POLITICAL AND MARKET EQUILIBRIA WITH INCOME TAXES

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ABSTRACT

In this thesis we explore political and market equilibria in worlds with income taxes. In part I we study individual and majority-rule choice of an income tax schedule in the context of a simple two-sector economy in which individuals respond to higher taxes by earning less taxable income and devoting more time to untaxed activities. If voters are concerned with the "fairness" of the distribution of after-tax incomes in society, then a majority-rule equilibrium tax schedule exists, and is linear. If voters care primarily about their own after-tax income however, then in general no such equilibrium exists, although equilibria may exist within special classes of taxes. In characterizing individual preferences we find that "middle-class" voters prefer sharply progressive schedules that impose low marginal tax rates on lower-income taxpayers and high marginal rates on upper-income taxpayers. This suggests that the observed preference for marginal-rate progression has little to do with "fairness," but results from the middle-class' successfully reducing its own tax burden.

In Part II we study the effects of income taxation on capital asset market equilibrium, using a popular model of asset pricing, the Arbitrage Pricing Theory (APT). We focus on two features found in many tax codes, the differential treatment of dividends and capital gains, and the different treatment of various types of investors. We show first that, with restrictions on the portfolios investors may hold, in general at *any* prices there will be some investor who can make unlimited arbitrage profits. Next we restrict portfolios, requiring that no investor borrow so much that her total dividend payment on short sales exceeds her total dividend income on the assets she owns. Given this restriction there exist prices at which no investors can make unlimited arbitrage profits. We show that if at least one investor faces a higher tax rate on capital gains than dividends (true for corporations in the U.S. today) then the prices must be different from those predicted by the APT without taxes.

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I. THE POLITICS OF INCOME TAXATION IN A TWO-SECTOR ECONOMY

1. Introduction

The taxation of personal incomes is a very direct and transparent redistributive mechanism, and hence potentially a source of great instability in democratic societies. In a world where majority coalitions form to impose heavy taxes on the minority and distribute the revenue to themselves, we might expect frequent, dramatic changes in the distribution of tax burdens and after-tax incomes; those in the minority would always have a strong incentive to change the current scheme by offering rewards to selected members of the majority in return for their support.

In fact, however, we do not see such chaos. Virtually all advanced industrial democracies impose taxes on incomes and use these taxes to redistribute income as well as raise revenue, and the tax schedules are stable over time. Changes in tax codes, while fairly frequent, generally address technical details rather than the overall structure of rates, and changes in tax schedules that do occur are often "technical" in nature as well, such as when rates are adjusted for inflation. Moreover, there are important patterns in the rate structures across countries. Taxes paid as a fraction of income increase with income, implying some redistribution from upper-income individuals to those with lower incomes; generally however, the amount of redistribution is quite modest. Marginal tax rates also rise with income, much more quickly than average rates, and thus in many countries large groups of taxpayers face very different marginal tax rates.

So far, theoretical studies have had mixed success in explaining these patterns. Such studies usually begin with the reasonable assumption that the individuals in society must choose a tax schedule, by majority voting, from some set of admissible schedules. The admissible schedules must satisfy an exogenous government revenue target. Taking before-tax incomes as independent of the tax schedule chosen, Foley [10] finds that if citizens are concerned with their own after-tax incomes then a majority-rule equilibrium exists if the set of admissible schedules is restricted to the class of linear schedules (i.e. schedules with a constant marginal tax rate). If the distribution of incomes is skewed to the left however, as is usually the case, then the equilibrium schedule has a marginal tax rate of 100%, and results in complete equalization of after-tax incomes. This is empirically implausible. Hamada [11] also takes before-tax incomes as fixed, but assumes that voters are "benevolent," concerned with the "fairness" of the after-tax distribution of incomes in society, as measured by Bergsonian-Samuelsonian social welfare functions (weighted averages of after-tax incomes). Given weighting functions that are concave in income, a majority-rule equilibrium exists even if no restrictions are placed on the set of admissible schedules; however, the equilibrium is the same as Foley's, yielding equal after-tax incomes.

An unrealistic assumption underlying these results is that incomes are exogenous, so individuals do not respond even to large changes in the tax structure. Aumann and Kurz [4], and Romer [18] analyze models which relax this assumption. In Aumann and Kurz [4], individuals have the option to destroy part or all of their income, and this "threat" limits the power of the majority. Voters are assumed to be egoistic, and the tax schedule is chosen by majority-rule. Treating the choice as a cooperative game and using the Harsanyi-Shapley-Nash value as their solution concept, they find that there exists a unique solution. However, little can be said in general about the shape of tax schedule associated with the solution; average and marginal rates may be progressive, regressive, or constant.

Romer [18] works in an even more realistic framework, the "labor-leisure" model popular in the literature on optimal income taxation, in which individuals can respond to high taxes by substituting untaxable leisure for taxable income. Voters are again egoistic, each voting to maximize a utility function that depends only on her own after-tax income and leisure. Romer shows that if voters have a Cobb-Douglas utility function and the set of admissible tax schedules is restricted to those linear schedules under which all individuals work some positive amount (i.e., no individual allocates all her time to leisure), then a majority-rule equilibrium exists. This equilibrium typically involves some redistribution, but does not result in complete equality of after-tax incomes, a more plausible result than that of Foley and Hamada. It is possible, however (depending on the parameters of the model), that the equilibrium tax has a positive intercept and thus be average-rate regressive. Also, it is probable that under a more general specification of the utility function, no voting equilibrium exists.

The "labor-leisure" framework is appealing in its generality, but it seems unlikely that it can be used to generate interesting results in the context of majority voting. Aside from the usual mathematical difficulties of optimal control, there are fundamental nonconvexities which arise from the structure of the problem, and it is quite likely that voter preferences over tax schedules (linear and nonlinear) will in general be very badly behaved.

We explore the issues of individual and collective choice of an income tax schedule in the context of a simple economic model which incorporates incentive effects which are similar in spirit, but different in detail, from those of the optimal taxation literature. We assume, conventionally, that individuals vary in their potential to earn income (i.e., their endowment of "ability," or "labor productivity"). However, in our model an individual faced with a high tax rate on his labor income responds, not by substituting untaxable leisure for taxable work effort, but rather by working in an untaxed "underground" economy, at a lower (but tax-free) wage rate. We take the government's budget constraint as given exogenously, and only compare taxes that raise enough revenue to satisfy it.

We assume that collective choices are made by simple majority rule. Since it is not clear what is the most reasonable way to model voters, or even whether there is a single model of voters that applies generally, we consider several different assumptions about voters' preferences. Studying different possibilities allows us to compare the outcomes under each. At the one extreme, we suppose that voters are purely "egoistic" when choosing among tax schedules, each looking only at how she personally will fare under a schedule to decide how well she likes it. At the other extreme we suppose that voters are purely selfless, or "benevolent," and rank tax schedules based on some social welfare measure of the whole distribution of after-tax incomes, preferring a large per capita income and more equality to less (as in Hamada [11]). We also examine intermediate cases, of voters who care both about their own after-tax income and the distribution of incomes in society.

In Section 2 we describe the model in detail, and develop some necessary preliminary results. In Section 3 we study linear tax schedules, and show that for all three type of voters (egoistic, benevolent, and those with more general, "mixed" preferences) there exists a majority-rule equilibrium in the set of linear schedules. We find that egoistic voters with small endowments, and benevolent voters with a strong preference for equality, prefer higher marginal tax rates. Unfortunately, it is in general impossible to say anything about the relation between the marginal tax rate of the equilibrium schedule in the different cases, since egoistic voters might choose a schedule with a higher or lower rate than benevolent voters, depending on the characteristics of the median voter.

As we are interested in explaining the apparently stable democratic preferences for income taxes with increasing marginal rates, in Section 4 we study nonlinear tax schedules. We first show that for benevolent voters, given any nonlinear schedule there exists a linear one which is *unanimously* preferred. This means that the majority-rule equilibrium linear tax is an equilibrium over the set of *all* taxes, and hence quite robust. The fact that such a stable schedule exists is a step in the right direction; however the fact that it is linear leaves marginal-rate progressivity unexplained.

For egoistic voters the story is quite different. If the government revenue constraint is not too high then egoistic voters favor tax schedules with sharply increasing marginal rates. We characterize the schedules most preferred by egoistic voters of varying abilities and find that for those with medium ability levels this schedule has two large "tax brackets"--it imposes a marginal rate of zero on lower incomes and a positive rate on higher incomes. Also, we show that if the government's revenue constraint is nonpositive (e.g., it is doing pure redistribution) then there exists a majority-rule equilibrium within this set, namely the most-preferred schedule of the median ability (and median income) voter, most likely a middle-class taxpayer. If the government revenue constraint is large then for high-ability individuals the most-preferred tax schedules are a bit more complicated, and the marginal and average rates actually *decrease* in income over part of their ranges. For individuals of "low enough" ability however, the most-preferred tax schedule is still of the increasing-rate, "two-bracket" type. It is quite possible that middle-range ability levels, including the median, are "low enough"; this is an empirical question. Also, if we restrict attention to convex schedules then the most-preferred schedules for voters of *all* ability levels will either be linear or have two brackets (and of course be marginal-rate progressive), and for some voters the marginal rate in the lower bracket may be positive. When we consider voters whose utility depends on both their own after-tax income and the distribution of after-tax incomes in society, the most-preferred convex schedules are again either linear or "two-bracket" schedules.

These results suggest that the observed stability and marginal-rate progressivity of income taxation in democratic societies have little to do with fairness or equity considerations, but arises from the success of the middle-class in minimizing its own tax burden, at the expense of upper and lower income taxpayers.

In Section 5 we relax the assumption, implicitly made throughout the above discussion, that voters are perfectly informed about all the relevant parameters in the economy. In particular, we suppose that they are uncertain about their own ability level. This is especially relevant if they are choosing a tax schedule that will be in effect for several years (say, because it is very costly to change the schedule often), since it is likely that the future income-earning potential of an individual is subject to considerably more variance than the underlying structure of the economy (i.e., the distribution of endowments and technology). If voters are risk-averse and either view their ability level as a random draw from the actual distribution in the economy, or are "pessimistic" about their chances of having a high ability, then they again prefer linear tax schedules. "Optimistic" voters, on the other hand, may prefer schedules that are marginal- and average-rate regressive; this is true, for example, if they are risk-neutral.

In Section 6 we compare our model with the "labor-leisure" framework of the optimal taxation literature, and clarify the role of some of the assumptions in driving results.

2. The Model and Some Preliminaries

Here we describe in detail the essential features of the economic and political parts of the model, and prove some necessary preliminary results, in particular, the existence of a market equilibrium in the economy.

2.1 The Economy

We assume a simple one-good economy with two sectors, a legal "taxed" sector and an untaxed "sheltered" sector. The agents are worker-consumers, each endowed with a fixed amount of labor to supply to the economy. Each agent allocates her labor between sectors so as to maximize her consumption, or after-tax income. Labor endowments vary across individuals, and the distribution of endowments is given by the probability distribution function F.[1] We assume that F is nonatomic, its support is an interval $[\underline{n},\bar{n}] \subseteq [0,1]$, and the average labor endowment (i.e., the total labor available per capita) is $N = \int_{\underline{n}}^{\bar{n}} n dF(n) > 0$. (For notational brevity, we will suppress the limits on integrals whenever the limits are $\underline{n}$ and $\bar{n}$.)

We assume that for each sector all units of labor supplied to the sector receive the same return, or "wage." The wage in the taxed sector is always 1. The wage in the untaxed sector may vary however, depending on the total labor supplied to the sector. Thus there exists a function $w_D : [0,N] \to (0,1)$,[2] where $w_D(L)$ gives the wage in the untaxed sector when L is the total labor used in the sector. (We denote the function by w_D because it may be interpreted as the inverse labor demand function in the untaxed sector.) We assume that w_D is continuous and strictly decreasing.[3] We will call w_D the "relative wage" function. Since w_D is strictly decreasing on $[0,N]$ it has an inverse over the interval $[w_D(N),w_D(0)]$, which we denote by L_D. We can then extend L_D naturally to $[0,1]$ by letting $L_D(w) = N$ for all $w < w_D(N)$ and $L_D(w) = 0$ for all $w > w_D(0)$. We call L_D the untaxed sector "labor demand" function.

The government taxes income earned in the taxed sector.[4] A ***tax schedule*** is a lower semi-continuous function $T : [0,1] \to [-1,1]$ satisfying $T(x) \leq x$ for all x. Thus, $T(x)$ is the net tax liability or credit due on a pretax (taxable) income of x. Lower semi-continuity ensures the existence of an optimal labor allocation for all individuals, and the second restriction prevents an individual's tax liability from exceeding her taxable income. We place no further restrictions on the form of T, so tax schedules may be discontinuous, increasing, decreasing, or whatever. We denote the set of tax schedules by $\mathbf{T}$.

2.2 Market Equilibrium and Preliminary Results

Given a relative wage in the untaxed sector w and tax schedule T, an individual with labor endowment n who allocates $x \in [0,n]$ of her work effort to the taxed sector and $n-x$ to the untaxed sector earns an after-tax income of

$$x - T(x) + w(n - x) = -[T(x) - (1 - w)x] + wn. \tag{2.1}$$

We assume that in choosing how to allocate their labor supply, individuals take the wage and tax schedule as fixed, and thus an optimal allocation is one which maximizes (2.1), or equivalently, which minimizes the quantity in square brackets.[5] Let

$$\hat{x}(n,w,T) = \{x' \in [0,n] \mid x' - T(x') + w(n - x') \geq \qquad (2.2)$$

$$x - T(x) + w(n - x) \text{ for } all\ x \in [0,n]\}$$

be the set of allocations which are optimal for n given w and T. In view of the observation above, evidently

Comment 2.1. $x' \in \hat{x}(n,w,T)$ if and only if x' minimizes $T(x) - (1 - w)x$ over $x \in [0,n]$.

From lower semi-continuity, $\hat{x}(n,w,T)$ is non-empty and compact for all n, w, and T. A more explicit characterization, which will be useful below, is as follows: suppose the function $T(x) - (1 - w)x$ possesses a minimum x' on some interval of the form $[0,x'')$. Let $\underline{x}$ be the smallest such minimum (by lower semi-continuity $\underline{x} = \min \{x' \mid x' = \underset{[0,x'')}{argmin}\ T(x) - (1 - w)x\}$ is well defined), and let $\bar{x}$ define the largest interval on which this is still a minimum (i.e., $\bar{x} = \sup\{x'' \mid T(x) - (1 - w)x \geq T(x') - (1 - w)x' \text{ for all } x \in [0,x'')\}$). We shall say the interval $[\underline{x},\bar{x})$ is *w-critical for T*. Clearly any T and w define a unique (possibly empty) set of disjoint critical intervals. (See Figure 2.1.)

Now for individuals with endowments $n \in [\underline{x},\bar{x})$, from Comment 2.1 evidently $x' \in \hat{x}(n,w,T)$ if and only if

$T(x') - (1 - w)x' = \min_{x \in [0,n]} T(x) + (1 - w)x = T(\underline{x}) + (1 - w)\underline{x}$; thus $\underline{x}$ is always optimal (though not necessarily uniquely so) for all such n. For individuals with $n = \bar{x}$, either $T(\bar{x}) - (1 - w)\bar{x} = T(\underline{x}) - (1 - w)\underline{x}$, in which case the above equality again defines the optima, or else there is a discontinuity at $\bar{x}$ with (from lower semi-continuity)

$T(\bar{x}) + (1 - w)\bar{x} < T(\underline{x}) + (1 - w)\underline{x}$, so from Comment 2.1, $\hat{x}(n,w,T) = \{\bar{x}\} = \{n\}$ uniquely. On the other hand, if n does not belong to the closure of any critical interval, then it must be true that $T(x) - (1 - w)x$ has no minimum on $[0,n)$. By lower semi-continuity it must have a minimum on $[0,n]$, which must therefore be at n, so $\hat{x}(n,w,T) = \{n\}$ uniquely, again by Comment 2.1. Summarizing, we have

Comment 2.2. The correspondence $\hat{x}(\cdot,\cdot,T)$ is as follows.

(i) if $n \notin [\underline{x},\bar{x}]$ for every w-critical interval $[\underline{x},\bar{x})$, then $\hat{x}(n,w,T) = \{n\}$,

(ii) if $n \in [\underline{x},\bar{x})$ for some such interval, then

$$\hat{x}(n,w,T) = \{x' \in [\underline{x},n] \mid T(x') + (1 - w)x' = T(\underline{x}) + (1 - w)\underline{x}\},$$

(iii.a) if $n = \bar{x}$ for some such interval and T is discontinuous at $\bar{x}$ with

$T(\bar{x}) - (1 - w)\bar{x} < T(\underline{x}) + (1 - w)\underline{x}$ then $\hat{x}(n,w,T)$ is given by (i), and

(iii.b) if $n = \bar{x}$ for some such interval and T is continuous at $\bar{x}$ then

$\hat{x}(n,w,T)$ is given by (ii).

Let $\hat{X}(w,T) = \{X \mid X = \int \hat{\hat{x}}(n)dF(n)$, *where* $\hat{\hat{x}} \in \hat{x}(\cdot,w,T)$ *is integrable*$\}$ be the set of possible aggregate labor supplies to the taxed sector given wage w and tax schedule T, and let $\hat{L}(w,T) = N - \hat{X}(w,T)$ be the possible aggregate labor supplies to the untaxed sector. Clearly, $\hat{L}(w,T)$ is non-empty and compact (and convex, from Richter's theorem) for all w and T. A *market equilibrium* for the tax schedule T is a wage w^* and an aggregate untaxed labor supply L^* satisfying $w_D(L^*) = w^*$ and $L^* \in \hat{L}(w^*,T)$. Our first proposition is that for any tax schedule T there exists a unique market equilibrium.

Proposition 2.1. For any tax schedule T, a market equilibrium (w^*, L^*) exists and is unique.

Proof. See Appendix A.

Denote this equilibrium by $(w^*(T), L^*(T))$, and let $X^*(T) = N - L^*(T)$. From individual maximization, an individual's optimal allocations must all yield the same after-tax income. Thus, let $y^*(n,T)$ be the after-tax income of an individual with endowment n under the tax schedule T at the equilibrium wage $w^*(T)$; i.e., $y^*(n,T) = x' - T(x') + w^*(T)(n - x')$ for all $x' \in \hat{x}(n, w^*(T), T)$. Let $x^*(n,T) = \hat{x}(n, w^*(T), T)$ be the labor supply of an individual with endowment n at the equilibrium wage. Of course, $x^*(n,T)$ need not be single-valued; thus, in general, before-tax incomes and taxes paid at the equilibrium wage are indeterminate. However, the aggregates are unique, since $L^*(T)$ is. Denote the equilibrium aggregate before-tax income by $Z^*(T)$ and the aggregate after-tax income by $Y^*(T)$. Then $Z^*(T) = X^*(T) + w^*(T)(N - X^*(T)) = (1 - w^*(T))X^*(T) + w^*(T)N$, and $Y^*(T) = \int y^*(n,T)dF(n)$. Notice that if two tax schedules T_1 and T_2 produce the same market equilibrium, then $Z^*(T_1) = Z^*(T_2)$. Denote the aggregate revenue collected under T by $R(T)$. Clearly, $R(T) = Z^*(T) - Y^*(T)$. Finally, for any G denote the set of tax schedules T such that $R(T) = G$ by $\mathbf{T}(G)$.

We call two schedules T_1 and T_2 ***equivalent*** if (i) $w^*(T_1) = w^*(T_2)$, (ii) $R(T_1) = R(T_2)$, and (iii) $y^*(n,T_1) = y^*(n,T_2)$ for all n. Since equivalent schedules induce the same after-tax income distribution and raise the same revenue, their welfare implications for any individual, or for society, are the same.

One last fact, which will be used often later, is

Comment 2.3. If T_1 and T_2 are tax schedules such that $T_1(x) = \alpha + T_2(x)$ for all x, then T_1 and T_2 generate the same market equilibrium, and $R(T_1) = \alpha + R(T_2)$.

Proof. At the equilibrim $(w^*(T_1), L^*(T_1))$, $N - L^*(T_1) = X^*(T_1) = \int \hat{\hat{x}}(n)dF(n)$

for some integrable selection $\hat{\hat{x}}$ of $x^*(\cdot, T_1)$. Now for any n, $\hat{\hat{x}}(n)$ is optimal given T_1 and $w^*(T_1)$. So by Comment 2.1, for all $x \in [0,n]$,

$\alpha + T_1(\hat{\hat{x}}(n)) - (1 - w^*(T_1))\hat{\hat{x}}(n) \leq \alpha + T_1(x) - (1 - w^*(T_1))x$, or equivalently,

$T_2(\hat{\hat{x}}(n)) - (1 - w^*(T_1))\hat{\hat{x}}(n) \leq \alpha + T_1(x) - (1 - w^*(T_1))x$. Thus $\hat{\hat{x}}(n)$ is also optimal for n under T_2 at wage $w^*(T_1)$. This implies that $X^*(T_1) \in \hat{X}(w^*(T_1), T_2)$, and hence that $(w^*(T_1), L^*(T_1))$ is also an equilibrium for T_2. To see that $R(T_1) = \alpha + R(T_2)$, note that for all n

$y^*(n, T_1) = \hat{\hat{x}}(n) - T_1(\hat{\hat{x}}(n)) + w^*(T_1)[n - \hat{\hat{x}}(n)]$

$= \hat{\hat{x}}(n) - \alpha - T_2(\hat{\hat{x}}(n)) + w^*(T_1)[n - \hat{\hat{x}}(n)] = y^*(n, T_2) - \alpha$, so $Y^*(T_1) = Y^*(T_2) - \alpha$. Also, since $X^*(T_1) = X^*(T_2)$, $Z^*(T_1) = Z^*(T_2)$, and thus $R(T_1) = Z^*(T_1) - Y^*(T_1) = Z^*(T_2) - Y^*(T_2) + \alpha = R(T_2) + \alpha$.

QED

2.3 A Note About Convex Tax Schedules

In the analysis to follow, we sometimes pay special attention to tax schedules that are convex functions of taxable income, i.e., schedules that have nondecreasing marginal tax rates. One nice feature of convex schedules is that they have at most one critical interval, and (if it exists) it is of the form $[\underline{x},1]$. This is clear since if $T(x_1) - (1 - w^*(T))x_1 \geq T(\underline{x}) - (1 - w^*(T))\underline{x}$ for some $x_1 \geq \underline{x}$, then for all $x_2 > x_1$ we have (by the convexity of T)

$$T(x_1) < T(\underline{x}) + \frac{T(x_2) - T(\underline{x})}{x_2 - \underline{x}}, \text{ or}$$

$$T(x_2) > T(\underline{x}) + \frac{1}{x_1}[T(x_1) - T(\underline{x})](x_2 - \underline{x})$$

$$> T(\underline{x}) + \frac{1}{x_1}(1 - w^*(T))(x_1 - \underline{x})(x_2 - \underline{x}), \text{ or}$$

$$T(x_2) - (1 - w^*(T))x_2 > T(\underline{x}) - (1 - w^*(T))\underline{x} + \frac{1}{x_1}(1 - w^*(T))(x_2\underline{x} - \underline{x}^{2)}$$

$$> T(\underline{x}) - (1 - w^*(T))\underline{x}.$$

Thus, if x_1 is in the critical interval defined by $\underline{x}$ then so are all points $x_2 > x_1$.

Let $[\underline{x}(T),1)$ be the $w^*(T)$-critical interval for T. (Of course, $\underline{x}(T)$ might equal 0 or 1.) Then for all $n \leq \underline{x}(T)$, $x^*(n,T) = n$, and for all $n > \underline{x}(T)$, $\underline{x}(T) \in x^*(T)$. Also, by the convexity of T, if $n_1 \in x^*(n,T)$ and $n_2 \in x^*(n,T)$ then $[n_1,n_2] \subset x^*(n,T)$. Clearly, $X^*(T) \geq \int_{\underline{n}}^{\underline{x}(T)} n dF(n)$ (else $(w^*(T),L^*(T))$ would not be the equilibrium), so there exists a unique

$n^*(T) \in [\underline{x}(T), \bar{n}]$ such that $X^*(T) = \int_{\underline{n}}^{n^*(T)} n dF(n) + \int_{n^*(T)}^{\bar{n}} [n - n^*(T)] dF(n)$ (since at $n^*(T) = \underline{x}(T)$ the right-hand side is $\int_{\underline{n}}^{n^*(T)} n dF(n) \leq X^*(T)$, at $n^*(T) = \bar{n}$ it is $\int n dF(n) = N \geq X^*(T)$, and it is continuous and strictly increasing in $n^*(T)$). Also, by the convexity of T, the labor choices

$$\hat{x}(n) = \begin{cases} n & \text{for } n \leq n^*(T) \\ n^*(T) & \text{for } n > n^*(T) \end{cases}$$

are optimal for all n. (See Figure 2.2.) Of course, this selection need not be the unique optimum, although if T is ***strictly*** convex then it is. In any case, when dealing with convex schedules we will always assume that these particular labor choices are made. And, abusing notation slightly, we will denote these choices by $x^*(n,T)$ for all n. Then $n^*(T)$ is a "threshold" level for taxable income--no one earns a taxable income higher than this.

A useful fact about convex tax schedules is

Comment 2.4. Given a tax schedule T, the schedule T_1 induces the same market equilibrium if either

(i) $w^*(T) = w_D(0)$ and $T_1^-(\bar{n}) \leq 1 - w_D(0)$, or

(ii) $w^*(T) = w_D(N)$ and $T_1^+(\underline{n}) \geq 1 - w_D(N)$, or

(iii) $w^*(T) \in (w_D(N), w_D(0))$ and $1 - w^*(T) \in [T_1^-(n^*(T)), T_1^+(n^*(T))]$,

(where $T_1^-(x)$ is the left-hand derivative, and $T_1^+(x)$ the right-hand derivative, of T_1 at x).

Proof. If (i) holds then $\hat{x}(n, w^*(T_1)) = \{n\} \ni x^*(n,T)$ for all n, so $\hat{L}(w^*(T), T_1) = \{0\} \ni L_D(w^*(T)) = L^*(T)$, and thus $(w^*(T), L^*(T))$ is a market equilibrium for T_1. Since the market equilibrium is unique, it is ***the*** market equilibrium for T_1. If (ii) holds, then $\hat{x}(n, w^*(T_1)) = \{0\} \ni x^*(n,T)$ for all n, so $\hat{L}(w^*(T),T_1) = \{N\} \ni L_D(w^*(T)) = L^*(T)$, and again $(w^*(T),L^*(T))$ is the market equilibrium for T_1. If (iii) holds then

$$\hat{x}(n,w^*(T),T_1) \begin{cases} = \{n\} & \text{for } n \leq n^*(T) \\ \ni \{n^*(T)\} & \text{for } n > n^*(T) \end{cases}$$

so $\hat{x}(n,w^*(T),T_1) \ni x^*(n,T)$ for all n, so $\hat{L}(w^*(T),T_1) \ni L^*(T) = L_D(w^*(T))$ and thus $(w^*(T),L^*(T))$ is again the market equilibrium for T_1.

QED

2.4 The Politics

As discussed above, we are looking for a "long-run" majority-rule equilibrium tax schedule, i.e., a schedule which would defeat or tie any other in a pairwise simple-majority vote. Also, we take the government's budget constraint as given, and thus restrict the set of allowable tax schedules to those that raise enough revenue to meet it. Thus, we do not analyze the joint problem of simultaneously choosing the level of government spending and the tax schedule.

It is well known that when the set of alternatives can be ordered along a one-dimensional space and voters' preferences over this space satisfy a certain "single-peakedness" condition, then such an equilibrium exists (see Black [5]). This result has been proved under a variety of assumptions, but none of the statements of it found in the literature (by us) can be directly applied here, as the assumptions always include at least one of the following: (i) there is a finite set of voters; (ii) voter preferences are ***strictly*** single-peaked, with no "large regions" of indifference ("single-peaked," in Black's [5] terminology). Since we sometimes wish to consider a continuum of voters --

for example, when all worker-consumers in the economy are voters -- we cannot impose (i). Also, it turns out that we must allow voter preferences to have large regions of indifference, and thus cannot impose (ii); in particular we must allow preferences of the form Black calls "single-peaked with a plateau on top." Thus, we prove the required result below. (In fact, we prove a more general result than required, which allows voter preferences to have "plateaus" below the top as well as at the top.)

Let the set of alternatives be a closed interval $A \subset \mathbb{R}$ and let each voter j have a continuous preference relation $\geq_j$ over A. Denote the strict part of $\geq_j$ by $>_j$, and for any $a' \in A$ let $P_j(a') = \{a \in A \mid a >_j a'\}$ and let $P_j^{-1}(a') = \{a \in A \mid a <_j a'\}$. We say that $\geq_j$ is ***weakly single-peaked*** if there exists an alternative $a_j \in A$ such that $a'' \geq_j a'$ for all $a' \leq a'' \leq a_j$, and $a' \geq_j a''$ for all $a_j \leq a' \leq a''$.

To allow for a continuum of voters we must describe the "set" of voters as a measure-space. Thus, let the voters be given by $(J, \mathbf{F}, \mu)$, where J is the set of voter types, $\mathbf{F}$ is a σ-algebra over J such that for all $a \in A$ and all closed intervals $B \subset A$, $J(a,B) = \{j \in J \mid B \subseteq P_j^{-1}(a)\} \in \mathbf{F}$, and μ is a countably additive probability measure on $(J, \mathbf{F})$. We say that J satisfies *weak single-peakedness* if for all $j \in J$, $\geq_j$ is weakly single-peaked.

We define the majority relation $\geq_M$ by $a >_M a'$ if and only if $\mu(\{j \in J \mid a >_j a'\}) > \frac{1}{2}$. (We show below, in the proof of Comment 2.5, that $\{j \in J \mid a >_j a'\} \in \mathbf{F}$.) For any $a' \in A$ let $P_M(a') = \{a \in A \mid a >_M a'\}$ and let $P_M^{-1}(a') = \{a \in A \mid a <_M a'\}$. A ***majority-rule equilibrium*** is an alternative $a^* \in A$ such that $P_M(a^*) = \phi$.

The result needed is

Comment 2.5. Let $(J, \mathbf{F}, \mu)$ describe the set of voters. If J satisfies weak single-peakedness then a majority-rule equilibrium exists.

Proof. Using a lemma due to Fan (see Theorem 7.2 of Border [7]) an equilibrium exists if (i) $P_M^{-1}(a)$ is open for all a, and (ii) $a \notin co(P_M(a))$ for all a (where $co(P_M(a))$ is the convex hull of $P_M(a)$).

To see that (i) holds, pick a such that $P_M^{-1}(a) \neq \phi$. Let $a' \in P_M^{-1}(a)$ be arbitrary. We show that there exists an open neighborhood V of a' such that $V \subseteq P_M^{-1}(a)$, and thus $P_M^{-1}(a)$ is open. Let $J_0 = \{j \in J \mid a >_j a'\}$, and for $n = 1,2,\ldots$ let $J_n = \{j \in J \mid [a' - \frac{1}{n}, a' + \frac{1}{n}] \subseteq P_j^{-1}(a)\}$. Then $J_n \subseteq J_0$ for all n. Also, $[a' - \frac{1}{n+1}, a' + \frac{1}{n+1}] \subset [a' - \frac{1}{n}, a' + \frac{1}{n}]$ so if $[a' - \frac{1}{n}, a' + \frac{1}{n}] \subseteq P_j^{-1}(a)$ then $[a' - \frac{1}{n+1}, a' + \frac{1}{n+1}] \subseteq P_j^{-1}(a)$, so $J_n \subseteq J_{n+1}$ for all n. Also, since $\geq_j$ is continuous, $P_j^{-1}(a)$ is open, so for each $j \in J_0$ there exists $n_j \leq \infty$ such that $[a' - \frac{1}{n_j}, a' + \frac{1}{n_j}] \subseteq P_j^{-1}(a)$, and thus $\bigcup_{n=1}^{\infty} J_n = J_0$. By assumption, $J_n \in \mathbf{F}$ for all n. Also, $J_0 \in \mathbf{F}$, since $J_0 = \bigcup_{n=1}^{\infty} J_n$. Then since μ is a countably additive probability (hence finite) measure, $\lim_{n \to \infty} \mu(J_n) = \mu(J_0) > \frac{1}{2}$ (see, for example, Theorem 1.2.7 of Ash [2]). Thus, there exists $m < \infty$ such that $\mu(J_m) > \frac{1}{2}$. Then $[a' - \frac{1}{m}, a' + \frac{1}{m}] \subseteq P_M^{-1}(a)$, so $(a' - \frac{1}{m}, a' + \frac{1}{m}) \subseteq P_M^{-1}(a)$. Also, $(a' - \frac{1}{m}, a' + \frac{1}{m})$ is clearly open, and contains a', so (i) holds.

To see that (ii) holds, note first that $a \notin P_M(a)$. Suppose there exists $a' \in P_M(a)$, such that $a' > a$. If $a'' \notin P_M(a)$ for all $a'' < a$ then $a \notin co(P_M(a))$. Since $a' \in P_M(a)$, $\mu(\{j \in J \mid a' >_j a\}) > \frac{1}{2}$. By the definition of weakly single-peaked, if $a' >_j a$ for $a' > a$ then $a \geq_j a''$ for all $a'' < a$.

Thus, for all $a'' < a$, $\mu(\{j \in J \mid a \geq_j a''\}) > \frac{1}{2}$ so $\mu(\{j \in J \mid a'' >_j a\}) < \frac{1}{2}$ and thus $a'' \notin P_M(a)$. The same reasoning applies if instead there exists $a' \in P_M(a)$ such that $a' < a$, so (ii) holds.

QED

To use Comment 2.5, the sets $J(a,B)$ must be measurable (i.e., in $\mathbf{F}$) for all $a \in A$ and all closed $B \subseteq A$. To prove that this is true for the measure-spaces of voters we define below, the following comment will be useful.

Comment 2.6. Suppose J is a complete, separable, metric space, and for each $j \in J$, $\geq_j$ is representable by a continuous function $\psi(j,\cdot) : A \to [0,1]$ such that for each $a \in A$, $\psi(\cdot,a)$ is $\mathbf{F}$-measurable. If $\mathbf{F}$ contains all closed subsets of J then $J(a,B) \in \mathbf{F}$ for all $a \in A$ and all closed $B \subseteq A$.

Proof. Define $v_a : J \times [0,1] \to [-1,1]$ by $\pi_a(j,a') = \psi(j,a') - \psi(j,a)$. Then $\pi_a(j,\cdot)$ is continuous for all $j \in J$, and $v_a(\cdot,a')$ is $\mathbf{F}$-measurable for all $a' \in A$. So, π is $\mathbf{F}\otimes\mathbf{BOR}[0,1]$-measurable, where $\mathbf{BOR}[0,1]$ denotes the Borel sets of $[0,1]$, and $\otimes$ denotes the product σ-algebra (see example (7), page 42, of Hildenbrand [12]). Now,
$J(a,B) = \{j \in J \mid \pi_a(j,a') < 0 \text{ for } all\ a' \in B\} = \{j \in J \mid \max_{a' \in B} \pi_a(j,a') < 0\}$. Let $\varphi : J \twoheadrightarrow J \times A$ be the correspondence defined by $\varphi(j) = \{j\} \times B$. The graph of φ is closed, and hence measurable. Let $\lambda : J \to [-1,1]$ be defined by $\lambda(j) = \max\{\pi(\varphi(j))\}$ (i.e., $\max\{p \mid p = \pi(j,a') \text{ for } some\ (j,a') \in \varphi(j)\}$). Then λ is $\mathbf{F}$-measurable (see Proposition 3, page 60, of Hildenbrand [12]), so $\{j \in J \mid \lambda(j) < 0\} \in \mathbf{F}$. But $\{j \in J \mid \lambda(j) < 0\} \in \mathbf{F} = \{j \in J \mid \max_{a' \in B} \pi_a(j,a') < 0\}$.

QED

We study several different assumptions about voters, with the objective of comparing the outcomes under each. First, suppose voters care only about their own after-tax incomes. Then each voter of type j can be characterized by her endowment n_j, and we can define her preference relation over tax schedules, $\succeq_j$, by

$$T_1 \succeq_j T_2 \text{ if and only if } y^*(n_j, T_1) \geq y^*(n_j, T_2).$$

In this case we call a voter of type j *egoistic*.[6] If all voters are egoistic then the set J of voter types is $[\underline{n}, \bar{n}]$, endowed with the Euclidean norm. Then J is a complete, separable, metric space. Let $\mathbf{F}$ be the Borel σ-algebra on J.

Second, suppose each voter cares only about the "fairness" of the distribution of after-tax incomes in society, as measured by a social welfare function which is a weighted average of the after-tax incomes. Thus, given a "weighting function" $W : [0,1] \rightarrow [0,1]$, the distribution of after-tax incomes H_1 is preferred to the distribution H_2 if and only if $\int W(y)dH_1(y) \geq \int W(y)dH_2(y)$. We can then define an indirect social welfare function over tax schedules, $S(\cdot, W)$, by $S(T, W) = \int W(y^*(n, T)dF(n)$.[7] If the preference relation $\succeq_j$ of a voter of type j is given by

$$T_1 \succeq_j T_2 \text{ if and only if } S(T_1, W_j) \geq S(T_2, W_j),$$

for some weighting function W_j, we call her ***benevolent***.

We consider two types of weighting functions, (i) strictly increasing, differentiable and strictly concave, and (ii) strictly increasing and linear.[8] In both cases we are assuming that each individual, no matter how large her income, is given some weight. In the first case, income equalization is positively valued; in particular, given two distributions with the same mean (i.e., the same size "pie") the more egalitarian distribution is strictly preferred. In the second case, it is only the mean income (the size of the "pie") that matters -- all income levels are given equal weight so no redistributing (in any direction) changes the value of the social welfare function as long as the average income remains constant.

If all voters are benevolent then the set J of voter types is the space of continuous functions taking $[0,1]$ into $[0,1]$ that satisfy either (i) or (ii) (a subspace of the set of all continuous functions on $[0,1]$). Endowing this with the sup norm, J is a complete, separable, metric space. Let $\mathbf{F}$ be the Borel σ-algebra on J.

