ON NONMIXED SYMMETRIC END-LOAD PROBLEMS IN ELASTIC WAVEGUIDES

Thesis by

Glenn Bruce Sinclair

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1973

(Submitted August 9, 1972)

To Pucci

ACKNOWLEDGMENTS

I most sincerely thank my advisor, Professor Julius Miklowitz, for valuable guidance, suggestions, and assistance given during the research and preparation of this thesis. Additional thanks of this nature are owed to faculty and fellow students in the Applied Mechanics Department.

Personal support received from the following sources is appreciated: Earle C. Anthony Fellowships, 1969-71; Graduate Teaching Assistantships, 1970-71, 1971-72; and a Special Fellowship from the Shell Foundation, 1972.

I also wish to express my deep gratitude to my parents, Piri and Alan, for their help and encouragement throughout my years of study.

ABSTRACT

This investigation deals with the response of the semi-infinite, linear elastic, homogeneous, isotropic plate in plane strain, subject to symmetric normal loads acting, in the absence of shear stress, on its edge. A double Laplace transform technique is used to obtain long-time information for two problems; a uniform load and a line-load. Near- and far-field approximations are found, the far-field approximations giving the integral of the Airy integral for both problems.

TABLE OF CONTENTS

INTRODUCTION

Analysis of semi-infinite waveguides based on the equations of motion for a linear elastic, homogeneous, isotropic medium, is a subject of long-standing interest. Miklowitz, in [1], gives a review of the area (up to 1969). As he points out there, contributions on nonmixed problems (stresses or displacements prescribed at the edge) are relatively uncommon compared to those for mixed problems (wherein a combination of stress and displacement components is specified on the edge). This is because of the nonseparable nature of the former.

Recently, Miklowitz extended a method used by Benthem, [2], for the stress analysis of semi-infinite elastostatic strips, to the dynamic case, thereby providing a means for handling these nonmixed problems. The method uses a Laplace transform on the propagation coordinate, coupled with a boundedness condition on the solution, to generate integral equations for the edge displacements and their gradients. Solving these integral equations then determines the formal solution to a problem. The method is demonstrated in [1]; the long-time solution to a nonmixed problem of the displacement type being treated there.

In the present work, this technique is used on nonmixed problems of the stress type; being employed to obtain long-time information for two problems involving symmetric end-loads acting, in the absence of shear stress, on the edge of the semi-infinite plate in plane strain. The problems are:

Problem A . . . a suddenly applied uniform normal stress;

and Problem B . . . a suddenly applied normal line-load.

Results in the long-time, near-field for Problem A reduce to elementary forms; for Problem B, however, the corresponding forms are quite complex, entailing singular terms and some numerics. The far-field approximations lead to integrals of the Airy integral for both problems - the same function that arises in mixed problems (see [1]). If the forces acting on the plate edges in Problems A and B are equal, these far-field responses are found to be identical.

1. FORMULATION, FORMAL SOLUTION, AND BOUNDEDNESS CONDITION

We now formulate the class of elastic, waveguide problems to be considered, essentially following Miklowitz in [1]. Plane rectangular cartesian coordinates are chosen for the plate such that the x-axis is in the direction of propagation whilst the y-axis is in the thickness direction (Fig. 1). $u = u(x, y, t)$ is the displacement in the x direction; $v = v(x, y, t)$ the displacement in the y direction. The plate is 2h thick.

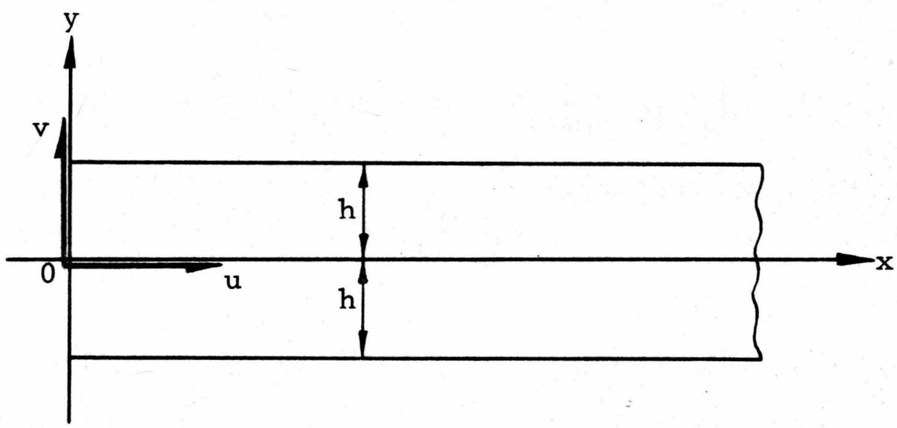

Fig. 1. Coordinates and displacements for the semi-infinite plate

Under the assumptions of plane strain, the displacement *equations of motion* for a linear elastic, homogeneous, isotropic medium are

