^{a}_{b}$. Choosing the function $f(a)$ to be supported at a particular value of a gives a generalized MPS state supported at this value of a.

The symmetry group G acts transitively on $H\backslash G$. This suggests that we are dealing with a phase where the symmetry G is spontaneously broken down to H, so that we get $|H\backslash G|$ sectors labeled by the index a. To confirm this, consider the partition function of this TQFT on a closed oriented 2-manifold Σ with a trivial G-bundle. After we choose a skeleton of Σ, we can represent this G-bundle by labeling every 1-simplex with the identity element of G. In addition, every 1-simplex is labeled by a pair of basis vectors of A. Since both the multiplication in the algebra A and the scalar product are pointwise in $H\backslash G$, the partition function receives contributions only from those labelings where all a labels are the same. Furthermore, turning on a gauge field which takes values in H does not destroy this property. We conclude that the theory has superselection sectors labeled by elements of $H\backslash G$, and each sector has unbroken symmetry H.

Twisted-sector states

Now let us not assume that $g_{i,i+1} = 1$, but instead allow the gauge field around the circle to have a nontrivial holonomy. Let us take $H = G$ first, i.e. the case of unbroken symmetry. Consider the MPS (3.39). Applying a gauge transformations (by $g_{1,2}g_{2,3}\cdots g_{k-1,k}$ at vertex k) to the boundary vertices it can be written as

$$\begin{aligned}\langle\psi_{T,g}| = \sum \mathrm{Tr}_{U\otimes W}[\mathcal{Q}(g)T(e_{i_1})\cdots T(e_{i_N})]\\ \otimes_{k=1}^{N} R(g_{1,2}\cdots g_{k-1,k})^{i_k}{}_{j_k}\langle j_k|\end{aligned} \tag{3.43}$$

where $g = g_{1,2}g_{2,3}\cdots g_{N,1}$ is the holonomy of the gauge field. This is LU equivalent to the state

$$\langle\psi_{T,g}| = \sum \mathrm{Tr}_{U\otimes W}[\mathcal{Q}(g)T(e_{i_1})\cdots T(e_{i_N})]\langle i_1\cdots i_N| \tag{3.44}$$

so we have effectively set $g_{i,i+1} = 1$ for all $i \neq N$ and $g_{N,1} = g$. Note that $\mathcal{Q} = Q\otimes S$, so the trace factors into a product of a trace over U and a trace over W. The latter gives us an overall factor, and we have

$$\begin{aligned}\langle\psi_{T,g}| = \mathrm{Tr}_W[S(g)]\sum \mathrm{Tr}_U[Q(g)T(e_{i_1})\cdots\\ \cdots T(e_{i_N})]\langle i_1\cdots i_N|.\end{aligned} \tag{3.45}$$

This state transforms under $h \in G$ into

$$\begin{aligned} R(h)^{\otimes N} \left\langle \psi_{T,g} \right| &= (\mathrm{Tr}_W[S(g)]) \sum \mathrm{Tr}\big[Q(g)T(e_{i_1})\cdots T(e_{i_n})\big] \langle (h^{-1}\cdot i_1)\cdots(h^{-1}\cdot i_n)| \\ &= (\mathrm{Tr}_W[S(g)]) \sum \mathrm{Tr}\big[Q(h)^{-1}Q(g)Q(h)T(e_{i_1})\cdots T(e_{i_n})\big] \langle i_1 \cdots i_n| \\ &= (\mathrm{Tr}_W[S(g)])\omega(g,h)\omega(h^{-1},gh) \sum \mathrm{Tr}\big[Q(h^{-1}gh)T(e_{i_1})\cdots T(e_{i_n})\big] \langle i_1 \cdots i_n| \end{aligned} \tag{3.46}$$

Note that the g-twisted sector becomes the hgh^{-1}-twisted sector.

Now suppose H is a proper subgroup of G. Since T acts pointwise in the a label, while G acts on $a \in H\backslash G$ by right translations, the annulus state vanishes unless the holonomy around the circle is in H. This confirms once again that H is the unbroken subgroup. Indeed, when the holonomy does not belong to the unbroken subgroup, there must be a domain wall somewhere on the circle. Its energy is nonzero in the thermodynamic limit, so the TQFT space of states must be zero-dimensional for holonomies not in H.

If $\mathcal{A}_g$ denotes the space of states in the g-twisted sector, the space $\mathcal{A} = \oplus_g \mathcal{A}_g$ has an automorphism $\alpha_h := R(h)^{\otimes N}$ for each $h \in G$ such that $\alpha_h(\mathcal{A}_g) \subset \mathcal{A}_{hgh^{-1}}$. $\mathcal{A}$ is the G-graded vector space underlying the G-crossed Frobenius algebra that defines the associated G-equivariant TQFT.(Turaev, 2010; G. W. Moore and Segal, 2006)

Morita equivalence

We have seen that to any semisimple G-equivariant algebra one can associate a G-equivariant 2d TQFT. But different algebras may give rise to the same TQFT. In particular, we would like to argue that the TQFT corresponding to an indecomposable algebra $A = (H, U, Q)$, where (U, Q) is a projective representation of H, depends only on the subgroup H and the 2-cocycle ω, but not on the specific choice of (U, Q).

To show this, note first of all that the partition function vanishes if the holonomy does not lie in H (this again follows from the fact that multiplication in the algebra A is pointwise with respect to the a index). Thus it is sufficient to consider oriented 2-manifolds with H-bundles. Further, if U and U' are projective representations of H with the same 2-cocycle, then $U' = U \otimes W$, where W is an ordinary representation of H. Thus we only need to show that the partition functions corresponding to algebras (H, U, Q) and $(H, U \otimes W, Q \otimes S)$ are the same, where $S : H \to \mathrm{E}nd(W)$ is a representation of H. But it is clear from the state sum construction that the two partition functions differ by a factor which is the partition function of two dimensional H-equivariant TQFT corresponding to the algebra (H, W, S).

We reduced the problem to showing that the H-equivariant TQFT constructed from the algebra (H, W, S) is trivial when (W, S) is an ordinary (not projective) representation of H. This is straightforward: the equation $S(h_1)\dots S(h_n) = S(h_1 \dots h_n)$ and the flatness condition for the H gauge field imply that the partition function is independent of the H-bundle, and for the trivial H-bundle the partition function is the same as for the trivial TQFT with $A = \mathbb{C}$.

From the mathematical viewpoint, G-equivariant algebras with the same H and ω are Morita-equivalent.[12](Ostrik, 2003) Thus we have shown that Morita-equivalent algebras lead to identical G-equivariant TQFTs.[13]

Stacking phases

Consider two gapped systems built from algebras A_1 and A_2. Recall from Section 3.1 that the stacked system (3.12) is built from the tensor product algebra $A_1 \otimes A_2$. Although we have not discussed parent Hamiltonians of G-equivariant MPS, an analogous stacking operation can be defined for G-symmetric gapped phases by way of the connection to TQFT. Now suppose A_1 and A_2 are G-equivariant algebras. It is clear from the G-equivariant state sum construction that the partition functions for the algebra $A_1 \otimes A_2$ are products of those for A_1 and A_2 and that the Hilbert spaces are tensor products. Thus the MPS ground states, which determine a phase and which are realized in TQFT, stack like the tensor product of G-equivariant algebras.

It is a tedious but straightforward exercise to check that the result of stacking the phase labeled by subgroup-cocycle pair (H, ω) with the phase (K, ρ) is the phase

$$(H \cap K, \omega|_{H\cap K} + \rho|_{H\cap K})^{\oplus[G:HK]} \tag{3.47}$$

where $\omega|_{H\cap K}$ denotes the restriction of ω to the intersection subgroup $H \cap K$ and $[G : HK]$ denotes the index of the subgroup HK in G, assuming H and K are normal in G.

Let us consider a simple example: take $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b\rangle$, where a and b are commuting elements of order 2. For the subgroup $H = G$, there are two

[12] More accurately, algebras with the same H and ω, *up to conjugation in* G, are Morita-equivalent. In physical contexts, however, it is typical to keep track of the embedding of the unbroken symmetry H in the full symmetry group G. Therefore, the classification of physical gapped G-symmetric phases is slightly more refined than that of Morita classes.

[13] Strictly speaking, we only showed this for closed 2d TQFTs, but the argument easily extends to the open-closed case.

(H,ω)	type of phase	name
$(\langle a,b\rangle,1)$	trivial	1
$(\langle a,b\rangle,\omega_1)$	symmetry-protected	ω
$(\langle a\rangle,1)$	broken symmetry	A
$(\langle b\rangle,1)$	broken symmetry	B
$(\langle ab\rangle,1)$	broken symmetry	C
$(1,1)$	broken symmetry	0

Figure 3.11: Indecomposable phase classification for the $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

cohomology classes $\omega \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$. Let ω_1 denote the nontrivial class. For each of the other subgroups $H = \langle a\rangle$, $\langle b\rangle$, $\langle ab\rangle$, 1, there is a unique cocycle. Thus the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant phases is like Figure 3.11.

According to (3.47), the stacking rules are

$$\begin{gathered}
1 \otimes 1 = 1, \quad 1 \otimes \omega = \omega, \quad 1 \otimes A = A, \quad 1 \otimes B = B, \quad 1 \otimes C = C, \quad 1 \otimes 0 = 0, \\
\omega \otimes \omega = 1, \quad \omega \otimes A = A, \quad \omega \otimes B = B, \quad \omega \otimes C = C, \quad \omega \otimes 0 = 0, \\
A \otimes A = A^{\oplus 2}, \quad B \otimes B = B^{\oplus 2}, \quad C \otimes C = C^{\oplus 2}, \\
A \otimes B = 0, \quad B \otimes C = 0, \quad C \otimes A = 0, \\
A \otimes 0 = 0^{\oplus 2}, \quad B \otimes 0 = 0^{\oplus 2}, \quad C \otimes 0 = 0^{\oplus 2}, \quad 0 \otimes 0 = 0^{\oplus 4}
\end{gathered}$$

Symmetry Protected Topological Phases

Finally, let us discuss the case of Short-Range Entangled (SRE) phases with symmetry G. According to one definition,(Kitaev, 2015) an SRE phase is one that is invertible under the aforementioned stacking operation. Such phases have a one-dimensional space of ground states for every G-bundle on a circle. Since the space of states of a decomposable TQFT on a circle with a trivial bundle has dimension greater than one, a TQFT corresponding to an SRE phase must be indecomposable. We showed that when H is a subgroup of G, the space of states is zero-dimensional whenever the holonomy does not lie in H. Hence an equivariant TQFT built from an indecomposable G-equivariant algebra (H, U, Q) cannot correspond to an SRE unless $H = G$.

These SRE phases are all Symmetry Protected Topological (SPT) phases - phases that are trivial if we ignore symmetry. A G-equivariant algebra of the form $\mathrm{E}nd(U)$, where U is a projective representation of G, is simply a matrix algebra if we ignore the G action. Hence the corresponding non-equivariant TQFT is trivial; the corresponding Hamiltonian is connected to the trivial one by a Local Unitary transformation. Hence SPT phases with symmetry G are labeled by 2-cocycles $\omega \in H^2(G, U(1))$. This is a well-known result. (X. Chen, Gu, and Wen, 2011a; X. Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011)

3.4 Spin-TQFTs

$\mathbb{Z}_2$-graded semi-simple algebras

The algebraic input for the fermionic state-sum construction is a $\mathbb{Z}_2$-graded semisimple Frobenius algebra A (Novak and Runkel, 2015; Gaiotto and Kapustin, 2016).[14] A Frobenius algebra is a finite-dimensional algebra over $\mathbb{C}$ with a non-degenerate symmetric scalar product $\eta : A \otimes A \to \mathbb{C}$ satisfying $\eta(a, bc) = \eta(ab, c)$ for all $a, b, c \in A$. A $\mathbb{Z}_2$-grading on A is a decomposition $A = A_+ \oplus A_-$ such that

$$A_+ \cdot A_+ \subset A_+, \quad A_- \cdot A_- \subset A_+, \quad A_- \cdot A_+ \subset A_-, \quad A_+ \cdot A_- \subset A_-. \tag{3.48}$$

Equivalently, a $\mathbb{Z}_2$-grading is an operator $\mathcal{F} : A \to A$ such that $\mathcal{F}^2 = 1$ and $\mathcal{F}(a) \cdot \mathcal{F}(b) = \mathcal{F}(a \cdot b)$. The operator $\mathcal{F}$ is called fermion parity and is traditionally denoted $(-1)^F$. We also assume that the scalar product η is $\mathcal{F}$-invariant:

$$\eta(\mathcal{F}(a), \mathcal{F}(b)) = \eta(a, b). \tag{3.49}$$

Note that $\mathcal{F}$ defines an action of $\mathbb{Z}_2$ on A which makes A into a $\mathbb{Z}_2$-equivariant algebra. This observation is the root cause of the bosonization phenomenon: there is a 1-1 map between 1+1d phases of bosons with $\mathbb{Z}_2$ symmetry and 1+1d phases of fermions. For now, we use this fact to describe the classification of $\mathbb{Z}_2$-graded simple algebras. Namely, since the only proper subgroup of $\mathbb{Z}_2$ is the trivial one, and $H^2(\mathbb{Z}_2, U(1)) = 0$, a simple $\mathbb{Z}_2$-graded algebra is isomorphic either to $\mathrm{E}nd(V)$ for some $\mathbb{Z}_2$-graded vector space $V = V_+ \oplus V_-$, or to $\mathrm{C}l(1) \otimes \mathrm{E}nd(V)$ for some purely even vector space $V = V_+$ (Kapustin, Turzillo, and You, 2017). Here $\mathrm{C}l(1)$ denotes the Clifford algebra with one generator, i.e. an algebra with an odd generator Γ satisfying $\Gamma^2 = 1$.

As explained in (Kapustin, Turzillo, and You, 2017), the bosonic phase depends only on the Morita-equivalence class of A. The choice of V does not affect the Morita-equivalence class of the algebra, so there are only two Morita equivalence classes of $\mathbb{Z}_2$-graded algebras: the trivial one, corresponding to the algebra $\mathbb{C}$, and the nontrivial one, corresponding to the algebra $\mathrm{C}l(1)$. In the bosonic case, the

[14] While it is possible to relax the semi-simplicity condition (Novak and Runkel, 2015), here we are interested in unitary TQFTs, and for such TQFTs one may assume that A is semi-simple (Kapustin, Turzillo, and You, 2017).

former one corresponds to the trivial gapped phase with a $\mathbb{Z}_2$ symmetry, while the latter one corresponds to the phase with a spontaneously broken $\mathbb{Z}_2$.

The fermionic interpretation is different. As briefly mentioned in (Gaiotto and Kapustin, 2016) and discussed in more detail below, the algebra $\mathrm{C}l(1)$ describes a gapped fermionic phase which is equivalent to the nontrivial Majorana chain. This is in accord with the intuition that fermion parity cannot be spontaneously broken.

Spin structures

A spin structure on an oriented manifold enables one to define a spin bundle. For a 1d manifold X, a spin bundle is a real line bundle L plus an isomorphism $L \otimes L \to TX$. Thus a spin bundle is a square root of the tangent bundle. Since TX is trivial, such L are classified by elements of $H^1(X, \mathbb{Z}_2)$. Since $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$, there are two possible spin structures on a circle, called the R (Ramond) an NS (Neveu-Schwarz) spin structures in the string theory literature. The R structure corresponds to a trivial L, while NS structure corresponds to the "Möbius band" L. In other words, if we give L a metric and compute the holonomy of the unique connection compatible with it along S^1, we get 1 for the R case, and -1 for the NS case.

For an oriented 2d manifold Σ, we can regard $T\Sigma$ as a complex line bundle, and then a spin bundle on Σ is a complex line bundle S equipped with an isomorphism $S \otimes S \to T\Sigma$. One can show that such an S always exists. If S and S' are two spin bundles, they differ by a line bundle which squares to a trivial line bundle on Σ. The latter are classified by elements of $H^1(\Sigma, \mathbb{Z}_2)$. Thus there are as many spin structures as there are elements of $H^1(\Sigma, \mathbb{Z}_2)$. But in general there is no natural way to identify elements of $H^1(\Sigma, \mathbb{Z}_2)$ with spin structures.[15]

It is easy to see that a spin structure s on an oriented 2d manifold Σ induces a spin structure on any oriented 1d manifold γ embedded into Σ. Define $\sigma_s(\gamma) = +1$ if the induced structure is of the NS type and $\sigma_s(\gamma) = -1$ if the induced structure is of the R type. That is, $\sigma_s(\gamma)$ is the negative of the holonomy of the connection corresponding to the induced spin structure. It is easy to show that $\sigma_s(\gamma)$ depends only on the homology class of γ and thus defines a function $\sigma_s : H_1(\Sigma, \mathbb{Z}_2) \to \mathbb{Z}_2$. With more work, one can show that this function satisfies

$$\sigma_s([\gamma] + [\gamma']) = \sigma_s([\gamma])\sigma_s([\gamma'])(-1)^{\langle [\gamma], [\gamma'] \rangle}. \tag{3.50}$$

[15] The case of a torus is an exception, since then $T\Sigma$ is trivial. This is why one can talk about periodic and anti-periodic spin structures on a torus.

That is, it is a quadratic $\mathbb{Z}_2$-valued function on $H_1(\Sigma, \mathbb{Z}_2)$ whose corresponding bilinear form is the intersection pairing on $H_1(\Sigma, \mathbb{Z}_2)$. In fact, it is a theorem of Atiyah (Atiyah, 1971) that for a closed Σ the spin structure is determined by such a quadratic function, and that any such quadratic function determines a spin structure. Note that the ratio of two such quadratic functions is a linear function on $H_1(\Sigma, \mathbb{Z}_2)$, or equivalently an element of $H^1(\Sigma, \mathbb{Z}_2)$. Thus we recover the result that two spin structures differ by an element of $H^1(\Sigma, \mathbb{Z}_2)$.

We record for future use another property of the function σ_s:

$$\sigma_{s+a}([\gamma]) = (-1)^{\int_\gamma a} \sigma_s([\gamma]), \tag{3.51}$$

where a is an arbitrary element of $H^1(\Sigma, \mathbb{Z}_2)$. Thus $\sigma_s([\gamma])$ is an affine-linear function of s and a quadratic function of $[\gamma]$.

We will also need a version of this result for the case when Σ has a nonempty boundary. As in the case of equivariant TQFT, it is convenient to choose, along with a spin structure s, a point on every connected component of $\partial\Sigma$ and a normalized basis vector for the real spin bundle L at this point. This simplifies the gluing of spin manifolds. We will denote by $\partial_0\Sigma$ the set of all marked points, and will call a spin structure on Σ together with a trivialization of L at $\partial_0\Sigma$ a spin structure on the pair $(\Sigma, \partial_0\Sigma)$. The group $H^1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$ acts freely and transitively on the set of spin structures on $(\Sigma, \partial_0\Sigma)$. Despite this, there is no canonical way to identify spin structures with elements of $H^1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$. To get an algebraic description of spin structures, one can proceed as follows (Segal, 2004). First, note that $H_1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$ can be identified with $H_1(\Sigma_*, \mathbb{Z}_2)$, where Σ_* is a closed oriented 2d manifold obtained by gluing a sphere with holes onto Σ. This identification depends on the choice of a cyclic order of the set of boundary circles of Σ. Thus the intersection form on $H_1(\Sigma_*, \mathbb{Z}_2)$ induces a non-degenerate symmetric bilinear form on $H_1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$. There is also an identification of the set of spin structures on $(\Sigma, \partial_0\Sigma)$ and the set of of spin structures on Σ^* (Segal, 2004). Thus the set of spin structures on $(\Sigma, \partial_0\Sigma)$ can be identified with the set of $\mathbb{Z}_2$-valued quadratic functions on $H_1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$ refining the intersection form. This identification still depends on a choice of a cyclic order on the set of boundary circles of Σ. One can determine which spin structure is induced on any particular connected component of $\partial\Sigma$ by evaluating this quadratic function on the closed curve wrapping that component.

State-sum construction of the spin-dependent partition function

To define the partition function of a spin-TQFT on a closed oriented 2-manifold Σ with a spin structure, we choose a skeleton of Σ, i.e. a trivalent graph Γ on Σ whose complement is homeomorphic to a disjoint union of disks. Equivalently, one may think of Γ as the Poincaré dual of a triangulation $\mathcal{T}$ of Σ.[16] For every vertex $v \in \Gamma$, let $\Gamma(v)$ denote the edges containing v. Orientation of Σ gives rise to a cyclic order on $\Gamma(v)$ for all v. This is sufficient to produce the partition function of a bosonic TQFT based on the algebra A, but in order to construct the fermionic partition function, we need to choose an actual order on $\Gamma(v)$. We can do it by picking one special edge $e_0(v) \in \Gamma(v)$ for every v. We also choose an orientation for each edge of Γ. (In Ref. (Gaiotto and Kapustin, 2016) both an orientation of edges and a choice of $e_0(v)$ arose from a branching structure on $\mathcal{T}$, but here we follow Ref. (Novak and Runkel, 2015) and choose them independently.) These choices are called a *marking* of Γ.

We also need to describe a choice of spin structure on Σ. This is a cellular 1-cochain s valued in $\mathbb{Z}_2$ (i.e. an assignment of elements of $\mathbb{Z}_2$ to edges of Γ) with coboundary a certain 2-cocycle w_2 whose cohomology class is the second Stiefel-Whitney class $[w_2](\Sigma)$. Following Ref. (Novak and Runkel, 2015), we write the constraint $\delta s = w_2$ as

$$(\delta s)(f) = 1 + K + D \text{ m}od\ 2. \tag{3.52}$$

where f is a particular cell in $\Sigma\backslash\Gamma$, K is the number of clockwise oriented edges in ∂f, and D is the number of vertices v for which the counterclockwise-oriented curve homologous to ∂f in Γ enters v through $e_0(v)$. Two solutions s, s' of this constraint are regarded equivalent, $s \sim s'$, if $s - s' = \delta t$ for some 0-cochain t. Two solutions s, s' define isomorphic spin structures on Σ if and only if $s \sim s'$ (Novak and Runkel, 2015; Gaiotto and Kapustin, 2016). Thus we recover the fact that the number of distinct spin structures on Σ is equal to $|H^1(\Sigma, \mathbb{Z}_2)|$.

One can give an explicit description of the holonomy function $\sigma_s(\gamma)$ corresponding to the 1-cochain s. Regard a closed curve γ embedded into the graph Γ as a 1-cycle. Then $\sigma_s(\gamma)$ is given in terms of the signs s and the marking of Γ along γ by eq. (3.45) of Ref. (Novak and Runkel, 2015). This expression simplifies greatly in the important case of when γ is a counterclockwise-oriented curve bounding a single cell in $\Sigma\backslash\Gamma$. Here $\sigma_s(\gamma) = -(-1)^{s(\gamma)}$; that is, -1 for each edge of γ oriented

[16] One can formulate the construction either in terms of triangulations or in terms of skeletons, but the latter approach gives a bit more flexibility when we allow Σ to have a nonempty boundary.

clockwise, times -1 for each vertex v such that γ enters v through $e_0(v)$. One can show that this function depends only on the homology class of γ and is a quadratic refinement of the intersection form.

Choose a basis e_i in A whose elements are eigenvectors of $\mathcal{F}$. Let $\eta_{ij} = \eta(e_i, e_j)$. Since η is non-degenerate, it has an inverse η^{ij}. Let $C^i{}_{jk}$ denote the structure constants of A. Define $C_{ijk} = \eta_{il} C^l{}_{jk}$. It can be shown that the tensor C_{ijk} is cyclically symmetric (Kapustin, Turzillo, and You, 2017). Denote by $(-1)^{\beta_i}$ the eigenvalue of $\mathcal{F}$ corresponding to e_i.

Now we can explain the recipe for computing the partition function for a surface Σ with a marked skeleton Γ and a spin structure s. Each edge of Γ is colored with a pair of basis vectors $e_i \in A$, and we have a factor of C_{ijk} for each vertex and η^{ij} for each edge. Since A is $\mathbb{Z}_2$-graded, η^{ij} vanishes unless $\beta_i = \beta_j$, and C_{ijk} vanishes unless $\beta_i + \beta_j + \beta_k = 0$. Hence the function $\beta : e_i \mapsto \beta_i$ on the set of edges of Γ defines a mod-2 1-cycle on Σ. The contribution of a particular coloring of Γ is the product of all C_{ijk} and η^{ij}, the spin-dependent sign factor

$$(-1)^{s(\beta)} = (-1)^{\sum_e s(e)\beta(e)}, \tag{3.53}$$

and the Koszul sign $\sigma_0(\beta)$. The partition function is obtained by summing over all colorings. Note that

$$Z_{\text{ferm}}(A, \eta) = \sum_{\beta} Z_{\text{bose}}(A, \beta)\sigma_s(\beta), \tag{3.54}$$

where $Z_{\text{bose}}(A, \beta)$ is the sum over all colorings with a fixed 1-cycle β. Using the isomorphism $H_1(\Sigma, \mathbb{Z}_2) \simeq H^1(\Sigma, \mathbb{Z}_2)$, one can interpret β as a $\mathbb{Z}_2$ gauge field on a dual triangulation and $Z_{\text{bose}}(A, \beta)$ as the partition function of a bosonic system with a global $\mathbb{Z}_2$ symmetry coupled to β. Equation (3.54) is a manifestation of the bosonization phenomenon.

It remains to explain how the Koszul sign $\sigma_0(\beta)$ is evaluated. Consider a vertex whose edges are labeled by i, j, k starting from the special edge and going counter-clockwise. Assign to it an element $C_v = C_{ijk} e_i \otimes e_j \otimes e_k$ in $A \otimes A \otimes A$. Tensoring over vertices, we get an element C_Γ of $A^{\otimes 3N}$, where N is the number of vertices of Γ. Now consider an oriented edge of Γ labeled by i, j. It corresponds to an ordered pair of factors in C_Γ. Permute the factors of C_Γ until these two are next to each other and in order, keeping track of the fermionic signs

$$e_i \otimes e_j \mapsto (-1)^{\beta_i \beta_j} e_j \otimes e_i \tag{3.55}$$

one incurs in the process, and then contract using the scalar product η. Continuing in this fashion, we are left with the product of all C_{ijk} and η^{ij} times a sign. This sign is the Koszul sign $\sigma_0(\beta)$. It is clear that it depends on the coloring of Γ only through the 1-cycle β. Note that the elements C_v as well as the pairs of factors for each edge are all even, so one does not need to order the set of vertices or the set of edges. One can also define $\sigma_0(\beta)$ as a Grassmann integral, as was originally done in (Gu and Wen, 2014). The product of the Koszul sign $\sigma_0(\beta)$ and the spin-dependent factor $(-1)^{s(\beta)}$ is nothing but the quadratic function $\sigma_s(\beta)$ (Gaiotto and Kapustin, 2016).

One can show (Novak and Runkel, 2015; Gaiotto and Kapustin, 2016) that the partition function thus defined depends only on the spin surface (Σ, s) and not the skeleton Γ, its marking, or the particular 1-cochain representing s. Finally, it is clear that if A is purely even, both the Koszul sign and the spin-dependent sign factor are trivial, and the partition function reduces to the bosonic partition function associated with A.

Stacking and the supertensor product

It is interesting to determine the behavior of the partition function under stacking systems together. Given a pair of fermionic systems encoded in a pair of $\mathbb{Z}_2$-graded Frobenius algebras A_1, A_2, stacking these systems together gives us a system with a partition function $Z_{\text{ferm}}(A_1, \eta) Z_{\text{ferm}}(A_2, \eta)$. It turns out that

$$Z_{\text{ferm}}(A_1, \eta) Z_{\text{ferm}}(A_2, \eta) = Z_{\text{ferm}}(A_1 \mathbin{\widehat{\otimes}} A_2, \eta), \tag{3.56}$$

where $\widehat{\otimes}$ is the supertensor product of $\mathbb{Z}_2$-graded algebras. Let us recall what this means. The usual tensor product of algebras $A_1 \otimes A_2$ obeys the multiplication rule

$$(a_1 \otimes a_2) \cdot (a_1' \otimes a_2') = (a_1 \cdot a_1') \otimes (a_2 \cdot a_2'). \tag{3.57}$$

If the algebras A_1, A_2 are $\mathbb{Z}_2$-graded, $A_1 \otimes A_2$ is also $\mathbb{Z}_2$-graded in an obvious way. On the other hand, for the supertensor product the multiplication is defined as follows:

$$(a_1 \mathbin{\widehat{\otimes}} a_2) \cdot (a_1' \mathbin{\widehat{\otimes}} a_2') = (-1)^{|a_2| \cdot |a_1'|} (a_1 \cdot a_1') \mathbin{\widehat{\otimes}} (a_2 \cdot a_2'), \tag{3.58}$$

where $(-1)^{|a|}$ is the fermionic parity of a.

To derive (3.56), we first note that

$$Z_{\text{bose}}(A_1, \beta_1) Z_{\text{bose}}(A_2, \beta_2) = Z_{\text{bose}}(A_1 \otimes A_2, \beta_1, \beta_2), \tag{3.59}$$

where we used the fact that the stacking of two bosonic systems with symmetry $\mathbb{Z}_2$ has a symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ and thus can be coupled to a pair of $\mathbb{Z}_2$ gauge fields β_1, β_2. Next, it is easy to see that

$$Z(A_1 \widehat{\otimes} A_2, \beta_1, \beta_2) = (-1)^{\langle [\beta_1],[\beta_2] \rangle} Z(A_1 \otimes A_2, \beta_1, \beta_2). \tag{3.60}$$

These two identities together with (3.50) imply (3.56).

As an illustration, consider $A = \mathbb{C}l(1)$. Since apart from 1 this algebra has a single odd basis element γ, β completely determines the coloring of Γ. With the proper normalization of Z_{bose}, one gets

$$Z_{\text{ferm}}(s) = 2^{-b_1(\Sigma)/2} \sum_{[\beta]} \sigma_s([\beta]). \tag{3.61}$$

The r.h.s. is called the Arf invariant of the spin structure s and is denoted $\mathrm{A}rf(s)$. One can show that it takes values ± 1. If we stack two such systems together, we will get the partition function which is 1 for all spin structures and all Σ, i.e. a trivial spin-TQFT.

It is easy to see that $\mathbb{C}l(1) \widehat{\otimes} \mathbb{C}l(1)$ is the Clifford algebra with two generators, $\mathbb{C}l(2)$. This algebra is non-trivial, but it is Morita-equivalent to the trivial algebra $\mathbb{C}$. One can show that, just as in the bosonic case (Kapustin, Turzillo, and You, 2017), spin-TQFT constructed from A depends only on the Morita equivalence class of A. This explains why the spin-TQFT corresponding to $\mathbb{C}l(2)$ is trivial.

We see that $A = \mathbb{C}l(1)$ corresponds to a nontrivial SRE phase in the fermionic case (it is its own inverse). On the other hand, $\mathbb{C}l(1) \otimes \mathbb{C}l(1)$ is a commutative algebra isomorphic to a sum of two copies of $\mathbb{C}l(1)$. Therefore the bosonic phase corresponding to $\mathbb{C}l(1)$ is not invertible. This example illustrates that bosonization does not preserve the stacking operation.

Including boundaries

When Σ has a non-empty boundary, Γ is allowed to have univalent vertices which all lie on the boundary $\partial\Sigma$. Let M be the number of boundary vertices. For every vertex v we color each element of $\Gamma(v)$ with a basis vector of A, so that a vertex on the boundary has only a single label. As before, the weight of each coloring is a product of three factors: the product of C_{ijk} over all trivalent vertices and η^{ij} over all edges, the Koszul sign, and the spin-dependent sign. When summing over colorings, the labels of the boundary vertices remain fixed. The result of the summation can be interpreted as a value of a map

$$Z_\Gamma(\Sigma) : A^{\otimes M} \to \mathbb{C}, \tag{3.62}$$

on a particular basis vector in $A^{\otimes M}$.

It is implicit here that the map depends on the spin structure on every connected component of $\partial\Sigma$. It can be read off from the function $\sigma_s(\gamma)$ evaluated on the boundary components. The spin structure is Neveu-Schwarz if $\sigma_s = 1$ and Ramond if $\sigma_s = -1$.

We can also consider open-closed spin-TQFT, i.e. spin-TQFT in the presence of topological boundary conditions (branes). Such boundary conditions are encoded in $\mathbb{Z}_2$-graded modules over A. A $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded algebra A is a $\mathbb{Z}_2$-graded vector space $U = U_+ \oplus U_-$ with the structure of an A-module such that $A_+ \cdot U_\pm \subseteq U_\pm$ and $A_- \cdot U_\pm \subseteq U_\mp$. Equivalently, U is an A-module equipped with an involution P such that $T(\mathcal{F}(a)) = PT(a)P^{-1}$.

For each boundary component of Σ, choose a $\mathbb{Z}_2$-graded A-module U and a homogeneous basis f_μ^U of U. Label each boundary edge with a basis vector of U. The weight of the coloring is a product of the C's and η's and a sign $\sigma_s(\beta)$, as well as a module tensor $T^\mu{}_{\nu i}$ for each boundary vertex. The sign is computed as before as a product of the spin-structure-dependent sign and the Koszul sign.

3.5 Fermionic MPS

Fermionic Matrix Product States and the annulus diagram

In this section, we will extract MPS wavefunctions from the spin-TQFT by considering the special case when Σ is an annulus. Take one of the boundary circles to be a source cut boundary and the other to be a brane boundary corresponding to a $\mathbb{Z}_2$-graded A-module U with action $T(a) \in \mathrm{E}nd(U)$. Choose a triangulation of Σ. It was shown in (Kapustin, Turzillo, and You, 2017) that one can deform the skeleton to look like Figure 3.12.

Give the skeleton a marking and spin signs that models the spin structure on Σ. It is convenient to make the choices shown in Figure 3.12. The sign on the N-to-1 edge is $+1$ if the spin structure induced on the boundary circles is NS and -1 if it is R. To get the sign (3.53), we insert a factor of P for each $+1$.

Following the procedure detailed in Section 3.4 to evaluate the diagram in Figure

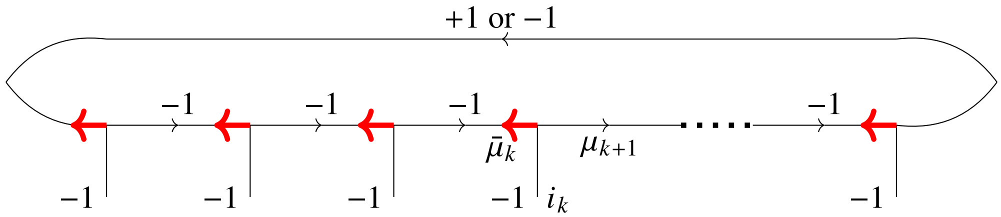


Figure 3.12: Black arrows are edge orientations, and red arrows are special edges. All of the spin signs are −1 except possibly the one on the N-to-1 edge, which is +1 in the NS sector and −1 in the R sector.

3.12, one finds

$$Z(\Sigma_{T,NS}) = \sum_{I=\{i_k,\mu_k,\bar{\mu}_k\}} \sigma_0(\beta_I) \times T^{\bar{\mu}_N i_1 \mu_1} T^{\bar{\mu}_1 i_2 \mu_2} \ldots T^{\bar{\mu}_{N-1} i_N \mu_N} \times \delta_{\mu_1\bar{\mu}_1}\delta_{\mu_2\bar{\mu}_2}\cdots P_{\mu_N\bar{\mu}_N} \langle i_1 i_2 \cdots i_N| \tag{3.63}$$

in the NS sector and

$$Z(\Sigma_{T,R}) = \sum_{I=\{i_k,\mu_k,\bar{\mu}_k\}} \sigma_0(\beta_I) \times T^{\bar{\mu}_N i_1 \mu_1} T^{\bar{\mu}_1 i_2 \mu_2} \ldots T^{\bar{\mu}_{N-1} i_N \mu_N} \times \delta_{\mu_1\bar{\mu}_1}\delta_{\mu_2\bar{\mu}_2}\cdots \delta_{\mu_N\bar{\mu}_N} \langle i_1 i_2 \cdots i_N| \tag{3.64}$$

in the R sector, where the Koszul sign is given as a Grassmann integral

$$\sigma_0(\beta_I) = \int d\theta_1^{|\mu_1|} d\bar{\theta}_1^{|\bar{\mu}_1|} d\theta_2^{|\mu_2|} d\bar{\theta}_2^{|\bar{\mu}_2|} \cdots d\theta_N^{|\mu_N|} d\bar{\theta}_N^{|\bar{\mu}_N|} d\theta_{i_1}^{|i_1|} d\theta_{i_2}^{|i_2|} \cdots d\theta_{i_N}^{|i_N|} \times \bar{\theta}_N^{|\bar{\mu}_N|} \theta_{i_1}^{|i_1|} \theta_1^{|\mu_1|} \bar{\theta}_1^{|\bar{\mu}_1|} \theta_{i_2}^{|i_2|} \theta_2^{|\mu_2|} \cdots \bar{\theta}_{N-1}^{|\bar{\mu}_{N-1}|} \theta_{i_N}^{|i_N|} \theta_N^{|\mu_N|}. \tag{3.65}$$

Evaluating the integral amounts to reordering the variables in the integrand to match the ordering in the measure while recording the sign

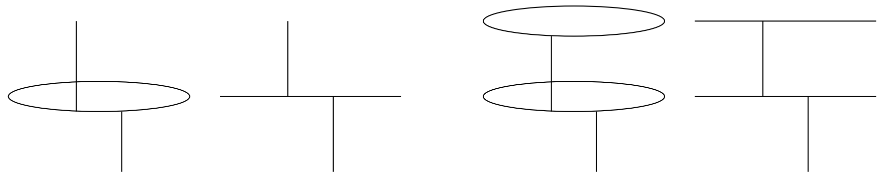
$$\theta_1^{s_1}\theta_2^{s_2} = (-1)^{s_1 s_2}\theta_2^{s_2}\theta_1^{s_1}. \tag{3.66}$$

Moving $\bar{\theta}_N^{|\bar{\mu}_N|}$ across the integrand gives a sign $(-1)^{|\bar{\mu}_N|}$. Then moving each $\theta_{i_k}^{|i_k|}$ to the right gives a sign +1. Therefore the total sign is

$$\sigma_0(\beta_I) = (-1)^{|\bar{\mu}_N|}. \tag{3.67}$$

Noting that $\delta_{\mu_n\bar{\mu}_N}(-1)^{|\bar{\mu}_N|} = P_{\mu_n\bar{\mu}_N}$, we find that the MPS wavefunctions take the forms

$$\langle \psi_{T,NS}| = Z(\Sigma_{T,NS}) = \sum_{i_1,i_2,\cdots,i_N} \mathrm{Tr}\left[T(e_{i_1})T(e_{i_2})\cdots T(e_{i_N})\right] \langle i_1 i_2 \cdots i_N| \tag{3.68}$$



(a) Skeleton of an annulus with cut boundaries

(b) An annulus with one brane and one cut boundary

Figure 3.13

and

$$\left\langle\psi_{T,R}\right| = Z(\Sigma_{T,R}) = \sum_{i_1,i_2,\cdots,i_N} \operatorname{Tr}\left[PT(e_{i_1})T(e_{i_2})\cdots T(e_{i_N})\right] \langle i_1 i_2 \cdots i_N| \,. \tag{3.69}$$

More general states, called *generalized MPS*, on the closed chain are obtained from the spin-TQFT by inserting a local observable on the brane boundary of the annulus. Such observables are parametrized by linear maps $X : U \to U$ and can be either even or odd; that is, $PX = XP$ or $PX = -XP$, respectively.

The NS sector MPS resulting from the insertion of X has conjugate wavefunction

$$\left\langle\psi^X_{T,NS}\right| = \sum_{i_1\cdots i_N} \operatorname{tr}\left[X^\dagger T(e_{i_1})\cdots T(e_{i_N})\right] \langle i_1 \cdots i_N| \,. \tag{3.70}$$

In the R sector,

$$\left\langle\psi^X_{T,R}\right| = \sum_{i_1\cdots i_N} \operatorname{tr}\left[PX^\dagger T(e_{i_1})\cdots T(e_{i_N})\right] \langle i_1 \cdots i_N| \,. \tag{3.71}$$

Note that the generalized MPS corresponding to the trivial observable $X = \mathbb{1}$ are the states $\langle\psi_T|$ (3.68)(3.69).

The state $\left|\psi^X_{T,NS/R}\right\rangle$ has the same fermionic parity as the observable X since

$$\begin{aligned}
\mathcal{F}^{\otimes N}\left\langle\psi^X_{T,NS(R)}\right| &= \sum \operatorname{Tr}\left[(P)X^\dagger T(\mathcal{F}\cdot e_{i_1})\cdots T(\mathcal{F}\cdot e_{i_n})\right]\langle i_1\cdots i_n| \\
&= \sum \operatorname{Tr}\left[(P)PX^\dagger PT(e_{i_1})\cdots T(e_{i_n})\right]\langle i_1\cdots i_n| \\
&= (-1)^{|X|}\left\langle\psi^X_{T,NS(R)}\right| .
\end{aligned} \tag{3.72}$$

Parent Hamiltonians

Hamiltonians appear in TQFT as cylinders. There is one for each of the NS and R sectors. To be precise, the Hamiltonian is the linear map

$$H_{NS(R)} = \mathbb{1} - Z(C_{NS(R)}), \tag{3.73}$$

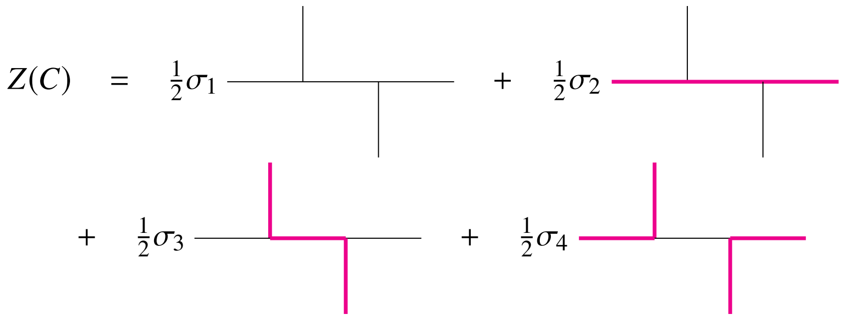


Figure 3.14: The cylinder partition sum $Z(C)$ factors as a signed sum of four colored diagrams: $\sigma(\beta_1)C_1 + \sigma(\beta_2)C_2 + \sigma(\beta_3)C_3 + \sigma(\beta_4)C_4 = C_1 + \eta C_2 + C_3 - \eta C_4$. Magenta lines indicate odd edges.

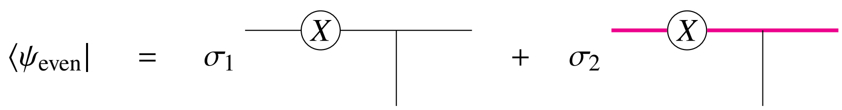


Figure 3.15: $\langle\psi_{\text{even}}| = \sigma(\beta_1)\,\langle\psi_1| + \sigma(\beta_2)\,\langle\psi_2| = \langle\psi_1| + \eta\,\langle\psi_2|$

Figure 3.16: $\langle\psi_{\text{odd}}| = \sigma(\beta_3)\,\langle\psi_3| + \sigma(\beta_4)\,\langle\psi_4| = \langle\psi_3| + \eta\,\langle\psi_4|$

where $C_{NS(R)}$ denotes the cylinder with NS (R) spin structure. The composition of two cylinder cobordisms is again a cylinder, so $Z(C)$ is a projector, and therefore so is H. Ground states are those with eigenvalue 1 under $Z(C)$. It is convenient to specialize to the case of a single site, $N = 1$. Since these Hamiltonians arise from a topologically-invariant theory, properties of the $N = 1$ system must hold more generally. Consider the skeleton of the cylinders depicted in Figure 3.13.

By exploiting (3.54), we will not need the full machinery of lattice spin structures to understand the Hamiltonians and their ground states. The path integrals for the cylinders can be expressed as a sum over the four relative 1-cycles $\beta_1, \ldots, \beta_4$ depicted in Figure 3.14. The first colored diagram corresponds to the trivial cycle β_1 and has no odd labels, so its sign is trivial, $\sigma_s(\beta_1) = 1$. The second one corresponds to the equator of the cylinder and comes with the sign $\sigma(\beta_2) := \eta$, which is $+1$ in the NS sector and -1 in the R sector. The relative cycles β_3 and β_4 sum to β_2 and have intersection number 1, where the intersection pairing is defined by gluing another annulus onto the annulus, to get a torus $C^* = T^2$, as explained in Section 3.4. Therefore (3.50) says there is a relative sign

$$\sigma_s(\beta_3)\sigma_s(\beta_4) = \sigma_s(\beta_3 + \beta_4)(-1)^{\langle\beta_3,\beta_4\rangle} = \sigma_s(\beta_2)(-1) = -\eta. \tag{3.74}$$

One can choose a spin structure on the closed space $C^* = T^2$ such that $\sigma_s(\beta_3) = 1$; this amounts to fixing trivializations of the spin structures induced on each component of ∂C at the univalent vertices.

Similarly, an even MPS can be expressed as the sum in Figure 3.15, where $\sigma_1 = 1$ and $\sigma_2 = \eta$, and an odd MPS as the sum in Figure 3.16 with $\sigma_1 = 1$ and $\sigma_2 = \eta$.

Now we are ready to argue that the parent Hamiltonian has a generalized MPS $\left\langle \psi_T^X \right|$ a ground state if X *supercommutes* with $T(a)$; that is, if an even observable satisfies

$$XT(a) = T(a)X \quad \forall a \in A, \tag{3.75}$$

and an odd observable satisfies

$$XT(a) = (-1)^{|a|}T(a)X \quad \forall a \in A. \tag{3.76}$$

Linear maps satisfying these conditions are called even and odd $\mathbb{Z}_2$-graded module endomorphisms.

The maps C_3 and C_4 correspond to diagrams with odd legs, and so annihilate even states $\langle \psi_{\text{even}}|$. Therefore

$$Z(C)\langle \psi_{\text{even}}| = \tfrac{1}{2}(C_1 + \eta C_2)(\langle \psi_1| + \eta \langle \psi_2|). \tag{3.77}$$

By the sequence of diagram moves depicted in Figures 3.17, C.1, C.2, and C.3, one can show that

$$C_1 \langle \psi_1| = \langle \psi_1|, \quad C_2 \langle \psi_1| = \eta_X \langle \psi_2|, \quad C_1 \langle \psi_2| = \langle \psi_2|, \quad C_2 \langle \psi_2| = \eta_X \langle \psi_1|, \tag{3.78}$$

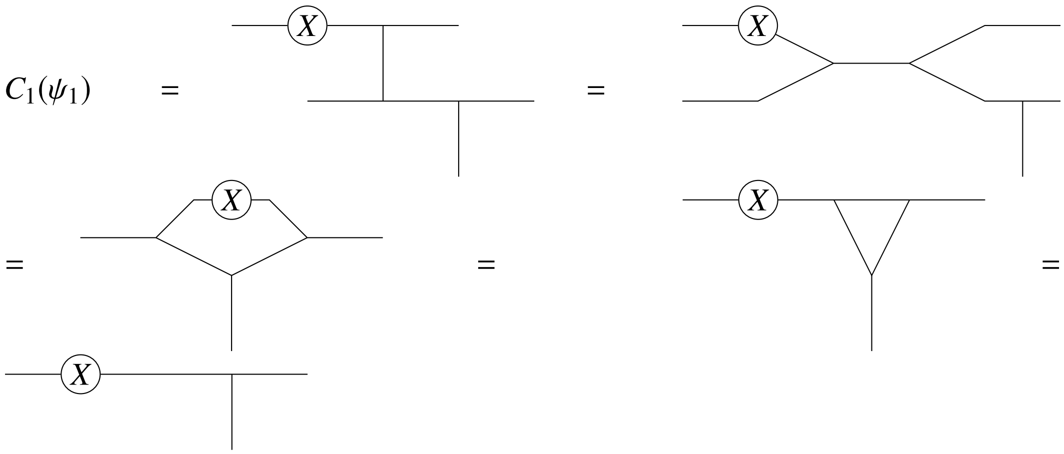
where η_X denotes the sign due to commuting X with odd $T(a)$. According to the rule (3.75), $\eta_X = 1$, so

$$Z(C)\langle \psi_{\text{even}}| = \tfrac{1}{2}(1 + \eta_X)\langle \psi_{\text{even}}| = \langle \psi_{\text{even}}|. \tag{3.79}$$

Similarly, the cylinder acts on odd states as

$$Z(C)\langle \psi_{\text{odd}}| = \tfrac{1}{2}(C_3 - \eta C_4)(\langle \psi_3| + \eta \langle \psi_4|). \tag{3.80}$$

Commuting X with the vertex gives $\langle \psi_4| = \eta_X \langle \psi_3|$, which means $\langle \psi_{\text{odd}}| = (1 + \eta\eta_X)\langle \psi_3|$. According to the rule (3.76), $\eta_X = -1$, so the only odd ground state in the NS sector is $\langle \psi| = 0$. This agrees with (G. W. Moore and Segal, 2006).



$= \psi_1$

Figure 3.17: Diagrammatic proof of $C_1 \langle \psi_1 | = \langle \psi_1 |$. The topmost line represents the physical boundary, with module indices living on it. The others are depicted in Appendix C.1.