In much of the "optimal taxation" literature, "optimal" taxes are defined as those that maximize a social welfare function of the type above. Thus, in characterizing a benevolent voter's favorite schedule we also characterize an "optimal" schedule in this sense. Like Hamada [11], however, we are interested in more than this. We find conditions under which a political equilibrium exists for a group of voters interested in promoting social welfare who may have different ideas about exactly what "social welfare" is.

The most general form of voter preferences that we study is a mixture of egoism and benevolence. In particular, we consider voters with utility functions defined over their own after-tax incomes and a social welfare measure of the distribution of after-tax incomes in society. Thus if y_j is the after-tax income of a voter of type j, S_j is her evaluation of the distribution of after-tax incomes in society, and u_j her utility function, then her utility is $u_j(y_j, S_j)$. We assume that S_j is derived from a social welfare function with weighting function W_j. Thus, each voter of type j is fully characterized by the triple (n_j, W_j, u_j), where n_j is her endowment, and her indirect utility function over tax schedules v_j is defined by $v_j(T) = u_j(y^*(n_j, T), S(T, W_j)$. In this case we say a voter of type j has ***generalized preferences***. We assume that u_j is quasiconcave, and that the weighting function W_j satisfies either (i) or (ii) above. Thus, if all voters have generalized preferences, the set of voter types J is the product space of $[\underline{n},\bar{n}]$, the space of admissible weighting functions, and the space of admissible utility functions. Endowing both the space of weighting functions and the space of utility functions with the sup norm, and endowing J with the product topology, J is a complete, separable, metric space. Let $\mathbf{F}$ be the Borel σ-algebra on J.

Modeling preferences this way seems to be the most natural way to extend egoistic and benevolent preferences, and purely egoistic or benevolent preferences are clearly special cases. Essentially, we treat the distribution of incomes in society as a public good, in the same way as we would treat national defense, expenditures on education, or other public goods. As Thurow [23] has argued, there are several reasons to believe that the distribution of incomes in society is a valuable public good. Besides purely aesthetic reasons, a more even distribution of income may cause less crime and social unrest by reducing poverty, less envy between citizens, and higher productivity due to more widespread (and perhaps more overall) education and a feeling among workers that the society is "just."

Implicit in all the above is the assumption that voters are perfectly informed about their own endowment, and about the distribution of endowments in society (and also about the technology in both sectors of the economy, the tax functon, etc.). We will examine what happens when voters have imperfect information about their endowment. Voters may be choosing a tax schedule that will apply for many years, or that will not go into effect immediately, and they may not know with certainty what their income-earning potential (endowment) will be in the future. Thus, we assume that a voter of type j has a subjective probability distribution F_j over endowments, and is an expected utility maximizer. Also, we assume the voter is egoistic, with a continuous, strictly increasing utility function U_j over after-tax income, so her indirect utility function over tax schedules is

$$EU_j(T) = \int U_j(y^*(n,T))dF_j(n).$$

If all voters have preferences of this sort, the set J of voter types is the product space of allowable probability distribution functions and utility functions. Endow the space of measures with the weak-* topology, and the space of utility functions with the sup norm. Then putting the product topology on J, J is again a complete, separable metric space. Let $\mathbf{F}$ be the Borel σ-algebra on J.

Notice that if $F_j \equiv F$ then EU_j is equivalent to the indirect social welfare function $S(\cdot, U_j)$. Also, strict risk aversion is equivalent to (i) above (i.e., U_j strictly concave), and risk neutrality is equivalent to (ii). Thus, the results proved below about benevolent voters apply to the case where $F_j \equiv F$ for all $j \in J$. In general, however, $F_j \not\equiv F$. We consider several different cases, including a form of "optimism" ("pessimism"), where the voter places a higher subjective probability on high (low) endowments than does F, defined in terms of first-order stochastic dominance.

3. Results for Linear Tax Schedules

In this section we study in detail voter preferences over the set of linear tax schedules. Linear schedules are of interest because of their simplicity, both mathematically and practically (many economists and politicians argue that much of the tax code's volume and complexity is due to having a nonlinear tax schedule). Also, as we will show in the next section, some linear schedules are optimal from a social welfare point of view, and under certain assumptions about voters' utility functions, some linear schedule will be a robust majority rule voting equilibrium.

3.1 After-tax Incomes under Linear Schedules

Let T be defined by $T(x) = \alpha + \beta x$ for all x, with $\beta \in [0,1]$. Also, since $T(x) \leq x$ for all x, $\alpha \leq 0$. Then for all n,

$$\hat{x}(n,w,T) = \begin{cases} n & \text{if } w < 1-\beta \\ [0,n] & \text{if } w = 1-\beta \\ 0 & \text{if } w > 1-\beta \,. \end{cases}$$

So,

$$\hat{L}(w,T) = \begin{cases} 0 & \text{if } w < 1-\beta \\ [0,N] & \text{if } w = 1-\beta \\ N & \text{if } w > 1-\beta \,. \end{cases}$$

To characterize the equilibrium after-tax incomes, there are three cases to consider.

(i) If $\beta < 1-w_D(0)$ then $w^*(T) = w_D(0)$, $L^*(T) = 0$, and $x^*(n,T) = n$ for all n. Thus $y^*(n,T) = (1-\beta)n - \alpha$ for all n, and $R(T) = \alpha + \beta N$.

(ii) If $\beta > 1-w_D(N)$ then $w^*(T) = w_D(N)$, $L^*(T) = N$, and $x^*(n,T) = 0$ for all

n. Thus $y^*(n,T) = w_D(N)n - \alpha$ for all n, and $R(T) = \alpha$.

(iii) If $\beta \in [1-w_D(0), 1-w_D(N)]$ then $w^*(T) = 1-\beta$, $L^*(T) = L_D(1-\beta)$, and $x^*(n,T) = [0,n]$ for all n. Thus, $\hat{x} = n$ is optimal at wage $w^*(T)$ for all n, so $y^*(n,T) = (1-\beta)n - \alpha$ for all n, and $R(T) = \alpha + \beta[N - L_D(1-\beta)]$.

For any G, let $\mathbf{L}(G)$ be the set of linear schedules T such that $R(T) = G$. Because of the constraint $\alpha \leq 0$, there exists a revenue level, G^M, which is the maximum amount that can be raised by any linear schedule (which, as shown later, is the maximum revenue that can be raised by *any* schedule).

Comment 3.1. There exists $G^M > 0$ such that $\mathbf{L}(G) = \phi$ for all $G > G^M$. Also, any schedule $T^M \in \mathbf{L}(G^M)$ satisfies $T^M(x) = \beta^M x$ for all x, for some $\beta^M \in [1-w_D(0), 1-w_D(N))$.

Proof. Consider the class of schedules $\mathbf{L}_0$ that are linear with intercept zero (i.e., if $T \in \mathbf{L}_0$ then $T(x) = \beta x$ for all x, for some $\beta \in [0,1]$). Then, letting $\tilde{R}_0$ be defined by $\tilde{R}_0(\beta) = R(T)$ where $T \in \mathbf{L}_0$ has slope β,

$$\tilde{R}_0(\beta) = \begin{cases} \beta N & \text{for } \beta < 1 - w_D(0) \\ \beta[N - L_D(1-\beta)] & \text{for } \beta \in [1-w_D(0), 1-w_D(N)] \\ 0 & \text{for } \beta > 1 - w_D(N) \,. \end{cases}$$

Since L_D is continuous on $[1-w_D(0), 1-w_D(N)]$, with $L_D(1-\beta) = 0$ at $\beta = 1-w_D(0)$ and $L_D(1-\beta) = N$ at $\beta = 1-w_D(N)$, $\tilde{R}_0$ is continuous on $[0,1]$. Thus $\tilde{R}_0$ achieves a maximum on $[0,1]$, say at β^M. Let $G^M = \tilde{R}_0(\beta^M)$. Now, if T is any linear tax with slope β and intercept α, and T_1 is linear with slope β and intercept 0, then by Comment 2.3, $R(T) \leq R(T_1)$, with equality if and only if $\alpha = 0$. Thus, for $G > G^M$, $R(T) < G$ for all linear taxes T, so

$\mathbf{L}(G) = \phi$. Also, any tax $T^M \in \mathbf{L}(G^M)$ must have intercept 0. To see that the slope β^M of such a tax must lie in $[1-w_D(0), 1-w_D(N))$, differentiate $\tilde{R}_0$. For $\beta < 1-w_D(0)$, $\tilde{R}_0'(\beta) = N > 0$, so $\beta^M \geq 1-w_D(0)$. For $\beta \in (1-w_D(0), 1-w_D(N))$, $\tilde{R}_0'(\beta) = N - L_D(1-\beta) + \beta L_D'(1-\beta)$ so

$$\lim_{\beta \to 1-w_D(N)_-} \tilde{R}_0'(\beta) = [1 - w_D(N)] \lim_{\beta \to 1-w_D(N)_-} L_D'(1-\beta) < 0, \text{ so } \beta^M < 1-w_D(N).$$

QED

Note that by Comment 2.3 the schedules in $\mathbf{L}(G)$ can be characterized completely by their slope parameter (if T_1 and T_2 are linear taxes with the same slope but different intercepts then $R(T_1) \neq R(T_2)$). Let $\tilde{\mathbf{L}}(G) \subseteq [0,1]$ be the set of β such that $\mathbf{L}(G)$ contains a tax schedule T with slope parameter β. Notice that for $G \leq 0$, $\tilde{\mathbf{L}}(G) = [0,1]$, and for $G > 0$, $\tilde{\mathbf{L}}(G) \subseteq (0, 1-w_D(N)]$ (since, if T has slope equal to zero or greater than $1-w_D(N)$ then $R(T) = \alpha \leq 0 < G$). Of course, for $G > G^M$, $\tilde{\mathbf{L}}(G) = \phi$.

For the remainder of this section we will assume that $G = 0$; we can prove similar results for $G \neq 0$ but this simply adds technical and notational details without altering the conclusions. (Basically, when stating and proving the results for $G \neq 0$ we must be careful to restrict attention to $\tilde{\mathbf{L}}(G)$.)

Let $\tilde{\alpha}(\beta)$ be the intercept parameter that makes the tax T defined by $T(x) = \tilde{\alpha}(\beta) + \beta x$ an element of $\mathbf{L}(0)$. Then

$$\tilde{\alpha}(\beta) = \begin{cases} -\beta N & \text{for } \beta < 1-w_D(0) \\ -\beta[N - L_D(1-\beta) & \text{for } \beta \in (1-w_D(0), 1-w_D(N)) \\ 0 & \text{for } \beta > 1-w_D(N) \, . \end{cases}$$

Let $\tilde{y}(n,\beta)$ be the equilibrium after-tax income to an individual with endowment n under the tax in $\mathbf{L}(0)$ with slope β. Then

$$\tilde{y}(n,\beta) = \begin{cases} (1-\beta)n + \beta N & \text{for } \beta < 1-w_D(0) \\ (1-\beta)n + \beta[N - L_D(1-\beta)] & \text{for } \beta \in [1-w_D(0), 1-w_D(N)] \\ nw_D(N) & \text{for } \beta > 1-w_D(N)\,. \end{cases} \quad (3.1)$$

Clearly, for fixed n, $\tilde{y}(n,\cdot)$ is continuous. Also, differentiating with respect to β,

$$\frac{\partial\tilde{y}}{\partial\beta}(n,\beta) = \begin{cases} -n + N & \text{for } \beta < 1-w_D(0) \\ -n + N - L_D(1-\beta) + \beta L_D'(1-\beta) & \text{for } \beta \in (1-w_D(0), 1-w_D(N)) \\ 0 & \text{for } \beta > 1-w_D(N) \end{cases} \quad (3.2)$$

$$\frac{\partial^2\tilde{y}}{\partial\beta^2}(n,\beta) = \begin{cases} 0 & \text{for } \beta < 1-w_D(0) \\ 2L_D'(1-\beta) + \beta L_D''(1-\beta) & \text{for } \beta \in (1-w_D(0), 1-w_D(N)) \\ 0 & \text{for } \beta > 1-w_D(N)\,. \end{cases} \quad (3.3)$$

Using these we can characterize $\tilde{y}(n,\cdot)$. For each n, let $\beta^M(n)$ be the set of β which maximize $\tilde{y}(n,\cdot)$.

Comment 3.2. If $L_D''(w) > \dfrac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$ then

(i) for all n, $\tilde{y}(n,\cdot)$ is strictly concave over $(1-w_D(0), 1-w_D(N))$, weakly concave

over $[0, 1-w_D(N))$, and constant over $[1-w_D(N), 1]$; and

(ii) for $n > N$, $\tilde{y}(n,\cdot)$ is strictly decreasing over $[0, 1-w_D(N)]$, and hence $\beta^M(n) = 0$,

(iii) for $n = N$, $\tilde{y}(n,\cdot)$ is constant over $[0, 1-w_D(0)]$ and strictly decreasing over $(1-w_D(0), 1-w_D(N))$, and hence $\beta^M(n) = [0, 1-w_D(0)]$, and

(iv) for $n < N$, $\tilde{y}(n,\cdot)$ is strictly increasing over $[0, 1-w_D(0))$, so $\beta^M(n)$ is unique

and an element of $[1-w_D(0), 1-w_D(N))$.

(See Figure 3.1.)

Proof. From (3.2), $\tilde{y}(n,\cdot)$ is linear over $[0,1-w_D(0))$ and constant over $[1-w_D(N),1]$. Also, if $L_D''(w) > \frac{2L_D'(w)}{1-w}$ then by (3.3), $\tilde{y}(n,\cdot)$ is strictly concave over $(1-w_D(0),1-w_D(N))$. To see that $\tilde{y}(n,\cdot)$ is weakly concave over $[0,1-w_D(N))$ we must show that $\frac{\partial \tilde{y}}{\partial \beta}(n,\cdot)$ is nonincreasing across the possible discontinuity at $\beta = 1-w_D(0)$. Recall that $L_D'(w) < 0$ for all $w \in (w_D(N), w_D(0))$. Thus, using (3.2),

$$\lim_{\beta \to 1-w_D(0)_+} \frac{\partial \tilde{y}}{\partial \beta}(n,\beta) = -n + N + [1-w_D(0)] \lim_{\beta \to 1-w_D(0)_+} L_D'(1-\beta)$$

$$\leq -n + N = \frac{\partial \tilde{y}}{\partial \beta}(n,\beta) \text{ for all } \beta \in [0,1-w_D(0)).$$

Thus, (i) holds.

If $n > N$ then from (3.2), $\frac{\partial \tilde{y}}{\partial \beta}(n,\beta) < 0$ for all $\beta \in [0,1-w_D(0))$ and all $\beta \in (1-w_D(0),1-wD(N))$. And $\tilde{y}(n,\cdot)$ is continuous, so (ii) holds. If $n = N$ then from (3.2), $\frac{\partial \tilde{y}}{\partial \beta}(n,\beta) = 0$ for all $\beta \in [0,1-w_D(0))$, and $\frac{\partial \tilde{y}}{\partial \beta}(n,\beta) < 0$ for all $\beta \in (1-w_D(0),1-w_D(N))$, so (iii) holds. For $n < N$, $\frac{\partial \tilde{y}}{\partial \beta}(n,\cdot) > 0$ for all $\beta \in [0,1-w_D(0))$, so $\tilde{y}(n,\cdot)$ is strictly increasing over this interval. Also, $\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{y}}{\partial \beta}(n,\beta) = -n + [1-w_D(N)] \lim_{\beta \to 1-w_D(N)_-} L_D'(1-\beta) < 0$, so any β that maximizes $\tilde{y}(n,\cdot)$ must be less than $1-w_D(N)$. Thus, since $\tilde{y}(n,\cdot)$ is strictly concave over $(1-w_D(0),1-w_D(N))$, it has a unique maximum, achieved at some $\beta \in [1-w_D(0),1-w_D(N))$, so (iv) holds.

QED

The condition $L_D''(w) > \frac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N),w_D(0))$ is rather weak, requiring only that the labor demand function not be "too concave." For example, if the labor demand function is linear or convex then the condition is clearly satisfied. We have just shown that, given this, $\tilde{y}(n,\cdot)$ is weakly single-peaked for all n. Thus if we consider egoistic voters, each of whom ranks tax schedules by her own after-tax income under the tax schedules, preferences over the set of linear schedules will satisfy weak single-peakedness as defined in Section 2.4 and there will be a majority-rule voting equilibrium.

Proposition 3.1. Given a measure-space of egoistic voters, if $L_D''(w) > -\frac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N),w_D(0))$ then there exists a majority-rule equilibrium tax schedule over the set $\mathbf{L}(0)$, and any such equilibrium schedule has slope parameter $\beta_E \in [0,1-w_D(N))$.

Proof. If we order the elements of $\mathbf{L}(0)$ along the line by their slope parameters then by Comment 3.2 the preference relations of the set J of voter types satisfies weak single-peakedness, as defined in Section 2.4. Also, $\tilde{y}(n,\cdot)$ is continuous for all n, and $\tilde{y}(\cdot,\beta)$ is continuous (hence measurable) for all β, so the conditions of Comment 2.6 hold. Thus, by Comment 2.5, there exists a majority-rule equilibrium over $\mathbf{L}(0)$. Clearly, there exists a schedule with slope parameter $\beta < 1-w_D(N)$ such that all voters prefer it to any schedule with slope parameter $\beta' \geq 1-w_D(N)$, so the equilibrium must have slope parameter $\beta_E \in [0, 1-w_D(N))$.

QED

Notice that if more than half of the voters have a higher endowment than average (i.e., greater than N) then $\beta_E = 0$. More likely however, the median endowment is below the average and thus $\beta_E \in [1-w_D(0), 1-w_D(N))$. Also, if the distribution of endowments stays constant but the set of voters changes so that the endowment of the median voter falls, then the slope of the majority-rule equilibrium tax schedule rises (or stays constant). This follows from the comment below.

Comment 3.3. If $n_2 \leq n_1$ then for all $\beta_1 \in \beta^M(n_1)$ and all $\beta_2 \in \beta^M(n_2)$, $\beta_2 \geq \beta_1$.

Proof. Differentiate $\frac{\partial \tilde{y}}{\partial \beta}$ with respect to n to get

$$\frac{\partial^2 \tilde{y}}{\partial \beta \partial n}(n, \beta) = \begin{cases} -1 & \text{for } \beta < 1-w_D(0) \\ -1 & \text{for } \beta \in (1-w_D(0), 1-w_D(N)) \\ 0 & \text{for } \beta > 1-w_D(N) \, . \end{cases}$$

Thus, if $\frac{\partial \tilde{y}}{\partial \beta}(n_2, \beta) \leq 0$ then $\frac{\partial \tilde{y}}{\partial \beta}(n_1, \beta) \leq 0$. Using Comment 3.2 this implies that any β that maximizes $\tilde{y}(n_1, \cdot)$ must be less than or equal to any β that maximizes $\tilde{y}(n_2, \cdot)$, as desired.

QED

Thus we have

Comment 3.4. Fix the distribution F of endowments in the economy and consider two measure-spaces of egoistic voters $(J_1, \mathbf{F}_1, \mu_1)$ and $(J_2, \mathbf{F}_2, \mu_2)$ with unique median endowments n_1 and n_2 respectively, $n_2 < N$. If $n_2 < n_1$ then the tax rate of the majority-rule equilibrium tax schedule for $(J_1, \mathbf{F}_1, \mu_1)$ is at least as great as that for $(J_2, \mathbf{F}_2, \mu_2)$.

Over time, as voting in the United States (and in other countries as well) has become more universal due both to changes in the law and changes in participation rates, the income earning potential (i.e., endowment) of the median voter has probably fallen relative to the average. Given that the shape of the distribution of endowments has stayed relatively constant, Comment 3.4 suggests that tax rates should have risen. It also suggests that given two (democratic) countries with similar technologies and distributions of endowments but different voting patterns, the country with a higher relative participation rate among low income people should have a higher average income tax rate.

3.2 Social Welfare Functions over Linear Schedules

In this subsection we characterize social welfare functions over linear tax schedules. Recall that given a weight function W we can define the function $S(\cdot, W) : \mathbf{T} \to I\!R$ by $S(T, W) = \int W(y^*(n, T))dF(n)$. For the set $\mathbf{L}(0)$ we can define a function over slope parameters, $\tilde{S}(\cdot, W) : \tilde{\mathbf{L}}(0) \to I\!R$ by $\tilde{S}(\beta, W) = \int W(\tilde{y}(n, \beta))dF(n)$. Of course, the generalization to $G \neq 0$ is straightforward. Using the results of the previous subsection we can easily characterize $\tilde{S}(\cdot, W)$. Let $\beta^S(W)$ be the set of β that maximize $\tilde{S}(\cdot, W)$.

Comment 3.5. If $L_D''(w) > \frac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$ then

(i) $\tilde{S}(\cdot, W)$ is constant over $[1-w_D(N), 1]$; and

(ii) if W is strictly concave then $\tilde{S}(\cdot, W)$ is strictly concave over $[0, 1-w_D(N))$,

so $\beta^S(W)$ is unique and an element of $[1-w_D(0), 1-w_D(N))$, and

(iii) if W is linear then $\tilde{S}(\cdot, W)$ is constant over $[0, 1-w_D(0)]$, strictly decreasing and concave over $(1-w_D(0), 1-w_D(N))$, and thus $\beta^S(W) = [0, 1-w_D(0)]$.

(See Figure 3.2.)

Proof. Clearly, $\tilde{S}(\cdot, W)$ is constant over $[1-w_D(N), 1]$ since by Comment 3.2 $\tilde{y}(n, \cdot)$ is constant over $[1-w_D(N), 1]$ for all n, so (i) holds.

Now, since W is increasing and $\tilde{y}(n, \cdot)$ is strictly increasing and weakly concave over $[0, 1-w_D(N))$ for all n, if W is strictly concave then $\tilde{S}(\cdot, W)$ is strictly concave over $[0, 1-w_D(N))$. Next, note that

$\frac{\partial \tilde{S}}{\partial \beta}(\beta, W) = \int W'(\tilde{y}(n, \beta)) \frac{\partial \tilde{y}}{\partial \beta}(n, \beta) dF(n)$, and thus

$\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{S}}{\partial \beta}(\beta, W) = \int W'(\tilde{y}(n, 1-w_D(N))) [\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{y}}{\partial \beta}(n, \beta)] dF(n)$. As shown in Comment 3.2, $\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{y}}{\partial \beta}(n, \beta) < 0$ for all n, so $\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{S}}{\partial \beta}(\beta, W) < 0$.

Thus, $\beta^S(W) < 1-w_D(N)$, and by the concavity of $\tilde{S}(\cdot, W)$, $\beta^S(W)$ is unique.

To see that $\beta^S(W) \geq 1-w_D(0)$, note that for $\beta < 1-w_D(0)$,

$$\lim_{\beta \to 1-w_D(0)_-} \frac{\partial \tilde{S}}{\partial \beta}(\beta, W) = \int W'(\tilde{y}(n, 1-w_D(0)))[-n + N]dF(n)$$

$$= \int_{\underline{n}}^{N} [N-n]W'(\tilde{y}(n,1-w_D(0)))dF(n) + \int_{N}^{\bar{n}} [N-n]W'(\tilde{y}(n,1-w_D(0)))dF(n).$$

For $n < N$, $\tilde{y}(n,1-w_D(0)) < \tilde{y}(N,1-w_D(0))$ so

$W'(\tilde{y}(n,1-w_D(0)) > W'(\tilde{y}(N,1-w_D(0))$, and for $n > N$,

$\tilde{y}(n,1-w_D(0)) > \tilde{y}(N,1-w_D(0))$, so $W'(\tilde{y}(n,1-w_D(0))) < W'(\tilde{y}(N,1-w_D(0)))$.

Thus,

$$\lim_{\beta \to 1-w_D(0)_-} \frac{\partial \tilde{S}}{\partial \beta}(\beta, W) > \int_{\underline{n}}^{N} [N-n]W'(\tilde{y}(N,1-w_D(0)))dF(n)$$

$$+ \int_{N}^{\bar{n}} [N-n]W'(\tilde{y}(N,1-w_D(0)))dF(n)$$

$$= W'(\tilde{y}(N,1-w_D(0))) \int [N-n]dF(n) = 0.$$

Since $\tilde{S}(\cdot, W)$ is concave over $[0,1-w_D(0))$, $\frac{\partial \tilde{S}}{\partial \beta}(\beta, W) > 0$ for all $\beta \in [0,1-w_D(0))$, and thus $\beta^S(W) \geq 1-w_D(0)$. So, (ii) holds.

If W is linear then for $\beta \in [0,1-w_D(0)]$,

$\tilde{S}(\beta, W) = \int W(n(1-\beta) + \beta N)dF(n) = W' \cdot [(1-\beta) \int ndF(n) + \beta N] = W'N$, so $\tilde{S}(\cdot, W)$ is constant over $[0,1-w_D(0)]$. Since W is increasing, and $\tilde{y}$ is increasing and strictly concave over $(1-w_D(0),1-w_D(N))$, $\tilde{S}(\cdot, W)$ is strictly concave over $(1-w_D(0),1-w_D(N))$. And, for $\beta \in (1-w_D(0),1-w_D(N))$,

$$\frac{\partial \tilde{S}}{\partial \beta}(\beta, W) = \int W' \cdot [-n + N - L_D(1-\beta) + \beta L_D'(1-\beta)]dF(n)$$

$= W' \cdot [-L_D(1-\beta) + \beta L_D'(1-\beta)] < 0$, so $\tilde{S}(\cdot, W)$ is strictly decreasing on $(1-w_D(0), 1-w_D(N))$. Thus, (iii) holds.

QED

Thus, if voters rank tax schedules by social welfare functions (i.e., they are benevolent, as defined in Section 2.4), their preferences over the set of linear schedules $\mathbf{L}(0)$ are weakly single-peaked, so we have a majority-rule equilibrium, just as in the case for egoistic voters above.

Proposition 3.2. Given a measure-space of benevolent voters, if $L_D''(w) > \dfrac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$ then there exists a majority-rule equilibrium tax schedule over the set $\mathbf{L}(0)$. and any such equilibrium schedule has slope parameter $\beta_E \in [0, 1-w_D(N))$.

Proof. If we order the elements of $\mathbf{L}(0)$ along the line by their slope parameters then by Comment 3.5 the preference relations of the set J of voter types satisfies weak single-peakedness, as defined in Section 2.4. Also, $\tilde{S}(\cdot, W)$ is continuous for all W, and $\tilde{S}(\beta, \cdot)$ is continuous (hence measurable) for all β, so the conditions of Comment 2.6 hold. Thus, by Comment 2.5, there exists a majority-rule equilibrium over $\mathbf{L}(0)$. Clearly, there exists a schedule with slope parameter $\beta < 1-w_D(N)$ such that all voters prefer it to any schedule with slope parameter $\beta' \geq 1-w_D(N)$, so the equilibrium must have slope parameter $\beta_E \in [0, 1-w_D(N))$.

QED

Note that if more than half of the voters have a linear weight function W then the majority-rule equilibrium schedule is not unique, but in fact any schedule in $\mathbf{L}(0)$ with slope $\beta \in [0,1-w_D(0)]$ is an equilibrium. (since with a linear weight function the median voter wants to maximize average income, and all schedules in $\mathbf{L}(0)$ with slope $\beta \in [0,1-w_D(0)]$ produce the same average income, namely N). On the other hand, if more than half of the voters have a strictly concave weight function, then the majority-rule equilibrium tax schedule is unique, and will have slope $\beta_E \in [1-w_D(0),1-w_D(N))$ (the schedule with slope $1-w_D(0)$ is preferred to any schedule with slope less that $1-w_D(0)$ by all voters with strictly concave weighting functions, who constitute a majority).

In general, the slope of the equilibrium linear tax schedule for egoistic voters may be greater than, less than, or equal to that for benevolent voters. In two special cases we can determine which is greater. First, if the egoistic voter with median endowment has an endowment greater than average and more than half of the benevolent voters have a strictly concave weighting function, then the equilibrium linear tax schedule for benevolent voters has a higher slope than that for egoistic voters (some $\beta > 1-w_D(0)$ versus 0). Second, if the egoistic voter with median endowment has an endowment less than average and more than half of the benevolent voters have a linear weighting function then the equilibrium linear tax schedule for benevolent voters has a lower slope than that for egoistic voters (any $\beta \leq 1-w_D(0)$ versus some $\beta \geq 1-w_D(0)$).

3.3 Linear Schedules and More General Voter Preferences

We now consider voters with more general utility functions, as described above in Section 2.4. Recall that a voter of type j is characterized by n_j,

W_j and u_j, and that her preferences over tax schedules are given by the function v_j, defined by $v_j(T) = u_j(y^*(n_j,T), S(T,W_j))$. For tax schedules in $\mathbf{L}(0)$ we can define the function $\tilde{v}_j$ over slope parameters by $\tilde{v}_j(\beta) = u_j(\tilde{y}(n_j,\beta), \tilde{S}(\beta,W_j))$ for all β.

As in the previous subsections we wish to characterize voters' preferences over $\mathbf{L}(0)$ and prove the existence of a majority-rule voting equilibrium. Fortunately, the results of the previous subsections make this an easy task

Comment 3.6. If $L_D''(w) > \dfrac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N),w_D(0))$ then $\tilde{v}_j$ is weakly single-peaked over $[0,1]$, and the set of β which maximize $\tilde{v}_j$ over $[0,1]$ is a closed interval in $[0,1-w_D(N))$.

Proof. Clearly, $\tilde{v}_j$ is constant over $[1-w_D(N),1]$ since $\tilde{y}(n_j,\cdot)$ and $\tilde{S}(\cdot,W_j)$ both are. Now, suppose u_j is concave in y_j and S_j. Then for all β_1 and β_2 in $[0,1-w_D(N))$,

$$\tilde{v}_j(\lambda\beta_1 + (1-\lambda)\beta_2) = u_j(\tilde{y}(n_j,\lambda\beta_1 + (1-\lambda)\beta_2), \tilde{S}(\lambda\beta_1 + (1-\lambda)\beta_2, W_j))$$

$$\geq u_j(\lambda\tilde{y}(n_j,\beta_1) + (1-\lambda)\tilde{y}(n_j,\beta_2), \lambda\tilde{S}(\beta_1,W_j) + (1-\lambda)\tilde{S}(\beta_2,W_j))$$

$$\geq \lambda u_j(\tilde{y}(n_j,\beta_1), \tilde{S}(\beta_1,W_j)) + (1-\lambda)u_j(\tilde{y}(n_j,\beta_2), \tilde{S}(\beta_2,W_j))$$

$$= \lambda\tilde{v}_j(\beta_1) + (1-\lambda)\tilde{v}_j(\beta_2).$$

The first inequality follows since $\tilde{y}(n_j,\cdot)$ and $\tilde{S}(\cdot,W_j)$ are weakly concave over $[0,1-w_D(N))$, and u_j is nondecreasing in y_j and S_j. The second inequality follows since u_j is assumed to be concave. Thus, $\tilde{v}_j$ is weakly concave over $[0,1-w_D(N))$.

Next, note that $\frac{\partial \tilde{v}_j}{\partial \beta}(\beta) = \frac{\partial u_j}{\partial y_j} \cdot \frac{\partial \tilde{y}}{\partial \beta}(n_j,\beta) + \frac{\partial u_j}{\partial S_j} \cdot \frac{\partial \tilde{S}}{\partial \beta}(\beta, W_j)$, so

$$\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{v}_j}{\partial \beta} = \frac{\partial u_j}{\partial y_j}(\tilde{y}(n_j,1-w_D(N)),\ \tilde{S}(1-w_D(N),W_j)) \cdot [\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{y}}{\partial \beta}(n_j,\beta)]$$

$$+ \frac{\partial u_j}{\partial S_j}(\tilde{y}(n_j,1-w_D(N)),\ \tilde{S}(1-w_D(N),W_j)) \cdot [\lim_{\beta \to 1-w_D(N)_-} \frac{\partial \tilde{S}}{\partial \beta}(\beta,W_j)].$$

Using the limit results from the proofs of Comments 3.2 and 3.5, plus the assumption that u_j is nondecreasing in y_j and S_j, and strictly increasing in at least one of them, this is less than zero. So $\tilde{v}_j$ is strictly decreasing in a neighborhood of $1-w_D(N)$, and thus it is weakly single-peaked on $[0,1]$, and the set of β which maximize it is a closed interval in $[0,1-w_D(N))$.

If u_j is not concave then let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function such that $\varphi(u_j)$ is concave. Such a φ exists since u_j is quasiconcave. Then, applying the result just proved for concave utility functions, $\varphi(\tilde{v}_j)$ is weakly single-peaked and the set of β which maximize it is a closed interval in $[0,1-w_D(N))$. However, since φ is strictly increasing, $\varphi(\tilde{v}_j(\beta_1)) \geq \varphi(\tilde{v}_j(\beta_2))$ if and only if $\tilde{v}_j(\beta_1) \geq \tilde{v}_j(\beta_2)$. Thus, the same properties hold for $\tilde{v}_j$.

QED

With this comment in hand, we have a third voting result.

Proposition 3.3. Given a measure-space of voters with generalized preferences, if $L_D''(w) > \frac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$ then there exists a majority-rule equilibrium tax schedule over the set $\mathbf{L}(0)$, and any such equilibrium schedule has slope parameter $\beta_E \in [0, 1-w_D(N))$.

Proof. If we order the elements of $\mathbf{L}(0)$ along the line by their slope parameters then by Comment 3.6 the preference relations of the set J of voter types satisfies weak single-peakedness, as defined in section 2.4. Letting $\psi : J \times [0,1] \to \mathbb{R}$ be defined by $\psi(j,\beta) = \tilde{v}_j(\beta)$, $\psi(j, \cdot)$ is continuous for all j, and $\psi(\cdot, \beta)$ is continuous (hence measurable) for all β, so the conditions of Comment 2.6 hold. Thus, by Comment 2.5, there exists a majority-rule equilibrium over $\mathbf{L}(0)$. Clearly, there exists a schedule with slope parameter $\beta < 1-w_D(N)$ such that all voters prefer it to any schedule with slope parameter $\beta' \geq 1-w_D(N)$, so the equilibrium must have slope parameter $\beta_E \in [0, 1-w_D(N))$.

QED

4. Nonlinear Schedules

Here, we expand our attention, and consider nonlinear tax schedules. First, we will define the notion of "simple" schedules, which will be quite useful in proving results.

4.1 Simple Schedules

Let us say a schedule T is *simple* if

$$T(x) = T(\underline{x}) + (1 - w^*(T))(x - \underline{x}) \text{ for all } x \in [\underline{x}, \bar{x}),$$

for every interval $[\underline{x}, \bar{x})$ that is critical for $w^*(T)$. Thus, a simple schedule is linear, with slope $(1 - w^*(T))$, over its $w^*(T)$-critical intervals. An alternative and equivalent characterization is as follows.

Comment 4.1. T is simple if and only if $\dfrac{T(x_2) - T(x_1)}{(x_2 - x_1)} \leq 1 - w^*(T)$ for all x_1, x_2 such that $0 \leq x_1 < x_2 \leq 1$.

Proof. If the inequality holds everywhere, it clearly holds (with equality) on every $w^*(T)$ critical interval, so T is simple. Conversely, if the inequality fails for some x_1, x_2, let $\underline{x} = \min\{x' \in [0, x_2] \mid T(x') - (1 - w^*(T))x' \leq T(x) - (1 - w^*(T))x \textit{ for all } x \in [0, x_2)\}$. Evidently, this defines a $w^*(T)$-critical interval $[\underline{x}, \bar{x})$ which contains x_2, and $T(\underline{x}) - (1 - w^*(T))\underline{x} \leq T(x_1) - (1 - w^*(T))x_1 < T(x_2) - (1 - w^*(T))x_2$, so T is not simple.

QED

Comment 4.1 implies, in particular, that the marginal tax rate of a simple schedule cannot exceed $1-w^*(T)$. Thus, for example, if T is linear then T is simple if and only if it has slope $\beta \leq 1-w_D(N)$. Simple schedules have the following convenient property.

Comment 4.2. If T is simple then $y^*(n,T) = n - T(n)$ for all n, and $R(T) = \int T(n)dF(n) - (1-w^*(T))L^*(T)$.

Proof. Suppose $n \in [\underline{x},\bar{x})$ for some $w^*(T)$-critical interval $[\underline{x},\bar{x})$. Since T is simple, $T(n) = T(\underline{x}) + (1 - w^*(T))(n - \underline{x})$, so by (ii) of Comment 2.2, $x' = n$ is optimal, whence $y^*(n,T) = n - T(n)$ for all such n. If n does not belong to any $w^*(T)$-critical interval then by the rest of Comment 2.2, $x' = n$ is optimal for n (uniquely optimal if n is not in the closure of any $w^*(T)$-critical intervals), again implying that $y^*(n,T) = n - T(n)$. To see that $R(T) = \int T(n)dF(n) - (1-w^*(T))L^*(T)$, recall that $Z^*(T) = N - (1-w^*(T))L^*(T)$ and $R(T) = Z^*(T) - Y^*(T)$ (see Section 2.2), integrate $y^*(n,T)$ to get $Y^*(T)$, and substitute.

QED

The importance of simple schedules lies in the following fact.

Comment 4.3. For any tax schedule T, there exists a simple tax schedule T_1 which is equivalent to T.

(See Figure 4.1.)