$$c_d^2 \frac{\partial^2 u}{\partial x^2}(x, y, t) + (c_d^2 - c_s^2)\frac{\partial^2 v}{\partial x \partial y}(x, y, t) + c_s^2 \frac{\partial^2 u}{\partial y^2}(x, y, t) = \frac{\partial^2 u}{\partial t^2}(x, y, t),$$

$$c_s^2 \frac{\partial^2 v}{\partial x^2}(x, y, t) + (c_d^2 - c_s^2)\frac{\partial^2 u}{\partial x \partial y}(x, y, t) + c_d^2 \frac{\partial^2 v}{\partial y^2}(x, y, t) = \frac{\partial^2 v}{\partial t^2}(x, y, t), \tag{1.1}$$

for $x > 0$, $-h < y < h$, $t > 0$. Here $c_d = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$ are, respectively, the dilational and equivoluminal body, wave speeds, λ and μ are the Lamé constants, and ρ is the mass density. The associated *stress-displacement relations* may be expressed by

$$\left.\begin{aligned}
\sigma_x(x, y, t) &= \mu\left[k^2 \frac{\partial u}{\partial x}(x, y, t) + (k^2 - 2)\frac{\partial v}{\partial y}(x, y, t)\right], \\
\sigma_y(x, y, t) &= \mu\left[(k^2 - 2)\frac{\partial u}{\partial x}(x, y, t) + k^2 \frac{\partial v}{\partial y}(x, y, t)\right], \\
\sigma_{xy}(x, y, t) &= \mu\left[\frac{\partial u}{\partial y}(x, y, t) + \frac{\partial v}{\partial x}(x, y, t)\right],
\end{aligned}\right\} \tag{1.2}$$

for $x > 0$, $-h < y < h$, $t > 0$. Here $\sigma_x, \sigma_y, \sigma_{xy}$ denote the rectangular components of stress in the usual way, and $k^2 = c_d^2/c_s^2$.

For quiescent *initial conditions* one has

$$u(x, y, 0) = \frac{\partial u}{\partial t}(x, y, 0) = v(x, y, 0) = \frac{\partial v}{\partial t}(x, y, 0) = 0, \tag{1.3}$$

for $x \geq 0$, $-h \leq y \leq h$. *Conditions at infinity* are taken as

$$\lim_{x \to \infty} \left\{ \begin{matrix} u(x, y, t), & \frac{\partial u}{\partial x}(x, y, t), & \text{etc.} \\ v(x, y, t), & \frac{\partial v}{\partial x}(x, y, t), & \text{etc} \end{matrix} \right\} = 0, \tag{1.4}$$

for $-h \le y \le h$, $t \ge 0$. *Plate-face conditions* for stress-free surfaces are

$$\sigma_y(x, h, t) = \sigma_{xy}(x, h, t) = 0, \tag{1.5}$$

for $x \ge 0$, $t \ge 0$. To these are added, for excitation of the plate which is symmetric with respect to the x-axis, the *symmetry criteria*,

$$v(x, 0, t) = 0, \quad \sigma_{xy}(x, 0, t) = 0, \tag{1.6}$$

for $x > 0$, $t \ge 0$. Finally we have our *edge conditions*, which consist of a normal stress suddenly applied on the edge $x = 0$, for $0 \le y < h$, in the absence of any shear stresses. Equations (1.1) through (1.6), in conjunction with the edge conditions, define the class of problems from which the two particular examples of the ensuing sections will be drawn.