In the Ramond sector, one can have nonzero odd ground states. The sequence of moves of Figures C.4 and C.5 shows

$$C_3 \langle \psi_3 | = \langle \psi_3 | , \quad C_4 \langle \psi_3 | = \langle \psi_3 | , \tag{3.81}$$

so

$$Z(C) \langle \psi_{\text{odd}} | = \tfrac{1}{2}(1 - \eta) \langle \psi_{\text{odd}} | = \langle \psi_{\text{odd}} | \text{ (in the R sector).} \tag{3.82}$$

Therefore $\left\langle \psi_T^X \right|$ is indeed a ground state of $H_{NS(R)}$ provided X is a $\mathbb{Z}_2$-graded module endomorphism.

Next we argue that every ground state of H of the form (3.70) or (3.71) for arbitrary X can be written as a generalized MPS where X supercommutes with T. A result of Ref. (G. W. Moore and Segal, 2006) (c.f. eq 3.18) implies that

$$Z(C_{NS}) |ij\rangle = (-1)^{|i||j|+|i|} Z(C_{NS}) |ji\rangle \tag{3.83}$$

and

$$Z(C_R) |ij\rangle = (-1)^{|i||j|} Z(C_R) |ji\rangle . \tag{3.84}$$

In Appendix C.2, we rederive this result in the Novak-Runkel formalism. Then, since $|X| = |i| + |j|$,

$$Z(C_{NS}) \operatorname{Tr}\big[XT(e_i)T(e_j)\big] |ij\rangle = (-1)^{|i||X|} Z(C_{NS}) \operatorname{Tr}\big[T(e_i)XT(e_j)\big] |ji\rangle \tag{3.85}$$

and

$$Z(C_R) \operatorname{Tr}\big[PXT(e_i)T(e_j)\big] |ij\rangle = (-1)^{|i||X|} Z(C_R) \operatorname{Tr}\big[PT(e_i)XT(e_j)\big] |ji\rangle . \tag{3.86}$$

For ground states, i.e. eigenstates of $Z(C)$ with eigenvalue 1, this means that X supercommutes with T.

It turns out that all ground states of H can be written as generalized MPS. As discussed in (Kapustin, Turzillo, and You, 2017), in a unitary theory T is an isometry with respect to some inner product on A and the standard inner product

$$\langle M|N\rangle = \mathrm{Tr}\left[M^\dagger N\right] \quad M, N \in \mathrm{E}nd(V) \tag{3.87}$$

on $\mathrm{E}nd(V)$. For an orthogonal basis $\{e_i\}$ of A, $\mathrm{Tr}\left[T(e_i)^\dagger T(e_j)\right] = \delta_{ij}$. Consider the case $N = 1$. An arbitrary state

$$\langle\psi| = \sum_i a_i \langle i| \tag{3.88}$$

can be written in generalized MPS form (3.70)(3.71) if one takes

$$X_{NS} = \sum_j a_j T(e_j)^\dagger \quad \text{or} \quad X_R = \sum_j a_j P T(e_j)^\dagger. \tag{3.89}$$

Thus generalized MPS with supercommuting X are the only ground states. Neither the number of generalized MPS nor the number of ground states depends on N; thus, the argument extends to all N.

A consequence of supercommutativity and (3.72) is that there are no odd ground states in the NS sector. Suppose that X is an odd observable. For $a \in A_-$, the matrix $X^\dagger$ anticommutes with $T(a)$, so the coefficient $\mathrm{Tr}\left[X^\dagger T(a)\right]$ vanishes. For $a \in A_+$, the matrix $X^\dagger T(a)$ maps $U_\pm$ to $U_\mp$ and so also vanishes in the trace. Therefore the state (3.70) is zero for odd X, which is to say that the NS sector does not support odd states. The argument fails for the state (3.71); generically, the R sector supports both even and odd states. The lack of odd states in the NS sector can also be seen directly from (3.83), which implies $|C\,|ij\rangle\,| = |i| + |j| = 0$.

Stacking fermionic MPS

Bosonization establishes a 1-1 correspondence between 1d bosonic systems with $\mathbb{Z}_2$ symmetry and 1d fermionic systems. In the gapped case, the corresponding topological phases are described by the same algebraic data, namely by a $\mathbb{Z}_2$-graded algebra A. But bosonization does not preserve a crucial physical structure: stacking systems together. From the mathematical viewpoint, either bosonic or fermionic topological phases of matter form a commutative monoid (a set with a commutative associative binary operation and a neutral element, but not necessarily with an inverse for every element), but bosonization does not preserve the monoid structure (i.e. it does not preserve the product). A well-known example is given by the fermionic SRE phases: the non-trivial fermionic SRE phase (the Majorana chain) is mapped to the bosonic phase with a spontaneously broken $\mathbb{Z}_2$. The former one is invertible, while the latter one is not. Both phases correspond to the algebra $\mathbb{C}l(1)$.

In the bosonic case, it was shown in (Kapustin, Turzillo, and You, 2017) that, given two algebras A_1 and A_2 with bosonic Hamiltonians H_1 and H_2, the tensor product system $A_1 \otimes A_2$ has a Hamiltonian $H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$. That is, stacking bosonic systems together corresponds to the tensor product of algebras.

On the other hand, in section 2.4 we have shown that for fermionic systems stacking corresponds to the supertensor product (3.58). We can now see that the supertensor product rule is consistent with the way fermionic generalized MPS are defined (while the usual tensor product is not).

Suppose H_1 is the Hamiltonian for the MPS system built from a $\mathbb{Z}_2$-graded algebra A_1 that acts on a $\mathbb{Z}_2$-graded module U_1 by T_1. Its ground states are parametrized by $\mathbb{Z}_2$-graded module endomorphisms X_1 of U_1. Consider stacking H_1 with a second system H_2 defined by $T_2 : A_2 \to \mathrm{E}nd(U_2)$ with ground states parametrized by X_2. The stacked system is the MPS system with physical space $A_1 \otimes A_2$ and Hamiltonian $H = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$. It has bond space $U_1 \otimes U_2$ and MPS tensor $T = T_1 \otimes T_2$.

The ground states are generalized MPS, and so correspond to $\mathbb{Z}_2$-graded endomorphisms of the module $U_1 \otimes U_2$. Since the MPS tensor is $T = T_1 \otimes T_2$, the state $\langle \psi_T^X |$ is trivial unless X is of the form $X_1 \otimes X_2$. We also know that X supercommutes with T:

$$(X_1 \otimes X_2)(T_1 \otimes T_2) = (-1)^{(|X_1|+|X_2|)(|T_1|+|T_2|)}(T_1 \otimes T_2)(X_1 \otimes X_2) \tag{3.90}$$

There are two ways one might define the composition of tensor products of operators[17]:

$$(X_1 \otimes X_2)(T_1 \otimes T_2) = X_1T_1 \otimes X_2T_2 \tag{3.91}$$

and

$$(X_1 \mathbin{\widehat{\otimes}} X_2)(T_1 \mathbin{\widehat{\otimes}} T_2) = (-1)^{|X_2||T_1|} X_1T_1 \mathbin{\widehat{\otimes}} X_2Y_2 \tag{3.92}$$

Since X_1 supercommutes with T_1 and X_2 with T_2, only the second notion (3.92) of composition is consistent with (3.90). The composition rule is an algebra structure

[17]These correspond to the two symmetric monoidal structures on the category of $\mathbb{Z}_2$-graded vector spaces.

on $\mathrm{E}nd(U_1) \otimes \mathrm{E}nd(U_2)$ and pulls back by T to an algebra structure on $A_1 \otimes A_2$ given by the rule (3.58).

An important assumption in this argument is that isomorphic TQFTs correspond to equivalent gapped phases. Assuming this is true, we can easily see that the group of fermionic SRE phases is isomorphic to $\mathbb{Z}_2$. Indeed, one can easily see that a phase which is invertible must correspond to an indecomposable algebra (i.e. the algebra which cannot be decomposed as a sum of algebras). Since all our algebras are semisimple, this means that invertible phases must correspond to simple algebras. It is well-known that there are exactly two Morita-equivalence classes of $\mathbb{Z}_2$-graded algebras: the trivial one and the class of $\mathrm{C}l(1)$. The square of the nontrivial class is the trivial class. Hence the group of invertible fermionic phases is isomorphic to $\mathbb{Z}_2$. In the next section we will show explicitly that $\mathrm{C}l(1)$ corresponds to the nontrivial Majorana chain.

3.6 Hamiltonians for fermionic SRE phases

The trivial SRE phase

An example of a system in the trivial phase is the trivial Majorana chain (Fidkowski and Kitaev, 2010). On a circle, this system has only bosonic states: one in the NS sector and one in the R sector. We will now demonstrate that this is the same phase as the MPS system built out of the Clifford algebra $\mathrm{C}l(2) = \mathrm{E}nd(\mathbb{C}^{1|1})$.

The algebra $A = C\ell(2)$ is expressed in terms of its odd generators as $\mathbb{C}[x, y]/(x^2 - 1, y^2 - 1, xy + yx)$. Let A act on $U = \mathbb{C}^{1|1}$ by

$$T : x \mapsto [\sigma_x]_\pm \quad , \quad y \mapsto [\sigma_y]_\pm \tag{3.93}$$

where $[\cdot]_\pm$ denotes a matrix in the homogeneous basis of U. This action is graded and faithful. The fermion parity operator P acts by σ_z.

The even ground states of this system are parametrized by matrices that commute with σ_x, σ_y, and σ_z. Thus X is proportional to the identity $\mathbb{1}$. The corresponding NS sector state has the wavefunction $\mathrm{Tr}\big[T(e_{i_1}) \cdots T(e_{i_N})\big]$. There is also an even state in the R sector given by $\mathrm{Tr}\big[PT(e_{i_1}) \cdots T(e_{i_N})\big]$.

The odd ground states are parametrized by matrices that commute with $T(a)$ – in particular, $T(xy) = \sigma_z$ – and anticommute with $P = \sigma_z$. This is impossible, so there are no odd states in either sector.

In summary, the ground states of the $A = \mathrm{C}l(2)$ MPS system are a bosonic one in the NS sector and a bosonic one in the R sector, just like the ground states of the trivial Majorana chain.

One can show that the MPS parent Hamiltonian (c.f. (Kapustin, Turzillo, and You, 2017; Schuch, Perez-García, and I. Cirac, 2010)) is a nearest-neighbor Hamiltonian with the two-body interaction $H_T = -\sum_{\alpha=1}^{4} |v_\alpha\rangle \langle v_\alpha|$ where

$$\begin{aligned} v_1 &= 1 \otimes 1 - x \otimes x - y \otimes y - xy \otimes xy \\ v_2 &= 1 \otimes x + x \otimes 1 + y \otimes xy - xy \otimes y \\ v_3 &= 1 \otimes y + y \otimes 1 + xy \otimes x - x \otimes xy \\ v_4 &= 1 \otimes xy + xy \otimes 1 + x \otimes y - y \otimes x \end{aligned} \tag{3.94}$$

It is not obvious that H_T is equivalent to the Hamiltonian of the trivial Majorana chain

$$H = \sum_j (a_j^\dagger a_j - 1) \tag{3.95}$$

but it should be possible to construct an LU transformation between the two Hamiltonians (after some blocking), as the systems have the same spaces of ground states and so lie in the same phase.

The nontrivial SRE phase

An example of a fermionic system in a nontrivial SRE phase is the Majorana chain with a two-body Hamiltonian (Fidkowski and Kitaev, 2010)

$$H_j = \frac{1}{2}\left(-a_j^\dagger a_{j+1} - a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}^\dagger + a_{j+1} a_j\right) \tag{3.96}$$

This system has one bosonic and one fermionic ground state on the interval arising from one Majorana zero mode at each end. In the continuum limit this system becomes a free Majorana fermion with a negative mass. In the NS sector there is a unique ground state which is bosonic, while in the R sector there is a unique ground state which is fermionic (this is most easily seen from the continuum field theory).

In order to get this phase from a spin TQFT, we let $A = C\ell(1)$. To see the full space of ground states, we need a faithful graded module over A. Let $U = U_+ \oplus U_-$, where each $U_\pm$ is spanned by a single vector $u_\pm$. Let A act on U by

$$T : \Gamma \mapsto [\sigma_x]_\pm = u_+ \otimes u_-^* + u_- \otimes u_+^*. \tag{3.97}$$

In other words, U is A regarded as a module over itself.

The even ground states of this system are parametrized by matrices that commute with $P = [\sigma_z]_\pm$ and $T(\Gamma) = [\sigma_x]_\pm$. Such matrices are proportional to $\mathbb{1}$. The corresponding NS sector state has wavefunction $\mathrm{Tr}\big[T(e_{i_1})\cdots T(e_{i_N})\big]$. There is no even state in the R sector as the trace $\mathrm{Tr}\big[PT(e_1)\cdots T(e_{i_N})\big]$ vanishes.

The odd ground states are parametrized by matrices that anticommute with P and $T(\Gamma)$. Such matrices X are all proportional to $[\sigma_y]_\pm$. By the general argument of Section 3.5, we know that the NS sector has no odd states. The wavefunction $\mathrm{Tr}\big[PX^\dagger T(e_{i_1})\cdots T(e_{i_N})\big]$ defines an odd state in the R sector.

In summary, the ground states of the $A = \mathrm{C}l(1)$ MPS system are a bosonic one in the NS sector and a fermionic one in the R sector, just like the ground states of the nontrivial Majorana chain.

We can also observe the equivalence of the two systems from the standpoint of Hamiltonians. We build the MPS parent Hamiltonian for the $A = C\ell(1)$ system by following Ref. (Kapustin, Turzillo, and You, 2017; Schuch, Perez-García, and I. Cirac, 2010). The adjoint $\mathcal{P} = T^\dagger$ is given by

$$\mathcal{P} : 2u_\pm \otimes u_\pm^* \mapsto 1 \otimes 1 + \Gamma \otimes \Gamma \quad , \quad 2u_\pm \otimes u_\mp^* \mapsto 1 \otimes \Gamma + \Gamma \otimes 1 \tag{3.98}$$

With respect to the inner products on A and U for which 1 and Γ and u_+ and u_- are unit vectors, the graded module structure T is an isometry, so the left inverse $\mathcal{P}^+$ is simply T. Putting these pieces together, we find

$$H_T = |11\rangle\langle\Gamma\Gamma| - |1\Gamma\rangle\langle\Gamma 1| - |\Gamma 1\rangle\langle 1\Gamma| + |\Gamma\Gamma\rangle\langle 11| \tag{3.99}$$

where $|ab\rangle\langle cd|$ denotes the element $a \otimes b \otimes c^* \otimes d^* \in \mathrm{E}nd(A \otimes A)$. In terms of the annihilation operators $a_j = \sqrt{2}\,|1\rangle\langle\Gamma|_j$ and their adjoints, the hopping (top row) and pairing (bottom) terms look like

$$\begin{aligned} a_j^\dagger \otimes a_{j+1} &= 2\,|\Gamma 1\rangle\langle 1\Gamma| \qquad & a_{j+1}^\dagger \otimes a_j &= 2\,|1\Gamma\rangle\langle\Gamma 1| \\ a_j^\dagger \otimes a_{j+1}^\dagger &= 2\,|\Gamma\Gamma\rangle\langle 11| \qquad & a_{j+1} \otimes a_j &= 2\,|11\rangle\langle\Gamma\Gamma| \end{aligned} \tag{3.100}$$

so the Hamiltonians (3.96) and (3.99) agree. The variables a_j satisfy fermionic anti-commutation relations. For example,

$$\{a_j, a_{j+1}\} = (a \otimes \mathbb{1})(\mathbb{1} \otimes a) + (\mathbb{1} \otimes a)(a \otimes \mathbb{1}) = a \otimes a + (-1)^{|a||a|} a \otimes a = 0 \tag{3.101}$$

if we are careful to use the fermionic tensor product (3.58). The other relations can be checked similarly.

3.7 Equivariant spin-TQFT and equivariant fermionic MPS

$(\mathcal{G}, p)$-equivariant algebras and modules

Let $(\mathcal{G}, p)$ be a finite supergroup, i.e. a finite group $\mathcal{G}$ with a distinguished involution $p \in \mathcal{G}$ called *fermion parity*. We assume the involution p is central in $\mathcal{G}$, which means that there are no supersymmetries. Every supergroup $(\mathcal{G}, p)$ arises as a central extension of a group $G_b \simeq \mathcal{G}/\mathbb{Z}_2$ of bosonic symmetries by $\mathbb{Z}_2 = \{1, p\}$; that is, there is an exact sequence

$$1 \to \mathbb{Z}_2 \xrightarrow{i} \mathcal{G} \xrightarrow{b} G_b \to 1. \tag{3.102}$$

A trivialization of $(\mathcal{G}, p)$ is a function $t : \mathcal{G} \to \mathbb{Z}_2$ such that $t \circ i$ is the identity on $\mathbb{Z}_2$. Given a trivialization, one can encode the multiplication rule for $\mathcal{G}$ in terms of the product on G_b and a $\mathbb{Z}_2$-valued group 2-cocycle ρ of G_b. Consider the following product on the set $G_b \times \mathbb{Z}_2$ (denoted $G_b \times_\rho \mathbb{Z}_2$). For $\bar{g}, \bar{h} \in G_b$, $f, f' \in \mathbb{Z}_2$,

$$(\bar{g}, f) \cdot (\bar{h}, f') = (\bar{g}\bar{h}, \rho(\bar{g}, \bar{h}) + f + f') \tag{3.103}$$

Denote $\bar{g} := b(g)$. The map $b \times_\rho t : g \mapsto (\bar{g}, t(g))$ defines a group isomorphism $\mathcal{G} \xrightarrow{\sim} G_b \times_\rho \mathbb{Z}_2$; that is,

$$g \cdot h = (\bar{g}, t(g)) \cdot (\bar{h}, t(h)) = (\bar{g}\bar{h}, \rho(\bar{g}, \bar{h}) + t(g) + t(h)) = (\bar{gh}, t(gh)) = gh, \tag{3.104}$$

if and only if

$$\rho(\bar{g}, \bar{h}) = t(gh) + t(g) + t(h). \tag{3.105}$$

Suppose t' is another trivialization. Since $t = t'$ on the image of i and the sequence (3.102) is exact, the map $t - t'$ defines a 1-cochain of G_b. Thus, upon replacing t with t', ρ is modified by the coboundary $\delta(t - t')$, so only the cohomology class $[\rho]$ of c is an invariant of the extension. If $[\rho]$ is trivial, $\mathcal{G}$ is isomorphic to the direct product group $G_b \times \mathbb{Z}_2$ and we say the extension *splits*; in general, this is not the case. Some discussions of fermionic phases in the physics literature assume that $(\mathcal{G}, p)$ is split, but we will consider both cases simultaneously. Note that (Fidkowski and Kitaev, 2010) considered both cases as well.

An action R of $(\mathcal{G}, p)$ on a vector space V endows it with a distinguished $\mathbb{Z}_2$-grading

$$V_\pm = \{v \in V : R(p)v = \pm v\}. \tag{3.106}$$

Centrality of p ensures that $R(g)$ is even with respect to this grading, for all $g \in \mathcal{G}$. A $(\mathcal{G}, p)$-equivariant Frobenius algebra is a Frobenius algebra (A, m, η) with an action of $(\mathcal{G}, p)$ that satisfies

$$m(R(g)a \otimes R(g)b) = R(g)m(a \otimes b) \tag{3.107}$$

and

$$\eta(R(g)a, R(g)b) = \eta(a, b) \tag{3.108}$$

for all $a, b \in A, g \in G$. As was true for the special case $\mathcal{G} = \mathbb{Z}_2$, there are two notions of tensor product of these algebras: the usual one that forgets the distinguished $\mathbb{Z}_2$-grading and a supertensor product (3.58) that remembers it. In both cases, the symmetry acts on the product as

$$R(g)(a_1 \otimes a_2) = R_1(g)a_1 \otimes R_2(g)a_2 \tag{3.109}$$

which is a special case of the rule

$$(\phi_1 \otimes \phi_2)(a_1 \otimes a_2) = (-1)^{|\phi_2||a_1|}\phi_1(a_1) \otimes \phi_2(a_2) \tag{3.110}$$

for $\phi_1 \otimes \phi_2 \in \mathrm{E}nd(A_1) \otimes \mathrm{E}nd(A_2)$, where we have taken $R(g) = R_1(g) \otimes R_2(g)$.

We have argued in (Kapustin, Turzillo, and You, 2017) that bosonic phases with symmetry G are classified by G-equivariant symmetric Frobenius algebras and that stacking of phases corresponds to the usual tensor product of their algebras. Here we will argue the fermionic analog: $(\mathcal{G}, p)$-equivariant symmetric Frobenius algebras classify fermionic phases with symmetry $(\mathcal{G}, p)$, for which stacking is governed by the supertensor product. In this language, bosonization means taking a $(\mathcal{G}, p)$-equivariant algebra to a $\mathcal{G}$-equivariant algebra by forgetting the distinguished involution p. Generically, if $\mathcal{G}$ has more than one central involution, this map is many-to-one.

An equivariant module over a $(\mathcal{G}, p)$-equivariant algebra A is vector space V with compatible actions of A and $(\mathcal{G}, p)$; that is, for every $a \in A$, we have a linear map $T(a) \in \mathrm{E}nd(V)$ such that $T(a)T(b) = T(ab)$, and for every $g \in G$, a linear map $Q(g)$ such that $Q(g)Q(h) = Q(gh)$. The compatibility condition reads

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1} \tag{3.111}$$

Note that T automatically respects the $\mathbb{Z}_2$-grading.

For a review of the classification of equivariant algebras and modules, we refer the reader to the prequel (Kapustin, Turzillo, and You, 2017), which compiles some algebraic facts from (Ostrik, 2003; Etingof, 2015). There are two classes of algebras that will be especially useful in the present context, as they describe fermionic SRE phases. One class of algebras is those of the form $\mathrm{E}nd(U)$ for a projective representation U of $\mathcal{G}$. Each pair (Q, U) has an associated class $[\omega] \in H^2(\mathcal{G}, U(1))$ that measures the failure of Q to be a homomorphism:

$$Q(g)Q(h) = \exp(2\pi i\omega(g, h))Q(gh). \tag{3.112}$$

Each $[\omega]$ defines a Morita class of algebras and therefore a phase. Equivariant modules over $\mathrm{E}nd(U)$ are all of the form $U \otimes W$, where W is a projective representation with class $-[\omega]$. When $\mathcal{G}$ can be written as $G_b \times \{1, p\}$ for some group G_b of bosonic symmetries, another class of equivariant algebras is those of the form $\mathrm{E}nd(U_b) \otimes \mathrm{C}l(1)$ for a projective representation (U_b, Q_b) of G_b. The group G_b acts by conjugation on $\mathrm{E}nd(U_b)$. It also acts on the generator of $\mathrm{C}l(1)$ by

$$\bar{g} : \Gamma \mapsto (-1)^{\beta(\bar{g})}\Gamma, \tag{3.113}$$

where $\beta : G_b \to \mathbb{Z}_2$ is a homomorphism. Up to Morita-equivalence, algebras of this type depend only on the 1-cocycle β and the 2-cocycle α on G_b corresponding to the projective representation Q_b. While the bosonic phases built from these algebras have a broken $\mathbb{Z}_2$, their fermionic duals are nonetheless SRE phases.

Equivariant fermionic MPS

Let $(\mathcal{G}, p)$ be a supergroup acting on the physical space A by a unitary representation R. A $(\mathcal{G}, p)$-invariant MPS tensor is a map $T : A \mapsto \mathrm{E}nd(U)$ such that $T(a)T(b) = T(ab)$ and

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1} \tag{3.114}$$

where the linear maps $Q(g) \in \mathrm{E}nd(U)$ form a projective representation of $(\mathcal{G}, p)$ on U. For $X \in \mathrm{E}nd(U)$ satisfying the supercommutation rule (3.75) or (3.76), the conjugate generalized MPS is

$$\left\langle\psi_T^X\right| = \mathrm{Tr}_U[XT(e_{i_1})\cdots T(e_{i_N})]\,\langle i_1 \cdots i_N| \tag{3.115}$$

in the NS sector and

$$\left\langle\psi_T^X\right| = \mathrm{Tr}_U[PXT(e_{i_1})\cdots T(e_{i_N})]\,\langle i_1 \cdots i_N| \tag{3.116}$$

in the R sector, where P denotes $Q(p)$. More generally, we can insert $Q(g)$ instead of P:

$$\left\langle\psi_T^X\right| = \mathrm{Tr}_U[Q(g)XT(e_{i_1})\cdots T(e_{i_N})]\,\langle i_1 \cdots i_N| \tag{3.117}$$

These are twisted sector states. When $\mathcal{G} = G_b \times \{1, p\}$, states with twist $Q(\bar{g}, 1)$ correspond to NS spin structure on a circle and a G_b gauge field of holonomy $\bar{g}$, while states with twist $Q(\bar{g}, p)$ correspond to the R spin structure on a circle and a G_b gauge field of holonomy $\bar{g}$. When $\mathcal{G}$ is non-split, one does not have spin structures and gauge fields, but a $\mathcal{G}$-Spin structure, as discussed in Section 3.7.

Note that $\mathrm{E}nd(U)$ carries a genuine (not projective) action of $(\mathcal{G}, p)$. By arguing as in (3.72), one can show that $\langle \psi_T^X |$ transforms under $(\mathcal{G}, p)$ in the same way as X.

Fermionic SRE phases and their group structure

In this section, we restrict our attention to fermionic SRE phases, i.e. topological fermionic phases that are invertible under the stacking operation. These phases form a group under stacking. According to (Fidkowski and Kitaev, 2010), if the symmetry group $\mathcal{G}$ splits as $G_b \times \mathbb{Z}_2$, each fermionic SRE phase corresponds to an element of the set

$$(\alpha, \beta, \gamma) \in H^2(G_b, U(1)) \times H^1(G_b, \mathbb{Z}_2) \times \mathbb{Z}_2. \tag{3.118}$$

If $G_b = \{1\}$, the two elements $(0,0,0)$ and $(0,0,1)$ correspond to the trivial and nontrivial Majorana chains, respectively. More generally, elements of the form $(\alpha, \beta, 0)$ correspond to fermionic SRE phases that remain invertible after bosonization, while the bosonic duals of the fermionic SREs $(\alpha, \beta, 1)$ are not SREs (they have a spontaneously broken $\mathbb{Z}_2$ but unbroken G_b).

If $\mathcal{G}$ does not split, we claim that fermionic SRE phases are classified by pairs (α, β), where $\beta \in H^1(G_b, \mathbb{Z}_2)$, and α is a 2-cochain on G_b with values in $U(1)$ satisfying $\delta\alpha = \frac{1}{2}\rho \cup \beta$, i.e. for $\bar{g}, \bar{h}, \bar{k}, \in G_b$,

$$\alpha(\bar{g}, \bar{h}) + \alpha(\bar{g}\bar{h}, \bar{k}) = \alpha(\bar{h}, \bar{k}) + \alpha(\bar{g}, \bar{h}\bar{k}) + \frac{1}{2}\rho(\bar{g}, \bar{h})\beta(\bar{k}) \tag{3.119}$$

Here ρ is the 2-cocycle on G_b which encodes the multiplication in $\mathcal{G}$. Certain pairs (α, β) correspond to equivalent SRE phases. Namely, adding to α an exact 2-cochain gives an equivalent SRE. Also, if we add to the 2-cocycle ρ a coboundary of a 1-cochain μ, α is shifted by $\frac{1}{2}\mu \cup \beta$.

This classification can be understood from the standpoint of bosonization. Recall that $\mathcal{G}$-invariant bosonic SREs are classified by group cohomology classes $[\omega] \in H^2(\mathcal{G}, U(1))$ and arise from algebras of the form $A = \mathrm{E}nd(U)$ where U is a projective representation of class $[\omega]$. Unlike the linear maps $R(g)$ of a genuine representation, the $Q(g)$ can be either even or odd with respect to $P := Q(p)$. Using (3.112) and the centrality of p, it can be shown that $Q(g)$ and $Q(gp)$ have the same parity $\omega(p, g) - \omega(g, p)$; thus, one can define $\beta(\bar{g}) := |Q(g)|$. The function β is clearly a homomorphism, and so defines a $\mathbb{Z}_2$-valued group 1-cocycle of G_b. Given a trivialization t, one can re-express ω in terms of β and a $U(1)$-valued group 2-cochain α of G_b satisfying $\delta\alpha = \frac{1}{2}\rho \cup \beta$ as follows:[18]

$$\omega(g,h) = \alpha(\bar{g},\bar{h}) + \frac{1}{2}t(g)\beta(\bar{h}). \tag{3.120}$$

Using (3.105), one can verify that (3.119) is equivalent to the cocycle condition for ω. We prove in Appendix C.3 that (3.120) defines an isomorphism between $H^2(\mathcal{G}, U(1))$ and the set of pairs (α, β), up to coboundaries.

When $\mathcal{G}$ does not split, it is impossible to break $\mathbb{Z}_2$ without breaking $G_b = \mathcal{G}/\mathbb{Z}_2$, so all fermionic SRE phases arise as fermionized bosonic SRE phases. Then the analysis above agrees with the result of (Fidkowski and Kitaev, 2010) that, in the non-split case, fermionic SREs are classified by elements of $H^2(\mathcal{G}, U(1))$ (modulo identifications).

But when $\mathcal{G}$ splits, it is possible to break $\mathcal{G}$ and still get an invertible fermionic phase. One can break $\mathcal{G}$ down to any subgroup H such that the quotient $\mathcal{G}/H$ is a $\mathbb{Z}_2$ generated by p. Any such subgroup takes the form $H_\beta = \{g \in \mathcal{G} : t(g) = \beta(\bar{g})\}$ for some homomorphism $\beta : G_b \to \mathbb{Z}_2$, and all homomorphisms give such a subgroup. This gives rise to a second class of fermionic SPTs - those whose bosonic duals are not invertible.

The algebras corresponding to these phases are of the form $A = \mathrm{E}nd(U_\beta) \otimes \mathrm{C}l(1)$ for some projective representation (U_β, Q_β) of H_β. Let $h \in H_\beta$, $M \in \mathrm{E}nd(U_\beta)$, $m \in \mathbb{Z}_2$. The subgroup and quotient act on A as

$$R(h) : M \otimes \Gamma^m \mapsto Q_\beta(h)^{-1} M Q_\beta(h) \otimes \Gamma^m, \tag{3.121}$$

$$R(p) : M \otimes \Gamma^m \mapsto (-1)^m M \otimes \Gamma^m \tag{3.122}$$

This action is a special case of the more general rule discussed in Section 4.3 of (Kapustin, Turzillo, and You, 2017). In terms of $\mathcal{G}$,

$$\begin{aligned} R(g) = R(\bar{g}, \beta(\bar{g})) \cdot R(p)^{t(g)+\beta(\bar{g})} :& M \otimes \Gamma^m \mapsto \\ & (-1)^{m(t(g)+\beta(\bar{g}))} Q_\beta(\bar{g}, \beta(\bar{g}))^{-1} M Q_\beta(\bar{g}, \beta(\bar{g})) \otimes \Gamma^m \end{aligned} \tag{3.123}$$

[18]When the extension splits, both α and β are cocycles, and their equivalence to ω can be seen from the Künneth theorem for homology and the fact that $H^2(\mathcal{G}, U(1))$ is the Pontryagin dual of $H_2(\mathcal{G}, \mathbb{Z})$.

as claimed in (3.113) (after setting $t(\bar{g},1) = 0$). Note that β, which encodes the action of the symmetry on fermions, can be offset by changing the trivialization t, i.e. the splitting isomorphism $\mathcal{G} \xrightarrow{\sim} G_b \times \mathbb{Z}_2$. As a projective representation, Q_β is characterized by a class $[\alpha] \in H^2(H, U(1)) \simeq H^2(G_b, U(1))$.

We have shown that $(\mathcal{G}, p)$-equivariant fermionic SRE phases can be characterized by pairs (α, β) and - if $\mathcal{G}$ is split - an additional $\mathbb{Z}_2$ label γ that represents a $C\ell(1)$ factor in the algebra. This parameterization is useful for discussing stacking of fermionic phases, which is different from the standard group structure on $H^2(\mathcal{G}, U(1))$ (the latter describes bosonic stacking). First, since $\mathrm{C}l(1) \mathbin{\widehat{\otimes}} \mathrm{C}l(1) \simeq \mathrm{C}l(2)$ is Morita-equivalent to $\mathbb{C}$, the γ parameters must simply add up under stacking. Second, if we consider two phases with parameters $(\alpha_1, \beta_1, 0)$ and $(\alpha_2, \beta_2, 0)$ corresponding to two $\mathcal{G}$-equivariant algebras (Q_1, U_1) and (Q_2, U_2), the supertensor product is a $\mathcal{G}$-equivariant algebra (Q, U), where $U = U_1 \mathbin{\widehat{\otimes}} U_2$ and $Q = Q_1 \mathbin{\widehat{\otimes}} Q_2$. We can easily compute:

$$
\begin{aligned}
Q(g)Q(h) &= (Q_1(g) \mathbin{\widehat{\otimes}} Q_2(g))(Q_1(h) \mathbin{\widehat{\otimes}} Q_2(h)) \\
&= (-1)^{\beta_2(\bar{g})\beta_1(\bar{h})} Q_1(g)Q_1(h) \mathbin{\widehat{\otimes}} Q_2(g)Q_2(h) \\
&= (-1)^{\beta_2(\bar{g})\beta_1(\bar{h})} \exp\big(2\pi i \alpha_1(\bar{g},\bar{h})\big)(-1)^{t(g)\beta_1(\bar{h})} \\
&\qquad\qquad \exp\big(2\pi i \alpha_2(\bar{g},\bar{h})\big)(-1)^{t(g)\beta_2(\bar{h})} Q_1(gh) \mathbin{\widehat{\otimes}} Q_2(gh) \\
&= \exp\left(2\pi i(\alpha_1 + \alpha_2 + \frac{1}{2}\beta_2 \cup \beta_1)\right)(\bar{g},\bar{h})(-1)^{t(g)(\beta_1+\beta_2)(\bar{h})} Q(gh).
\end{aligned}
\tag{3.124}
$$

Thus the group structure in this case is

$$
(\alpha_1, \beta_1, 0) + (\alpha_2, \beta_2, 0) = (\alpha_1 + \alpha_2 + \frac{1}{2}\beta_1 \cup \beta_2, \beta_1 + \beta_2, 0). \tag{3.125}
$$

Note that $\beta_1 \cup \beta_2$ differs from $\beta_2 \cup \beta_1$ by an exact term, and thus the difference between them is inessential. Based on these two special cases it is easy to guess that the group structure induced by stacking is

$$
(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2 + \frac{1}{2}\beta_1 \cup \beta_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2). \tag{3.126}
$$

This is verified in Appendix C.4

The set of triples (α, β, γ) with this group law is isomorphic to the spin-cobordism group $\Omega^2_{Spin}(BG_b)$ (Gaiotto and Kapustin, 2016). This agrees with the proposal of (Kapustin, Thorngren, et al., 2015) about the classification of fermionic SRE phases. In the non-split case, the group structure is given by the same formulas, except that γ is set to zero, and α is not closed, but satisfies the equation $\delta\alpha = \frac{1}{2}\rho \cup \beta$.

If $\mathcal{G}$ splits, the isomorphism $\mathcal{G} \simeq G_b \times \mathbb{Z}_2$ may be taken as part of the physical data. This means that one fixes the action of G_b on fermions as well as on bosons. Alternatively, if one regards this isomorphism as unphysical, one only fixes the action of G_b on bosons, while the action on fermions is fixed only up certain signs. So far we have been taking the former viewpoint. If we take the latter viewpoint, we also need to understand how the parameters (α, β, γ) change when we change the action of G_b on fermions. Given a particular action of $\bar{g} \in G_b$, any other action which acts in the same way on bosons differs from it by $p^{\mu(\bar{g})}$, where p is fermion parity and $\mu : G_b \to \mathbb{Z}_2$ is a homomorphism. If we define $\tilde{Q}(\bar{g}) = Q(\bar{g})P^{\mu(\bar{g})}$, we have

$$\tilde{Q}(\bar{g})\tilde{Q}(\bar{h}) = \exp\big(2\pi i\alpha(\bar{g}, \bar{h})\big)(-1)^{\mu(\bar{g})\beta(\bar{h})}\tilde{Q}(\bar{g}\bar{h}), \tag{3.127}$$

and

$$P\tilde{Q}(\bar{g})P^{-1} = (-1)^{\beta(\bar{g})}\tilde{Q}(\bar{g}). \tag{3.128}$$

This implies that for $\gamma = 0$ the parameter β is unchanged, while $\alpha \mapsto \alpha + \frac{1}{2}\mu \cup \beta$. For $\gamma = 1$ the situation is different, since fermion parity acts trivially on U, and thus α is not modified. But it acts nontrivially on the generator of $Cl(1)$, so that the new G_b transformation multiplies it by $(-1)^{\beta(\bar{g})+\mu(\bar{g})}$. Thus $\beta \mapsto \beta + \mu$. Thus if we do not fix the action of G_b on fermions, all fermionic SRE phases with $\gamma = 1$ and a fixed $[\alpha]$ are equivalent. This agrees with (Fidkowski and Kitaev, 2010).

Two examples with $G_b = \mathbb{Z}_2$

Let us consider the case $G_b = \mathbb{Z}_2 = \{1, b\}$. There are two extensions of G_b by fermionic parity $\mathbb{Z}_2^{\mathcal{F}} = \{1, p\}$: one is $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}[b]/(b^2) \times \mathbb{Z}[p]/(p^2)$; the other is $\mathbb{Z}_4 = \mathbb{Z}[b, p]/(b^2 - p)$.

First take $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider algebras of the form $A = \mathrm{E}nd(U)$, where U is a projective representation of $\mathcal{G}$. Each is characterized by a class $[\omega] \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. The two options for $[\omega]$ have cocycle representatives

$$\omega_0(g, h) = 0 \quad \text{ and } \quad \omega_1(g, h) = \frac{1}{2}g_2 h_1 \tag{3.129}$$

where $g = (g_1, g_2)$, $h = (h_1, h_2)$. On the bosonic side of the duality, we think of ω_0 as describing the trivial phase and ω_1 as describing a nontrivial SRE. Alternatively, one can replace each ω by a pair (α, β). There is only the trivial $[\alpha] \in H^2(\mathbb{Z}_2, U(1))$. There are two β's: $\beta_0(b) = 0$ and $\beta_1(b) = 1$. These correspond to ω_0 and ω_1, respectively, as

$$\omega_i(g, h) = \frac{1}{2}t(g)\beta_i(b(h)) \tag{3.130}$$

where $t(g) = g_2$ and $b(h) = h_1$. On the fermionic side, β_0 describes a trivial phase and β_1 a nontrivial SRE.

Now consider breaking the symmetry down to any of the three $\mathbb{Z}_2$ subgroups of $\mathcal{G}$; this means considering algebras $A = \mathrm{Ind}_H^{\mathcal{G}}(\mathrm{E}nd(U))$ for projective representations U of the unbroken $H = \mathbb{Z}_2$. Since $H^2(\mathbb{Z}_2, U(1))$ is trivial, the only possibility (up to Morita equivalence) is $A = C\ell(1)$, graded by $\mathcal{G}/H$. On the bosonic side, each choice of H is a different non-invertible phase. As fermionic phases, the G_b-graded $C\ell(1)$ is a symmetry-broken phase, while the $\mathbb{Z}_2^{\mathcal{F}}$-graded $C\ell(1)$ is a nontrivial Majorana-chain phase $(0, \beta_0, 1)$. Breaking down to the diagonal $\mathbb{Z}_2$ gives a p-graded $C\ell(1)$ on which the bosonic symmetry acts non-trivially, i.e. $(0, \beta_1, 1)$.

Now take $\mathcal{G} = \mathbb{Z}_4$. The extension class is represented by the 2-cocycle $\rho(b, b) = 1$. There is only the trivial class $[\omega] \in H^2(\mathbb{Z}_4, U(1)) = \{1\}$. Meanwhile, there are two β's: β_0 and β_1 as before. They satisfy $\rho \cup \beta_0 = 0$ and $\rho \cup \beta_1(b, b, b) = 1$. The trivial α is the unique solution to $\delta\alpha = \rho \cup \beta_0$, and one can show that there are no solutions to $\delta\alpha = \rho \cup \beta_1$. In summary, there is only one pair (α, β) - it's the trivial one.

Consider breaking the only subgroup $\mathbb{Z}_2^{\mathcal{F}}$. The corresponding algebra is the G_b-graded $C\ell(1)$, which, as before, describes a symmetry-broken phase in both the bosonic and fermionic pictures.

bosonic	(H, ω)	(α, β, γ)	fermionic
trivial	$(\mathcal{G}, \omega_0)$	$(0, \beta_0, 0)$	trivial
BSRE	$(\mathcal{G}, \omega_1)$	$(0, \beta_1, 0)$	FSRE
SB	$(G_b, 1)$	$(0, \beta_0, 1)$	FSRE
SB	$(\langle bp \rangle, 1)$	$(0, \beta_1, 1)$	FSRE
SB	$(\mathbb{Z}_2^{\mathcal{F}}, 1)$	n/a	SB
SB	(1,1)	n/a	SB

(a) Phases with $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$

bosonic	(H, ω)	(α, β)	fermionic
trivial	$(\mathcal{G}, \omega_0)$	$(0, \beta_0)$	trivial
SB	$(\mathbb{Z}_2^{\mathcal{F}}, 1)$	n/a	SB
SB	(1,1)	n/a	SB

(b) Phases with $\mathcal{G} = \mathbb{Z}_4$

Figure 3.18: Phase classification for the $G_b = \mathbb{Z}_2$ symmetry groups

State-sum for the equivariant fermionic theory

In Section 3.5, we observed that fermionic MPS arise from the state-sum for a spin-TQFT evaluated on an annulus diagram. A similar story can be told about equivariant fermionic MPS. Now we will define a state-sum for equivariant spin-TQFTs and recover the MPS (3.117) as states on an annulus.

We will focus on the case where the total symmetry group $\mathcal{G}$ splits as a product of G_b and $\mathbb{Z}_2$ and then indicate the modifications needed in the non-split case. A

G_b-equivariant spin-TQFT is defined in the same way as an ordinary spin TQFT, except that spin manifolds are replaced with spin manifolds equipped with principal G_b-bundles. Since G_b is finite, a G_b-principal bundle is completely characterized by its holonomies on non-contractible cycles. We will denote by $\mathcal{A}$ the collection of all holonomies. When working on manifolds with boundaries, it is convenient to fix a marked point and a trivialization of the bundle at the this point on each boundary, so that the holonomy around each of these circles is a well-defined element of G_b rather than a conjugacy class.

The algebraic input for the state-sum construction is $G_b \times \mathbb{Z}_2$-equivariant semisimple Frobenius algebra A. The geometric data are a closed oriented two-dimensional manifold Σ equipped with a G_b-bundle and a spin structure. To define the state-sum, we also choose a marked skeleton Γ, then a trivialized G_b-bundle can be represented as a decoration of each oriented edge with an element $g \in G_b$. Reversing an edge orientation replaces g with g^{-1}. We impose a flatness condition: the product of group labels around the boundary of each 2-cell is the identity. Equivalently, we can use the dual triangulation Γ^*: each dual edge is labeled by a group element, and the flatness condition says that the cyclically-ordered product of group elements on dual edges meeting at each dual vertex is the identity. One can think of the dual edges as domain walls and the dual edge labels as the G_b transformations due to moving across them.

The state-sum is defined as follows. Given a skeleton with a principal bundle, color the edges with pairs of elements e_i of some homogeneous basis of A. The weight of a coloring is the product of structure constants C_{ijk} over vertices (with indices cyclically ordered by orientation) and terms $R(g)^i{}_k \eta^{kj}$ over edges times the spin-dependent Koszul sign σ_s. The partition sum is the sum of the weights over colorings; the holonomies $\mathcal{A}$, which represent a background gauge field, are not summed over.

To incorporate brane boundaries, choose a $G_b \times \mathbb{Z}_2$-equivariant A-module U for each boundary component. Color the boundary edges by pairs of elements f^U_μ of a homogeneous basis of U - one for each vertex sharing the edge. The weight of a coloring is the usual weight times a factor of $T^{i\mu}{}_\nu$ for each boundary vertex and $Q(g)^\mu{}_\nu$ for each boundary edge.

As in the non-equivariant case, the partition sum is a spin-topological invariant. It also does not depend on the choice of trivialization of the principal bundle; in other words, it is gauge invariant. Invariance is ensured by the equivariance conditions

(3.107), (3.108), and (3.111). In fact, one can evaluate the partition function in a closed form when the boundary is empty. Let $A = \mathrm{E}nd(U) \otimes \mathrm{C}l(1)$ for some projective representation of G_b with a 2-cocycle α, and the action of G_b on $\mathrm{C}l(1)$ determined by a homomorpism $\beta : G_b \to \mathbb{Z}_2$. It is easy to see that the partition function factorizes into a product of the partition function corresponding to $\mathrm{E}nd(U)$ and the partition function corresponding to $\mathrm{C}l(1)$. The former factor is the partition function of a bosonic SRE phases, i.e. $\exp\left(2\pi i \int_\Sigma \alpha\right)$ (Kapustin, Turzillo, and You, 2017). The latter one is essentially the Arf invariant, modified by additional signs from the edges e for which $\beta(e) = 1$:

$$2^{-b_1(\Sigma)/2} \sum_{[a] \in H_1(\Sigma, \mathbb{Z}_2)} \sigma_s(a)(-1)^{\sum_{e \in a} \beta(\mathcal{A}(e))}. \tag{3.131}$$

Using the property (3.51), the definition of the Arf invariant, and the identity $\mathrm{A}rf(s + a) = \mathrm{A}rf(s)\sigma_s(a)$ (Atiyah, 1971), we can write this as

$$\mathrm{A}rf(s + \beta(\mathcal{A})) = \mathrm{A}rf(s)\sigma_s(\beta(\mathcal{A})). \tag{3.132}$$

Thus partition function of the fermionic SRE with the parameters $(\alpha, \beta, 1)$ is

$$\exp\left(2\pi i \int_\Sigma \alpha\right)\sigma_s(\beta(\mathcal{A}))\mathrm{A}rf(s). \tag{3.133}$$

Tensoring with another copy of $\mathrm{C}l(1)$ multiplies this by another factor $\mathrm{A}rf(s)$, so that the partition function of the fermionic SRE with the parameters $(\alpha, \beta, 0)$ is

$$\exp\left(2\pi i \int_\Sigma \alpha\right)\sigma_s(\beta(\mathcal{A})). \tag{3.134}$$

We can also recover the equivariant MPS wavefunctions from the state sum. First suppose $A = \mathrm{E}nd(U)$, i.e. the parameter $\gamma = 0$. An equivariant module over A is of the form $M = U \otimes W$, where (U, Q) and (W, S) have projective actions of $\mathcal{G}$ characterized by opposite cocycles. Consider the annulus where one boundary is a brane boundary labeled by M and the other is a cut boundary. We work with a skeleton on the annulus such that each boundary is divided into N intervals, and let $g_{i,i+1}$ denote the group label between vertices i and $i + 1$. A computation similar to that of Section 3.5 gives the state

$$\langle \psi_T | = \sum \mathrm{Tr}_{U \otimes W}[T(e_{i_1})Q(g_{12}) \cdots T(e_{i_N})Q(g_{N1})] \langle i_1 \cdots i_N | \tag{3.135}$$

which, after performing gauge transformations and LU transformations, can be put in the form

$$\langle \psi_T | = \sum \mathrm{Tr}_{U \otimes W}[Q(g)T(e_{i_1}) \cdots T(e_{i_N})] \langle i_1 \cdots i_N | \tag{3.136}$$

where $g = g_{12} \cdots g_{N1}$. Since $\mathcal{Q} = Q \otimes S$ and $T(e_i)$ has the form $T(e_i) \otimes \mathbb{1}_W$, the trace factorizes:

$$\langle \psi_T | = \mathrm{Tr}_W[S(g)] \sum \mathrm{Tr}_U[Q(g)T(e_{i_1}) \cdots T(e_{i_N})] \langle i_1 \cdots i_N | \, . \tag{3.137}$$

Up to normalization, this is the MPS (3.117).

The case $A = \mathrm{E}nd(U_\beta) \otimes C\ell(1)$ is similar. An indecomposable module over A is of the form $U \otimes W \otimes V$, where U and W carry projective H_β actions of opposite cocycles and $V = \mathbb{C}^{1|1}$ is the $C\ell(1)$-module considered in Section 3.6. The action of $\mathcal{G}$ is determined by $\mathcal{Q}(h) = Q_\beta(h) \otimes S(h) \otimes \mathbb{1}$ and $\mathcal{Q}(p)(M \otimes u_\pm) = \pm M \otimes u_\pm$. The argument proceeds as before, with the trace over W factoring out. We are left with an expression of the form (3.117) where the trace is over $U \otimes V$, the most general indecomposable MPS tensor over A.