Proof. We construct T_1 by linearizing T over its $w^*(T)$-critical intervals. Without loss of generality we can suppose there is just one such interval

$[\underline{x},\bar{x})$. T_1 is then defined by

$$T_1(x) = \begin{cases} T(\underline{x}) + (1 - w^*(T))(x - \underline{x}) & \text{for all } x \in [\underline{x},\bar{x}) \\ T(x) & \text{otherwise .} \end{cases}$$

Evidently $T_1(x) - (1 - w^*(T))x = T_1(\underline{x}) - (1 - w^*(T))\underline{x}$ for all $x \in [\underline{x},\bar{x})$, and $[\underline{x},\bar{x})$ is the unique $w^*(T)$-critical interval of T_1. Comment 2.2 then implies that

$$\hat{x}(n,w^*(T),T_1) = \begin{cases} [\underline{x},n] \supseteq \hat{x}(n,w^*(T),T) & \text{for } n \in [\underline{x},\bar{x}), \\ \{n\} = \hat{x}(n,w^*(T),T) & \text{for } n \notin [\underline{x},\bar{x}], \text{ and either} \\ [\underline{x},n] \supseteq \hat{x}(n,w^*(T),T) & \text{or} \\ \{n\} = \hat{x}(n,w^*(T),T) & \text{for } n = \bar{x}\,. \end{cases}$$

Thus, if $\hat{x}(n)$ is optimal for n at $w^*(T)$ under T then it is also optimal under T_1 at the same wage, so $\hat{X}(w^*(T),T_1) \supseteq \hat{X}(w^*(T),T)$, implying $w^*(T_1) = w^*(T)$ and $X^*(T_1) = X^*(T)$. This implies that $[\underline{x},\bar{x})$ is the unique $w^*(T_1)$-critical interval of $T1$, and that T_1 is simple.

We next show that both schedules induce the same after-tax income distribution. By (ii) of Comment 2.2, for $n \in [\underline{x},\bar{x})$, $\hat{x}(n) = \underline{x}$ is optimal under T at $w^*(T)$ and under T_1 at at $w^*(T_1)$ (since $[\underline{x},\bar{x})$ is critical in either case) and $T_1(\underline{x}) = T(\underline{x})$ by construction, so $y^*(n,T) = \underline{x} - T(\underline{x}) + w^*(T)(n - \underline{x}) = \underline{x} - T_1(\underline{x}) + w^*(T_1)(n - \underline{x}) = y^*(n,T_1)$. Similarly, for $n \notin [\underline{x},\bar{x}]$, $\hat{x}(n) = n$ is (uniquely) optimal under T at $w^*(T)$ and under T_1 at $w^*(T_1)$, and $T(n) = T_1(n)$, so $y_T(n) = n - T(n) = n - T(n) = y^*(n,T_1)$ for all such n. The same conclusion is readily verified for $n = \bar{x}$, whence $y^*(n,T) = y^*(n,T_1)$ for all n.

Hence, $Y^*(T) = Y^*(T_1)$, and since $X^*(T) = X^*(T_1)$ it follows that $Z^*(T) = Z^*(T_1)$ and $R(T) = R(T_1)$. So, T_1 is equivalent to T. It remains to show that T_1 is a tax schedule; i.e., that $T_1(x) \leq x$ for all x. But T is a tax schedule (by assumption), so this follows immediately from the construction of $T1$, since $T_1(x) = T(x)$ for $x \notin [\underline{x},\bar{x})$, and
$T_1(x) = T(\underline{x}) + (1 - w^*(T))(x - \underline{x}) \leq T(x)$ for $x \in [\underline{x},\bar{x})$.

QED

It is nearly, but not quite, true that every schedule has a ***unique*** simple equivalent. In particular, if $\bar{n} < 1$ the portion of any schedule which applies to $x \in (\bar{n},1]$ is irrelevant, since taxable incomes in this range cannot occur. Thus if T is equivalent to T_1 as constructed above, it is also equivalent to every simple schedule which coincides with T_1 on $(0,\bar{n}]$. If we define a "canonical" simple schedule as a simple schedule T_1 such that $T_1(x) = T(\bar{n}) + (1 - w^*(T_1))(x - \bar{n})$ for $x > \bar{n}$, however, it follows from the construction above that every schedule is equivalent to a unique canonical simple schedule.

As an example of how using simple schedules simplifies matters, we prove the folllowing comment, which will be used later. In Comment 3.1 we showed that there exists a linear schedule $T^M \in \mathbf{L}^0$ that maximizes government revenue over the set of all linear schedules $\mathbf{L}$. Here we extend that result, and show that T^M maximizes revenue over the class of ***all*** tax schedules.

Comment 4.4. Let T^M be defined as in Comment 3.1. Then for all $T \in \mathbf{T}$, $R(T) \leq R(T^M) = G^M$.

Proof. From Comment 4.3 we may confine attention to simple schedules (if T

is not simple then take a simple equivalent). For any simple schedule T, let T_1 be defined by $T_1(x) = (1-w^*(T))x$ for all x. Clearly, $T_1(x) \geq T(x)$ for all x, using Comment 4.1 and the fact that $T(0) \leq 0$ (since it is an admissible schedule). Moreover, since $T_1 \in \mathbf{L}$ with slope $1-w^*(T)$, $\hat{L}(w^*(T), T_1) = [0,N] \ni L^*(T)$, so T_1 induces the same market equilibrium as T; i.e., $w^*(T_1) = w^*(T)$ and $L^*(T_1) = L^*(T)$. Thus T_1 is simple, so by Comment 4.2, $R(T_1) - R(T) = \int T_1(n)dF(n) - \int T(n)dF(n) \geq 0$. And by Comment 3.1, $R(T_1) \leq R(T^M)$, so $R(T) \leq R(T^M)$.

QED

4.2 Social Welfare Functions and Nonlinear Schedules

Here we show that for any nonlinear tax schedule there exists a linear schedule which raises the same amount of revenue and is ranked higher by *any* indirect social welfare function whose weighting function is strictly concave. For social welfare functions with linear weights, the linear tax will be ranked higher than or equal to the nonlinear schedule. Thus, given any set of benevolent voters, the linear schedule will be ***unanimously*** preferred to the nonlinear one (perhaps with some voters indifferent). Combined with the results from section 3.2, this implies that for any set of benevolent voters there exists a majority-rule equilibrium tax schedule, and that schedule is (up to equivalence) linear. It also implies that, regardless of the particular indirect social welfare function chosen, the "optimal" tax schedule is (again, up to equivalence) linear.

The key is a result by Atkinson [3], which in the present context says roughly that given two distributions of income with the same mean, if one distribution can be obtained from the other by redistributing income from richer to poorer individuals, then it is ranked higher by any social welfare function with a strictly concave weighting function. We can describe the result in terms of the indirect social welfare functions over tax schedules as follows.

Comment 4.5. Let T_1 and T_2 be tax schedules such that $R(T_1) = R(T_2)$ and $Y^*(T_1) = Y^*(T_2)$. If there exists an $n_e \in (\underline{n}, \overline{n})$ such that

$$y^*(n, T_1) \geq y^*(n, T_2) \quad \text{for all } n < n_e, \text{ and}$$
$$y^*(n, T_1) \leq y^*(n, T_2) \quad \text{for all } n > n_e,$$

then for any strictly increasing concave W, $\int W(y^*(n, T_1) dF(n) \geq \int W(y^*(n, T) dF(n)$. If W is strictly concave, and the above inequalities hold strictly on some set of positive (F) measure, then $\int W(y^*(n, T_1) dF(n) > \int W(y^*(n, T) dF(n)$.

Proof. See Atkinson [3].

Recall that for indirect social welfare functions with linear weighting functions, if $Y(T_1) = Y(T_2)$ then T_1 and T_2 are ranked the same regardless of how the income is distributed.

Proposition 4.1. If T is a simple tax schedule which is not linear over $(\underline{n}, \overline{n})$ then there exists a (simple) linear tax schedule T_1 with $R(T_1) = R(T)$ such that for any $S(\cdot, W)$ with W strictly concave, $S(T_1, W) > S(T, W)$.

Proof. Let T_1 be defined by $T_1(x) = \alpha + \beta x$, with $\beta = 1 - w^*(T)$ and $\alpha = w^*(T)N - Y^*(T)$. It is readily verified that this schedule induces the same equilibrium (the interval $[0,1)$ is $w^*(T)$-critical for T_1, so

$\hat{X}(w^*(T),T_1) = [0,N]$, whence $X^*(T) \in \hat{X}(w^*(T),T_1)$; thus $w^*(T_1) = w^*(T)$ and $X^*(T_1) = X^*(T)$), so $Z^*(T_1) = Z^*(T_2)$. Also, T_1 is simple, so $y^*(n,T_1) = n - T_1(n) = w^*(T)(n - N) + Y^*(T)$ for all n (from Comment 4.2), and thus $Y^*(T_1) = \int [w^*(T)(n - N) + Y^*(T)] = Y^*(T)$, and $R(T) = Z^*(T) - Y^*(T) = Z^*(T_1) - Y^*(T_1) = R(T_1)$.

Since T is simple, for any x_1 and x_2 with $0 \le x_1 < x_2 \le 1$, $T(x_2) \le T(x_1) + (1 - w^*(T))(x_2 - x_1)$ (by Comment 4.1), so $T_1(x_2) - T(x_2) \ge T_1(x_1) - T(x_1) + (1 - w^*(T))(x_2 - x_1)$. Thus $T_1 - T$ is nondecreasing on $[0,1]$. Moreover, since T is nonlinear over $(\underline{n},\bar{n})$ by hypothesis, there must exist $n' \in (\underline{n},\bar{n})$ such that $T_1(x) - T(x) > T_1(\underline{n}) - T(\underline{n})$ for all $x \in (n',\bar{n})$. Clearly, if $T_1(\underline{n}) - T(\underline{n}) \ge 0$ then $T_1(x) - T(x) \ge 0$ for all $x > \underline{n}$, with strict inequality for $x > n'$, which from Comment 4.2 would imply that $Y^*(T_1) < Y^*(T)$, a contradiction. Hence $T_1(\underline{n}) < T(\underline{n})$. Similarly, $T_1(\bar{n}) - T(\bar{n}) \le 0$ would imply that $Y^*(T_1) < Y^*(T)$, again a contradiction, so $T_1(\bar{n}) > T(\bar{n})$. Thus there must exist $n_e \in (\underline{n},\bar{n})$ such that $T_1(n) \le T(n)$, whence $y^*(n,T_1) \ge y^*(n,T)$ (from Comment 4.2) for $n < n_e$, and $y^*(n,T_1) \le y^*(n,T)$ for $n \ge n_e$ (using lower semi-continuity). Moreover, the inequality must be strict for a neighborhood of $\underline{n}$ and a neighborhood of $\bar{n}$. Thus, by Comment 4.5, the proposition holds.

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Thus, the voting result of Section 3.2 extends to nonlinear tax schedules; i.e., if voters are benevolent then there exists a voting equilibrium among the class of *all* tax schedules that raise the same amount of revenue.

Proposition 4.2. Given a measure-space of benevolent voters, if $L_D''(w) > \frac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$ then there exists a linear majority-rule equilibrium tax schedule over the set $\mathbf{T}(0)$. If at least half of the voters have a strictly concave weighting function then any such equilibrium schedule is equivalent to a linear schedule with slope $\beta_E \in [0, 1-w_D(N))$.

Proof. From Proposition 3.2 there exists a majority-rule equilibrium schedule $T_E \in \mathbf{L}(0)$ over the set $\mathbf{L}(0)$ of linear tax schedules, and has slope parameter $\beta_E \in [0, 1-w_D(N))$. Suppose there exists a schedule $T_1 \in \mathbf{T}(0)$ which defeats T_E in a pairwise majority-rule vote, and let $K \subseteq J$ be the set of voter types that strictly prefer T_1 to T_E. Then K constitutes a majority (i.e., $\mu(K) > \frac{1}{2}$). By Proposition 4.1 there exists a linear schedule $T_2 \in \mathbf{L}(0)$ such that *all* voter types in J, and hence all voter types in K, prefer T_2 to T_1. By the transitivity of individual preferences, all voter types in K also prefer T_2 to T_E. But this means that $T_2 \geq_M T_E$, contradicting the assumption that T_E is a majority-rule equilibrium over $\mathbf{L}(0)$. Thus, no such T_2 exists, so T_E is a majority-rule equilibrium over $\mathbf{T}(0)$. The fact that when more than half of the voters have strictly concave weighting functions every majority-rule equilibrium must be equivalent to a linear schedule follows directly from Proposition 4.1.

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4.3 After-tax Incomes Under Nonlinear Schedules

While there is a robust political equilibrium for benevolent voters, no such equilibrium exists if voters are egoistic. In general, if we condider large classes of nonlinear taxes, there will be severe voting cycles. Thus, to obtain a political equilibrium for egoistic voters we must either put more structure on the political game or change our equilibrium concept, or look at restricted sets of taxes. The results in Section 3.1 are an example of this, where we found a voting equilibrium over the set of linear taxes. In this section we consider a particular class of nonlinear tax schedules, namely the set of "individually optimal" tax schedules (to be defined shortly). This set is of special interest because it helps point out who the "winners" and "losers" are under different types of taxes. Thus, even though there is no robust majority-rule equilibrium, we are able to suggest likely outcomes depending on how political power is distributed.

In defining the tax schedules that are optimal for a particular individual, we restrict attention to ***nondecreasing*** schedules. We denote the set of nondecreasing tax schedules by $\mathbf{N}$, and for each G the set of nondecreasing schedules that raise revenue equal to G by $\mathbf{N}(G)$.[9] For $n \in [\underline{n}, \bar{n}]$, we say that a tax schedule T_n is ***optimal for type n given G*** if it solves $\max_{T \in \mathbf{N}(G)} \dot{y}^*(n, T)$. Let $\mathbf{B}(G) = \{T : T$ ***is optimal for type n, for some n***$\}$ be the set of all such individually optimal schedules.

Before discussing majority voting over $\mathbf{B}(G)$, we must characterize its elements. The characterization itself is rather simple and intuitive, but proving that it is in fact true is a nontrivial task, involving tedious proofs. Therefore, we state the results first, leaving the proofs for later. For n and

G "small enough," T_n will be (up to equivalence) of the form

$$T_n(x) = \begin{cases} \alpha & \text{for } x \leq n \\ \alpha + \beta(x - n) & \text{for } x > n \end{cases}$$

for some $\alpha \leq 0$ and β. If $G \leq 0$ then T_n will be of this form for all n. For $G > 0$ and n "too large" (to be made precise below), no function of this form will be an admissible tax schedule--any such function T that satisfies $R(T) = G$ must have $\alpha > 0$, and thus violates the condition that $T(x) \leq x$ for all x. In this case T_n is of the form

$$T_n(x) = \begin{cases} \beta x & \text{for } x \leq \frac{\tau}{\beta} \\ \tau & \text{for } x \in (\frac{\tau}{\beta}, n] \\ \tau + \beta(x - n) & \text{for } x > n \end{cases}$$

for some $\tau \geq 0$ and β. (See Figure 4.2.)

To make this precise and prove that it is true, we must first make a few definitions. For simplicity we assume that the distribution F of abilities has a density f and that its support is $[\underline{n}, \bar{n}] = [0,1]$.

Let $\mathbf{S}_0 = \{(\tau,\beta,n) \mid n \in [0,1], \tau \in [0,(1-w_D(N))n], \textit{and } \beta \in [\frac{\tau}{n},1]\}$, for all n, let $\mathbf{S}_0(n) = \{(\tau,\beta) \mid (\tau,\beta,n) \in \mathbf{S}_0\}$, and let $\mathbf{S}_0' = \{(\tau,n) \mid n \in [0,1]$ *and* $\tau \in [0,(1-w_D(N))n]\}$. Fix $n_i \in [0,1]$ and consider the set $\mathbf{S}(n_i)$ of tax schedules of the form

$$T(x) = \begin{cases} \beta x & \text{for } 0 \leq x \leq \frac{\tau}{\beta} \\ \tau & \text{for } \frac{\tau}{\beta} < x \leq n_i \\ \tau + \beta(x - n_i) & \text{for } n_i < x \leq 1 \end{cases}$$

for some $(\tau,\beta) \in \mathbf{S}_0(n_i)$. Also let $\mathbf{S}(n_i)$ contain the schedule T defined by

$T(x) = 0$ for all x. Note that each $T \in \mathbf{S}(n_i)$ is completely characterized by the two parameters τ and β (let the parameters for the schedule $T \equiv 0$ be $(\tau,\beta) = (0,0)$), and there is a 1-1 mapping from $\mathbf{S}_0(n_i)$ onto $\mathbf{S}(n_i)$. Let $R_0 : \mathbf{S}_0 \to [0,1]$ be defined by $R_0(\tau,\beta,n_i) = R(T)$, where $T \in \mathbf{S}(n_i)$ has parameters (τ,β). In Appendix B we prove that $R_0(\cdot,\cdot,n_i)$ is continuous on $\mathbf{S}_0(n_i)$ for all n_i. (See Lemma B.3.)

We now define a set of schedules $\bar{\mathbf{S}}(n_i,G)$ that are optimal for n_i given the government's revenue constraint G. First, note that the revenue maximizing linear schedule T^M, defined by $T^M(x) = \beta^M x$ for all x, is an element of $\mathbf{S}(n_i)$, and has parameters $(\beta^M n_i,\beta^M)$. By Comment 4.4 $R(T^M) = \max_{T \in \mathbf{T}} R(T) = G^M$. Thus, we have $R_0(0,0,n_i) = 0$ and $R_0(\beta^M n_i, \beta^M, n_i) = G^M$, so by the continuity of $R_0(\cdot,\cdot,n_i)$, for any $G \in [0,G^M]$ there exists $(\tau,\beta) \in \mathbf{S}_0(n_i)$ such that $R_0(\tau,\beta,n_i) = G$.

Since $[0,1]$ is compact, the continuity of $R_0(\cdot,\cdot,n_i)$ guarantees that there exists $\beta_H(0,n_i)$ that solves $\max_{\beta \in [0,1]} R_0(0,\beta,n_i)$. Let $G_i = R_0(0, \beta_H(0,n_i), n_i)$ be the maximum value of $R_0(0,\cdot,n_i)$. Also, since $R_0(0,\beta,n_i) = R_0(0,\beta_e(0,n_i),n_i)$ for $\beta > \beta_e(0,n_i)$, there is a solution $\beta_H(0,n_i) \in [0,\beta_e(0,n_i)]$. Let $\hat{\beta}_i$ be the set of all such solutions. Clearly, if T_i is a schedule in $\mathbf{S}(n_i)$ with parameters $(0,\beta_i)$, $\beta_i \in \hat{\beta}_i$, then T_i is simple and solves $\max_{T \in \mathbf{S}(n_i)} R(T)$ s.t. $T(n_i) = 0$. Let $\mathbf{P}(n_i)$ be the set of tax schedules of the form $\alpha + T$ (i.e., $T_1 \in \mathbf{P}(n_i)$ is defined by $T_1(x) = \alpha + T(x)$ for all x) where $\alpha \leq 0$ and $T \in \mathbf{S}(n_i)$ with parameters $(0,\beta_i)$, $\beta_i \in \hat{\beta}_i$. For $G \leq G_i$, let $\bar{\mathbf{S}}(n_i,G)$ be the elements of $\mathbf{P}(n_i)$ with $\alpha = G - G_0$. (Note that if $\hat{\beta}_i$ has more than one element, then so does $\bar{\mathbf{S}}(n_i,G)$.)

Next, note that $\{(\tau,\beta) \in \mathbf{S}_0(n_i) \mid R_0(\tau,\beta,n_i) = G\}$ is closed (since $\{G\}$ is closed and R_0 is continuous) and bounded (since $\mathbf{S}_0(n_i)$ is bounded), hence compact. So, there exists a solution $(\bar{\tau}(n_i,G),\bar{\beta}(n_i,G))$ to $\min_{(\tau,\beta) \in \mathbf{S}_0(n_i)} \tau$ s.t. $R_0(\tau,\beta,n_i) = G$. Since $R_0(\tau,\beta,n_i) = R_0(\tau,\beta_e(\tau,n_i),n_i)$ for all $\beta \geq \beta_e(\tau,n_i)$ and all τ, there is a solution such that $\bar{\beta}(n_i,G) \in [\bar{\tau}(n_i,G)/n_i, \beta_e(\bar{\tau}(n_i,G),n_i)]$. Clearly $\bar{\tau}$ is nondecreasing in G for any n_i, so $(\bar{\tau}(n_i,G),\bar{\beta}(n_i,G))$ also solves $\min_{(\tau,\beta) \in \mathbf{S}_0(n_i)} \tau$ s.t. $R_0(\tau,\beta,n_i) \geq G$. Let $\bar{\mathbf{S}}_0(n_i,G)$ be the set of such solutions. If T_i is any schedule in $\mathbf{S}(n_i)$ with parameters $(\tau_0(n_i,\beta_i) \in \bar{\mathbf{S}}_0(n_i,G)$ then T_i is simple and solves

$$\min_{T \in \mathbf{S}(n_i)} T(n_i) \quad \text{s.t.} \quad R(T) \geq G.$$

For $G > G_i$ let $\bar{\mathbf{S}}(n_i,G)$ be the set of such solutions.

To facilitate the proof that the schedules in $\bar{\mathbf{S}}(n_i,G)$ are optimal for n_i given G, we use the following lemma.

Lemma 4.1. Fix $n_i \in [0,1]$. Let T be simple with $T(n_i) \geq 0$, and let T_1 be defined by

$$T_1(x) = \begin{cases} (1 - w^*(T))x & \text{for } 0 \leq x \leq \dfrac{T(n_i)}{1-w^*(T)} \\ T(n_i) & \text{for } \dfrac{T(n_i)}{1-w^*(T)} < x \leq n_i \\ T(n_i) + (1 - w^*(T))(x - n_i) & \text{for } n_i < x \leq 1 \end{cases}$$

Then $T_1 \in \mathbf{S}(n_i)$ and $R(T_1) \geq R(T)$. Furthermore, if $T \equiv T_1$ then $R(T_1) > R(T)$.

Proof. See Appendix B.

We are now ready to prove

Proposition 4.3. For any $n_i \in [0,1]$ and $G \in [0, G^m]$, the schedules in $\bar{\mathbf{S}}(n_i, G)$ are optimal for n_i given G, and any T which is optimal for n_i given G is equivalent to some element of $\bar{\mathbf{S}}(n_i, G)$.

Proof. By definition, T_i is optimal for n_i given G if and only if it solves $\max_{T \in \mathbf{N}(G)} y^*(n_i, T)$ s.t. $R(T) \geq G$. By Comment 4.3 we may restrict attention to simple schedules (if T is not simple, pick its unique simple equivalent), for which $y^*(n_i, T) = n_i - T(n_i)$. Then T_i is optimal for n_i given G if and only if it solves $\min_{T \in \mathbf{N}(G)} T(n_i)$ s.t. $R(T) \geq G$.

Fix n_i, and for each G let T_i^G be a selection from $\bar{\mathbf{S}}(n_i, G)$. Consider first $G \leq G_i$. Let T be any (simple) schedule such that $R(T) \geq G$. We show that $T(n_i) \geq T_i^G(n_i) = G - G_i$. Let $T_2 \equiv T - T(n_i)$ (i.e., define T_2 by $T_2(x) = T(x) - T(n_i)$ for all x). Then by Comment 2.3, $w^*(T_2) = w^*(T)$ and $R(T_2) = R(T) - T(n_i)$. Let $T_1 \in \mathbf{S}(n_i)$ have parameters $(0, 1 - w^*(T))$ (see Figure 4.3). Then by Lemma 4.1, $R(T_1) \geq R(T_2)$. But by definition $R(T_2) \leq G_i$, so $G_i \geq R(T_2) = R(T) - T(n_i) \geq G - T(n_i)$, or $T(n_i) \geq G - G_i = T_i^G(n_i)$ as desired. If $T_2 \not\equiv T_1$, then by Lemma 4.1 $R(T_1) > R(T_2)$, whence $T(n_i) > T_i^G(n_i)$ and T is not optimal. Or, if $R(T_2) < G_i$ then again $T(n_i) > T_i^G(n_i)$ and T is not optimal. Thus, if T is optimal for n_i given G then $T_2 \equiv T_1$ and $R(T_2) = G_i$, so $T \equiv T_2 + T(n_i)$ and thus $T \in \bar{\mathbf{S}}(n_i, G)$.

Next, consider $G > G_i$. Again, let T be any (simple) schedule such that $R(T) \geq G$. First, we show that $T(n_i) > 0$. Suppose not, and as above let $T_2 \equiv T - T(n_i)$ and $T_1 \in \mathbf{S}(n_i)$ have parameters $(0, 1 - w^*(T))$. Then

$R(T_1) \geq R(T_2)$. But $G_i \geq R(T_1)$, so we have $G_i \geq R(T_2) = R(T) - T(n_i)$ $\geq G - T(n_i) \geq G$, contradicting $G > G_i$. Now, let T_3 be the schedule in $\mathbf{S}(n_i)$ with parameters $(T(n_i), 1 - w^*(T))$. Then $T_3(n_i) = T(n_i)$ and by Lemma 4.1 $R(T_3) \geq R(T) \geq G$. Thus, by definition $T_i^G(n_i) \leq T_3(n_i)$, so $T_i^G(n_i) \leq T(n_i)$ as desired. Again, if $T_3 \not\equiv T$ then by Lemma 3.4 $R(T_3) > R(T) \geq G$ and the schedule $T_4 \equiv T_3 + R(T) - R(T_3)$ satisfies $T_4(n_i) < T_3(n_i) = T(n_i)$, and $R(T_4) = R(T) \geq G$, so T is not optimal. Or, if $T_3 \notin \bar{\mathbf{S}}(n_i, G)$ then $T_i^G(n_i) < T_3(n_i) = T(n_i)$ and T is not optimal. So, if T is optimal for n_i given G then $T \in \bar{\mathbf{S}}(n_i, G)$.

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Next we show that under the same restriction on L_D as in the previous sections (namely, that L_D not be "too concave" in w), egoistic voters' preferences over the set of individually optimal schedules are weakly single-peaked for any government revenue constraint $G \leq 0$.

We prove the result as follows. Given the restriction on LD, $\bar{\mathbf{S}}(n_i, G)$ is single-valued for all n_i and G; i.e., there exits a ***unique*** optimal schedule for n_i given G. Denote this schedule by T_i^G. and for each n, n_i and G let $\bar{T}(n_i, n, G) = T_i^G(n)$ and let $\bar{y}(n_i, n, G) = y^*(n_i, T_i^G)$. We show that $\bar{T}(\cdot, n, G)$ is "weakly single-troughed" (at n) for any fixed n; i.e., $n_i < n_1 \leq n$ implies that $\bar{T}(n_i, n, G) \leq \bar{T}(n_1, n, G)$, and $n \leq n_i < n_1$ implies that $\bar{T}(n_i, n, G) \geq \bar{T}(n_1, n, G)$. Then since T_i^G is simple, $\bar{y}(n_i, n, G) = n - \bar{T}(n_i, n, G)$, so $\bar{y}(\cdot, n, G)$ is weakly single-peaked (at n). Thus, if we order the schedules by n_i along a line and consider any set of egoistic voters, voter preferences will satisfy weak single-peakedness, as defined in Section 2.4, and thus there will exist a voting equilibrium over the set of individually optimal schedules.

To show that $\bar{T}(\cdot, n, G)$ is weakly single-troughed we must first characterize the individually optimal schedules more precisely. In particular, we must describe the parameters β and τ as functions of n_i. Fix $n_i \in [0,1)$ and $\tau \in [0,(1 - w_D(N)n_i]$, and solve

$$\max_{\beta} R_0(\tau,\beta,n_i) \text{ s.t. } \beta \in [\tau/n_i, 1]. \tag{4.1}$$

As noted above, a solution $\beta_H(\tau,n_i)$ exists since $R_0(\tau, \cdot, n_i)$ is continuous in β and $[\tau/n_i,1]$ is compact. Now we characterize β_H and find the condition under which it is unique.

From above, for $(\tau,n_i) \in \mathbf{S}_0'$

$$R_0(\tau,\beta,n_i) =$$

$$\begin{cases} \beta\underline{M}(\tau/\beta) + \tau\int_{\tau/\beta}^{1} dF(n) + \beta\bar{N}(n_i) & \text{for } \frac{\tau}{n_i} \le \beta < 1 - w_D(0) \\ \beta\underline{M}(\tau/\beta) + \tau\int_{\tau/\beta}^{1} dF(n) + \beta\bar{N}(n_i) - \beta L_D(1-\beta) & \text{for } \frac{\tau}{n_i} \le \beta < 1 - w_D(0) \\ \beta_e(\tau,n_i)\underline{M}(\tau/\beta_e(\tau,n_i)) + \tau\int_{\tau/\beta_e(\tau,n_i)}^{1} dF(n) & \\ \quad + \beta_e(\tau,n_i)\bar{N}(n_i) - \beta_e(\tau,n_i)L_D(1-\beta_e(\tau,n_i)) & \text{for } \beta_e(\tau,n_i) < \beta \le 1 \end{cases}$$

so, $\frac{\partial R_0}{\partial \beta}(\tau,\beta,n_i) =$

$$\begin{cases} \underline{M}(\tau/\beta) + \bar{N}(n_i) & \text{for } \frac{\tau}{n_i} \le \beta < 1 - w_D(0) \\ \underline{M}(\tau/\beta) + \bar{N}(n_i) - L_D(1-\beta) + \beta L_D'(1-\beta) & \text{for } \max\left(\frac{\tau}{n_i}, 1 - w_D(0)\right) \le \beta < \beta_e(\tau,n_i) \\ 0 & \text{for } \beta_e(\tau,n_i) < \beta \le 1 . \end{cases} \tag{4.2}$$

For $\tau < \tau_0(n_i)$, $\frac{\partial R_0}{\partial \beta}(\tau, \cdot, n_i) > 0$ on $(\frac{\tau}{n_i}, 1 - w_D(0))$. So there are no solutions in $[\frac{\tau}{n_i}, 1 - w_D(0))$. Also, the left-hand derivative $\frac{\partial R_0^-}{\partial \beta}(\tau,\beta_e(\tau,n_i),n_i)$

$= \beta \frac{dL_D}{dw}(1 - \beta_e(\tau,n_i)) < 0$ so there are no solutions in $[\beta_e(\tau,n_i),1]$. Thus, the only solutions lie in $[\max(\frac{\tau}{n_i},1 - w_D(0)), \beta_e(\tau,n_i))$. If R_0 is strictly concave in β over this interval then the solution is unique. For

$\beta \in (\max(\frac{\tau}{n_i}, 1 - w_D(0)),\beta_e(\tau,n_i))$,

$\frac{\partial^2 R_0}{\partial \beta^2}(\tau,\beta,n_i) = -\frac{\tau^2}{\beta^3}f(\frac{\tau}{\beta}) + 2L_D'(1-\beta) - \beta L_D''(1-\beta)$. This is negative if and only if $-\frac{\tau^2}{\beta^4}f(\frac{\tau}{\beta}) + \frac{2}{\beta}L_D'(1-\beta) < L_D''(1-\beta)$. Now $\frac{\tau^2}{\beta^4}f(\frac{\tau}{\beta}) \geq 0$, so if

$$\frac{2}{\beta}L_D'(1-\beta) < L_D''(1-\beta) \quad \text{for all } \beta \tag{4.3}$$

then for all $\tau < \tau_0(n_i)$, $R_0(\tau, \cdot ,n_i)$ is strictly concave over the interval $[\max(\frac{\tau}{n_i},1 - w_D(0)), \beta_e(\tau,n_i)]$, and there is a unique solution $\beta_H(\tau,n_i)$ in the interval.

For $\tau \geq \tau_0(n_i)$, $\beta_e(\tau,n_i) = \frac{\tau}{n_i}$ and $R_0(\tau,\beta,n_i) = R_0(\tau, \tau/n_i, n_i)$ for all $\beta \in [\tau/n_i, 1]$, so any $\beta \in [\tau/n_i, 1]$ solves equation (4.1). When $\beta > \beta_e(\tau)$ however, the tax schedule with parameters (τ,β) is not simple, and is equivalent to the schedule with parameters $(\tau,\beta_e(\tau,n_i))$. So for $\tau > \tau_0(n_i)$ pick $\beta_H(\tau,n_i) = \frac{\tau}{n_i}$. (See Figure 4.4.)

Let $R_H(\tau,n_i) = R_0(\tau,\beta_H(\tau,n_i),n_i)$. We now characterize $\beta_H(0, \cdot)$ and $R_H(0, \cdot)$ as functions of n_i.

Lemma 4.2. There exists an $n' \in [0,1]$ such that $\beta_H(0, \cdot)$ is strictly decreasing on $[0,n')$ and $\beta_H(0,n_i) = 1 - w_D(0)$ for $n \in [n',1]$. Also $\beta_H(0,0) = \beta^M$. $R_H(0,\cdot)$ is strictly decreasing on $[0,1]$ with $\frac{\partial R_H}{\partial n_i} = \beta_H(0,n_i)\bar{N}'(n_i)$ for all n_i. $R_H(0,0) = G^M$, and $R_H(0,1) = 0$.

Proof. At $\tau = 0$, equation (4.2) becomes

$$\frac{\partial R_0}{\partial \beta}(0,\beta,n_i) = \begin{cases} \bar{N}(n_i) & \text{for } 0 < \beta < 1 - w_D(0) \\ \bar{N}(n_i) - L_D(1-\beta) - \beta \dfrac{dL_D}{dw}(1-\beta) & \text{for } 1 - w_D(0) < \beta < \beta_e(0,n_i) \\ 0 & \text{for } \beta_e(0,n_i) < \beta < 1. \end{cases}$$

Now $\bar{N}$ is a continuous, strictly decreasing function of n_i on $[0,1]$ with $\bar{N}(0) = N$ and $\bar{N}(1) = 0$, so either there exists an n' such that $\bar{N}(n') = -[1 - w_D(0)] \cdot L_D'(w_D(0))$ or else $N < -[1 - w_D(0)] \cdot L_D'(w_D(0))$. In the latter case, let $n' = 0$. Then for $n_i \in [0,1]$,

$$\beta_H(0,n_i) \begin{cases} \in (1 - w_D(0), \beta_e(0,n_i)) & \text{for } n_i < n' \\ = 1 - w_D(0) & \text{for } n_i \geq n' \end{cases}$$

and

$$\frac{\partial R_0}{\partial \beta}(0,\beta_H(0,n_i),n_i) \begin{cases} = 0 & \text{for } n_i \leq n' & (4.4)(a) \\ < 0 & \text{for } n_i > n'. & (4.4)(b) \end{cases}$$

Now $\frac{\partial R_H}{\partial n_i}(0,n_i) = \frac{\partial R_0}{\partial \beta}(0,\ \beta_H(0,n_i),\ n_i) \cdot \frac{\partial \beta_H}{\partial n_i}(0,n_i) + \frac{\partial R_0}{\partial n_i}(0,\ \beta_H(0,n_i),\ n_i)$. So, for $n_i \in [0,n')$, $\frac{\partial R_H}{\partial n_i}(0,n_i) = \frac{\partial R_0}{\partial n_i}(0,\ \beta_H(0,n_i),\ n_i) = \beta_H(0,n_i)\bar{N}'(n_i) < 0$ (this is an example of the "envelope theorem"). Also, differentiating both sides of (4.4)(a) yields $\frac{\partial^2 R_0}{\partial \beta^2} \cdot \frac{\partial \beta_H}{\partial n_i} + \frac{\partial^2 R_0}{\partial \beta \partial n_i} = 0$, or $\frac{\partial \beta_H}{\partial n_i} = -\frac{\partial^2 R_0}{\partial \beta \partial n_i} \Big/ \frac{\partial^2 R_0}{\partial \beta^2}$. Now $\frac{\partial^2 R_0}{\partial \beta \partial n_i} = \bar{N}'(n_i) < 0$ and $\frac{\partial^2 R_0}{\partial \beta^2} < 0$ because of equation (4.3), so $\frac{\partial \beta_H}{\partial n_i} < 0$ for all $n_i \in (0,n')$. For $n_i \in [n',1)$, $\beta_H(0,n_i)$ is constant, so $\frac{\partial \beta_H}{\partial n_i} = 0$, and $R_H(0,n_i) = \beta_H(0,n_i)\bar{N}(n_i)$ so $\frac{\partial R_H}{\partial n_i} = \beta_H(0,n_i)\bar{N}'(n_i) < 0$. Clearly, $\beta_H(0,0) = \beta^M$, $R_H(0,0) = G^M$ and $R_H(0,1) = 0$, so $\beta_H(0,\cdot)$ and $R_H(0,\cdot)$ are as described.

QED

Thus we can show

Comment 4.6. If $L_D''(w) > \frac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$, and $G \le 0$, then for all $n \in [0,1]$, $\bar{y}(\cdot, n, G)$ is weakly single-peaked (at n).

Proof. Since $G \le 0$, $R_H(0,n_i) = R(T_i) \ge G$, so $T_i^G \in \mathbf{P}(n_i)$ for all n_i. So

$$\bar{T}(n_i,n,G) = \begin{cases} G - R_H(0,n_i) + \beta_H(0,n_i)[n_i - n] & \text{for } 0 \le n_i < n \\ G - R_H(0,n_i) & \text{for } n \le n_i < 1 . \end{cases}$$

If $n' \ge n$ then

$$\frac{\partial \bar{T}}{\partial n_i}(n_i,n,G) = \begin{cases} -\frac{\partial R_H}{\partial n_i}(0,n_i) + \frac{\partial \beta_H}{\partial n_i}(0,n_i)[n - n_i] - \beta_H(0,n_i) & \text{for } 0 < n_i < 1 \\ -\frac{\partial R_H}{\partial n_i}(0,n_i) & \text{for } n < n_i < 1 . \end{cases}$$

Now $\frac{\partial R_H}{\partial n_i}(0,n_i) = \beta \bar{N}'(n_i) = -\beta \int_{n_i}^{1} dF(n)$, so for $n_i < n$,

$\frac{\partial \bar{T}}{\partial n_i}(n_i,n,G) = \frac{\partial \beta_H}{\partial n_i}(0,n_i)[n - n_i] + \beta_H(0,n_i)\int_{n_i}^{1} dF(n) - 1 < 0$ since $\int_{n_i}^{1} dF(n) < 1$. Clearly, for $n_i > n$, $\frac{\partial \bar{T}}{\partial n_i}(n_i,n,G) > 0$. Thus $\bar{T}(\cdot, n, G)$ is weakly single-troughed at n. If $n' < n$ then

$$\frac{\partial \bar{T}}{\partial n_i}(n_i,n,G) = \begin{cases} -\frac{\partial R_H}{\partial n_i}(0,n_i) + \frac{\partial \beta_H}{\partial n_i}(0,n_i)[n - n_i] - \beta(0,n_i) & \text{for } 0 < n_i < n' \\ -\frac{\partial R_H}{\partial n_i}(0,n_i) - \beta_H(0,n_i) & \text{for } n' < n_i < n \\ -\frac{\partial R_H}{\partial n_i}(0,n_i) & \text{for } n < n_i < 1 . \end{cases}$$

Then as above, $\frac{\partial \bar{T}}{\partial n_i}(n_i,n,G) < 0$ for $n_i < n$, $n_i \ne n'$, and $\frac{\partial \bar{T}}{\partial n_i}(n_i,n,G) > 0$

for $n_i > n$. At $n_i = n'$, $\frac{\partial \bar{T}}{\partial n_i}(n_i, n, G)$ does not exist; however, since T_G is continuous, it must be strictly decreasing at n' also (it simply has a "kink"), and again $\bar{T}(\cdot, n, G)$ is weakly single-troughed at n. Hence, $\bar{y}(\cdot, n, G)$ is weakly single-peaked at n.