We now focus our attention on a formal solution for this class of problems, defining the Laplace transforms on x, parameter s, and t, parameter p, by

$$\tilde{f}(s) = \int_0^\infty f(x)e^{-sx}\,dx, \qquad \bar{f}(p) = \int_0^\infty f(t)e^{-pt}\,dt. \tag{1.7}$$

Double transformation of (1.1), followed by the application of (1.2), (1.3), (1.5), (1.6), transformed appropriately, and the attendant inversion formulae for (1.7), produces the forms that follow in equations (1.8) through (1.16). First, the pertinent inversion integrals, namely

$$\left\{\begin{matrix} u(x,y,t) \\ \\ v(x,y,t) \end{matrix}\right\} = \frac{1}{2\pi i}\int_{Br_p} e^{pt}\left[\frac{1}{2\pi i}\int_{Br_s} e^{sx}\left\{\begin{matrix} \tilde{\tilde{u}}(s,y,p) \\ \\ \tilde{\tilde{v}}(x,y,p) \end{matrix}\right\} ds\right] dp, \tag{1.8}$$

in which Br_p and Br_s are, respectively, the Bromwich contours in the p- and s- planes. Here

$$\left.\begin{aligned} \tilde{\tilde{u}}(s,y,p) &= \tilde{\tilde{u}}^p(s,y,p) + \tilde{\tilde{u}}^c(s,y,p), \\ \tilde{\tilde{v}}(s,y,p) &= \tilde{\tilde{v}}^p(s,y,p) + \tilde{\tilde{v}}^c(s,y,p), \end{aligned}\right\} \tag{1.9}$$

where the superscripts indicate the source of the contribution to the transformed displacement field; p signifying the particular integrals for the ordinary differential equations arising from the transformation of (1.1) while c signifies the complementary functions of the same. Consequently

$$\left.\begin{aligned} \tilde{\tilde{u}}^p(s,y,p) = \frac{1}{k_s^2}\int_0^y \Big\{&\Big[\frac{s^2}{\alpha}\sinh\alpha(y-y') + \beta\sinh\beta(y-y')\Big]g(s,y',p) \\ &+ s[\cosh\alpha(y-y') - \cosh\beta(y-y')]h(s,y',p)\Big\}dy', \\ \tilde{\tilde{v}}^p(s,y,p) = \frac{1}{k_s^2}\int_0^y \Big\{&\Big[\alpha\sinh\alpha(y-y') + \frac{s^2}{\beta}\sinh\beta(y-y')\Big]h(s,y',p) \\ &+ s[\cosh\alpha(y-y') - \cosh\beta(y-y')]g(s,y',p)\Big\}dy', \end{aligned}\right\} \tag{1.10}$$

and

$$\left.\begin{aligned}\tilde{\bar{u}}^{c}(s, y, p) &= C_1(s, p)\cosh\alpha y + C_2(s, p)\cosh\beta y,\\ \tilde{\bar{v}}^{c}(s, y, p) &= \frac{\alpha}{s} C_1 (s, p)\sinh\alpha y - \frac{s}{\beta} C_2(s, p)\sinh\beta y,\end{aligned}\right\} \quad (1.11)$$

where

$$\left.\begin{aligned}C_1(s, p) &= -s\left[(2s^2-k_s^2)I(s, p)\sinh\beta h+2s\beta J(s, p)\cosh\beta h\right]/R(s, p),\\ C_2 (s, p) &= -\beta\left[2s\alpha I(s, p)\sinh\alpha h - (2s^2-k_s^2)J(s, p)\cosh\alpha h\right]/R(s, p),\end{aligned}\right\} \quad (1.12)$$

$$R(s, p) = (2s^2-k_s^2)^2\cosh\alpha h\sinh\beta h + 4s^2\alpha\beta \sinh\alpha h\cosh\beta h, \quad (1.13)$$

$$\left.\begin{aligned}I(s, p) &= \frac{1}{k_s^2}\int_0^h\left\{\frac{s}{\alpha}\left[(2s^2-k_s^2)\sinh\alpha(h-y')+2\alpha\beta\sinh\beta(h-y)\right]g(s,y', p)+\right.\\ &\left.\left[(2s^2-k_s^2)\cosh\alpha(h-y')-2s^2\cosh\beta(h-y')\right]h(s, y', p)\right\}dy' + (k^2-2)\bar{u}(0,h,p),\\ J(s, p) &= \frac{1}{k_s^2}\int_0^h\left\{\left[2s^2\cosh\alpha(h-y')-(2s^2-k_s^2)\cosh\beta(h-y')\right] g(s, y',p) +\right.\\ &\left.\frac{s}{\beta}\left[2\alpha\beta\sinh\alpha(h-y') + (2s^2-k_s^2)\sinh\beta(h-y')\right]h(s, y',p)\right\} dy'-\bar{v}(0, h, p),\end{aligned}\right\} \quad (1.14)$$