Let us now discuss the non-split case. If $\mathcal{G}$ is a nontrivial extension of G_b by fermion parity, it is no longer true that a $\mathcal{G}$-equivariant algebra defines a G_b-equivariant spin-TQFT. Rather, it defines a $\mathcal{G}$-Spin TQFT (Kapustin, Thorngren, et al., 2015). A $\mathcal{G}$-Spin structure on a manifold X is a G_b gauge field $\mathcal{A}$ on X together with a trivialization of the $\mathbb{Z}_2$-valued 2-cocycle $w_2 - \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is the pull-back of ρ from BG_b to X and w_2 is a 2-cocycle representing the 2$^{\text{nd}}$ Stiefel-Whitney class of X. Now, if X is a Riemann surface Σ, $[w_2]$ is always zero, so $[\rho(\mathcal{A})]$ must be trivial too. Instead of choosing a trivialization of $w_2 - \rho(\mathcal{A})$, we can choose a trivialization s of w_2 and a trivialization τ of $\rho(\mathcal{A})$. That is, we choose $\mathbb{Z}_2$-valued 1-cochains s and τ such that $\delta s = w_2$ and $\delta \tau = \rho(\mathcal{A})$. These data are redundant: we can shift both s and τ by $\psi \in H^1(\Sigma, \mathbb{Z}_2)$.

We can now proceed as in the split case. Instead of a triple (α, β, γ) we have a pair (α, β) where $\beta \in H^1(G_b, \mathbb{Z}_2)$ and α is a 2-cochain on G_b with values in $U(1)$ satisfying $\delta\alpha = \frac{1}{2}\rho \cup \beta$. These data parameterize a 2-cocycle on $\mathcal{G}$. As shown above, the pairs (α, β) and $(\alpha + \frac{1}{2}\mu \cup \beta, \beta)$ correspond to the same 2-cocycle on $\mathcal{G}$, for any $\mu \in H^1(G_b, \mathbb{Z}_2)$. The partition function is evaluated exactly in the same way as in the split case, except that α is no longer closed, and an extra correction factor is needed to ensure the invariance of the partition function under a change of triangulation or a G_b gauge transformation. This correction factor is

$$(-1)^{\int_\Sigma \tau \cup \beta(\mathcal{A})} \tag{3.138}$$

where τ is a trivialization of $\rho(\mathcal{A})$ which is part of the definition of the $\mathcal{G}$-Spin structure on Σ. Thus the partition function is

$$\exp\left(2\pi i \int_\Sigma \alpha(\mathcal{A})\right)(-1)^{\int_\Sigma \tau\cup\beta(\mathcal{A})}\sigma_s(\beta(\mathcal{A})). \tag{3.139}$$

Using (3.51) one can easily see that the partition function is invariant under shifting both τ and s by any $\psi \in H^1(\Sigma, \mathbb{Z}_2)$. One can also see that the partition function is invariant under shifting α by $\frac{1}{2}\mu \cup \beta$ for any $\mu \in H^1(G_b, \mathbb{Z}_2)$ if we simultaneously shift $\tau \mapsto \tau + \mu(\mathcal{A})$.

Returning to the split case, we can examine the effect of treating the isomorphism $\mathcal{G} \simeq G_b \times \mathbb{Z}_2$ as unphysical. Every two such isomorphisms differ by a homomorphism $\mu : G_b \to \mathbb{Z}_2$. The effect this has on the data (α, β, γ) has been described in section 3.7:

$$\alpha \mapsto \alpha + (1-\gamma)\frac{1}{2}\mu \cup \beta, \qquad \beta \mapsto \beta + \gamma\mu, \qquad \gamma \mapsto \gamma. \tag{3.140}$$

Using the properties of σ_s and the Arf invariant, it is easy to check that the partition function is unaffected by these substitutions if we simultaneously shift the spin structure:

$$s \mapsto s + \mu(\mathcal{A}). \tag{3.141}$$

This can be interpreted as a special case of an equivalence relation between different spin structures which define the same $\mathcal{G}$-Spin structure.

3.8 Fermionic MPS

Unitary symmetries

We begin by briefly recalling the fMPS formalism, leaving many of the details to Ref. (Kapustin, Turzillo, and You, 2018).

The symmetries of a fermionic system form a finite supergroup $(\mathcal{G}, p)$; that is, a finite group $\mathcal{G}$ with a distinguished central involution $p \in \mathcal{G}$ called *fermion parity*. Every supergroup arises as a central extension of a group $G_b \simeq \mathcal{G}/\mathbb{Z}_2$ of bosonic symmetries by $\mathbb{Z}/2 = \{1, p\}$:

$$\mathbb{Z}_2 \underset{t}{\overset{i}{\rightleftharpoons}} \mathcal{G} \underset{s}{\overset{b}{\rightleftharpoons}} G_b. \tag{3.142}$$

Such extensions are classified by cohomology classes $[\rho] \in H^2(G_b, \mathbb{Z}/2)$. A trivialization $t : \mathcal{G} \to \mathbb{Z}/2$ defines a representative $\rho = \delta t$ of this class. In the context of fermionic phases,[19] when $\mathcal{G}$ splits, the choice of splitting t is part of the physical data, as it determines the action of G_b on fermions.

[19] On the contrary, the trivialization is unphysical from the perspective of TQFT. This subtlety is explored in (Kapustin, Turzillo, and You, 2018).

MPS is an ansatz for constructing one-dimensional gapped systems with such a symmetry. A translation-invariant $\mathcal{G}$-symmetric MPS system consists of the following data: a physical one-site Hilbert space A with a unitary action R of $\mathcal{G}$, a virtual space V with a projective action Q of $\mathcal{G}$, and an injective MPS tensor $T : A \to \mathrm{E}nd V$ that satisfies an equivariance condition: for all $a \in A, g \in \mathcal{G}$,

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1}. \tag{3.143}$$

From this data, one may construct a gapped, symmetric, and frustration-free lattice Hamiltonian (Schuch, Perez-García, and I. Cirac, 2010).

For our purposes, we are only interested in MPS at an RG fixed point, where a system exhibits the physics universal to its gapped phase. Physical sites are blocked together under real-space RG. At a fixed point, this procedure endows the space A with a product $m : A \otimes A \to A$, making A into a finite-dimensional associative algebra and the space V into a faithful module over A with structure tensor T (Kapustin, Turzillo, and You, 2017); that is,

$$T(a)T(b) = T(m(a \otimes b)) \tag{3.144}$$

for all $a, b \in A$. It follows that $T(m(R(g)a \otimes R(g)b)) = T(R(g)m(a \otimes b))$. Since T is injective, m satisfies

$$R(g)m(a \otimes b) = m(R(g)a \otimes R(g)b). \tag{3.145}$$

We say A is a $\mathcal{G}$-equivariant algebra. Because T can be put into a canonical form, A is semisimple (Kapustin, Turzillo, and You, 2017). It can also be viewed as the algebra of linear operators on the space of low energy boundary states (Fidkowski and Kitaev, 2010).

Our approach is to work with ground states of MPS systems, rather than their Hamiltonians. Every ground state of an MPS Hamiltonian has the form of a generalized MPS, which we now describe. Given a $(\mathcal{G}, p)$-equivariant MPS system described by an algebra A and tensor $T : A \to \mathrm{E}nd(V)$, one obtains a generalized MPS by choosing an observable $X \in \mathrm{E}nd(V)$ that *supercommutes* with $T(a)$;[20] that is

$$XT(a) = (-1)^{|a||X|}T(a)X \tag{3.146}$$

[20] Herein lies the difference between the bosonic and fermionic MPS formalisms. In the bosonic case, X commutes with $T(a)$ regardless of its charge under P (which is not a distinguished symmetry). For this reason, the twisted sector state spaces and how they are acted on by symmetries typically differ between a fermionic system and its Jordan-Wigner transform.

for all $a \in A$ such that $R(p)a = (-1)^{|a|}a$, where the parity of X is defined by $PXP^{-1} = (-1)^{|X|}X$, for $P := Q(p)$. A linear map X satisfying this condition is called an even or odd $\mathbb{Z}/2$-graded module endomorphism, depending on its parity.

For X satisfying the supercommutation rule (3.146), the generalized MPS has conjugate wavefunction

$$\left\langle \psi^X_{T,g} \right| = \sum_{i_1,\ldots,i_N} \mathrm{Tr}_U[Q(g)XT(e_{i_1})\cdots T(e_{i_N})] \left\langle i_1 \cdots i_N \right| \tag{3.147}$$

in the g-twisted sector. When $\mathcal{G}$ splits as $G_b \times \{1, p\}$, the g-twisted sector for $t(g) = 0$ consists of states on the circle with NS spin structure and a G_b gauge field of holonomy $b(g)$, while the g-twisted sector for $t(g) = 1$ consists of states on the circle with R spin structure and a G_b gauge field of holonomy $b(g)$. We emphasize that one must choose a splitting t in order to make sense of the the g-twisted sector as a $b(g)$-twisted NS or R sector. When $\mathcal{G}$ is non-split, one cannot speak of independent spin structures and gauge fields.[21]

The MPS description of a gapped system is not unique. For this reason, fixed point systems – and therefore gapped phases – are classified not by the algebras themselves but by their Morita classes (Kapustin, Turzillo, and You, 2017). After all, it is the module category (the T's and X's) of A that determines the system's ground states.

Equivariant algebras for fermionic SRE phases

For the remainder of the paper, we will restrict our attention to SRE phases. The algebras that correspond to fixed points in such phases are either of the form $\mathrm{E}nd(U)$ for U a projective representation Q of $\mathcal{G}$ or of the form $\mathrm{E}nd(U_b) \otimes \mathbb{C}\ell(1)$, for U_b a projective representation of G_b, with the following actions of $\mathcal{G}$ (Kapustin, Turzillo, and You, 2018). We refer to algebras of the first (second) type as "even" ("odd"). Odd algebras are only present when the extension for $\mathcal{G}$ splits.

The action of $\mathcal{G}$ on an even algebra $A = \mathrm{E}nd(U)$ is simply

$$R(g) \cdot M = Q(g)MQ(g)^{-1}. \tag{3.148}$$

Two even algebras are Morita equivalent if their projective representations have the same $[\omega] \in H^2(\mathcal{G}, U(1))$ (Ostrik, 2003). It is shown in the appendix that $[\omega]$ is equivalent to a pair $(\alpha, \beta) \in C^2(G_b, U(1)) \times C^1(G_b, \mathbb{Z}/2)$ that satisfies $\delta\alpha = {}^1\!/\!{}_2\beta \cup \rho$

[21] The appropriate geometric structure – the $\mathcal{G}$-Spin structure – is discussed in Ref. (Kapustin, Turzillo, and You, 2018).

and $\delta\beta = 0$, up to coboundaries.[22] In particular, when $\mathcal{G}$ splits, ρ is trivial and the equivalence classes defining the phase are $[\alpha], [\beta] \in H^2(G_b, U(1)) \times H^1(G_b, \mathbb{Z}/2)$. The invariant $[\alpha]$ is simply $[\omega]$ pulled back by s to G_b, while β measures the parity of $Q(g)$:

$$PQ(g)P^{-1} = e^{i\pi\beta(b(g))}Q(g). \tag{3.149}$$

Let Γ denote the generator of $\mathbb{C}\ell(1)$ with $\Gamma^2 = +1$. The action of $\mathcal{G}$ on an odd algebra is

$$R(g) \cdot M \otimes \Gamma^m = (-1)^{(\beta(g)+t(g))m} Q(g) M Q(g)^{-1} \otimes \Gamma^m \tag{3.150}$$

Morita classes of odd algebras are classified by the class $[\alpha] \in H^2(G_b, U(1))$ that describes the projective G_b-action Q and the class $[\beta] \in H^1(G_b, \mathbb{Z}/2)$ that describes the action of G_b on Γ.

It is apparent from (3.150) that changing the trivialization t shifts β by an arbitrary $\mu = (t' - t) \in Z^1(G_b; \mathbb{Z}/2)$. Therefore, unless t is fixed, β is not a well-defined invariant of odd algebras. On the other hand, for even algebras, redefining the action of g_b to $Q(g_b)P^{\mu(g_b)}$ leaves β unchanged but shifts α by $1/2\beta \cup \mu$.

In summary, when $\mathcal{G}$ splits as $G_b \times \mathbb{Z}_2^F$, the $\mathcal{G}$-symmetric fermionic SRE phases are classified by $[\alpha], [\beta], [\gamma] \in H^2(G_b, U(1)) \times H^1(G_b, \mathbb{Z}/2) \times \mathbb{Z}/2$ where $\gamma \in \mathbb{Z}/2$ tells us whether the algebra is even ($\gamma = 0$) or odd ($\gamma = 1$). When $\mathcal{G}$ does not split, only the invariants $[\alpha]$ (not a cocycle) and $[\beta]$ are present.

A system in an SRE phase has exactly one state per twisted sector. To see this from the algebra, count independent solutions X. The unique simple module over an even algebra $\mathrm{E}nd(U)$ is U, and by Schur's lemma its only endomorphism is $X = \mathbb{1}$. The unique faithful simple module over an odd algebra $\mathrm{E}nd(U_b) \otimes \mathbb{C}\ell(1)$ is $U_b \otimes \mathbb{C}^{1|1}$. It has two endomorphisms - an even one $X = \mathbb{1} \otimes \mathbb{1}$ and an odd one $X = \mathbb{1} \otimes \sigma_y$. The former appears in the wavefunction of the NS sector MPS state and the latter for the R sector MPS state.

Invariants of fermionic SRE phases

The invariants α, β, and γ can also be extracted from an SRE fermionic MPS system without reference to the algebra A. Below we give a physical interpretation of these invariants as observable quantities.

[22]In our notation, α takes values in $\mathbb{R}/\mathbb{Z} = [0, 1)$ and $1/2\beta \cup \rho$ in $\{0, 1/2\} \subset [0, 1)$. Crucially, α is defined up to a $\mathcal{G}$-coboundary with arguments in G_b; only when $\mathcal{G}$ splits is this a G_b-coboundary. This subtlety is relevant when $\mathcal{G} = \mathbb{Z}_4^{FT}$, see Section 3.10.

We begin by studying how the MPS in the g-twisted sector transforms under the action of a unitary symmetry $h \in \mathcal{G}_0$. Let ω be the cocycle that characterizes the projective action Q on the module. Then

$$\begin{aligned}R(h)\cdot \operatorname{Tr}\left[Q(g)XT^i\right]\langle i| \\ &= \operatorname{Tr}\left[Q(g)XQ(h)^{-1}T^iQ(h)\right]\langle i| \\ &= e^{2\pi i(\omega(h,g)+\omega(hg,h^{-1})-\omega(h,h^{-1}))} \\ &\qquad \operatorname{Tr}\left[Q(hgh^{-1})[Q(h)XQ(h)^{-1}]T^i\right]\langle i|\,.\end{aligned} \tag{3.151}$$

We have used the fact that

$$\omega(h,h^{-1}) = \omega(h^{-1},h), \tag{3.152}$$

which follows from the cocycle condition.[23] We see that under the action of a unitary symmetry h,

1. The g-twisted sector maps to the hgh^{-1}-twisted sector.
2. The operator X is conjugated by $Q(h)$.
3. States also pick up a phase of

$$e^{2\pi i(\omega(h,g)+\omega(hg,h^{-1})-\omega(h,h^{-1}))}. \tag{3.153}$$

We are now ready to interpret the three invariants.

Gamma.

Suppose $h = p$ and $g \in \{1,p\}$. Then the phase (3.153) vanishes, but there is still a sign coming from the conjugation of X by P. It is always $+1$ if the algebra is of the from $\mathrm{E}nd(U)$ (i.e. if $\gamma = 0$). If the algebra is of the form $\mathrm{E}nd(U_b) \otimes \mathbb{C}\ell(1)$ (i.e. if $\gamma = 1$), this sign is $+1$ in the NS sector and -1 in the R sector. Therefore we can conclude that the invariant $(-1)^\gamma$ is detected as the fermion parity (p-charge) of the R sector state.

Beta.

Continuing to take $h = p$, in the g-twisted sector the phase (3.153) becomes

$$\tfrac{1}{2}\beta(g) := \omega(p,g) - \omega(g,p). \tag{3.154}$$

[23] We always work in a gauge $Q(1) = \mathbb{1}$.

This term satisfies $\beta(pg) = \beta(g)$ and takes values in $\{0, 1/2\}$; in fact, it defines a $\mathbb{Z}/2$-valued cocycle of G_b. See the appendix for a proof. When $\gamma = 0$, the sign $(-1)^{\beta(g_b)}$ is the fermion parity of the g-twisted sector for g with $b(g) = g_b$. If $\mathcal{G}$ splits, one can equivalently say that $(-1)^{\beta(g)}$ is the parity of the $b(g)$-twisted NS and R sectors. If $\mathcal{G}$ splits, it is possible that $\gamma = 1$. In this case, one must choose a splitting to make sense of β. Then $(-1)^{\beta(g)}$ is still the parity of the $b(g)$-twisted NS sector, but the parity of the $b(g)$-twisted R sector receives a contribution of -1 from conjugation of X by P, in addition to the $\beta(g)$ term.

Note that $\beta(g)$ also describes the g-charge of the p-twisted (Ramond) sector for systems with $\gamma = 0$. This is no coincidence: the phase (3.153) agrees with Equation 4.11 of Ref. (Kapustin and Turzillo, 2017), where it was derived from bosonic (i.e. $X = \mathbb{1}$) TQFT. If g and h commute, one can sew together the ends of the cylinder to create a torus with holonomies g and h around its cycles. This torus evaluates to the phase

$$\omega(h, g) + \omega(hg, h^{-1}) - \omega(h, h^{-1}) = \omega(h, g) - \omega(g, h). \tag{3.155}$$

This surface can also be evaluated as a torus with holonomies h and g^{-1}, respectively, yielding

$$\begin{aligned}\omega(g^{-1}, h) + &\omega(g^{-1}h, g) - \omega(g^{-1}, g) \\ &= \omega(h, g) + \omega(g^{-1}, hg) - \omega(g^{-1}, g) \\ &= \omega(h, g) - \omega(g, h)\end{aligned} \tag{3.156}$$

These are equal, as is required by consistency of the TQFT. In terms of states, the h-charge of the g-twisted sector is the same as the g^{-1}-charge of the h-twisted sector, as long as g and h commute. There is no analogous statement for systems with $\gamma = 1$. Recall that $\beta(g)$ measures whether or not g acts as σ_z on the second factor of $\mathrm{E}nd(U) \otimes \mathbb{C}\ell(1)$. Then $Q(g)$ anticommutes with $X = \mathbb{1} \otimes \sigma_y$, and so the state picks up an extra charge of $\beta(g)$ which cancels with the sign (3.153) for a total g-charge of $+1$ in the R sector.

Alpha.

Consider the MPS state on a circle with two adjacent domain walls, parametrized by bosonic symmetries $g_b, h_b \in G_b$, as in Figure 3.19. Upon fusing them, the state picks up a phase:

$$\begin{aligned}\mathrm{Tr}\big[Q(s(g_b))Q(s(h_b))T^i\big]\, \langle i| \\ = e^{2\pi i \omega(s(g_b), s(h_b))}\, \mathrm{Tr}\big[Q(s(g_b)s(h_b))T^i\big]\, \langle i|\end{aligned} \tag{3.157}$$

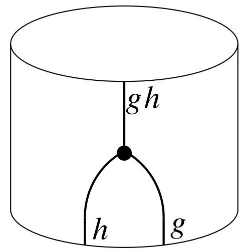


Figure 3.19: fusion of domain walls

These phases define a G_b-cochain

$$\alpha(g_b, h_b) = \omega(s(g_b), s(h_b)). \tag{3.158}$$

If $\mathcal{G}$ splits, then the fact that ω is a cocycle implies that α is as well. If the extension $\mathcal{G}$ is instead defined by a nontrivial ρ, then α has coboundary $1/2\beta \cup \rho$. See the appendix for details. Redefining each $X = \mathbb{1}$ by a sector-dependent phase shifts α by a $\mathcal{G}$-coboundary with arguments in G_b, as expected.

Note that when β and γ are trivial, there are no fermionic states and the system is insensitive to spin structure. In this sense, α captures purely bosonic features of the system.

In summary.

- $(-1)^\gamma$ is the fermion parity of the untwisted R sector.
- If $\gamma = 0$, $(-1)^{\beta(g_b)}$ is the fermion parity of the g-twisted sector for either of the two g's with $b(g) = g_b$. Alternatively, $(-1)^{\beta(g_b)}$ is the g-charge of the untwisted R sector. If $\gamma = 1$, $(-1)^{\beta(g_b)}$ is the fermion parity of the g_b-twisted NS sector, as determined by the choice of splitting.
- $e^{2\pi i\alpha(g_b,h_b)}$ is the phase due to fusing g_b and h_b domain walls.

Anti-unitary and orientation-reversing symmetries

More generally, a fermionic system may be invariant under anti-unitary symmetries as well as unitary ones. In this case, the full symmetry group $\mathcal{G}$ is a central extension by $\mathbb{Z}_2^F$ of a bosonic symmetry group G_b, which is itself an extension of $\mathbb{Z}_2^T$ by a finite group G_0, as in Figure 3.20. The symmetry class $(\mathcal{G}, p, x)$ is determined by a central $p \in \mathcal{G}$ and a map $x : G_b \to \mathbb{Z}/2$ that encodes whether a bosonic symmetry is unitary

$$\begin{tikzcd} & \mathbb{Z}_2^F \arrow[d] & \\ & \mathcal{G} \arrow[d, "b"] & \\ G_0 \arrow[r] & G_b \arrow[r, "x"] & \mathbb{Z}_2^T \end{tikzcd}$$

Figure 3.20: symmetry data

or anti-unitary. Note that the composition $x \circ b$, which we also call 'x,' satisfies $x(p) = 0$. Let $\mathcal{G}_0$ denote its kernel.

A fixed point MPS system of symmetry class $(\mathcal{G}, p, x)$ consists of a finite-dimensional semisimple associative algebra A and a faithful module $T : A \to \mathrm{E}nd(V)$, satisfying the equivariance conditions (3.145), (3.143) as before, only now the group action may be anti-unitary. In particular, the projective action on V is given by a unitary operator $Q(g)$ for each $g \in \mathcal{G}_0$ and an anti-unitary operator $Q(g)$ for each $g \notin \mathcal{G}_0$ that satisfy

$$Q(g)Q(h) = e^{2\pi i \omega(g,h)} Q(gh) \tag{3.159}$$

for phases $\omega(g, h)$. By comparing $[Q(g)Q(h)]Q(k)$ and $Q(g)[Q(h)Q(k)]$, we find the x-twisted cocycle condition:

$$\omega(g, h) + \omega(gh, k) = (-1)^{x(g)} \omega(h, k) + \omega(g, hk). \tag{3.160}$$

Redefining each $Q(g)$ by a g-dependent phase corresponds to shifting ω by an x-twisted coboundary. Therefore the action of $\mathcal{G}$ on the module V is characterized by a twisted cohomology class $[\omega] \in H^2(\mathcal{G}, U(1)_T)$. The group action R on A is defined via (3.143). It will be convenient to define linear maps $M(g)$ by

$$M(g) = \begin{cases} Q(g) & g \in \mathcal{G}_0 \\ Q(g)K & g \notin \mathcal{G}_0 \end{cases} \tag{3.161}$$

where K denotes complex conjugation.

Unitary symmetries that reverse the orientation of one-dimensional space can also be described in this language. Let x measure whether a symmetry reverses orientation. The natural generalization of (3.143) is

$$\begin{aligned} T(R(g)a) &= M(g)T(a)M(g)^{-1} \qquad \text{for } g \in \mathcal{G}_0 \\ T(R(g)a) &= M(g)T(a)^T M(g)^{-1} \qquad \text{for } g \notin \mathcal{G}_0. \end{aligned} \tag{3.162}$$

Let us introduce the following shorthand. For a matrix $O \in \mathrm{E}nd(V)$, define

$$\begin{aligned} O^{T0} &= O, \qquad & O^{T1} &= O^T, \\ \{O\}^0 &= O, \qquad & \{O\}^1 &= [O^{-1}]^T. \end{aligned} \tag{3.163}$$

Since R is a group homomorphism,

$$\begin{aligned} M(g)\{M(h)\}^{x(g)}&T(a)^{Tx(gh)}M(h)^{Tx(g)}M(g)^{-1} \\ &= T(R(g)R(h)a) \\ &= T(R(gh)a) \\ &= M(gh)T(a)^{Tx(gh)}M(gh)^{-1}. \end{aligned} \tag{3.164}$$

This implies there exists a number $\omega(g,h) \in \mathbb{R}/\mathbb{Z}$ such that

$$M(g)\{M(h)\}^{x(g)} = e^{2\pi i\omega(g,h)}M(gh). \tag{3.165}$$

By comparing the two equal expressions

$$M(g)\{M(h)\}^{x(g)}\{M(k)\}^{x(gh)} \qquad \text{and} \qquad M(g)\{M(h)M(k)^{x(h)}\}^{x(g)},$$

one recovers the x-twisted cocycle condition (3.160) for ω.

From the perspective of two-dimensional spacetime, it is not surprising that time-reversal[24] and space-reversal should be treated similarly. To make the connection more explicit, note that the physical Hilbert space carries the action of an anti-linear involution $*$, which we regard as CPT (see Ref. (Kapustin, Turzillo, and You, 2017)). Using equivariance of the multiplication and (anti-)unitarity of $R(g)$ with respect to the inner product on the Hilbert space, it may be shown that $*$ commutes with $R(g)$ for all $g \in \mathcal{G}$. With respect to the product on A, this map is an anti-automorphism. If $R(g)$ denotes the action of a time-reversing symmetry, $R(g)*$ is a unitary symmetry that reverses the orientation of space. Then

$$T(R(g) * a) = M(g)\overline{T(*a)}M(g)^{-1} = M(g)T(a)^T M(g)^{-1}. \tag{3.166}$$

Moreover, since $*$ commutes with $R(g)$, the equivariance condition (3.143) implies that $M(g)$ is unitary (up to a phase), so (3.159) and (3.165) are equivalent (up to a coboundary). For the remainder of the paper, we suppress $*$ and simply write R to denote a time-reversing or space-reversing symmetry.

[24] By a well-known result of Wigner, an anti-unitary symmetry reverses the direction of time.

Invariants of fermionic SRE phases with anti-unitary symmetries

As in the case of unitary symmetries, fermionic SRE systems at fixed points correspond to even algebras of the form $\mathrm{E}nd(U)$ and odd algebras of the form $\mathrm{E}nd(U_b) \otimes \mathbb{C}\ell(1)$. However, when the symmetries may act anti-unitarily, the cohomology class characterizing the Morita class (and hence the SRE phase) is twisted.

We now discuss the meaning of the invariants α, β, and γ in the anti-unitary context, following the previous analysis. The form of the MPS conjugate wavefunction is (3.147) as before. Consider the action of an anti-unitary symmetry $h \notin \mathcal{G}_0$ on an MPS in the g-twisted ($g \in \mathcal{G}_0$) sector:

$$\begin{aligned}
R(h)\cdot \operatorname{Tr}&\left[Q(g)XT^i\right] \langle i| \\
&= \operatorname{Tr}\left[M(g)XM(h^{-1})(T^i)^T M(h^{-1})^{-1}\right] \langle i| \\
&= \operatorname{Tr}\left[M(h^{-1})^T X^T M(g)^T M(h^{-1})^{-1T} T^i\right] \langle i| \\
&= e^{2\pi i\omega(h,h^{-1})} \operatorname{Tr}\left[M(h^{-1})^T M(hg)^{-1} M(hg)\right. \\
&\qquad\qquad \left. X^T [M(h)M(g)^{-1T}]^{-1} T^i\right] \langle i| \\
&= e^{2\pi i(\omega(h,h^{-1})-\omega(h,g))} \operatorname{Tr}\left[[M(hg)M(h^{-1})^{-1T}]^{-1}\right. \\
&\qquad\qquad \left. [M(hg)X^T M(hg)^{-1}]T^i\right] \langle i| \\
&= e^{2\pi i(\omega(h,g^{-1})+\omega(hg^{-1},h^{-1})+\omega(g,g^{-1})-\omega(h,h^{-1}))} \\
&\qquad \operatorname{Tr}\left[Q(hg^{-1}h^{-1})[M(hg)X^T M(hg)^{-1}]T^i\right] \langle i|
\end{aligned} \tag{3.167}$$

where in the last line we use the fact that

$$\begin{aligned}
\omega(&hg^{-1}h^{-1}, hgh^{-1}) \\
&= -\omega(h,g) - \omega(hg,h^{-1}) - \omega(hg^{-1},h^{-1}) \\
&\qquad - \omega(h,g^{-1}) - \omega(g^{-1},g) - 2\omega(h^{-1},h),
\end{aligned} \tag{3.168}$$

which can be verified by repeated application of the twisted cocycle condition. We see that under the action of an anti-unitary symmetry h,

1. The g-twisted sector maps to the $hg^{-1}h^{-1}$-twisted sector.

2. The operator X is transposed, then conjugated by $M(hg)$.[25]

3. States also pick up a phase of

$$e^{2\pi i(\omega(h,g^{-1})+\omega(hg^{-1},h^{-1})+\omega(g,g^{-1})-\omega(h,h^{-1}))}. \tag{3.169}$$

[25]If X is Hermitian, this is the same as X being conjugated by the anti-linear operator $Q(hg)$.

The phase matches Equation 4.12 of Ref. (Kapustin and Turzillo, 2017). In particular, when g acts on the R sector, it is

$$\sfrac{1}{2}\beta(g) := \omega(g,p) - \omega(p,g) + \omega(p,p), \qquad g \notin \mathcal{G}_0. \tag{3.170}$$

This phase satisfies $\beta(pg) = \beta(g)$, takes values in $\mathbb{Z}/2$, and, together with (3.154), is a G_b-cocycle. Refer to the appendix for a proof. When $\gamma = 0$, this is the g-charge of the R sector. However, when $\gamma = 1$, the charge receives an additional contribution from the transformation of X. Similar to in the unitary case detailed above, the total charge is the β-independent quantity $(-1)^{x(g)}$, so this interpretation of β fails.

The invariant β also has an interpretation in terms of edge states, like (3.149).[26] A time-reversing symmetry $g \notin G_0$ maps V to its dual space V^*, on which p acts as P^{-1}, so the parity of $Q(g)$ is read off of

$$P^{-1}Q(g)P^{-1} = e^{i\pi\beta(\bar{g})}Q(g), \qquad g \notin \mathcal{G}_0 \tag{3.171}$$

A similar interpretation holds if g reverses the orientation of space. Let $V^* \otimes V$ represent the tensor product of left and right edge state spaces. On this space, g acts as

$$\begin{aligned} &\psi_L \otimes \psi_R \mapsto \\ &\qquad Q(g)^{-1}(\psi_L \otimes \psi_R)^T Q(g) = Q(g)^{-1}\psi_R \otimes Q(g)\psi_L. \end{aligned} \tag{3.172}$$

Then β appears as the result of acting by $P \otimes P^{-1}$, g, then $P^{-1} \otimes P$:

$$\psi \otimes 1 \mapsto 1 \otimes PQ(g)P\psi = e^{i\pi\beta(\bar{g})}(1 \otimes \psi). \tag{3.173}$$

The meaning of α (3.158) is more difficult to describe in Hamiltonian language.[27] The lack of twisted sectors for anti-unitary symmetries means that $\alpha(g_b, h_b)$ has an interpretation as the phase due to fusing domain walls only when g_b and h_b are unitary. The rest of α appears in other places. It is convenient to first describe the invariant ω. For two unitary symmetries $g, h \in \mathcal{G}_0$, the phase $\omega(g,h)$ is due to fusing domain walls. It was shown in Ref. (Kapustin and Turzillo, 2017) that two extra families of phases – which we now describe – together with ω restricted to $\mathcal{G}_0$, determine the full ω on $\mathcal{G}$. The first family is the phases (3.169) due to acting on the g-twisted sector by an anti-unitary symmetry h. The second family consists

[26] If g is anti-linear, the expression (3.149) is not invariant under the change of gauge $\omega \mapsto \omega + \delta\Lambda$.
[27] In the Lagrangian picture, we expect α to be related to trivalent junctions of possibly orientation-reversing domain walls.

of the relative phases due to comparing, for each anti-unitary symmetry $g \notin \mathcal{G}_0$, the crosscap state (see Refs. (Shiozaki and Ryu, 2016; Kapustin and Turzillo, 2017)) $\mathrm{Tr}\big[Q(g)Q(g)T^i\big]\,\langle i|$ to the MPS state in the g^2-twisted sector. These phases have the simple form $\omega(g,g)$. Note that these data are not gauge invariant, and the equivalence classes of them under shifting ω by a twisted coboundary do not take a simple form. Now that we have described the full ω, the full α can be recovered by restricting to G_b. As we demonstrate in the appendix, the result is a G_b cochain whose x-twisted coboundary is $\beta \cup \rho$.

Finally, γ is the fermion parity of the untwisted Ramond sector, as in the unitary case.

3.9 The fermionic stacking law

Gapped fermionic phases form a commutative monoid under the operation of stacking. The result of stacking fixed point systems corresponding to algebras A_1 and A_2 is the system corresponding to the supertensor product $A_1 \widehat{\otimes} A_2$, defined by the multiplication law $(a_1 \widehat{\otimes} a_2)(b_1 \widehat{\otimes} b_2) = (-1)^{|a_2||b_1|} a_1 b_1 \widehat{\otimes} a_2 b_2$ (Bultinck et al., 2017a; Kapustin, Turzillo, and You, 2018). SRE phases are precisely those that are invertible under stacking, and so they form a group. The goal of this section is to derive this group structure on the set of SRE phases in terms of the invariants α, β, and γ. Our plan is to follow the argument presented in Appendix D of Ref. (Kapustin, Turzillo, and You, 2018), while taking into account that $Q(g)$ is anti-linear when $x(g) = 1$. We summarize the results at the end of the section.

The following discussion relies on a result proven in the appendix: that one can choose a gauge such that the twisted cocycle ω is related to α and β by, for all $g, h \in \mathcal{G}$, where $\bar{g}$ is short for $b(g)$,

$$\omega(g,h) = \alpha(\bar{g},\bar{h}) + {}^1\!/\!{}_2\beta(\bar{g})t(h). \tag{3.174}$$

There are three cases to consider: the stacking of 1) two even algebras, 2) an even and an odd algebra, 3) two odd algebras. When $\mathcal{G}$ does not split, there are no odd algebras so we need only consider the first case.

Even-Even Stacking.

Consider the even algebras $\mathrm{E}nd(U_1)$ and $\mathrm{E}nd(U_2)$. Their tensor product is $\mathrm{E}nd(U_1 \widehat{\otimes}$

U_2), where $U_1 \mathbin{\widehat{\otimes}} U_2$ carries a projective representation $Q = Q_1 \mathbin{\widehat{\otimes}} Q_2$. Then

$$\begin{aligned} Q(g)Q(h) &= \big(Q_1(g) \mathbin{\widehat{\otimes}} Q_2(g)\big)\, \big(Q_1(h) \mathbin{\widehat{\otimes}} Q_2(h)\big) \\ &= (-1)^{\beta_2(\bar g)\beta_1(\bar h)} Q_1(g)Q_1(h) \mathbin{\widehat{\otimes}} Q_2(g)Q_2(h) \\ &= (-1)^{(\beta_2\cup\beta_1)(\bar g,\bar h)} e^{2\pi i(\alpha_1(\bar g,\bar h)+{}^1\!/_2\beta_1(\bar g)t(h))} \\ &\qquad\qquad e^{2\pi i(\alpha_2(\bar g,\bar h)+{}^1\!/_2\beta_2(\bar g)t(h))} Q_1(gh) \mathbin{\widehat{\otimes}} Q_2(gh) \\ &= e^{2\pi i(\alpha_1+\alpha_2+{}^1\!/_2\beta_2\cup\beta_1)(\bar g,\bar h)+{}^1\!/_2(\beta_1+\beta_2)(\bar g)t(h)} Q(gh). \end{aligned} \tag{3.175}$$

Thus the invariants of the stacked phase are $\alpha = \alpha_1 + \alpha_2 + {}^1\!/_2(\beta_1 \cup \beta_2)$,[28] and $\beta = \beta_1 + \beta_2$. Since the stacked algebra is again even, $\gamma = 0$. The presence of anti-unitary symmetries does not affect even-even stacking.

Even-Odd Stacking.

Now consider the even algebra $A_1 = \mathrm{E}nd(U_1)$, where U_1 carries a projective representation Q_1 of $\mathcal{G}$, and the odd algebra $A_2 = \mathrm{E}nd(U_2) \otimes \mathbb{C}\ell(1)$, where U_2 carries a projective representation Q_2 of G_b. Their tensor product $\mathrm{E}nd(U_1) \mathbin{\widehat{\otimes}} (\mathrm{E}nd(U_2) \otimes \mathbb{C}\ell(1))$ is isomorphic as an algebra to the odd algebra $\mathrm{E}nd(U_1 \otimes U_2) \otimes \mathbb{C}\ell(1)$ by the map

$$JW : M_1 \mathbin{\widehat{\otimes}} (M_2 \otimes \Gamma^m) \mapsto M_1 P^m \otimes M_2 \otimes \Gamma^{m+|M_1|}, \tag{3.176}$$

which has inverse

$$JW^{-1} : M_1 \otimes M_2 \otimes \Gamma^m \mapsto M_1 P^{m+|M_1|} \mathbin{\widehat{\otimes}} (M_2 \otimes \Gamma^{m+|M_1|}), \tag{3.177}$$

where the parity of M_1 is defined by Q_1: $P_1 M_1 P_1 = (-1)^{|M_1|} M_1$. This isomorphism respects the $\mathbb{Z}/2$-grading defined by the standard action of fermion parity on even and odd algebras.

It remains to determine the G_b action on the odd algebra. For $g \in \mathcal{G}$ with $t(g) = 0$,

$$\begin{aligned} &JW \circ g \circ JW^{-1} \cdot (M_1 \otimes M_2 \otimes \Gamma^m) \\ &\quad = JW \circ g \cdot \Big(M_1 P^{m+|M_1|} \mathbin{\widehat{\otimes}} (M_2 \otimes \Gamma^{m+|M_1|})\Big) \\ &\quad = JW \cdot \Big(Q_1(g) M_1 P^{m+|M_1|} Q_1(g)^{-1} \mathbin{\widehat{\otimes}} \\ &\quad (Q_2(\bar g) M_2 Q_2(\bar g)^{-1} \otimes (-1)^{(m+|M_1|)\beta_2(\bar g)} \Gamma^{m+|M_1|})\Big) \\ &\quad = (-1)^{(m+|M_1|)(\beta_1(\bar g)+\beta_2(\bar g))} Q_1(g) M_1 Q_1(g)^{-1} \\ &\qquad\quad \otimes Q_2(\bar g) M_2 Q_2(\bar g)^{-1} \otimes \Gamma^m \end{aligned} \tag{3.178}$$

[28]We have used the fact that $\beta_2 \cup \beta_1$ is cohomologous to $\beta_1 \cup \beta_2$ in $\mathbb{Z}/2$.

In order to read off the invariants from this group action, we must rewrite it in the standard form by defining $\tilde{Q}_1(g) = Q_1(g)P^{\beta_1(g)+\beta_2(g)}$[29] and $Q(g) = \tilde{Q}_1(g) \otimes Q_2(\bar{g})$. Then, continuing from (3.178),

$$\begin{aligned} g \cdot (M_1 \otimes M_2 \otimes \Gamma^m) \\ &= (-1)^{m(\beta_1(\bar{g})+\beta_2(\bar{g}))}(\tilde{Q}_1(g) \otimes Q_2(\bar{g}))M_1 \\ &\quad \otimes M_2(\tilde{Q}_1(g)^{-1} \otimes Q_2(\bar{g})^{-1}) \otimes \Gamma^m, \end{aligned} \tag{3.179}$$

from which we read off the stacked invariant $\beta = \beta_1 + \beta_2$. And

$$\begin{aligned} Q(g)Q(h) &= \left(\tilde{Q}_1(g) \otimes Q_2(\bar{g})\right)\left(\tilde{Q}_1(h) \otimes Q_2(\bar{h})\right) \\ &= Q_1(g)P^{\beta_1(\bar{g})+\beta_2(\bar{g})}Q_1(h)P^{\beta_1(\bar{h})+\beta_2(\bar{h})} \otimes Q_2(\bar{g})Q_2(\bar{h}) \\ &= (-1)^{\beta_1(\bar{h})(\beta_1(\bar{g})+\beta_2(\bar{g}))} \\ &\qquad Q_1(g)Q_1(h)P^{\beta_1(\bar{gh})+\beta_2(\bar{gh})} \otimes Q_2(\bar{g})Q_2(\bar{h}) \\ &= e^{2\pi i(\alpha_1(g,h)+\alpha_2(g,h)+1/2(\beta_2\cup\beta_1)(g,h)+1/2(\beta_1\cup\beta_1)(g,h))}Q(gh), \end{aligned} \tag{3.180}$$

from which we see $\alpha = \alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2 + 1/2\beta_1 \cup \beta_1$. There is no asymmetry: the $1/2\beta_1 \cup \beta_1$ term always comes from the β of the even algebra.[30] Finally, $\gamma = 1$ since the stacked algebra is odd.

Odd-Odd Stacking.

Consider the odd algebras $A_1 = \mathrm{E}nd(U_1) \otimes \mathbb{C}\ell(1)$, where U_1 carries a projective representation Q_1 of G_b, and $A_2 = \mathrm{E}nd(U_2) \otimes \mathbb{C}\ell(1)$, where U_2 carries a projective representation Q_2 of G_b. Their tensor product is given by $A_1 \widehat{\otimes} A_2 \simeq \mathrm{E}nd(U_1 \otimes U_2 \otimes \mathbb{C}^{1|1})$, since $\mathbb{C}\ell(1) \widehat{\otimes} \mathbb{C}\ell(1) \simeq \mathbb{C}\ell(2) \simeq \mathrm{E}nd(\mathbb{C}^{1|1})$, via an isomorphism

$$(M_1 \otimes \Gamma_1^m) \widehat{\otimes} (M_2 \otimes \Gamma_2^n) \mapsto M_1 \otimes M_2 \otimes \sigma_1^m \sigma_2^n, \tag{3.181}$$

where σ_1 and σ_2 are any two distinct Pauli matrices. With respect to the action of fermion parity on the $\mathrm{E}nd(\mathbb{C}^{1|1})$ factor as conjugation by $\sigma_3 = -i\sigma_1\sigma_2$, this map is an isomorphism of $\mathbb{Z}/2$-graded algebras.

[29] Adding a phase factor to $\tilde{Q}$ would have shifted the resulting 2-cocycle α by an irrelevant coboundary. For example, if we had chosen a factor $i^{\beta_1(g)}$ as in Ref. (Kapustin, Turzillo, and You, 2018), we would have gotten $\beta_1 \cup x$ instead of $\beta_1 \cup \beta_1$ in the final answer.

[30] Note that while $\beta_1 \cup \beta_1$ is an ordinary coboundary and hence could be ignored for phases without time-reversal, it is not a twisted coboundary and so cannot be ignored when time-reversing symmetries are present. By adding a twisted coboundary, we can put it in the form $\beta_1 \cup x$, which makes the dependence on time-reversal symmetry manifest.

One choice[31] of G_b-action Q on $U_1 \otimes U_2 \otimes \mathbb{C}^2$, with respect to which (3.181) is equivariant, is

$$g : u_1 \otimes u_2 \otimes v \mapsto Q_1(\bar{g})u_1 \otimes Q_2(\bar{g})u_2 \otimes \sigma_1^{\beta_2(\bar{g})}\sigma_2^{\beta_1(\bar{g})}K^{x(\bar{g})}v, \tag{3.182}$$

for $g \in \mathcal{G}$ with $t(g) = 0$, where K denotes complex conjugation in a basis in which σ_1 and σ_2 are real. Then

$$\begin{aligned}
&Q(\bar{g})Q(\bar{h}) \\
&\quad= \Big(Q_1(\bar{g}) \otimes Q_2(\bar{g}) \otimes \sigma_1^{\beta_2(\bar{g})}\sigma_2^{\beta_1(\bar{g})}K^{x(\bar{g})}\Big) \\
&\qquad\qquad \Big(Q_1(\bar{h}) \otimes Q_2(\bar{h}) \otimes \sigma_1^{\beta_2(\bar{h})}\sigma_2^{\beta_1(\bar{h})}K^{x(\bar{h})}\Big) \\
&\quad= e^{2\pi i\alpha_1(\bar{g},\bar{h})}Q_1(\bar{gh}) \otimes e^{2\pi i\alpha_2(\bar{g},\bar{h})}Q_2(\bar{gh})\otimes \\
&\qquad\qquad (-1)^{\beta_1(\bar{g})\beta_2(\bar{h})}\sigma_1^{\beta_1(\bar{gh})}\sigma_2^{\beta_2(\bar{gh})}K^{x(\bar{gh})} \\
&\quad= e^{2\pi i(\alpha_1+\alpha_2+{}^1\!/\!{}_2\beta_1\cup\beta_2)(\bar{g},\bar{h})}Q(\bar{gh}),
\end{aligned} \tag{3.183}$$

from which we see that $\alpha = \alpha_1 + \alpha_2 + {}^1\!/\!{}_2\beta_1 \cup \beta_2$. Since $U_1 \otimes U_2$ is purely even, the parity of Q comes from

$$\begin{aligned}
&P\sigma_1^{\beta_2(\bar{g})}\sigma_2^{\beta_1(\bar{g})}K^{x(\bar{g})}P \\
&\quad= (-i\sigma_1\sigma_2)\sigma_1^{\beta_2(\bar{g})}\sigma_2^{\beta_1(\bar{g})}K^{x(\bar{g})}(-i\sigma_1\sigma_2) \\
&\quad= (-1)^{\beta_2(\bar{g})+\beta_1(\bar{g})}\sigma_1^{\beta_2(\bar{g})}\sigma_2^{\beta_1(\bar{g})}K^{x(\bar{g})}(-1)^{x(\bar{g})}.
\end{aligned} \tag{3.184}$$

We read off $\beta = \beta_1 + \beta_2 + x$. Finally, the stacked algebra is even, so $\gamma = 0$.

In Summary.

The stacking law for the invariants (α, β, γ) is given by

$$\begin{aligned}
(\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 0) &= (\alpha_1 + \alpha_2 + {}^1\!/\!{}_2\beta_1 \cup \beta_2, \beta_1 + \beta_2, 0) \\
(\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 1) &= (\alpha_1 + \alpha_2 + {}^1\!/\!{}_2\beta_1 \cup \beta_2 + {}^1\!/\!{}_2\beta_1 \cup \beta_1, \beta_1 + \beta_2, 1) \\
(\alpha_1, \beta_1, 1) \cdot (\alpha_2, \beta_2, 1) &= (\alpha_1 + \alpha_2 + {}^1\!/\!{}_2\beta_1 \cup \beta_2, \beta_1 + \beta_2 + x, 0).
\end{aligned} \tag{3.185}$$

This group law inherits the properties of commutativity and associativity from the tensor product of algebras. When $\mathcal{G}$ does not split, γ is not present, and the stacking law is simply

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2 + {}^1\!/\!{}_2\beta_1 \cup \beta_2, \beta_1 + \beta_2). \tag{3.186}$$

[31] Again, had we chosen a different G_b-action on $\mathbb{C}^{1|1}$ compatible with the action $g_b : \Gamma_i \mapsto (-1)^{\beta_i(g_b)}\Gamma_i$ on the $\mathbb{C}\ell(1)$ factors, the 2-cocycle α would be shifted by a twisted coboundary.

We emphasize that while data $[\alpha, \beta]$ are equivalent to $[\omega] \in H^2(\mathcal{G}, U(1)_T)$, the group structure on $H^2(\mathcal{G}, U(1)_T)$ differs from (3.185). On the other hand, the stacking of bosonic SRE phases, which are also characterized by classes $[\omega]$, is described by the usual group structure on $H^2(\mathcal{G}, U(1)_T)$.

3.10 Examples

Class BDI fermions: $\mathcal{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2^T$

Let us consider SRE phases with symmetry $\mathcal{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2^T$. The two classes $\alpha \in H^2(\mathbb{Z}_2^T, U(1)_T) = \mathbb{Z}/2$, two classes $\beta \in H^1(\mathbb{Z}_2^T, \mathbb{Z}/2) = \mathbb{Z}/2$, and two classes $\gamma \in \mathbb{Z}/2$ make for a total of eight phases. A straightforward application of the general stacking law (3.185) reveals that these phases stack like the cyclic group $\mathbb{Z}/8$. In this section, we will reproduce this group law by exploiting the relationship between $\mathcal{G}$-equivariant algebras, real super-division algebras, and Clifford algebras, which have Bott periodicity $\mathbb{Z}/8$.

We begin by describing simple $\mathcal{G}$-equivariant algebras. The matrix algebra $M_2\mathbb{C}$ represents the sole Morita class of simple complex algebras. This algebra has a unitary structure $*$ given by conjugate transposition. Its action fixes a basis $\{\mathbb{1}, X, Y, Z = -iXY\}$. On $C\ell_2\mathbb{C} \simeq M_2\mathbb{C}$, $*$ acts by Clifford transposition and complex conjugation of coefficients with respect to a pair of generators that square to $+1$.