QED

With this in hand, we can prove a voting result analogous to that in Proposition 3.1.

Proposition 4.4. Given a measure-space of egoistic voters, if $L_D''(w) > \frac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$ and $G \le 0$, then there exists a majority-rule equilibrium tax schedule over the set of individually optimal schedules for G.

Proof. If we order the individually optimal simple schedules along the line by n_i then by Comment 4.6 the preference relations of the set J of voter types satisfies weak single-peakedness, as defined in Section 2.4. Also, $\bar{y}(\cdot, n, G)$ is continuous for all n, and $\bar{y}(\beta, \cdot, G)$ is continousе (hence measurable) for all β, so the conditions of Comment 2.6 are satisfied. Thus, by Comment 2.5, there exists a majority-rule equilibrium over this set.

QED

While the voting result here (like those of Section 3) is limited in that it holds only over a restricted class of schedules, the characterization of the individual optima is quite informative. For G not too large and individuals with very small endowments, the optimal simple schedule is linear over most of its range, with a positive slope. For individuals with large endowments, the optimal schedule has a marginal rate of zero over most of its range, and thus is essentially a "lump-sum" tax, except at the highest (and for large G also the lowest) income levels. For individuals in the middle, however, the optimal schedule is sharply progressive in that there is a large range of incomes that face a low marginal rate (zero, in fact), and a large range that face a higher marginal rate. This suggests that if some group in the middle income range is "getting their way," say because of their strategic position in the disribution of voters, then we would see income tax schedules that are marginal-rate progressive.

4.4 After-tax Incomes and Convex Schedules

In this section we again study individually optimal tax schedules, but we restrict the choice set to schedules that are nondecreasing convex functions of income, i.e., schedules that are marginal-rate progressive. An argument for doing this is that citizens may have such strong beliefs about the "unfairness" of marginal-rate regressive tax schedules that no such schedule could be sustained (a similar argument was implicit in the previous section, regarding decreasing tax schedules).

From Proposition 4.3, for any n_i and any $G \leq G_i$, the individually optimal schedules $\bar{\mathbf{S}}(n_i, G)$ over *all* nondecreasing schedules are convex, and hence will also be optimal among the set of convex schedules. For $G > G_i$ however, the schedules in $\bar{\mathbf{S}}(n_i, G)$ are not convex, and thus cannot be optimal among the set of convex schedules. As we show below, for $G > G_i$

the set of simple optimal schedules for n_i given G are of the form

$$T_i(x) = \begin{cases} \gamma_i x & \text{for } x \le n_i \\ \gamma_i n_i + \beta_i(x - n_i) & \text{for } x > n_i \end{cases} \tag{4.5}$$

for some $\gamma_i \le \beta_i$.

Proposition 4.5. If T_i is a simple, convex schedule that is optimal for n_i given G then it satisfies equation (4.5), and any other schedule which is also optimal for n_i given G is equivalent to such a T_i.

Proof. Suppose T is simple and convex. Then $T(x) = T(n^*(T)) + (1-w^*(T))(x - n^*(T))$ for all $x > n^*(T)$ (if not then T violates either simplicity or convexity). Thus, if T is nonlinear over $(n_i, 1]$ then $n_i < n^*(T)$ and T is nonlinear over $(n_i, n^*(T)]$, so $T^+(n_i) < T^+(n^*(T)) \le 1-w^*(T)$. In this case, let T_1 be defined by

$$T_1(x) = \begin{cases} T(x) & \text{for } x \le n_i \\ T(n_i) + (1-w^*(T))(x - n_i) & \text{for } x > n_i \, . \end{cases}$$

Then T_1 is convex (since T is) and $T_1'(n^*(T)) = 1 - w^*(T)$, so by Comment 2.4, T_1 induces the same market equilibrium as T, and thus is simple. Also, by continuity $T^+(x) < 1 - w^*(T) = T_1'(x)$ for some neighborhood $(n_i, x_i]$ of n_i and thus $T(n) = T(n_i) + \int_{n_i}^{n} T^+(x)dx < T(n_i) + \int_{n_i}^{n} T_1'(x)dx = T_1(n)$ for all $n \in (n_i, n^*(T)]$, so $R(T_1) > R(T)$. If T is optimal for n_i given G then $R(T) \ge G$; but then $R(T_1) > G$, so there exists a schedule $T_2 \equiv T_1 - \alpha$ for some $\alpha < 0$ such that $R(T_2) = G$. By Comment 2.3, T_2 generates the same equilibrium as T_1 and hence is simple (since T_1 is), so $y^*(n_i, T_2) = n_i - T_2(n_i) = n_i - T_1(n_i) + \alpha = n_i - T(n_i) + \alpha$ $= y^*(n_i, T) + \alpha > y^*(n_i, T)$, contradicting the assumption that T is optimal for n_i given G. So, T must be linear over $(n_i,1]$.

Next, suppose T is not linear over $[0,n_i)$. If $n_i < n^*(T)$ then define T_1 by

$$T_1(x) = \begin{cases} \dfrac{T(n_i)}{n_i}x & \text{for } x \leq n_i \\ T(x) & \text{for } x \leq n_i . \end{cases}$$

Then T_1 is convex and induces the same market equilibrium as T, and hence is simple. Also, since T is convex, $T(\lambda n_i) \leq \lambda T(n_i) + T(0)$ for all $\lambda \in [0,1]$, or letting $\lambda = \frac{x}{n_i}$, $T(x) \leq \frac{T(n_i)}{n_i}x + T(0)$ for all $x \in [0,n_i]$. Since T is a tax schedule, $T(0) \leq 0$, and since T is assumed nonlinear on $[0,n_i)$, $T(x') < \frac{T(n_i)}{n_i}x'$ for some $x' \in [0,n_i)$ and hence for some open neighborhood of x'. Thus $R(T) < R(T_1)$, so again if T is optimal for n_i given G then $R(T) \geq G$, and we define $T_2 \equiv T_1 - \alpha$ for $\alpha > 0$ such that $R(T_2) = G$. Then $y^*(n_i,T_2) > y^*(n_i,T)$, so T could not be optimal, and thus T must be linear over $[0,n_i)$.

If $n_i > n^*(T)$ then define T_1 by

$$T_1(x) = \begin{cases} \dfrac{T(n^*(T))}{n^*(T)}x & \text{for } x \leq n_i \\ T(x) & \text{for } x > n_i \end{cases}$$

and repeat the above exercise. Note in this case that the optimal schedule for n_i given G will be linear (i.e., λ must equal β).

That the marginal tax rate over $[0,n_i)$ cannot exceed the rate over $(n_i,1]$ follows directly from convexity.

QED

Of course, this merely characterizes an optimum *if* one exists, and does not prove existence. To do so, we can use the same strategy as in section 4.3. That is, given any n_i and G, we define the two-parameter family of schedules that satisfy equation (4.5) and $R(T) \geq G$, define the corresponding parameter space, show that the parameter space is compact and that the individual's after-tax income is continuous over it, and apply the theorem of the maximum. Since the work is tedious and rather unenlightening, we omit it here.

4.5 Convex Schedules and More General Preferences

In the previous sections we characterized the optimal schedules for benevolent and egoistic voters, finding that for a benevolent voter the taxes are always linear and for an egoistic voter they are in general "two-part" or "three-part" schedules with two marginal rates. In this section we characterize the optimal tax schedule for voters with generalized preferences. This is especially interesting since the shape of the optimal schedules for a benevolent voter is so different from that for an egoistic voter. The question is, how is the difference resolved by a voter whose preferences are a mixture of the two. We provide a simple and intuitive answer.

As in Section 4.4, we restrict attention to tax schedules that are nondecreasing, convex functions of income. The characterization we shall prove is

Proposition 4.6. Let a voter of type j have preferences over tax schedules given by $v_j(T) = u_j(y^*(n_j, T), S(T, W_j))$, as defined in Section 2.4, with W_j strictly concave and u_j strictly increasing in S. If T_j is convex, simple, and optimal for j given G then T_j satisfies

$$T_j(x) = \begin{cases} \alpha_j + \gamma_j x & \text{for } x \le n_j \\ \alpha_j + \gamma_j n_j + \beta_j(x - n_j) & \text{for } x > n_j . \end{cases}$$

Proof. We proceed exactly as in the proof of Proposition 4.5. Suppose T is not linear over $(n_j, 1]$. Then $n_j < n^*(T)$ and T is not linear over $(n_j, n^*(T)]$. Let T_1 be defined by

$$T_1(x) = \begin{cases} T(x) & \text{for } x \le n_j \\ T(n_j) + (1 - w^*(T))(x - n_j) & \text{for } x > n_j . \end{cases}$$

Then T_1 is simple and convex, generates the same equilibrium as T, and $R(T_1) = R(T)$. Let T_2 be defined by $T_2(x) = T_1(x) + [R(T) - R(T_1)]$ for all x. (See Figure 4.5.) Then $y^*(n, T_2) > y^*(n, T_1) = y^*(n, T)$ for all $n \le n_j$, and $R(T_2) = R(T)$. So there must exist $n_e \in (n_j, n^*(T))$ such that $y^*(n, T_2) > y^*(n, T)$ for all $n < n_e$, and $y^*(n, T_2) < y^*(n, T)$ for all $n > n_e$. Then by Comment 4.5, $S(T_2, W_j) > S(T, W_j)$, and since $y^*(n_j, T_2) > y^*(n_j, T)$, $v_j(T_2) = u_j(y^*(n_j, T2), S(T_2, W_j)) > u_j(y^*(n_j, T), S(T, W_j) = v_j(T)$, and T is not optimal for a voter of type j.

Next, suppose $0 < n_j < n^*(T)$ and T is not linear on $[0, n_j)$, so $T^+(0) < T^-(n_j)$. For any $\beta \in [(T(n_j) - T(0))/n_j, T^-(n_j)]$, define the schedule $T^j(\cdot, \beta)$ by

$$T^j(x, \beta) = \begin{cases} T(n_j) - (n_j - x)\beta & \text{for } x \le n_j \\ T(x) & \text{for } x > n_j . \end{cases}$$

(See Figure 4.6.) Then $T^j(\cdot, \beta)$ is simple and convex, and generates the same equilibrium as T. Let $R^j(\beta) = R(T^j(\cdot\,,\beta))$ for all β. Then for $\beta = (T(n_j) - T(0))/n_j$, $T^j(x,\beta) > T(x)$ for all $x \in (0,n_j)$ (since T is convex), so $R^j(\beta) > R(T)$. And for $\beta = T^-(n_j)$, $T^j(x,\beta) < T(x)$ for all x in some open neighborhood of 0, so $R^j(\beta) < R(T)$. Clearly R^j is continuous, so there exists $\beta' \in ((T(n_j) - T(0))/n_j,\ T^-(n_j))$ such that $R^j(\beta') = R(T)$. And for $T^j(\cdot\,,\beta')$ there exists $n' \in (0,n_j)$ such that $y^*(n,T^j(\cdot\,,\beta')) > y^*(n,T)$ for all $n < n'$, and $y^*(n,T^j(\cdot\,,\beta')) < y^*(n,T)$ for all $n \in (n',n_j)$. Thus by Comment 4.5, $S(T^j(\cdot\,,\beta'),W_j) > S(T,W_j)$. Also, $y^*(n_j,T^j(\cdot\,,\beta')) = y^*(n_j,T)$, so since u_j is strictly increasing in S, $v_j(T^j(\cdot\,,\beta')) = u_j(y^*(n_j,T^j(\cdot\,,\beta')),\ S(T^j(\cdot\,,\beta'),W_j)) > u_j(y^*(n_j,T),\ S(T,W_j)) = v_j(T)$, and T is not optimal for j.

If $n_j > n^*(T)$ the argument proceeds in the same fashion, with $T^j(\cdot\,,\beta)$ defined by

$$T^j(x,\beta) = \begin{cases} T(n^*(T)) - (n^*(T) - x)\beta & \text{for } x \le n^*(T) \\ T(x) & \text{for } x > n^*(T)\,. \end{cases}$$

QED

In general the conclusions from Sections 4.4. and 4.5 hold, although with some modifications. For voters with very small endowments the optimal schedule is again linear over most of its range. For voters with very large endowments the optimal schedule is also linear over most of its range, but the slope is positive if they care about the distribution of after-tax incomes, rather than zero as in the case of purely egoistic voters when G is small. Thus, the conclusion that when G is small the optimal tax for voters with large endowment is essentially "lump-sum" is true only for egoistic voters--it does not hold for voters with generalized preferences. For voters with "average" endowments the optimal schedule again imposes different marginal tax rates on two wide ranges of incomes, and thus is rather marginal-rate progressive. The tax rate on lower incomes is not zero, however, as in the case of egoistic voters when G is small, but positive, and thus the "degree" of progressivity, as measured, say, by the difference between the two marginal tax rates, is probably less than in the egoistic case. For large G ($G > G_i$) the shape of the optimal schedule for voters with generalized preferences is the same as that for egoistic voters (when we restrict the choice set to convex schedules), although for different reasons.

5. Tax Schedules and Uncertain Voters

In the previous sections we have assumed that each voter is perfectly informed about her own endowment, the distribution of endowments in the economy, the production functions in each sector (or at least the demand function for labor in the untaxed sector), and the tax function. Here we relax the assumption that voters know with perfect certainty their own endowments. One reason for doing this is that we may imagine voters choosing a tax schedule that will apply in the future, and they may not have perfect foresight about their future productivity.

Thus, let a voter of type j be egoistic with utility function U_j over her after-tax income, and let her have a subjective probability distribution function F_j over the interval of possible endowments $[\underline{n},\bar{n}]$. Her expected utility given a tax schedule T is then $EU_j(T) = \int U_j(y^*(n,T)dF_j(n)$. We wish to characterize the schedules T that maximize this, subject to the government's revenue constraint.

If $F_j \equiv F$ (where F, as above, is the distribution of endowments in the economy) and U_j is either linear or strictly concave then clearly a type j voter's expected utility function has the same form as that of a benevolent voter with weighting function U_j, so the Propositions of sections 3.2 and 4.2 regarding benevolent voters apply here. Thus,

Comment 5.1. Given a measure-space of uncertain egoistic voters such that each voter has a subjective probability distribution over her endowments equal to F, if $L_D''(w) > \dfrac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N),w_D(0))$ then there exists a majority-rule equilibrium tax schedule over the set $\mathbf{T}(0)$ which is linear. If more than half of the voters are strictly risk-averse, then any equilibrium is equivalent to a linear schedule.

Recall that a linear utility function means risk-neutrality, and a strictly concave utility function means strict risk aversion.

The voters in Comment 5.1 have rather "neutral" beliefs, in that they essentially view their own endowment as a random draw from the distribution in society ("tomorrow I could be anyone"). We might imagine instead that voters are "pessimistic" ("optimistic") and believe it more likely (less likely) that they will have a high endowment than if their endowment was drawn randomly from F. One natural way to capture this is to call a voter *pessimistic* (*optimistic*) if F_j is first-order stochastically dominated by (first-order stochastic dominates) F. Then for pessimistic voters we have the following result.

Proposition 5.1. For any pessimistic, egoistic voter there exists a simple linear schedule which maximizes her expected utility over the set $\mathbf{T}(0)$. If she is strictly risk-averse then any simple schedule that maximizes her expected utility over $\mathbf{T}(0)$ is linear.

Proof. Let the voter be of type j, with endowment $n \subset j$ and subjective probability distribution function F_j. First, we show that if T is any simple schedule that is not linear on $[\underline{n}, \bar{n}]$ then there exists a (simple) linear schedule T_1 with $R(T_1) = R(T)$ such that the voter prefers T_1 to T. Let T be simple and nonlinear on $[\underline{n}, \bar{n}]$ and let T_1 be the linear schedule with slope $1-w^*(T)$ and intercept $w^*(T)N - Y^*(T)$. Then, as shown in the proof of Proposition 4.1, T_1 is simple and generates the same equilibrium as T,

$R(T_1) = R(T)$ and hence $Y^*(T_1) = Y^*(T)$, and there exists $n_e \in (\underline{n}, \bar{n})$ such that $y^*(n, T_1) \geq y^*(n, T)$ for all $n \leq n_e$ and $y^*(n, T_1) \leq y^*(n, T)$ for all $n \geq n_e$, with strict inequality on some open neighborhoods of $\underline{n}$ and $\bar{n}$.

In fact, the function $y^*(\cdot, T_1) - y^*(\cdot, T)$ is nonincreasing on $[\underline{n}, \bar{n}]$. To see this, note first that by simplicity $\dfrac{T(x_2) - T(x_1)}{x_1 - x_2} \leq 1 - w^*(T)$ for all $x_1 < x_2$. And, by construction, $\dfrac{T_1(x_2) - T_1(x_1)}{x_1 - x_2} = 1 - w^*(T)$ for all $x_1 < x_2$ so $T(x_1) - T_1(x_1) \geq T(x_2) - T_1(x2)$ for all $x_1 < x_2$. Thus, since T and T_1 are both simple, $y^*(n_1, T1) - y^*(n_1, T) = [n_1 - T_1(n_1)] - [n_1 - T(n_1)] = T(n_1) - T_1(n_1) \geq T(n_2) - T1(n_2) = y^*(n_2, T_1) - y^*(n_2, T)$, for all $n_1 < n_2$, as desired.

Then, since F_j is stochastically dominated by F, $\int [y^*(n, T_1) - y^*(n, T)] dF_j(n) \geq \int [y^*(n, T_1) - y^*(n, T)] dF(n) = 0$. Let $\gamma = \int [y^*(n, T_1) - y^*(n, T)] dF_j(n) \geq 0$, and define the schedule T_2 by $T_2(x) = T_1(x) + \gamma$ for all x. (See Figure 5.1.) Then, by Comment 2.3, T_2 generates the same equilibrium as T_1 so $y^*(n, T_2) = y^*(n, T_1) - \gamma \leq y^*(n, T_1)$ for all n and $\int y^*(n, T_2) dF(n) = \int y^*(n, T) dF(n)$. Also, $\gamma < y^*(0, T_1) - y^*(0, T)$ (else $\gamma > \int [y^*(n, T_1) - y^*(n, T)] dF_j(n)$ contrary to assumption) so there exists $n_e' \in (\underline{n}, \bar{n})$ such that $y^*(n, T2) \geq y^*(n, T)$ for all $n \leq n_e$ and $y^*(n, T2) \leq y^*(n, T)$ for all $n \geq n_e$, with strict inequality for some open intervals of $\underline{n}$ and $\bar{n}$. Then, by Comment 4.5, $EU_j(T_2) = \int U_j(y^*(n, T_2)) dF_j(n) \geq \int U_j(y^*(n, T)) dF_j(n) = EU_j(T)$. And, since U_j is increasing, $U_j(y^*(n, T_1) \geq U_j(y^*(n, T_2))$ for all n, so $EU_j(T_1) \geq EU_j(T_2)$, and thus $EU_j(T_1) \geq EU_j(T)$ as desired.

Since F_j is stochastically dominated by F, $F_j(n) \geq F(n)$ for all n and thus F_j has positive measure on a neighborhood of $\underline{n}$ (since F does, by assumption). Thus, since $y^*(\cdot, T_2) < y^*(\cdot, T)$ on an open neighborhood of $\underline{n}$, if U_j is strictly concave then $EU_j(T_2) > EU_j(T)$ and hence $EU_j(T_1) > EU_j(T)$ and T is not optimal. So, when the voter is strictly risk averse, any optimal schedule for j must be linear over $[\underline{n}, \overline{n}]$.

Now, we show that an optimal linear schedule exists. Let $\tilde{EU}_j : \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{EU}_j(\beta) = EU_j(T)$ where $T \in \mathbf{L}(0)$ has slope β. Then from equation (3.1), $\tilde{y}(n, \cdot)$ is continuous for all n, so $\tilde{EU}_j$ is continuous. And, for $\beta \notin [0, 1-w_D(N)]$, $\tilde{EU}_j(\beta) \leq \tilde{EU}_j(0)$ so we can restrict our search for an optimal β to $[0, 1-w_D(N)]$. This is compact, and thus a β_j that maximizes $\tilde{EU}_j$ exists. Then the linear tax schedule $T_j \in \mathbf{T}(0)$ with slope β_j is optimal for the voter given $G = 0$.

QED

We have a voting result analogous to that for benevolent voters.

Proposition 5.2. Given a measure-space of pessimistic, egoistic voters who all have the same utility function U, if $L_D'' > \dfrac{2L_D'(w)}{1-w}$ for all $w \in (w_D(N), w_D(0))$ then there exists a majority-rule equilibrium tax schedule over the set $\mathbf{T}(0)$. If the voters are strictly risk-averse, then any such equilibrium schedule is equivalent to a linear schedule.

Proof. Using the same argument as in the proof of Comment 3.5 (with F_j substituted for F) we can show that the preferences of each voter type are weakly single-peaked over the set $\mathbf{L}(0)$ when we order the elements of $\mathbf{L}(0)$ by their slope parameter. Thus, the set of voter preferences satisfy weak single-peakedness, as defined in Section 2.4. Also, the conditions of Comment 2.6 are satisfied, so by Comment 2.5 there exists a majority-rule equilibrium tax schedule T_E over $\mathbf{L}(0)$. Using the same argument as in the proof of Proposition 4.2 we can extend this to $\mathbf{T}(0)$, so T_E is an equilibrium over $\mathbf{T}(0)$. If more than half of the voters are strictly risk-averse then by Proposition 5.1 any equilibrium must be equivalent to a linear schedule.

QED

In general, we cannot easily characterize the optimal schedules for "optimistic" voters. In the special case of risk-neutrality however, we can, and the result is somewhat surprising, at least at first glance.

Comment 5.2. Given an egoistic, risk-neutral, optimistic voter, for any $G \leq < 0$ the schedule T_G defined by $T_G(x) = G$ for all x is optimal for the voter given G, over the set $\mathbf{N}(G)$.

Proof. Let the voter be of type j. Clearly, if T is any simple schedule in $\mathbf{N}(G)$ then the function $T - T_G$ is nondecreasing on $[\underline{n}, \overline{n}]$, so $y^*(\cdot, T_G) - y^*(\cdot, T)$ is also (since both T and T_G are simple). Then since F_j stochastically dominates F,

$\int [y^*(n, T_G) - y^*(n, T)]dF_j(n) \geq \int [y^*(n, T_G) - y^*(n, T)]dF(n)$. And

$X^*(T_G) = N \geq X^*(T)$, so $Y^*(T_G) = N - R(T_G) = N - G \geq Y^*(T)$, so

$\int [y^*(n, T_G) - y^*(n, T)]dF(n) \geq 0$. Thus

$EU_j(T_G) = \int y^*(n, T_G)dF_j(n) \geq \int y^*(n, T)dF_j(n) = EU_j(T)$, so T_G is optimal for j.

QED

Notice that T_G is the schedule that maximizes the after-tax income of the individual with the highest ability level (over the set $\mathbf{N}(G)$). It is easily checked that the same result holds for $G > 0$; i.e., the schedule that maximizes the after-tax income of the most able individual over $\mathbf{N}(G)$ is optimal for j over $\mathbf{N}(G)$. Of course, there may be other optimal schedules, depending on the relationship between F_j and F (where F_j and F coincide, there is lots of "room for maneuver").

6. A Comparison with the "Labor-Leisure" Framework

Here we compare our two-sector model to the "labor-leisure" model formulated (or at least formalized) by Mirrlees [17], which is quite popular and has been used extensively in the optimal tax literature, to see how the assumptions differ and what role the differences play in driving results. In the Mirrlees framework, each worker-consumer has a utility function u over consumption (income) and leisure, and has 1 unit of time to divide between earning income and leisure. An individual's "ability level" (endowment) affects his marginal productivity in earning income, but does not affect his "productivity" in converting leisure into utility. Thus, an individual of ability n who works t units of time and consumes $1-t$ units of leisure gets utility

$$u(nt - T(nt), 1-t) \tag{6.1}$$

where it is assumed that the wage is 1 and T is the income tax schedule (only income can be taxed, not leisure or ability levels).

In our model, if we interpret endowments as "ability levels," and give each worker-consumer 1 unit of time to divide between sectors (as in Note 2.1), we can rewrite equation (2.1) as

$$nt - T(nt) + wn(1-t)\ . \tag{6.2}$$

Interpreting the untaxed sector earnings as leisure, this looks very much like a special case of the utility function above, since if $u(c,l) = c + wl$ for all c and l then equation (6.1) becomes

$$nt - T(nt) + w(1-t)\ . \tag{6.3}$$

Note, however, the difference that n does not directly affect the marginal utility of leisure time in equation (6.3), as it does in (6.2). Also, in the Mirrlees framework all individuals are assumed to have the same utility function u (hence the same w, if u is linear), so we cannot make the marginal utility of leisure depend on n by having u depend on n. Another difference between the models is that in the Mirrlees model (with a linear utility function) the marginal utility of leisure time for any individual, w, is fixed, while in our model the marginal utility of time spent in the untaxed sector varies, depending on the aggregate supply of time in that sector.

It is clear that while the models are slightly different in these respects, the "linearity" in our model of each individual's indirect utility function over labor plays a key role in driving some results, particularly regarding the shape of the optimal tax schedules, which do not come out so "cleanly" in the Mirrlees model. Basically, in the two-sector model we need only worry about the marginal tax rate at "a few" income levels--the lower endpoints of critical intervals--because each individual either makes her labor choice to satisfy a marginal condition at one of those points, or else will "corner," and supply all her labor to the taxed sector (i.e "consume no leisure"), while in the Mirrlees model with highly nonlinear indirect utility functions over labor choices, the marginal tax rate is important at almost all income levels because all individuals who work choose an "interior" point where they are balancing the marginal utility gain from working (which depends on the marginal tax rate) with the marginal utility of leisure.

Some of the other conclusions of our model are due not to the absence of a nonlinear "labor-leisure" tradeoff, but to assumptions made about the untaxed sector. In particular, the disturbing prediction that for any strictly convex tax schedule T no individuals earn a taxable income higher than

$n^*(T)$ and there is a large "clump" of individuals who earn taxable income exactly $n^*(T)$ (see Section 2.3), is due to the assumption that the untaxed sector is perfectly competitive; i.e., that only one wage prevails, and that labor is homogenous. To prove this, we show below that this same conclusion holds even in the "labor-leisure" setting if a competitive untaxed sector exists.

As in the standard Mirrlees model, let each individual have a utility function u over money income and leisure. Suppose however, that individuals can split the time they work between a taxed and an untaxed sector, and that the labor markets in both sectors are perfectly competitive so that in each sector all units of labor are paid the same wage. Let the wage in the taxed sector be 1, and that in the untaxed sector w. Then an individual of ability n who works x units of time in the taxed sector, z units in the untaxed sector, and consumes $1-x-z$ units of time as leisure, receives utility

$$v(x,z,n) = u(nx - T(nx) + nwz,\ 1-x-z)$$

where T is the tax due as a function of taxable income. As we are proving a proposition about strictly convex tax schedules, assume T is strictly convex. Also, for simplicity, assume T is twice differentiable.

Individuals choose x and z to maximize v, subject to $x \in [0,1]$, $z \in [0,1]$, and $x + z \leq 1$. Assume, as in Mirrlees, that $\lim_{c \to 0} \frac{\partial u}{\partial c} = +\infty$ and $\lim_{l \to 0} \frac{\partial u}{\partial l} = +\infty$, so no individuals choose to consume either zero leisure or zero income. Differentiating v yields

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial c} n\left(1 - \frac{dT}{dI}(nx)\right) - \frac{\partial u}{\partial l} = 0 \text{ if and only if } 1 - \frac{dT}{dI}(nx) = \frac{\frac{\partial u}{\partial c}}{n \cdot \frac{\partial u}{\partial l}}, \text{ and}$$

$$\frac{\partial v}{\partial z} = \frac{\partial u}{\partial c} nw - \frac{\partial u}{\partial l} = 0 \text{ if and only if } w = \frac{\frac{\partial u}{\partial c}}{n \cdot \frac{\partial u}{\partial l}}.$$

Thus, both $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial z}$ only if $1 - \frac{dT}{dI}(nx) = w$. This is the same condition as that for convex schedules in our simple two-sector model--the individual supplies labor to the taxed sector up to the point at which the after-tax marginal wage in the taxed sector equals the wage in the untaxed sector. (See Figure 2.2.) If the individual is unable to allocate her *working* time between sectors to achieve this condition, then she will "corner," and spend all her working time in one sector.

To clarify this, let $x^*(n), z^*(n)$ be an optimal labor choice pair for n; thus $l^*(n) = 1 - x^*(n) - z^*(n)$ is her leisure choice. If these are optimal, then no reallocation of labor x, z such that $x + z = l^*(n)$ can increase her after-tax income. Thus,

(i) If $\frac{dT}{dI}(0) \geq 1 - w$ then $z^*(n) = 1 - l^*(n)$ and $x^*(n) = 0$; i.e., she allocates all her labor to the untaxed sector (else shifting labor from the taxed sector to the untaxed would increase her after-tax income).

(ii) If $\frac{dT}{dI}(nl^*(n)) \leq 1 - w$ then $x^*(n) = 1 - l^*(n)$ and $z^*(n) = 0$; i.e., she allocates all her labor to the taxed sector (else shifting labor from the taxed sector to the taxed would increase her after-tax income).

(iii) Otherwise, $x^*(n) \in (0, l^*(n))$ satisfies the marginal condition above,

$\frac{dT}{dI}(nx^*(n)) = 1-w$.

So, letting $I(w,T)$ solve $\frac{dT}{dI}(I(w,T)) = 1-w$ (such an $I(w,T)$ exists as long as $\frac{dT}{dI}(0) < 1-w$ and $\frac{dT}{dI}(I) > 1-w$ for some I), it is clear that given w and T no individual ever earns a taxable income higher than $I(w,T)$, and all individuals who spend a positive amount of time in the untaxed sector earn a taxable income of ***exactly*** $I(w,T)$. This holds for all w, and thus for $w^*(T)$ in particular (note, we do not need to specify how $w^*(T)$ is determined).

Thus, it is the assumption of a purely competitive untaxed sector, and homogenous labor, that generates the "maximal" taxable income and "clumping." Of course, if the untaxed sector is modelled differently--for example, if each individual has an untaxed sector production function that depends on her own labor input to the sector--or if there are several types of labor which differ with respect to their relative productivity in the taxed and untaxed sectors, then "clumping" of taxable incomes might not occur.

Notes for Part I

1. Equivalently, we could assume that all individuals supply the same amount of labor, but productivity per unit varies across individuals.

2. Actually, to get interesting results we only need the weaker condition $w_D(N) < 1$. However, assuming $w_D(L) \in (0,1)$ for all $L \in [0,N]$ simplifies notation and avoids some technical details.

3. There are several simple stories that would support these assumptions; we give two here.

   (1) We may imagine that in each sector there is a production function which depends on the total labor supplied to the sector. Suppose that there are constant returns to scale in the taxed sector, and choose units so that the output per unit of labor in that sector is 1. Suppose that production in the untaxed sector is always "less efficient" than that in the taxed sector, and exhibits decreasing returns to scale. That is, suppose there exists a function $q : [0,N] \rightarrow RR$, where $q(L)$ is the output given total labor input of L, with $q'(L) \in (0,1)$ and $q''(L) < 0$ for all L. If the labor market in each sector is perfectly competitive then the the return to each unit of labor must be equal to the marginal product of labor, so the wage in the taxed sector is 1 and that in the untaxed sector is $w_D(L) = q'(L)$. Then $w_D(L) \in (0,1)$ and $w_D'(L) < 0$ for all L. There is a further complication in this story however -- given decreasing returns to scale, there are profits earned on all inframarginal units of labor in the untaxed sector, and it is unclear who receives these profits. Thus, to eliminate profits we must imagine some

sort of negative production externality, a kind of "crowding", so that $q'(L)$ gives the productivity of all units of labor in the untaxed sector when the total labor used in that sector is L.

(2) Suppose that the production function is the same in both sectors, and satisfies constant returns to scale. Again, choose units so the marginal product (and thus the average product) per unit of labor is 1. Suppose that the government polices the untaxed sector and punishes labor in that sector by confiscating any output it discovers. Suppose also that the government is more effective at policing, the greater the amount of labor supplied to the untaxed sector. (Note that we are taking the government's policing "effort" as exogenous.) Thus, let $p(L) \in (0,1)$ be the probability that the government discovers a unit of labor in the untaxed sector given a total untaxed labor supply of L, with $p'(L) > 0$ for all L. Again suppose that labor markets in both sectors are perfectly competitive, so the wage in the taxed sector is 1. The *expected* wage in the untaxed sector is then $p(L)$ ($p(L) \cdot 1 + (1-p(L)) \cdot 0$), so if we let w_D be the expected wage in the untaxed sector, $w_D(L) \in (0,1)$) and $w_D'(L) < 0$ for all L as desired.

4. Thus, the government cannot tax labor in the sheltered sector, nor can it tax labor directly.

5. Alternatively, we could assume that each individual i in the economy, with endowment n_i, has a utility function u_i over her own after-tax income and some public good. Thus, if y_i is her after-tax income and G is the amount of public good provided, her utility is $u_i(y_i, G)$. If the level of public good is fixed at G, and $u_i(\cdot, G)$ is strictly increasing in

y_i, then $u_i(\cdot, G)$ is maximized if and only if y_i is. Thus, an individual i with $n_i = n$ will maximize the expression in equation (2.1). This is interesting because we wish to consider some taxes that raise nonzero net revenue to the government, the assumption being that the government does not throw the money away, but spends it on public goods.

6. We could instead assume that voters have utility functions over both their own income and the amount of public good (as in note 2.5), and are choosing a tax schedule taking the level of public good as ***fixed*** at some level G. (And thus only consider tax schedules T such that $R(T) = G$.)

7. Again, we could suppose (as in note 2.5) that individuals in the economy have utility functions u_i over y_i and G, and voters are choosing a tax schedule taking the level of public good provided, G, as fixed. (And thus only consider tax schedules T such that $R(T) = G$.) If all individuals have the same utility function u then a benevolent voter j with weighting function W_j has indirect utility function over tax schedules $S(\cdot, W_j)$ defined by $S(T, W_j) = \int W_j(u(y^*(n, T), G))dF(n)$.

8. We could deal with intermediate cases as well, where W is only weakly concave but not linear, but doing so merely adds technical details without additional insights.

9. If we allow decreasing schedules then the optimal schedule given G for an individual with endowment n is of the form below, with a sharp

discontinuity at n. (See Figure N.1.)

$$T(x) = \begin{cases} \beta x & \text{for } x < n \\ T(n) + \beta x & \text{for } x \geq n \end{cases}$$

where $\beta \in [1 - w_D(0), 1 - w_D(N)]$ and

$$T(n) = \frac{-\beta N + \beta L_D(1 - \beta) + G}{\int_n^1 dF(m)}$$

(or any equivalent schedule).

Appendix A

Here we prove the existence and uniqueness of a market equilibrium. In what follows, $\underline{x}(n,w,T) = \min \hat{x}(n,w,T)$, $\bar{x}(n,w,T) = \max \hat{x}(n,w,T)$, $\underline{X}(w,T) = \min \hat{X}(w,T)$ and $\bar{X}(w,T) = \max \hat{X}(w,T)$.

Lemma A.1. For each $n \in [0,1]$, $\underline{x}(n,\cdot,T)$ and $\bar{x}(n,\cdot,T)$ satisfy
(i) if $w_1 \leq w_2$ then $\underline{x}(n,w_2,T) \leq \bar{x}(n,w_2,T) \leq \underline{x}(n,w_1,T) \leq \bar{x}(n,w_1,T)$, and
(ii) $\underline{x}(n,\cdot,T)$ is right-hand continuous, and $\bar{x}(n,\cdot,T)$ is left-hand continuous.

Proof. For any w, $\underline{x}(n,w,T) \leq \bar{x}(n,w,T)$ by definition. For notational convenience, write $x_m = \underline{x}(n,w_1,T)$. We show that $T(x) - (1 - w_2)x > T(x_m) - (1 - w_2)x_m$ for all $x \in (x_m,n]$ and thus $\bar{x}(n,w_2,T) \leq x_m$, from which (i) follows. If $x \in (x_m,n]$, and $w_2 > w_1$, then $x > x_m$ implies $w_2x - w_2x_m > w_1x - w_1x_m$, so

$$T(x) - T(x_m) + x_m - x + w_2x - w_2x_m >$$

$$T(x) - T(x_m) + x_m - x + w_1x - w_1x_m, \text{ so}$$

$$[T(x) - (1 - w_2)x] - [T(x_m) - (1 - w_2)x_m] >$$

$$[T(x) - (1 - w_1)x] - [T(x_m) - (1 - w_1)x_m].$$

But x_m minimizes $T(x') - (1 - w_1)x'$ over $[0,n]$, so the right side is nonnegative, and $T(x) - (1 - w_2)x > T(x_m) - (1 - w_2)x_m$ as desired.