with

$$\left.\begin{aligned}g(s, y, p) &= k^2\left[s\bar{u}(0, y, p) + \frac{\partial\bar{u}}{\partial x}(0, y, p)\right]+(k^2-1)\frac{\partial\bar{v}}{\partial y}(0, y, p),\\ h(s, y, p) &= s\bar{v}(0, y, p) + \frac{\partial\bar{v}}{\partial x}(0, y, p) + (k^2-1)\frac{\partial\bar{u}}{\partial y}(0, y, p),\end{aligned}\right\} \quad (1.15)$$

and

$$\alpha = \sqrt{k_d^2 - s^2}, \quad \beta = \sqrt{k_s^2 - s^2}, \quad k_d^2 = p^2/c_d^2, \; k_s^2 = p^2/c_s^2. \tag{1.16}$$

The set of results numbered (1.8) to (1.16) correspond to equation (19) of [1]. Details of the derivation for this set appear in Miklowitz, [3], Section 2.

To avoid the awkwardness of equal and opposite (hence cancelling) singularities on performing the integration for the particular integrals and complementary functions in the problem treated in Section 3, these forms are now combined. (1.9) through (1.14), together, imply

$$\left.\begin{aligned}
\tilde{\bar{u}}(s,y,p) &= \frac{1}{k_s^2}\Big[\Big\{\frac{s}{\alpha}\Lambda(s,p;0,h)[\Phi(s,p)\cosh\alpha y - 2\alpha\beta(2s^2-k_s^2)\cosh\beta y] - \\
&\Pi(s,p;0,h)[\Psi(s,p)\cosh\beta y - 2s^2(2s^2-k_s^2)\cosh\alpha y] + \\
&k_s^2[\beta\phi^u(s,y,p)\bar{v}(0,h,p) - (k^2-2)s\psi^u(s,y,p)\bar{u}(0,h,p)]\Big\}\Big/R(s,p) \\
&-\frac{s}{\alpha}[\Lambda(s,p;0,y)\sinh\alpha y - \Theta(s,p;y,h)\cosh\alpha y] \\
&+\Pi(s,p;0,y)\sinh\beta y + \Omega(s,p;y,h)\cosh\beta y\Big], \\
\tilde{\bar{v}}(s,y,p) &= \frac{1}{k_s^2}\Big[\Big\{\Lambda(s,p;0,h)[\Phi(s,p)\sinh\alpha y + 2s^2(2s^2-k_s^2)\sinh\beta y] \\
&+\frac{s}{\beta}\Pi(s,p;0,h)[\Psi(s,p)\sinh\beta y + 2\alpha\beta(2s^2-k_s^2)\sinh\alpha y] + \\
&k_s^2[s\,\phi^v(s,y,p)\bar{v}(0,h,p) - (k^2-2)\alpha\psi^v(s,y,p)\bar{u}(0,h,p)]\Big\}\Big/R(s,p) \\
&-\frac{s}{\beta}[\Pi(s,p;0,y)\cosh\beta y + \Omega(s,p;y,h)\sinh\beta y] \\
&-\Lambda(s,p;0,y)\cosh\alpha y + \Theta(s,p;y,h)\sinh\alpha y\Big],
\end{aligned}\right\} \tag{1.17}$$

in which

$$
\left.\begin{aligned}
\Lambda(s,p;a,b) &= \int_a^b [\alpha h(s,y',p)\sinh\alpha y' - s g(s,y',p)\cosh\alpha y']dy', \\
\Pi(s,p;a,b) &= \int_a^b [s h(s,y',p)\sinh\beta y' + \beta g(s,y',p)\cosh\beta y']dy', \\
\Theta(s,p;a,b) &= \int_a^b [s\, g(s,y',p)\sinh\alpha y' - \alpha h(s,y',p)\cosh\alpha y']dy', \\
\Omega(s,p;a,b) &= \int_a^b [\beta\, g(s,y',p)\sinh\beta y' + s\, h(s,y',p)\cosh\beta y']dy',
\end{aligned}\right\} \quad (1.18)
$$