There are two distinct real structures on $M_2\mathbb{C}$ given by complex conjugation T on the second component of $M_2\mathbb{C} \simeq M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ and $M_2\mathbb{C} \simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. The unitary structure $*$ of $M_2\mathbb{C}$ acts by transposition on $M_2\mathbb{R}$, complex conjugation on $\mathbb{C}$, and inversion of the generators $\hat{\imath}$ and $\hat{\jmath}$ of $\mathbb{H}$; that is, its fixed bases are

$$\begin{gathered} \{\mathbb{1} \otimes 1, X \otimes 1, iY \otimes i, Z \otimes 1\} \in M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}, \\ \{\mathbb{1} \otimes 1, \hat{\imath} \otimes i, \hat{\jmath} \otimes i, \hat{k} \otimes i\} \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}. \end{gathered} \tag{3.187}$$

These bases have the same T-eigenvalues as they do $*T$-eigenvalues, where $*T$ acts as transposition on $M_2\mathbb{R}$ and inversion of generators on $\mathbb{H}$. Under the algebra isomorphisms $M_2\mathbb{R} \simeq C\ell_{1,1}\mathbb{R} \simeq C\ell_{2,0}\mathbb{R}$ and $\mathbb{H} \simeq C\ell_{0,2}\mathbb{R}$, $*T$ acts by inverting the generators and products of generators that square to -1.

Let us derive the invariants α of these real structures. Pulled back from $M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ to $M_2\mathbb{C}$, T acts like complex conjugation and $*T$ like transposition; that is $M(t) = \mathbb{1}$. Then

$$M(t)M(t)^{-1T} = \mathbb{1}, \tag{3.188}$$

which means $\alpha(t,t) = \omega(t,t) = 0$. Pulled back from $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, T acts like complex conjugation and conjugation by Y, while $*T$ acts like transposition and conjugation by Y; that is $M(t) = Y$. Then $\alpha(t,t) = {}^1\!/\!_2$ since

$$M(t)M(t)^{-1T} = e^{i\pi}\mathbb{1}. \tag{3.189}$$

By the Skolem-Noether theorem, a superalgebra structure on $M_2\mathbb{C}$ is given by conjugation by an element that squares to one. If this element is $\mathbb{1}$, the $\mathbb{Z}/2$-grading is purely even; otherwise, it has two even dimensions and two odd. All structures of the latter type are isomorphic in the absence of a real structure.

In the presence of the real structure $M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, there are three distinct gradings. First, there is the purely even grading, given by $P = \mathbb{1}$. This structure has ${}^1\!/\!_2\beta(t) = \omega(t,p) = 0$. Second, there is conjugation by Z (or X), which gives $M_2\mathbb{R}$ the superalgebra structure of $C\ell_{1,1}\mathbb{R}$. Again, $\beta(t) = 0$ since $P = Z$ means

$$PM(t)P^T = Z\mathbb{1}Z^T = \mathbb{1}. \tag{3.190}$$

The matching of the invariants alludes to the fact that the real superalgebra structures $M_2\mathbb{R}$ and $C\ell_{1,1}\mathbb{R}$ are graded Morita equivalent. Third, there is conjugation by Y; that is, $P = Y$. Then $\beta(t) = 1$ since

$$PM(t)P^T = Y\mathbb{1}Y^T = e^{i\pi}\mathbb{1}. \tag{3.191}$$

The corresponding real Clifford algebra is $C\ell_{2,0}\mathbb{R}$ and represents a distinct Morita class.

On the real structure $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, there are two distinct gradings. First, there is the purely even grading $P = \mathbb{1}$, which has $\beta(t) = 0$. The second grading is given by conjugation by Z (or X or Y) on $M_2\mathbb{C}$ and gives $\mathbb{H}$ the superalgebra structure of $C\ell_{0,2}\mathbb{R}$. Then $\beta(t) = 1$ since

$$PM(t)P^T = ZYZ^T = e^{i\pi}Y. \tag{3.192}$$

Now consider algebras of the form $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$. The second component $C\ell_1\mathbb{C}$ has a unitary structure $*$ given by complex conjugation of coefficients of the generator Γ that squares to $+1$. There are two distinct real structures on $C\ell_1\mathbb{C}$ given by complex conjugation T on the second component of $C\ell_{1,0}\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}$ and $C\ell_{0,1}\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}$. The unitary structure $*$ of $C\ell_1\mathbb{C}$ acts by complex conjugation on $\mathbb{C}$ and inversion of generators that square to -1; that is, the fixed bases are $\{1 \otimes 1, \gamma \otimes 1\}$ for $C\ell_{1,0}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ and

$\{1 \otimes 1, \gamma \otimes i\}$ for $C\ell_{0,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$. The map $*T$ is trivial on $C\ell_{1,0}\mathbb{R}$ and inversion of the generator on $C\ell_{0,1}\mathbb{R}$. Therefore, pulled back from $C\ell_{1,0}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ to $C\ell_1\mathbb{C}$, T is complex conjugation and $*T$ is trivial. From $C\ell_{0,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, $*T$ is inversion of Γ.

As discussed in Section 3.8 and in Ref. (Kapustin, Turzillo, and You, 2018), we need only to consider a single $\mathbb{Z}/2$-grading on $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$ - the one where $M_2\mathbb{C}$ is purely even and the generator of $C\ell_1\mathbb{C}$ is odd. The algebra $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$ has four real structures: a choice of $M_2\mathbb{R}$ or $\mathbb{H}$ for the first component and $C\ell_{1,0}\mathbb{R}$ or $C\ell_{0,1}\mathbb{R}$ for the second. As was true for even algebras, the first choice determines whether $M(t)$ is $\mathbb{1}$ or Y; that is, whether $\alpha(t,t)$ is 0 or $1/2$. The second choice determines whether $*T$ inverts the odd generator; this is $\beta(t)$.

Due to the Morita equivalence $M_2\mathbb{R} \sim \mathbb{R}$, several of the eight Morita classes are represented by algebras of lower dimension; for example, $C\ell_{1,0}\mathbb{R}$ instead of $M_2\mathbb{R} \otimes_{\mathbb{R}} C\ell_{1,0}\mathbb{R}$. Up to this substitution, the eight real-structured superalgebras we found are complexifications of the eight *central real super-division algebras* – real superalgebras with center $\mathbb{R}$ that are invertible under supertensor product up to graded Morita equivalence (Wall, 1964; Trimble, 2005). They constitute a set of representatives of the eight graded Morita classes of real superalgebras. These algebras appear in second column of Figure 3.21, next to their invariants in the third column.

Another set of Morita class representatives is the Clifford algebras $C\ell_{n,0}\mathbb{R}$. In terms of these algebras, stacking is simple, as

$$C\ell_{n,0}\mathbb{R} \mathbin{\widehat{\otimes}} C\ell_{m,0}\mathbb{R} \simeq C\ell_{n+m,0}\mathbb{R} \tag{3.193}$$

and

$$C\ell_{n,0}\mathbb{R} \sim C\ell_{m,0}\mathbb{R} \qquad \text{for } n \equiv m \bmod 8. \tag{3.194}$$

Each central super-division algebra can be matched with the Clifford algebra $\mathbb{C}\ell_{n,0}\mathbb{R}$, $n < 8$ in its Morita class (Trimble, 2005), as in the first column of Figure 3.21. This determines a $\mathbb{Z}/8$ stacking law on central super-division algebras and their invariants that agrees with the more general law (3.185).

Physically speaking, the $\mathbb{Z}/8$ classification is generated by the time-reversal-invariant Majorana chain (Fidkowski and Kitaev, 2010; Fidkowski and Kitaev, 2009). While the symmetry protects pairs of dangling Majorana zero modes from being gapped out, turning on interactions can gap out these modes in groups of eight. Fidkowski and Kitaev formulate their stacking law in terms of three invariants that are equivalent to α, β, and γ. Their results match ours.

For contrast, we list the invariants of the corresponding bosonic phases in the rightmost column of Figure 3.21. There, H denotes the subgroup of unbroken symmetries and ω denotes 2-cocycle characterizing the SPT order. These invariants can be obtained from the fermionic invariants (Kapustin, Turzillo, and You, 2018). We observe that invertibility is not preserved by bosonization; in particular, only the fermionic SREs with $\gamma = 0$ become bosonic SREs. The four bosonic SRE phases have a $\mathbb{Z}/2 \times \mathbb{Z}/2$ stacking law. We also include the two *non-central* super-division algebras $\mathbb{C}$ and $\mathbb{C}\ell_1$ at the bottom of the table. These correspond to symmetry-breaking (SB) phases.

$C\ell_{n,0}$	A_{div}	α, β, γ	fermionic	bosonic	(H, ω)
0	$\mathbb{R}$	$0, 0, 0$	trivial	trivial	$(\mathcal{G}, 0)$
1	$C\ell_{1,0}$	$0, 0, 1$	SRE	SB	$(\mathbb{Z}_2^T, 0)$
2	$C\ell_{2,0}$	$0, 1, 0$	SRE	SPT	$(\mathcal{G}, \omega_1)$
3	$\mathbb{H} \otimes C\ell_{0,1}$	$1, 1, 1$	SRE	mixed	$(\mathbb{Z}_2^{\text{diag}}, \alpha)$
4	$\mathbb{H}$	$1, 0, 0$	SRE	SPT	$(\mathcal{G}, \omega_2)$
5	$\mathbb{H} \otimes C\ell_{1,0}$	$1, 0, 1$	SRE	mixed	$(\mathbb{Z}_2^T, \alpha)$
6	$C\ell_{0,2}$	$1, 1, 0$	SRE	SPT	$(\mathcal{G}, \omega_1 + \omega_2)$
7	$C\ell_{0,1}$	$0, 1, 1$	SRE	SB	$(\mathbb{Z}_2^{\text{diag}}, 0)$
-	$\mathbb{C}$	-	SB	SB	$(\mathbb{Z}_2^F, 0)$
-	$\mathbb{C}\ell_1$	-	SB	SB	$(1, 0)$

Figure 3.21: the 10-fold way of $\mathbb{Z}_2^F \times \mathbb{Z}_2^T$-symmetric fermionic phases

Class DIII fermions: $\mathcal{G} = \mathbb{Z}_4^{FT}$

In the following, $\mathcal{G} = \mathbb{Z}_4^{FT}$ denotes the non-trivial extension of $G_b = \mathbb{Z}_2^T$ by fermion parity. Let us consider fermionic SRE phases with this symmetry. There are two distinct classes $\beta \in H^1(\mathbb{Z}_2^T, \mathbb{Z}/2)$, determined by $\beta(t) = 0$ and $\beta(t) = 1$. The trivial β has a single α, the trivial one, that satisfies $\delta_T \alpha = {}^1\!/\!_2 \beta \cup \rho$, up to the proper equivalence.[32] The nontrivial β also has a single compatible α, up to equivalence: $\alpha(t, t) = {}^1\!/\!_4$.

The trivial phase is represented by the algebra $\mathbb{C}$ with trivial actions of p and t, as

[32]The cocycle $\alpha(t, t) = {}^1\!/\!_2$ is nontrivial in $H^2(G_b, U(1)_T)$ but is trivialized by adding a 2-coboundary on $\mathcal{G}$ satisfying the proper conditions. See the Appendix for details.

always. For the nontrivial phase, consider $A = \mathrm{E}nd(U)$, where P and T act on U as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad M(t) = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}. \tag{3.195}$$

Then the invariants can be recovered:

$$M(t)M(t)^{-1T} = e^{2\pi i/4} P \qquad \Rightarrow \qquad \alpha(t,t) = {}^1\!/\!_4 \tag{3.196}$$

$$PM(t)P^T = e^{i\pi} M(t) \qquad \Rightarrow \qquad \beta(t) = 1 \tag{3.197}$$

According to the rule (3.185), stacking two copies of this phase results in the trivial phase:

$$({}^1\!/\!_4, 1) \cdot ({}^1\!/\!_4, 1) = ({}^1\!/\!_4 + {}^1\!/\!_4 + {}^1\!/\!_2 \cdot 1 \cdot 1, 1 + 1) = (0, 0). \tag{3.198}$$

We find that fermionic SRE phases with symmetry $\mathbb{Z}_4^{FT}$ have a $\mathbb{Z}/2$ classification, in agreement with the condensed matter literature (Ryu et al., 2010; Kitaev, 2009a). The nontrivial phase appears as a Majorana chain with two dangling modes protected by the symmetry.

Unitary $\mathbb{Z}/2$ symmetry

As a last set of examples, let us consider systems with a unitary bosonic symmetry group $G_0 = \mathbb{Z}/2$, in addition to time-reversal and fermion parity. There are many ways to organize these symmetries into a full symmetry class $(\mathcal{G}, p, x)$. Here, we consider the five abelian possibilities, which are listed with their fermionic and bosonic phase classifications in Figure 3.22. The first three have $G_b = \mathbb{Z}_2 \times \mathbb{Z}_2^T$, the last two $G_b = \mathbb{Z}_4^T$. In the two cases where the central extension of G_b by $\mathbb{Z}_2^F$ splits, we use a superscript γ to denote the subgroup of the fermionic classification that contains the odd phases.

symmetry class	fermionic	bosonic
$\mathbb{Z}_2 \times \mathbb{Z}_2^T \times \mathbb{Z}_2^F$	$\mathbb{Z}_4 \times \mathbb{Z}_8^\gamma$	$(\mathbb{Z}_2)^4$
$\mathbb{Z}_2 \times \mathbb{Z}_4^{FT}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^T \times \mathbb{Z}_4^F$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^F \times \mathbb{Z}_4^T$	$\mathbb{Z}_2 \times \mathbb{Z}_4^\gamma$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_8^{FT}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Figure 3.22: fermionic phases with unitary and anti-unitary symmetries

Chapter 4

DIAGRAMMATIC STATE SUMS FOR 2D PIN-MINUS TQFTS

Turzillo, A. (2018). "Diagrammatic state sums for two dimensional pin-minus topological quantum field theories". In: arXiv: `1811.12654`.

Forward

In this chapter, the two dimensional state sum models of Barrett and Tavares are extended to unoriented spacetimes. The input to the construction is an algebraic structure dubbed half twist algebras, a class of examples of which is real separable superalgebras with a continuous parameter. The construction generates pin-minus TQFTs, including the root invertible theory with partition function the Arf-Brown-Kervaire invariant. Decomposability, the stacking law, and Morita invariance of the construction are discussed.

Background and Overview

State sum constructions of quantum field theories extend Feynman's formulation of the time-sliced quantum mechanical path integral to theories of positive spatial dimension. They are closely related to lattice models, which are expected to generate all consistent[1] quantum field theories by a continuum limit. In the case of topological theories – those which are sensitive only to the spacetime topology (rather than a metric) – the study of state sums has been particularly fruitful, with applications in mathematics – perhaps most famously, to knot theory (Kauffman, 1987) – as well as in physics. An advantage of this approach is that the algebraic structure used to define a topological state sum is simpler than the continuum data. This, however, comes at the cost of redundancy, as lattice realizations are not unique. As we will see, this trade-off essentially reflects the difference between certain algebraic structures and their Morita classes.

Topological quantum field theories (TQFTs) have recently gained prominence in condensed matter physics due to their connection to topological phases of matter. It is claimed that the field theories encode the universal, long-distance effective behavior – the "phase" – of gapped quantum systems, which means characterizing their

[1] Free of anomalies, such as the framing anomalies suffered by Reshetikhin-Turaev theories with nonzero chiral central charge.

responses to topological probes and reproducing the ground state expectation values of nonlocal order parameters (Shiozaki, Shapourian, et al., 2017). Topological state sums are related to the gapped lattice models that live at renormalization group fixed points (Kapustin, Turzillo, and You, 2017; Shiozaki and Ryu, 2016; Kapustin, Turzillo, and You, 2018). In this program, a sensitivity of the theory to a spin structure, in addition to topology, captures the response of massive fermions in the gapped system to boundary conditions. Such field theories are known as spin-TQFTs. When a gapped system has a time-reversal symmetry, its effective field theory is insensitive to the orientation of spacetime and is defined on all unoriented spacetimes.[2] When fermions transform under time-reversal symmetry with $T^2 = \mp 1$, the appropriate geometric structure is a pin$^\pm$ structure. Of particular physical relevance are pin$^-$ theories in two (spacetime) dimensions and their relationship with time-reversal-invariant Majorana chains, which have been known for some time to have an interesting interacting gapped phase classification (Fidkowski and Kitaev, 2010).

Given the advantages of state sum models for purely topological theories, it is natural to ask whether spin- and pin$^\pm$-TQFTs yield state sums as well. A challenge in realizing such theories discretely is that spin and pin$^\pm$ structures are in a sense "global," while state sums are inherently local. The case of spin theories in two spacetime dimensions was recently studied by Barrett and Tavares (Barrett and Tavares, 2015) (see also Ref. (Novak and Runkel, 2015)). They exploit the relation between spin structures on a surface M and immersions of M into $\mathbb{R}^3$ to construct, for each spin surface, a ribbon diagram, the twists and crossings of which keep track of the spin structure.

The main result of our paper is a state sum construction for two dimensional pin$^-$ theories. Our approach extends that of Ref. (Barrett and Tavares, 2015) to unoriented spacetimes. The state sums amount to discretizations of all invertible (and perhaps all unitarizable) field theories with this structure, in particular the Arf-Brown-Kervaire theory, which was recently studied in Ref. (Debray and Gunningham, 2018). A broad class of them has a simple algebraic characterization in terms of certain real superalgebras. From this perspective, the eight distinct powers of the Arf-Brown-Kervaire theory (the eight phases of time-reversal-invariant Majorana chains) arise from the eight Morita classes of real central simple superalgebras,[3] a

[2]The path integral on nonorientable spacetimes computes the time-reversal symmetry protected trivial (SPT) order (Shiozaki, Shapourian, et al., 2017).

[3]As discussed below, the state sums for the non-central algebras describe the two symmetry-

connection which has been noted previously in the context of tensor network states (Fidkowski and Kitaev, 2010; Turzillo and You, 2019). In topological theories, the state sum data has an interpretation as the space of states on the interval (Lauda and Pfeiffer, 2006); similarly, the real Clifford algebras $C\ell_{n,0}\mathbb{R}$, $n = 0, \dots, 7$, whose state sums are the eight invertible pin$^-$ theories, have to do with Majorana zero modes localized at the endpoints of the open chain.

The structure of the paper is as follows. In Section 4.1, we review some elementary facts about pin structures on closed surfaces and cobordisms and their relation to codimension one immersions and quadratic enhancements. Diffeomorphism classes of these objects and their classification by the Arf-Brown-Kervaire invariant are discussed. We also derive a simple expression for the evaluation of the quadratic enhancement on an embedded curve in terms of its ribbon diagram. In Section 4.2, we show how to construct a ribbon diagram from an immersed surface and evaluate its state sum. Imposing invariance under re-triangulation and regular homotopy, we derive the defining axioms of a half twist algebra. The state spaces of the associated pin$^-$-TQFT are constructed as well. In Section 4.3, we specialize to a class of half twist algebras related to real superalgebras. Decomposability and stacking are understood on the level of these algebras, and it is shown that Morita equivalent algebras define the same theory. We explicitly compute the path integrals for the Euler and Arf-Brown-Kervaire theories and discuss the classification of invertible pin$^-$-TQFTs.

4.1 Pin Geometry in Two Dimensions

Pin structures, immersions, and quadratic enhancements

The goal of this section is to review the following equivalences:

$$\left\{\begin{array}{c}\text{pin}^- \text{ structures / isotopy} \\ / \\ \text{pin}^-\text{-diffeomorphism} \\ = \\ \text{pin}^-\text{-diffeo. classes}\end{array}\right\} \begin{array}{c}\leftrightarrow \\ \\ \leftrightarrow \\ \\ \leftrightarrow\end{array} \left\{\begin{array}{c}\text{quadratic enhancements} \\ / \\ \text{lin. aut. with } q' = q \circ \alpha \\ = \\ \text{quadratic enh. / equiv.}\end{array}\right\} \begin{array}{c}\leftrightarrow \\ \\ \leftrightarrow \\ \\ \leftrightarrow\end{array} \left\{\begin{array}{c}\text{immersions / reg. homot.} \\ / \\ \text{diffeo. with } f = g \circ \phi \\ = \\ \text{imm. surf. / reg. homot.}\end{array}\right\}$$

Pin structures generalize spin structures to unoriented smooth manifolds. The structure group $O(n)$[4] of an unoriented manifold has two double covers $Pin^-(n)$ and $Pin^+(n)$, which differ in the behavior of the lifts $\tilde{r}$ of odd reflections $r \in O(n)$: in $Pin^\pm(n)$, they square to $\tilde{r}^2 = \pm 1$. A pin$^\pm$ structure on an unoriented manifold is a principal $Pin^\pm(n)$ bundle with a 2-fold covering of the orthogonal frame bundle that

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broken theories.

[4] A Riemannian metric is required to reduce the structure group from $GL_n\mathbb{R}$ to $O(n)$.

restricts to the double cover $\rho : Pin^{\pm}(n) \to O(n)$ on fibers. In terms of an open cover on M, it is a global lift of the $O(n)$-valued transition functions t_{ij} to $s_{ij} \in Pin^{\pm}(n)$. The triple overlap condition $t_{ij}t_{jk}t_{ki} = 1$ ensures that any local lifts $\rho : s_{ij} \mapsto t_{ij}$ satisfy $s_{ij}s_{jk}s_{ki} = o_{ijk} \in \ker\rho \simeq \mathbb{Z}/2$. By looking at the quadruple overlap, one sees that the signs o_{ijk} form a Čech 2-cocycle. Local lifts are acted on transitively by $\ker\rho$-valued 1-cochains A as $s_{ij} \mapsto s_{ij}A_{ij}$, which shifts o by the coboundary δA. The class $[o] \in H^2(M;\mathbb{Z}/2)$ is the obstruction to a global lift, or pin$^{\pm}$ structure, and is $w_2 + w_1^2$ for pin$^-$ and w_2 for pin$^+$. Two pin$^{\pm}$ structures are regarded as isotopic if they are related by a transformation $s_{ij} \mapsto \lambda_i s_{ij}(\lambda_j)^{-1}$, $\lambda_i \in Pin^{\pm}(n)$. If A is closed[5] and s is a pin$^{\pm}$ structure, the lift sA is again a pin$^{\pm}$ structure, and the two are isotopic iff A is a coboundary $\delta\lambda$; thus, assuming $[o]$ vanishes, isotopy classes of pin$^{\pm}$ structures on M form a torsor for $H^1(M;\mathbb{Z}/2)$. Our focus will be on surfaces and their pin$^-$ structures, or simply "pin structures." The obstruction class vanishes in two dimensions, so each surface supports exactly $|H^1(M;\mathbb{Z}/2)|$ pin structures, up to isotopy.

Another characterization of pin structures on a surface M can be given in terms of immersions of M into $\mathbb{R}^3$. Two immersions are said to be *regular homotopic* if they are connected by a smooth 1-parameter family of immersions. Immersions of a surface M into $\mathbb{R}^3$ fall into $|H^1(M;\mathbb{Z}/2)|$ regular homotopy classes (James and Tomas, 1966; Schlichting, 1977), one for each isotopy class of pin structure on M. The pin structure corresponding to a immersion is obtained by pulling back the standard pin structure on $\mathbb{R}^3$ by the immersion (R. C. Kirby and L. R. Taylor, 1989). Two immersions f, g are *equivalent* if there exists a diffeomorphism ϕ of M such that $f = g \circ \phi$, and these equivalence classes, called *immersed surfaces*, are said to be regular homotopic if their representative immersions are. Equivalence of immersions corresponds to pin diffeomorphism of the corresponding pin surfaces.

Pin structures on surfaces have a third characterization: up to isotopy, they are in bijective correspondence with *quadratic enhancements of the intersection form* (R. C. Kirby and L. R. Taylor, 1989); that is, functions

$$q : H_1(M;\mathbb{Z}/2) \to \mathbb{Z}/4 \tag{4.1}$$

such that

$$q(x + y) = q(x) + q(y) + 2 \cdot \langle x, y\rangle, \tag{4.2}$$

[5]Čech cocycles $A \in Z^1(M;\mathbb{Z}/2)$ are often referred to as $\mathbb{Z}/2$-gauge fields.

where $2\cdot$ embeds $\mathbb{Z}/2$ into $\mathbb{Z}/4$ as a subgroup and $\langle\cdot,\cdot\rangle$ denotes the intersection form on M. In Ref. (R. C. Kirby and L. R. Taylor, 1989) Kirby and Taylor demonstrate how to build a quadratic enhancement from a pin structure, while in Ref. (Pinkall, 1985) Pinkall does the same from its associated immersion. Since the constructions are similar, below we will focus solely on the latter. Every quadratic enhancement arises from both a pin structure and an immersion, and the constructions are isotopy and regular homotopy invariant, respectively. We say that two quadratic enhancements q, q' are *equivalent* if they are related as $q' = q \circ \alpha$ by a linear automorphism α of $H_1(M;\mathbb{Z}/2)$. As all linear automorphisms α that preserve the intersection form are induced by diffeomorphisms of M (Pinkall, 1985; III and Patrusky, 1978), all equivalences of quadratic enhancements arise from equivalences of immersions. A pin diffeomorphism that covers a diffeomorphism ϕ of the base space M induces an equivalence $q' = q \circ \phi_*$ on the associated quadratic forms. Quadratic enhancements form a torsor for $H^1(M;\mathbb{Z}/2)$ by the action $q \mapsto q + 2\cdot A$, with respect to which the correspondence with pin structures is equivariant (R. C. Kirby and L. R. Taylor, 1989).

The quadratic enhancement as a self-linking number

Let us now follow Ref. (Pinkall, 1985) in constructing a quadratic enhancement from an immersion. Begin by defining a function $\tilde{q}_f$ that takes closed loops in M to their self-linking numbers. To be precise, $\tilde{q}_f$ is defined on smooth embeddings $\gamma : S^1 \to M$ such that $f \circ \gamma : S^1 \to \mathbb{R}^3$ is also an embedding. Images of such embeddings have embedded tubular neighborhoods ("ribbons") N_γ. The self-linking number is given by the linking number of the loop $f \circ \gamma$ with the loop obtained by pushing $f \circ \gamma$ along N_γ:

$$\tilde{q}_f(\gamma) = \mathrm{link}(f \circ \gamma, f(\partial N_\gamma)). \tag{4.3}$$

Under regular homotopy, $\tilde{q}_f$ is stable only modulo 4; moreover, it depends only on the $\mathbb{Z}/2$-homology class $[\gamma] \in H_1(M;\mathbb{Z}/2)$ and defines a map q_f on $H_1(M;\mathbb{Z}/2)$ satisfying the quadratic enhancement condition (4.2).

By projecting a ribbon onto $\mathbb{R}^2$ and obtaining a ribbon diagram, its self-linking number may be computed by a local algorithm. As is discussed in greater detail in Section 4.2, one may use regular homotopy so that the projection $\mathbb{R}^3 \to \mathbb{R}^2$ onto the xy-plane is an immersion of N_γ at all but finitely many points where the ribbon makes a half twist (left or right handed). The image of the curve γ may be taken to cross itself transversely and away from these points. Away from the twists and

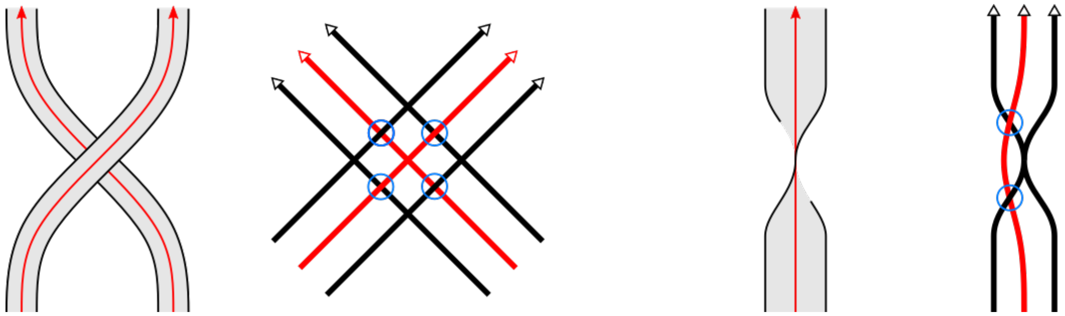

Figure 4.1: Following Pinkall (Pinkall, 1985), give the core (red) and edges (black) of the ribbon a particular orientation. Then compute the linking number of red lines with the black lines. The crossing has four red-black intersections, all of the same parity. The half twist has two red-black intersections of the same parity.

crossings, the self-linking number is zero. As demonstrated in Figure 4.1, each right handed half twist contributes $+1$ to $\tilde{q}_f(\gamma)$; likewise, each left handed half twist contributes -1. Each crossing contributes ± 2. In total,

$$\tilde{q}_f(\gamma) = (\#\text{ r.h. twists}) - (\#\text{ l.h. twists}) + 2 \cdot (\#\text{ crossings}) \mod 4. \tag{4.4}$$

The Arf-Brown-Kervaire invariant

The Arf-Brown-Kervaire (ABK) invariant of a surface M with quadratic enhancement q is defined as

$$\mathrm{ABK}(M, q) = \frac{1}{\sqrt{|H_1(M; \mathbb{Z}/2)|}} \sum_{x \in H_1(M;\mathbb{Z}/2)} e^{i\pi q(x)/2}. \tag{4.5}$$

It is valued in eighth roots of unity and has the nice property that two quadratic enhancements on M have the same ABK invariant if and only if they are equivalent (E. H. Brown, 1972). In order words, the ABK invariant is well-defined on diffeomorphism classes of pin surfaces as well as on immersed surfaces. The ABK invariant determines the pin bordism class of the pin surface and so defines an isomorphism $\Omega_2^{\mathrm{pin}}(\mathrm{pt}) \xrightarrow{\sim} \mathbb{Z}/8$.

Decomposition of pin surfaces

Every closed unoriented surface may be decomposed as a connect sum of tori and real projective planes. Each of these building blocks has two diffeomorphism classes of pin structures. On the torus, there are four isotopy classes of pin structures given by a choice of NS or R boundary conditions around each independent 1-cycle. Pin diffeomorphisms covering Dehn twists relate the NS-NS, NS-R, and R-NS classes.

To see this, note that a Dehn twist induces a map $\{x', y'\} = \{x, x+y\}$ on a basis of $H_1(T^2; \mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2$. Then use the rule (4.2): the NS-NS pin structure $q(x) = 0, q(y) = 0$ becomes the NS-R pin structure

$$q(x') = q(x) = 0, \qquad q(y') = q(x+y) = q(x) + q(y) + 2 \cdot \langle x, y \rangle = 2. \tag{4.6}$$

These pin structures are distinct from the R-R pin structure. One may also use (4.5) to see that the NS-NS, NS-R, and R-NS pin structures have ABK invariant $+1$ (and so are diffeomorphic to each other), while the R-R pin structure has ABK invariant -1. Moreover, since the ABK invariant determines the bordism class, this calculation shows that the NS-NS pin structure bounds a solid torus, while the R-R pin structure is non-bounding. On the real projective plane, there are two isotopy classes of pin structure. To see this, note that $H_1(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2$, the generator z of which is represented by 1-sided (i.e. orientation-reversing) curve and has self-intersection $\langle z, z \rangle = 1$. Since $q(0) = 0$, the rule (4.2) says

$$0 = q(z) + q(z) + \langle z, z \rangle = 2q(z) + 1, \tag{4.7}$$

so there are two isotopy classes of pin structures given by $q(z) = 1$ and $q(z) = 3$. These are non-diffeomorphic since they have ABK invariants $\exp(i\pi/4)$ and $\exp(7i\pi/4)$, respectively. Call them $\mathbb{R}P^2_1$ and $\mathbb{R}P^2_7$.

The pin structures on other surfaces may be readily understood from their connect sum decompositions. For example, the Klein bottle decomposes as $K \simeq \mathbb{R}P^2 \# \mathbb{R}P^2$. Let z_1, z_2 denote the generating 1-(co)cycles of the real projective planes. In this basis, the four quadratic enhancements are $q = (1,1), (1,3), (3,1), (3,3)$. In the familiar basis of $H_1(K; \mathbb{Z}/2)$ given by the orientation-preserving curve $x = z_1 + z_2$ and orientation-reversing curve $y = z_2$, the possibilities are $q = (2,1), (0,3), (0,1), (2,3)$. They have ABK invariants $+i$, $+1$, $+1$, and $-i$, so there are three diffeomorphism classes of pin structures on K, one of which is null-bordant.

Pin bordism and TQFT

Our discussion so far has focused on closed surfaces. To define pin TQFTs, it is necessary to also understand pin one manifolds and the bordisms between them. There are two connected one dimensional pin manifolds given by the antiperiodic (NS) and periodic (R) spin structures on the circle. A pin manifold with boundary induces a pin structure on its boundary, and a pin bordism between pin one manifolds S_0 and S_1 is a pin surface M whose boundary, with induced pin structure, is $S_0 \sqcup S_1$.

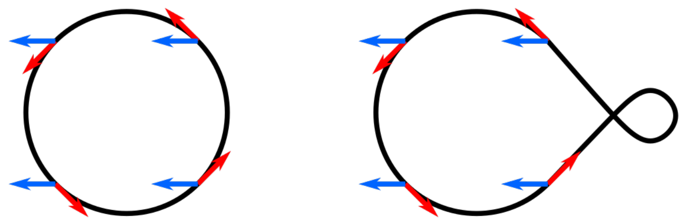

Figure 4.2: Two examples of immersions of the circle in the plane, with turning numbers 1 (left) and 0 (right) defined as the winding of a tangent frame (red) relative to a constant vector field (blue). This number mod 2 determines the induced (s)pin structure on the circle: NS for odd, R for even.

Each of the two pin structures on the circle is related to a class of immersed circles in the plane, depicted in Figure 4.2. Fix two planes $\mathbb{R}^2_0, \mathbb{R}^2_1$ normal to the y-axis. An immersion of the cobordism (S_0, S_1, M) is an immersion of M such that S_0, S_1 lie in $\mathbb{R}^2_0, \mathbb{R}^2_1$, respectively. A regular homotopy of the immersions of the cobordism is again a 1-parameter family of immersions. We emphasize that at each value of the parameter, the boundaries S_0, S_1 are pinned to the planes $\mathbb{R}^2_0, \mathbb{R}^2_1$.

The theory of quadratic enhancements associated to pin surfaces with boundary requires more care than we will give it here. The idea is to extend the discussion of Ref. (Segal, 2004). Choose a set of basepoints $\partial_0 M$ – one on each connected component of ∂M, and let a pin structure on $(M, \partial_0 M)$ be a pin structure on M together with a trivialization of the $Pin^-(1) = \mathbb{Z}/4$ bundle over $\partial_0 M$. Such pin structures should be (non-canonically) identified with quadratic enhancements of the intersection form on $H_1(M_*; \mathbb{Z}/2) \simeq H_1(M, \partial_0 M; \mathbb{Z}/2)$, where M_* is a closed pin surface obtained by sewing a punctured sphere into M.

A pin TQFT assigns state spaces $\mathcal{A}_{NS}, \mathcal{A}_R$ to the circles S^1_{NS}, S^1_R and linear maps to the pin bordisms between them. In particular, the mapping cylinders associated to elements of the pin mapping class group of the circles defines a supervector space structure on the state spaces. A complete algebraic characterization of pin TQFTs would resemble the discussions of Ref.'s (G. W. Moore and Segal, 2006; Turaev and Turner, 2006). We will not give one here; instead our focus will be on the pin TQFTs that arise from the diagrammatic state sum construction introduced below.

4.2 Ribbon Diagrams and Half Twist Algebras

A state sum model provides a combinatorial description of a theory, such as a TQFT or, in the present case, a pin TQFT, typically defined on the continuum. Let us first discuss partition functions of closed manifolds before extending to cobordisms in Section 4.2. The idea is to first define an invariant of discretized spaces, given as a weighted sum over colorings of a discretization. The weight assigned to a coloring is typically computed "locally" from the contributions of the local elements of the discretization. The requirement that the invariant is independent of the discretization imposes structure on the weights.

For example, Ref. (Fukuma, Hosono, and Kawai, 1994) studies two-dimensional *topological* state sums, which are defined on triangulated surfaces and whose weights receive contributions from the faces and edges of the triangulation. Topological invariance – that is, lack of dependence on the triangulation – imposes Pachner move conditions on this algebraic data. The result is that the local tensors assigned to faces and edges form a separable algebra.

State sum models for pin TQFTs have a similar logic. A discretization of a pin surface is a triangulation together with an additional combinatorial structure representing a pin structure. Finding these structures and the equivalence relations under which they represent the same continuum structure is not easy. One approach is to find a *local* combinatorial structure, or *marking*, as Ref. (Novak and Runkel, 2015) does for spin structures. This paper follows a different path, one based on the connection between pin structures and immersions into $\mathbb{R}^3$. In the following, a discretization is a triangulation together with a choice of immersed surface. The construction is automatically invariant under equivalence of immersion, and invariance under regular homotopy is enforced by hand. The weights are products of tensors assigned to (nonlocal) elements of the discretization. The requirement of invariance under change of discretization (Pachner moves and regular homotopy) means that these tensors satisfy several relations. The resulting algebraic structure is what we dub a *half twist algebra* and extends the separable algebras of Ref. (Fukuma, Hosono, and Kawai, 1994) to allow for the theory's sensitivity to pin structure.

Ribbon diagrams

We now construct a ribbon diagram from a triangulation of an immersed surface. Dual to the triangulation of the surface is a graph, which may be enlarged to a *ribbon graph* by taking a regular neighborhood, the compliment of which in M is one or more disks. Any immersion of M is regular homotopic to one that is an embedding on the ribbon graph (Pinkall, 1985). This embedded ribbon graph

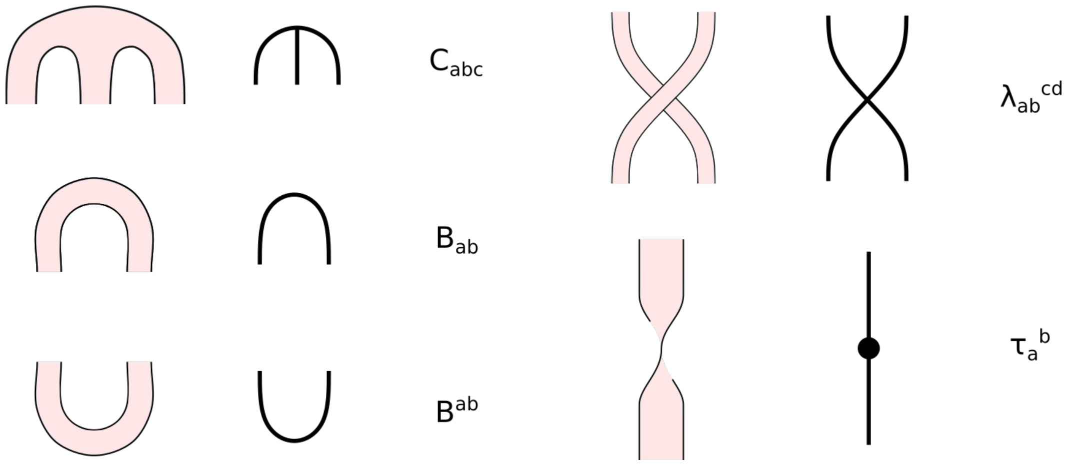

Figure 4.3: The five building blocks of ribbon diagrams satisfying the regularity conditions.

is passed through the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ onto the xy-plane.[6] By regular homotopy, the projection can be made to satisfy certain regularity conditions. First, the projection is an immersion of the ribbon graph at all but finitely many points where the ribbon makes a *half twist* (Kauffman, 1990). Second, the edges of the graph intersect transversely in the image of p. Third, the graph is parallel to the x-direction at only finitely many "critical points" (nodes, caps, cups) where either all legs exit above the x-parallel or all legs exit below (no saddle points). Fourth, each node of the graph is located at a critical point with its three legs exiting below. Fifth, at most one of the following can occur at any point: a half twist, a crossing, and a critical point. In addition to the image of the projection, the helicities of the half twists (right or left handed) are recorded. Unlike diagrams typical in knot theory, ours do not record whether one strand crosses over or under the other at a crossing, as these two configurations are related by regular homotopy. A ribbon diagram satisfying the regularity conditions is composed of the five building blocks – nodes, caps, cups, crossings, and half twists – depicted in Figure 4.3.

If two ribbon diagrams are built from the same regular homotopy class of ribbon graphs, they are related by the set of moves depicted in Figure 4.4. The moves[7] show that a left handed twist is related by regular homotopy to a sequence of three right handed twists. This means, by replacing each left handed half twist by three

[6] The ribbon diagrams associated to any two projections are related by rotation of the immersed surface in $\mathbb{R}^3$, which is a regular homotopy. Since the state sum is, by construction, regular homotopy invariant, the choice of p does not matter.

[7] Two half twists is a full twist, and the ribbon Reidemeister moves show that a pair of full twists can be undone.

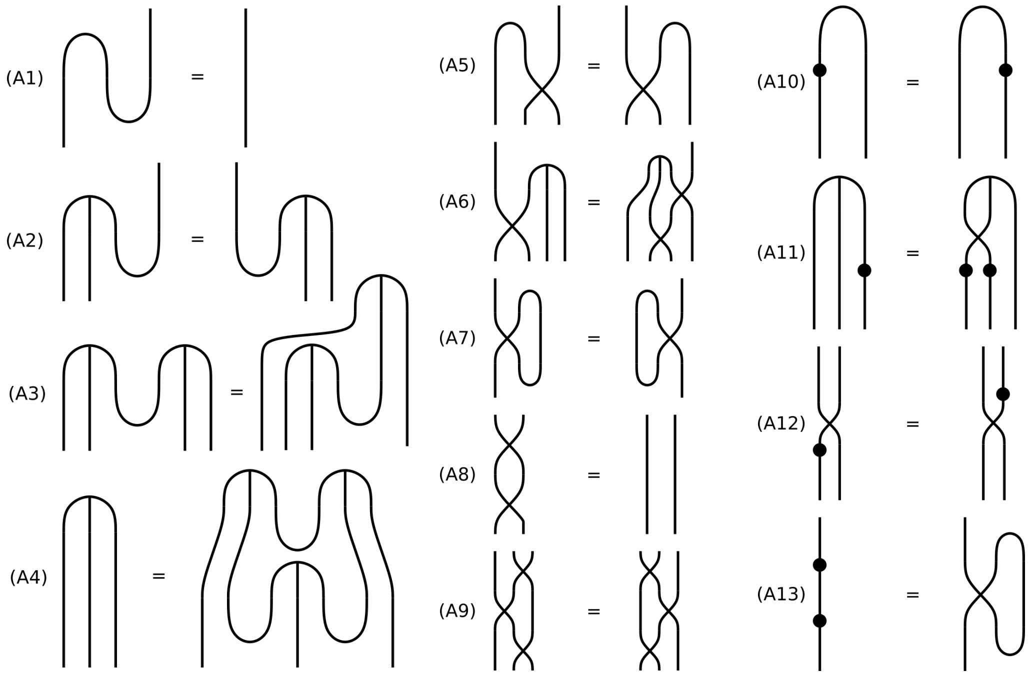

Figure 4.4: Ribbon diagrams for the conditions (4.8)–(4.20), each due to regular homotopy or Pachner moves.

right handed half twists, one obtains a ribbon diagram where the half twists are all right handed. In the following, we simplify the algebra by assuming that all half twists are right handed. Two of the moves, which may be more difficult to visualize, are depicted in ribbon form in Figure 4.5.

Any two triangulations on M are related by the 2-2 and 3-1 Pachner moves, also depicted in Figure 4.4.

Algebraic structure

We now show how to evaluate a partition function for a regular homotopy class of immersed surfaces. Begin with a ribbon diagram, decomposed into the five building blocks. Color the diagram by labeling the legs of each block by elements in a finite set $\mathcal{I}$. The blocks are assigned the following $\mathbb{C}$-valued weights:

1. Nodes labeled left to right by $a, b, c \in \mathcal{I}$ receive a weight C_{abc}.

2. Caps labeled left to right by $a, b \in \mathcal{I}$ receive a weight B_{ab}, while cups receive a weight B^{ab}.

3. Crossings labeled as in Figure 4.3 by $a, b, c, d \in \mathcal{I}$ receive a weight $\lambda_{ab}{}^{cd}$.

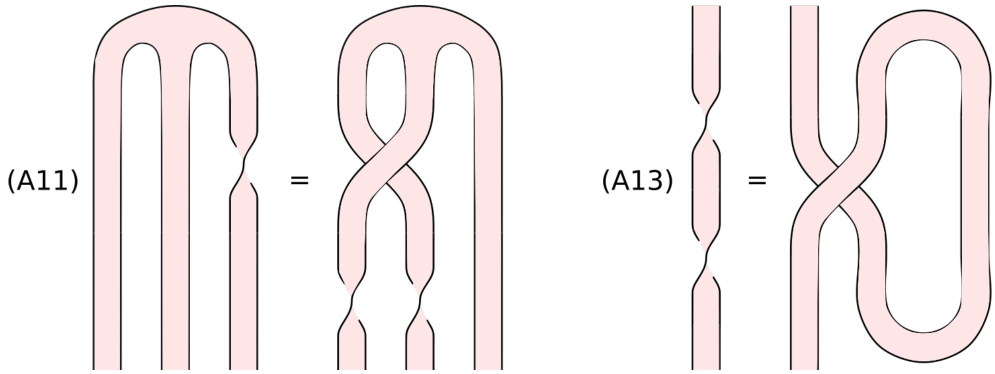

Figure 4.5: The moves (4.18) and (4.20) as ribbon diagrams.

4. (Right handed) half twists labeled bottom to top by $a, b \in \mathcal{I}$ receive a weight ${\tau_a}^b$.

5. Vertices[8] of the triangulation receive a weight R.

The weight of the colored diagram is the product of the weights of the pieces in its decomposition, and the partition function for a diagram is a sum of the weights of its colorings.

For the partition function to be independent of the discretization, it must be invariant under the moves of Figure 4.4. By evaluating them according to our procedure, we find the following algebraic conditions:

(Snake)	define η	$B_{ac}B^{cb} = \delta_a^b$	(4.8)
(Cyclicity)	define m	$C_{abd}B^{dc} = B^{cd}C_{dab}$	(4.9)
(Pachner 2-2)	m associative	$C_{abe}B^{ef}C_{fcd} = C_{bce}B^{ef}C_{afd}$	(4.10)
(Pachner 3-1)	η special	$C_{abc} = R\, C_{ade}B^{df}C_{fbg}B^{gh}C_{ihc}B^{ei}$	(4.11)
(Crossing at a critical point)		$B_{ae}{\lambda_{bc}}^{ed} = {\lambda_{ab}}^{de}B_{ec}$	(4.12)
(Crossing at a node)		${\lambda_{ab}}^{ef}C_{fcd} = C_{aeg}{\lambda_{bc}}^{ef}{\lambda_{fd}}^{ge}$	(4.13)
(Modified Reidemeister I)		$B^{cd}B_{ce}{\lambda_{da}}^{eb} = {\lambda_{ac}}^{bd}B^{ce}B_{de}$	(4.14)
(Reidemeister II)		${\lambda_{ab}}^{ef}{\lambda_{ef}}^{cd} = \delta_a^c\delta_b^d$	(4.15)
(Reidemeister III)		${\lambda_{ag}}^{di}{\lambda_{bc}}^{gh}{\lambda_{ih}}^{ef} = {\lambda_{ab}}^{gh}{\lambda_{hc}}^{if}{\lambda_{gi}}^{de}$	(4.16)
(Twist at a critical point)	$\eta(\mathbb{1} \otimes \tau) = \eta(\tau \otimes \mathbb{1})$	$B_{ac}{\tau_b}^c = {\tau_a}^c B_{cb}$	(4.17)
(Twist at a node)	$\tau m = m\lambda(\tau \otimes \tau)$	$C_{abd}{\tau_c}^d = {\tau_a}^d{\tau_b}^e{\lambda_{de}}^{fg}C_{fgc}$	(4.18)
(Twist at a crossing)	$\lambda(\tau \otimes \mathbb{1}) = (\mathbb{1} \otimes \tau)\lambda$	${\tau_a}^e{\lambda_{eb}}^{cd} = {\lambda_{ab}}^{ce}{\tau_e}^d$	(4.19)
(Two half twists)	$\tau^2 = \phi$	${\tau_a}^c{\tau_c}^b = {\lambda_{ac}}^{bd}B^{ce}B_{de}\quad (= {\lambda_{ac}}^{bd}{\sigma_d}^c = {\phi_a}^b)$	(4.20)

[8] Surfaces with boundary are discussed in Section 4.2. In this more general case, only *internal* vertices receive a weight R.

The conditions (4.8)–(4.11) define a special Frobenius algebra (A, m, η); that is, an a unital, associative algebra (A, m) with a non-degenerate bilinear form η satisfying the Frobenius condition $\eta(xy, z) = \eta(x, yz)$, $x, y, z \in A$, and the specialness[9] condition $m \circ \eta^{-1} = R^{-1}\,1$. This algebra is defined on the vector space with basis $\{e_a\}$, $a \in \mathcal{I}$, has product $m(e_a \otimes e_b) = C_{ab}{}^c e_c$ given by associative structure coefficients $C_{ab}{}^c = C_{abd} B^{dc}$, unit $1 = B^{ab} C_{bcd} B^{cd} e_a$, and non-degenerate bilinear form $\eta(e_a, e_b) = B_{ab}$. Ref. (Barrett and Tavares, 2015) shows that the conditions (4.8)–(4.11) enforce the axioms of a special Frobenius algebra and, conversely, that a special Frobenius algebra defines tensors C_{abc} and B_{ab} that satisfy these conditions. If η is taken to be the unique (up to R) *symmetric* special Frobenius form, this result reduces to the familiar case studied by Fukuma, Hosono, and Kawai (Fukuma, Hosono, and Kawai, 1994).