We now show that $\bar{x}(n,\cdot,T)$ is left-hand continuous; the proof for $\underline{x}(n,\cdot,T)$ is analogous. Let $w \in (0,1)$, write $\bar{x}(n,w,T) = x^m$, and fix $\varepsilon > 0$. We must find δ such that $w' \in (w - \delta,w)$ implies $\bar{x}(n,w',T) - x^m < \varepsilon$ (by (i),

$\bar{x}(n,w',T) \geq x^m$ for all $w' < w$). Now if $x^m + \varepsilon > n$ then clearly $\bar{x}(n,w',T) < x^m + \varepsilon$ for all $w' < w$ and we may choose any δ, so suppose $x^m + \varepsilon \leq n$. Let $s(\varepsilon) = \min\limits_{[x^m + \varepsilon, n]} \dfrac{T(x) - T(x^m)}{x - x^m}$. Clearly $s(\varepsilon)$ exists, since $[x^m + \varepsilon, n]$ is compact and $\dfrac{T(x) - T(x^m)}{x - x^m}$ is continuous. Also, for all $x \in (x^m, n]$, $T(x) - (1 - w)x > T(x^m) - (1 - w)x^m$, or $\dfrac{T(x) - T(x^m)}{x - x^m} > 1 - w$, so $s(\varepsilon) > 1 - w$. Let $\delta = s(\varepsilon) - 1 + w > 0$, and let $w' \in (w - \delta, w) = (1 - s(\varepsilon), w)$. Then for all $x \in [x^m + \varepsilon, n]$, $\dfrac{T(x) - T(x^m)}{x - x^m} \geq s(\varepsilon) > 1 - w$, or $T(x) - (1 - w')x > T(x^m) - (1 - w')x^m$. So, $\bar{x}(n,w',T) < x^m + \varepsilon$ and $\bar{x}(n, \cdot, T)$ is left-hand continuous at w. And $w \in (0,1)$ was arbitrary, so $\bar{x}(n, \cdot, T)$ is left-hand continuous on (0,1).

QED

Lemma A.2. $\underline{X}(\cdot, T)$ and $\bar{X}(\cdot, T)$ satisfy

(i) if $w_1 \leq w_2$ then $\underline{X}(w_2, T) \leq \bar{X}(w_2, T) \leq \underline{X}(w_1, T) \leq \bar{X}(w_1, T)$, and

(ii) $\underline{X}(\cdot, T)$ is right-hand continuous, and $\bar{X}(\cdot, T)$ *is* left-hand continuous.

Proof. For all $w \in (0,1)$, $\underline{X}(w,T) = \int \underline{x}(n,w,T)dF(n)$ and $\bar{X}(w,T) = \int \bar{x}(n,w,T)dF(n)$, so (i) follows directly from Lemma A.1. To see that $\bar{X}(\cdot, T)$ is left-hand continuous, let $w \in (0,1)$ and let $\{w_i\}$ be a sequence in $(0,w]$ with $w_i \to w$. Then $\lim\limits_{w_i \to w} \bar{x}(w_i, T)$

$= \lim\limits_{w_i \to w} \int \bar{x}(n,w_i,T)dF(n) = \int \lim\limits_{w_i \to w} \bar{x}(n,w_i,T)dF(n) = \int \bar{x}(n,w,T)dF(n)$

$= \bar{X}(w,T)$. The second equality follows from the Lesbegue Dominated Convergence Theorem, and the third follows from Lemma A.1. A similar argument shows that $\underline{X}(\cdot, T)$ is right-hand continuous.

QED

Recall that an equilibrium is a pair (w^*, L^*) satisfying $L^* \in \hat{L}(w^*, T)$ and $w_D(L^*) = w^*$.

Proposition A.1. For any tax schedule T, a market equilibrium (w^*, L^*) exists and is unique.
(See Figure A.1)

Proof. As noted in the paper (following (3)) $\hat{X}(\cdot, T) : (0,1) \to\to [0,N]$ has nonempty compact, convex values. Hence $\hat{X}(w,T) = [\underline{X}(w,T), \bar{X}(w,T)]$ for all w, so by Lemma A.2 $\hat{X}(\cdot, T)$ clearly has closed graph (it is upper hemicontinous and has compact range), so $\hat{L}(\cdot, T) \equiv N - \hat{X}(\cdot, T)$ does also. By assumption, the wage function $w_D : [0,N] \to (0,1)$ is continuous, so by the Von Neumann Intersection Lemma, an equilibrium (w^*, L^*) exists. To prove uniqueness, let (w', L') be another equilibrium, and suppose $w' \neq w^*$. If $w' > w^*$ then $L' < L^*$ since w_D is a strictly decreasing function, so $X^* < X'$. But by Lemma A.2 $\bar{X}(w', T) \le \underline{X}(w^*, T)$, so $X' \in \hat{X}(w^*, T)$ implies that $X' \le \underline{X}(u^*, T) \le X^*$, a contradiction. Similarly, we cannot have $w' < w^*$, so $w' = w^*$. And, since w_D is strictly decreasing, $X' = X^*$.

QED

Appendix B

We must show that $R_0(\cdot,\cdot,n_i)$ is continuous on $\mathbf{S}_0(n_i)$. This requires some preliminary work.

If $T \in \mathbf{S}(n_i)$ has parameters (τ,β) then

$$\hat{L}(w,T) = N - \hat{X}(w,T) = \begin{cases} 0 & \text{for } w \in [0,1-\beta) \\ [0,\underline{M}(\frac{\tau}{\beta}) + \bar{N}(n_i)] & \text{for } w = 1-\beta \\ \underline{M}(\frac{\tau}{(1-w)}) + \bar{N}(n_i) & \text{for } w \in (1-\beta, 1-\frac{\tau}{n_i}) \\ [\underline{M}(n_i) + \bar{N}(n_i), N] & \text{for } w = 1-\frac{\tau}{n_i} \\ N & \text{for } w \in (1-\frac{\tau}{n_i}, 1] \end{cases} \quad \text{(B.1)}$$

where for any $n \in [0,1]$, $\underline{M}(n) = \int_0^n m dF(m)$ and $\bar{N}(n) = \int_n^1 (m-n)dF(m)$. Let $\tau_0(n_i) = n_i[1 - w_D(\underline{M}(n_i) + \bar{N}(n_i))]$. Then

$$\underline{M}(n_i) + \bar{N}(n_i) = L_D(1 - \frac{\tau_0(n_i)}{n_i}),$$

$$\underline{M}(n_i) + \bar{N}(n_i) > L_D(1 - \frac{\tau}{n_i}) \quad \text{for } \tau < \tau_0(n_i), \text{ and}$$

$$\underline{M}(n_i) + \bar{N}(n_i) < L_D(1 - \frac{\tau}{n_i}) \quad \text{for } \tau > \tau_0(n_i).$$

Let $\beta_e : \mathbf{S}_0{}' \to [1 - w_D(0), 1 - w_D(N)]$ be defined by

$$\beta_e(\tau,n_i) \text{ solves } \underline{M}(\frac{\tau}{\beta}) + \bar{N}(n_i) = L_D(1-\beta) \quad \text{for } \tau \in [0,\tau_0(n_i)]$$

$$\beta_e(\tau,n_i) = \frac{\tau}{n_i} \quad \text{for } \tau \in (\tau_0(n_i), (1-w(N))n_i].$$

(See Figure B.1.)

We now prove that β_e is well-defined and has several convenient properties.

Lemma B.1. The function β_e exists, is continuous and strictly increasing in τ for any $n_i \in [0,1]$, and satisfies

$\beta_e(\tau,n_i) \in [\max(1 - w_D(0), \frac{\tau}{n_i}),\ 1 - w_D(N)]$ for all $(\tau,n_i) \in \mathbf{S}_0'$.

Proof. Consider $\tau \in [0,\tau_0(n_i)]$. For $\tau > (1 - w_D(0))n_i$ and $\beta = \frac{\tau}{n_i}$, $\underline{M}(\frac{\tau}{\beta}) + \overline{N}(n_i) = \underline{M}(n_i) + \overline{N}(n_i) \geq L_D(1 - \frac{\tau}{n_i}) = L_D(1 - \beta)$. For $\tau \leq (1 - w_D(0))n_i$ and $\beta = 1 - w_D(0)$, $\underline{M}(\frac{\tau}{\beta}) + \overline{N}(n_i) = \underline{M}(\frac{\tau}{1-w_D(0)})$ $+ \overline{N}(n_i) > 0 = L_D(1 - \beta)$. For any $\tau \in [0,\tau_0(n_i)]$ and $\beta = 1 - w_D(N)$, $\underline{M}(\frac{\tau}{\beta}) + \overline{N}(n_i) = \underline{M}(\frac{\tau}{1-w_D(N)}) + \overline{N}(n_i) < \overline{N} = L_D(1 - \beta)$. Now, $\underline{M}(\frac{\tau}{\beta}) + \overline{N}(n_i)$ is decreasing and continuous in β on $[\max(1 - w_D(0), \frac{\tau}{n_i}),\ 1 - w_D(N)]$, and $L_D(1 - \beta)$ is strictly increasing and continuous in β on $(1-w_D(0), 1-w_D(N))$, so there exists a unique $\beta_e(\tau,n_i)$ that solves $\underline{M}(\frac{\tau}{\beta}) + \overline{N}(n_i) = L_D(1-\beta)$, and $\beta(\tau,n_i) \in [\max(1-w_D(0), \frac{\tau}{n_i}),\ 1-w_D(N)]$. For $\tau \in (\tau_0(n_i),\ (1-w_D(N))n_i]$, $\beta_e(\tau,n_i) = \frac{\tau}{n_i} \leq 1 - w_D(N)$, so $\beta_e(\tau n_i) \in [\max(1-w_D(0), \frac{\tau}{n_i}),\ 1-w_D(N)]$. Thus, the function β_e defined above exists, and satisfies

$\beta_e(\tau,n_i) \in [\max(1-w_D(0), \frac{\tau}{n_i}),\ 1-w_D(N)]$.

Write the equation $\underline{M}(\frac{\tau}{\beta}) + \overline{N}(n_i) = L_D(1-\beta)$ as $\overline{N}(n_i) = L_D(1-\beta) - \underline{M}(\frac{\tau}{\beta})$ $= g(\tau,\beta)$ and differentiate to get $\frac{\partial g}{\partial \beta}(\tau,\beta) = \frac{\tau^2}{\beta^3} f(\frac{\tau}{\beta}) - \frac{\partial L_D}{\partial w}(1-\beta) > 0$ for $\beta \in (1-w_D(0),\ 1-w_D(N))$. Thus, by the implicit function theorem $\beta_e(\cdot,n_i)$ is continuous (in fact, differentiable) on $[0,\tau_0(n_i)]$ for any n_i. Obviously, $\beta_e(\cdot,n_i)$ is continuous over $(\tau_0(n_i),\ (1-w_D(N))n_i]$. From above

$\beta_e(\tau_0(n_i),n_i) = \dfrac{\tau_0(n_i)}{n_i}$, and $\lim_{\tau \to \tau_0(n_i)} \beta_e(\tau,n_i) = \dfrac{\tau_0(n_i)}{n_i}$, so $\beta_e(\cdot,n_i)$ is continuous on $[0, (1-w_D(N))n_i]$.

Clearly, $\beta_e(\cdot,n_i)$ is strictly increasing on $(\tau_0(n_i), (1-w_D(N))n_i)$. For $\tau \in (0,\tau_0(n_i))$, $g(\tau,\beta_e(\tau)) = \bar{N}(n_i)$, so $\dfrac{\partial g}{\partial \tau} + \dfrac{\partial g}{\partial \beta}\cdot\dfrac{\partial \beta}{\partial \tau} = 0$, or $\dfrac{\partial \beta}{\partial \tau} = -\dfrac{\frac{\partial g}{\partial \tau}}{\frac{\partial g}{\partial \beta}}$. Now $\dfrac{\partial g}{\partial \beta} > 0$ from above, and $\dfrac{\partial g}{\partial \tau} = -\dfrac{\tau}{\beta^2} f(\dfrac{\tau}{\beta}) < 0$, so $\dfrac{\partial \beta}{\partial \tau} > 0$. Thus, $\beta_e(\cdot,n_i)$ is strictly increasing on $[0,(1-w_D(N))n_i]$.

QED

The function $\beta_e(\cdot,n_i)$ defines the "boundary" between the simple and nonsimple schedules in $\mathbf{S}(n_i)$. That is,

Lemma B.2. Let $T \in \mathbf{S}(n_i)$ have parameters (τ,β). If $\beta \in [\dfrac{\tau}{n_i}, \beta_e(\tau,n_i)]$ then $w^*(T) = min(1-\beta, w_D(0))$ and T is simple. If $\beta \in (\beta_e(\tau,n_i),1]$ then $w^*(T) = 1-\beta_e(\tau,n_i)$ and T is not simple, but T is equivalent to the simple schedule $T_1 \in \mathbf{S}(n_i)$ with parameters $(\tau,\beta_e(\tau, n_i))$.
(See Figure B.2.)

Proof. Fix $n_i \in [0,1]$ and $\tau \in [0,(1-w_D(N))n_i)$. For $\beta \in [\dfrac{\tau}{n_i}, 1-w_D(0))$ (if any such β exist), $N - X^*(w_D(0),T) = 0 = L_D(w_D(0))$ so $w^*(T) = w_D(0)$ and T is clearly simple. For $\beta \in [\max(1-w_D(0),\dfrac{\tau}{n_i}), \beta_e(\tau,n_i)]$, $N - \hat{X}(1-\beta,T) =$

$[0, \underline{M}(\dfrac{\tau}{\beta}) + \bar{N}(n_i)] \supseteq [0, \underline{M}(\dfrac{\tau}{\beta_e(\tau,n_i)}) + \bar{N}(n_i)] \ni L_D(1-\beta_e(\tau,n_i))$. Since L_D is strictly decreasing, $L_D(1-\beta) < L_D(1-\beta_e(\tau,n_i))$ and thus $L_D(1-\beta) \in N - \hat{X}(1-\beta,T)$, so $w^*(T) = 1-\beta$, so again T is simple. For

$\beta \in (\beta_e(\tau,n_i),1]$, if $\tau \le \tau_0(n_i)$ then $N - \hat{X}(1-\beta_e(\tau,n_i),T) \ni$ $\underline{M}(\frac{\tau}{\beta_e(\tau,n_i)}) + \bar{N}(n_i) = L_D(1-\beta_e(\tau,n_i))$ so $w^*(T) = 1-\beta_e(\tau,n_i)$, and if $\tau > \tau_0(n_i)$ then $N - \hat{X}(1-\beta_e(\tau,n_i),T) = N - \hat{X}(1 - \frac{\tau}{n_i},T) = [\underline{M}(n_i) + \bar{N}(n_i),1] \ni$ $L_D(1 - \frac{\tau}{n_i}) = L_D(1-\beta_e(\tau,n_i))$ so again $w^*(T) = 1 - \beta_e(\tau,n_i)$. Then T is not simple. However, the schedule $T_1 \in \mathbf{S}(n_i)$ with parameters $(\tau,\beta_e(\tau,n_i))$ is simple, and following the proof of Comment 4.3 it is straightforward to show that T_1 is equivalent to T. (T_1 is T "properly linearized" over its critical intervals.)

QED

Thus, we finally have

Lemma B.3. For any $n_i \in [0,1]$, $R_0(\cdot,\cdot,n_i)$ is continuous on $S(n_i)$.

Proof. Fix $n_i \in [0,1]$. In view of Comment 4.2 and Lemma B.2, for $\{(\tau,\beta) \mid \tau \in [0,(1-w_D(N))n_i]$ and $\beta \in [\frac{\tau}{n_i}, 1-w_D(0)), \beta \neq 0\}$,

$$R_0(\tau,\beta,n_i) = \beta\underline{M}(\frac{\tau}{\beta}) + \tau\int_{\tau/\beta}^{1} dF(n) + \beta\bar{N}(n_i);$$ for

$\{(\tau,\beta) \mid \tau \in [0, (1-w_D(N))n_i]$ and $\beta \in [\max(1-w_D(0), \beta_e(\tau,n_i)\}$,

$$R_0(\tau,\beta,n_i) = \beta\underline{M}(\frac{\tau}{\beta}) + \tau\int_{\tau/\beta}^{1} dF(n) + \beta\bar{N}(n_i) - \beta L_D(1-\beta);$$ and for

$\{(\tau,\beta) \mid \tau \in [0,(1 - w_D(N))n_i]$ and $\beta \in (\beta_e(\tau,n_i),1]\}$, $R_0(\tau,\beta,n_i) = R_0(\tau,\beta_e(\tau,n_i),n_i)$. Since $\beta_e(\cdot,n_i)$ is continuous by Lemma B.1 and L_D is continuous by assumption, $R_0(\cdot,\cdot,n_i)$ is continuous over $\mathbf{S}_0(n_i) - \{(0,0)\}$. Also, $\lim_{\beta\to 0} R_0(0,\beta,n_i) = \lim_{\beta\to 0} \beta\int_{n_i}^{1}(n - n_i)dF(n) = 0 = R_0(0,0,n_i)$, so $R_0(\cdot,\cdot,n_i)$ is also continuous at $\{(0,0)\}$. So $R_0(\cdot,\cdot,n_i)$ is continuous on

$\mathbf{S}_0(n_i)$.

QED

Here we prove Lemma 4.1.

Lemma 4.1 Fix $n_i \in [0,1]$. Let T be simple with $T(n_i) \geq 0$, and let T_1 be defined by

$$T_1(x) = \begin{cases} (1 - w^*(T))x & \text{for } 0 \leq x \leq \dfrac{T(n_i)}{1-w^*(T)} \\ T(n_i) & \text{for } \dfrac{T(n_i)}{1-w^*(T)} < x \leq n_i \\ T(n_i) + (1 - w^*(T))(x - n_i) & \text{for } n_i < x \leq 1 . \end{cases}$$

Then $T_1 \in \mathbf{S}(n_i)$ and $R(T_1) \geq R(T)$. Furthermore, if $T \not\equiv T_1$ then $R(T_1) > R(T)$.

Proof. Since T is simple, $T(n_i) \leq (1 - w^*(T))n_i$, so $(1 - w^*(T)) \in \left[\dfrac{T(n_i)}{n_i}, 1\right]$ and thus $T_1 \in \mathbf{S}(n_i)$. Since T is also nondecreasing, $T(x) \leq (1 - w^*(T))x = T_1(x)$ for $0 \leq x \leq \dfrac{T(n_i)}{(1-w^*(T))}$, $T(x) \leq T(n_i) = T(x)$ for $\dfrac{T(n_i)}{1-w^*(T)} < x \leq n_i$, and $T(x) \leq T(n_i) + (1 - w^*(T))(x - n_i) = T_1(x)$ for $n_i < x \leq 1$. If $1 - w^*(T) \in \left[\dfrac{T(n_i)}{n_i}, \beta_e(T(n_i),n_i)\right]$ then by Lemma B.2, T_1 is simple and $w^*(T_1) = w^*(T)$, so using Comment 4.2,

$R(T_1) = \int T_1(n)dF(n) - (1 - w^*(T_1))L_D(w^*(T_1))$

$\geq \int T(n)dF(n) - (1 - w^*(T))L_D(w^*(T)) = R(T)$. If $T(x) \neq T_1(x)$ for some x, then by continuity $T < T_1$ over some open neighborhood of x, so $\int T_1(n)dF(n) > \int T(n)dF(n)$, and thus $R(T_1) > R(T)$.

Otherwise $1 - w^*(T) \in (\beta_e(T(n_i),n_i), 1]$, so by Lemma B.2 T_1 is not simple, but it is equivalent to the simple schedule $T_2 \in \mathbf{S}(n_i)$ with parameters $(T(n_i), \beta_e(T(n_i),n_i))$. For $T(n_i) \in [0,\tau_0(n_i)]$,
$L_D(1 - \beta_e(T(n_i),n_i)) = M(T(n_i)/\beta_e(T(n_i),n_i)) + \bar{N}(n_i)$ and
$R_0(T(n_i), \beta_e(T(n_i),n_i)) = T(n_i)\int_{T(n_i)/\beta_e(T(n_i),n_i)}^{1} dF(n)$. Since T is simple,

$$
\begin{aligned}
R(T) &= \int T(n)dF(n) - (1 - w^*(T))L_D(w^*(T)) \\
&\le \int T_1(n)dF(n) - (1 - w^*(T))L_D(w^*(T)) \\
&= (1 - w^*(T))\underline{M}(T(n_i)/(1-w^*(T))) + T(n_i)\int_{T(n_i)/(1-w^*(T))}^{1} dF(n) \\
&\quad + (1 - w^*(T))\bar{N}(n_i) - (1 - w^*(T))L_D(w^*(T)) \\
&= (1 - w^*(T))\underline{M}(T(n_i)/(1-w^*(T))) + T(n_i)\int_{T(n_i)/(1-w^*(T))}^{T(n_i)/\beta_e(T(n_i),n_i)} dF(n) \\
&\quad + T(n_i)\int_{T(n_i)/\beta_e(T(n_i),n_i)}^{1} dF(n) + (1 - w^*(T))\bar{N}(n_i) \\
&\quad - (1 - w^*(T))L_D(w^*(T)) .
\end{aligned}
$$

For $n \in \left[\dfrac{T(n_i)}{1-w^*(T)}, \dfrac{T(n_i)}{\beta_e(T(n_i),n_i)}\right]$, $T(n_i) \le (1 - w^*(T))n$, so

$$
\begin{aligned}
R(T) &\le (1-w^*(T))\underline{M}(T(n_i)/1-w^*(T)) + (1-w^*(T))\int_{T(n_i)/(1-w^*(T))}^{T(n_i)/\beta_e(T(n_i),n_i)} n dF(n) \\
&\quad + (1-w^*(T))\bar{N}(n_i) - (1-w^*(T))L_D(w^*(T)) + T(n_i)\int_{T(n_i)/\beta_e(T(n_i),n_i)}^{1} dF(n)
\end{aligned}
$$

$$= (1-w^*(T))\cdot\left[\underline{M}(T(n_i)/\beta_e(T(n_i),n_i) + \bar{N}(n_i) - (1-w^*(T))L_D(w^*(T))\right]$$

$$+ T(n_i)\int_{T(n_i)/\beta_e(T(n_i),n_i)}^{1} dF(n) .$$

(See Figure B.3.)

Since $1 - w^*(T) > \beta_e(T(n_i),n_i)$, $L_D(w^*(T)) > L_D(1 - \beta_e(T(n_i),n_i))$, so the term in brackets is strictly negative, and

$R(T) < T(n_i)\int_{T(n_i)/\beta_e(T(n_i),n_i)}^{1} dF(n) = R(T_1)$ as desired.

For $T(n_i) > \tau_0(n_i)$, $\beta_e(T(n_i),n_i) = \dfrac{T(n_i)}{n_i}$, and $L_D(1 - \beta_e(T(n_i),n_i)) =$ $\underline{M}(n_i) + \bar{N}(n_i) + \int_{n_i}^{n_1} dF(n) > \underline{M}(n_i) + \bar{N}(n_i)$ and $R_0(T(n_i),\beta_e(T(n_i),n_i)) =$ $T(n_i)\int_{n_1}^{1} dF(n)$ for some $n_1 > n_i$. Using a similar argument to that above,

$$R(T) \le (1-w^*(T))\underline{M}(T(n_i)/(1-w^*(T))) + T(n_i)\int_{T(n_i)/(1-w^*(T))}^{1} dF(n)$$

$$+ (1-w^*(T))\bar{N}(n_i) - (1-w^*(T))L_D(w^*(T))$$

$$\le (1-w^*(T))\cdot\left[\underline{M}(n_i) + \int_{n_i}^{n_1} dF(n) + \bar{M}(n_i) - L_D(w^*(T))\right] + T(n_i)\int_{n_1}^{1} dF(n).$$

(See Figure B.4.)

Again, since $1 - w^*(T) > \beta_e(T(n_i),n_i)$, $L_D(w^*(T)) > L_D(1 - \beta_e(T(n_i),n_i))$ so the term in brackets is negative, and $R(T) < T(n_i)\int_{n_1}^{1} dF(n) = R(T_1)$ as desired.

QED

II. INCOME TAXES IN THE ARBITRAGE PRICING THEORY

7. Introduction

A necessary condition for a market to be in equilibrium is that there are no opportunities for any trader to make an arbitrarily large profit. This condition often imposes limitations on what an equilibrium must look like, particularly when the markets involved function "smoothly;" i.e., when information and transaction costs are low. Capital asset markets seem "smooth," or at least they are often modelled as if they were, and thus the condition that no one can make limitless profits is quite powerful in restricting the prices at which the markets will be in equilibrium.

It is quite possible, of course, that for ***all*** price vectors there is at least one trader who has the opportunity to make limitless profits. In "perfect" (very smooth) capital markets this may be true when different investors receive different profits from the same investment. One reason investors may receive different profits is taxes--if different investors and assets are taxed differently then their profits ***after taxes*** may be different even though their profits ***before taxes*** are the same.

The tax codes in many industrialized nations do in fact treat different taxpayers differently. Different types of investors are distinguished, such as private individuals, pension funds, corporations, nonprofit institutions and insurance companies, and for various reasons are taxed in different ways and at different rates. Different types of income are distinguished as well, such as wages and salaries, royalties, gifts, interest, dividends and capital gains, and these may be taxed differently. Thus, for example, in the U.S. today corporations in the top bracket face a (nominal) marginal tax rate of 6.9% on dividend income (the top bracket is 46%, and 85% of dividend income is excluded) and 28% on capital gains, while for an individual in the 50% tax bracket the rates are 50% on dividends (dividends are fully taxed, after a small exclusion) and 20% on long-term capital gains (60% of long-term capital gains are excluded). And many pension funds are not taxed at all on investment income, so for such investors the rates are zero.

Using some of these features it is easy to construct, in highly simplified worlds, situations in which there always exists at least one investor with an opportunity for making limitless profits. For example, consider two investors, Ms. T who is taxed at 50% on all net income (and all net payments) and Ms. N who is not taxed. Suppose there are (at least) two riskless assets traded, bond t which is taxable, and bond n which is not. Suppose these bonds are available in limitless quantities, there are no transactions costs, and there are no limitations on buying and short-sales for either investor. If the return per dollar invested in bond t, r_t, is not equal to the return per dollar invested in bond n, r_n, then Ms. N can make unlimited profits by buying the bond with the higher return and selling short equal amounts (in dollars) of the other. If, say, $r_t > r_n$ then she earns $r_t - r_n > 0$ for each dollar of matching short and long positions. And the transaction costs her nothing (her short and long positions cancel) so she can make it limitlessly. A transaction involving no net investment (such as Ms. N's) is called an "arbitrage" transaction, and the ensuing profits are called "arbitrage profits." Because arbitrage transactions involve zero net investment, the scale of such transactions is not limited by the investor's net wealth. As a result, if r_n and r_t remain fixed, Ms. N can earn any amount of arbitrage profits that she desires. On the other hand, if r_n does equal r_t then Ms. T can make unlimited arbitrage profits by buying bond n and selling short bond t in equal amounts. On each dollar of this (costless) transaction she earns $r_n - .5r_t = .5r_n > 0$. Thus, given any pattern of rates of return, at least one of the investors can make arbitrarily large profits. Other examples can be found in Schaefer [21].

Schaefer [21] points out that if capital asset markets were perfect then opportunities such as this would abound, and thus equilibrium in these markets must be generated by "frictions;" i.e., market imperfections of some sort. Thus, to understand the nature of the equilibrium we must understand what the frictions are and what the relationship is between them and the equilibrium.

Using the Capital Asset Pricing Model (CAPM) as formulated by Sharpe [22], Black [6] and others, Brennan [8], Long [16], Litzenberger and Ramaswamy [15] derived various conditions under which income taxes cause, or do not cause, the relative rates of return (i.e.,, relative prices)[1] of assets to be different from the prices without taxes. In this part we study similar issues using a more recent and comprehensive model of capital asset pricing, the Arbitrage Pricing Theory (APT) formulated by Ross [19],[20]. We define a simple extension of the standard APT (in which there are no taxes) with an income tax, paying special attention to the differential treatment of capital gains and dividends. First we assume that capital markets are perfect, and find that only under very strong conditions will there exist prices at which *no* investor can make limitless profits. Then we consider a particular type of market imperfection, portfolio restrictions on borrowing, lending and short sales. We find that, in general, there exist prices at which no investors have any arbitrage opportunities, but the pricing relation will often differ from that in the no-tax world. That is, the taxes and market imperfection considered affect the relative prices of assets.

To concentrate on the intuition behind the results, in Section 8 we deal with a highly simplified version of the APT in which there are no idiosyncratic (asset-specific) risks. This allows us to state and prove results simply, without the need to use limiting arguments as in Huberman [13], Ingersoll [14], and Ross [20]. In Section 9 we do the asymptotic work, and prove results analogous to those of Section 8 for the APT with idiosyncratic risks.

8. Income Taxes and the APT with No Idiosyncratic Risks

Here we assume that there are no asset-specific risks. In Section 8.1 we derive necessary and sufficient conditions under which the condition of "no arbitrage" is satisfied simultaneously for all investors. With no restrictions on the portfolios investors may hold, these conditions seem quite strong. They can be stated in various ways as constraints on tax rates, asset factor weights (betas), and dividend returns. One straightforward implication of the conditions is that if there is at least one taxpayer whose tax rates on capital gains and dividends are the same (for example, tax exempt investors) then the same linear pricing relationship as in the no-tax APT must hold for asset pretax returns. At the end of the section we relate these conditions to those derived in Long [16].

In Section 8.2 we investigate the effects of portfolio restrictions in the form of constraints on short sales or "borrowing," similar to those in Litzenberger and Ramaswamy [15]. We find that in the special case considered the form of the no arbitrage asset pricing relation will probably be different from that of the usual no-tax APT. If the conditions that prevent arbitrage with no portfolio restrictions are not met, then an asset pricing relation of the same form as that in the no-tax APT is a necessary and sufficient condition for no arbitrage to exist if and only if all investors face a tax rate on dividends no less than their tax rate on capital gains. (Recall, this is not true for corporations in the the U.S. today.) We derive a different no arbitrage pricing relation (or rather, a set of them) assuming that there are restrictions on borrowing. In this relation, an asset's expected returns depend not only on its factor weights but also on its dividend payment.

8.1 The Model with Perfect Markets

In the standard APT there is a fixed number of assets whose returns are generated by a K-factor linear model, i.e.,

$$\tilde{R}_j = E_j + \boldsymbol{\beta}_j'\tilde{\boldsymbol{\gamma}} + \tilde{\varepsilon}_j \text{ , for } j = 0,1,\ldots,J \tag{8.1}$$

where $J+1$ is the number of assets, $\tilde{R}_j$ is the jth asset's (random) realized return and E_j its expected return, $\boldsymbol{\beta}_j' = (\beta_{j1}, \ldots, \beta_{jK})$ is the jth asset's vector of factor weights, and $\tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_K)$ is the (random) realized vector of factor values. It is assumed that $E(\tilde{\gamma}_k) = 0$ for all k, $E(\tilde{\varepsilon}_j) = 0$ and $var(\tilde{\varepsilon}_j) < \infty$ for all j, and $cov(\tilde{\varepsilon}_j, \tilde{\varepsilon}_l) = 0$ for all $j \neq l$. To obtain results, one must have an economy with an infinite number of assets, so that it is possible to form portfolios in which the variance caused by idiosyncratic risks (the $\tilde{\varepsilon}_j$ terms) is negligible. Thus Huberman [13] and Ross [20] consider sequences of economies in which the number of assets goes to infinity, and prove results for the limit economy.

To keep the arguments as transparent as possible, we ignore the asset-specific risk terms entirely. Thus, we assume that there are $J+1$ assets, $J > K$, with returns generated by the K-factor linear model

$$\tilde{R}_j = E_j + \boldsymbol{\beta}_j'\tilde{\boldsymbol{\gamma}} \text{ , for } j = 0,\ldots,J \tag{8.2}$$

where $\tilde{R}_j$, E_j, $\boldsymbol{\beta}_j'$ and $\tilde{\boldsymbol{\gamma}}$ are as in (8.1). $\tilde{R}_j$ is the pretax realized return on asset j, and E_j the pretax expected return. We assume that $\tilde{R}_j$ consists partly of dividends and partly of capital gains. That is, $\tilde{R}_j = \tilde{D}_j + \tilde{G}_j$, where $\tilde{D}_j = f_j(\tilde{R}_j)$ are the dividends, and $\tilde{G}_j = \tilde{R}_j - f_j(\tilde{R}_j)$ the capital gains, paid by asset j.

Investors are taxed on their returns, and their tax rates on capital gains and dividends may be different. Let $s^i \in [0,1)$ be investor i's tax rate on capital gains, and $t^i \in [0,1)$ her tax rate on dividends. Then i's after-tax rate of return on asset j is

$$\tilde{R}_j^i = (1 - t^i)f_j(\tilde{R}_j) + (1 - s^i)[\tilde{R}_j - f_j(\tilde{R}_j)]$$

$$= (s^i - t^i)f_j(\tilde{R}_j) + (1 - s^i)\tilde{R}_j \ .$$

Note that each investor's tax rates do not depend on the realized return; i.e., individuals do not change tax brackets. (This is similar to the tax treatment in Brennan [8] and Long [16].) In the real world taxes are often progressive, so an investor's tax rates may change with her income. While progressive taxation raises a number of interesting questions, we do not address them here. What is important for our purposes is that there be investors who face different relative tax rates on capital gains versus dividends regardless of their incomes, as is true for corporations and individuals in many countries.

In the general form written above, it is difficult to proceed much further, so we assume that

$$f_j(\tilde{R}_j) = d_j + e_j\tilde{R}_j \ , \text{ for } j = 0,1,\ldots,J. \tag{8.3}$$

This linear form is easily manipulated, and if $e_j = 0$ then it reduces to the case considered in Brennan [8], Litzenberger and Ramaswamy [15] and Long [16], where dividends are known with certainty. Given (8.3), investor i's after-tax return on asset j is

$$\tilde{R}_j^i = (s^i - t^i)(d_j + e_j\tilde{R}_j) + (1 - s^i)\tilde{R}_j$$

$$= (s^i - t^i)d_j + [(s^i - t^i)e_j + 1 - s^i]\tilde{R}_j \, .$$

Letting $(s^i - t^i)e_j + 1 - s^i = v_j^i$ and substituting from (8.2),

$$\tilde{R}_j^i = (s^i - t^i)d_j + v_j^i\tilde{R}_j$$

$$= (s^i - t^i)d_j + v_j^iE_j + v_j^i\beta_j'\tilde{\gamma} \, .$$

Note that v_j^i is investor i's marginal after-tax return on asset j per dollar of the pretax return. If $v_j^i < 0$ then investor i's after-tax return on j falls as the pretax return on j rises, a peculiar state of affairs which we will assume never to occur. In fact, under reasonable assumptions about the dividend payment functions, $v_j^i \in (0,1]$ for all j and i. To see this, note that v_j^i is a linear function of e_j, with $v_j^i = 1 - s^i$ for $e_j = 0$ and $v_j^i = 1 - t^i$ for $e_j = 1$, so v_j^i lies between $1 - s^i$ and $1 - t^i$ for all $e_j \in [0,1]$. Since s^i and t^i are restricted to lie in $[0,1)$, $v_j^i \in (0,1]$ for all such e_j. If $e_j < 0$ then asset j pays a lower dividend the higher its realized return, and if $e_j > 1$ then asset j pays more that \$1 in dividends for each additional \$1 it earns. Both of these seem unrealistic so it may be reasonable to assume that $e_j \in [0,1]$.

If $\mathbf{x} = (x_0, \ldots, x_J)'$ is any portfolio (x_j is the dollar amount held in asset j) then the after-tax return on $\mathbf{x}$ to investor i is

$$\tilde{R}_x^i = \sum_{j=0}^{J} x_j\tilde{R}_j^i$$

$$= (s^i - t^i)\sum_{j=0}^{J} x_jd_j + \sum_{j=0}^{J} x_jv_j^iE_j + \sum_{j=0}^{J} x_jv_j^i\beta_j'\tilde{\gamma} \, . \tag{8.4}$$

In what follows, it is convenient to use matrix notation. Thus, we write (8.4) as

$$\tilde{R}_x^i = \mathbf{x}'\tilde{\mathbf{R}}^i$$

$$= (s^i - t^i)\mathbf{x}'\mathbf{d} + \mathbf{x}'\mathbf{V}^i\mathbf{E} + \mathbf{x}'\mathbf{V}^i\mathbf{B}\tilde{\gamma}\,, \qquad (8.4)'$$

where $\tilde{\mathbf{R}}^i = (\tilde{R}_0^i, \ldots, \tilde{R}_J^i)'$ is the vector of realized returns, $\mathbf{d} = (d_1, \ldots, d_J)'$ is the vector of dividend line intercept parameters, $\mathbf{V}^i$ is a $(J+1)\times(J+1)$ diagonal matrix whose (j,j)th element is v_j^i, $\mathbf{E} = (E_0, \ldots, E_J)'$ is the vector of expected asset returns, and $\mathbf{B}$ is the $(J+1)\times K$ matrix of factor weights with (j,k)th element β_{jk}.

Portfolio $\mathbf{x}$ uses no wealth if (A1) $\mathbf{x}'\mathbf{1} = 0$, where $\mathbf{1}$ is a vector of $J+1$ ones,[2] and $\mathbf{x}$ has no "systematic risk" for investor i (in the present model, no risk at all, since there is no asset-specific risk) if (A2) $\mathbf{x}'\mathbf{V}^i\boldsymbol{\beta}_k = 0$ for all $k = 1,\ldots,K$, where $\boldsymbol{\beta}_k = (\beta_{1k}, \ldots, \beta_{Jk})'$ is the kth column of $\mathbf{B}$. We call a portfolio that satisfies (A1) and (A2) an *arbitrage portfolio for i*. The after-tax return to i of such a portfolio is

$$\tilde{R}_x^i = (s^i - t^i)\mathbf{x}'\mathbf{d} + \mathbf{x}'\mathbf{V}^i\mathbf{E} \qquad (8.5)$$

$$= \mathbf{x}'[(s^i - t^i)\mathbf{d} + \mathbf{V}^i\mathbf{E}] \qquad (8.6)$$

since the last term in (8.4)' is zero by (A2). If $\tilde{R}_x^i > 0$ then i will wish to hold infinite amounts of such a portfolio, given rather mild assumptions about her utility function. We call the existence of such a portfolio (i.e., an $\mathbf{x}$ satisfying (A1), (A2) and $\tilde{R}_x^i > 0$) an *arbitrage opportunity for i*.