$$
\left.\begin{aligned}
\Phi(s,p) &= (2s^2-k_s^2)^2 \sinh\alpha h \sinh\beta h + 4s^2\alpha\beta \cosh\alpha h \cosh\beta h, \\
\Psi(s,p) &= (2s^2-k_s^2)^2 \cosh\alpha h \cosh\beta h + 4s^2\alpha\beta \sinh\alpha h \sinh\beta h,
\end{aligned}\right\} \quad (1.19)
$$

$$
\left.\begin{aligned}
\psi^u(s,y,p) &= (2s^2-k_s^2)\sinh\beta h \cosh\alpha y + 2\alpha\beta \sinh\alpha h \cosh\beta y, \\
\phi^u(s,y,p) &= 2s^2 \cosh\beta h \cosh\alpha y - (2s^2-k_s^2)\cosh\alpha h \cosh\beta y, \\
\psi^v(s,y,p) &= (2s^2-k_s^2)\sinh\beta h \sinh\alpha y - 2s^2 \sinh\alpha h \sinh\beta y, \\
\phi^v(s,y,p) &= 2\alpha\beta\cosh\beta h \sinh\alpha y + (2s^2-k_s^2)\cosh\alpha h \sinh\beta y.
\end{aligned}\right\} \quad (1.20)
$$

Observe that $\tilde{\bar{u}}(s, y, p)$, $\tilde{\bar{v}}(s, y, p)$ of (1.17) involve, via $g(s, y, p)$ and $h(s, y, p)$ of (1.15), the six *edge unknowns:* $\bar{u}(0, y, p)$, $\bar{v}(0, y, p)$, $\frac{\partial \bar{u}}{\partial y}(0, y, p)$, $\frac{\partial \bar{v}}{\partial y}(0, y, p)$, $\frac{\partial \bar{u}}{\partial x}(0, y, p)$, $\frac{\partial \bar{v}}{\partial x}(0, y, p)$. Differentiation with respect to y of the first pair of these quantities yields the second pair. Application of the edge conditions further reduces the number of the edge unknowns to be found to two. It remains for us to utilize the only condition left in the preceding formulation, to wit the infinity condition (1.4), to obtain a means of ascertaining these last two edge unknowns.

Note that the denominator of the complementary functions given in (1.11), (1.12), is R(s, p) of (1.13). Now R(s, p) set equal to zero is a generalized form of the *Rayleigh-Lamb frequency equation* for symmetric waves in an infinite elastic plate. Hence as Miklowitz points out in [1], R(s, p) has an infinite set of zeros in each quadrant of the complex s-plane and, in particular, there exists $s = s_j(p)$ such that

$$R(s_j(p), p) = 0, \qquad \text{Re } s_j(p) > 0. \tag{1.21}$$

Accordingly, a residue evaluation of the inner integral in (1.8) would lead to exponentially unbounded waves as $x \to \infty$, a violation of (1.4). Thus, to eliminate these transcendentally large terms the numerators of $C_1(s, p)$ and $C_2(s, p)$ in (1.12) are set equal to zero for $s = s_j(p)$ satisfying (1.21), yielding

$$J(s_j, p) - Y(s_j, p)I(s_j, p) = 0, \tag{1.22}$$

where

$$Y(s_j, p) = \frac{2s_j\alpha_j}{(2s_j^2 - k_s^2)} \tanh\alpha_j h = -\frac{(2s_j^2 - k_s^2)}{2s_j\beta_j} \tanh\beta_j h, \tag{1.23}$$

α_j, β_j, being α, β, of (1.16) for $s = s_j(p)$. On expansion, (1.22) becomes

$$\left.\begin{aligned} &\frac{1}{k_s^2}\int_0^h\left\{\left[2s_j^2\frac{\cosh\alpha_j y'}{\cosh\alpha_j h} - (2s_j^2 - k_s^2)\frac{\cosh\beta_j y'}{\cosh\beta_j h}\right]g(s_j, y', p)\right. \\ &\left.-\frac{s_j}{\beta_j}\left[2\alpha_j\beta_j\frac{\sinh\alpha_j y'}{\cosh\alpha_j h} + (2s_j^2 - k_s^2)\frac{\sinh\beta_j y'}{\cosh\beta_j h}\right]h(s_j, y', p)\right\}dy' \\ &- \bar{v}(0, h, p) - (k^2 - 2)Y(s_j, p)\bar{u}(0, h, p) = 0. \end{aligned}\right\} \tag{1.24}$$

(1.24), the *boundedness condition*, corresponds to equation (23) of [1] and is the requisite system of integral equations for completing the evaluation of the edge unknowns.