The conditions (4.12)–(4.16) imply other relations like $B^{be} \lambda_{ea}{}^{cd} = \lambda_{ae}{}^{bc} B^{ed}$. The existence of a *symmetric structure* $\lambda : A \otimes A \to A \otimes A$, satisfying the axioms, is also a constraint on η. The *Nakayama automorphism*

$$\sigma_a{}^b = B_{ac} B^{bc}, \qquad \eta(a, b) = \eta(\sigma(b), a) \tag{4.21}$$

measures the failure of η to be symmetric. Ref. (Barrett and Tavares, 2015) demonstrates that conditions (4.8)–(4.16) imply

$$B_{ac} B^{bc} = B_{ca} B^{cb}, \qquad \sigma^2 = 1, \tag{4.22}$$

equivalently, that η decomposes as a sum of symmetric and antisymmetric parts. Define the *full twist*

$$\phi_a{}^b = \lambda_{ac}{}^{bd} B^{ce} B_{de} = \lambda_{ac}{}^{bd} \sigma_d{}^c. \tag{4.23}$$

Ref. (Barrett and Tavares, 2015) argues from these conditions that ϕ is a Frobenius algebra automorphism; that is,

$$\phi \circ m(a \otimes b) = m(\phi(a) \otimes \phi(b)), \qquad \eta(\phi(a), \phi(b)) = \eta(a, b). \tag{4.24}$$

Moreover, ϕ is an involution and so defines a $\mathbb{Z}/2$-grading on A: on homogeneous elements,

$$\phi(a) = (-1)^{|a|} a, \qquad |a| \in \{0, 1\}. \tag{4.25}$$

The data (C, B, λ) satisfying these axioms is what Ref. (Barrett and Tavares, 2015) use to define their spin state sums.

[9] Ref. (Lauda and Pfeiffer, 2006) discusses a generalization of the oriented state sum construction to non-special Frobenius algebras, where *window elements* $a^{-1} \nsim 1$ are attached to vertices. In their language, we always take $a^{-1} = R\,1$ with $R \in \mathbb{C}$.

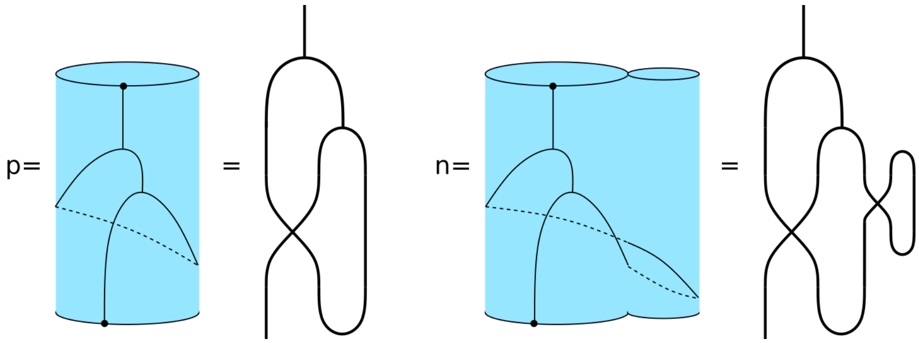

Figure 4.6: Ribbon diagrams for the projectors p and n. We have been careful to account for the half twists that appear when the ribbon turns a "corner," setting them up to cancel.

Other relations like $B^{cb}\tau_c{}^a = B^{ad}\tau_d{}^b$ and $\tau_b{}^e\lambda_{ae}{}^{cd} = \lambda_{ab}{}^{fd}\tau_f{}^c$ follow from the conditions (4.17)–(4.20). We will refer to the data (C, B, λ, τ) as a *half twist algebra*. It is the input for our state sum construction.

State spaces and bordisms

The construction has so far focused on closed surfaces. In order to define a TQFT, it must also assign state spaces $\mathcal{A}_0, \mathcal{A}_1$ to one dimensional closed pin manifolds S_0, S_1 and linear maps $Z(M) : \mathcal{A}_0 \to \mathcal{A}_1$ to the pin bordisms M between them. Given an immersion of M, set up according to Section 4.1, form its ribbon diagram as usual. Suppose there are n edges in the triangulation of S_0 and m in that of S_1. Then the state sum over internal colorings defines a map $\otimes^n A \to \otimes^m A$. This map has a clear dependence on the triangulation, as re-triangulating may change n and m. It is also non-invariant under regular homotopy, as crossing the external legs over each other introduces single factors of the crossing map λ. The following discussion shows that both of these problems are solved by composing each end with a certain projector.

Consider the ribbon diagrams depicted in Figure 4.6, which arise from immersions of cylindrical topologies. One diagram corresponds to a cylinder with boundary circles of NS type, the other R.[10] Since the cylinder defines a regular homotopy between the input and output circles, they are immersed in the same way.

It has been argued by Ref. (Barrett and Tavares, 2015) (see also (G. W. Moore and

[10]The ribbon diagrams for cylinders of circles with rotation numbers $n, n+2$ are related by the ribbon Reidemeister moves.

Segal, 2006)) that these diagrams define projectors p and n onto subspaces

$$\mathrm{im}\, p = \mathcal{A}_{NS} = \{a \in A : m(b \otimes a) = m \circ \lambda(b \otimes a), \forall b \in A\} \tag{4.26}$$

$$\mathrm{im}\, n = \mathcal{A}_{R} = \{a \in A : m(b \otimes a) = m \circ \lambda(\phi(b) \otimes a), \forall b \in A\} \tag{4.27}$$

The maps assigned to other ribbon diagrams with cylindrical topology are related to these by composition with some power of τ, and we will not consider them here. By gluing a copy of p into each NS-type connected component of S_0, S_1 and a copy of n into each R-type component, the map $\otimes^n A \to \otimes^m A$ becomes

$$\mathcal{Z}(M) : \mathcal{Z}(S_0) \to \mathcal{Z}(S_1), \tag{4.28}$$

where $\mathcal{Z}(S_0)$ consists of a copy of $\mathcal{A}_{NS}, \mathcal{A}_R$ for each NS-type component and R-type component, respectively, and likewise for $\mathcal{Z}(S_1)$. This solves the problem of triangulation-dependence.

One must check whether composition with p and n is independent of the way in which the cylindrical ribbon diagrams are glued into the cobordism. Regular homotopy has been used to push the legs of the cylindrical ribbon diagrams to the "front" (positive z-coordinate) side of the cylinders, so it must also be checked that our construction of $\mathcal{Z}(M)$ is independent of the way in which this was done. Both of these checks follow from (4.26) and (4.27), which show that p and n are unchanged by cyclic permutation of the legs, as in Figure 4.7. The only ambiguity that remains is due to reordering the boundary components, which introduces factors of λ. These terms reflect the fact that the product assigned to the pair-of-pants cobordism is not commutative, but twisted-commutative. To obtain a definite $\mathcal{Z}(M)$, one must fix an ordering of the boundary components; this is a characteristic of the continuum pin TQFT and not a relic of the state sum construction. For the special class of theories discussed in Section 4.3, the product is graded-commutative with respect to the supervector structure on $\mathcal{A}_{NS}, \mathcal{A}_R$. In this case, $\mathcal{Z}(M)$ may be interpreted as a map $\wedge_i \mathcal{Z}(S^1_{0,i}) \to \wedge_i \mathcal{Z}(S^1_{1,i})$ of exterior algebras, where $S^1_{0,i}, S^1_{1,i}$ denote boundary components.

An axiom of (pin) TQFT requires that gluing two bordisms M_1, M_2 along their cut boundaries amounts to composing the linear maps assigned to them. This is true of the present construction. Leaving off the projectors, the bordisms are assigned matrices $\otimes^n A \to \otimes^m A$ and $\otimes^m A \to \otimes^l A$. The amplitude for the composite bordism is a sum over colorings of the internal edges of M_1, M_2 as well as the edges of the glued boundary, weighted the product of the weights for M_1, M_2. This

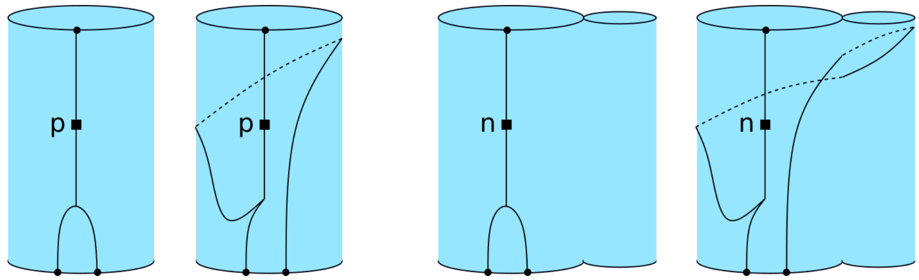

Figure 4.7: Gluing independence. Since p and n project onto certain twisted centers of A, according to (4.26) and (4.27), an external leg may be pulled around the circle without affecting the state sum.

is matrix multiplication. To complete the argument, add back the projectors; by re-triangulation invariance, this does nothing at the glued boundary.

A *Hermitian structure* on a pin TQFT is a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{Z}(S)$ for each closed one dimensional pin manifold S, with respect to which $\mathcal{Z}(M)$ and $\mathcal{Z}(-M)$ are adjoint for any cobordism (M, S_0, S_1) (Turaev, 2010; Freed and Hopkins, 2016). Here, $-M$ denotes the "opposite" pin cobordism from S_1 to S_0. In terms of immersed surfaces, $-M$ is obtained from M by reflecting over an xz-plane. A *unitary structure* is a Hermitian structure for which the sesquilinear form is positive definite (an inner product).

4.3 Real Superalgebras and the Arf-Brown-Kervaire TQFT

The remainder of this paper focuses on a special class of half twist algebras closely related to separable real superalgebras, the state sum models associated to which constitute a broad class of interesting examples such as the Arf-Brown-Kervaire theory. To be precise, these state sums take as a input a symmetric special Frobenius real superaglebra or, equivalently, a separable real superalgebra with a continuous parameter α.[11]

Real superalgebras

A real superalgebra is an algebra (A_r, m) over $\mathbb{R}$ with a linear involution $\phi : a \mapsto (-1)^{|a|}a$, with respect to which the product m is equivariant, as in (4.24). Superalgebras inherit the natural symmetric structure

$$\lambda : a \otimes b \mapsto (-1)^{|a||b|} b \otimes a \tag{4.29}$$

[11] Sometimes we neglect α and speak only of the superalgebra; this is because α's contribution is just an Euler term.

from the symmetric monoidal category of supervector spaces. Separability means there is a symmetric[12] special Frobenius inner product η, unique up the real scalar α, given by the *trace form*

$$\eta(x, y) = \alpha\, \mathrm{Tr}[L(x)L(y)], \tag{4.30}$$

where $L : A \longrightarrow \mathrm{E}nd(A)$ denotes left multiplication. The real algebra A_r is equivalent to its complexification $A = A_r \otimes_{\mathbb{R}} \mathbb{C}$ together with an antilinear automorphism T of A, called a *real structure*, that fixes A_r.

By virtue of being special Frobenius, the complex algebra A is separable as a superalgebra. This means it is a direct sum of simple superalgebras ("blocks"), of which there are two types: matrix algebras $\mathbb{C}(p|q)$ and odd algebras $\mathbb{C}(n) \otimes \mathbb{C}\ell(1)$. Each block has an involutive antilinear anti-automorphism given by conjugate transposition of $\mathbb{C}(p|q)$ or the $\mathbb{C}(n)$ factor.[13] The direct sum of these is a map $*$ on A. Its composition with the real structure is a linear involutive anti-automorphism $t = *T$.

The structures m, η, λ, and ϕ of A_r extend linearly onto A, where the map t satisfies

$$\eta(tx, ty) = \eta(x, y), \quad tm(x \otimes y) = m(ty \otimes tx), \quad \lambda(t \otimes \mathbb{1}) = (\mathbb{1} \otimes t)\lambda, \quad t^2 = \mathbb{1}. \tag{4.31}$$

These relations resemble the four half twist axioms (4.17)–(4.20) but are not quite the same: while t is η-orthogonal, τ is η-symmetric; while t is an anti-automorphism, τ is a λ-twisted-automorphism; while t is an involution, τ squares to ϕ. Outside of these differences, A is much like a half twist algebra: its involution ϕ is determined by the symmetric structure λ as $\phi_a{}^b = \lambda_{ac}{}^{bc}$, and it is straightforward to verify that m, η, and λ are compatible in the sense that they satisfy the first nine axioms (4.8)–(4.16).

To make A into a genuine half twist algebra, we would like to construct a half twist τ, satisfying (4.17)–(4.20), out of the involutive linear anti-automorphisms t (associated with T), satisfying (4.31). If $s(x) \in \{0, 1\}$ is any grading of the algebra that shares an eigenbasis with ϕ (such as $s = 0$), we may define

$$\tau : x \mapsto (-1)^{s(x)} i^{|x|} t(x). \tag{4.32}$$

[12] Here we mean "symmetric" in the usual sense, as a Frobenius algebra object in Vect, not sVect.

[13] There may exist other such maps, but our construction uses this canonical one. In any basis $\{e_{ij}\}$ where $e_{ij}e_{jk} = +e_{ik}$, "conjugate transposition" is unambiguously defined as the map $e_{ij} \mapsto e_{ji}$.

It is straightforward to verify that τ squares to ϕ and is η-symmetric. Moreover, t is a λ-twisted-automorphism:

$$\begin{aligned} m \circ \lambda(\tau(x) \otimes \tau(y)) &= (-1)^{|x||y|} m(\tau(y) \otimes \tau(x)) \\ &= (-1)^{s(x)+s(y)} i^{|x|+|y|-2|x||y|} m(t(y) \otimes t(x)) \\ &= (-1)^{s(m(x \otimes y))} i^{|m(x \otimes y)|} t \circ m(x \otimes y) \\ &= \tau \circ m(x \otimes y). \end{aligned} \tag{4.33}$$

The choice of s has to do with the decomposability of the state sum and is discussed in Section 4.3. A half twist algebra constructed from a real superalgebra is not generic. In particular, its crossing map is given by Eq. (4.29) and its half twist satisfies $*\tau* = \tau^{-1}$. The symmetry of η is not an independent condition, as the special form of λ means that the Nakayama automorphism (4.21) is trivial.

It is worth noting at this point that our separable superalgebras come with an sesquilinear form

$$\langle x, y \rangle = \eta(*x, y). \tag{4.34}$$

In fact $\langle \cdot, \cdot \rangle$ is positive definite and so defines an inner product. By (4.30) it is clear that η vanishes if x and y are supported on different blocks. On an even block, $\langle M, N \rangle = \mathrm{Tr}\left[M^\dagger N\right]$, which is positive definite. On an odd block, $\langle M \otimes \gamma^i, N \otimes \gamma^j \rangle = \delta^{ij}\, \mathrm{Tr}\left[M^\dagger N\right]$, which is also positive definite.

In any theory, the circles S^1_{NS} and S^1_R have macaroni bordisms,[14] whose partition functions define bilinear forms $\eta_{NS} : \mathcal{A}_{NS} \otimes \mathcal{A}_{NS} \to \mathbb{C}$ and $\eta_R : \mathcal{A}_R \otimes \mathcal{A}_R \to \mathbb{C}$. Evaluating ribbon diagrams for the macaroni bordisms gives these maps in terms of the superalgebra data: $\eta_{NS} = \eta(p, p)$ and $\eta_R = \eta(n, n)$. Inserting the map $*$, as in (4.34), one may define sesquilinear forms $\langle , \rangle_{NS} = \eta_{NS}(*,)$ and $\langle , \rangle_R = \eta_R(*,)$. The form on an arbitrary closed one dimensional pin manifold S is given as a tensor product of these forms.

We would like to show that state sum pin TQFTs associated with real separable superalgebras are unitary in the sense of Section 4.2. It remains to check adjointness. Due to the form of η (4.30), this condition reads $*\mathcal{Z}(M)* = \mathcal{Z}(-M)^T$. In terms of ribbon diagrams in $\mathbb{R}^2$, reflection across the y axis must have the effect of acting on

[14] Macaroni bordisms are built by gluing the cap bordism into a pair-of-pants. Accounting for spin structures, there are two distinct such bordisms on S^1_R. Choose one. The other is related by composition with a cylinder.

each external leg by $*$. The conditions on each building blocks read

$$\begin{aligned} *m(*a \otimes *b) &= m(b \otimes a), \qquad \eta(*a \otimes *b) = \eta(b \otimes a), \\ (* \otimes *)\lambda(*a \otimes *b) &= \lambda(b \otimes a), \qquad * \tau(*a) = \tau^{-1}(a). \end{aligned} \tag{4.35}$$

The first condition follows from the fact that $*$ is an anti-automorphism, the second and third from symmetry of η (4.30) and λ (4.29), and the fourth from the antilinearity of $*$ and the i factor in (4.32). Unitarity also requires $R \in \mathbb{R}$, which follows from $\alpha \in \mathbb{R}$. Therefore theories associated to real separable superalgebras are unitary.

A useful construction on superalgebras A, B is the *supertensor product* $A \mathbin{\widehat{\otimes}} B$. This superalgebra has underlying vector space $A \otimes B$ with grading $\phi_{A \widehat{\otimes} B} = \phi_A \mathbin{\widehat{\otimes}} \phi_B$ and associative product

$$(a \widehat{\otimes} b)(a' \widehat{\otimes} b') = (-1)^{ba'} aa' \widehat{\otimes} bb', \qquad \text{i.e.} \qquad m_{A \widehat{\otimes} B} = (m_A \widehat{\otimes} m_B)(1 \widehat{\otimes} \lambda \widehat{\otimes} 1). \tag{4.36}$$

The special symmetric Frobenius form is $\eta_{A \widehat{\otimes} B} = (\eta_A \mathbin{\widehat{\otimes}} \eta_B)(1 \mathbin{\widehat{\otimes}} \lambda \mathbin{\widehat{\otimes}} 1)$. It is helpful to interpret the product rule (4.36) diagrammatically. In Figure 4.8, the products on A and B are represented by trivalent nodes of red and blue lines, respectively. The product on $A \mathbin{\widehat{\otimes}} B$ has a red-blue crossing, contributing the sign λ. More generally, one may consider diagrams that consist of a red ribbon diagram superimposed on a blue ribbon diagram such that the usual regularity conditions are met. Color the red diagram by basis elements e_a of A and the blue diagram by basis elements f_i of B. The weight of this double coloring is the weight of the red coloring, according to A, times the weight of the blue coloring, according to B, times signs $|e_a||f_i|$ at each red-blue crossing. It is invariant under the usual moves (4.8)-(4.20) of each of the red and blue diagrams. Due to the graded products on A and B, the weight is also invariant under these same moves where some of the ribbons are red and some are blue. In particular, the weight is unchanged by pulling a red-blue crossing across a critical point, node, or half twist, and satisfies colored versions of the ribbon Reidemeister moves. This sort of representation will prove useful in Section 4.3 when we discuss the state sum for $A \mathbin{\widehat{\otimes}} B$.

Example: Clifford algebras

In this section, we define the Clifford algebras $C\ell_{p,q}\mathbb{R}$ and $C\ell_n\mathbb{C}$ and discuss their associated half twist algebras, from which one can extract the state sum data (C, B, λ, τ). As will be shown in Section 4.3, the significance of these examples is that they generate all theories associated to separable real superalgebras.[15]

[15] We leave open the question of whether there exist pin-TQFTs that do not arise via our state sum construction.

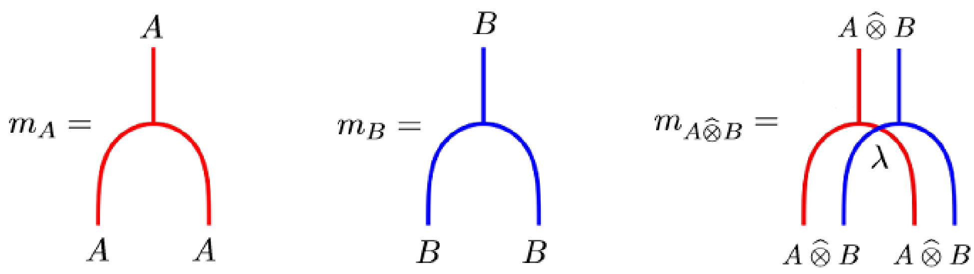

Figure 4.8: A diagrammatic representation of the supertensor product of superalgebras A and B.

The real Clifford algebra $A = C\ell_{p,q}\mathbb{R}$ is generated by anticommuting elements $\gamma_1, \ldots, \gamma_p$ with $\gamma_j^2 = +1$ and $\gamma_{p+1}, \ldots, \gamma_{p+q}$ with $\gamma_j^2 = -1$. It has a basis $\{\gamma_1^{N_1} \cdots \gamma_n^{N_n}\}$ for $N_j = 0, 1$, $n = p + q$. The form $\eta = \epsilon \circ m$ given by the counit

$$\epsilon(\gamma_1^{N_1} \cdots \gamma_n^{N_n}) = \begin{cases} \alpha\, 2^{n/2} & N_j = 0, \forall j \\ 0 & \text{else} \end{cases} \tag{4.37}$$

is Frobenius, symmetric, and special with $R = \alpha\, 2^{-n/2}$. The grading is given by the standard involution

$$\phi(\gamma_1^{N_1} \cdots \gamma_n^{N_n}) = (-1)^{\sum_j N_j} \gamma_1^{N_1} \cdots \gamma_n^{N_n}. \tag{4.38}$$

For the element $x = \gamma_1^{N_1} \cdots \gamma_n^{N_n}$, let $\{x\} = \sum_j N_j$, which is to say $|x| = \{x\} \bmod 2$.

The corresponding half twist algebra is defined on the complexification $C\ell_{p+q}\mathbb{C} = C\ell_{p,q}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, which comes with a real structure T that fixes the γ-basis and complex conjugates its coefficients. Let us define new generators $\Gamma_j = \gamma_j$ for $1 < j \leq p$ and $\Gamma_j = i\gamma_j$ for $p < j \leq p + q$, so that $\Gamma_j^2 = +1$. The basis element $x = \Gamma_1^{N_1} \cdots \Gamma_n^{N_n}$ has T-eigenvalue $(-1)^{|x|_q}$, where $|x|_q = \sum_{i>p} N_i \bmod 2$. It remains to construct the half twist τ. The Clifford algebra has a natural Hermitian structure given by the conjugate transpose map

$$*(\Gamma_1^{N_1} \cdots \Gamma_n^{N_n}) = \Gamma_n^{N_n} \cdots \Gamma_1^{N_1} = (-1)^{\{x\}(\{x\}-1)/2} \Gamma_1^{N_1} \cdots \Gamma_n^{N_n}. \tag{4.39}$$

The composition $t = *T$ fails the condition (4.20); however, it can be corrected, as in Eq. (4.32) with $s = 0$. Define

$$\tau(x) = i^{|x|} t(x) = i^{|x|} (-1)^{|x|_q} (-1)^{\{x\}(\{x\}-1)/2} x = i^{\{x\}} (-1)^{|x|_q} x. \tag{4.40}$$

The general discussion in Section 4.3 shows that the half twist axioms are satisfied.

The complex Clifford algebra $C\ell_n\mathbb{C}$ is generated by anticommuting elements $\gamma_1, \cdots, \gamma_n$ with $\gamma_j^2 = +1$ and central ι with $\iota^2 = -1$.[16] On basis elements $\gamma_1^{N_1} \cdots \gamma_n^{N_n} \iota^M$, the counit is $\alpha\, 2^{(n+2)/2}$ if $N_j = M = 0$ and 0 otherwise. The form $\eta = \epsilon \circ m$ is Frobenius, symmetric, and special with $R = \alpha\, 2^{-n/2}$. The central element ι is ϕ-even, while the γ_j are ϕ-odd, so $|x| = \{x\} \bmod 2$ where $\{x\} = \sum_j N_j$. The complexification $\mathbb{C}\ell_n\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has real structure T that fixes the γ_j and ι. The structure $*$ is again given by conjugate transposition. According to (4.32) with $s(x) = M$, the half twist is the composition $\tau(x) = (-1)^M i^{\{x\}} x$.

State sum for the Arf-Brown-Kervaire TQFT

The pin state sum construction discussed in Section 4.2 amounts to choosing a discretization of a pin surface M, building an associated ribbon diagram, and performing a weighted sum over colorings of the ribbon diagram. The state sum associated to a separable real superalgebra has the special property that it can be written as a sum over colorings *of the graph* dual to the triangulation of M. These colorings are a special type of coloring of the ribbon diagram where all segments of a ribbon from node to node have the same label, as in Figure 4.9. A pin state sum localizes to these colorings if the amplitudes for all other colorings vanish. This means that B is symmetric and there is a basis of τ eigenstates in which ${\lambda_{ab}}^{cd} = \lambda(a,b)\delta_a^d \delta_b^c$ for some values $\lambda(a,b) \in \mathbb{C}$. By (4.12) and (4.15), $\lambda(a,b) = \lambda(b,a) \in \{\pm 1\}$, and by definition of the full twist $\lambda(a,a) = (-1)^{|a|}$. The half twist algebra associated to separable real superalgebra satisfies these conditions with $\lambda(a,b) = (-1)^{|a||b|}$. The collection of edges labeled by ϕ-odd basis elements forms a 1-chain x with $\mathbb{Z}/2$ coefficients for the triangulation of M. Since the product m is ϕ-equivariant (4.24), a coloring contributes zero amplitude to the state sum unless the number of odd labels surrounding each node of the graph is even; that is, unless x is a cycle. Thus the sum over colorings reduces to a sum over cycles x:

$$\mathcal{Z} = \sum_{x \in C_1(M;\mathbb{Z}/2)} \mathcal{Z}(x). \tag{4.41}$$

Consider the half twist algebra A corresponding to $C\ell_{1,0}\mathbb{R}$. It is spanned by 1 and the ϕ-odd generator Γ with $\Gamma^2 = +1$. In this basis, the tensor B_{ab} is $\alpha\sqrt{2}\,\delta_{ab}$, while $C_{abc} = {C_{ab}}^d B_{dc}$ is $\alpha\sqrt{2}$ if an $|a| + |b| + |c| = 0 \mod 2$ and 0 otherwise. The half twist has $\tau(1) = 1$, $\tau(\Gamma) = i\Gamma$. The constant R is $\alpha/\sqrt{2}$.

[16] This algebra is graded-isomorphic to one with $\tilde{\gamma}_j^2 = -1$ for some j by the identification $\tilde{\gamma}_j = \gamma_j \iota$.

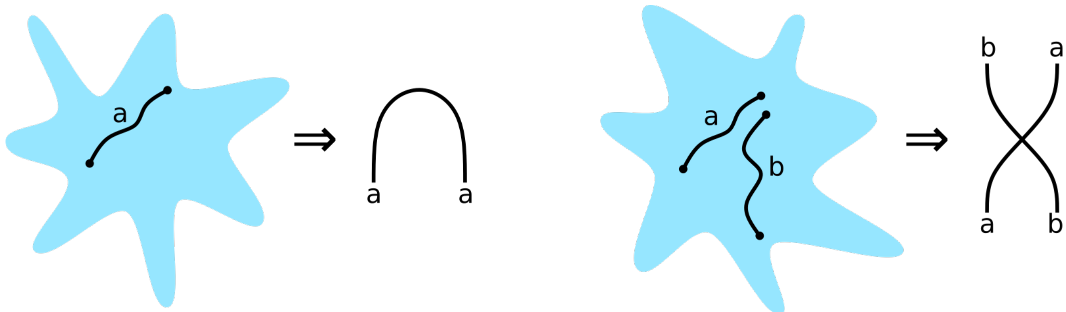

Figure 4.9: If B is symmetric, any coloring of the cap that has nonzero amplitude arises from a coloring of an edge in the ribbon graph. Likewise, if λ is of the form $\lambda : a \otimes b \mapsto \lambda(a,b) b \otimes a$, any coloring of the crossing that has nonzero amplitude arises from a coloring of two edges in the ribbon graph.

Each cycle x is represented by a collection $\{\gamma_i\}_i$ of disjoint loops in the graph. Let us first consider the case of a single loop γ. Form a ribbon diagram and assign a weight to γ using the data of the half twist algebra. Without loss of generality, take the legs of each C to point downward and those of each B upward. The tensors C_{abc} and B^{ab} contribute $\alpha\sqrt{2}$ and $(\alpha\sqrt{2})^{-1}$, respectively, since there are an even number of Γ labels at each node, cap, and cup. Since the number of C's is the number $|V|$ of vertices of the graph and the number of B's is the number $|E|$ of edges, these contributions give an overall factor of $(\alpha\sqrt{2})^{|V|-|E|}$. Each half twist traversed by γ contributes i, while each self-crossing of γ contributes ${\lambda_{\Gamma\Gamma}}^{\Gamma\Gamma} = -1$. Therefore, the contribution of γ to the state sum is $i^{\tilde{q}(\gamma)}$, where $\tilde{q}$ counts the number of half twists plus twice the number of crossings. It was observed in Section 4.1 Eq. (4.4) that this $\tilde{q}$ is the quadratic enhancement associated to the pin structure on M. Now allow for multiple loops. If the images of distinct loops intersect, they must do so at an even number of points, so the factor due to their crossing vanishes. The contribution to the state sum is $i^{\sum_j \tilde{q}(\gamma_j)}$. Since the loops are disjoint and so have intersection number zero, it follows from (4.2) that the exponent is $\sum_j \tilde{q}(\gamma_j) = q(x)$, the quadratic enhancement evaluated on the cycle x associated to $\{\gamma_j\}_j$. The contributions of two homologous chains differ by that of a boundary, which must be $i^{q(x)} = 1$. This means that the sum over x reduces to a sum over homology classes $[x]$ times the number of boundaries. This number is $2^{|F|-1}$ where $|F|$ is the number of faces of the graph.[17] The full state sum is

$$\begin{aligned}
\mathcal{Z}_{C\ell_{1,0}\mathbb{R}}(M,s) &= \left(\alpha/\sqrt{2}\right)^{|F|}\left(\alpha\sqrt{2}\right)^{|V|-|E|}\sum_{x\in C_1(M;\mathbb{Z}/2)} e^{i\pi q_s(x)/2} \\
&= \frac{\alpha^{\chi(M)}}{\sqrt{2^{2-\chi(M)}}}\sum_{[x]\in H_1(M;\mathbb{Z}/2)} e^{i\pi q_s([x])/2} \\
&= \alpha^{\chi(M)}\mathrm{ABK}(M,s),
\end{aligned} \tag{4.42}$$

since $|V|-|E|+|F|$ is the Euler characteristic $\chi(M)$ and $2^{2-\chi(M)} = |H_1(M;\mathbb{Z}/2)|$.

Using the expressions (4.26) and (4.27), we find $\mathcal{A}^{NS}_{C\ell_{1,0}\mathbb{R}} = \mathbb{C}^{1|0}$, spanned by 1, while $\mathcal{A}^{R}_{C\ell_{1,0}\mathbb{R}} = \mathbb{C}^{0|1}$, spanned by Γ. In other words, the NS sector is even (as always), while the R sector is odd (unlike the trivial theory).

Here is a good place to discuss the theory associated to the real superalgebra $C\ell_1\mathbb{C}$. It is convenient to work in a basis of complex central idempotents $E_\pm = (1 \pm i\imath)/2$ and elements $\gamma E_\pm$. In this basis, B_{ab} is $\alpha\sqrt{2}\,\delta_{ab}$, while C_{abc} vanishes if the three $\pm$ indices do not agree or if there are an odd number of γ's and is otherwise $\alpha\sqrt{2}$. The half twist exchanges E_+ with E_- and γE_+ with γE_- while multiplying the latter two by i. This means that, if any loop in the ribbon diagram has an odd number of half twists, there is no way to color the edges such that the amplitude is nonzero. This happens if and only if M is nonorientable; thus, the partition function vanishes on nonorientable surfaces. For orientable surfaces, it is always possible to remove all half twists from the ribbon diagram. Then, for colorings with nonzero amplitude, either all of the edges are labeled by $E_+, \gamma E_+$ or they are all labeled by $E_-, \gamma E_-$. In each case, such colorings are given by disjoint loops labeled by γE with all other edges labeled by E. As above, these configurations contribute factors of $i^{q(x)}$. The contributions of the B and C tensors are the same as before. In total,

$$\mathcal{Z}_{C\ell_1\mathbb{C}}(M,s) = \begin{cases} 2\,\alpha^{\chi(M)}\mathrm{Arf}(M,s) & M \text{ orientable} \\ 0 & M \text{ nonorientable} \end{cases} \tag{4.43}$$

The factor of 2 comes from the equal contributions of the $E_+, \gamma E_+$ sector and the $E_-, \gamma E_-$ sector, and Arf is the name for the ABK invariant restricted to orientable surfaces. One may compute the state spaces $\mathcal{A}^{N}_{C\ell_1\mathbb{C}}S = \mathbb{C}^{2|0}$, spanned by 1 and $\imath$, and $\mathcal{A}^{R}_{C\ell_1\mathbb{C}} = \mathbb{C}^{0|2}$, spanned by γ and $\gamma\imath$.

The vanishing of the partition function on nonorientable surfaces reflects the fact that the time reversal symmetry of the corresponding lattice model has been broken.

[17] Assuming M is connected, the boundary map on 2-cells has a two element kernel.

This interpretation is also compatible with the two dimensional state spaces, which appear as ground state degeneracies in the lattice model.

Decomposability, stacking, and Morita equivalence

A TQFT $\mathcal{Z}$ is said to be *decomposable* if there exist TQFTs $\mathcal{Z}_1, \mathcal{Z}_2$ such that $\mathcal{Z} \simeq \mathcal{Z}_1 \oplus \mathcal{Z}_2$ on all spaces and cobordisms. The previous subsection demonstrated how the data of a separable real superalgebra A defines a pin TQFT $\mathcal{Z}_A$. We now argue that if A decomposes as $A_1 \oplus A_2$ the TQFT $\mathcal{Z}_A$ decomposes as $\mathcal{Z}_{A_1} \oplus \mathcal{Z}_{A_2}$. This result motivates us to restrict our attention to indecomposable algebras.

It is clear that the circle state spaces, found in Section 4.2 to be certain twisted centers of A, decompose as $\mathcal{A}_{NS} = \mathcal{A}_{NS,1} \oplus \mathcal{A}_{NS,2}$ and $\mathcal{A}_R = \mathcal{A}_{R,1} \oplus \mathcal{A}_{R,2}$. Thus $\mathcal{Z}(S) \simeq \mathcal{Z}_1(S) \oplus \mathcal{Z}_2(S)$. A coloring of a ribbon diagram by elements in a basis of $A_1 \oplus A_2$ has zero amplitude unless either all of the labels (internal and external) are from A_1 or they are all from A_2. This is the case because it holds for the building blocks C, B, and τ. Therefore, $\mathcal{Z}$ acts as $\mathcal{Z}_1(M)$ on the subspaces $\mathcal{Z}_1(S)$ and as $\mathcal{Z}_2(M)$ on $\mathcal{Z}_2(S)$, so $\mathcal{Z}(M) \simeq \mathcal{Z}_1(M) \oplus \mathcal{Z}_2(\mathrm{M})$, as claimed. In particular, when M is a closed surface, $\mathcal{Z}(M) = \mathcal{Z}_1(M) + \mathcal{Z}_2(M) \in \mathbb{C}$.

There is another operation on pin TQFTs called *stacking*. The result of stacking $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is the theory defined by the graded tensor product $\mathcal{Z} \simeq \mathcal{Z}_1 \mathbin{\widehat{\otimes}} \mathcal{Z}_2$. We now argue that $\mathcal{Z}_{A \widehat{\otimes} B} \simeq \mathcal{Z}_A \mathbin{\widehat{\otimes}} \mathcal{Z}_B$.

Recall that $\mathcal{A}_{NS} = \operatorname{im} p$ (4.26) and $\mathcal{A}_R = \operatorname{im} n$ (4.27). If $a \in \mathcal{A}_{NS}, b \in \mathcal{B}_{NS}$, then for all $a \in A, b \in B$,

$$(a \mathbin{\widehat{\otimes}} b)(a' \mathbin{\widehat{\otimes}} b') = (-1)^{ba'} aa' \mathbin{\widehat{\otimes}} bb' = (-1)^{ba'+aa'+bb'} a'a \mathbin{\widehat{\otimes}} b'b = (-1)^{(a+b)(a'+b')}(a' \mathbin{\widehat{\otimes}} b')(a \mathbin{\widehat{\otimes}} b), \tag{4.44}$$

so $a \mathbin{\widehat{\otimes}} b \in \operatorname{im} p_{A \widehat{\otimes} B}$. The same argument shows the converse. Similarly, if $a \in \mathcal{A}_R, b \in \mathcal{B}_R$,

$$(a \mathbin{\widehat{\otimes}} b)(a' \mathbin{\widehat{\otimes}} b') = (-1)^{(a+b)(a'+b')+(a'+b')}(a' \mathbin{\widehat{\otimes}} b')(a \mathbin{\widehat{\otimes}} b). \tag{4.45}$$

Therefore, $\mathcal{Z}_{A \widehat{\otimes} B}(S^1_\alpha) \simeq \mathcal{Z}_A(S^1_\alpha) \mathbin{\widehat{\otimes}} \mathcal{Z}_B(S^1_\alpha)$ for $\alpha = \mathrm{NS}, \mathrm{R}$. On a one dimensional closed pin manifold,

$$\mathcal{Z}_{A \widehat{\otimes} B}(S) = \widehat{\bigotimes}_i \mathcal{Z}_{A \widehat{\otimes} B}(S^1_i) = \widehat{\bigotimes}_i \mathcal{Z}_A(S^1_i) \mathbin{\widehat{\otimes}} \mathcal{Z}_B(S^1_i), \tag{4.46}$$

which is isomorphic to $\mathcal{Z}_A(S) \mathbin{\widehat{\otimes}} \mathcal{Z}_B(S)$ by a sign arising from the rule (4.29). Therefore $\mathcal{Z}_{A \widehat{\otimes} B} \simeq \mathcal{Z}_A \mathbin{\widehat{\otimes}} \mathcal{Z}_B$ on the level of state spaces. Note that this argument demonstrates that the supertensor product, rather than the ordinary tensor product, is the correct stacking operation.

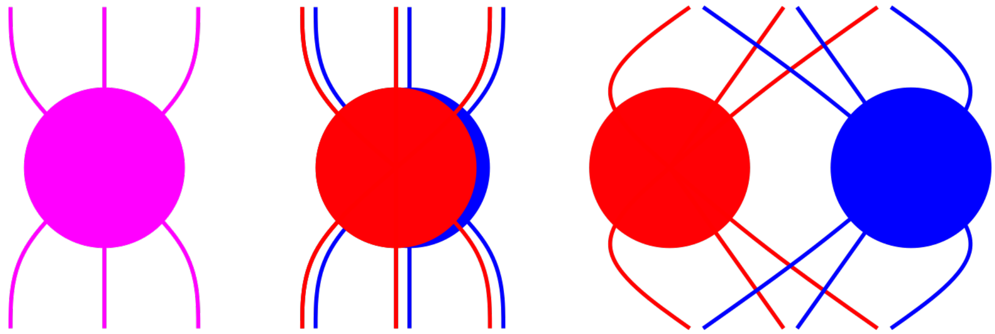

Figure 4.10: A ribbon diagram for the supertensor product algebra $A \widehat{\otimes} B$ (purple) may be split into a ribbon diagram for A (red) superimposed on a ribbon diagram for B (blue). Then they may be separated.

The state sum for $\mathcal{Z}_{A\widehat{\otimes}B}$ is given by a sum over colorings of a ribbon diagram by basis elements $e_a \widehat{\otimes} f_i$. One may represent these colorings as follows. Add to the ribbon diagram (in red) a copy of itself (in blue), shifted a small distance in the x-direction, as in Figure 4.10. The weight of this red-blue diagram, discussed in Section 4.3, reproduces the weight (4.36) at nodes as well as the correct weights for the other building blocks in $A \widehat{\otimes} B$. Now observe that the two diagrams may be pulled apart. This is allowed due to red-blue versions of the half twist axioms leaving the weight invariant. If M is closed, we are done, as the weights for the $A \widehat{\otimes} B$ theory are the products of those of the A and B theories. If M has cut boundaries, we may assume that each connected component of the boundary has a single leg. Pulling apart the diagrams costs signs due to the crossings of these external legs, but these signs are precisely those in the isomorphism $\mathcal{Z}_{A\widehat{\otimes}B}(S) \simeq \mathcal{Z}_A(S) \widehat{\otimes} \mathcal{Z}_B(S)$. We conclude that $\mathcal{Z}_{A\widehat{\otimes}B} \simeq \mathcal{Z}_A \widehat{\otimes} \mathcal{Z}_B$ on the level of amplitudes as well.

Two indecomposable separable real superalgebras A, B are said to be *Morita equivalent* if they are related by stacking with a matrix algebra; that is, $B \simeq A \widehat{\otimes} \mathbb{R}(p|q)$ for some $p, q \in \mathbb{N}$. It is easy to see that the operation of stacking is compatible with Morita equivalence, so that one may speak of stacking Morita classes: $[A] \widehat{\otimes} [B] \simeq [A \widehat{\otimes} B]$. It will be shown in Section 4.3 that the pin TQFT corresponding to the algebras $\mathbb{R}(p|q)$, with $\alpha = 1$, is the unit in the monoid of pin TQFTs under stacking; in particular, it has state spaces $\mathcal{Z}(S^1_{NS}) = \mathcal{Z}(S^1_R) = \mathbb{C}^{1|0}$ and partition function $\mathcal{Z}(M) = 1$ for any closed pin surface M.

We conclude that Morita equivalent algebras $A \sim B$ define the same TQFT, $\mathcal{Z}_A \simeq \mathcal{Z}_B$, up to an Euler term. This motivates us to focus on certain convenient representatives from each Morita class. There are ten Morita classes of simple real superalgebras. Eight of them are central simple and form a group $\mathbb{Z}/8$ under stacking. The real Clifford superalgebra $C\ell_{p,q}\mathbb{R}$ – discussed in Section 4.3 – lives in Morita class number $p-q$. The remaining two Morita classes are non-central and do not have inverses under stacking. They are represented by the complex Clifford superalgebras $C\ell_n\mathbb{C}$, with n mod 2 being Morita invariant.

It is worth emphasizing that the state sum construction takes as input a real superalgebra. Forgetting the graded structure identifies many of these, as does complexifying and forgetting the real structure.

In light of the result of Section 4.3 that the $C\ell_{1,0}\mathbb{R}$ theory has partition function ABK, our discussion of stacking and Morita equivalence means that the algebra $C\ell_{p,q}\mathbb{R}$ has partition function ABK^{p-q}.

Let us make one additional comment about decomposability. The converse – that indecomposability of A implies that of $\mathcal{Z}_A$ – of the statement above is not true in the generality of Eq. (4.32); however, it holds for the examples considered in Section 4.3 due to our careful choices of the grading s. The careful choice of s for generic A is the following. Decompose A_r as a direct sum of Clifford algebras tensored with matrix algebras and choose $s=0$ on each real Clifford algebra, $s=M$ on each complex Clifford algebra, and $s=0$ on each matrix algebra. The complex algebra A splits into blocks by orthogonal central idempotents E_i. With these choices, τ fixes an E_i if and only if T does.[18] The meaning of T fixing an E_i is that A_r decomposes along this block, while the meaning of τ fixing an E_i is that the state sum decomposes. This is because, for colorings with nonzero weight, each of the three edges at a node must be colored in a single block, and so, unless τ exchanges blocks between nodes, the coloring of all edges of the ribbon diagram must be in a single block.

Invertible pin TQFTs

An invertible pin TQFT is one whose state spaces are one dimensional and whose partition functions on closed pin spacetimes are nonzero. Invertible theories have a special property: not only are they completely determined by their partition functions on closed pin manifolds, these partition functions must be a cobordism invariant – a power of the ABK invariant – times an Euler term α^χ for $\alpha\in\mathbb{C}^\times$ (Freed

[18] In the example of $C\ell_1\mathbb{C}$, the elements $E_\pm$ are fixed by neither τ nor T when $s=M$ but are fixed by τ when $s=0$.

and G. W. Moore, 2006) (see also (Yonekura, 2018), in the unitary case).[19] In particular, if $\mathcal{Z}(S^2) = \alpha^2 = 1$,[20] the partition functions are cobordism-invariant and multiplicative under the appropriate notion of connect sum. Consider the unitary case, where $\alpha \in \mathbb{R}_{>0}$. Since $\mathrm{ABK}^k(\mathbb{R}P^2_1) = \exp(k\pi i/4)$ and $\mathrm{ABK}^8 = 1$, the partition function on $\mathbb{R}P^2_1$ (alternatively, $\mathbb{R}P^2_7$) determines k and therefore the full pin TQFT.[21] In the following, we will compute the partition functions of $\mathbb{R}P^2_1$ for the theories associated to the real superalgebras $\mathbb{R}(p|q)$ and $C\ell_{p,q}\mathbb{R}$ and find that they are $+1$ and $\exp((p-q)\pi i/4)$, respectively, up to Euler terms. Since these theories are invertible and unitary, this demonstrates that the state sum for matrix algebras is trivial – as claimed in Section 4.3 – while that for $C\ell_{p,q}\mathbb{R}$ is the ABK^{p-q} theory – in agreement with the findings of Section 4.3.

A ribbon diagram for $\mathbb{R}P^2_1$ is depicted in Figure 4.11. It evaluates to

$$\mathcal{Z}(\mathbb{R}P^2_1) = R\,\eta(\mathbb{1} \otimes \tau)\eta^{-1}. \tag{4.47}$$

The matrix algebra $\mathbb{R}(p|q)$ is spanned by a basis of matrices e_{ij} with $0 < i, j \leq p + q = n$. The trace form is

$$\eta(e_{ij}, e_{kl}) = \alpha\,\mathrm{Tr}\left[e_{ij}e_{kl}\right] = \alpha\,\delta_{jk}\delta_{il}, \qquad \eta^{-1} = \alpha^{-1}\sum_{i,j} e_{ij} \otimes e_{ji}, \qquad R = \alpha/n. \tag{4.48}$$

Let $|i|$ be 1 if $i > p$ and 0 otherwise. The grading on $\mathbb{R}(p|q)$ is given by $|e_{ij}| = |i| + |j| - |i||j|$. T acts trivially in this basis, and $\mathbb{R}(p|q)$ has a Hermitian structure given by conjugate transposition: $*e_{ij} = e_{ji}$. Therefore, by the discussion in Section 4.3, the half twist is $\tau(e_{ij}) = i^{|i|+|j|+|i||j|}e_{ji}$. Then compute

$$\mathcal{Z}_{\mathbb{R}(p|q)}(\mathbb{R}P^2_1) = \frac{1}{n}\sum_{i,j}\eta(e_{ij} \otimes \tau(e_{ji})) = \frac{1}{n}\sum_{i,j} i^{|i|+|j|+|i||j|}\eta(e_{ij} \otimes e_{ij}) = \frac{\alpha}{n}\sum_{i,j} i^{|i|+|j|+|i||j|}\delta_{ij} = \alpha, \tag{4.49}$$

as claimed. Meanwhile $C\ell_{p,q}\mathbb{R}$ was discussed in Section 4.3. Let $|x|_p = |x| - |x|_q$

[19] Invertible pin TQFTs *do not* generate a complete set of pin diffeomorphism invariants, as the bounding torus and bounding Klein bottle cannot be distringuished: they have both ABK and χ trivial.

[20] When $\alpha = \pm 1$, the Euler term $(\pm 1)^\chi = (\pm 1)^{w_2} = (\pm 1)^{w_1^2} = \mathrm{ABK}^{2\mp 2}$ is cobordism-invariant.

[21] If unitarity is not assumed, α may be an eighth root of unity. Then, in order to determine the full theory, one must also evaluate the partition function on a Klein bottle with one of the nontrivial pin structures.