Notice that the set of arbitrage portfolios for investor i is a linear subspace, and the dimension of the space is $(J+1)$ minus the rank of $\mathbf{V}^i\mathbf{B}$

(thus if the vectors $\mathbf{1},\mathbf{V}^i\boldsymbol{\beta}_1,\ldots,\mathbf{V}^i\boldsymbol{\beta}_K$ are linearly independent, the dimension is $(J+1)-(K+1) = J-K$). Also, while in general the set of arbitrage portfolios will be different for different investors, for any investor i with tax rates satisfying $s^i = t^i$, $v_j^i = (1-s^i)$ for all j, so the set of arbitrage portfolios is the set of $\mathbf{x}$ (simultaneously) orthogonal to $\mathbf{1},\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_K$. Thus the set of arbitrage portfolios is the same for all investors who face equal tax rates on dividends and capital gains, and is equal to the set of arbitrage portfolios for the tax exempt investor. Note also that even if the set of arbitrage portfolios is the same for two investors, the set of arbitrage opportunities may not be. That is, a portfolio $\mathbf{x}$ may simultaneously satisfy $\mathbf{x}'\mathbf{1} = 0$, $\mathbf{x}'\mathbf{V}^i\boldsymbol{\beta}_k = 0$ and $\mathbf{x}'\mathbf{V}^m\boldsymbol{\beta}_k = 0$ for $k = 1,\ldots,K$, $\tilde{R}_x^i > 0$ and $\tilde{R}_x^m \le 0$, so that $\mathbf{x}$ is an arbitrage portfolio for both i and m, but an arbitrage opportunity only for i.

It is reasonable to assume that for the market to be in equilibrium, there can be no arbitrage opportunities for any investor. We call this state of affairs *no arbitrage*. We now consider the implications of this condition on the expected return, dividend, tax rate and factor sensitivity parameters, assuming there are no restrictions on the portfolios investors may hold.

If there is no arbitrage opportunity for i then $\tilde{R}_x^i \le 0$ for any $\mathbf{x}$ that is an arbitrage portfolio for i. Since, if $\mathbf{x}$ is an arbitrage portfolio for i then $-\mathbf{x}$ is also, $\tilde{R}_x^i$ must *equal* zero for any such arbitrage portfolio. From (8.6) this means that the vector $(s^i - t^i)\mathbf{d} + \mathbf{V}^i\mathbf{E}$ must be orthogonal to any $\mathbf{x}$ that satisfies (A1) and (A2), and thus $(s^i - t^i)\mathbf{d} + \mathbf{V}^i\mathbf{E}$ must be in the linear span (i.e., the set of all linear combinations) of $\mathbf{1},\mathbf{V}^i\boldsymbol{\beta}_1,\ldots,\mathbf{V}^i\boldsymbol{\beta}_K$. We write this as $(s^i - t^i)\mathbf{d} + \mathbf{V}^i\mathbf{E} \in span(\mathbf{1},\mathbf{V}^i\boldsymbol{\beta}_1,\ldots,\mathbf{V}^i\boldsymbol{\beta}_K)$. So, there exist constants

$q_0^i, \ldots, q_K^i$ such that

$$(s^i - t^i)\mathbf{d} + \mathbf{V}^i\mathbf{E} = q_0^i\mathbf{1} + q_1^i\mathbf{V}^i\boldsymbol{\beta}_1 + \cdots + q_K^i\mathbf{V}^i\boldsymbol{\beta}_K, \text{ or} \tag{8.7}$$

$$\mathbf{E} = q_0^i[\mathbf{V}^i]^{-1}\mathbf{1} + q_1^i\boldsymbol{\beta}_1 + \cdots + q_K^i\boldsymbol{\beta}_K - (s^i - t^i)[\mathbf{V}^i]^{-1}\mathbf{d}, \tag{8.8}$$

where $[\mathbf{V}^i]^{-1}$, the inverse of $\mathbf{V}^i$, is a diagonal matrix whose (j,j)th element is $1/v_j^i$. Thus, $\mathbf{E}$ must lie in the linear span of $[\mathbf{V}^i]^{-1}\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K$, translated by the vector $(s^i - t^i)[\mathbf{V}^i]^{-1}\mathbf{d}$. It is a necessary and sufficient condition for there to be no arbitrage opportunities for i. So we have

Comment 8.1. With no restrictions on allowable portfolios, there is no arbitrage if and only if for all investors i there exist $q_0^i, \ldots, q_K^i$ such that $\mathbf{E} = q_0^i[\mathbf{V}^i]^{-1}\mathbf{1} + q_1^i\boldsymbol{\beta}_1 + \cdots + q_K^i\boldsymbol{\beta}_K - (s^i - t^i)[\mathbf{V}^i]^{-1}\mathbf{d}$.

As noted above, for an investor with $s^i = t^i$, $v_j^i = (1 - s^i)$ for all j, so $\mathbf{V}^i = (1 - s^i)\mathbf{I}$ and $[\mathbf{V}^i]^{-1} = \left[\frac{1}{1 - s^i}\right]\mathbf{I}$, where $\mathbf{I}$ is the $(J+1)\times(J+1)$ identity matrix. Thus (8.8) becomes

$$\mathbf{E} = \left[\frac{q_0^i}{1 - s^i}\right]\mathbf{1} + q_1^i\boldsymbol{\beta}_1 + \cdots + q_K^i\boldsymbol{\beta}_K$$

$$= r^i\mathbf{1} + q_1^i\boldsymbol{\beta}_1 + \cdots + q_K^i\boldsymbol{\beta}_K. \tag{8.9}$$

Since the vectors $\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K$ are independent of i, the coefficients $r^i, q_1^i, \ldots, q_K^i$ are also independent of i, and we can write (8.9) as

$$\mathbf{E} = r\mathbf{1} + q_1\boldsymbol{\beta}_1 + \cdots + q_K\boldsymbol{\beta}_K = r\mathbf{1} + \mathbf{B}\mathbf{q}, \tag{8.10}$$

where $\mathbf{q} = (q_1, \ldots, q_K)$. This is the familiar pricing relationship of the standard no-tax APT. As usual, r is the expected return on any "zero-beta" asset, and also the return on the risk-free asset if one exists. Thus we have

Comment 8.2. If there exists at least one investor with equal tax rates on capital gains and dividends then (8.10) is a necessary condition for there to be no arbitrage.

In the U.S. there are many tax exempt investors, such as pension funds, and for such investors Comment 8.2 clearly applies. Thus, since $\mathbf{E}$ is the vector of pretax expected returns, empirical tests of the no-tax APT are also partial tests of the APT with income taxes. Also, if all investors face equal tax rates on capital gains and dividends then (8.10) is sufficient as well as necessary for there to be no arbitrage.[3ud]

Consider now the implications of the requirement that there are no arbitrage opportunities for an investor m with $s^m \neq t^m$, assuming there is some investor i with $s^i = t^i$. Substituting (8.10) into (8.7) yields

$$(s^m - t^m)\mathbf{d} + \mathbf{V}^m[r\mathbf{1} + q_1\boldsymbol{\beta}_1 + \cdots + q_K\boldsymbol{\beta}_K] = q_0^m\mathbf{1} + q_1^m\mathbf{V}^m\boldsymbol{\beta}_1 + \cdots + q_K^m\mathbf{V}^m\boldsymbol{\beta}_K .$$

Or, using $\mathbf{V}^m\mathbf{1} = (s^m - t^m)\mathbf{e} + (1 - s^m)\mathbf{1}$, where $\mathbf{e} = (e_0, \ldots, e_J)$,

$$\mathbf{d} + r\mathbf{e} = \left[\frac{q_0^m - r(1 - s^m)}{s^m - t^m}\right]\mathbf{1} + \left[\frac{q_1^m - q_1}{s^m - t^m}\right]\mathbf{V}^m\boldsymbol{\beta}_1 + \cdots + \left[\frac{q_K^m - q_K}{s^m - t^m}\right]\mathbf{V}^m\boldsymbol{\beta}_K$$

$$= r_0^m\mathbf{1} + r_1^m\mathbf{V}^m\boldsymbol{\beta}_1 + \cdots + r_K^m\mathbf{V}^m\boldsymbol{\beta}_K . \qquad (8.11)$$

This says that $\mathbf{d} + r\mathbf{e} \in span(\mathbf{1}, \mathbf{V}^m\boldsymbol{\beta}_1, \ldots, \mathbf{V}^m\boldsymbol{\beta}_K)$. Given that the pricing relation (8.10) holds, (8.11) is a necessary and sufficient condition for there to be no arbitrage opportunities for m. Thus we have

Proposition 8.1. If there are no restrictions on portfolios and there exists at least one investor i with $s^i = t^i$ then there is no arbitrage if and only if (8.10) holds and (8.11) holds for every investor m such that $s^m \neq t^m$.

Thus $\mathbf{d} + r\mathbf{e}$ must be in the intersection of all the linear subspaces spanned by vectors $\mathbf{1}, \mathbf{V}^m\boldsymbol{\beta}_1, \ldots, \mathbf{V}^m\boldsymbol{\beta}_K$ such that $s^m \neq t^m$ for some investor m. This intersection contains the line through $\mathbf{0}$ (a vector of $J+1$ zeros) and $\mathbf{1}$, but it appears that the conditions under which it contains anything else are quite strong. Consider the case where $K=1$, and suppose there are two investors i and m such that $s^i \neq t^i$ and $s^m \neq t^m$. Then (8.11) is satisfied for both investors and the intersection of $span(\mathbf{1}, \mathbf{V}^i\boldsymbol{\beta}_1)$ and $span(\mathbf{1}, \mathbf{V}^m\boldsymbol{\beta}_1)$ is of dimension greater than one only if $\mathbf{V}^i\boldsymbol{\beta}_1 \in span(\mathbf{1}, \mathbf{V}^m\boldsymbol{\beta}_1)$. So there must exist constants α^{im} and δ^{im} such that $[(s^i - t^i)e_j + (1 - s^i)]\beta_{j1} = \alpha^{im} + \delta^{im}[(s^m - t^m)e_j + (1 - s^m)]\beta_{j1}$ for all j, or letting $c_1^{im} = [(s^i - t^i) - \delta^{im}(s^m - t^m)]$ and $c_2^{im} = [(1 - s^i) - \delta^{im}(1 - s^m)]$, $c_1^{im}e_j\beta_{j1} + c_2^{im}\beta_{j1} = \alpha^{im}$ for all j. If $c_1^{im} = 0$ and $c_2^{im} \neq 0$ then $\beta_{j1} = \alpha^{im}/c_2^{im}$ for all j, so all assets have exactly the same risk characterstics and hence are effectively identical. If $c_1^{im} \neq 0$ then

$$e_j = \frac{\alpha^{im} - c_2^{im}\beta_{j1}}{c_1^{im}\beta_{j1}}$$ for all j such that $\beta_{j1} \neq 0$. Unless there are some economic forces that might cause this functional relationship between asset dividend line slopes and factor betas, it is extremely unlikely that it is satisfied. The last possibility is that $c_1^{im} = c_2^{im} = 0$, in which case there is no restriction imposed on $\mathbf{e}$ and $\boldsymbol{\beta}_1$. However, $c_1^{im} = c_2^{im} = 0$ if and only if $\frac{s^i - t^i}{1 - s^i} = \frac{s^m - t^m}{1 - s^m}$, a strong restriction on tax rates. For $K>1$ the conditions are more difficult to interpret, and work explaining them needs to

be done, but we expect the results to be similar.

On the other hand, if the intersection of the linear spans described above contains only the line through **0** and **1** then $\mathbf{d} + r\mathbf{e} = h\mathbf{1}$ for some h. This has an easily recognized economic interpretation--it says that all asset dividend return lines cross at $\tilde{R}_j = r$; i.e., all assets pay the same dividend if they earn the zero-beta (riskless) rate of return.

Another interesting case is when $\mathbf{e} = e\mathbf{1}$, i.e., all assets' dividend return lines have the same slope. (A special case of this is that considered in Brennan [8], Litzenberger and Ramaswamy [15] and Long [16], where all assets' dividends are known with certainty so $\mathbf{e} = \mathbf{0}$.) Then for each investor m

$$v_j^m = (s^m - t^m)e + (1 - s^m) = v^m \quad \text{for } j = 0,1,\ldots,J, \tag{8.12}$$

so $\mathbf{V}^m = v^m I$ and $\mathbf{V}^m\mathbf{B} = v^m\mathbf{B}$. Thus the set of arbitrage portfolios is the same for all investors, and $span(1,\mathbf{V}^m\boldsymbol{\beta}_1, \ldots ,\mathbf{V}^m\boldsymbol{\beta}_K) = span(1,\boldsymbol{\beta}_1, \ldots ,\boldsymbol{\beta}_K)$ for all investors, so $\bigcap_m span(1,\mathbf{V}^m\boldsymbol{\beta}_1, \ldots ,\mathbf{V}^m\boldsymbol{\beta}_K) = span(1,\boldsymbol{\beta}_1, \ldots ,\boldsymbol{\beta}_K)$. Writing (8.11) as

$$\begin{aligned}\mathbf{d} &= (r_0^m - re)1 + r_1^m v^m \boldsymbol{\beta}_1 + \cdots + r_K^m v^m \boldsymbol{\beta}_K \\ &= p_0^m 1 + p_1^m \boldsymbol{\beta}_1 + \cdots + p_K^m \boldsymbol{\beta}_K\end{aligned} \tag{8.13}$$

we have

Comment 8.3. If there are no restrictions on portfolios, if there exists at least one investor with equal tax rates on capital gains and dividends and one investor with unequal tax rates on capital gains and dividends, and if $\mathbf{e} = e\mathbf{1}$, then there is no arbitrage only if (8.10) holds and $\mathbf{d} \in span(1,\boldsymbol{\beta}_1, \ldots ,\boldsymbol{\beta}_K)$.

8.2 Portfolio Restrictions

So far we have assumed that investors may hold any portfolio of assets. Actually, for at least some investors, there are limits on such transactions as borrowing and shorts sales, which restrict the set of portfolios they can acquire. As Schaefer [21] argues, it may be these restriction ("frictions") that prevent arbitrage and induce a market equilibrium. This equilibrium need not be the same as the equilibrium without restrictions (if one exists), and probably will be sensitive to the form of the restrictions imposed.

Economists have employed several kinds of restrictions in past work. Black [6] analyzed the CAPM assuming that investors could not borrow or lend at the riskless rate of return. Schaefer [21] studied two types of constraints, (C1) no short sales ($\mathbf{x} \geq \mathbf{0}$) and (C2) dollar limits on short sales ($\mathbf{x} \geq \mathbf{a}$ for some fixed $\mathbf{a}$). Litzenberger and Ramaswamy [15] also impose restrictions on short sales, but on the aggregate level of short sales, not on individual assets. In their model there is a risk-free asset whose return is all dividends, and the dividends of all risky assets are known with certainty. Their first restriction is that an investor's "interest" payments (dividend payments on short sales of the riskless asset) cannot exceed her total dividend income on risky assets; i.e., (C3) $\sum_{j=1}^{J} x_j d_j \geq -x_0 E_0 = -x_0 d_0$, where asset 0 is riskless. Noting that (C3) can be written as $\sum_{j=0}^{J} x_j d_j \geq 0$ (or, in vector notation, $\mathbf{x}'\mathbf{d} \geq \mathbf{0}$), an equivalent statement of the constraint is that an investor's total dividend and interest income must be nonnegative. The second restriction is that an investor's holdings of risky assets cannot exceed some fixed fraction of her total wealth; i.e., (C4) $\sum_{j=1}^{J} x_j \leq b \sum_{j=0}^{J} x_j$, or

$\frac{1-b}{b}\sum_{j=0}^{J} x_j \le x_0$. As Litzenberger and Ramaswamy note, (C4) is a type of "margin requirement" and (C3) an "income requirement." Since lending institutions seem to be concerned about the wealth and income of potential borrowers (e.g., loan applications require such information), (C3) and (C4) may be empirically relevant.

Clearly (C1) is sufficient to prevent arbitrage, because along with (A1) it implies that the only allowable arbitrage portfolio for any investor is $\mathbf{x} = \mathbf{0}$. Of course, in this case the no arbitrage condition is rather empty, as it imposes no restrictions on asset expected returns. If $\mathbf{a} < \mathbf{0}$ then (C2) allows investors to hold at least small amounts of any arbitrage portfolio opportunities that exist without portfolio restrictions. However, like (C1) it is not useful for deriving conditions on arbitrage pricing.

The most interesting restriction is (C3). In the model here, where dividends may not be known with certainty, it has a slightly more awkward interpretation than in Litzenberger and Ramaswamy [15]: when $\mathbf{e} \neq \mathbf{0}$, it says that an investor's total interest payments cannot exceed the total dividend income she receives from risky assets when all risky assets realize a return of zero. This seems somewhat strange--it would perhaps be more natural to suppose that borrowing is restricted so that an investor's interest payments cannot exceed her dividend income when, say, all assets realize their expected return; i.e., $\sum_{j=1}^{J} x_j(d_j + E_j e_j) \ge -x_0 E_0$. However, in the special case we consider, our model points to a condition like (C3).

Suppose that $\mathbf{e} = e\mathbf{1}$ and consider the constraint (C5) $\mathbf{x}'(\mathbf{d} + \mathbf{y}) \ge \mathbf{0}$, where $\mathbf{y}$ is any vector in the linear span of $\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K$. When $\mathbf{y} = \mathbf{0}$ (C5) reduces to (C3). Also, given the restriction on $\mathbf{e}$ we can choose $\mathbf{y} = r\mathbf{e}$

where r is the zero-beta rate of return. Then (C5) becomes $\mathbf{x}'(\mathbf{d} + r\mathbf{e}) \geq 0$, which says that an investor's total dividend income must be nonegative when all assets realize the zero-beta return. Or, if asset 0 is riskless, then writing (C5) as $\sum_{j=1}^{J} x_j(d_j + re) \geq -x_0(d_0 + re)$, we can interpret it as saying that an investor's interest payments cannot exceed her dividend income on risky assets when all assets realize the riskless return. Note also that (C5) and (C3) can be considered as restrictions on either pretax or after-tax dividends, since all dividends (and interest) are taxed at the same rate for any investor--for example, $\sum_{j=1}^{J} x_j d_j \geq -x_0 d_0$ if and only if $\sum_{j=1}^{J} (1 - t^i)x_j d_j \geq -(1 - t^i)x_0 d_0$.

Since $\mathbf{e} = e\mathbf{1}$, (8.12) applies so for each individual i, $v_j^i = v^i$ for all j. Let n solve

$$\max_i \frac{s^i - t^i}{v^i}.$$

Also, Comment 8.3 applies. We can now state and prove

Proposition 8.2. Suppose $e_j = e$ for all j, $v^i > 0$ for all i, all allowable portfolios satisfy (C5), and $\mathbf{d} \not\in span(\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K)$. Then there exists no arbitrage if and only if

$$\mathbf{E} = r\mathbf{1} + q_1\boldsymbol{\beta}_1 + \cdots + q_K\boldsymbol{\beta}_K - p\mathbf{d} \tag{8.14}$$

for some $(r, q_1, \ldots, q_K)$ and some $p \geq \dfrac{s^n - t^n}{v^n}$.

Proof. Suppose $\mathbf{E}$ satisfies (8.14). Fix i and consider any allowable arbitrage portfolio $\mathbf{x}$ for i. Then from (8.5),

$$\tilde{R}^i_x = (s^i - t^i)\mathbf{x}'\mathbf{d} + \mathbf{x}'\mathbf{V}^i\mathbf{E} = (s^i - t^i)\mathbf{x}'\mathbf{d} + v^i\mathbf{x}'\mathbf{E}\,. \qquad (8.15)$$

Substituting (8.14) into (8.15) yields

$$\tilde{R}^i_x = (s^i - t^i)\mathbf{x}'\mathbf{d} + rv^i\mathbf{x}'\mathbf{1} + q_1 v^i\mathbf{x}'\boldsymbol{\beta}_1 + \cdots + q_K v^i\mathbf{x}'\boldsymbol{\beta}_K - pv^i\mathbf{x}'\mathbf{d}$$

$$= (s^i - t^i - pv^i)\mathbf{x}'\mathbf{d}$$

$$= (s^i - t^i - pv^i)\mathbf{x}'(\mathbf{d} + \mathbf{y})$$

using (A1), (A2) and the assumption that $\mathbf{y}$ is a linear combination of $\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K$. Now $v^i > 0$, so we can write

$$\tilde{R}^i_x = v^i\left[\frac{s^i - t^i}{v^i} - p\right]\mathbf{x}'\mathbf{d} \le 0$$

since $\dfrac{s^i - t^i}{v^i} \le \dfrac{s^n - t^n}{v^n} \le p$ by the definitions of n and p, and $\mathbf{x}'\mathbf{d} \ge 0$ by (C5). So, there is no arbitrage opportunity for i. Since i was chosen arbitrarily, there is no arbitrage.

Now suppose (8.14) does not hold. Then either (i) $\mathbf{E} = r\mathbf{1} + q_1\boldsymbol{\beta}_1 + \cdots + q_K\boldsymbol{\beta}_K - p\mathbf{d}$ for some $(q_1, \ldots, q_K)$ and $p < \dfrac{s^n - t^n}{v^n}$, or (ii) $\mathbf{E}$ is not in the linear span of $\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K, \mathbf{d}$. Suppose (i) holds. Project $\mathbf{E} + \left[\dfrac{s^n - t^n}{v^n}\right]\mathbf{d}$ onto $span(\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K)$. Let $\mathbf{F} = a_0\mathbf{1} + a_1\boldsymbol{\beta}_1 + \cdots + a_K\boldsymbol{\beta}_K$ be the projection, and let $\mathbf{x} = \mathbf{E} + \left[\dfrac{s^n - t^n}{v^n}\right]\mathbf{d} - \mathbf{F}$. Then $\mathbf{x}'\mathbf{1} = 0$ and $\mathbf{x}'\boldsymbol{\beta}_k = 0$ for all $k = 1, \ldots, K$, so $\mathbf{x}$ is an arbitrage portfolio. Also,

$$\mathbf{x}'(\mathbf{d} + \mathbf{y}) = \mathbf{x}'\mathbf{d}$$

$$= \left[\frac{s^n - t^n}{v^n} - p\right]^{-1} \mathbf{x}'[\mathbf{x} - (r - a_0)\mathbf{1} - (q_1 - a_1)\boldsymbol{\beta}_1 - \ldots - (q_K - a_K)\boldsymbol{\beta}_K]$$

$$= \left[\frac{s^n - t^n}{v^n} - p\right]^{-1} \mathbf{x}'\mathbf{x} \geq 0, \tag{8.16}$$

so $\mathbf{x}$ is allowable. Since $p \neq \frac{s^n - t^n}{v^n}$ and $\mathbf{d}$ is not a linear combination of $\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K$, $\mathbf{x} \neq \mathbf{0}$. Thus, by (8.16), $\mathbf{x}'\mathbf{d} > 0$. Choosing investor n and using (8.15) and (i), $\tilde{R}^n_{\mathbf{x}} = v^n\left[\frac{s^n - t^n}{v^n} - p\right]\mathbf{x}'\mathbf{d} > 0$, so $\mathbf{x}$ is an arbitrage opportunity for n, contradicting the assumption of no arbitrage. Thus, (i) cannot hold.

So, suppose (ii) holds, and project $\mathbf{E}$ onto $span(\mathbf{1}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K, \mathbf{d})$. Let $\mathbf{F} = a_0\mathbf{1} + a_1\boldsymbol{\beta}_1 + \cdots + a_K\boldsymbol{\beta}_K + b\mathbf{d}$ be the projection, and let $\mathbf{x} = \mathbf{E} - \mathbf{F}$. Then $\mathbf{x}'\mathbf{1} = 0$, $\mathbf{x}'\boldsymbol{\beta}_k = 0$ for all $k = 1,\ldots,K$ and $\mathbf{x}'\mathbf{d} = 0$, so $\mathbf{x}$ is an allowable arbitrage portfolio for any investor i. Also, $\mathbf{x} \neq \mathbf{0}$ by the assumption of (ii), so

$$\tilde{R}^i_{\mathbf{x}} = (s^i - t^i)\mathbf{x}'\mathbf{d} + v^i\mathbf{x}'\mathbf{E}$$

$$= (s^i - t^i)\mathbf{x}'\mathbf{d} + v^i\mathbf{x}'[\mathbf{x} + a_0\mathbf{1} + a_1\boldsymbol{\beta}_1 + \cdots + a_K\boldsymbol{\beta}_K + b\mathbf{d}]$$

$$= v^i\mathbf{x}'\mathbf{x} > 0\ ,$$

and $\mathbf{x}$ is an arbitrage opportunity for i, again contradicting the assumption of no arbitrage. Thus (8.14) must hold.

QED

There are several things to note about the pricing relation (8.14). Since

$v^i > 0$ for all i, $\frac{s^i - t^i}{v^i} > 0$ if and only if $s^i > t^i$.[4] Thus, if there is some investor i with $s^i > t^i$, $p > 0$ and the form of the asset pricing relation is different from that in the no-tax APT. Assets must be priced to prevent arbitrage opportunities for the investors with $s^i > t^i$. On the other hand, if $s^i \leq t^i$ for all i then $p = 0 \geq \frac{s^n - t^n}{v^n}$ so the pricing relation with taxes *could* have (but does not need to have) the same form as that in the world with no taxes and no portfolio restrictions.[5] For small corporations in the U.S. today, the tax rate on capital gains is higher than the rate on dividends, so the first case seems to apply.

8.3 A Note on Testing the Model

Econometric tests of the model presented here are as straightforward (or difficult, depending on one's view) as tests of the APT with no taxes. The basic results to test are Proposition 8.1 and Proposition 8.2. Assuming that $\mathbf{e} = e\mathbf{1}$, this means testing the linear relationships (8.10), (8.13) and (8.14). One powerful method of testing whether or not a group of data lie on a line is the "bilinear paradigm" used in Brown and Weinstein [9]. If (8.10) or (8.13) is rejected, this is evidence against the model with no restrictions on portfolios. Assuming that there exists some investor whose tax rate on capital gains exceeds her tax rate on dividends, we have two tests of the model with portfolio restrictions--that the asset returns lie on a line like (8.14), and that the sign of the coefficient on $\mathbf{d}$ is negative. Testing the model with taxes and portfolio restrictions against the "standard" APT (with no taxes or portfolio restrictions) is also quite simple. It involves comparing (8.10) versus (8.14), which can be done by a t-test on the coefficient of **d** in (8.14). If we cannot reject the hypothesis that the coefficient is zero, this is evidence in favor of our model with taxes and portfolio restrictions over the simpler APT.

9. Income Taxes and the APT with Idiosyncratic Risks

In this section we prove propositions analagous to those in the previous section for the APT with asset-specific risks. As in previous work on the APT (e.g., Ross [20], Huberman [13], Ingersoll [14]), our propositions are "limit approximation results" that bound the total deviation in asset expected returns from a given linear relation. In stating and proving our propositions we follow closely the paper by Huberman [13].

9.1 Assumptions and Definitions

We consider an infinite sequence of economies $J = 1,2,\ldots$ with an increasing number of assets. As in Huberman [13] and Ingersoll [14] we will consider both the general case in which the sets of assets in different economies need not bear any particular relation to each other, and the special case where the sets of assets are nested--i.e., where all assets in the Jth economy also appear in any economy L with $L > J$. We present the general case first. In the Jth economy there are J assets whose pretax returns are generated by a K-factor linear model of the form

$$\tilde{\mathbf{R}}^J = \mathbf{E}^J + \mathbf{B}^J\tilde{\boldsymbol{\gamma}}^J + \tilde{\boldsymbol{\varepsilon}}^J,$$

where $\tilde{\mathbf{R}}^J = (\tilde{R}_1^J, \ldots, \tilde{R}_J^J)'$ is the vector of (random) realized rates of return, $\mathbf{E}^J = (E_1^J, \ldots, E_J^J)'$ is the vector of expected rates of return, $\mathbf{B}^J$ is the $J \times K$ matrix of factor weights ("betas") whose (j,k)th element is β_{jk}^J, $\tilde{\boldsymbol{\gamma}}^J = (\tilde{\gamma}_1^J, \ldots, \tilde{\gamma}_K^J)'$ is the vector of (random) realized factor values, and $\tilde{\boldsymbol{\varepsilon}}^J = (\tilde{\varepsilon}_1^J, \ldots, \tilde{\varepsilon}_J^J)'$ is the vector of (random) realized asset-specific risk terms.

We assume that

$$E[\tilde{\gamma}_k^J] = 0 \quad \text{for all } k, \tag{9.1)(i}$$

$$E[\tilde{\varepsilon}_j^J] = 0 \quad \text{for all } j, \tag{9.1)(ii}$$

$$E[(\tilde{\varepsilon}_j^J)^2] \le \sigma^2 < \infty \quad \text{for all } j, \text{ and} \tag{9.1)(iii}$$

$$E[\tilde{\varepsilon}_j^J \tilde{\varepsilon}_l^J] = 0 \quad \text{for all } j \ne l\,. \tag{9.1)(iv}$$

These are the same assumptions as in Huberman [13]. Note that (9.1)(iii) says that the variances of asset-specific risks are uniformly bounded, and (9.1)(iv) says that asset-specific risks are uncorrelated between assets. It is straightforward to relax (9.1)(iv) as done in Ingersoll [14] and consider sets of assets with correlated idiosyncratic risks. We will write the kth column of $\mathbf{B}^J$ as $\boldsymbol{\beta}_k^J$. Also, we will let $\mathbf{1}^J$ be a vector of J ones, and denote the $J\times(K+1)$ matrix $[\mathbf{1}^J \mid \mathbf{B}^J]$ by $\hat{\mathbf{B}}^J$.

Each asset j's return is divided into dividends $\tilde{D}_j^J$ and capital gains $\tilde{G}_j^J$ according to

$$\tilde{D}_j^J = d_j^J + \tilde{R}_j^J e_j^J \quad \text{and}$$

$$\tilde{G}_j^J = \tilde{R}_j^J - \tilde{D}_j^J.$$

For simplicity assume that the set of investors is fixed. (There are various ways to weaken this, but they shed no light on the problem.) As before, each investor i is taxed on her dividend income at a rate $t^i \in [0,1)$ and on her capital gains at a rate $s^i \in [0,1)$. Denote i's after-tax rate of return on asset j by $\tilde{R}_j^{iJ}$. Then

$$\tilde{R}_j^{iJ} = (1 - t^i)(d_j^J + \tilde{R}_j^J e_j^J) + (1 - s^i)(\tilde{R}_j^J - d_j^J - \tilde{R}_j^J e_j^J)$$

$$= [(s^i - t^i)e_j^J + 1 - s^i]\tilde{R}_j^J + (s^i - t^i)d_j^J$$

$$= v_j^{iJ}\tilde{R}_j^J + (s^i - t^i)d_j^J \tag{9.2}$$

$$= v_j^{iJ}E_j^J + (s^i - t^i)d_j^J + \sum_{k=1}^{K} \beta_{jk}^J \tilde{\gamma}_k^J + v_j^{iJ}\tilde{\varepsilon}_j^J .$$

From (9.2) it is evident that v_j^{iJ} is the marginal after-tax rate of return to investor i on asset j, when the after-tax rate of return is viewed as a function of the asset's pretax return. As noted in section 8.1, under reasonable assumptions $v_j^{iJ} \in (0,1]$ for all i and j. Here we assume that

$$0 < v^l \le v_j^{iJ} \le v^u < \infty \quad \text{for all } i \text{ and } j;$$

i.e., the v_j^{iJ} are uniformly bounded away from zero and infinity. Letting $\mathbf{V}^{iJ}$ be the $J \times J$ diagonal matrix with (j,j)th element v_j^{iJ}, we can write the vector of after-tax returns for investor i, $\tilde{\mathbf{R}}^{iJ} = (\tilde{R}_1^{iJ}, \ldots, \tilde{R}_J^{iJ})'$, as

$$\tilde{\mathbf{R}}^{iJ} = \mathbf{V}^{iJ}\mathbf{E}^J + (s^i - t^i)\mathbf{d}^J + \mathbf{V}^{iJ}\mathbf{B}^J\tilde{\boldsymbol{\gamma}}^J + \mathbf{V}^{iJ}\tilde{\boldsymbol{\varepsilon}}^J .$$

If $\mathbf{x}^J = (x_1^J, \ldots, x_J^J)'$ is any portfolio of assets, the after-tax rate of return of $\mathbf{x}^J$ to investor i is

$$\tilde{R}^{iJ}(\mathbf{x}^J) = \mathbf{x}^{J\prime}\tilde{\mathbf{R}}^{iJ}$$

$$= \mathbf{x}^{J\prime}\mathbf{V}^{iJ}\mathbf{E}^J + (s^i - t^i)\mathbf{x}^{J\prime}\mathbf{d}^J + \mathbf{x}^{J\prime}\mathbf{V}^{iJ}\mathbf{B}^J\tilde{\boldsymbol{\gamma}}^J + \mathbf{x}^{J\prime}\mathbf{V}^{iJ}\tilde{\boldsymbol{\varepsilon}}^J. \tag{9.3}$$

We must modify our definitions of arbitrage portfolios and arbitrage opportunities slightly from those used in Section 8. Let an ***arbitrage portfolio*** in the Jth economy be any portfolio $\mathbf{x}^J$ satisfying $\mathbf{x}^{J\prime}\mathbf{1}^J = 0$, i.e., any portfolio that uses no wealth. Let an ***arbitrage opportunity for i*** be any sequence of portfolios $\{\mathbf{x}^J\}_{J=1}^{\infty}$, where $\mathbf{x}^J$ is an arbitrage portfolio in the Jth economy, such that there exists a subsequence $\{\mathbf{x}^L\} \subseteq \{\mathbf{x}^J\}_{J=1}^{\infty}$ with

$$\lim_{L\to\infty} E[\tilde{R}^{iL}(\mathbf{x}^L)] = +\infty \quad \text{and}$$

$$\lim_{L\to\infty} var[\tilde{R}^{iL}(\mathbf{x}^L)] = 0 .$$

There is *no arbitrage* if there are no arbitrage opportunities for any investor.

9.2 Perfect Markets Again

Here we prove results that parallel the results in Section 8.1.

Proposition 9.1. If there is no arbitrage then for each investor i there exist $A^i < \infty$ and a sequence $\{(q_0^{iJ}, \ldots, q_K^{iJ})\}_{J=1}^{\infty}$ such that for all J

$$\sum_{j=1}^{J} \left[v_j^{iJ} E_j^J + (s^i - t^i) d_j^J - q_0^{iJ} - v_j^{iJ} \sum_{k=1}^{K} q_k^{iJ} \beta_{jk}^J \right]^2 \le A^i. \tag{9.4}$$

Proof. Fix i and for each J project $\mathbf{V}^{iJ}\mathbf{E}^J + (s^i - t^i)\mathbf{d}^J$ orthogonally onto $span(1^J, \mathbf{V}^{iJ}\beta_1^J, \ldots, \mathbf{V}^{iJ}\beta_K^J)$. Let $\varphi^{iJ} = q_0^{iJ} 1^J + \sum_{k=1}^{K} q_k^{iJ} \beta_k^J$ be the projection, and let

$$\mathbf{x}^{iJ} = \mathbf{V}^{iJ}\mathbf{E}^J + (s^i - t^i)\mathbf{d}^J - \varphi^{iJ} \tag{9.5}$$

Then

$$\mathbf{x}^{iJ\prime} 1^J = 0 , \quad \text{and} \tag{9.6}$$

$$\mathbf{x}^{iJ\prime} \mathbf{V}^{iJ} \beta_k^J = 0 \quad \text{for all } k. \tag{9.7}$$

Consider the sequence of portfolios $\{\alpha^{iJ}\mathbf{x}^{iJ}\}_{J=1}^{\infty}$ where

$$\alpha^{iJ} = (\mathbf{x}^{iJ\prime}\mathbf{x}^{iJ})^{-3/4} \quad \text{for all } J.$$

By (9.6), $\alpha^{iJ}\mathbf{x}^{iJ\prime}\mathbf{x}^J = 0$, so $\alpha^{iJ}\mathbf{x}^{iJ}$ is an arbitrage portfolio in the Jth

economy. By (9.3), $\tilde{R}^{iJ}(\alpha^{iJ}\mathbf{x}^{iJ}) =$
$\alpha^{iJ}[\mathbf{x}^{iJ\prime}\mathbf{V}^{iJ}\mathbf{E}^J + (s^i - t^i)\mathbf{x}^{iJ\prime}\mathbf{d}^J + \mathbf{x}^{iJ\prime}\mathbf{V}^{iJ}\mathbf{B}^J\tilde{\gamma}^J + \mathbf{x}^{iJ\prime}\mathbf{V}^{iJ}\tilde{\varepsilon}^J]$. Substituting for $\mathbf{V}^{iJ}\mathbf{E}^J$ from (9.5) and using (9.6) and (9.7), $\tilde{R}^{iJ}(\alpha^{iJ}\mathbf{x}^{iJ}) = \alpha^{iJ}\mathbf{x}^{iJ\prime}\mathbf{x}^{iJ} + \alpha^{iJ}\mathbf{x}^{iJ\prime}\mathbf{V}^{iJ}\tilde{\varepsilon}^J$, so

$$E[\tilde{R}^{iJ}(\alpha^{iJ}\mathbf{x}^{iJ})] = \alpha^{iJ}\mathbf{x}^{iJ\prime}\mathbf{x}^{iJ} = (\mathbf{x}^{iJ\prime}\mathbf{x}^{iJ})^{1/4}, \quad \text{and}$$

$$\begin{aligned} var[\tilde{R}^{iJ}(\alpha^{iJ}\mathbf{x}^{iJ})] &= (\alpha^{iJ})^2\mathbf{x}^{iJ\prime}\mathbf{V}^{iJ}(E[\tilde{\varepsilon}^J\tilde{\varepsilon}^{J\prime}])\mathbf{V}^{iJ\prime}\mathbf{x}^{iJ} \\ &\leq (\alpha^{iJ})^2(v^u)^2\sigma^2\mathbf{x}^{iJ\prime}\mathbf{x}^{iJ} \\ &= (v^u)^2\sigma^2(\mathbf{x}^{iJ\prime}\mathbf{x}^{iJ})^{-1/2}. \end{aligned}$$

By the definition of $\mathbf{x}^{iJ}$, $\mathbf{x}^{iJ\prime}\mathbf{x}^{iJ} = \sum_{j=1}^{J}\left[v_j^{iJ}E_j^J + (s^i - t^i)d_j^J - q_0^{iJ} - v_j^{iJ}\sum_{k=1}^{K} q_k^{iJ}\beta_{jk}^J\right]^2$.