2. PROBLEM A: UNIFORM LOAD

In this section we use the boundedness condition of Section 1 to find the formal solution valid for small p, thence the long-time solution, for the specific problem of a uniform normal stress suddenly applied to the end of our waveguide (Fig. 2). Thus σ_A is a positive quantity with the dimensions of stress; U(t) is the *unit step function.*

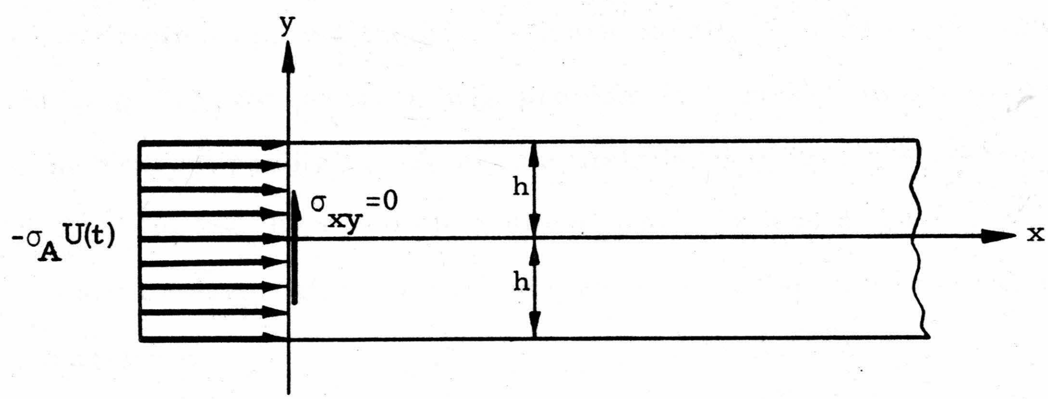

Fig. 2. Edge conditions for Problem A

This particular loading is expressed through the stress-displacement relations, (1.2), (assuming the displacement field to be continuously differentiable on the edge), by

$$\left.\begin{aligned}\bar{\sigma}_x(0, y, p) &= \mu\left[k^2 \frac{\partial \bar{u}}{\partial x}(0, y, p) + (k^2-2)\frac{\partial \bar{v}}{\partial y}(0, y, p)\right] = -\frac{\sigma_A}{p}, \\ \bar{\sigma}_{xy}(0, y, p) &= \mu\left[\frac{\partial \bar{u}}{\partial y}(0, y, p) + \frac{\partial \bar{v}}{\partial x}(0, y, p)\right] = 0,\end{aligned}\right\} \quad (2.1)$$

for $0 \le y < h$, $p > 0$; wherein the quantities concerned have been transformed with respect to time so as to involve the edge unknowns.

The long-time behaviour of the waveguides we are currently studying can be derived by approximating $\tilde{\bar{u}}(s, y, p)$ and $\tilde{\bar{v}}(s, y, p)$ of (1.17) for small p once the edge unknowns have been established. It follows that determination, via the boundedness condition, of the edge unknowns satisfying (2.1) for small p, will provide the formal long-time solution for the present problem. To accomplish this we first turn to the *elementary theory* of compressional waves in a plate as our modus operandi for estimating the time-dependence of the edge unknowns in the long-time.

For the elementary theory one derives the equation of motion directly on assuming the stress is uniform over any cross-section of the plate perpendicular to the direction of propagation of the long waves (the x direction). Setting the second of (1.2) equal to zero, and substituting the result found thereon for $\frac{\partial v}{\partial y}(x, y, t)$ in the first of (1.2), furnishes the associated constitutive relations. On combining these expressions, one has the standard, elementary theory, field equations,

$$\left.\begin{aligned}
\frac{\partial^2 u^e}{\partial x^2}(x,t) &= \frac{1}{c_p^2}\frac{\partial^2 u^e}{\partial t^2}(x,t), \\
\frac{\partial v^e}{\partial y}(x,y,t) &= -\left(\frac{k^2-2}{k^2}\right)\frac{\partial u^e}{\partial x}(x,t), \\
\sigma_x^e(x,t) &= \frac{4\mu(k^2-1)}{k^2}\frac{\partial u^e}{\partial x}(x,t),
\end{aligned}\right\} \tag{2.2}$$