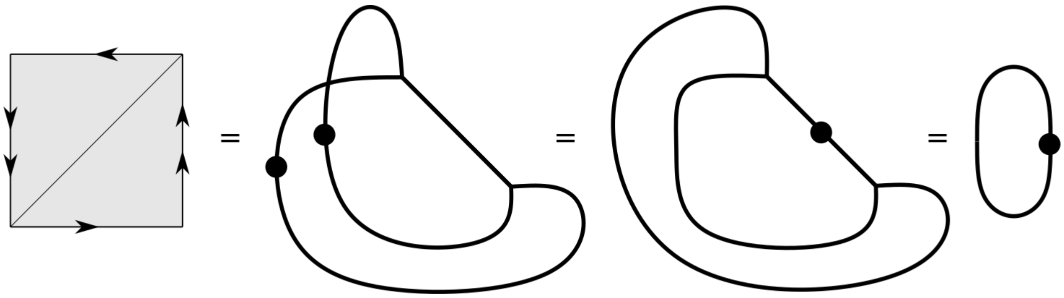

Figure 4.11: A ribbon diagram for $\mathbb{R}P^2$ is obtained from the graph dual to a triangulation of its fundamental square and then simplified using the moves (4.18) and (4.11).

mod 2. Then compute

$$\begin{aligned}
\mathcal{Z}_{C\ell_{p,q}\mathbb{R}}(\mathbb{R}P^2_1) &= \frac{1}{2^{(p+q)}} \sum_{N_i} \eta(\mathbb{1} \otimes \tau) \left(\Gamma_1^{N_1} \cdots \Gamma_n^{N_n} \otimes *(\Gamma_1^{N_1} \cdots \Gamma_n^{N_n}) \right) \\
&= \frac{1}{2^{(p+q)}} \sum_{N_i} i^{|x|} (-1)^{|x|_q} \eta \left(\Gamma_1^{N_1} \cdots \Gamma_n^{N_n} \otimes \Gamma_1^{N_1} \cdots \Gamma_n^{N_n} \right) \\
&= \frac{\alpha}{2^{(p+q)/2}} \sum_{N_i} i^{|x|} (-1)^{|x|_q} (-1)^{\{x\}(\{x\}-1)/2} \\
&= \frac{\alpha}{2^{(p+q)/2}} \sum_{N_i} i^{\{x\}} (-1)^{|x|_q} \qquad (4.50) \\
&= \frac{\alpha}{2^{(p+q)/2}} \sum_{N_i} i^{|x|_p} (-i)^{|x|_q} \\
&= \alpha \left(\frac{1}{2^{p/2}} \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} i^k \right) \left(\frac{1}{2^{q/2}} \sum_{l=0}^{q} \begin{pmatrix} q \\ l \end{pmatrix} (-i)^l \right) \\
&= \alpha \exp((p-q)\pi i/4).
\end{aligned}$$

This completes our argument. As a consistency check, one may evaluate the state sums on other closed pin manifolds and verify that they yield powers of the ABK invariant. This was done in Ref. (Barrett and Tavares, 2015) for orientable pin (spin) surfaces. They show that $C\ell_{1,0}\mathbb{R}$ yields partition function $\mathcal{Z}(M^{\text{or}}) \sim \text{Arf}(M^{\text{or}}) = \text{ABK}(M^{\text{or}}) \in \{\pm 1\}$.

Chapter 5

FREE AND INTERACTING SRE PHASES OF FERMIONS: BEYOND THE TEN-FOLD WAY

Chen, Y-A. et al. (2018). "Free and interacting short-range entangled phases of fermions: beyond the ten-fold way". In: arXiv: 1809.04958.

Forward

It is well-known that sufficiently strong interactions can destabilize some SPT phases of free fermions, while others remain stable even in the presence of interactions. It is also known that certain interacting phases cannot be realized by free fermions. This chapter systematically studies both of these phenomena in low dimensions and determine the map from free to interacting SPT phases for an arbitrary unitary symmetry G. In particular, in dimension zero and one we describe precisely which SPT phases can be realized by free fermions. We show that in dimension three there are no non-trivial free fermionic SPT phases with a unitary symmetry. We also describe how to compute invariants characterizing interacting phases for free band Hamiltonians with symmetry G (in any dimension) using only representation theory.

Background and Overview

It is well-known by now that short-range-entangled (SRE) phases of free fermions on a lattice can be classified using K-theory (Kitaev, 2009b), or equivalently using the topology of symmetric spaces (Schnyder et al., n.d.; Ryu et al., 2010). Originally, the classification was done in the framework of the ten-fold way, where the only allowed symmetries are charge conservation, time-reversal, particle-hole symmetry, or a combination thereof. But the K-theory framework can also be extended to systems with more general symmetries, both on-site and crystallographic (Teo, Fu, and Kane, 2008; Mong, Essin, and J. E. Moore, 2010; Fu, 2011; Kruthoff et al., 2017; Freed and G. W. Moore, 2013; Ando and Fu, 2015). The answer is encoded in an abelian group, with the group operation corresponding to the stacking of phases.

When interactions of arbitrary strength are allowed, the classification of SRE phases of fermions is much more complicated, but in low dimensions[1] the answer is known

[1] An answer in an arbitrary number of dimensions was conjectured in (Kapustin, 2014b).

for an arbitrary finite on-site symmetry G (X. Chen, Gu, and Wen, 2010; Fidkowski and Kitaev, 2010; Gu and Wen, 2014; Bhardwaj, Gaiotto, and Kapustin, 2017; C. Wang, Lin, and Gu, 2017; Kapustin, Turzillo, and You, 2018; Q. Wang and Gu, 2018; Kapustin and Thorngren, 2017). It is also given by an abelian group, where the group operation is given by stacking.

It is natural to ask how the free and interacting classifications are related. Every free fermionic SRE phase can be regarded as an interacting one, and this gives a homomorphism from the abelian group of free SRE phases to the abelian group of interacting ones (with the same symmetry). In general, this homomorphism is neither injective not surjective. The homomorphism may have a non-trivial kernel because some non-trivial free SRE phases can be destabilized by interactions. It may fail to be onto because some interacting SRE phases cannot be realized by free fermions. The simplest example of the former phenomenon occurs in 1d systems of class BDI (Fidkowski and Kitaev, 2010): while free SRE phases in this symmetry class are classified by $\mathbb{Z}$, the interacting ones are classified by $\mathbb{Z}_8$. An example of the latter phenomenon apparently occurs in dimension 6, where the cobordism classification of systems in class D predicts $\mathbb{Z}\times\mathbb{Z}$, while the free phases in the same symmetry class are classified by $\mathbb{Z}$.

The main goal of this paper is to study both phenomena more systematically in low dimensions. In particular, we will see that already in zero and one dimensions there exist fermionic SRE phases protected by a unitary symmetry which cannot be realized by free fermions.

To address such questions, it is very useful to have an efficient way to compute the "interacting" invariants of any given band Hamiltonian with any on-site symmetry G. One of the results of our paper is the computation of these invariants for arbitrary 0d and 1d band Hamiltonians. We also propose a partial answer in the 2d case. In the 1d case, we identify one of the invariants as a charge-pumping invariant.

Another goal of this paper is to describe the classification of free fermionic SRE phases with a unitary on-site symmetry G in arbitrary dimensions. We show that in any dimension representation-theoretic considerations reduce the problem to classifying systems of class D, A, and C. The solution of the latter problem is well-known. The key step in the derivation is reduction from a general symmetry G to a tenfold symmetry class. Such a reduction is not new and has been described in detail in Ref. (Heinzner, Huckleberry, and Zirnbauer, 2005). But since the authors of (Heinzner, Huckleberry, and Zirnbauer, 2005) work with complex fermions, and for our purposes it is more convenient to use Majorana fermions, we give a new proof of the reduction.

When we consider systems with symmetries other than the ten-fold way symmetries, it is no longer useful to adopt the ten-fold way nomenclature. For example, a fermionic system with a $U(1) \times G$ symmetry, where the generator of $U(1)$ is the fermion number, can equally well be regarded as a symmetry-enriched class A system and as a symmetry-enriched class D system. On the other hand, the distinction between unitary and anti-unitary symmetries remains important. If we denote by $\hat{G}$ the total symmetry group (including the fermion parity $\mathbb{Z}_2^F$), this information is encoded in a homomorphism

$$\rho : \hat{G} \to \mathbb{Z}_2. \tag{5.1}$$

We also need to specify an element $P \in \hat{G}$ which generates the subgroup $\mathbb{Z}_2^F$. This elements satisfies $P^2 = 1$ and is central.[2] Since P is unitary, we must have $\rho(P) = 1$ (here we identify $\mathbb{Z}_2$ with the set $\{1, -1\}$). The symmetry of a fermionic system is encoded in a triplet $(\hat{G}, P, \rho)$. For example, class D systems correspond to a triplet $(\mathbb{Z}_2, -1, \rho_0)$, where ρ_0 is the trivial homomorphism (sends the whole $\hat{G}$ to the identity), while class A systems correspond to a triplet $(U(1), -1, \rho_0)$. In this paper we study only systems with unitary symmetries, i.e. we always set $\rho = \rho_0$. We allow $\hat{G}$ to be an arbitrary compact Lie group, with the exception of section 3.3, where $\hat{G}$ is assumed to be finite.

A mathematically sophisticated reader might notice that many of our results on the classification of free systems can be naturally expressed in terms of equivariant K-theory. The connection between free systems with an arbitrary (not necessarily on-site or unitary) symmetry and equivariant K-theory has been studied in detail in (Freed and G. W. Moore, 2013). However, in this paper we prefer to use more elementary methods, such as representation theory of compact groups. This has the advantage of making clear the physical meaning of K-theory invariants, which is crucial for the purpose of comparison with interacting systems.

The content of the paper as follows. In Section 5.1, we derive the classification of free SRE phases with a unitary symmetry $\hat{G}$ in an arbitrary number of dimensions. In particular, we show that for $d = 3$ all such phases are trivial. In Section 5.2 we describe the map from free to interacting SRE phases for $d = 0, 1,$ and 2. Appendices D.1 and D.2 contain some mathematical background. In Appendix D.3 we show that

[2]Centrality is equivalent to the assumption that all symmetries are bosonic, i.e. do not change fermions into bosons or vice versa.

one of the invariants for free 1d SPT systems can be interpreted as a charge-pumping invariant.

5.1 Free fermionic systems with a unitary symmetry

Reduction to the ten-fold way

In this section we show that the classification of free fermionic SRE systems with a unitary symmetry $\hat{G}$ in dimension d reduces to the classification of systems of class D, A, and C in the same dimension. For simplicity, we first show this for 0d systems. The general case is easily deduced from the 0d case. The group G is assumed to be a compact Lie group. This includes finite groups as a special case.

Consider a general quadratic 0d Hamiltonian

$$H = \frac{i}{2} A_{IJ} \Gamma^I \Gamma^J, \tag{5.2}$$

where A_{IJ}, $I, J = 1, \ldots, 2N$ is a real skew-symmetric matrix and Γ^I are Majorana fermions satisfying

$$\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}. \tag{5.3}$$

Suppose the Hamiltonian is invariant under a linear action of a group $\hat{G}$:

$$\hat{g} : \Gamma^I \mapsto \widehat{R}(\hat{g})^I_J \Gamma^J. \tag{5.4}$$

This defines a homomorphism $\widehat{R} : \hat{G} \to O(2N)$.

Let us decompose $\widehat{R}$ into real irreducible representations of $\hat{G}$. Suppose the irreducible representation r_α enters with multiplicity n_α. The sum of all these copies of r_α will be called a block. It is clear that the Hamiltonian can only couple the fermions in the same block, so the matrix A is block-diagonal.

Let us focus on a particular block corresponding to an irreducible real representation r. There are three kinds of real irreducibles which are distinguished by the commutant of the set of matrices $r(\hat{g})$, $\hat{g} \in \hat{G}$ (Bröcker and Dieck, 1985). By Schur's lemma, this commutant must be a real division algebra, so we have irreducibles of type $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, corresponding to the algebras of real numbers, complex numbers, and quaternions. The corresponding block A_r can be thought of as an operator on the space $r \otimes \mathbb{R}^n$, where n is the multiplicity of r. $\hat{G}$-invariance of the Hamiltonian implies that this operator commutes with the $\hat{G}$-action. The resulting constraint on A_r depends on the type of the representation r.

If r is of $\mathbb{R}$-type, only scalar matrices commute with all $r(\hat{g})$. (Hence $r \otimes_{\mathbb{R}} \mathbb{C}$ is a complex irreducible representation of $\hat{G}$. This is an equivalent characterization of

$\mathbb{R}$-type irreducibles.) Hence A_r must have the form

$$A_r = 1 \otimes Y_r, \tag{5.5}$$

where Y_r is a real skew-symmetric matrix of size $n \times n$. There are no further constraints on Y, so such a block can be thought of as describing d$im\, r$ copies of a system of class D.

If r is of $\mathbb{C}$-type, then the algebra of matrices commuting with all $r(\hat{g})$ is spanned by 1 and an element J satisfying $J^2 = -1$. Since J^T must be proportional to J, this means that $J^T = -J$. The most general $\hat{G}$-invariant A_r must have the form

$$A_r = 1 \otimes \mathcal{A} + J \otimes \mathcal{B}, \tag{5.6}$$

where $\mathcal{A}$ is skew-symmetric and $\mathcal{B}$ is symmetric. We can equivalently parameterize such a Hamiltonian by a complex Hermitian matrix $h = \mathcal{B} + i\mathcal{A}$. Upon complexification, we can decompose r into eigenspaces of J with eigenvalues $\pm i$. These eigenspaces are complex irreducible representations of $\hat{G}$, and it is clear that they are conjugate to each other. We will denote them q and $\bar{q}$. (An equivalent definition of a $\mathbb{C}$-type representation is that $r \otimes_{\mathbb{R}} \mathbb{C}$ is a sum of two complex irreducible representations q and $\bar{q}$ which are complex-conjugate and inequivalent). The $n \cdot$ d$im\, r$ Majorana fermions can be equivalently described by $\frac{1}{2}n \cdot$ d$im\, r$ complex fermions Ψ^a_k, $a = 1, \ldots, n$, $k = 1, \ldots, \frac{1}{2}$d$im\, r$ satisfying the commutation relations

$$\{\Psi^a_k, \bar{\Psi}^b_l\} = \delta^a_b \delta^l_k. \tag{5.7}$$

In terms of these fermions, the Hamiltonian takes the form

$$H = \sum_{k,a,b} \bar{\Psi}^b_k h^b_a \Psi^a_k. \tag{5.8}$$

Thus a $\mathbb{C}$-type block can be thought of as describing d$im\, q = \frac{1}{2}$d$im\, r$ copies of a system of class A.

If r is of $\mathbb{H}$-type, then the algebra of matrices commuting with all $r(\hat{g})$ is spanned by 1 and three elements I, J, K which are skew-symmetric and obey the relations

$$I^2 = J^2 = K^2 = -1, \quad IJ = K. \tag{5.9}$$

Accordingly, A_r must have the form

$$X_r = 1 \otimes \mathcal{A} + I \otimes \mathcal{B} + J \otimes \mathcal{C} + K \otimes \mathcal{D}, \tag{5.10}$$

where $\mathcal{A}$ is skew-symmetric and $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are symmetric. Equivalently, we can introduce a Hermitian $2n \times 2n$ matrix

$$Z = \begin{pmatrix} \mathcal{C} + i\mathcal{A} & \mathcal{B} + i\mathcal{D} \\ \mathcal{B} - i\mathcal{D} & -(\mathcal{C} + i\mathcal{A})^T \end{pmatrix}. \tag{5.11}$$

This is the most general Hermitian matrix satisfying the particle-hole (PH) symmetry condition

$$C^\dagger Z^T C = -Z, \tag{5.12}$$

where $C = i\sigma_2 \otimes 1$. Since $C^* C = -1$, such a PH-symmetric system belong to class C.

To make this relationship with class C systems explicit, we again decompose $r \otimes_{\mathbb{R}} \mathbb{C}$ into a pair of complex-conjugate representations q and $\bar{q}$. These two representations are equivalent, with the intertwiner being given by the tensor I. We also can think of I as a non-degenerate skew-symmetric pairing $q \otimes q \to \mathbb{C}$. This implies that d$im\, q$ is divisible by 2 (and hence d$im\, r$ is divisible by four). As in the $\mathbb{C}$-type case, we can describe the system by $n \cdot$ d$im\, q$ complex fermions. However, the presence of an $\hat{G}$-invariant tensor I means that the most general $\hat{G}$-invariant Hamiltonian is

$$H = \bar{\Psi}(1 \otimes h)\Psi + \frac{1}{2}\left(\Psi^T (I \otimes Y)\Psi + h.c.\right), \tag{5.13}$$

where h is a Hermitian matrix, and Y is a complex symmetric matrix. This is a BdG Hamiltonian, which can be re-written in terms of Dirac-Nambu fermions

$$\Phi = \begin{pmatrix} \Psi \\ (I \otimes 1)\bar{\Psi}^T \end{pmatrix}. \tag{5.14}$$

The Dirac-Nambu spinors are defined so that the upper and lower components transform in the same way under $\hat{G}$. They take values in $q \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$, where $\mathbb{C}^2$ is the Dirac-Nambu space. The particle-hole (PH) symmetry acts by

$$C : \Phi \mapsto (I \otimes \sigma_1 \otimes 1)\, \bar{\Phi}^T \tag{5.15}$$

and satisfies $C^2 = -1$. In terms of Dirac-Nambu spinors, the Hamiltonian takes the form

$$H = \bar{\Phi}(1 \otimes Z)\Phi, \tag{5.16}$$

where

$$Z = \frac{1}{2}\begin{pmatrix} h & -Y^\dagger \\ -Y & -h^T \end{pmatrix}. \tag{5.17}$$

Such matrices describe the most general class C system. Thus an $\mathbb{H}$-type block can be thought of as describing $\dim q = \frac{1}{2}\dim r$ copies of a system of class C.

For systems of dimension $d > 0$, the Majorana fermions have an additional index (the coordinate label). Accordingly, all matrices except $r(\hat{g})$ become infinite. However, since the symmetry is on-site, all representation-theoretic manipulations remain valid, and the conclusions are unchanged.

Classification of free SRE phases with a unitary symmetry

We always assume that the generator of $\mathbb{Z}_2^F$ acts on all fermions by negation, i.e.

$$\widehat{R}(P) = -1. \tag{5.18}$$

The same must be true for all irreducible representations r_α which appear with nonzero multiplicity. We will call such irreducible representations *allowed*. The set of all irreducible real representations of a compact group $\hat{G}$ will be denoted $\mathrm{I}rr(\hat{G})$, while the set of all allowed irreducible real representations will be denoted $\mathrm{I}rr'(\hat{G})$. The set of allowed irreducible representations of type K ($K = \mathbb{R}, \mathbb{C}, \mathbb{H}$) will be denoted $\mathrm{I}rr'(\hat{G}, K)$. If $\hat{G} = \mathbb{Z}_2^F \times G$, we can identify $\mathrm{I}rr'(\hat{G}, K)$ with the set $\mathrm{I}rr(G, K)$.

Let us recall the classification of class D, A, and C systems from the periodic table. Here we are listing only the "strong" invariants which do not depend on translational invariance.

	0	1	2	3	4	5	6	7
Class D ($\mathbb{R}$-type)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	
Class A ($\mathbb{C}$-type)	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
Class C ($\mathbb{H}$-type)			$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

These results together with those of the previous subsection allow us deduce the classification of free fermionic SREs with an arbitrary unitary symmetry $\hat{G}$. In the physically interesting dimensions $d \leq 3$,

	free phase classification
$d = 0$	$\oplus_{r\in \mathrm{I}rr'(\hat{G},\mathbb{R})}\mathbb{Z}_2 \times \oplus_{r\in \mathrm{I}rr'(\hat{G},\mathbb{C})}\mathbb{Z}$
$d = 1$	$\oplus_{r\in \mathrm{I}rr'(\hat{G},\mathbb{R})}\mathbb{Z}_2$
$d = 2$	$\oplus_{r\in \mathrm{I}rr'(\hat{G})}\mathbb{Z}$
$d = 3$	trivial

This does not contradict the fact that there are interesting interacting fermionic 3d SREs.

In what follows, an invariant attached to a particular irreducible representation r_α will be denoted ϱ_α. Depending on the spatial dimension and the type of r_α, ϱ_α will take values either in $\mathbb{Z}_2$ or $\mathbb{Z}$. An invariant of a free SRE phases will thus be a "vector" with components ϱ_α. If $\hat{G}$ is finite, then the number of allowed irreducible representations is finite, and the "vector" has a finite length. If $\hat{G}$ is a compact Lie group, the number of allowed irreducible representations may be infinite, and then the space of "vectors" has infinite dimension (although all but a finite number of ϱ_α are zero for a particular SRE phase). These vectors can be interpreted as elements of the (twisted) equivariant K-theory, whose relevance to the classification of gapped band Hamiltonians is explained in (Freed and G. W. Moore, 2013).

The above results can be simplified a bit when $\hat{G}$ is a product of G and $\mathbb{Z}_2^F$. In this case the sums over allowed representations of $\hat{G}$ can be replaced with the sums over all representations of G.

The $\mathbb{Z}$ and $\mathbb{Z}_2$ invariants that appear in K-theory are relative invariants; that is, they detect something non-trivial about the junction between two phases. If one chooses a phase to regard as trivial (typically the phase containing the product state ground state in dimension $d > 0$), the invariant for the junction of a phase $[H]$ with the trivial phase may be regarded as an absolute invariant of $[H]$.

Examples

Let us consider a few examples.

$\hat{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2$. The action of $\mathbb{Z}_2^F$ on fermions is fixed, so we only need to choose the action of the second $\mathbb{Z}_2$. Overall, there are two allowed irreducible representations, both of them of $\mathbb{R}$-type. Thus free phases with this symmetry are classified by $\mathbb{Z}_2 \times \mathbb{Z}_2$ in 0d and 1d, and by $\mathbb{Z} \times \mathbb{Z}$ in 2d.

$\hat{G} = \mathbb{Z}_4$, where the $\mathbb{Z}_2$ subgroup is fermion parity. $\mathbb{Z}_4$ has three irreducible real representations, of dimensions 1, 1, and 2, but only the 2-dimensional representation is allowed. It is of $\mathbb{C}$-type, hence free 0d and 2d phases with this symmetry are classified by $\mathbb{Z}$, while those in 1d have a trivial classification.

$\hat{G} = \mathbb{Z}_2^F \times \mathbb{Z}_4$. Allowed irreducible representations of $\hat{G}$ are equivalent to the 1, 1, and 2 dimensional irreducible representations of $G = \mathbb{Z}_4$. Therefore the 0d classification is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$, the 1d classification is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the 2d classification is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

$\hat{G} = U(1)$, with the obvious $\mathbb{Z}_2^F$ subgroup. There is one real representation for every non-negative integer, but only odd integers are allowed. All of these representations are of $\mathbb{C}$-type, so free 0d phases with this symmetry are classified by $\mathbb{Z}^{\mathbb{N}}$, that is, by a product of countably many copies of $\mathbb{Z}$. Note that although the symmetry is the same as for class A insulators, the classification is different. This is because it is usually assumed that complex fermions have charge 1 with respect to $U(1)$, while we allow arbitrary odd charges. In 1d, there are no free phases with this symmetry, while in 2d there is again a $\mathbb{Z}^{\mathbb{N}}$ classification.

$\hat{G} = SU(2)$ with $\mathbb{Z}_2^F$ being the center. In this case, only representations of half-integer spin are allowed. All these representations are of $\mathbb{H}$-type, hence all free Hamiltonians with this symmetry are in the same (trivial) phase in both 0d and 1d. In 2d, the classification is $\mathbb{Z}^{\mathbb{N}}$.

5.2 Interacting invariants of band Hamiltonians

Zero dimensions

The only invariant of a general gapped fermionic 0d Hamiltonian with a unique ground state and symmetry $\hat{G}$ is the charge of the ground state

$$\omega \in H^1(\hat{G}, U(1)). \tag{5.19}$$

As usual, this charge suffers from ambiguities, so it is better to consider the relative charge of two ground states. Let us compute this relative charge for the free Hamiltonian corresponding to a representation $\widehat{R}$. We decompose it into irreducibles, compute the charge in each sector separately, and then add up the results.

Let us start with $\mathbb{C}$-type representations. The corresponding Hamiltonian is described by a non-degenerate Hermitian matrix h of size $n_r \times n_r$. Suppose we are given two such matrices h and h', with the number of negative eigenvalues m_r and m'_r. We can consider a path deforming h' to h. Every time an eigenvalue of h' changes from a positive one to a negative one, the ground state is multiplied by an operator

$$\prod_a \bar{\Psi}_i^a v^i, \tag{5.20}$$

where v^i is the corresponding eigenvector of h. Since $\bar{\Psi}_i^a$ transforms under $\hat{g} \in \hat{G}$ as

$$\bar{\Psi}_i^a \mapsto \bar{q}(\hat{g})_b^a \bar{\Psi}_i^b, \tag{5.21}$$

the above operator has charge $\det \bar{q}(\hat{g})$. Thus a $\mathbb{C}$-type irreducible representation r_α contributes a relative charge

$$(\det \bar{q}_\alpha(\hat{g}))^{\varrho_\alpha}, \tag{5.22}$$

where $\varrho_\alpha = m_\alpha - m'_\alpha \in \mathbb{Z}$ is the relative topological invariant of a pair of gapped class A Hamiltonians.

For an $\mathbb{R}$-type representation r, the Hamiltonian is described by a non-degenerate skew-symmetric real matrix $A_{r,ij}$ of size $n_r \times n_r$. Any two such matrices A_r and A'_r are related by

$$A_r = O^T A'_r O, \quad O \in O(n_r). \tag{5.23}$$

To compute the relative charge of the ground states, we recall that the orthogonal group is generated by hyperplane reflections. Without loss of generality, we can assume that the hyperplane is orthogonal to the 1st coordinate axis. Let us compute the change in the ground state charge due to a reflection of the 1st coordinate axis. This corresponds to the following map on fermions:

$$\Gamma^1_a \mapsto -\Gamma^1_a, \quad a = 1, \ldots, \mathrm{d}im\, r, \tag{5.24}$$

while the rest of the fermions remain invariant. We need to treat separately the cases when d$im\, r$ is even and when it is odd.

If d$im\, r$ is even, the map on fermions is in $SO(n_r \cdot \mathrm{d}im\, r)$, even though it arises from an element of $O(n_r)$ with determinant -1. On the Hilbert space, this map is represented by a bosonic operator proportional to

$$\prod_{a=1}^{\mathrm{d}im\, r} \Gamma^1_a. \tag{5.25}$$

This operator carries charge $\det r(\hat{g})$ under $\hat{g} \in \hat{G}$, hence the relative charge of the ground state corresponding to a hyperplane reflection is $\det r(\hat{g})$.

If d$im\, r$ is odd, the map on fermions is an orthogonal transformation with determinant -1, and thus must be represented on the Hilbert space by a fermionic operator. This fermionic operator is proportional to

$$\prod_{j=2}^{n_r} \prod_{a=1}^{\mathrm{d}im\, r} \Gamma^j_a. \tag{5.26}$$

It carries charge $(\det r(\hat{g}))^{n_r - 1} = \det r(\hat{g})$ under $\hat{g} \in \hat{G}$. Hence the relative charge of the ground state is again $\det r(\hat{g})$.

We conclude that when $O \in O(n_r)$ is a hyperplane reflection, the relative charge of the ground state under $\hat{g} \in \hat{G}$ is det $r(\hat{g})$. Since det $r(\hat{g}) = \pm 1$ and every element of $SO(n_r)$ is a product of an even number of hyperplane reflections, this implies that the relative charge is trivial when $O \in SO(n_r)$. Since every element of $O(n_r)$ is a product of a hyperplane reflection and an element of $SO(n_r)$, the relative charge of the ground state for an O which is not in $SO(n_r)$ is det $r(\hat{g})$.

To summarize, the relative charge contribution from an $\mathbb{R}$-type representation r_α is

$$(\det\, r_\alpha(\hat{g}))^{\varrho_\alpha}, \tag{5.27}$$

where $\varrho_\alpha \in \mathbb{Z}_2$ is the relative invariant of a pair of gapped class D Hamiltonians.

Finally, $\mathbb{H}$-type representations do not contribute to the relative charge since all 0d class C systems are deformable into each other.

In summary, the map from free to interacting phases in 0d is

$$\{\varrho_\alpha\} \mapsto \omega(\hat{g}) = \prod_{\alpha \in Irr'(\hat{G},\mathbb{R})} (\det r_\alpha(\hat{g}))^{\varrho_\alpha} \prod_{\alpha \in Irr'(\hat{G},\mathbb{C})} (\det \bar{q}_\alpha(\hat{g}))^{\varrho_\alpha}. \tag{5.28}$$

In what follows, we often find it more convenient to identify $U(1)$ with $\mathbb{R}/\mathbb{Z}$, i.e. write the abelian group operation on 1-cocycles additively rather than multiplicatively. This amounts to taking the logarithm of both sides of (5.28) and dividing by $2\pi i$. Then ω becomes as sum of two terms, $\omega = \omega_1 + \omega_2$. The first term

$$\omega_1(\hat{g}) = \sum_{\alpha \in \mathrm{I}rr'(\hat{G},\mathbb{R})} \frac{1}{2\pi i} \varrho_\alpha \log\det r_\alpha(\hat{g}) \tag{5.29}$$

can be interpreted as the weighted sum of the 1st Stiefel-Whitney classes of the representations r_α (see Appendix D.2 for an explanation of this terminology). More precisely, the 1st Stiefel-Whitney class $w_1(r_\alpha)$ is an element of $H^1(\hat{G}, \mathbb{Z}_2)$, while ω_1 involves the corresponding class in $H^1(\hat{G}, \mathbb{R}/\mathbb{Z})$ which we denote $w_1^{U(1)}(r_\alpha)$:

$$\omega_1 = \sum_{\alpha \in \mathrm{I}rr'(\hat{G},\mathbb{R})} \varrho_\alpha w_1^{U(1)}(r_\alpha) \in H^1(\hat{G}, \mathbb{R}/\mathbb{Z}). \tag{5.30}$$

The 2nd term which arises from $\mathbb{C}$-type representations can be interpreted in terms of the 1st Chern class of the complex representations q_α:

$$\omega_2 = \sum_{\alpha \in \mathrm{I}rr'(\hat{G},\mathbb{C})} \beta^{-1}(c_1(\varrho_\alpha q_\alpha)) \in H^1(\hat{G}, \mathbb{R}/\mathbb{Z}). \tag{5.31}$$

Here β^{-1} is the inverse of the Bockstein isomorphism $\beta : H^1(\hat{G}, \mathbb{R}/\mathbb{Z}) \to H^2(\hat{G}, \mathbb{Z})$. In the 0d case, it seems superfluous to express determinants in terms of Stiefel-Whitney and Chern classes, but in higher dimensions characteristic classes of representations become indispensable. They are briefly reviewed in Appendix D.2.

It is clear that the map from $\{\varrho_\alpha\}$ to ω is many-to-one for almost all $\hat{G}$. In fact, for Lie group symmetries, such as $U(1)$ or $SU(2)$, a single interacting phase corresponds to an infinite number of free phases.

More surprisingly, the map may fail to be surjective. A class $\omega \in H^1(\hat{G}, U(1))$ defines a one-dimensional complex representation q of $\hat{G}$. If this representation is allowed (i.e. if $\omega(P) = -1$), we can take a complex fermion $\bar{\Psi}$ and its Hermitian conjugate Ψ and let them transform in the representations q and $\bar{q}$, respectively. Now the two $\hat{G}$-invariant Hamiltonians

$$H_\pm = \pm\left(\bar{\Psi}\Psi - \frac{1}{2}\right) \tag{5.32}$$

have relative ground-state charge ω. But if the representation q is not allowed, $\omega(P) = 1$, then the situation is more complicated. For certain $\hat{G}$ there are no allowed one-dimensional representations at all, but one could try to use higher-dimensional allowed representation to get the relative ground-state charge ω.

Let us exhibit an example of a group $\hat{G}$ where certain relative charges ω cannot be obtained from free systems. This shows that the map from free to interacting 0d phases is not surjective, in general. Consider extending the group $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ by $\mathbb{Z}_2$. If the extension class in $H^2(G, \mathbb{Z}_2)$ maps to a non-trivial element in $H^2(G, U(1))$, the group $\hat{G}$ may be presented in terms of generators A, B, P, where P is central and

$$P^2 = A^4 = B^4 = 1 \qquad \text{and} \qquad AB = PBA. \tag{5.33}$$

The group of one-dimensional representations of $\hat{G}$ is then the same as the group of one-dimensional representations of G, i.e. $\mathbb{Z}_4 \times \mathbb{Z}_4$, defined by $q(A), q(B) \in \{\pm 1, \pm i\}$. All sixteen of these are disallowed, as $q(P) = +1$. Up to equivalence, only four irreducible representations remain. They are two-dimensional and of the form $q(P) = -\mathbb{1}_2$ (allowed), $q(A) = i^a \sigma_z$, and $q(B) = i^b \sigma_x$, for $a, b \in \{0, 1\}$. Each is related to a complexification of a real irreducible representation r by $r_{\mathbb{C}} = q \oplus \bar{q}$ and has $\det q(\hat{g}) \in \{\pm 1\}$. This means that twelve out of sixteen cocycles (those with $\omega(\hat{g}) = \pm i$ for some $\hat{g}$) do not arise from free systems.

One dimension

Let us begin by recalling invariants of interacting fermionic SRE phases in 1d and their interpretation in terms of boundary zero modes. Any fermionic 1d SRE phase has an invariant $\gamma \in \mathbb{Z}_2$ (Fidkowski and Kitaev, 2010). (From now on, we will write $\mathbb{Z}_2$ additively, i.e. identify it with the set $\{0,1\}$, unless stated otherwise.) It tells us whether the number of fermionic zero modes on the boundary is even or odd. Algebraically, if $\gamma = 0$, the algebra of boundary zero modes A_b is a matrix algebra, while for $\gamma = 1$ it is a sum of two matrix algebras. In both cases A_b is simple provided we regard it as a $\mathbb{Z}_2^F$-graded algebra. In the case $\gamma = 0$, the graded center of A_b is isomorphic to $\mathbb{C}$, while for $\gamma = 1$ it is isomorphic to $\mathbb{C}l(1)$. The odd generator of $\mathbb{C}l(1)$ is denoted $\hat{Z}$.

If the system also has a unitary symmetry $\hat{G}$, then there are further invariants whose form depends on the value of γ (Fidkowski and Kitaev, 2010). If $\gamma = 0$, the additional invariant is $\hat{\alpha} \in H^2(\hat{G}, U(1))$. If $\gamma = 1$, the additional invariants are a homomorphism $\mu : \hat{G} \to \mathbb{Z}_2$ such that $\mu(P) = 1$ (the generator of $\mathbb{Z}_2$) and $\alpha \in H^2(G, U(1))$. A homomorphism μ allows one to define an isomorphism $\hat{G} \simeq G \times \mathbb{Z}_2^F$ as follows:

$$\hat{g} \mapsto (g, \mu(\hat{g})). \tag{5.34}$$

So if $\hat{G}$ is not isomorphic to the product $G \times \mathbb{Z}_2^F$, the case $\gamma = 1$ is impossible.

Note that there is a homomorphism $H^2(\hat{G}, U(1)) \to H^1(G, \mathbb{Z}_2)$ whose kernel is non-canonically isomorphic to $H^2(G, U(1))$. To see this, let us define the group law on $\hat{G}$ using a $\mathbb{Z}_2$-valued 2-cocycle ρ on G:

$$(g, \varepsilon) \circ (g', \varepsilon') = (gg', \varepsilon + \varepsilon' + \rho(g, g')), \quad g, g' \in G, \quad \varepsilon, \varepsilon' \in \{0, 1\}. \tag{5.35}$$

Then $\hat{\alpha}$ can be parameterized by a pair of cochains $(\alpha, \beta) \in C^2(G, U(1)) \times C^1(G, \mathbb{Z}_2)$ satisfying $\delta\beta = 0$ and $\delta\alpha = \frac{1}{2}\rho \cup \beta$, modulo $\alpha \mapsto \alpha + \delta\lambda$, $\lambda \in C^1(G, U(1))$. The map from $H^2(\hat{G}, U(1))$ to $H^1(G, \mathbb{Z}_2)$ sends the pair (α, β) to β.

The boundary interpretation of the additional invariants also depends on whether $\gamma = 0$ or $\gamma = 1$. For $\gamma = 0$, the algebra A_b is a matrix algebra, and therefore $\hat{G}$ acts on it by conjugation:

$$\hat{g} : a \mapsto V(\hat{g}) a V(\hat{g})^{-1}, \quad a \in A_b. \tag{5.36}$$

One can even choose the invertible elements $V(\hat{g}) \in A_b$ to be unitary (A_b is actually a C^*-algebra, so the notion of a unitary element makes sense). The elements $V(\hat{g})$

are well-defined up to a $U(1)$ factor and satisfy

$$V(\hat{g})V(\hat{g}') = \hat{\alpha}(\hat{g}, \hat{g}')V(\hat{g}\hat{g}'), \tag{5.37}$$

where $\hat{\alpha}$ is a 2-cocycle on $\hat{G}$.

On the other hand, if $\gamma = 1$, then the same considerations apply to the even part of the graded algebra A_b, and one gets an invariant $\alpha \in H^2(G, U(1))$ in the same way. In addition, one can ask how the group $\hat{G}$ acts on the odd central element $\hat{Z} \in A_b$. One must have

$$\hat{g} : \hat{Z} \mapsto (-1)^{\mu(\hat{g})}\hat{Z}, \tag{5.38}$$

where $\mu : \hat{G} \to \mathbb{Z}_2$ is a homomorphism satisfying $\mu(P) = 1$.

As explained above, free SRE 1d systems with symmetry $\hat{G}$ are classified by a sequence of invariants $\varrho_\alpha \in \mathbb{Z}_2$, one for each real irreducible representation of $\hat{G}$ of $\mathbb{R}$-type. The physical meaning of ϱ_α is simple. The group $\hat{G}$ acts on the boundary zero modes (assumed to form a Clifford algebra) via a real representation[3]

$$\mathcal{R} = \oplus \nu_\alpha r_\alpha. \tag{5.39}$$

The integer ν_α reduced modulo 2 is the free topological invariant ϱ_α discussed in Section 5.1.

Let us now describe the map from free to interacting invariants. For a free system, the algebra of boundary zero modes is $A_b = \mathrm{C}l(M)$, so one has $\gamma = M \,\mathrm{m}od\, 2$. Equivalently, using the decomposition (5.39), we get

$$\gamma = \sum_\alpha \varrho_\alpha \,\mathrm{d}im\, r_\alpha \mod 2. \tag{5.40}$$

Now let us determine the remaining invariants. Consider the case $\gamma = 0$ first. Then $O(M)$ is a non-trivial extension of $SO(M)$ by $\mathbb{Z}_2$. We can interpret $A_b = \mathrm{C}l(M)$ as the algebra of operators on a Fock space of dimension $2^{M/2}$, and the group $\hat{G}$ acts projectively on this space. The cohomology class of the corresponding cocycle is $\hat{\alpha}$. Clearly, it is completely determined by the representation $\mathcal{R} : \hat{G} \to O(M)$.

From the group-theoretic viewpoint, a projective action of $\hat{G}$ on the Fock space is the same as a homomorphism $\hat{G} \to Pin_c(M)$, where $Pin_c(M)$ is a certain non-trivial

[3] One should not confuse the "boundary" representation $\mathcal{R}$ with the on-site representation $\widehat{R}$. The former can be odd-dimensional, while the latter is always even-dimensional. Also, $\widehat{R}$ takes values in $SO(2N)$, while $\mathcal{R}$ in general takes values in the orthogonal group.

extension of $O(M)$ by $U(1)$. $Pin_c(M)$ and related groups are reviewed in Appendix D.1. Thus $\hat{\alpha}$ is the obstruction to lifting $\mathcal{R}$ to a homomorphism $\hat{G} \to Pin_c(M)$. As discussed in Appendix D.2, this obstruction is the image of the 2nd Stiefel-Whitney class of $\mathcal{R}$ under the homomorphism $H^2(\hat{G}, \mathbb{Z}_2) \to H^2(\hat{G}, U(1))$. We denote it $w_2^{U(1)}(\mathcal{R})$. The Whitney formula for Stiefel-Whitney classes says

$$w_2(\mathcal{R}) = w_2\left(\oplus \nu_\alpha r_\alpha\right) = \sum_\alpha \varrho_\alpha w_2(r_\alpha) + \sum_{\alpha<\beta} \varrho_\alpha \varrho_\beta w_1(r_\alpha) \cup w_1(r_\beta). \tag{5.41}$$

Therefore

$$\hat{\alpha} = w_2^{U(1)}(\mathcal{R}) = \sum_\alpha \varrho_\alpha w_2^{U(1)}(r_\alpha) + \sum_{\alpha<\beta} \frac{1}{2} \varrho_\alpha \varrho_\beta w_1(r_\alpha) \cup w_1(r_\beta). \tag{5.42}$$

Note that $Pin_c(M)$ is a $\mathbb{Z}_2$-graded group, i.e. it is equipped with a homomorphism to $\mathbb{Z}_2$. The value of this homomorphism tells us if $V(\hat{g})$ is an even or odd element in $\mathrm{C}l(M)$. It is easy to see that this homomorphism is precisely $\beta(g)$. On the other hand, as explained in Appendix D.2, the said homomorphism is simply $\det \mathcal{R}(\hat{g})$. Thus

$$\beta = w_1(\mathcal{R}) = \sum_\alpha \varrho_\alpha w_1(r_\alpha). \tag{5.43}$$

In Appendix D.3 we give an alternate characterization of β as a charge-pumping invariant.

Now consider the case $\gamma = 1$, where $A_b \simeq \mathrm{C}l(M)$ with odd M. In agreement with (Fidkowski and Kitaev, 2010), the map $\hat{g} \mapsto \det \mathcal{R}(\hat{g})$ defines a splitting of $\hat{G}$, i.e. an isomorphism $G \times \mathbb{Z}_2^F \simeq \hat{G}$. This means

$$\mu = w_1(\mathcal{R}) = \sum_\alpha \varrho_\alpha w_1(r_\alpha). \tag{5.44}$$

We can define a new representation $\tilde{\mathcal{R}} : G \to SO(M)$ by

$$\tilde{\mathcal{R}}(g) = \mathcal{R}(\hat{g}) \det \mathcal{R}(\hat{g}). \tag{5.45}$$

Here $\hat{g} \in \hat{G}$ is any lift of $g \in G$. Thus we get a homomorphism

$$G \times \mathbb{Z}_2^F \to O(M) \simeq SO(M) \times \mathbb{Z}_2^F, \quad (g, \varepsilon) \mapsto (\tilde{\mathcal{R}}(g), \varepsilon). \tag{5.46}$$

By definition, α is the obstruction for lifting $\tilde{\mathcal{R}}$ to a homomorphism $G \to Spin_c(M)$. Thus

$$\alpha = w_2^{U(1)}(\tilde{\mathcal{R}}). \tag{5.47}$$

Using a formula for Stiefel-Whitney classes of a tensor product (see Appendix D.2), one can show that $w_2(\tilde{\mathcal{R}}) = w_2(\mathcal{R})$, and thus one can also write

$$\alpha = w_2^{U(1)}(\mathcal{R}). \tag{5.48}$$

We note that the map from free to interacting 1d SRE phases is compatible with the stacking law derived in (Kapustin, Turzillo, and You, 2018; Gaiotto and Kapustin, 2016). For example, if we consider for simplicity the case $\gamma = 0$, then the stacking law takes the form

$$\hat{\alpha} \circ \hat{\alpha}' = \hat{\alpha} + \hat{\alpha}' + \tfrac{1}{2}\beta \cup \beta' \tag{5.49}$$

On the other hand, stacking two SRE systems characterized by representations $\mathcal{R}$ and $\mathcal{R}'$ gives an SRE system corresponding to the representation $\mathcal{R} \oplus \mathcal{R}'$. If we set $\alpha = w_2^{U(1)}(\mathcal{R}) = \frac{1}{2}w_2(\mathcal{R})$ and $\beta = w_1(\mathcal{R})$, then the stacking law (5.49) follows from the Whitney formulas

$$w_1(\mathcal{R} \oplus \mathcal{R}') = w_1(\mathcal{R}) + w_1(\mathcal{R}'), \tag{5.50}$$

$$w_2(\mathcal{R} \oplus \mathcal{R}') = w_2(\mathcal{R}) + w_2(\mathcal{R}') + w_1(\mathcal{R}) \cup w_1(\mathcal{R}'). \tag{5.51}$$

It is clear that the map from free to interacting phases is not injective. Let us discuss surjectivity. We have seen that for free systems the invariants $\hat{\alpha}$ and α are always of order 2. Hence to get an example of a fermionic SRE phase which cannot be realized by free fermions, it is sufficient to pick a $\hat{G}$ and a non-trivial 2-cocycle which is not of order 2. For example, if we take $\hat{G} = \mathbb{Z}_2^F \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and take α be any non-trivial element of $H^2(\mathbb{Z}_3 \times \mathbb{Z}_3, U(1)) = \mathbb{Z}_3$, then such a phase cannot be realized by free fermions.

One might hope that perhaps every $\hat{\alpha}$ or α of order 2 can be realized by free fermions, but this is not the case either. The reason for this is that for any orthogonal representation $\mathcal{R}$ of $\hat{G}$, the 2-cocycle $w_2(\mathcal{R})$ satisfies some relations (Strickland, n.d.). This is explained in Appendix D.2. These relations need not hold for a general 2-cocycle on $\hat{G}$. Unfortunately, the simplest example of $\hat{G}$ for which this happens is rather non-trivial (Gunarwardena, Kahn, and Thomas, 1989).

While not every fermionic 1d SRE phase can be realized by free fermions, every fermionic 1d SRE phase with $\hat{G} \simeq G \times \mathbb{Z}_2^F$ can be realized by stacking bosonic 1d SRE phases with free fermions. First, we can change γ of an SRE phase at will by stacking with the Kitaev chain. If we make $\gamma = 0$ by such stacking, then we can change β at will by stacking with two copies of the Kitaev chain on which the group G acts by

$$(\gamma_1, \gamma_2) \mapsto ((-1)^{\beta(g)}\gamma_1, \gamma_2). \tag{5.52}$$

Finally, since α is an arbitrary element of $H^2(G, U(1))$ in this case, one can change it at will by stacking with bosonic SRE phases with symmetry G.

Two dimensions

To every fermionic 2d SRE phase one can attach an integer invariant κ. It measures the chiral central charge for the boundary CFT.

If the SRE has a unitary symmetry $\hat{G}$, there are further invariants. For simplicity, let us assume that we are given an isomorphism $\hat{G} \simeq G \times \mathbb{Z}_2^F$. We will also assume that G is finite, rather than merely compact. Then the invariants are a 1-cocycle $\gamma \in H^1(G, \mathbb{Z}_2)$, a 2-cocycle $\beta \in H^2(G, \mathbb{Z}_2)$, and a 3-cochain $\alpha \in C^3(G, U(1))$ satisfying

$$\delta\alpha = \frac{1}{2}\beta \cup \beta. \tag{5.53}$$

There are certain non-trivial identifications on these data, see (Gu and Wen, 2014; Gaiotto and Kapustin, 2016). The abelian group structure corresponding to stacking the systems is also quite non-trivial. We just note for future use that if we ignore α, the group law is

$$(\beta, \gamma) + (\beta', \gamma') = (\beta + \beta' + \gamma \cup \gamma', \gamma + \gamma'). \tag{5.54}$$

The physical meaning of these invariants is somewhat complicated, with the exception of $\gamma(g)$: it measures the number of Majorana zero modes on a g-vortex, reduced modulo 2.

On the other hand, a free 2d SRE is characterized by a sequence of invariants $\varrho_\alpha \in \mathbb{Z}$, one for each real irreducible representation of G.

It is easy to determine the chiral central charge κ for such a free SRE. A basic system of class D has $\kappa = 1/2$. For example, a $p + ip$ superconductor has a single chiral Majorana fermion on the boundary which has chiral central charge $1/2$.[4] A basic system of class A has $\kappa = 1$. For example, the basic Chern insulator has a single chiral complex fermion on the boundary which has chiral central charge 1. Two

[4] In the literature on fermionic SRE phases, it is common to re-write systems of class D, which only have a $\mathbb{Z}_2^F$ symmetry, as systems with both a $U(1)$ symmetry and a particle-hole symmetry (Bernevig, 2013; Chiu et al., 2016). This entails doubling the number of degrees of freedom, and therefore doubling κ.

basic class C systems[5] have chiral central charge 2. For example, two copies of the basic Chern insulator can be regarded as the basic class C system tensored with a two-dimensional representation of $SU(2)$, and thus has $\kappa = 2$. Consequently, the chiral central charge is given by

$$\kappa = \frac{1}{2} \sum_{r_\alpha \in \mathrm{I}rr(G)} \varrho_\alpha \mathrm{d}im\, r_\alpha \tag{5.55}$$

The other interacting invariants are harder to deduce. We will propose natural candidates for γ and β based on experience with lower-dimensional cases.

Given an orthogonal representation $r : G \to O(n)$ we can define a 1-cocycle

$$\det r(g) \in H^1(G, \mathbb{Z}_2). \tag{5.56}$$

It is sometimes called the 1st Stiefel-Whitney class of r, for reasons explained in Appendix D.2. We will denote it $w_1(r)$. For irreducible representations of type $\mathbb{C}$ and $\mathbb{H}$ it is trivial.[6]

Similarly, we can define the 2nd Stiefel-Whitney class of G as an obstruction to lifting $r : G \to O(n)$ to $\tilde{r}_+ : G \to Pin_+(n)$. One can lift each $r(g)$ to an element $\tilde{r}_+(g) \in Pin_+$, but the composition law will only hold up to a 2-cocycle $\lambda(g, g')$ with values in ± 1. Thus we get a well-defined element $w_2(r) \in H^2(G, \mathbb{Z}_2)$. One might also consider an obstruction to lifting r to a homomorphism $\tilde{r}_- : G \to Pin_-(n)$, but it is expressed in terms of $w_2(r)$ and $w_1(r)$ (namely, the Pin_- obstruction is $w_2 + w_1^2$).