Suppose (9.4) does not hold. Then for any sequence $\{(q_0^{iJ}, \ldots, q_K^{iJ})\}_{J=1}^{\infty}$ there exists a subsequence $\{(q_0^{iL}, \ldots, q_K^{iL})\}$ such that

$$\lim_{L\to\infty}\sum_{j=1}^{J}\left[v_j^{iL}E_j^L + (s^i - t^i)d_j^L - q_0^{iL} - v_j^{iL}\sum_{k=1}^{K} q_k^{iL}\beta_{jk}^L\right]^2 = +\infty; \quad \text{i.e.,}$$

$$\lim_{L\to\infty}\mathbf{x}^{iL\prime}\mathbf{x}^{iL} = +\infty .$$

But then $\lim_{L\to\infty} E[\tilde{R}^{iL}(\alpha^{iL}\mathbf{x}^{iL})] = +\infty$ and $\lim_{L\to\infty} var[\tilde{R}^{iL}(\alpha^{iL}\mathbf{x}^{iL})] = 0$, so $\{\alpha^{iJ}\mathbf{x}^{iJ}\}_{J=1}^{\infty}$ is an arbitrage portfolio for i, contradicting the assumption of no arbitrage. Thus (9.4) must hold for all i.

QED

Proposition 9.1 is analogous to Theorem 1 of Huberman [13] and the proof is nearly identical.

Let a *stationary* sequence of economies be a sequence where the sets of assets are nested, so that we may write $E_j^J = E_j$, $\beta_{jk}^J = \beta_{jk}$, $d_j^J = d_j$, $e_j^J = e_j$, and $v_j^{iJ} = v_i^j$ for all $j \leq J$. (Simply order the assets so that asset 1 is the first asset in all economies, asset 2 is the second asset in all economies with $J \geq 2$, etc.) Then $\tilde{R}_j^J = Ej + \sum_{k=1}^{K} \beta_{jk}\tilde{\gamma}_k^J + \tilde{\varepsilon}_j^J$ for all $j \leq J$, for all J.

Proposition 9.2. Given a stationary sequence of economies, for each investor i there exist $q_0^i, \ldots, q_K^i$ such that

$$\sum_{j=1}^{\infty} \left[v_j^i Ej + (s^i - t^i)d_j - q_0^i - v_j^i \sum_{k=1}^{K} q_k^i \beta_{jk} \right]^2 < \infty. \qquad (9.8)$$

Proof. Fix i. For any matrix B let $r(B)$ be its rank. Then for all $J \geq 1$, $1 \leq r(\mathbf{V}^{iJ}\hat{\mathbf{B}}^J) \leq r(V^{i,J+1}\hat{\mathbf{B}}^{J+1}) \leq K+1$, so there exists L such that $r(\mathbf{V}^{iJ}\hat{\mathbf{B}}^J) = r(\mathbf{V}^{iL}\hat{\mathbf{B}}^L)$ for all $J \geq L$. Permute the columns of $\hat{\mathbf{B}}^J$ so that the last $K+1-r(\mathbf{V}^{iJ}\hat{\mathbf{B}}^J)$ columns are linear combinations of the first $r(\mathbf{V}^{iJ}\hat{\mathbf{B}}^J)$ columns. Let

$$H^{iJ} = \{(q_0, \ldots, q_K) \text{ s.t. } \sum_{j=1}^{J} \left[v_j^i Ej + (s^i - t^i)d_j - q_0 - v_j^i \sum_{k=1}^{K} q_k \beta_{jk} \right]^2 \leq A^i,$$
$$\text{and } q_k = 0 \text{ for } r(\mathbf{V}^{iJ}\hat{\mathbf{B}}^J) < k \leq K+1 \},$$

where A^i is the upper bound whose existence is guaranteed by Proposition 9.1. Then H^{iJ} is nonempty for all J (by Proposition 9.1) and compact for $J \geq L$, and $H^{i,J+1} \subseteq H^{iJ}$ for all J. Thus $\bigcap_{J=1}^{\infty} H^{iJ} \neq \phi$. And any element $(q_0^i, \ldots, q_K^i) \in \bigcap_{J=1}^{\infty} H^{iJ}$ satisfies (9.8).

QED

Proposition 9.2 is analogous to Theorem 2 of Huberman [13], and the proof is nearly identical.

If we pick i such that $s^i = t^i$ then the results in Propositions 9.1 and

9.2 reduce exactly to Theorems 1 and 2 of Huberman [13].

Corollary 9.1. If there is at least one investor i with $s^i = t^i$ and no arbitrage then there exist $A < \infty$ and $\{(q_0^J, \ldots, q_K^J)\}_{J=1}^{\infty}$ such that

$$\sum_{j=1}^{J} [E_j^J - q_0^J - \sum_{k=1}^{K} q_k^J \beta_{jk}^J]^2 \leq A \quad \text{for all } J. \tag{9.9}$$

Proof. Pick investor i and note that $v_j^{iJ} = (1 - s^i)$ for all $j \leq J$, for all J. Applying Proposition 9.1, there exist $A^i < \infty$ and $\{(q_0^{iJ}, \ldots, q_K^{iJ})\}_{J=1}^{\infty}$ such that

$$\sum_{j=1}^{J} \left[(1 - s^i)E_j^J - q_0^{iJ} - (1 - s^i)\sum_{k=1}^{K} q_k^{iJ}\beta_{jk}^J\right]^2 \leq A^i \quad \text{for all } J.$$

Dividing by $1 - s^i$ and setting $q_0^J = \dfrac{q_0^{iJ}}{1 - s^i}$, $q_k^J = q_k^{iJ}$ for all k, and $A = \dfrac{A^i}{(1 - s^i)^2}$, this reduces to (9.9).

QED

Corollary 9.2. Given a stationary sequence of economies, if there is at least one investor i with $s^i = t^i$ and no arbitrage then there exists $q_0, \ldots, q_K$ such that

$$\sum_{j=1}^{\infty} \left[Ej - q_0 - \sum_{k=1}^{K} q_k \beta_{jk}\right]^2 < \infty. \tag{9.10}$$

Proof. Pick investor i and apply Proposition 9.2; then proceed as in the proof of Corollary 9.2.

As usual, q_0 is the rate of return on the riskless asset if it exists, and the rate of return on any zero-beta portfolio. Note that Corollaries 9.1 and 9.2 do not predict a unique vector or sequence of vectors. Thus, if $\{(q_0^J, \ldots, q_K^J)\}_{J=1}^{\infty}$ is any sequence that satisfies (9.9), we call the sequence

$\{q_0^J\}_{J=1}^{\infty}$ a *sequence of zero-beta parameters*, and if $(q_0, \ldots, q_K)$ is any vector satisfying (9.10), we call q_0 a *stationary zero-beta parameter*.

Proposition 9.3. If there is at least one investor i such that $s^i = t^i$ and there is no arbitrage, then for any investor m with $s^m \neq t^m$ there exist $C^m < \infty$ and $\{(p_0^{mJ}, \ldots, p_K^{mJ})\}_{J=1}^{\infty}$ such that for all J

$$\sum_{j=1}^{J}\left[(r^J e_j^J + d_j^J) - p_0^{mJ} - v_j^{mJ}\sum_{k=1}^{K} p_k^{mJ}\beta_{jk}^J\right]^2 \leq C^m, \tag{9.11}$$

where $\{r^J\}_{J=1}^{\infty}$ is any sequence of zero-beta parameters.

Proof. Let A, $\{(r^J, q_1^J, \ldots, q_K^J)\}_{J=1}^{\infty}$ satisfy (9.9) and let A^m, $\{(q_0^{mJ}, \ldots, q_K^{mJ})\}_{J=1}^{\infty}$ satisfy (9.4). These sequences exist by Corollary 9.1 and Proposition 9.1, respectively. Let $\mathbf{x}^J = \mathbf{E}^J - r^J\mathbf{1}^J - \sum_{k=1}^{K} q_k^J\boldsymbol{\beta}_k^J$ and let $\mathbf{x}^{mJ} = \mathbf{V}^{mJ}\mathbf{E}^J + (s^m - t^m)\mathbf{d}^J - q_0^{mJ}\mathbf{1}^J - \sum_{k=1}^{K} q_k^{mJ}\mathbf{V}^{mJ}\boldsymbol{\beta}_k^J$. Solving the first of these equations for $\mathbf{E}^J$ and substituting into the second yields

$$\mathbf{x}^{mJ} - \mathbf{V}^{mJ}\mathbf{x}^J =$$
$$(s^m - t^m)(r^J\mathbf{e}^J + \mathbf{d}^J) - (q_0^{mJ} - 1 + s^m)\mathbf{1}^J - \sum_{k=1}^{K}(q_k^{mJ} - q_k^J)\mathbf{V}^{mJ}\boldsymbol{\beta}_k^J, \text{ so}$$

$$(\mathbf{x}^{mJ} - \mathbf{V}^{mJ}\mathbf{x}^J)'(\mathbf{x}^{mJ} - \mathbf{V}^{mJ}\mathbf{x}^J) =$$
$$\sum_{j=1}^{J}\left[(s^m - t^m)(r^J e_j^J + d_j^J) - (q_0^{mJ} - 1 + s^m) - \sum_{k=1}^{K}(q_k^{mJ} - q_k^J)\mathbf{V}^{mJ}{}_j\beta_{jk}^J\right]^2.$$

By the Minkowski inequality, $[(\mathbf{x}^{mJ} - \mathbf{V}^{mJ}\mathbf{x}^J)'(\mathbf{x}^{mJ} - \mathbf{V}^{mJ}\mathbf{x}^J)]^{1/2} \leq [\mathbf{x}^{mJ\prime}\mathbf{x}^{mJ}]^{1/2} + [(\mathbf{V}^{mJ}\mathbf{x}^J)'(\mathbf{V}^{mJ}\mathbf{x}^J)]^{1/2}$. By the definition of $\mathbf{x}^{mJ}$, $\mathbf{x}^{mJ\prime}\mathbf{x}^{mJ} \leq A^m$ for all J. Also, $(\mathbf{V}^{mJ}\mathbf{x}^J)'(\mathbf{V}^{mJ}\mathbf{x}^J) = \mathbf{x}^{J\prime}\mathbf{V}^{mJ\prime}\mathbf{V}^{mJ}\mathbf{x}^J \leq (v^u)^2\mathbf{x}^{J\prime}\mathbf{x}^J$, so by the definition of $\mathbf{x}^J$,

$(\mathbf{V}^{mJ}\mathbf{x}^J)'(\mathbf{V}^{mJ}\mathbf{x}^J) \le (v^u)^2 A$ for all J. Thus,

$$\sum_{j=1}^{J}\left[(s^m - t^m)(r^J e_j^J + d_j^J) - (q_0^{mJ} - 1 + s^m) - \sum_{k=1}^{K}(q_k^{mJ} - q_k^J)\mathbf{V}^{mJ}{}_j\beta_{jk}^J\right]^2$$

$$\le [(A^m)^{1/2} + v^u A^{1/2}]^2.$$

Letting $p_0^{mJ} = \dfrac{q_0^{mJ} - 1 + s^m}{s^m - t^m}$, $p_k^{mJ} = \dfrac{q_k^{mJ} - q_k^J}{s^m - t^m}$ for all k, and

$C^m = \left[\dfrac{(A^m)^{1/2} + v^u A^{1/2}}{s^m - t^m}\right]^2$, this reduces to (9.11).

QED

Corollary 9.3. Given a stationary sequence of economies, if there is at least one investor i such that $s^i = t^i$ and there is no arbitrage, then for any investor m with $s^m \ne t^m$ there exists $(p_0^m, \ldots, p_K^m)$ such that

$$\sum_{j=1}^{J}\left[(re_j + d_j) - p_0^m - v_j^m \sum_{k=1}^{K} p_k^m \beta_{jk}\right]^2 < \infty$$

where r is any stationary zero-beta parameter.

Proof. Similar to that of Proposition 9.3, using the vectors whose existence is guaranteed by Proposition 9.2 and Corollary 9.2.

Proposition 9.3 and Corollary 9.3 are analogous to Proposition 8.1, asserting that a necessary condition for no arbitrage in the absence of portfolio restrictions is that $r^J\mathbf{e}^J + \mathbf{d}^J$ is approximately in the linear span of $(\mathbf{1}^J, \mathbf{V}^{mJ}\beta_1^J, \ldots, \mathbf{V}^{mJ}\beta_K^J)$ for all investors m with $s^m \ne t^m$. As argued in Section 8.1, this is a very strong condition, not likely to be met in the real world.

9.3 Portfolio Restrictions

Here we consider the short sales ("borrowing") constraint used in Section 8.2, namely that any allowable portfolio $\mathbf{x}^J$ in the Jth economy must satisfy $\mathbf{x}^{J\prime}\mathbf{d}^J \geq 0$. We prove a proposition analogous to Proposition 8.2.

Suppose $\mathbf{e}^J = e^J \mathbf{1}^J$ for all J; i.e., all asset dividend lines have the same slope. Then $v_j^{iJ} = (s^i - t^i)e^J + 1 - s^i = v^{iJ}$ for all $j \leq J$ for all J, for all i, and (9.11) becomes

$$\sum_{j=1}^{J} \left[d_j^J - (p_0^{mJ} - r^J e^J) - \sum_{k=1}^{K} \mathbf{V}^{mJ} p_k^{mJ} \beta_{jk}^J \right]^2 \leq C^m,$$

or letting $p_0^J = p_0^{mJ} - r^J e^J$, and $p_k^J = \mathbf{V}^{mJ} p_k^{mJ}$ for all k,

$$\sum_{j=1}^{J} \left[d_j^J - p_0^J - \sum_{k=1}^{K} p_k^J \beta_{jk}^J \right]^2 \leq C \quad \text{for all } J. \tag{9.12}$$

That is, when $\mathbf{e}^J = e^J \mathbf{1}^J$ for all J, Proposition 9.3 asserts the existence of $\{(p_0^J, \ldots, p_K^J)\}_{J=1}^{\infty}$ such that (9.12) holds.

Let n solve $\max\limits_i \dfrac{s^i - t^i}{v^{iJ}}$. Note that, since $\dfrac{s^i - t^i}{v^{iJ}} \geq \dfrac{s^m - t^m}{v^{mJ}}$ if and only if $\dfrac{s^i - t^i}{1 - s^i} \geq \dfrac{s^m - t^m}{1 - s^m}$, n is independent of J.

For a nonstationary sequence of economies, the result we can prove is not very useful. However, we state and prove it for completeness, and use it to prove the more interesting result for stationary sequences of economies.

Lemma 9.1. If $\mathbf{e}^J = e^J \mathbf{1}^J$ for all J, and any allowable portfolio $\mathbf{x}^J$ in the Jth economy satisfies $\mathbf{x}^{J\prime}\mathbf{d}^J \geq 0$, and (9.12) is not satisfied, then there exist

$A < \infty$ and $\{(q_0^J, \ldots, q_K^J, h^J)\}_{J=1}^{\infty}$ such that

$$\sum_{j=1}^{J} \left[E_j^J + h^J d_j^J - q_0^J - \sum_{k=1}^{K} q_k^J \beta_{jk}^J \right]^2 \leq A \quad \text{for all } J. \tag{9.13}$$

Furthermore, $\{h^J\}_{J=1}^{\infty}$ must satisfy

$$\limsup_{J \to \infty} v^{nJ} h^J \geq s^n - t^n. \tag{9.14}$$

Proof. The proof that (9.13) holds is similar to the proof of Proposition 9.1. For each J, project $\mathbf{E}^J$ onto $span(\mathbf{1}^J, \boldsymbol{\beta}_1^J, \ldots, \boldsymbol{\beta}_K^J, \mathbf{d}^J)$, let $\boldsymbol{\varphi}^J = q_0^J \mathbf{1}^J + \sum_{k=1}^{K} q_k^J \boldsymbol{\beta}_k^J - h^J \mathbf{d}^J$ be the projection, and let $\mathbf{x}^J = \mathbf{E}^J - \boldsymbol{\varphi}^J$. Then $\mathbf{x}^{J\prime} \mathbf{1}^J = 0$, $\mathbf{x}^{J\prime} \boldsymbol{\beta}_k^J = 0$ for all k, and $\mathbf{x}^{J\prime} \mathbf{d}^J = 0$. If (9.13) does not hold, then there exists a subsequence $\{\mathbf{x}^L\} \subset \{\mathbf{x}^J\}_{J=1}^{\infty}$ such that $\lim_{L \to \infty} \mathbf{x}^{L\prime} \mathbf{x}^L = +\infty$. For any scalar α^J, $\alpha^J \mathbf{x}^{J\prime} \mathbf{1}^J = 0$ and $\alpha^J \mathbf{x}^{J\prime} \mathbf{d}^J = 0$, so $\alpha^J \mathbf{x}^J$ is an allowable arbitrage portfolio in the Jth economy. By (9.3), for any investor i, $\tilde{R}^{iJ}(\alpha^J \mathbf{x}^J) = \alpha^J \mathbf{x}^{J\prime} \mathbf{x}^J + \alpha^J \mathbf{x}^{J\prime} \tilde{\boldsymbol{\varepsilon}}^J$, so choosing $\alpha^J = (\mathbf{x}^{J\prime} \mathbf{x}^J)^{-3/4}$, we have $E[\tilde{R}^{iJ}(\alpha^J \mathbf{x}^J)] = \alpha^J \mathbf{x}^{J\prime} \mathbf{x}^J = (\mathbf{x}^{J\prime} \mathbf{x}^J)^{1/4}$ and $var[\tilde{R}^{iJ}(\alpha^J \mathbf{x}^J)] \leq \sigma^2 (\alpha^J)^2 \mathbf{x}^{J\prime} \mathbf{x}^J = \sigma^2 (\mathbf{x}^{J\prime} \mathbf{x}^J)^{-1/2}$. So, if (9.13) does not hold then $\lim_{L \to \infty} E[\tilde{R}^{iL}(\alpha^L \mathbf{x}^L)] = +\infty$ and $\lim_{L \to \infty} var[\tilde{R}^{iL}(\alpha^L \mathbf{x}^L)] = 0$. But then $\{\mathbf{x}^J\}_{J=1}^{\infty}$ is an arbitrage opportunity for i, contradicting the assumption of no arbitrage. Thus (9.13) holds.

To see that (9.14) holds, for each J project $\mathbf{d}^J$ onto $span(\mathbf{1}^J, \boldsymbol{\beta}_1^J, \ldots, \boldsymbol{\beta}_K^J)$, let $\boldsymbol{\varphi}^J = p_0^J \mathbf{1}^J + \sum_{k=1}^{K} p_k^J \boldsymbol{\beta}_k^J$ be the projection, and let $\mathbf{x}^J = \mathbf{d}^J - \boldsymbol{\varphi}^J$. Again, consider the sequence of portfolios $\{\alpha^J \mathbf{x}^J\}_{J=1}^{\infty}$, where $\alpha^J = (\mathbf{x}^{J\prime} \mathbf{x}^J)^{-3/4}$ for all J. Then $\alpha^J \mathbf{x}^{J\prime} \mathbf{1}^J = 0$ (so $\alpha^J \mathbf{x}^J$ is an arbitrage portfolio in the Jth economy), $\alpha^J \mathbf{x}^{J\prime} \boldsymbol{\beta}_k^J = 0$ for all k, and

$\alpha^J \mathbf{x}^{J\prime}\mathbf{d}^J = \alpha^J \mathbf{x}^{J\prime}\mathbf{x}^J + \alpha^J \mathbf{x}^{J\prime}\varphi^J = \alpha^J \mathbf{x}^{J\prime}\mathbf{x}^J \geq 0$ (so $\alpha^J \mathbf{x}^J$ is an allowable portfolio in the Jth economy). By (9.3) the rate of return to investor n on $\alpha^J \mathbf{x}^J$ is

$$\tilde{R}^{nJ}(\alpha^J \mathbf{x}^J) = \alpha^J\,[v^{nJ}\mathbf{x}^{J\prime}\mathbf{E}^J + (s^n - t^n)\mathbf{x}^{J\prime}\mathbf{d}^J + v^{nJ}\mathbf{x}^{J\prime}\mathbf{B}^J\tilde{\gamma}^J + v^{nJ}\mathbf{x}^{J\prime}\tilde{\varepsilon}^J]$$

$$= \alpha^J\,[v^{nJ}\mathbf{x}^{J\prime}\mathbf{E}^J + (s^n - t^n)\mathbf{x}^{J\prime}\mathbf{d}^J + v^{nJ}\mathbf{x}^{J\prime}\tilde{\varepsilon}^J]. \tag{9.15}$$

Let $\mathbf{c}^J = \mathbf{E}^J + h^J\mathbf{d}^J - q_0^J 1^J - \sum_{k=1}^{K} q_k^J \beta_k^J$ for all J. Substituting for $\mathbf{E}^J$ in (9.15) and simplifying, $\tilde{R}^{nJ}(\alpha^J \mathbf{x}^J) = \alpha^J\,[v^{nJ}\mathbf{x}^{J\prime}\mathbf{c}^J + (s^n - t^n - v^{nJ}h^J)\mathbf{x}^{J\prime}\mathbf{x}^J + v^{nJ}\mathbf{x}^{J\prime}\tilde{\varepsilon}^J]$, so

$$E[\tilde{R}^{nJ}(\alpha^J \mathbf{x}^J)] = \alpha^J\,[v^{nJ}\mathbf{x}^{J\prime}\mathbf{c}^J + (s^n - t^n - v^{nJ}h^J)\mathbf{x}^{J\prime}\mathbf{x}^J]$$

$$= v^{nJ}(\mathbf{x}^{J\prime}\mathbf{x}^J)^{-3/4}\mathbf{x}^{J\prime}\mathbf{c}^J + (s^n - t^n - v^{nJ}h^J)(\mathbf{x}^{J\prime}\mathbf{x}^J)^{1/4}, \text{ and}$$

$$var[\tilde{R}^{nJ}(\alpha^J \mathbf{x}^J)] \leq \sigma^2 (v^u)^2 \mathbf{x}^{J\prime}\mathbf{x}^J$$

$$= \sigma^2 (v^u)^2 (\mathbf{x}^{J\prime}\mathbf{x}^J)^{-1/2}. \tag{9.16}$$

By the Cauchy-Schwarz inequality, $|\,\mathbf{x}^{J\prime}\mathbf{c}^J\,| \leq (\mathbf{x}^{J\prime}\mathbf{x}^J)^{1/2}(\mathbf{c}^{J\prime}\mathbf{c}^J)^{1/2}$. By (9.13) and the definition of $\mathbf{c}^J$, $(\mathbf{c}^{J\prime}\mathbf{c}^J) \leq A$ for all J, so $\mathbf{x}^{J\prime}\mathbf{c}^J \geq -A^{1/2}(\mathbf{x}^{J\prime}\mathbf{x}^J)^{1/2}$ for all J. Then since $v^{nJ} \leq v^u$ for all J,

$$E[\tilde{R}^{nJ}(\alpha^J \mathbf{x}^J)] \geq -v^u A^{1/2}(\mathbf{x}^{J\prime}\mathbf{x}^J)^{-1/4} + (s^n - t^n - v^{nJ}h^J)(\mathbf{x}^{J\prime}\mathbf{x}^J)^{1/4}.$$

If (9.12) is not satisfied then there exists a subsequence $\{\mathbf{x}^L\} \subset \{\mathbf{x}^J\}_{J=1}^{\infty}$ such that $\lim_{L\to\infty} \mathbf{x}^{L\prime}\mathbf{x}^L = +\infty$. Then by (9.16), for this subseqence $\lim_{L\to\infty} var[\tilde{R}^{nL}(\alpha^L \mathbf{x}^L)] = 0$. If $\limsup_{L\to\infty} (s^n - t^n - v^{nL}h^L) > 0$, then there exists a subsequence $\{\mathbf{x}^M\} \subset \{\mathbf{x}^L\}$ such that

$$\lim_{M\to\infty} E[\tilde{R}^{nM}(\alpha^M \mathbf{x}^M)] \geq v^u A^{1/2} \lim_{M\to\infty}(\mathbf{x}^{M\prime}\mathbf{x}^M)^{-1/4} + [\lim_{M\to\infty}(s^n - t^n - v^{nM}h^M)]\,[\lim_{M\to\infty}(\mathbf{x}^{M\prime}\mathbf{x}^M)^{1/2}] = +\infty.$$

Then $\{\mathbf{x}^J\}_{J=1}^{\infty}$ would be an arbitrage opportunity for n, contradicting the assumption of no arbitrage. Thus, $\limsup_{L\to\infty} (s^n - t^n - v^{nL}h^L) \leq 0$, or $\liminf_{L\to\infty} v^{nL}h^L \geq s^n - t^n$. And $\limsup_{J\to\infty} v^{nJ}h^J \geq \liminf_{L\to\infty} v^{nL}h^L$, so (9.14) holds.

QED

In a stationary sequence of economies we have a much more elegant and useful result. In a stationary sequence $e^J = e$ for all J, so $v^{iJ} = (s^i - t^i)e^J + 1 - s^i = (s^i - t^i)e + 1 - s^i = v^i$ for all J.

Proposition 9.4. Consider a stationary sequence of economies. If $e_j = e$ for all j and any allowable portfolio $\mathbf{x}^J$ in the Jth economy satisfies $\mathbf{x}^{J\prime}\mathbf{c}^J \geq 0$, and (9.12) is not satisfied, then there exists $q_0, \ldots, q_K, h$ such that

$$\sum_{j=1}^{J} \left[E_j + hd_j - q_0 - \sum_{k=1}^{K} q_k \beta_{jk}\right]^2 < \infty . \tag{9.17}$$

Furthermore, h must satisfy

$$h \geq \frac{s^n - t^n}{v^n}. \tag{9.18}$$

Proof. The proof that (9.17) holds is similar to the proof of Proposition 9.2: for each J rearrange the columns of the matrix $[\hat{\mathbf{B}}^J \mid \mathbf{d}^J]$ appropriately and define

$$H^J = \{(q_0,\ldots,q_K,q_{K+1}) \ \ s.t. \ \sum_{j=1}^{J}\left[E_j + q_{K+1}d_j - q_0 - \sum_{k=1}^{K} q_k\beta_{jk}\right]^2 < A,$$

$$\text{and} \ \ q_k = 0 \text{ for } r(\hat{\mathbf{B}}^J) \le k \le K+2 \ \},$$

where A is the upper bound in (9.13) of Lemma 9.1. Then $\bigcap_{J=1}^{\infty} H^J \neq \phi$, so (9.17) holds, where $h = q_{K+1}$. To see that (9.18) holds, simply note that $\limsup_{J\to\infty} v^n h = v^n h$, and thus by (9.14) of Lemma 9.1, $v^n h \ge s^n - t^n$.

QED

Notes for Part II

1. Some tax features, like investment tax credits for some subset of assets, might alter the conditions under which a portfolio uses no wealth.

2. As in much of the finance literature, we refer to both "prices" and "rates of return." The relationship is as follows: consider an asset that yields a certain dollar profit (return) next period of x; then letting r denote the rate of return and p the price, $r = x/p$. Clearly, one knows the price if and only if one knows the rate of return.

3. Many economists argue that (for efficiency reasons) the preferential treatment of capital gains should be eliminated, and capital gains taxed at the same rate as dividends.

4. The function $\frac{s^i - t^i}{vi}$ does not have a simple economic interpretation. Except for its sign, it is not directly related to the gap between an investor's tax rates, $s^i - t^i$. However, if we assume as in Long [16] that all combinations of tax rates (s,t) inside some box $[0,T]\times[0,T]$ are possible, then $\frac{s^i - t^i}{vi}$ is maximized at $(s_n, t_n) = (T, 0)$.

5. Litzenberger and Ramaswamy [15] assume that $s^i = 0$ for all i, and thus that $s^i \leq t^i$ for all i.

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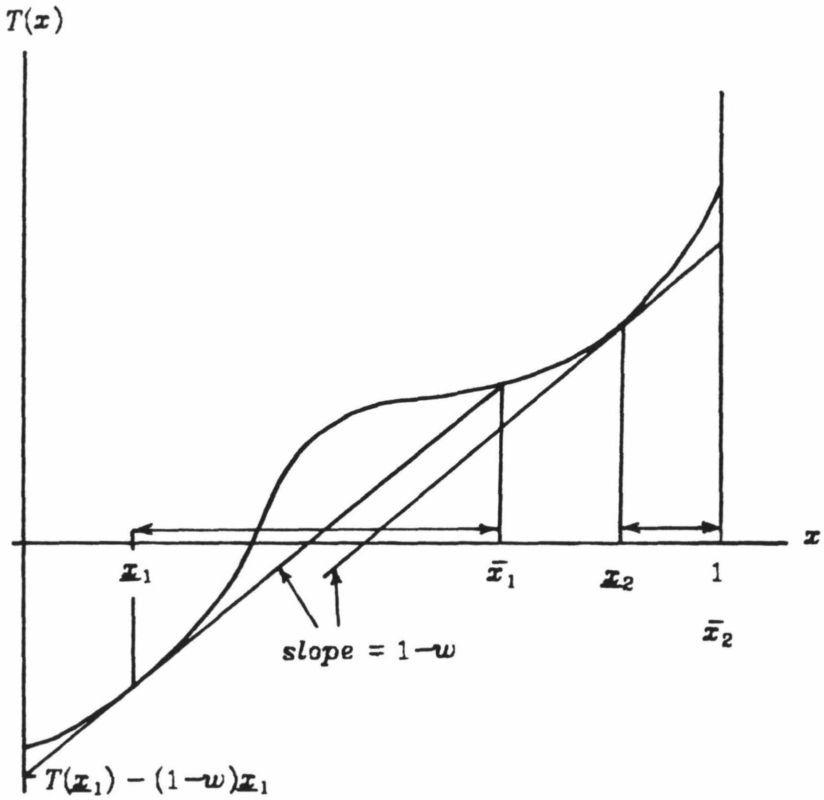


T(x)
x
$\underline{x}_1$
$\bar{x}_1$
$\underline{x}_2$
1
$\bar{x}_2$
slope = 1−w
$T(\underline{x}_1) - (1-w)\underline{x}_1$

Figure 2.1

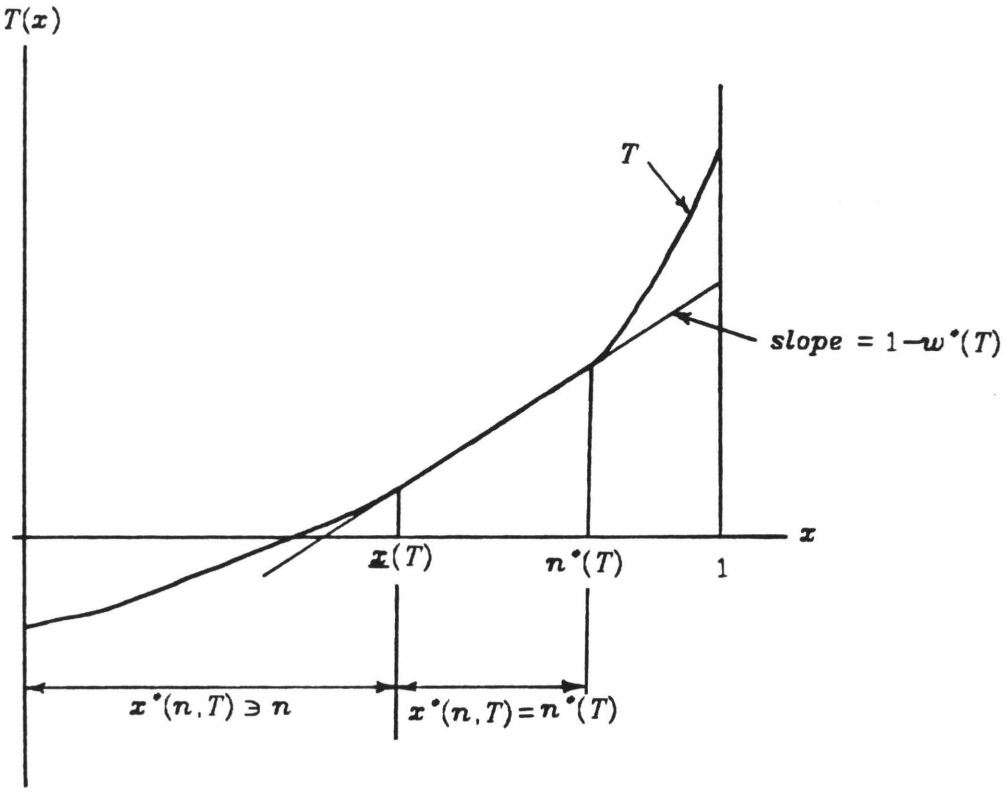
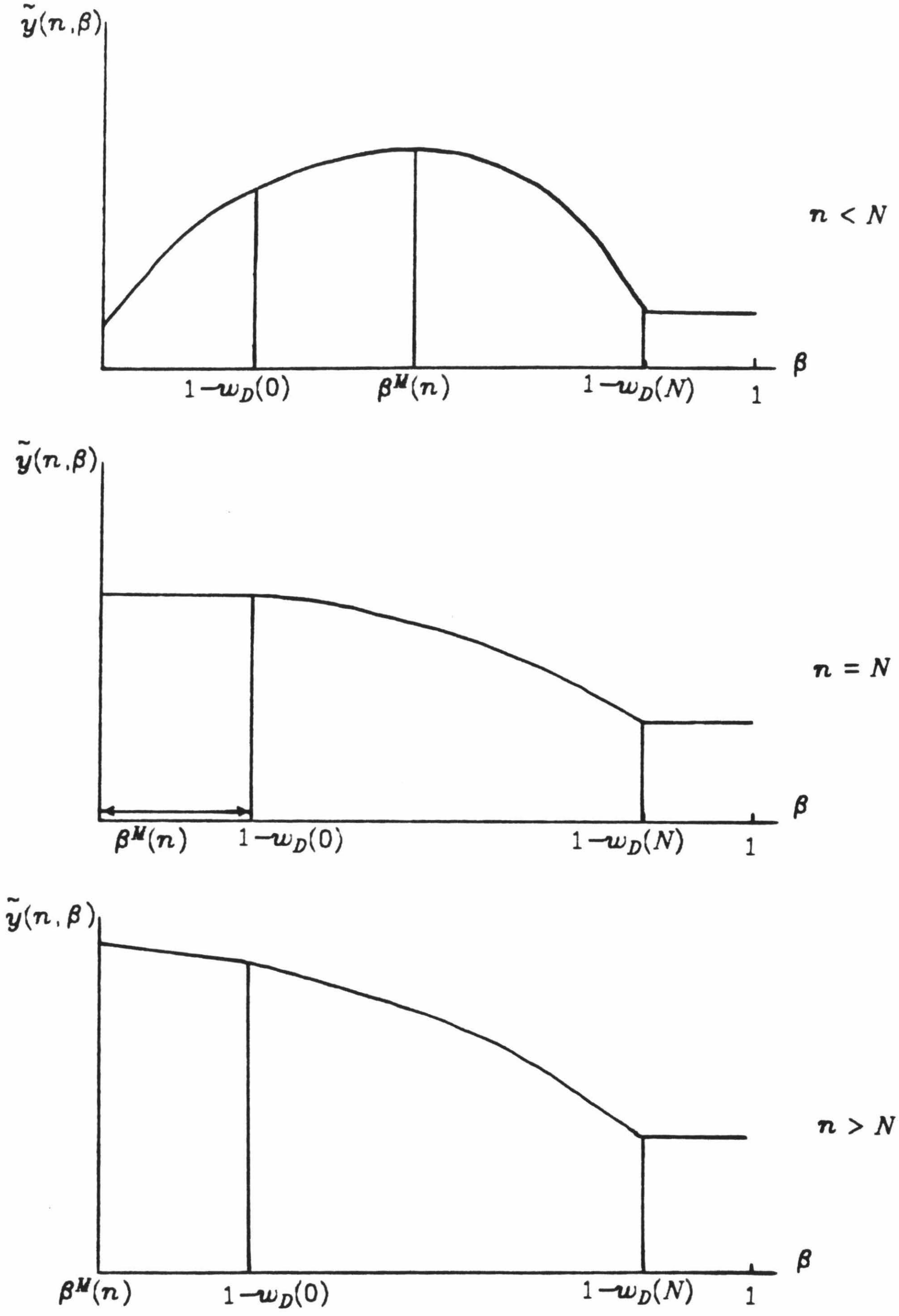
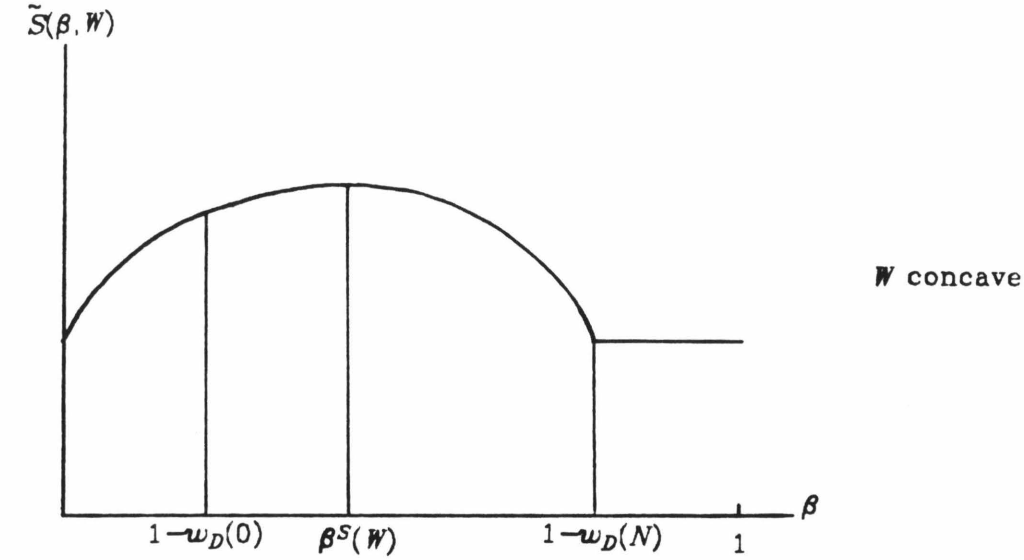