holding on $x > 0$, $-h < y < h$, for $t > 0$. Here $c_p = \sqrt{\frac{4(k^2-1)}{k^2}}\, c_s$ is the plate, wave speed and the e atop quantities serves to intimate their geneses. It should be noted that (2.2) cannot, in general, be realized as a consistent specialization of the equations of motion and the stress-displacement relations quoted in Section 1. Further, the relationship of the frequency, ω, to the wave number, $\varkappa$, obtained on substituting $u^e(x,t) = e^{i(\varkappa x - \omega t)}$ in the first of (2.2), is

$$\omega = c_p \varkappa . \tag{2.3}$$

Now approximation of the Rayleigh-Lamb frequency equation contained in (1.13), (1.21), for small p and s gives

$$s = \pm \frac{p}{c_p}\left[1 - \frac{1}{6}\left(\frac{k^2-2}{k^2}\,\frac{hp}{c_p}\right)^2\right] + O(p^5), \text{ as } p \to 0, \tag{2.4}$$

whence on setting $s = i\varkappa$, $p = i\omega$, one recovers (2.3) as the dominant term for the lowest mode in the frequency spectrum of the plate. Consequently our elementary theory should show close agreement with the exact theory for the very long-time response of plates to low frequency inputs and hence be a suitable approximation for Problem A.

To pose the analogous problem to Problem A for the elementary theory, one takes a plate of unit width at rest, impinges on it a force, of magnitude $2\sigma_A h$, in the positive x direction on its end, $x = 0$, and thereafter retains only outward travelling waves (in this instance, those propagating in the positive x direction). For a plate subjected to these conditions, application of the Laplace transform on time of (1.7) to (2.2) readily produces, at the edge, $x = 0$,

$$\left.\begin{aligned} \bar{u}^e(0, p) &= \frac{\sigma_A c_p}{4\mu p^2}\left(\frac{k^2}{k^2-1}\right), \\ \frac{\partial \bar{u}^e}{\partial x}(0, p) &= \frac{-\sigma_A}{4\mu p}\left(\frac{k^2}{k^2-1}\right), \\ \frac{\partial \bar{v}^e}{\partial y}(0, p) &= \frac{\sigma_A}{4\mu p}\left(\frac{k^2-2}{k^2-1}\right), \end{aligned}\right\} \quad (2.5)$$

for $-h < y < h$, $p > 0$.

To the estimates in (2.5) we adjoin a supplementary set of edge unknowns - distinguished by the superscript a - to account for any differences between the exact and elementary theories. For these additional contributions we employ the technique used first be Benthem, for the stress analysis of elastostatic strips, and later by Miklowitz, in dynamic problems, and represent them by Fourier series in y with the p dependence incorporated into the series coefficients.

For $\frac{\partial \bar{u}^a}{\partial y}(0, y, p)$, $\frac{\partial \bar{v}^a}{\partial y}(0, y, p)$ continuous on the interval $-h < y < h$, as functions of y, the symmetry criteria, (1.6), imply that $\frac{\partial \bar{u}^a}{\partial y}(0, y, p)$ is an odd function of y on $-h < y < h$ whilst $\frac{\partial \bar{v}^a}{\partial y}(0, y, p)$ is an even one. Accordingly, for $\frac{\partial \bar{u}^a}{\partial y}(0, y, p)$ and $\frac{\partial \bar{v}^a}{\partial y}(0, y, p)$ continuously differentiable in y on the open interval $]0, h[$, we have sine and cosine series representations in y, respectively. Contingent upon these two quantities being smoothly extended as periodic functions of y onto the infinite interval, $-\infty < y < \infty$, the coefficients $a_n'(p)$ and $b_n'(p)$, of their respective series must go down faster than $1/n$ as $n \to \infty$.† Thus, to secure expansions which are acceptable from a convergence standpoint, we seek trigonometric series capable of such extensions.