A natural guess for the contribution of an irreducible r_α to γ is $\varrho_\alpha w_1(r_\alpha)$. Assuming this, the formula for the invariant γ is

$$\gamma = \sum_{r_\alpha \in \mathrm{I}rr(G,\mathbb{R})} \varrho_\alpha w_1(r_\alpha) = w_1(\mathcal{R}), \tag{5.57}$$

where we defined a "virtual representation"[7]

$$\mathcal{R} = \oplus_\alpha \varrho_\alpha r_\alpha. \tag{5.58}$$

Note that only $\mathbb{R}$-type representations contribute to γ, since only those representations can have nonzero $w_1(r)$. On the other hand, $\mathcal{R}$ includes all representations.

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[5] Since $\mathrm{d}im\, q$ is even for $\mathbb{H}$-type representations, only an even number of class C systems can occur.

[6] For $\mathbb{C}$-type representations, we have $\det r(g) = \det q(g) \det\{\bar{q}\}(g) = 1$, while for $\mathbb{H}$-type representations $\det q(g) = 1$ since $q(g)$ takes values in the unitary symplectic group.

[7] The word "virtual" reflects the fact that the numbers ϱ_α can be both positive and negative. Thus $\mathcal{R}$ is best thought of as an element of the K-theory of the representation ring of G.

There are two natural guesses for the contribution of a single irreducible r to β: $w_2(r)$ or $\tilde{w}_2(r) = w_2(r) + w_1(r)^2$. To derive β for a general virtual representation $\mathcal{R}$, we note that the Whitney formula for Stiefel-Whitney classes says

$$w_2(\mathcal{R} + \mathcal{R}') = w_2(\mathcal{R}) + w_2(\mathcal{R}') + w_1(\mathcal{R}) \cup w_1(\mathcal{R}'). \tag{5.59}$$

The same formula applies to $\tilde{w}_2(r)$. This formula looks just like the stacking law for β and γ, if we identify γ with w_1 and β with w_2 (or $\tilde{w}_2$). Hence for a general $\mathcal{R}$ we have either $\beta(\mathcal{R}) = w_2(\mathcal{R})$ or $\beta(\mathcal{R}) = w_2(\mathcal{R}) + w_1(\mathcal{R})^2$.

A non-trivial check on both of these candidates is that they are compatible with the group supercohomology equation. This equation implies that $\beta \cup \beta \in H^4(G, \mathbb{Z}_2)$ maps to a trivial class in $H^4(G, U(1))$. This is automatically satisfied for both $\beta = w_2(\mathcal{R})$ and $\beta = w_2(\mathcal{R}) + w_1(\mathcal{R})^2$, as shown in Appendix D.2.

Is there any way to decide between the two candidates for β? Not without understanding better the physical meaning of β. Indeed, formally, a change of variables $\beta \mapsto \beta + \gamma \cup \gamma$ is an automorphism of the group of fermionic SRE phases in 2d. This automorphism maps one candidate for β to the other one. Thus formally there are equally good. One can pick one over another only if one assigns β a particular physical meaning. The same is even more true about $\alpha \in C^3(G, U(1))$, since it depends on various choices in a complicated way.

Let us make a few remarks about surjectivity of the map from free to interacting SRE phases in the 2d case. It is clear that every value of the parameter $\gamma \in H^1(G, \mathbb{Z}_2)$ can be realized by free fermionic systems. One can just take two copies of the basic system of class A with opposite values of the chiral central charge κ (for example, a $p + ip$ superconductor stacked with a $p - ip$ superconductor) and let G act only on the first copy via a 1-dimensional real representation of G given by the 1-cocycle γ. This construction was used in Ref. (Gu and Levin, 2014) for the case $G = \mathbb{Z}_2$.

One can also ask if every β that solves the supercohomology equation can be realized by free fermions. The answer appears to be no (Strickland, n.d.), for a sufficiently complicated G. The reason is again some highly non-trivial relations satisfied by Stiefel-Whitney classes. Thus not all supercohomology phases in 2d can be realized by free fermions. At the moment we do not know how to find a concrete example of a finite group G for which this happens. It would be interesting to study this question further and in particular determine both α and β for a general 2d band Hamiltonian with symmetry G.

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Appendix A

APPENDIX FOR CHAPTER 1

In this appendix we discuss the quantization of the coefficient of the gravitational Chern-Simons action. For all topological facts used here, the reader may consult (Milnor and Stasheff, 1974). Let X be an oriented 3-manifold whose tangent bundle is equipped with a connection ω. We can take ω to be a Levi-Civita connection for some Riemannian metric on X, so ω can be thought of as an $SO(3)$ connection.

We define the gravitational Chern-Simons action to be

$$S_{grav}(\omega) = \frac{\kappa}{192\pi}\int_M \mathrm{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right).$$

The choice of the normalization coefficient will be explained shortly. This formula is only schematic, since ω is not a globally-defined 1-form, in general. A more precise definition requires choosing a compact oriented 4-manifold M whose boundary is X (this is always possible, since $\Omega_3^{SO}(pt) = 0$). We also extend ω to X and define

$$S^X_{grav}(\omega) = \frac{k}{192\pi}\int_X \mathrm{Tr}\, R \wedge R.$$

We need to ensure that $\exp\left(iS^X_{grav}(\omega)\right)$ does not depend on the choice of X or the way ω is extended from M to X. If we choose another X' with the same boundary M, the difference between the two ways of defining the gravitational Chern-Simons action is

$$\frac{k}{192\pi}\int_{X'\cup\bar{X}} \mathrm{Tr}\, R(\omega) \wedge R(\omega),$$

where $\bar{X}$ is X with orientation reversed, and $R(\omega)$ is the curvature 2-form of ω. This expression can be rewritten as

$$\frac{\pi k}{24} p_1(X' \cup \bar{X}) = \frac{\pi k}{8}\sigma(X' \cup \bar{X}). \tag{A.1}$$

Here $p_1(Y)$ denotes the first Pontryagin number of a closed oriented 4-manifold Y, $\sigma(Y)$ denotes its signature, and we used the Hirzebruch signature theorem $p_1(Y) = 3\sigma(Y)$. Since the signature is an integer, we conclude that $\exp\big(iS_{grav}(\omega)\big)$ is well-defined provided k is an integer multiple of 16. This determines the quantization of the thermal Hall conductivity for $d = 3$ bosonic SPTs with time-reversal symmetry.

Now suppose M is given a spin structure. We can exploit it to define $\exp(iS_{grav})$ for arbitrary integral k. We merely require the spin structure to extend to X. It is always possible to find such an X, since $\Omega_3^{Spin}(pt) = 0$. The difference between $S_{grav}^X(\omega)$ and $S_{grav}^{X'}(\omega)$ is again given by (A.1). Since now $X' \cup \bar{X}$ is a closed spin 4-manifold, we can appeal to the Rohlin theorem which says that the signature of a closed spin 4-manifold is divisible by 16, and conclude that $\exp(iS_{grav}(\omega))$ is well-defined if k is integral. This determines the quantization of the thermal Hall conductivity for $d = 3$ fermionic SPTs with time-reversal symmetry. Note that in the fermionic case the quantum of conductivity is 16 times smaller than in the bosonic case.

Appendix B

APPENDICES FOR CHAPTER 2

B.1 Proof of Proposition 1

We have already shown that an unoriented equivariant $D = 1$ TQFT with symmetry (G, ρ) has an underlying ρ-twisted G-crossed algebra $(\mathcal{A}, \theta_g, \alpha_h)$. Oriented cobordisms and bundle isomorphisms constitute a G_0-crossed algebra $\mathcal{A} = \oplus_{g\in G_0}\mathcal{A}_g$, while crosscaps correspond to states $\theta_g \in \mathcal{A}_{g^2}$ and orientation-reversing homeomorphisms to algebra anti-automorphisms $\alpha_h : \mathcal{A}_g \to \mathcal{A}_{hg^{-1}h^{-1}}$. It remains to show the converse: that from each such algebra we can construct an unoriented equivariant TQFT with this underlying algebra. We generalize the approaches of (G. W. Moore and Segal, 2006) and (Turaev and Turner, 2006) to unoriented equivariant theories.

We begin by defining the vector spaces assigned to simple objects $P_{[g],x,t}$ of the source category C. To each circle S equipped with principal G-bundle $P_{[g]}$, basepoint x, local trivialization $t : P_{[g]}|_x \to G$, and global trivialization of $o(S) \otimes \rho(P_{[g]})$, assign the vector space $\mathcal{H}(P_{[g],x,t}) \cong \mathcal{A}_g$ where g is the holonomy of $P_{[g]}$ around S with respect to x and t. Any object E can be factored into simple objects $\sqcup_i P_{[g_i],x_i,t_i}$ and assigned a vector space $\mathcal{H}(E) \cong \otimes_i \mathcal{H}(P_{[g_i],x_i,t_i})$. It is clear that $\mathcal{H}(E)$ does not depend on the factorization of E.

Next we consider the linear maps assigned to morphisms of simple objects. One type of morphism $\tilde{\alpha}_k : P_{[g],x,t} \to P_{[g],y,s}$ arises from an isomorphism f of the bundle $P_{[g]}$ where $(y, s) = (f(x), (f^{-1})^* t)$. Realized as its mapping cylinder, f must have a global trivialization of $o(S \times I) \otimes \rho(f)$. Since $o(S \times I)$ is trivial, so must be $\rho(f)$, and so the holonomy of $P_{[g]}$ along a positive path from (x, t) to (y, s) is an element $k \in G_0$. We assign the linear map $\alpha_k : \mathcal{A}_g \to \mathcal{A}_{kgk^{-1}}$ to this morphism. The other type of morphism $\tilde{\alpha}_h : P_{[g],x,t} \to P_{[g^{-1}],y,s}$ arises from a bundle anti-isomorphism $P_{[g]} \to P_{[g^{-1}]}$ whose restriction to the base circle is not isotopic to the trivial homeomorphism. Since a bundle map of this type exchanges the sheets of $o(S)$, the holonomy of $P_{[g^{-1}]}$ from (x, t) to (y, s) is an element $h \notin G_0$. We assign the linear map $\alpha_h : \mathcal{A}_g \to \mathcal{A}_{hg^{-1}h^{-1}}$ to $\tilde{\alpha}_h$. This assignment is well defined for isomorphism classes of bundles, as the cylinder $\tilde{\alpha}_k\tilde{\alpha}_g$, related to $\tilde{\alpha}_k$ by a Dehn twist, is assigned the linear map $\alpha_k\alpha_g$, which equals α_k when restricted to $\mathcal{A}_g$ by (2.6).

Now we wish to define linear maps for cobordisms (W, E_0, E_1). The strategy will be to decompose W as a sequence of n elementary cobordisms (W^i, E_0^i, E_1^i), sewn along bundle (anti-)isomorphisms $s_i : E_1^i \to E_0^{i+1}$ with $E_0^0 = E_0$ and $E_1^n = E_1$. After assigning a linear map to each W^i, we assign their composition $\tau(W)$ to W. We must then verify that $\tau(W)$ does not depend on the decomposition. Begin by considering the cobordism of base spaces (N, M_0, M_1). By Sard's lemma, there exists a smooth function $f : N \to I$ such that $f^{-1}(0) = M_0$, $f^{-1}(1) = M_1$, and f is Morse; that is, the gradient df vanishes at finitely many critical points x_i, the Hessian $d^2 f$ is a non-degenerate quadratic form at all x_i, and the critical values $c_i = f(x_i)$ are distinct and not equal to 0 or 1. The index $\mathrm{ind}(x_i)$ is the number of negative eigenvalues of $d^2 f$ at x_i. Choose $t_i \in I$ such that $0 = t_0 < c_1 < t_1 < \cdots < c_n < t_n = 1$. By the implicit function theorem, each $M_{t_i} = f^{-1}(t_i)$ is a disjoint union of m_i circles, and $\Sigma_i = f^{-1}([t_{i-1}, t_i])$ is a cobordism from $M_{t_{i-1}}$ to M_{t_i} with a single critical point. The classification of surfaces tells us that Σ_i is homeomorphic to a disjoint union of cylinders and one of five possibilities: a cap, a pair-of-pants, their adjoints, and a twice-punctured real projective plane.

These spaces are base spaces for five classes of cobordisms W. Since any G-bundle over the disk is trivial, there is a unique cobordism over the cap, to which we assign the linear map $\eta : \mathcal{A}_1 \to \mathbb{C}$. A G-bundle over the pair-of-pants, based and trivialized at the critical point, is almost determined by the holonomies k and l around the legs of the pants. We assign to it the linear map $m_{k,l} : \mathcal{A}_k \otimes \mathcal{A}_l \to \mathcal{A}_{kl}$. The orderings are related by conjugation $\alpha_l : \mathcal{A}_{kl} \to \mathcal{A}_{lk}$, and consistency requires that $m_{k,l}(\psi_k \otimes \psi_l) = \alpha_k m_{l,k}(\psi_l \otimes \psi_k)$, which is enforced by the axioms (2.6) and (2.7) of the G_0-crossed algebra $\mathcal{A}$. The holonomies determine the bundle up to cylinders $\tilde{\alpha}_k$ sewn to the boundary circles, which were assigned maps α_k above. The next two maps are fixed by adjunction. The adjoint of η distinguishes a state $\psi_\eta \in \mathcal{A}_1$ with the property that $\eta(\psi_\eta) = 1$. The adjoint pair-of-pants is assigned a map $\Delta_{k,l}(\psi_{kl}) = \sum_i \psi_{kl}\phi^i \otimes \phi_i$ where $\{\phi^i\}$ is a basis for $\mathcal{A}_l$ and $\{\phi_i\}$ is a dual basis for $\mathcal{A}_{l^{-1}}$. A G-bundle over the crosscap is specified (up to cylinders) by a holonomy $g \notin G_0$ around the orientation-reversing loop. We assign to it the linear map $\psi_k \mapsto m_{g^2,k}(\theta_g \otimes \psi_k)$, determined by the distinguished state $\theta_g \in \mathcal{A}_{g^2}$.

One may worry about a redundancy in the assignment of linear maps to composite cobordisms. Whenever an elementary cobordism W^i and its sewing maps s_{i-1} and s_i can be modified in a way that preserves the composite cobordism W, consistency requires that $\tau(W)$ is also preserved. The map s_i used to sew a cap or its adjoint into another cobordism does not affect the composite cobordism. The consistency of the algebraic description follows from the fact that α_k and α_h preserve η. Let W^i be a pair-of-pants sewn along s_{i-1} and s_i. Sewing instead along $(\tilde{\alpha}_k \otimes \tilde{\alpha}_k) \circ s_{i-1}$ and $s_i \circ \tilde{\alpha}_k^{-1}$ does not change W. Since α_k is an automorphism of $\mathcal{A}$, $\tau(W)$ is also preserved. Let R be the bundle isomorphism that exchanges two circles. Then $(\tilde{\alpha}_h \otimes \tilde{\alpha}_h) \circ R \circ s_{i-1}$ and $s_i \circ \tilde{\alpha}_h^{-1}$ yield the same W. We require $\alpha_h^{-1} m(\alpha_h(\psi_l) \otimes \alpha_h(\psi_k)) = m_{k,l}(\psi_k \otimes \psi_l)$, which is enforced by axiom (2.18). Let (W^i, E_0^i, E_1^i) be a twice-punctured real projective plane with holonomy g realized as a cobordism from $s_{i-1} : P_{[k]} \to E_0^i$ to $s_i : E_1^i \to P_{[g^2 k]}$. There is a bundle isomorphism, covering a Dehn twist of the base space, between this cobordism and a twice-punctured real projective plane with holonomy $g^{-1}k^{-1}$ with sewing maps s_{i-1} and $s_i \circ \tilde{\alpha}_g$. By axioms (2.18) and (2.20), the consistency condition $\alpha_g m(\theta_{g^{-1}k^{-1}} \otimes \psi_k) = m_{g^2,k}(\theta_g \otimes \psi_k)$ is fulfilled. Now consider the Möbius strip with holonomy $g \notin G_0$ constructed by sewing a cap into the twice-punctured real projective plane with holonomy g. Sewing this cobordism into another along s_i yields the same composite cobordism related to the Möbius strip with holonomy $hg^{-1}h^{-1}$ sewn along $\tilde{\alpha}_{h^{-1}} \circ s_i$ by a bundle isomorphism that covers a Y-homeomorphism of the base space. Axiom (2.19) encodes this relation in the algebraic data.

The linear map $\tau(W)$ assigned to an arbitrary cobordism W is given by the composition of maps assigned to its factors under Morse decomposition. It remains to show that $\tau(W)$ does not depend on the choice of Morse function. Any two Morse functions f_0 and f_1 are related by a smooth family of functions f_s that are Morse at all but finitely many values of s. One possibility is that two critical points merge and annihilate for some s. Then f_s has a degenerate critical point. This situation only occurs when deforming a pair-of-pants and an adjoint cap into a cylinder. For $\tau(W)$ to be consistent over the deformation, we require $m_{k,1}(\psi_k \otimes \psi_\eta) = \psi_k$. This condition is enforced by the axioms of $\mathcal{A}$. The remaining possibility is that two critical values coincide for some non-Morse value of s. We must check, for each composition W of two elementary cobordisms, that all factorizations give the same linear map. This situation occurs when both critical points have index 1, in which case W has Euler characteristic $\chi(W) = \sum_i (-1)^{\mathrm{ind}(x_i)} = -2$. Hence W is one of seven cobordisms: a genus zero oriented cobordism from three circles to one, its adjoint, a genus zero oriented cobordism from two circles to two, a twice-punctured torus from one circle to one, a crosscap-pants cobordism from two circles to one, its adjoint, and a twice-punctured Klein bottle from one circle to one.

The consistency of the first two cobordisms follows immediately from associativity of multiplication. The remaining two oriented conditions have been proven in Appendix A.3 of (G. W. Moore and Segal, 2006) and follow from the oriented axioms, notably (2.8). The next condition says that moving a crosscap from the "torso" to a leg of the pair-of-pants is a consistent deformation and also follows from associativity of multiplication. The Klein bottle has a decomposition as a pair-of-pants glued along its two legs to an adjoint pair-of-pants as well as a decomposition as a sphere with two crosscaps. The composite linear maps assigned to these realizations are equal to the others by axiom (2.21). We have assigned a linear map to each cobordism in terms of a Morse function f and have seen that this map is independent of the choice of f. This completes the proof of Proposition 1.

B.2 Proof of Proposition 2

Consider the map from 2-cochains $a \in C^2(G, U(1))$ to TQFT data defined in (2.25)-(2.28). If we restrict to the set $Z^2(G, U(1)_\rho)$ of 2-cochains satisfying the ρ-twisted 2-cocycle condition (2.24), we obtain a map f from twisted cocycles to TQFT data. We show that numbers in the image of f satisfy axioms (2.18)-(2.21), and so give rise to a consistent invertible unoriented equivariant TQFT.

For an invertible theory, these axioms can be written as

$$\begin{aligned}
w(h, kl)b(k, l) &= w(h, k)w(h, l)b(hl^{-1}h^{-1}, hk^{-1}h^{-1}) \\
w(h, g^2)\theta(g) &= \theta(hg^{-1}h^{-1}) \\
b(g^2, k)\theta(g) &= b(gk^{-1}g^{-1}, gkgk)w(g, k)\theta(gk) \\
b(g^2hg^{-1}, gh)w(g, h^{-1}g^{-1}) &= \theta(g)\theta(h)b(g^2, h^2)b(h^{-1}g^{-1}, gh)
\end{aligned}$$

It will be useful to impose a "cyclic-symmetric gauge" on the restriction of the cocycle a to G_0:

$$a(k, k^{-1}) = 1, \quad a(k, l) = a(l^{-1}, k^{-1})^{-1}, \quad \forall k, l \in G_0.$$

We also fix some $T \in G$ and impose the condition $a(k, T) = 1$, $k \in G_0$.

Axiom (2.18):

$$\begin{aligned}
w(h,kl)b(k,l) &= \frac{a(h,l^{-1}k^{-1})a(hl^{-1}k^{-1},h^{-1})a(kl,l^{-1}k^{-1})}{a(h,h^{-1})}a(k,l)\\
&= \frac{a(h,l^{-1}k^{-1})}{a(h,h^{-1})}\frac{a(hl^{-1},k^{-1}h^{-1})}{a(l^{-1},k^{-1})a(hl^{-1},k^{-1})a(k^{-1},h^{-1})}\\
&= \frac{a(hl^{-1},k^{-1}h^{-1})}{a(h,h^{-1})a(k^{-1},h^{-1})}a(h,l^{-1})\\
&= \frac{a(h,l^{-1})}{a(h,h^{-1})a(k^{-1},h^{-1})}\frac{a(hl^{-1}h^{-1},hk^{-1}h^{-1})a(h,k^{-1}h^{-1})}{a(hl^{-1}h^{-1},h)}\\
&= \frac{a(h,l^{-1})a(hl^{-1}h^{-1},hk^{-1}h^{-1})a(h,k^{-1}h^{-1})}{a(h,h^{-1})a(k^{-1},h^{-1})}a(h^{-1},h)a(hl^{-1},h^{-1})\\
&= \frac{a(h,l^{-1})a(hl^{-1},h^{-1})a(hl^{-1}h^{-1},hk^{-1}h^{-1})}{a(h,h^{-1})a(h,h^{-1})}a(h,k^{-1})a(hk^{-1},h^{-1})\\
&= w(h,k)w(h,l)b(hl^{-1}h^{-1},hk^{-1}h^{-1})
\end{aligned}$$

Axiom (2.19):

$$\begin{aligned}
w(h,g^2)\theta(g) &= \frac{a(h,g^{-2})a(hg^{-2},h^{-1})a(g^2,g^{-2})}{a(h,h^{-1})}a(g,g)\\
&= \frac{a(hg^{-2},h^{-1})}{a(h,h^{-1})}a(hg^{-1},g^{-1})a(h,g^{-1})a(g,g)a(g^{-1},g^{-1})\\
&= \frac{a(h,g^{-1})a(g,g)a(g^{-1},g^{-1})}{a(h,h^{-1})}a(g^{-1},h^{-1})a(hg^{-1},g^{-1}h^{-1})\\
&= \frac{a(hg^{-1}h^{-1},hg^{-1}h^{-1})a(h,g^{-1})a(g,g)a(g^{-1},g^{-1})}{a(h,h^{-1})a(hgh^{-1},h)a(h,g^{-1}h^{-1})}a(g^{-1},h^{-1})\\
&= \frac{a(hg^{-1}h^{-1},hg^{-1}h^{-1})a(g,g)a(g^{-1},g^{-1})}{a(h,h^{-1})a(hgh^{-1},h)a(hg^{-1},h^{-1})}\frac{a(h,g^{-1})a(g^{-1},h^{-1})}{a(h,g^{-1})a(g^{-1},h^{-1})}\\
&= a(hg^{-1}h^{-1},hg^{-1}h^{-1})\\
&= \theta(hg^{-1}h^{-1})
\end{aligned}$$

Axiom (2.20):

$$\begin{aligned}
\theta(g)a(g^2,k) &= \theta(g)a(g^2,k)\frac{a(g,k)a(g,k^{-1})a(k,k^{-1})}{a(g,g^{-1})}\frac{a(gk^{-1},k)}{a(g^{-1},gk)}\\
&= \theta(g)a(g^2,k)a(g,k)\frac{a(g,k^{-1})a(gk^{-1},g^{-1})}{a(g,g^{-1})}a(gk^{-1}g^{-1},gk)\\
&= \frac{a(g,k^{-1})a(gk^{-1},g^{-1})}{a(g,g^{-1})}a(g,gk)a(gk^{-1}g^{-1},gk)\\
&= a(gk^{-1}g^{-1},gkgk)\frac{a(g,k^{-1})a(gk^{-1},g^{-1})}{a(g,g^{-1})}a(gk,gk)\\
&= b(gk^{-1}g^{-1},gkgk)w(g,k)\theta(gk)
\end{aligned}$$

Axiom (2.21):

$$\begin{aligned}
b(g^2hg^{-1}, gh)w(g, h^{-1}g^{-1}) &= a(g^2hg^{-1}, gh)\frac{a(g, gh)a(g^2h, g^{-1})a(h^{-1}g^{-1}, gh)}{a(g, g^{-1})} \\
&= \frac{a(g, gh)}{a(g, g^{-1})}\frac{a(g^2h, h)}{a(g^{-1}, gh)}a(h^{-1}g^{-1}, gh) \\
&= \frac{a(g^2, h)a(g, g)a(g, h)}{a(g, g^{-1})a(g^{-1}, gh)}a(g^2h, h)a(h^{-1}g^{-1}, gh) \\
&= \frac{a(g, g)a(g, h)}{a(g, g^{-1})a(g^{-1}, gh)}a(g^2, h^2)a(h, h)a(h^{-1}g^{-1}, gh) \\
&= a(g, g)a(h, h)a(g^2, h^2)a(h^{-1}g^{-1}, gh) \\
&= \theta(g)\theta(h)b(g^2, h^2)b(h^{-1}g^{-1}, gh)
\end{aligned}$$

We have shown that data in the image of f define consistent invertible unoriented equivariant TQFTs. Both $Z^2(G, U(1)_\rho)$ and the set of invertible unoriented equivairant TQFTs are groups, and it is easy to see that f is a group homomorphism.

It remains to show that f is injective and surjective. Let (g, h, k) denote the twisted cocycle condition (2.24). We construct a cocycle that solves (2.25)-(2.28), an inverse to f.

Consider the twisted cocycle condition for (k, T, T^{-1}):

$$a(k, T)a(kT, T^{-1}) = a(T, T^{-1}).$$

Taking into account $a(k, T) = 1$, we get $a(kT, T^{-1}) = a(T, T^{-1})$. This also implies $a(Tk, T^{-1}) = a(T, T^{-1})$. So in this gauge we get $w(T, k) = a(T, k^{-1})$. Next consider the twisted cocycle condition for (l, k, T):

$$a(l, k)a(lk, T) = a(l, kT)a(k, T).$$

Taking into account $a(k, T) = 1$, we get $a(l, kT) = a(l, k)$. Since $T^{-2} \in G_0$, this implies $a(k, T^{-1}) = a(k, T^{-2})$. Next consider the twisted cocycle condition for (T, k, T^{-1}):

$$a(T, k)a(Tk, T^{-1})a(k, T^{-1}) = a(T, kT^{-1}).$$

Using previous results, this is equivalent to

$$a(T, kT^{-1}) = a(k, T^{-2})a(T, T^{-1})a(T, k)$$

Next consider the twisted cocycle condition for (T, l, k):

$$a(Tl, k)a(T, l)a(l, k) = a(T, lk)$$

Recall also that in our gauge $a(T,l) = w(T,l^{-1})$. Then

$$a(Tl,k)a(l,k)w(T,l^{-1}) = w(T,k^{-1}l^{-1})$$

Since $\alpha_g\alpha_h = \alpha_{gh}$ and by axiom (2.18), we see

$$a(Tl,k) = w(T,k^{-1})a(TlT^{-1},TkT^{-1}).$$

We have determined the components of the twisted cocycle where one argument is in G_0 and the other is not. We have also determined $a(Tk,T^{-1})$ and $a(T,kT^{-1})$ up to a single term $a(T,T^{-1})$. We can determine $a(Tl,kT^{-1})$ by requiring that a satisfies the twisted cocycle condition (T,l,kT^{-1}):

$$a(Tl,kT^{-1})a(T,l)a(l,kT^{-1}) = a(T,lkT^{-1})$$

By construction, a is a 2-cochain that satisfies (2.25)-(2.28) as well as the (k,T,T^{-1}), (l,k,T^{-1}), (T,k,T^{-1}), (T,l,k), (l,k,m), and (T,l,kT^{-1}) cocycle conditions. The component $a(Tl,mT^{-1})$ is also determined by (Tl,k,T^{-1}), and equality of the two expressions must hold if a is a cocycle:

$$a(Tl,kT^{-1}) = a(Tlk,T^{-1})a(Tl,k)a(k,T^{-1})$$

In the above expression, apply the (T,lk,T^{-1}) condition to the first term to obtain $\frac{a(T,lkT^{-1})}{a(T,lk)a(lk,T^{-1})}$. Hit the second term with (T,l,k) to obtain $\frac{a(T,lk)}{a(l,k)a(T,l)}$. Hit $a(lk,T^{-1})$ with (l,k,T^{-1}) to get $\frac{a(l,kT^{-1})a(k,T^{-1})}{a(l,k)}$. After cancellation, we are left with the first expression for $a(Tl,kT^{-1})$.

To see injectivity of f, consider the trivial TQFT with b, w, θ trivial. The cocycle solution has $a(k,l) = 1$ and $a(k,lT) = 1$. We have $a(Tl,k) = \frac{w(T,k^{-1}l^{-1})}{a(l,k)w(T,l^{-1})} = 1$ as well as

$$a(Tl,kT^{-1}) = a(Tlk,T^{-1})a(Tl,k)a(k,T^{-1}) = \theta(T^{-1}) = 1$$

so the only the trivial cocycle corresponds to the trivial theory.

It remains to show that a satisfies the cocycle condition for all possible combinations of arguments; in particular, we must show the (k,l,mT), (kT,l,m), (k,lT,m), (k,lT,mT), (kT,l,mT), (kT,lT,m), and (kT,lT,mT) conditions. Consider the first condition:

$$a(k,l)a(kl,mT) = a(l,mT)a(l,kmT)$$

Since $a(k,lT) = a(k,l)$ for all $k,l \in G_0$ in our gauge, this follows from the G_0 cocycle condition. Now consider the third:

$$a(kT,l)a(kTl,m)a(l,m) = a(lT,km)$$

Apply the (T, k, l) condition to the first term to get $\frac{a(T,kl)}{a(k,l)a(T,k)}$, the (T, kl, m) condition to the second term to get $\frac{a(T,klm)}{a(kl,m)a(T,kl)}$, and the (T, k, lm) condition to the third term to get $\frac{a(T,lkm)a(T,m)}{a(k,lm)}$. The desired condition is reduced to a known condition.

Now consider (Tk, l, mT^{-1}):

$$a(Tk, l)a(Tkl, mT^{-1})a(l, mT^{-1}) = a(Tk, lmT^{-1})$$

The first term becomes $\frac{a(T,kl)}{a(T,k)a(k,l)}$ after (T, k, l), the second $\frac{a(T,klmT^{-1})}{a(T,kl)a(kl,mT^{-1})}$ after (T, kl, mT^{-1}), the third $(a(k, l)a(kl, mT^{-1}))^{-1}$ after (l, m, T^{-1}), and the fourth $\frac{a(T,klmT^{-1}}{a(T,k)a(k,lmT^{-1})}$ after (T, k, lmT^{-1}). Everything cancels.

Since $a(kT, T^{-1}) = a(T, T^{-1})$, we get the (l, lT, T^{-1}) condition by applying (kl, T, T^{-1}) to $a(klT, T^{-1})$. Then (k, lT, mT^{-1}) reads

$$a(k, lT)a(klt, mT^{-1}) = a(lT, mT^{-1})a(k, lTmT^{-1})$$

The last term is just $a(k, lTm)$ in our gauge and becomes $\frac{a(k,lT)a(klT,m)}{a(lT,m)}$ after (k, lT, m). $a(klT, mT^{-1})$ becomes $\frac{a(klTm,T^{-1})a(klT,m)}{a(m,T^{-1})}$ after (klT, m, T^{-1}), and $a(lT, mT^{-1})$ becomes $\frac{a(lTm,T^{-1})a(lT,m)}{a(m,T^{-1})}$ after (lT, m, T^{-1}). We have seen that $a(klTm, T^{-1}) = a(T, T^{-1}) = a(lTm, T^{-1})$ so we are done.

The condition (k, lT, T^{-1}) is shown by noting that $a(k, lT) = a(k, l)$ and $a(klT, T^{-1}) = a(T, T^{-1}) = a(lT, T^{-1})$. Consider the (k, Tl, mT^{-1}) condition:

$$a(k, Tl)a(kTl, mT^{-1}) = a(Tl, mT^{-1})a(k, TlmT^{-1})$$

Hit the second term with (kTl, m, T^{-1}) to get $a(kTlm, T^{-1})a(m, T^{-1})a(kTl, m)$ and the fourth term with (k, Tlm, T^{-1}) to get $\frac{a(kTlm,T^{-1})a(k,Tlm)}{a(Tlm,T^{-1})}$. Then $a(kTl, m)$ becomes $\frac{a(k,Tlm)a(Tl,m)}{a(k,TL)}$ by (k, Tl, m) and $a(Tlm, T^{-1})$ becomes $\frac{a(Tl,m)a(m,T^{-1})}{a(Tl,mT^{-1})}$ by (T, lm, T^{-1}).

Consider (T, T, T):

$$a(T^2, T)a(T, T)a(T, T) = a(T, T^2)$$

The first term vanishes, and we are left with $\theta(T)^2 = w(T, T^{-2})$, which is true by axiom (2.21) with $g = h = T$.

Consider (lT^{-1}, T, T):

$$a(lT^{-1}, T)a(T, T) = a(lT^{-1}, T^2)$$

The first term is just $\frac{a(T^{-1},T)}{a(l,T^{-1})}$ by (l, T^{-1}, T). The third is $\frac{a(T^{-1},T^2)}{a(l,T^{-1})}$. The condition then follows from (T, T, T). Consider (T, mT^{-1}, T):

$$a(T, mT^{-1}) = a(T, m)a(mT^{-1}, T)$$

The first term becomes $a(Tm, T^{-1})a(T, m)a(m, T^{-1})$ by (T, m, T^{-1}) and the second becomes $\frac{a(T^{-1},T)}{a(m,T^{-1})}$. We are left with $a(Tm, T^{-1})a(T^{-1}, T) = 1$. This is $\theta(T^{-1})\theta(T)$, which vanishes by axiom (2.21). Now consider (kT, lT^{-1}, T):

$$a(kT, lT^{-1})a(lT^{-1}) = a(kT, l)$$

The first term is $a(kTl, T^{-1})a(kT, l)a(l, T^{-1})$ by (kT, l, T^{-1}) and the second is $\frac{a(T^{-1},T)}{a(l,T^{-1})}$ by (l, T^{-1}, T). We are left with $a(kTl, T^{-1})a(T^{-1}, T) = 1$ which holds as before.

Start with the (T^{-1}, T, mT^{-1}) cocycle condition:

$$a(T^{-1}, T)a(T, mT^{-1}) = a(T^{-1}, TmT^{-1})$$

Apply (T, m, T^{-1}) to the third term. It becomes $a(Tm, T^{-1})a(T, m)a(m, T^{-1})$. Note that $a(T, m) = \frac{w(T,m^{-1})a(T,T^{-1})}{a(Tm,T^{-1})}$ and that $a(T^{-1}, TmT^{-1}) = \frac{w(T^{-1},Tm^{-1}T^{-1})a(T^{-1},T)}{a(mT^{-1},T)}$. By (m, T, T^{-1}), we have $a(mT^{-1}, T) = \frac{a(T^{-1},T)}{a(m,T^{-1})}$ The first equation becomes

$$w(T, m^{-1}) = w(T^{-1}, Tm^{-1}T^{-1})$$

Since $\alpha_T\alpha_{T^{-1}} = 1$, this becomes $w(T, m^{-1})w(T, m) = 1$ which is true by axiom (2.18). This proves the (T^{-1}, T, mT^{-1}) cocycle condition.

Now consider the (lT^{-1}, T, m) condition:

$$a(lT^{-1}, T)a(l, m)a(T, m) = a(lT^{-1}, Tm)$$

Hit the first term with (l, T^{-1}, T) to get $\frac{a(T^{-1},T)}{a(l,T^{-1})}$ and the fourth term with (l, T^{-1}, Tm) to get $\frac{a(l,m)a(T^{-1},Tm)}{a(l,T^{-1})}$. Apply the new result (T^{-1}, T, m) to $a(T^{-1}, Tm)$ to get $a(T, m)a(T^{-1}, T)$. Everything cancels. This proves (lT^{-1}, T, m).

Now consider the (lT, kT, m) condition:

$$a(lT, kT)a(lTkT, m)a(kT, m) = a(lT, kTm)$$

Hit the first term with (lT, k, T), the second term with the new result (lTk, T, m), the third term with (k, T, m), and the fourth term with (lT, k, Tm). Everything cancels.

Finally, check $(kT, T^{-1}l, mT)$:

$$a(kT, T^{-1}l)a(kl, m)a(T^{-1}l, mT) = a(kT, T^{-1}lmT)$$

The last term becomes $a(kT, T^{-1}lm)a(T^{-1}lm, T)$ by $(kT, T^{-1}lm, T)$. $a(kT, T^{-1}lm)$ becomes $a(kl, m)a(kT, T^{-1}l)a(T^{-1}l, m)$ by $(kT, T^{-1}l, m)$ and $a(T^{-1}lm, T)$ becomes

$\frac{a(T^{-1}l,mT)}{a(T^{-1}l,m)}$ by $(T^{-1}l, m, T)$. Everything cancels, proving the last cocycle condition (kT, lT, mT).

This proves that each invertible unoriented equivariant TQFT arises from a twisted 2-cocycle. Since this twisted 2-cocycle gives an inverse to f, we have shown that f is surjective. This completes the proof of Proposition 2.

Appendix C

APPENDICES FOR CHAPTER 3

C.1 Diagrams for the ground states

These diagrams are used in the argument of Section 3.5.

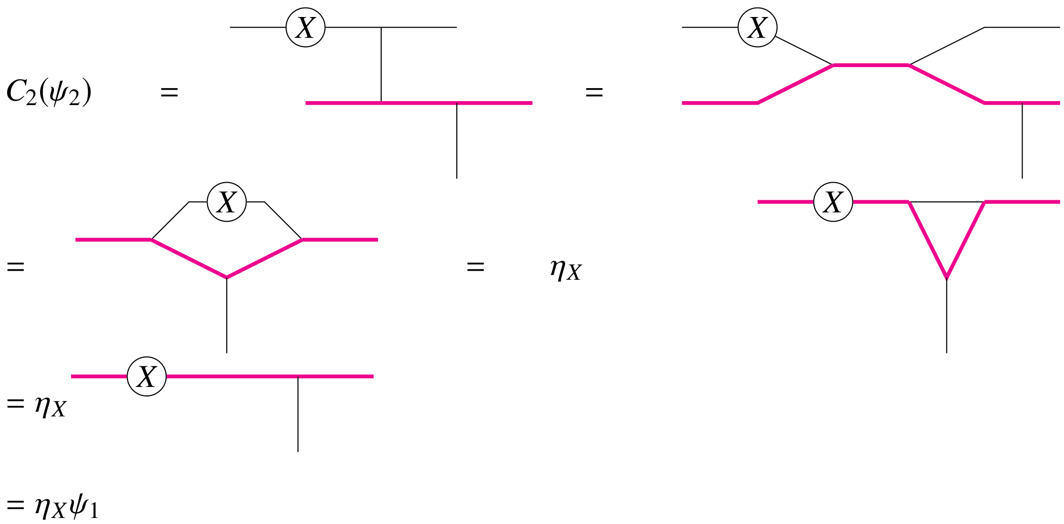


Figure C.1: Diagrammatic proof of $C_2\langle\psi_1| = \eta_X\langle\psi_2|$.

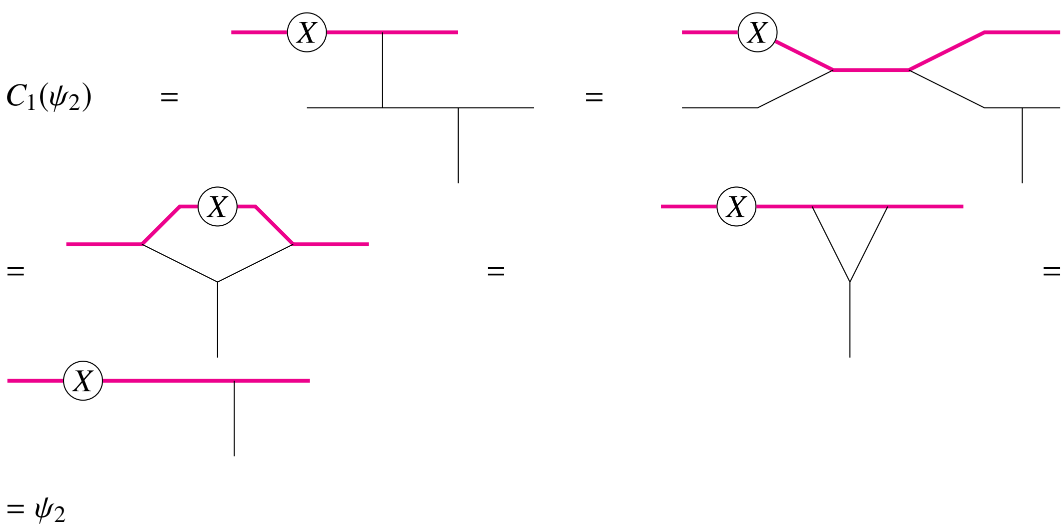
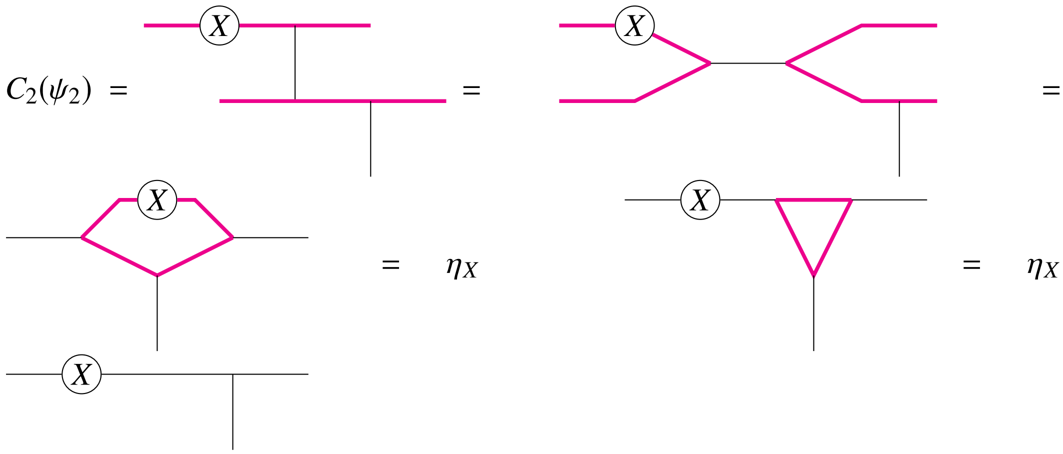
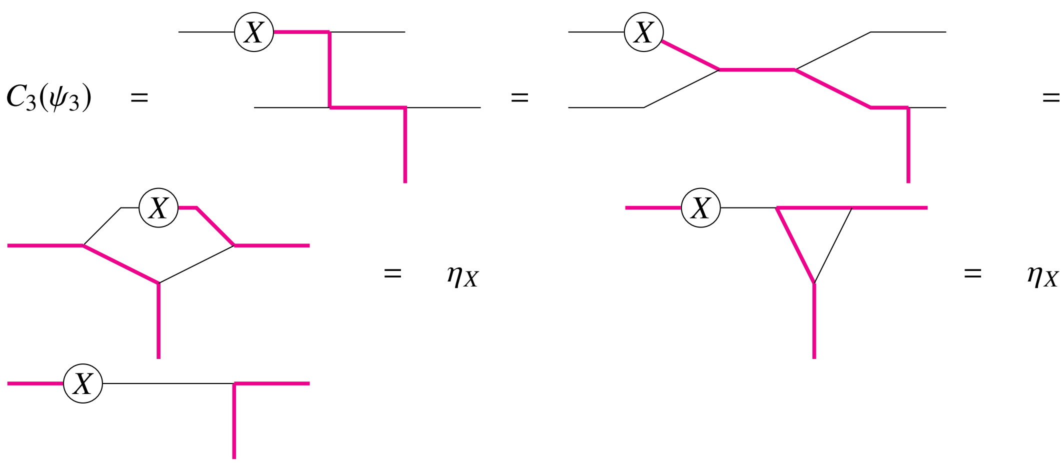


Figure C.2: Diagrammatic proof of $C_1\langle\psi_2| = \langle\psi_2|$.



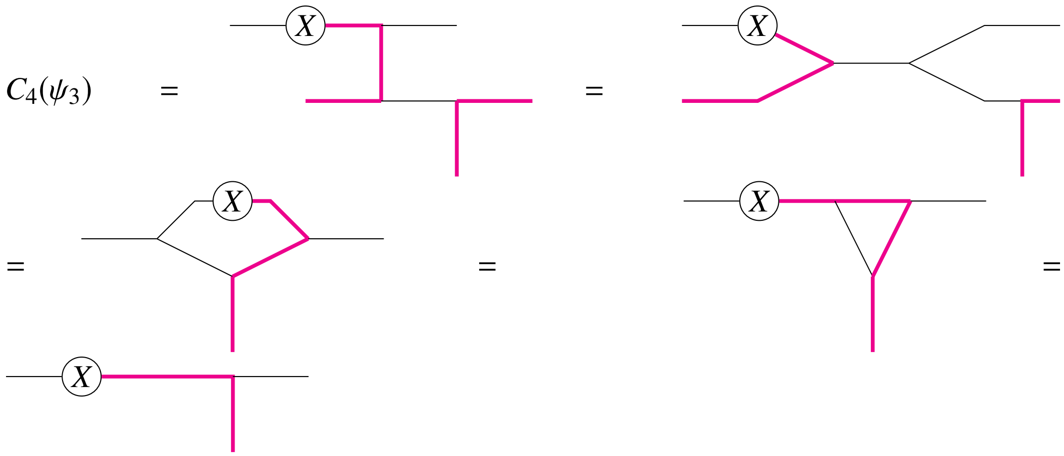
$= \eta_X \psi_1$

Figure C.3: Diagrammatic proof of $C_2\langle\psi_2| = \eta_X\langle\psi_1|$.



$= \eta_X \psi_4 = \psi_3$

Figure C.4: Diagrammatic proof of $C_3\langle\psi_3| = \langle\psi_3|$.

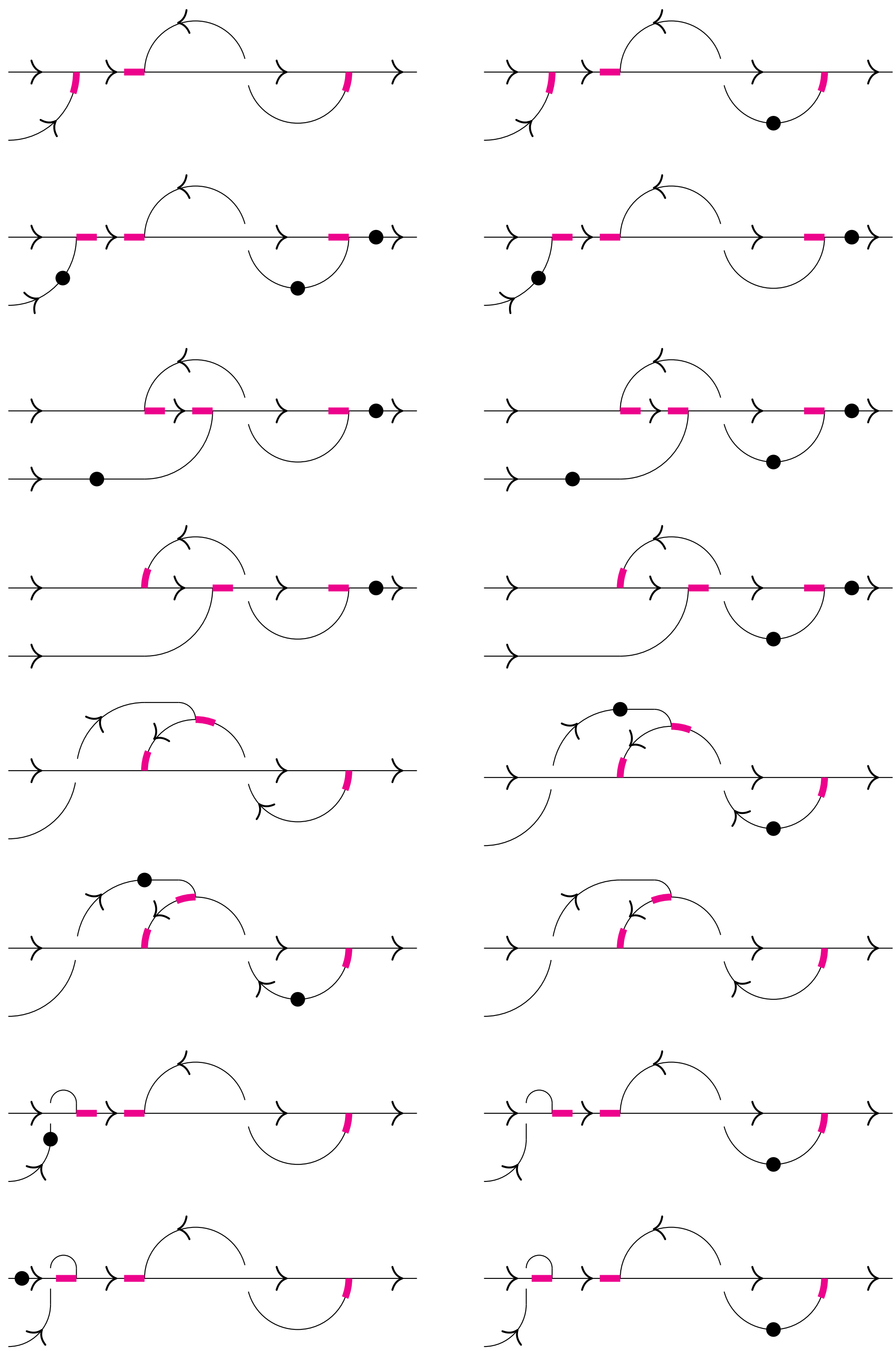


$= \psi_3$

Figure C.5: Diagrammatic proof of $C_4\langle\psi_3| = \langle\psi_3|$.