Figure 2.2



ỹ(n,β)
n < N
β
1−w_D(0)
β^M(n)
1−w_D(N)
1
ỹ(n,β)
n = N
β
β^M(n)
1−w_D(0)
1−w_D(N)
1
ỹ(n,β)
n > N
β
β^M(n)
1−w_D(0)
1−w_D(N)
1

Figure 3.1


$\tilde{S}(\beta, W)$
β
$1-w_D(0)$
$\beta^S(W)$
$1-w_D(N)$
1
W concave

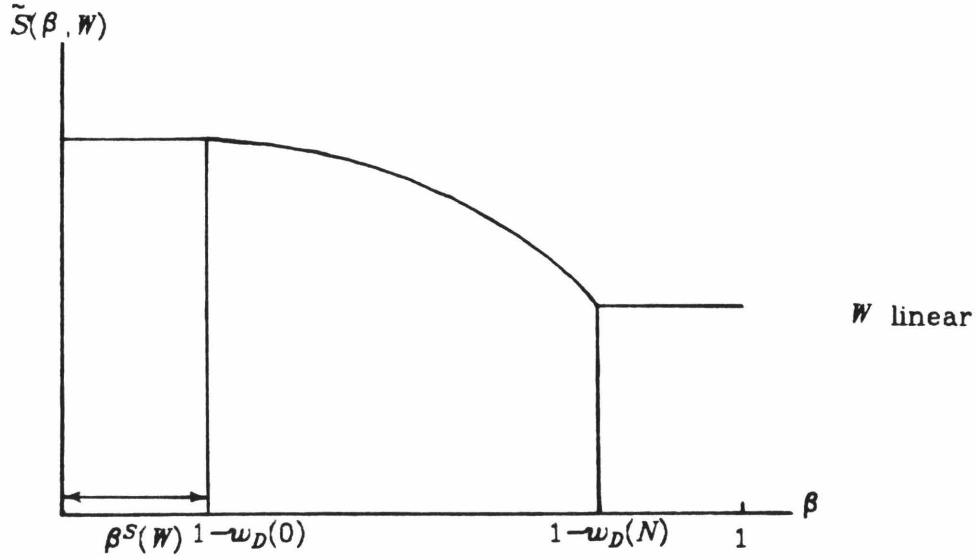

$\tilde{S}(\beta, W)$
β
$\beta^S(W)$
$1-w_D(0)$
$1-w_D(N)$
1
W linear

Figure 3.2

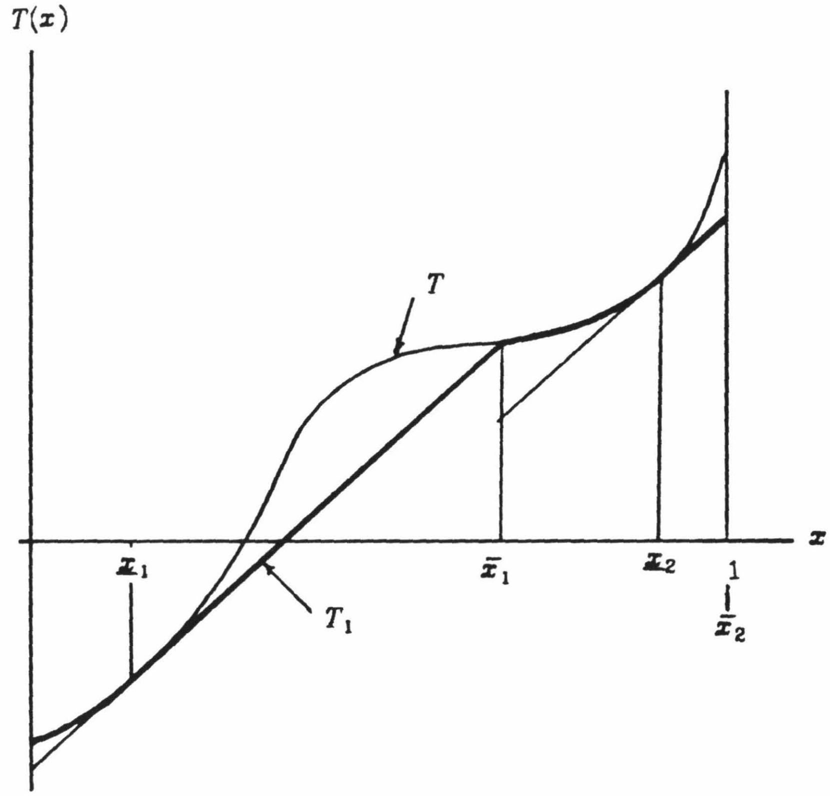


Figure 4.1

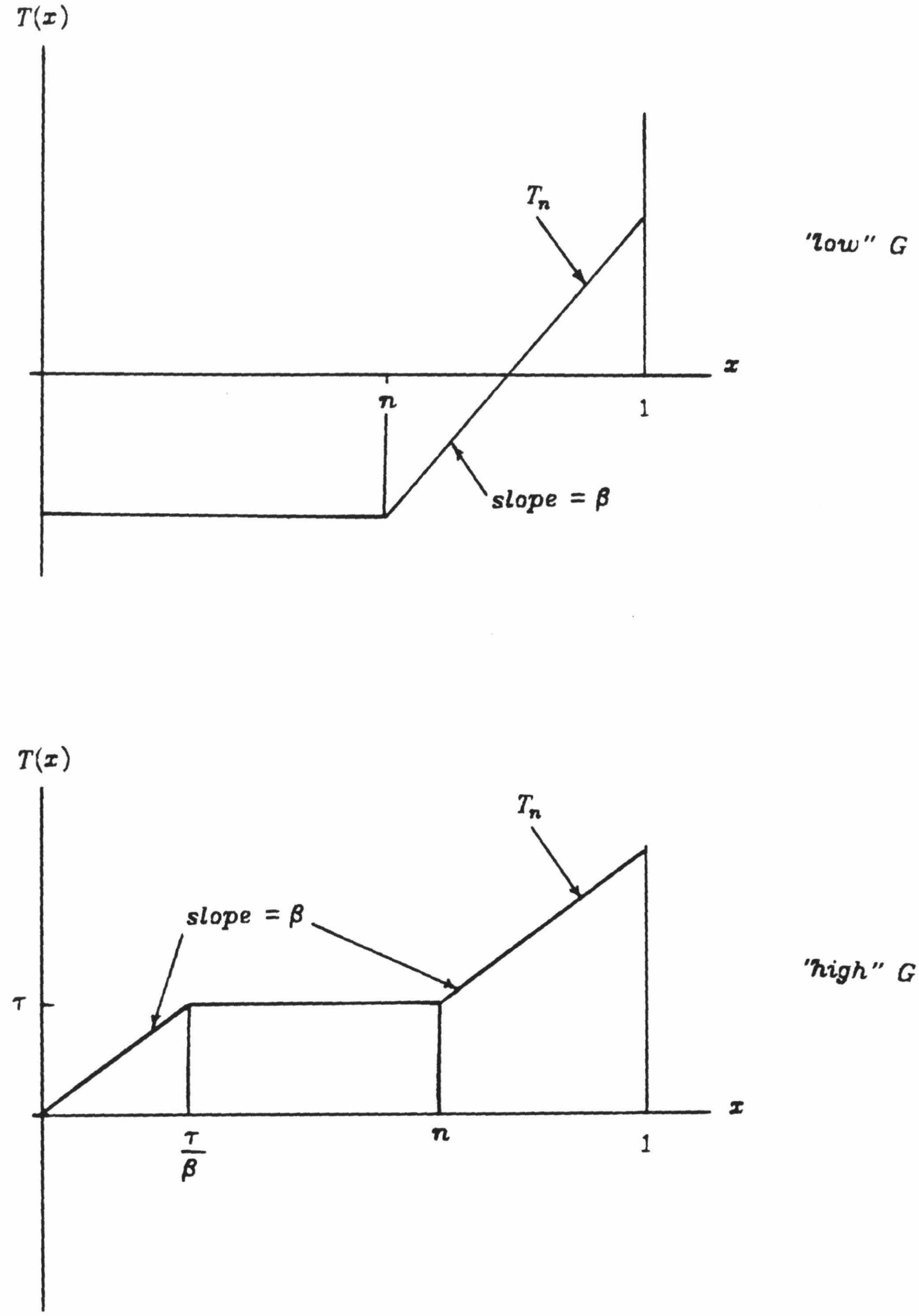


Figure 4.2

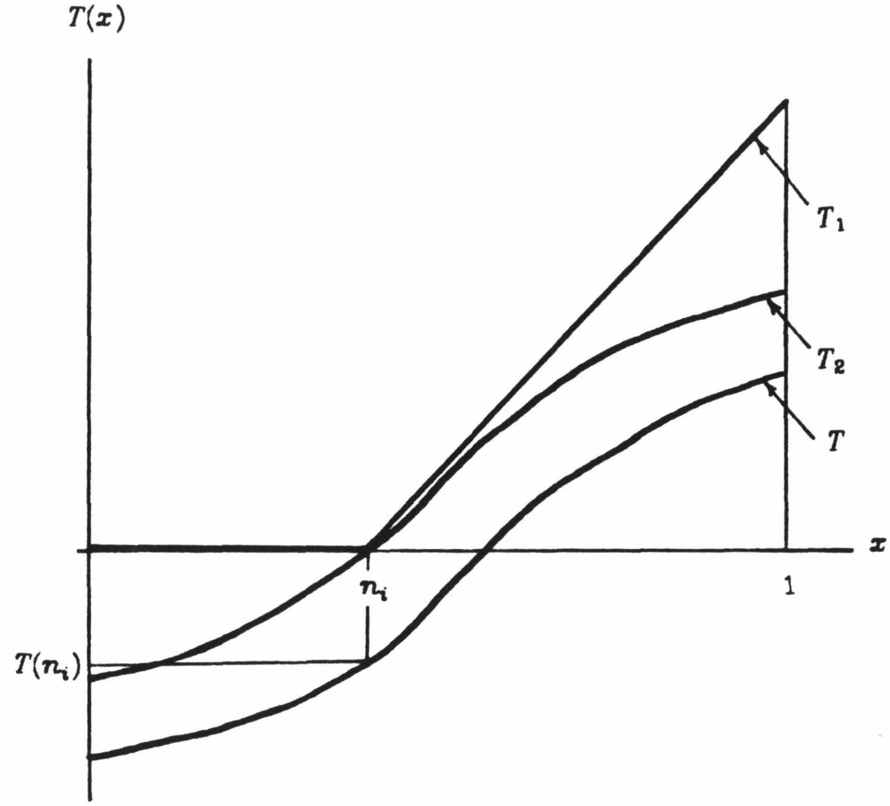
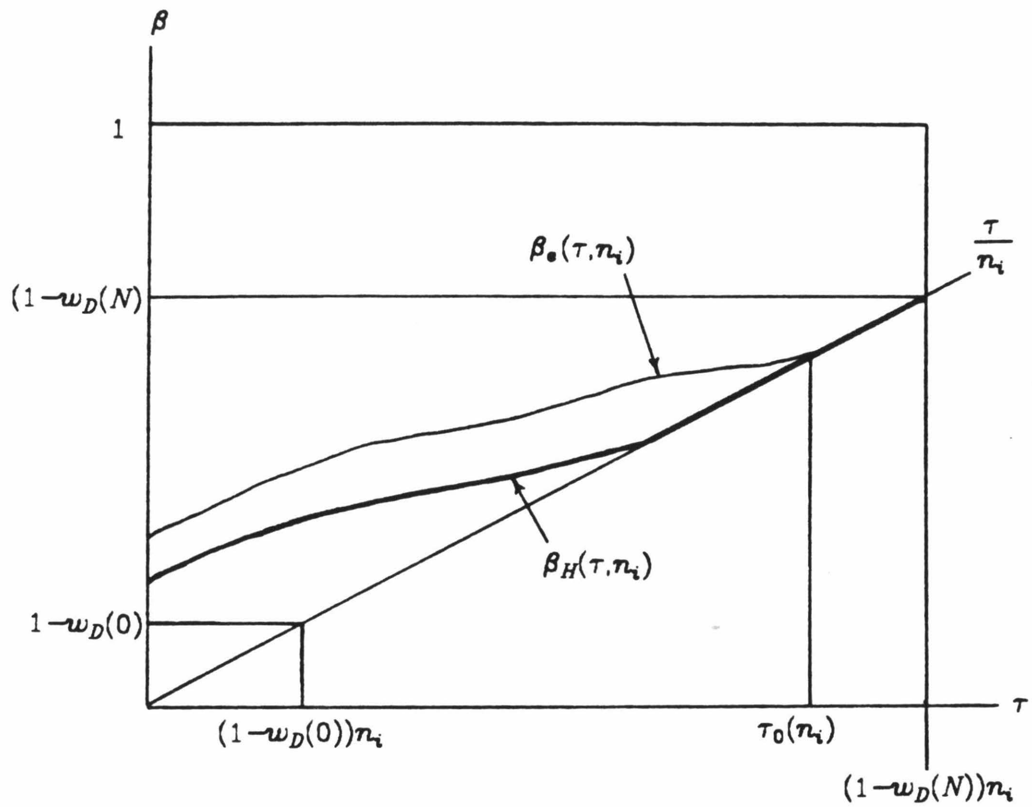


Figure 4.3



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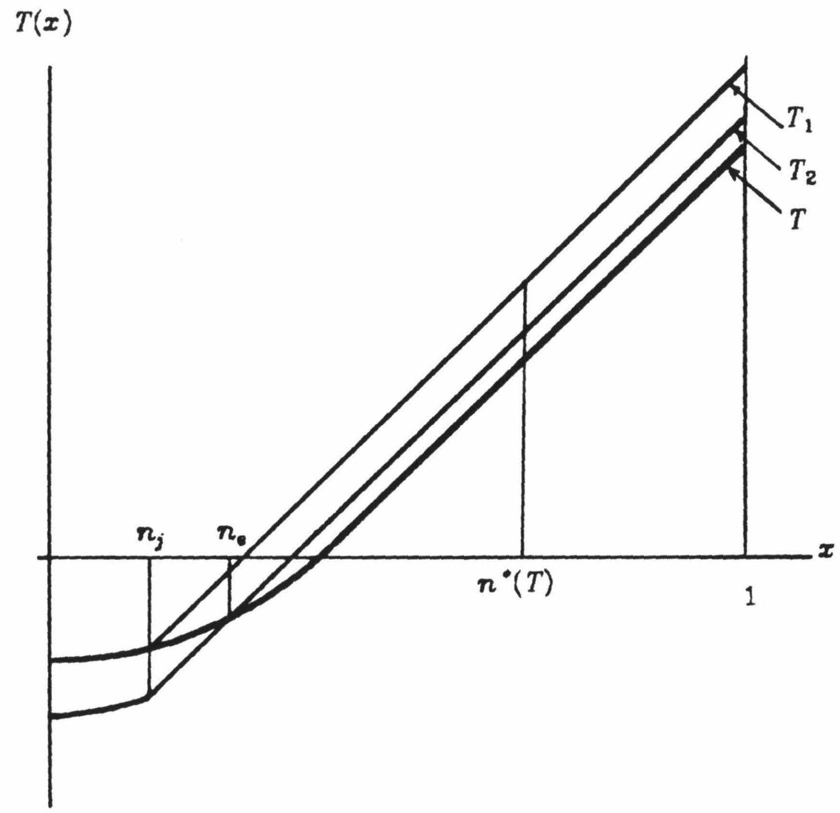

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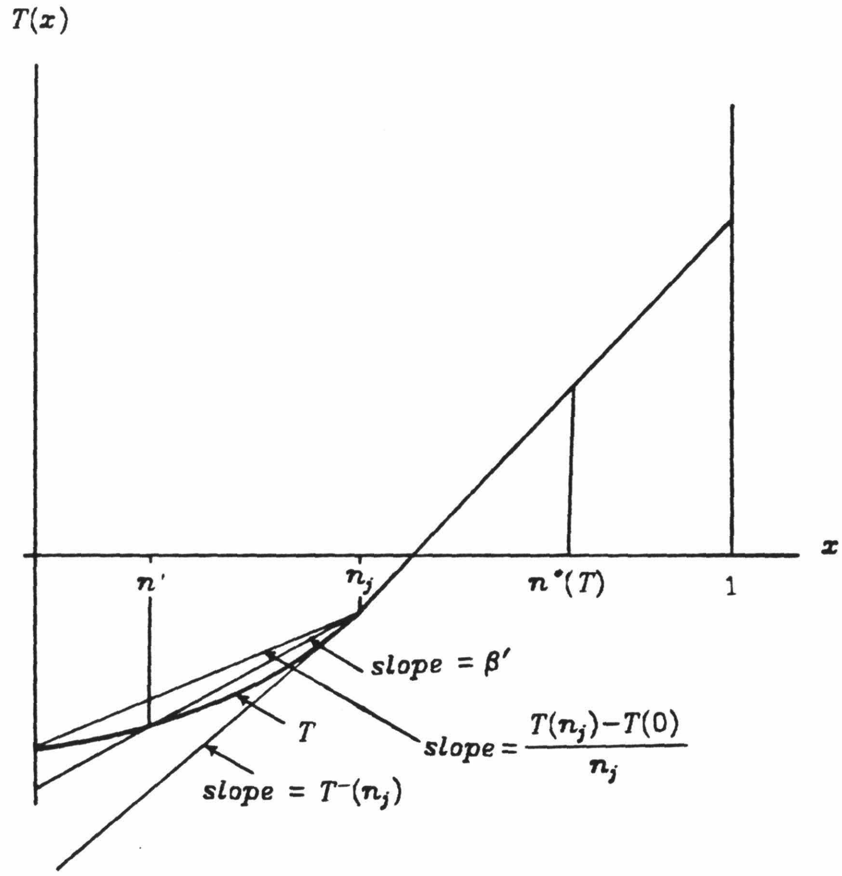

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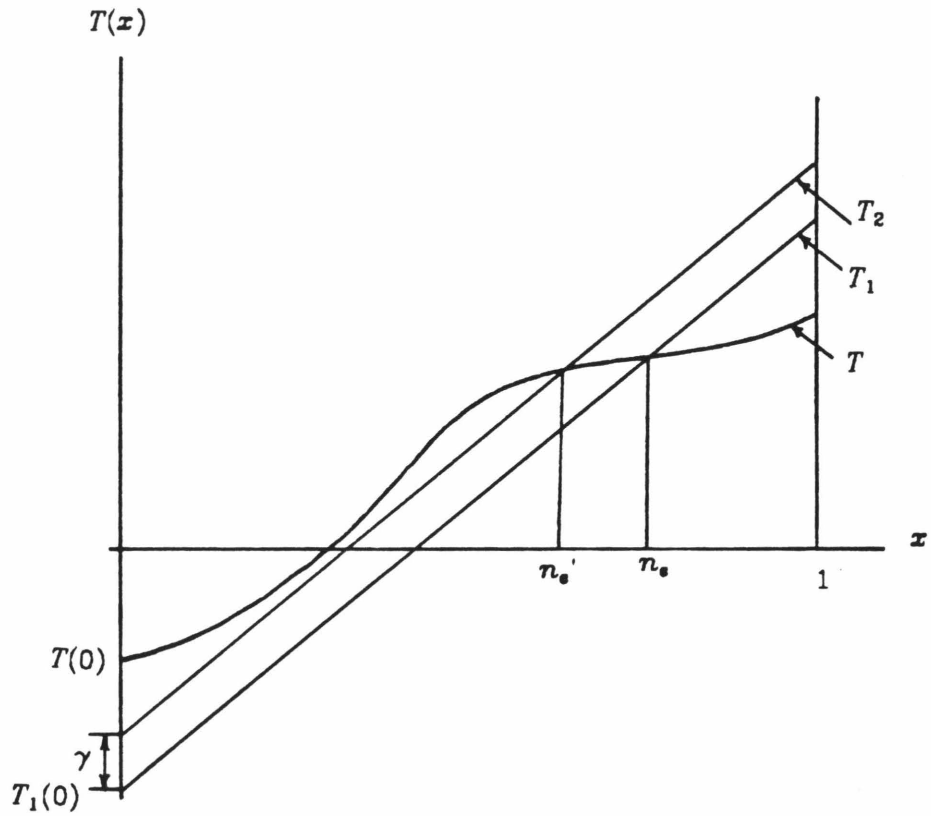


Figure 5.1

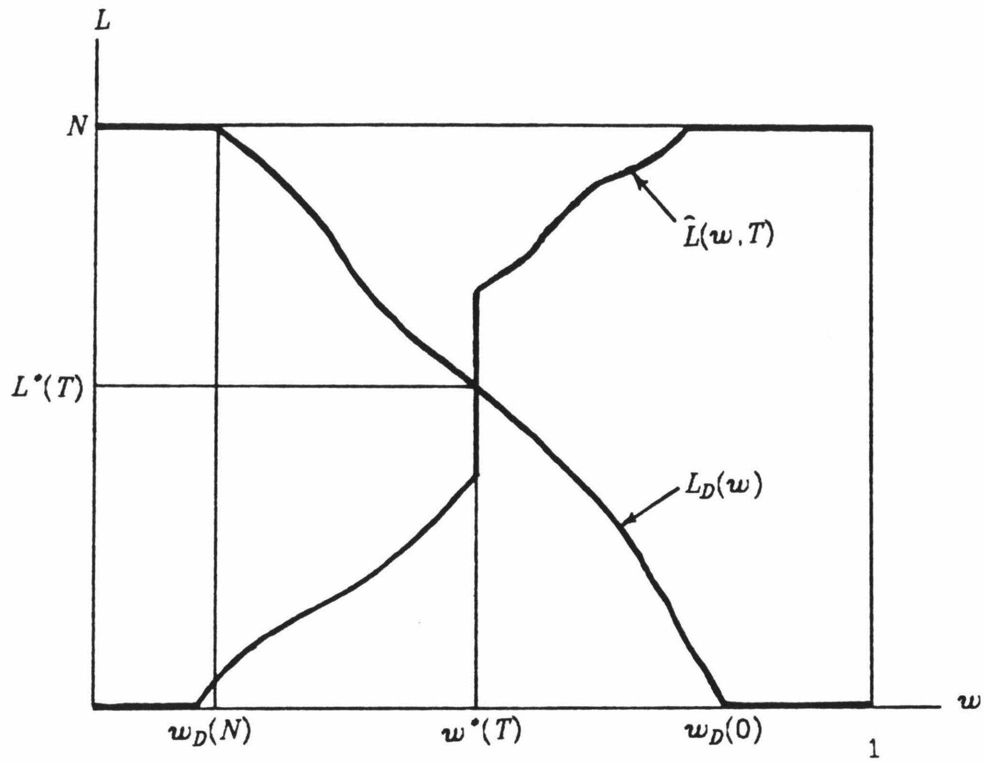
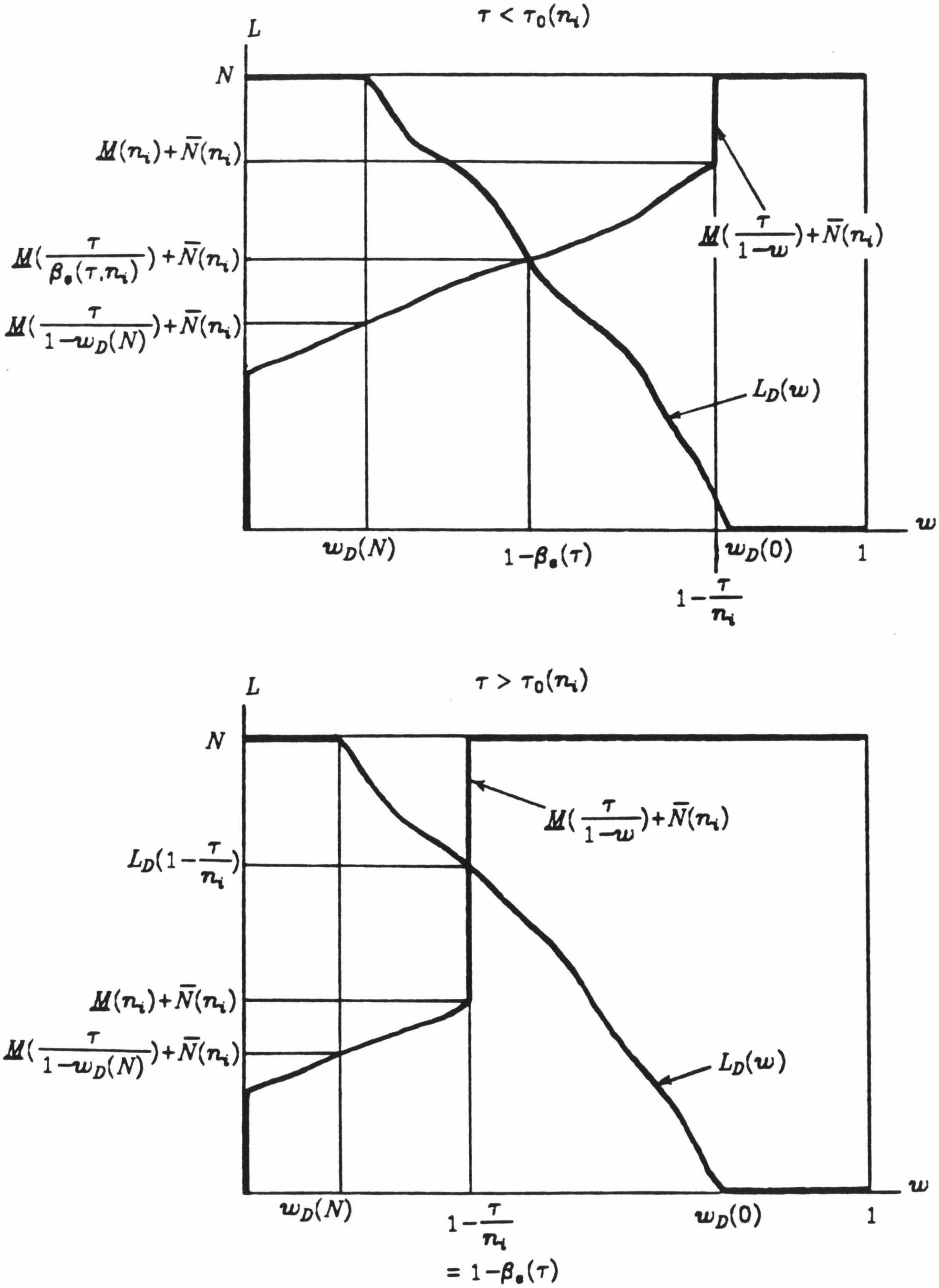


Figure A.1



$\tau < \tau_0(n_i)$
L
N
$\underline{M}(n_i)+\bar{N}(n_i)$
$\underline{M}(\frac{\tau}{\beta_e(\tau,n_i)})+\bar{N}(n_i)$
$\underline{M}(\frac{\tau}{1-w_D(N)})+\bar{N}(n_i)$
$\underline{M}(\frac{\tau}{1-w})+\bar{N}(n_i)$
$L_D(w)$
w
$w_D(N)$
$1-\beta_e(\tau)$
$w_D(0)$
1
$1-\frac{\tau}{n_i}$
$\tau > \tau_0(n_i)$
L
N
$L_D(1-\frac{\tau}{n_i})$
$\underline{M}(n_i)+\bar{N}(n_i)$
$\underline{M}(\frac{\tau}{1-w_D(N)})+\bar{N}(n_i)$
$\underline{M}(\frac{\tau}{1-w})+\bar{N}(n_i)$
$L_D(w)$
w
$w_D(N)$
$1-\frac{\tau}{n_i}$
$= 1-\beta_e(\tau)$
$w_D(0)$
1

Figure B.1

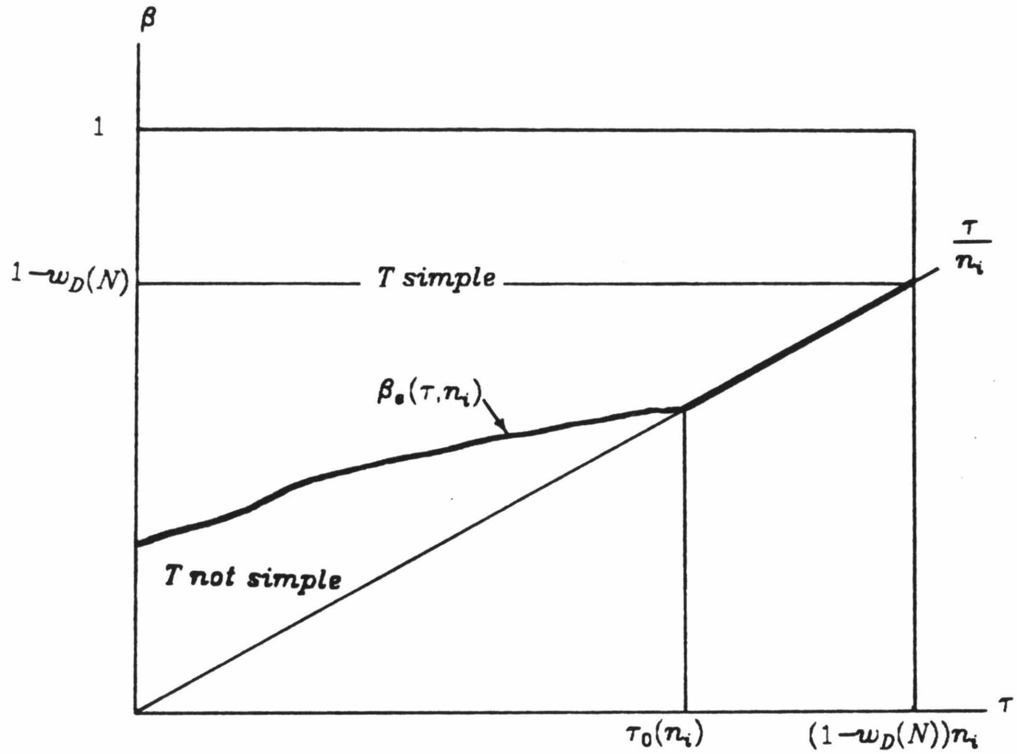


Figure B.2

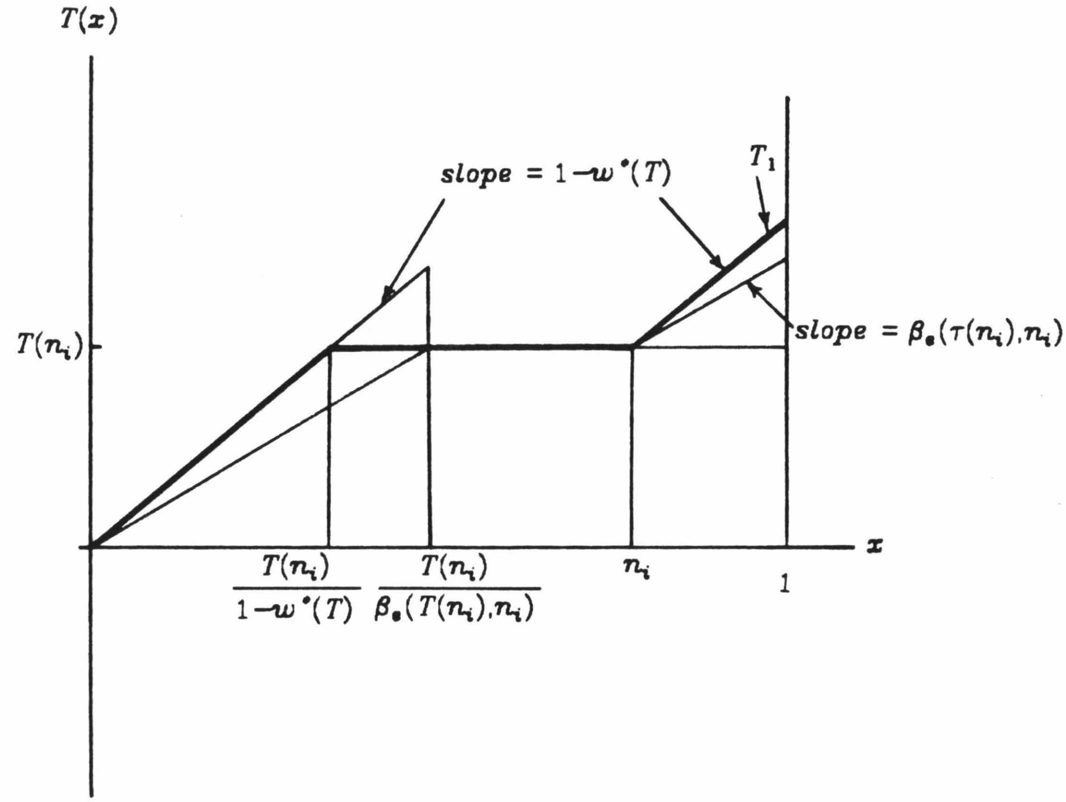


Figure B.3

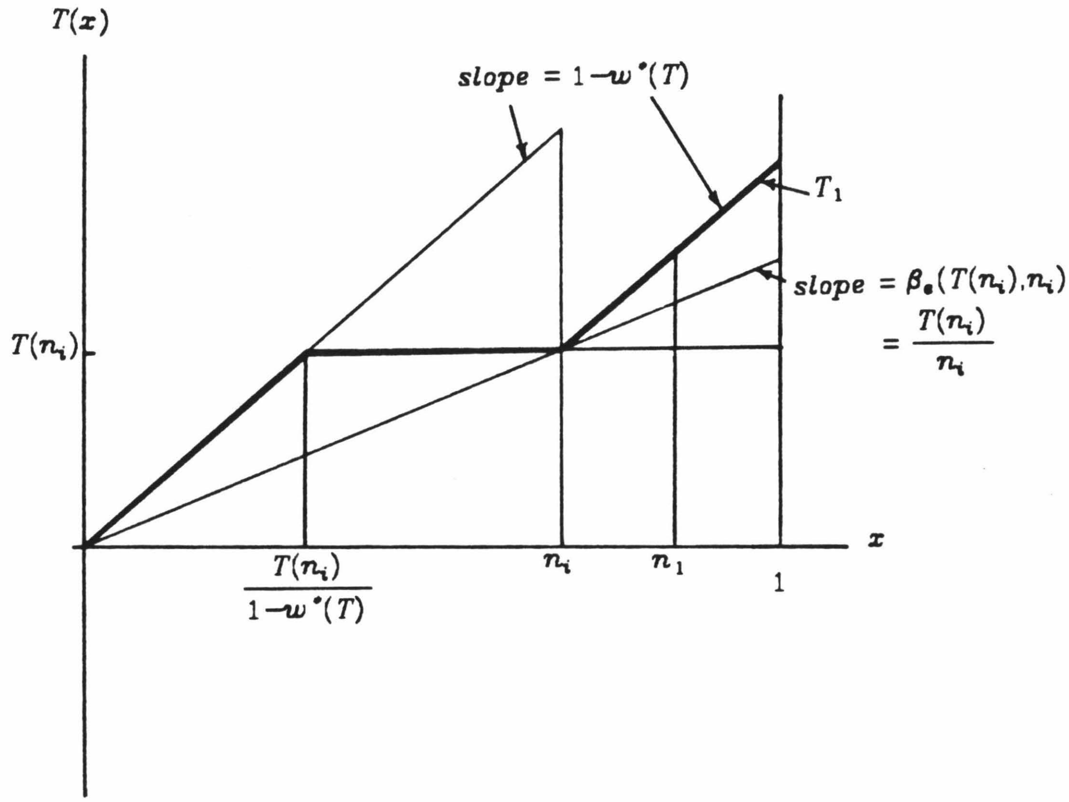


Figure B.4

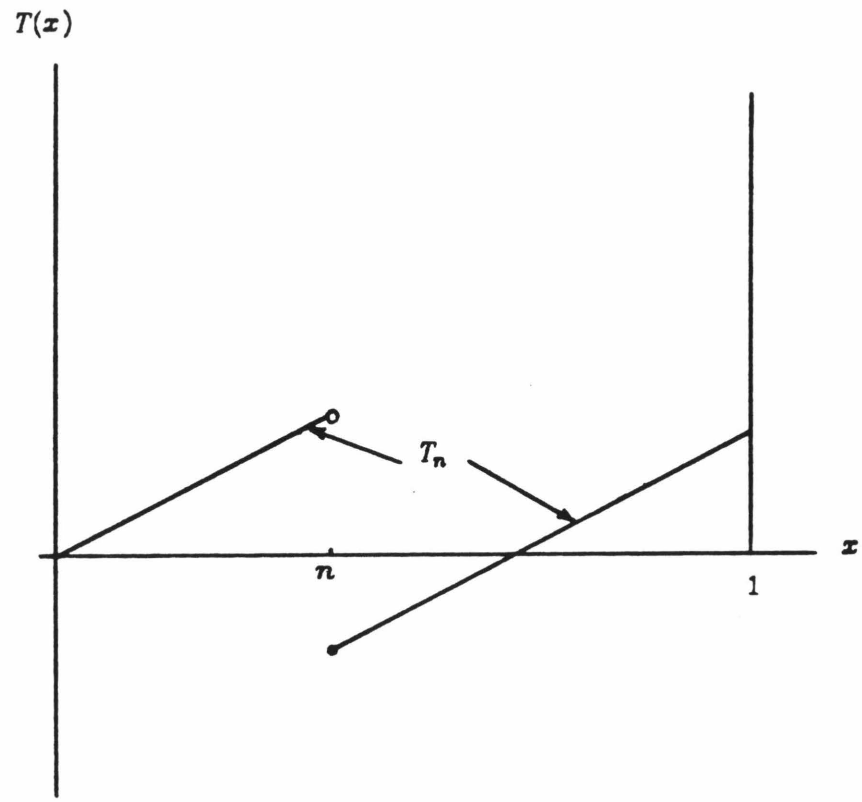


Figure N.1