C.2 Necessity of supercommutativity

This appendix is a derivation the results (3.83) and (3.84) from the lattice spin formalism introduced in Section 3.4. Consider acting on the state $|ij\rangle$ with the cylinder map $Z(C)$; this is represented in the top diagram of each column of Figure C.6. To manipulate these diagrams into the diagrams at the bottom of each column, one applies a series of "moves" that are like Pachner moves but are compatible with the lattice spin structure (see (Novak and Runkel, 2015) for details). Finally, one unbraids the legs at the cost of a sign $(-1)^{|i||j|}$.



(a) NS sector: $C_{ij} = (-1)^{|i||j|+|i|} C_{ji}$ (b) R sector: $C_{ij} = (-1)^{|i||j|} C_{ji}$

Figure C.6: A proof of equations (3.83) and (3.84). Arrows denote edge directions, magenta line segments denote special edges, and black dots denote spin signs +1, i.e. insertions of $\mathcal{F}$.

C.3 Appendix: Description of ω in terms of pairs (α, β)

Start with some $[\omega] \in H^2(\mathcal{G}, U(1))$. We denote by $\bar{g}$ either an element of G_b or the corresponding element in $\mathcal{G}$ whose $t(g) = 0$, i.e. $(\bar{g}, 0)$. A general element of $\mathcal{G}$ then takes the form of either $\bar{g}$ or $\bar{g}p$.

Given an arbitrary ω, we can shift it by a coboundary δB where $B \in C^1(\mathbb{Z}_2, U(1))$ such that $B(0) = 0$ and $B(p) = \frac{1}{2}\omega(p,p)$ so that our new ω satisfies $\omega(p,p) = 0$. Then we can add a coboundary δA with $A \in C^1(\mathcal{G}, \mathbb{Z}_2)$ satisfying $A(\bar{g}p) = A(\bar{g}) - \omega(\bar{g}, p)$ to ω to make $\omega(\bar{g}, p) = 0$ for all $\bar{g} \in G_b$.

Evaluating the 3-cochain $\delta\omega$ on $(\bar{g}, p, p)$, $(\bar{g}, \bar{h}, p)$, and $(\bar{g}p, \bar{h}, p)$, and using the fact that $\delta\omega = 0$, we see that changing the second argument of ω by p does not affect its value, i.e. $\omega(g, h) = \omega(g, hp)$, $\forall g, h \in \mathcal{G}$.

Then, evaluating $\delta\omega$ on $(\bar{g}, p, \bar{h})$ gives $\omega(\bar{g}p, \bar{h}) = \omega(\bar{g}, \bar{h}) + \omega(p, \bar{h})$. Defining $\alpha(\bar{g}, \bar{h}) := \omega(\bar{g}, \bar{h})$ and $\beta(\bar{g}) := \omega(p, \bar{g})$, $\omega = \alpha + t \cup \beta$, and we can check that $\delta\beta = 0$ and hence $\delta\alpha = -\delta t \cup \beta = \rho \cup \beta$. With our gauge choice, one can show that this definition of β is consistent with $\beta(\bar{g}) = |Q(g)|$. The residual gauge freedom which shifts ω by a coboundary $\delta\lambda$ for λ which is a pull-back from G_b. This leaves β invariant but shifts α by a G_b-coboundary. Hence $\alpha \sim \alpha + \delta\lambda$, and we see that equivalence classes of ω correspond to equivalence classes of pairs (α, β) satisfying $\delta\alpha = \rho \cup \beta$ and $\delta\beta = 0$ with $(\alpha, \beta) \sim (\alpha + \delta\lambda, \beta)$.

When $\mathcal{G}$ splits, ρ is trivial and we have $\delta\alpha = 0$, so the set of equivalence classes of α is $H^2(G_b, U(1))$. The set of equivalence classes of β is of course $H^1(G_b, \mathbb{Z}_2)$. This confirms $H^2(\mathcal{G}, U(1)) \simeq H^2(G_b, U(1)) \times H^1(G_b, \mathbb{Z}_2)$, which we already knew from more abstract arguments.

C.4 Derivation of the group law for fermionic SRE phases

In the body of the paper we derived the supertensor product of two $\mathcal{G}$-graded algebras of the form $\mathrm{E}nd(U_i)$, $i = 1, 2$, where (Q_i, U_i) is a projective representation of $\mathcal{G} = G_b \times \mathbb{Z}_2$. This allowed us to determine the group law for $\gamma = 0$ SRE phases. Here we compute the supertensor product for $\mathcal{G}$-equivariant algebras involving a $\mathrm{C}l(1)$ factor and determine the group law in the remaining cases.

Let (Q_1, U_1) be a projective representation of $\mathcal{G}$ with a 2-cocycle parameterized by a pair $(\alpha_1, \beta_1) \in Z^2(G_b, U(1)) \times Z^1(G_b, \mathbb{Z}_2)$. We will denote $Q_1(p) = P$, so that

$$Q_1(g)Q_1(h) = \exp(2\pi i \alpha_1(g,h)) Q_1(gh), \quad P Q_1(g) P^{-1} = (-1)^{\beta_1(\bar{g})} Q_1(g). \quad \text{(C.1)}$$

Let (Q_2, U_2) be a projective representation of G_b with a 2-cocycle $\alpha_2 \in Z^2(G_b, U(1))$, i.e.

$$Q_2(g)Q_2(h) = \exp(2\pi i \alpha_2(g,h))Q_2(gh) \tag{C.2}$$

The vector space U_2 is regarded as purely even. Let $\beta_2 : G_b \to \mathbb{Z}_2$ be a homomorphism. Let A_1 be the algebra $\mathrm{E}nd(U_1)$ with the obvious $\mathcal{G}$ action. Let $A_2 = \mathrm{E}nd(U_2) \otimes \mathrm{C}l(1)$, and define a $\mathcal{G}$ action on it as follows:

$$g : M \otimes \Gamma^m \mapsto (-1)^{m\beta_2(g)} Q_2(g) M Q_2(g)^{-1} \otimes \Gamma^m, \tag{C.3}$$

and

$$p : M \otimes \Gamma^m \mapsto (-1)^m M \otimes \Gamma^m. \tag{C.4}$$

where $M \in \mathrm{E}nd(U_2), m \in \mathbb{Z}_2$.

The first claim is that $A_1 \mathbin{\widehat{\otimes}} A_2$ is isomorphic (as a $\mathbb{Z}_2$-graded algebra) to $A_{12} = \mathrm{E}nd(U_1 \otimes U_2) \otimes \mathrm{C}l(1)$, where both U_1 and U_2 are regarded as purely even. The isomorphism is given by

$$JW : M_1 \mathbin{\widehat{\otimes}} M_2 \mathbin{\widehat{\otimes}} \Gamma^m \mapsto M_1 P^m \otimes M_2 \otimes \Gamma^{m+|M_1|} \tag{C.5}$$

We denoted it JW to indicate that it is a version of the Jordan-Wigner transformation. It is easy to check that the map preserves the product as well as grading, and its inverse is

$$JW^{-1} : M_1 \otimes M_2 \otimes \Gamma^m \mapsto M_1 P^{m+|M_1|} \mathbin{\widehat{\otimes}} M_2 \mathbin{\widehat{\otimes}} \Gamma^{m+|M_1|} \tag{C.6}$$

Thus the parameter γ for A_{12} is 1.

Next we compute the action of G_b on A_{12} induced by the isomorphism JW. We get:

$$JW \circ g \circ JW^{-1} : M_1 \otimes M_2 \otimes \Gamma^m \mapsto (-1)^{(\beta_1(\bar{g})+\beta_2(\bar{g}))(m+|M_1|)} Q_1(g) M_1 Q_1(g)^{-1} \otimes Q_2(g) M_2 Q_2(g)^{-1} \otimes \Gamma^m. \tag{C.7}$$

To bring this G_b-action to the standard form, we define $\tilde{Q}_1(g) = Q_1(g) P^{\beta_1(g)+\beta_2(g)} i^{\beta_1(g)}$.[1] Then the G_b-action on $\mathrm{E}nd(U) \otimes \mathrm{C}l(1)$ takes the form

$$M_1 \otimes M_2 \otimes \Gamma^m \mapsto (-1)^{m(\beta_1(\bar{g})+\beta_2(\bar{g}))} \tilde{Q}_1(g) M_1 \tilde{Q}_1(g)^{-1} \otimes Q_2(g) M_2 Q_2(g)^{-1} \otimes \Gamma^m. \tag{C.8}$$

Thus the parameter β for A_{12} is $\beta_1 + \beta_2$. Finally, it is easy to check that the matrices $\tilde{Q}_1(g) \otimes Q_2(g)$ form a projective representation of G_b with a 2-cocycle

$$\alpha(g,h) = \alpha_1(g,h) + \alpha_2(g,h) + \frac{1}{2}\beta_1(h)\beta_2(g). \tag{C.9}$$

[1] If the factor of $i^{\beta_1(g)}$ is omitted, α shifts by a coboundary $\delta(i^{\beta_1}) = (-1)^{\beta_1 \cup \beta_1}$ but its class is unchanged.

We conclude that the group law for the parameters (α, β, γ) obeys

$$(\alpha_1, \beta_1, 0) + (\alpha_2, \beta_2, 1) = (\alpha_1 + \alpha_2 + \frac{1}{2}\beta_1 \cup \beta_2, \beta_1 + \beta_2, 1). \tag{C.10}$$

The last case to consider is $\gamma_1 = \gamma_2 = 1$. The algebras to be tensored are $A_1 = \mathrm{E}nd(U_1) \otimes \mathrm{C}l(1)$ and $A_2 = \mathrm{E}nd(U_2) \otimes \mathrm{C}l(1)$, where (Q_1, U_1) and (Q_2, U_2) are projective representations of G_b with 2-cocycles α_1 and α_2. The group G_b acts as follows on the generators of the two Clifford algebras:

$$g : \Gamma_i \mapsto (-1)^{\beta_i(\bar{g})}\Gamma_i, \quad i = 1, 2. \tag{C.11}$$

It is easy to see that $\mathrm{C}l(1) \mathbin{\widehat{\otimes}} \mathrm{C}l(1) = \mathrm{C}l(2)$, and that $\mathrm{C}l(2) \simeq \mathrm{E}nd(\mathbb{C}^2)$. The isomorphism sends Γ_i to σ_i, $i = 1, 2$, and the action of p on $\mathbb{C}^2$ is given by the Pauli matrix $\sigma_3 = -i\Gamma_1\Gamma_2$. Thus

$$A_{12} = A_1 \mathbin{\widehat{\otimes}} A_2 \simeq \mathrm{E}nd(U_1 \otimes U_2 \otimes \mathbb{C}^2), \tag{C.12}$$

where U_1 and U_2 are regarded as purely even. Thus the γ parameter for A_{12} is 0.

The group G_b acts on $U_1 \otimes U_2$ by $Q_1 \otimes Q_2$. This is a projective action, with a 2-cocycle $\alpha_1 + \alpha_2$. There is no canonical choice of the projective G_b action on $\mathbb{C}^2$ which induces the action (C.11) on $\mathrm{C}l(2) \simeq \mathrm{E}nd(\mathbb{C}^2)$. One possible choice is

$$g : v \mapsto \Gamma_1^{\beta_2(\bar{g})}\Gamma_2^{\beta_1(\bar{g})}v, \quad v \in \mathbb{C}^2. \tag{C.13}$$

Any other choice differs from this one by a scalar factor $\exp(\lambda(g))$ which changes the corresponding 2-cocycle by a coboundary. Using the action (C.13), the corresponding 2-cocycle is $\frac{1}{2}\beta_1(\bar{g})\beta_2(\bar{h})$. The net result is that the G_b action on $U_1 \otimes U_2 \otimes \mathbb{C}^2$ is projective with a 2-cocycle $\alpha_1 + \alpha_2 + \frac{1}{2}\beta_1 \cup \beta_2$. We also compute:

$$(-i\Gamma_1\Gamma_2)\Gamma_1^{\beta_2(\bar{g})}\Gamma_2^{\beta_1(\bar{g})} = (-1)^{\beta_1(\bar{g})+\beta_2(\bar{g})}\Gamma_1^{\beta_2(\bar{g})}\Gamma_2^{\beta_1(\bar{g})}(-i\Gamma_1\Gamma_2). \tag{C.14}$$

This implies that the parameter β for A_{12} is $\beta_1 + \beta_2$.

We have shown that for the special case $\gamma_1 = \gamma_2 = 1$ the group law says

$$(\alpha_1, \beta_1, 1) + (\alpha_2, \beta_2, 1) = (\alpha_1 + \alpha_2 + \frac{1}{2}\beta_1 \cup \beta_2, \beta_1 + \beta_2, 0) \tag{C.15}$$

This completes the proof of (3.126).

C.5 Relations between bosonic and fermionic invariants

Lemma C.5.1. *For a twisted cocycle $\omega \in Z^2(\mathcal{G}, U(1)_T)$, the 1-cochain defined by*

$$\begin{aligned} {}^1\!/\!_2\beta(g) &:= \omega(g,p) - \omega(p,g) + x(g)\omega(p,p) \\ &= \begin{cases} \omega(g,p) - \omega(p,g) & g \in G_0 \\ \omega(g,p) - \omega(p,g) + \omega(p,p) & g \notin G_0 \end{cases} \end{aligned} \tag{C.16}$$

is gauge-invariant, satisfies $\beta(gp) = \beta(g)$, takes values in $\{0, {}^1\!/\!_2\}$, and defines a G_b-cocycle.

Proof. First, ${}^1\!/\!_2\beta(g)$ picks up a factor of

$$\begin{aligned} (L(g) + (-1)^{x(g)}L(p) - L(gp)) - (L(p) + L(g) - L(gp)) \\ + x(g)(L(p) + L(p) - L(1)) \\ = -2x(g)L(p) + 2x(g)L(p) - x(g)L(1) = 0 \end{aligned} \tag{C.17}$$

under a transformation $\omega \mapsto \omega + \delta_T L$ for some 1-cochain L of $\mathcal{G}$ satisfying $L(1) = 0$.[2]

Second,

$$\begin{aligned} {}^1\!/\!_2\beta(gp) &= \omega(gp,p) - \omega(p,gp) + x(gp)\omega(p,p) \\ &= \omega(g,p) - \omega(p,g) + x(g)\omega(p,p) - \delta_T\omega(p,g,p) \\ &= {}^1\!/\!_2\beta(g). \end{aligned} \tag{C.18}$$

Third,

$$\begin{aligned} &\omega(g,p) - \omega(p,g) \\ &\quad = (-1)^{x(g)}\omega(p,p) - \omega(g,p) - \omega(p,gp) \\ &\qquad\qquad - (\delta_T\omega)(g,p,p) + (\delta_T\omega)(p,g,p) \\ &\quad = (-1)^{x(g)}\omega(p,p) - \omega(g,p) - \omega(p,p) \\ &\qquad\qquad + \omega(p,g) + (\delta_T\omega)(p,p,g) \\ &\quad = -\omega(g,p) + \omega(p,g) - (1 - (-1)^{x(g)})\omega(p,p) \end{aligned} \tag{C.19}$$

means that ${}^1\!/\!_2\beta$ takes values in the $\mathbb{Z}/2$ subgroup of $U(1)$.

[2] This condition on L ensures that $Q(1) = \mathbb{1}$ is preserved.

Therefore ${}^1\!/\!{}_2\beta$ defines a $\beta \in C^1(G_b, \mathbb{Z}/2)$. Let $g_b, h_b \in G_b$ and choose any lifts g, h to $\mathcal{G}$. Fourth,

$$
\begin{aligned}
&(\delta\beta)(g_b, h_b)\\
&\quad = {}^1\!/\!{}_2(\beta(g) + \beta(h) - \beta(ghp^{\rho(\bar{g},\bar{h})}))\\
&\quad = {}^1\!/\!{}_2(\beta(g) + \beta(h) - \beta(gh))\\
&\quad = \omega(g,p) - \omega(p,g) + x(g)\omega(p,p)\\
&\qquad\quad + \omega(h,p) - \omega(p,h) + x(h)\omega(p,p)\\
&\qquad\quad - \omega(gh,p) + \omega(p,gh) - x(gh)\omega(p,p)\\
&\quad = \omega(g,p) - \omega(p,g) + \omega(h,p) - \omega(p,h)\\
&\qquad\quad + \omega(g,h) - \omega(g,hp) - (-1)^{x(g)}\omega(h,p)\\
&\qquad\quad - \omega(g,h) + \omega(p,g) + \omega(pg,h) + 2x(g)x(h)\omega(p,p)\\
&\quad = \omega(g,p) + 2x(g)\omega(h,p) - \omega(p,h)\\
&\qquad\quad + (-1)^{x(g)}\omega(p,h) - \omega(g,p) + 2x(g)x(h)\omega(p,p)\\
&\quad = 2x(g)(\omega(h,p) - \omega(p,h) + x(h)\omega(p,p))\\
&\quad = 2x(g)\cdot {}^1\!/\!{}_2\beta(h)\\
&\quad = 0.
\end{aligned}
\tag{C.20}
$$

□

Lemma C.5.2. *Each cohomology class $H^2(\mathcal{G}, U(1)_T)$ contains an element ω that satisfies, for all $g, h \in \mathcal{G}$,*

$$\omega(pg, h) = \omega(g, h) \tag{C.21}$$

$$\omega(g, ph) = \omega(g, h) + \omega(g, p). \tag{C.22}$$

Proof. For an arbitrary 2-cocycle $W \in Z^2(\mathcal{G}, U(1)_T)$, define

$$\omega = W - \delta_T L \tag{C.23}$$

where $L \in C^1(\mathcal{G}, U(1)_T)$ satisfies

$$
\begin{aligned}
L(1) &= 0\\
L(p) &= {}^1\!/\!{}_2W(p,p) \ \text{ or } \ {}^1\!/\!{}_2W(p,p) + {}^1\!/\!{}_2\\
L(p\bar{g}) &= L(\bar{g}) - W(p,\bar{g}) + L(p).
\end{aligned}
\tag{C.24}
$$

Here, we abuse notation by letting $\bar{g}$ denote a $g \in \mathcal{G}$ with $t(g) = 0$. This implies $L(p) = {}^1\!/\!{}_2W(p,p)$. We have fixed $L(p\bar{g})$ in terms of $L(\bar{g})$ and $L(p)$ but left $L(\bar{g})$ undetermined, while $L(p)$ is fixed up to a ${}^1\!/\!{}_2$.

We see that

$$\begin{aligned}\omega(p,p) &= W(p,p) - (-1)^{x(p)}L(p) - L(p) + L(1)\\ &= W(p,p) - 2\cdot {}^1\!/\!_2 W(p,p) = 0\end{aligned} \tag{C.25}$$

and

$$\begin{aligned}\omega(p,\bar g) &= W(p,\bar g) - \Big((-1)^{x(p)}L(\bar g) + L(p) - L(p\bar g)\Big)\\ &= W(p,\bar g) - W(p,\bar g) + {}^1\!/\!_2 W(p,p) - {}^1\!/\!_2 W(p,p) = 0.\end{aligned} \tag{C.26}$$

Next we show that any ω satisfying (C.25) and (C.26) must also satisfy the gauge conditions (C.21) and (C.22). First,

$$\begin{aligned}\omega(p,p\bar g) = -\delta_T\omega(p,p,\bar g) &- (-1)^x(p)\omega(p,\bar g)\\ &+ \omega(p,p) + \omega(1,\bar g) = 0.\end{aligned} \tag{C.27}$$

Similarly, computing $0 = \delta_T\omega(p,\bar g,\bar h)$ shows that $\omega(\bar g p,\bar h) = \omega(\bar g,\bar h)$ and computing $0 = \delta_T\omega(p,\bar g,\bar h p)$ shows that $\omega(\bar g p,\bar h p) = \omega(\bar g,\bar h p)$. Putting these together, we see that (C.21) is satisfied.

Now we compute $0 = \delta_T\omega(\bar g,p,\bar h)$ which shows that $\omega(\bar g,p\bar h) = \omega(\bar g,\bar h) + \omega(\bar g,p)$ and $0 = \delta_T\omega(\bar g,p,p\bar h)$ which shows that $\omega(\bar g,\bar h) = \omega(\bar g,p\bar h) + \omega(\bar g,p)$. Putting these together, we see that (C.22) is satisfied.

□

Lemma C.5.3. *Given a trivialization t, the map*

$$\omega(g,h) = \alpha(\bar g,\bar h) + {}^1\!/\!_2\beta(\bar g)t(h) \tag{C.28}$$

defines a bijection from pairs $(\alpha,\beta) \in C^2(G_b,U(1)_T)\times C^1(G_b,\mathbb{Z}/2)$ that satisfy $\delta_T\alpha = {}^1\!/\!_2\beta\cup\rho$ and $\delta\beta = 0$ (where ${}^1\!/\!_2\beta$ is regarded as a $U(1)_T$-valued cocycle) to twisted cocycles $\omega \in Z^2(\mathcal{G},U(1)_T)$ that satisfy (C.21) *and* (C.22) *for all $g,h\in\mathcal{G}$. In particular, for all $g_b,h_b\in G_b$, this map has an inverse*

$$\begin{aligned}\alpha(g_b,h_b) &= \omega(s(g_b),s(h_b))\\ {}^1\!/\!_2\beta(g_b) &= \omega(s(g_b),p).\end{aligned} \tag{C.29}$$

Proof. First we show that ω is a twisted cocycle:

$$\begin{aligned}
&(\delta_T\omega)(g,h,k) \\
&= (-1)^{x(g)}\omega(h,k) + \omega(g,hk) - \omega(g,h) - \omega(gh,k) \\
&= (-1)^{x(g)}\alpha(\bar{h},\bar{k}) + \alpha(\bar{g},\bar{hk}) - \alpha(\bar{g},\bar{h}) - \alpha(\bar{gh},\bar{k}) \\
&\qquad + {}^1\!/\!_2(-1)^{x(g)}\beta(\bar{h})t(k) + {}^1\!/\!_2\beta(\bar{g})t(hk) \\
&\qquad - {}^1\!/\!_2\beta(\bar{g})t(h) - {}^1\!/\!_2\beta(\bar{gh})t(k) \\
&= (\delta_T\alpha)(\bar{g},\bar{h},\bar{k}) + {}^1\!/\!_2(\delta_T\beta)(\bar{g},\bar{h})t(k) - {}^1\!/\!_2\beta(\bar{g})(\delta t)(h,k) \\
&= 0.
\end{aligned} \tag{C.30}$$

Next we verify that ω satisfies the gauge conditions:

$$\begin{aligned}
\omega(pg,h) &= \alpha(\bar{pg},\bar{h}) + {}^1\!/\!_2\beta(\bar{pg})t(h) \\
&= \alpha(\bar{g},\bar{h}) + {}^1\!/\!_2\beta(\bar{g})t(h) \\
&= \omega(g,h)
\end{aligned} \tag{C.31}$$

$$\begin{aligned}
\omega(g,ph) &= \alpha(\bar{g},\bar{ph}) + {}^1\!/\!_2\beta(\bar{g})t(ph) \\
&= \alpha(\bar{g},\bar{h}) + {}^1\!/\!_2\beta(\bar{g})(t(h)+1) \\
&= \omega(g,h) + \omega(g,p).
\end{aligned} \tag{C.32}$$

Then we check the conditions for α and β. For these two calculations, let $\bar{g}$ denote g_b and g denote $s(g_b)$. Note that $s(\bar{gh}) = p^{\rho(\bar{g},\bar{h})}gh$. Then

$$\begin{aligned}
&(\delta_T\alpha)(\bar{g},\bar{h},\bar{k}) \\
&= (-1)^{x(g)}\alpha(\bar{h},\bar{k}) + \alpha(\bar{g},\bar{hk}) - \alpha(\bar{g},\bar{h}) - \alpha(\bar{gh},\bar{k}) \\
&= (-1)^{x(g)}\omega(h,k) + \omega(g,p^{\rho(\bar{h},\bar{k})}hk) \\
&\qquad - \omega(g,h) - \omega(p^{\rho(\bar{g},\bar{h})}gh,k) \\
&= (\delta_T\omega)(g,h,k) + \{\text{terms of the form } \omega(p^-,-)\} \\
&\qquad + \omega(g,p^{\rho(\bar{h},\bar{k})}) + (\delta_T\omega)(p^{\rho(\bar{g},\bar{h})},gh,k) \\
&\qquad - (\delta_T\omega)(g,p^{\rho(\bar{h},\bar{k})},hk) + (\delta_T\omega)(p^{\rho(\bar{h},\bar{k})},g,hk) \\
&= {}^1\!/\!_2\beta(\bar{g})\rho(\bar{h},\bar{k}).
\end{aligned} \tag{C.33}$$

The object ${}^1\!/\!_2\beta$ defined in (C.29) is the gauge-fixed form of (C.16). Then, by C.5.1, it defines a $\beta \in Z^1(G_b,\mathbb{Z}/2)$.

It remains to show that these maps are indeed inverses. Since ${}^1\!/\!_2\beta$ is the image of a $\mathbb{Z}/2$-valued cocycle β, ω can be written with a minus sign like $\omega = \alpha - {}^1\!/\!_2\beta \cup t$.

Note also that $s(\bar{g}) = p^{t(g)} g$. Then

$$\begin{aligned} \omega(g,h) &= \alpha(\bar{g},\bar{h}) - {}^1\!/\!_2\beta(\bar{g})t(h) \\ &= \omega(p^{t(g)}g, p^{t(h)}h) - \omega(p^{t(g)}g, p)t(h) \\ &= \omega(g,h), \end{aligned} \tag{C.34}$$

$$\begin{aligned} \alpha(g_b,h_b) &= \omega(s(g_b), s(h_b)) \\ &= \alpha(g_b,h_b) + {}^1\!/\!_2\beta(g_b)t(s(g_b)) \\ &= \alpha(g_b,h_b), \end{aligned} \tag{C.35}$$

$$\begin{aligned} {}^1\!/\!_2\beta(g_b) &= \omega(s(g_b), p) \\ &= \alpha(g_b, 1) + {}^1\!/\!_2\beta(g_b)t(p) \\ &= {}^1\!/\!_2\beta(g_b). \end{aligned} \tag{C.36}$$

□

Theorem C.5.4. *$H^2(\mathcal{G}, U(1)_T)$ equals, as a set, the set of pairs (α,β) (see C.5.3) modulo the equivalence $(\alpha',\beta) \sim (\alpha,\beta)$ if $\alpha' = \alpha + \delta_T\lambda$ with λ a cochain in $C^1(\mathcal{G}, U(1)_T)$ satisfying $\lambda(s(g_b)p) = \lambda(s(g_b)) + \lambda(p)$.*[3]

Proof. The preceding lemmas show that the set of twisted cocycles ω satisfying the gauge conditions (C.21) and (C.22) is equivalent to the set of pairs (α,β). After transforming ω into this gauge, there remains freedom to choose $L(g)$ for each $g \in \mathcal{G}$ such that $t(g) = 0$, and to shift $L(p)$ by ${}^1\!/\!_2$. We have already seen that β is invariant under an arbitrary gauge transformation. However, there is some residual gauge freedom for α.

Let $\omega' = \omega + \delta_T\lambda$ be another 2-cocycle satisfying the gauge conditions. It takes the form $W - \delta_T L'$, with L' possibly differing from L in its values on $s(g_b)$ and p. We see from $\delta_T\lambda = \omega' - \omega = \delta_T(L' - L)$ that $\lambda = L' - L + \kappa$ where κ is a twisted 1-cocycle. Then, by (C.24), $\lambda(s(g_b)p) = \lambda(s(g_b)) + \lambda(p)$. The quantities $L(p)$, $L'(p)$, $\kappa(p)$, and therefore $\lambda(p)$, can each be chosen to be 0 or ${}^1\!/\!_2$. Finally, by (C.29), this freedom in gauge-fixed ω translates into the desired freedom in α.

□

[3]Had we chosen a different representative $\rho' = \rho + \delta\mu$ of $[\rho]$ to describe the extension of G_b by $\mathbb{Z}_2^F$, we would have considered a different set of cochains α (modulo coboundaries), shifted by ${}^1\!/\!_2\beta \cup \mu$, but their counting would be the same.

Appendix D

APPENDICES FOR CHAPTER 5

D.1 Appendix: Pin groups

Here we review the definition and some properties of *Pin* groups following Ref. (Atiyah, Bott, and Shapiro, 1963). Just as $Spin(M)$ is a non-trivial extension of $SO(M)$ by $\mathbb{Z}_2$, $Pin_+(M)$ and $Pin_-(M)$ are extensions of $O(M)$ by $\mathbb{Z}_2$. Since $O(M)$ has two connected components, so do $Pin_\pm(M)$. The connected component of the identity for both $Pin_+(M)$ and $Pin_-(M)$ is $Spin(M)$.

The groups $Pin_\pm(M)$ can be defined using the Clifford algebra $\mathrm{C}l(M)$. To define $Pin_+(M)$, one considers the Clifford algebra for the positive metric:

$$\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}, \quad I, J = 1, \dots, M. \tag{D.1}$$

This is a $\mathbb{Z}_2$-graded algebra. For any $a \in Cl(M)$ we let $\varepsilon(a) = a$ if a is even and $\varepsilon(a) = -a$ if a is odd. Invertible elements in the Clifford algebra from a group. $Pin_+(M)$ is a subgroup generated by elements of the form $\not{v} = \Gamma^I v^I$, where v_I is a unit vector in $\mathbb{R}^M$. To define the homomorphism $Pin_+(M) \to O(M)$, consider the "twisted conjugation map"

$$\Gamma^J \mapsto \varepsilon(a)\Gamma^J a^{-1}, \quad a \in Cl(M). \tag{D.2}$$

If $a = \not{v}$, then this map becomes

$$\Gamma^J \mapsto -\not{v}\Gamma^J \not{v}^{-1} = \Gamma^J - 2v^J \not{v}. \tag{D.3}$$

This is a hyperplane reflection on the space spanned by Γ^J. Since the whole group $O(M)$ is generated by hyperplane reflections, twisted conjugation by elements of $Pin_+(M)$ gives a surjective homomorphism from $Pin_+(M)$ to $O(M)$. The kernel of this map is the $\mathbb{Z}_2$ generated by -1. The subgroup $Spin(M) \subset Pin_+(M)$ consists of products of an even number of hyperplane reflections. Note that every hyperplane reflection $\not{v}$ squares to the identity in $Pin_+(M)$.

The group $Pin_-(M)$ is defined similarly, except that one starts with the "negative" Clifford algebra

$$\{\Gamma^I, \Gamma^J\} = -2\delta^{IJ}, \quad I, J = 1, \dots, M. \tag{D.4}$$

In this case, hyperplane reflections $\not{v}$ square to -1, which generates the kernel of the homomorphism $Spin(M) \to SO(M)$. In other words, for $Pin_-(M)$, hyperplane reflections square to fermion parity.

Finally, the group $Pin_c(M)$ is defined as $(Pin_+(M) \times U(1))/\mathbb{Z}_2^{diag}$, and its subgroup $Spin_c(M) \subset Pin_c(M)$ is defined as $(Spin(M) \times U(1))/\mathbb{Z}_2$. $Pin_c(M)$ is an extension of $O(M)$ by $U(1)$, while $Spin_c(M)$ is an extension of $SO(M)$ by $U(1)$. It is easy to show that the group $(Pin_-(M) \times U(1))/\mathbb{Z}_2$ is isomorphic to $Pin_c(M)$. The significance of $Pin_c(M)$ is the following: if we regard the complexification of the Clifford algebra as the algebra of observables of a fermionic system, then $Pin_c(M)$ can be identified with the subgroup of those unitaries which act linearly on the generators of the Clifford algebra. Thus lifting a real linear action of a group G on the Clifford generators Γ^I to a unitary action on the Fock space is equivalent to lifting the corresponding homomorphism $G \to O(M)$ to a homomorphism $G \to Pin_c(M)$. Similarly, if we are given a homomorphism $G \to SO(M)$, lifting it to a unitary action on the Fock space is the same as lifting it to a homomorphism $G \to Spin_c(M)$.

D.2 Appendix: Characteristic classes of representations of finite groups

The theory of characteristic classes of vector bundles (a classic reference is (Milnor and Stasheff, 1974)) is familiar to physicists. A version of this construction also gives rise to characteristic classes of representations of a finite group which take values in cohomology of the said group (Atiyah, 1961). Real representations give rise to Stiefel-Whitney and Pontryagin classes, while complex representations give rise to Chern classes.

To define these classes, it is best to think of a real representation of G of dimension n as a homomorphism $R : G \to O(n)$, which then induces a continuous map of classifying spaces $\tilde{R} : BG \to BO(n)$. The map $\tilde{R}$ is defined up to homotopy only, but this suffices to define cohomology classes on BG by pull-back from $BO(n)$. Any cohomology class ω on $BO(n)$ thus defines a cohomology class $\tilde{R}^*\omega$ on BG. Cohomology classes on $BO(n)$ are precisely characteristic classes of real vector bundles, and their pull-backs via $\tilde{R}$ are called characteristic classes of the representation R. Similarly, given a complex representation $R : G \to U(n)$, we get a continuous map $\tilde{R} : BG \to BU(n)$, and can define Chern classes of R by pull-back.

In low dimensions, these classes have a concrete representation-theoretic interpretation. For example, the 1st Stiefel-Whitney class $w_1(R) \in H^1(G, \mathbb{Z}_2)$ of a real representation R is the obstruction for $R : G \to O(n)$ to descend to homomorphism

$R' : G \to SO(n)$. Obviously $w_1(r)(g)$ is given by $\det R(g)$.

Similarly, the 1st Chern class $c_1(R) \in H^2(G, \mathbb{Z})$ of a complex representation R can be interpreted as an obstruction for R to descend to $R' : G \to SU(n)$. The obstruction det $R(g)$ is a 1-cocycle on G with values in $U(1)$. The corresponding class in $H^2(G, \mathbb{Z})$ is obtained by applying the Bockstein homomorphism (which for finite groups is an isomorphism). Explicitly:

$$c_1(R)(g, h) = \frac{1}{2\pi i} (\log \det R(gh) - \log \det R(g) - \log \det R(h)) . \tag{D.5}$$

The 2nd Stiefel-Whitney class $w_2(R) \in H^2(G, \mathbb{Z}_2)$ is an obstruction to lifting R to a homomorphism $R' : G \to Pin_+(n)$. One can always define R' as a projective representation, and the corresponding 2-cocycle represents $w_2(R)$. The image of $w_2(R)$ in $H^2(G, U(1))$ under the embedding $\mathbb{Z}_2 \to U(1)$ is an obstruction to lifting R to a homomorphism $R' : G \to Pin_c(n)$. In the main text, it is denoted $w_2^{U(1)}(R)$. By the isomorphism $H^2(G, U(1)) \simeq H^3(G, \mathbb{Z})$ (valid for finite groups), this class can be interpreted as an element of $H^3(G, \mathbb{Z})$. Then it is known as the 3rd integral Stiefel-Whitney class W_3.

By functoriality, known relations between cohomology classes of $BO(n)$ and $BU(n)$ lead to relations between characteristic classes of representations. Let us describe those of them which we have used in the main text. First of all, the Whitney formula expresses Stiefel-Whitney (or Chern) classes of $R + R'$ in terms of Stiefel-Whitney (or Chern) classes of R and R':

$$w_k(R + R') = \sum_{p=0}^{k} w_p(R) \cup w_{k-p}(R'). \tag{D.6}$$

There are also more complicated formulas expressing characteristic classes of $R \otimes R'$ in terms of those of R and R' (Milnor and Stasheff, 1974). We will only need a particular case: let R be a real representation of odd dimension M, and L be a one-dimensonal real representation, then

$$w_2(R \otimes L) = w_2(R). \tag{D.7}$$

In Section 5.2, we propose that given a gapped 2d band Hamiltonian, the invariant $\beta \in H^2(G, \mathbb{Z}_2)$ of 2d fermionic SRE phases with symmetry $G \times \mathbb{Z}_2^F$ is given either by $w_2(R)$ or $w_2(R) + w_1(R)^2$, where R is a certain representation of G. The supercohomology equation implies that $\beta \cup \beta \in H^4(G, \mathbb{Z}_2)$ maps to a trivial class in $H^4(G, U(1))$. To show that this is automatically the case for our two candidates, we note that for finite groups $H^4(G, U(1)) \simeq H^5(G, \mathbb{Z})$. The class in $H^5(G, \mathbb{Z})$ corresponding to $\beta \cup \beta$ can be obtained by applying the Bockstein homomorphism $H^4(G, \mathbb{Z}_2) \to H^5(G, \mathbb{Z})$. A mod-2 class is annihilated by the Bockstein homomorphism if and only if it is a mod-2 reduction of an integral class. Now recall the well-known relation between Stiefel-Whitney classes and Pontryagin classes (Milnor and Stasheff, 1974):

$$w_2^2 = p_1 \; mod \, 2. \tag{D.8}$$

Hence w_2^2 is indeed annihilated by the Bockstein homomorphism. The same is true if we replace w_2 with $w_2 + w_1^2$. Indeed, since

$$(w_2 + w_1^2)^2 = w_2^2 + w_1^4, \tag{D.9}$$

it is sufficient to show that w_1^4 maps to a trivial class in $H^4(G, U(1))$. Now we recall that w_1^2 is cohomologous to $\delta\omega/2$, where ω is an integral lift of w_1. Therefore w_1^2 is cohomologous to $\frac{1}{2}\delta\omega \cup \frac{1}{2}\delta\omega$, which is a coboundary of $\frac{1}{4}\omega \cup \delta\omega$.

In Section 5.2, we show that for a band Hamiltonian, the invariant $\hat{\alpha} \in H^2(\hat{G}, U(1))$ of 1d fermionic SRE phases with symmetry $\hat{G}$ is equal to the image of $w_2(R)$ under the map $\iota : H^2(\hat{G}, \mathbb{Z}_2) \to H^2(\hat{G}, U(1))$, for a particular representation R. Obviously, any element in the image of ι has order 2, so in general not every element in $H^2(\hat{G}, U(1))$ can be realized by a band Hamiltonian. But we claimed that for some $\hat{G}$, even certain elements of order 2 in $H^2(\hat{G}, U(1))$ cannot be realized by band Hamiltonians. This happens because not every element in $H^2(\hat{G}, \mathbb{Z}_2)$ arises as $w_2(R)$ for some representation R. The reason is again the relation (D.8). It implies that for any representation R of $\hat{G}$, the Bockstein homomorphism annihilates $w_2(R)^2$. On the other hand, a generic element of $H^2(\hat{G}, \mathbb{Z}_2)$ need not have this property. An example of a finite group $\hat{G}$ for which some elements of $H^2(\hat{G}, \mathbb{Z}_2)$ do not arise as $w_2(R)$ for any R is given in (Gunarwardena, Kahn, and Thomas, 1989).

D.3 Appendix: Beta as a charge pumping invariant

As discussed in Section 5.2, fermionic SRE phases in 1d with symmetry $\hat{G}$ have an invariant $\beta \in H^1(G, \mathbb{Z}_2)$. More precisely, this invariant is defined if the invariant γ (the number of boundary fermionic zero modes modulo 2) vanishes. The definition of β given in Ref. (Fidkowski and Kitaev, 2010) relies on the properties of boundary zero modes. Namely, $\beta(g) = 1$ (resp. $\beta(g) = 0$) if $g \in G$ acts on the boundary Hilbert space by a fermionic (resp. bosonic) operator. Here we explain an alternative formulation of $\beta \in H^1(G, \mathbb{Z}_2)$ as a charge pumping invariant. Any symmetry $\hat{g} \in \hat{G}$

gives rise to a loop in the space of 1d band Hamiltonians. The net fermion parity pumped through any point is a $\mathbb{Z}_2$-valued invariant of the loop. This is a special case of the Thouless pump (Teo and Kane, 2010; J. E. Moore and Balents, 2007).

Given $\hat{g} \in \hat{G}$ which is a symmetry of a band Hamiltonian $H(k)$, we can define a loop in the space of band Hamiltonians as follows. Since $SO(2N)$ is a connected group, we can choose a path $\eta : [0,1] \to SO(2N)$ such that $\eta(0) = 1$ and $\eta(1) = \widehat{R}(\hat{g})$. Next we define $H(k,t) = \eta(t)H(k)\eta(t)^{-1}$. Since $\widehat{R}(\hat{g})$ commutes with $H(k)$, $H(k,1) = H(k,0)$. Thus $H(k,t)$ is a loop in the space of 1d band Hamiltonians. A general argument (Teo and Kane, 2010; J. E. Moore and Balents, 2007) shows that the net fermion parity $(-1)^{B(\hat{g})}$ pumped through one cycle of this loop does not depend on the choice of path η. This immediately implies that $B(\hat{g}\hat{g}') = B(\hat{g}) + B(\hat{g}')$. Thus $B(\hat{g})$ defines an element of $H^1(\hat{G}, \mathbb{Z}_2)$.

To evaluate $B(\hat{g})$, we apply the general formula from Ref. (Teo and Kane, 2010) for Hamiltonians in class D. One simplification is that locally in k, t the Berry connection can be taken as $\eta^{-1}\partial_t \eta$, and thus its curvature vanishes. Then

$$B(\hat{g}) = \frac{1}{2\pi} \int \mathrm{Tr}\left[(P_+(0) - P_+(\pi))\eta(t)^{-1}\partial_t \eta(t)\right] dt \qquad \text{(D.10)}$$

where $P_+(k)$ is the projector to positive-energy states at momentum k.

Next we decompose $\widehat{R}$ into real irreducible representations r_α. Obviously, each representation contributes independently to $B(\hat{g})$. Representations of $\mathbb{C}$-type and $\mathbb{H}$-type do not contribute at all, since the corresponding Hamiltonians can be deformed to trivial ones. A Hamiltonian $A_{r,ij}$ corresponding to an $\mathbb{R}$-type representation r_α is of class D and can be deformed either to a trivial one or to a trivial one stacked with a single Kitaev chain. In the former case, both the boundary invariant $(-1)^{\beta(\hat{g})}$ and the charge-pumping invariant $B(\hat{g})$ are trivial (equal to 1). In the latter case, we get a single Majorana zero mode for each of the $d_r = \mathrm{d}im\, r$ basis vectors of r, so the boundary invariant $(-1)^{\beta(\hat{g})}$ is equal to $\det r(g)$. We just need to verify that $B(\hat{g})$ is also equal to $\det r(g)$ for d_r copies of the Kitaev chain. The on-site representation of $\hat{G}$ is given by $\widehat{R} = r \oplus r$ in this case.

For d_r copies of the Kitaev chain, the projector to positive-energy states is

$$P_+(k) = \tfrac{1}{2}\left(\mathbb{1}_2 - \sigma_y \sin k + \sigma_z \cos k\right) \otimes \mathbb{1}_{d_r}, \qquad \text{(D.11)}$$

which commutes with $\widehat{R}(\hat{g}) = \mathbb{1}_2 \otimes r(\hat{g})$ and satisfies $P_+(0)\widehat{R}(\hat{g}) = r(\hat{g}) \oplus 0$ and $P_+(\pi)\widehat{R}(\hat{g}) = 0 \oplus r(\hat{g})$. Let $\eta(t)$ be a path in $SO(2d_r)$ from 1 to $\widehat{R}(\hat{g})$. We may choose it to belong to the $U(d_r)$ subgroup of matrices that commute with $P_+(0)$ and $P_+(\pi)$. Then $\eta(t) = q(t) \oplus \bar{q}(t)$ for a path $q(t)$ through $U(d_r)$ from $\mathbb{1}$ to $r(\hat{g})$.

Substituting all this into (D.10), we get

$$B(\hat{g}) = \frac{1}{2\pi} \int \mathrm{Tr}\Big((P_+(0) - P_+(\pi))\eta(t)^{-1}\partial_t \eta(t)\Big)\, dt = \frac{1}{2\pi} \int \mathrm{Tr}\Big(q(t)^{-1}\partial_t q(t) - \bar{q}(t)^{-1}\partial_t \bar{q}(t)\Big)\, dt. \tag{D.12}$$

Note that this vanishes whenever $q(t) = \bar{q}(t)$ at all t. We now show how to recover $(-1)^{B(\hat{g})} = \det r(\hat{g})$.

If $r(\hat{g})$ has determinant $+1$, it lives in $SO(d_r)$, which is path-connected. Hence the path $q(t)$ from $\mathbb{1}$ to $r(\hat{g})$ may be taken to lie in $SO(d_r) \subset U(d_r)$. Therefore $q(t) = \bar{q}(t)$ is real, and so $B(\hat{g}) = 0$.

If $r(\hat{g})$ has determinant -1, we construct $q(t)$ as follows. First connect $\mathbb{1}$ to $\mathrm{diag}(-1, +1, +1, \ldots, +1)$ by $\mathrm{diag}(\exp(it), +1, +1, \ldots, +1)$. Now that the determinant is -1, we may get to $r(\hat{g})$ through a real path in the identity-disconnected component of $O(d_r)$. This second segment of the path contributes nothing to $B(\hat{g})$. It remains to compute the contribution of the first segment, where $q(t) = \exp(it) \oplus \mathbb{1}$:

$$B(\hat{g}) = \frac{1}{2\pi} \int \left(e^{-it}\partial_t e^{it} - e^{it}\partial_t e^{-it}\right)\, dt = 1. \tag{D.13}$$

This completes the proof that $B(\hat{g}) = \beta(\hat{g})$. In particular, $B(P) = 0$, i.e. B is really a homomorphism from $G = \hat{G}/\mathbb{Z}_2^F$ to $\mathbb{Z}_2$.

The interpretation of $\beta(g)$ in terms of a fermion-parity pump has the following intuitive reason. Assume that one can make a “Wick rotation” of the pump. Then the twist by $\hat{g}$ along the “time” direction gets reinterpreted as a twist along the spatial direction. The invariant $B(\hat{g})$ can be re-interpreted as the fermionic parity of the ground state of the system with an $\hat{g}$-twist, or equivalently as the fermionic parity of the $\hat{g}$ domain wall. On the other hand, it is known (Kapustin, Turzillo, and You, 2018) that this is yet another interpretation of the invariant β.

To conclude this section, we show how to compute $B(g) = \beta(g)$ from the holonomy of the Berry connection between $k = 0$ and $k = \pi$. This makes the topological nature of $B(g)$ explicit. Recall first how the holonomy is defined. If there are $2N$ Majorana fermions per site, a free 1d Hamiltonian can be described by a non-degenerate $2N \times 2N$ matrix $X(k)$, where k is the momentum (Chiu et al., 2016). At $k = 0$ and $k = \pi$ this matrix is real and skew-symmetric. We can bring $X(0)$ to the standard form X_0 using an orthogonal transformation $O(0) \in O(2N)$. Similarly, we use $X(\pi)$ to define $O(\pi) \in O(2N)$. The holonomy of the Berry connection is $O = O(\pi)O(0)^{-1}$. The invariant $(-1)^{\gamma}$ is equal to the sign of $\det O$ (Budich and Ardonne, 2013). If γ vanishes, then $\det O(0)$ and $\det O(\pi)$ have the same sign, and by a choice of basis we may assume that both $O(\pi)$ and $O(0)$ lie in $SO(2N)$.

To define a topological invariant associated to an element $\hat{g} \in \hat{G}$, we choose a path $\eta(t) : [0,1] \to SO(2N)$ from the identity to $\widehat{R}(\hat{g})$. Consider now the following map from $[0,1]$ to $SO(2N)$:

$$\Pi(t) = \begin{cases} \eta(2t), & 0 \leq t \leq 1/2, \\ O\eta(2-2t)O^{-1}, & 1/2 \leq t \leq 1. \end{cases} \tag{D.14}$$

Since $O \equiv O(\pi)O(0)^{-1}$ is the holonomy of the Berry connection from 0 to π, it commutes with all symmetries of the Hamiltonian, including $\widehat{R}(\hat{g})$ for all $\hat{g} \in \hat{G}$. This implies that $\Pi(t)$ is a loop in $SO(2N)$. We claim that $B(\hat{g})$ is the class of this loop in $\pi_1(SO(2N)) = \mathbb{Z}_2$.

This definition is independent of the path from 1 to $\widehat{R}(\hat{g})$. Any two paths differ (in the sense of homotopy theory) by a loop in $SO(2N)$. Thus changing the path will result in composing $\Pi(t)$ with a loop and its conjugation by O. Since these two loops are homotopic, the homotopy class $[\Pi]$ is unchanged.

To prove that the homotopy class of the loop Π coincides with $B(\hat{g})$, one can follow the same strategy as before: use homotopy-invariance to reduce to the case of a single Kitaev chain, and then compute the invariant by choosing a particularly convenient path.