Now introducing the edge-strain estimates of the elementary theory, set down in (2.5), into the first of the edge conditions in (2.1), reveals that these elementary theory quantities do produce the normal

† To exemplify this claim we consider f(y); a continuously differentiable odd function on the open interval $]-2h, 2h[$ with a continuous extension as a function of period 4h onto $]-\infty, \infty[$. Suppose f(y) admits representation by the series $\sum_{n=1,3,5,\ldots}^{\infty} s_n \sin\frac{n\pi y}{2h}$, where $s_n = \frac{1}{2h}\int_{-2h}^{2h} f(y)\sin\frac{n\pi y}{2h}\,dy$. Modelling our treatment after that of Kantorovich and Krylov, [4] (p. 79), we integrate the defining equation for s_n by parts to obtain

$$s_n = \left.\frac{-f(y)}{n\pi}\cos\frac{n\pi y}{2h}\right|_{-2h}^{2h} + \frac{1}{n\pi}\int_{-2h}^{2h}\frac{df(y)}{dy}\cos\frac{n\pi y}{2h}\,dy.$$

The first term on the right-hand side of the above is zero by virtue of the continuity and perodicity of f(y); the second term is $o(1/n)$, for $n \to \infty$, as a consequence of the Riemann-Lebesgue lemma. Thus we have an example of the coefficient decay advertised.

stress acting on the end of the plate in Problem A. Therefore, the additional contributions to the edge strains for Problem A must satisfy

$$k^2 \frac{\partial \bar{u}^a}{\partial x}(0, y, p) + (k^2-2)\frac{\partial \bar{v}^a}{\partial y}(0, y, p) = 0, \tag{2.6}$$

for $0 \le y < h$, $p > 0$. On approaching the corner $x = 0$, $y = h$, one sees, in the light of (2.6) and the plate-face conditions (1.5), that

$$\lim_{\substack{y \to h \\ y < h}} \left\{ \frac{\partial \bar{v}^a}{\partial y}(0, y, p) \right\} = 0. \tag{2.7}$$

This *corner condition* suggests that the cosine series chosen for $\frac{\partial \bar{v}^a}{\partial y}(0, y, p)$ be $\sum_{n=1,3,5,\ldots}^{\infty} b_n'(p) \cos \frac{n\pi y}{2h}$. The absence of any such corner condition for $\frac{\partial \bar{u}^a}{\partial y}(0, y, p)$ implies $\frac{\partial \bar{u}^a}{\partial y}(0, y, p)$ will not be zero at $y = h$ in general and thus dictates that the companion sine series be taken for this term. We then have

$$\left.\begin{aligned} \frac{\partial \bar{u}^a}{\partial y}(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} a_n'(p) \sin \frac{n\pi y}{2h}, \\ \frac{\partial \bar{v}^a}{\partial y}(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} b_n'(p) \cos \frac{n\pi y}{2h}, \end{aligned}\right\} \tag{2.8}$$

for $0 \le y \le h$, $p > 0$. The accompanying extensions for these *quarter-range Fourier series* † representations of $\frac{\partial \bar{u}^a}{\partial y}(0, y, p)$

† It is apparent from the continuations specified that the series representations in (2.8) can be drawn from the half-range series on [0, 2h] in much the same manner that the half-range representations on [0, h] can be extracted from the full Fourier series (sine and cosine terms) on [-h, h]. Thus a Fourier theorem holds true for the series of (2.8) and the name, quarter-range Fourier series, is appropriate.

and $\frac{\partial \bar{v}^a}{\partial y}(0, y, p)$ are defined, by:

on the closed interval [h, 2h],

$$\frac{\partial \bar{u}^a}{\partial y}(0, h+y, p) = \frac{\partial \bar{u}^a}{\partial y}(0, h-y, p) \text{ and}$$

$$\frac{\partial \bar{v}^a}{\partial y}(0, h+y, p) = -\frac{\partial \bar{v}^a}{\partial y}(0, h-y, p), \quad (0 \le y \le h);$$

on [-2h, 2h],

$$\frac{\partial \bar{u}^a}{\partial y}(0, -y, p) = -\frac{\partial \bar{u}^a}{\partial y}(0, y, p) \text{ and}$$

$$\frac{\partial \bar{v}^a}{\partial y}(0, -y, p) = \frac{\partial \bar{v}^a}{\partial y}(0, y, p), \quad (0 \le y \le 2h);$$

and on]-∞, ∞[,

$$\frac{\partial \bar{u}^a}{\partial y}(0, 4mh+y, p) = \frac{\partial \bar{u}^a}{\partial y}(0, y, p) \text{ and}$$

$$\frac{\partial \bar{v}^a}{\partial y}(0, 4mh+y, p) = \frac{\partial \bar{v}^a}{\partial y}(0, y, p), \quad (-2h \le y \le 2h),$$

m any integer. As a result of the corner condition, this is a continuous definition of these functions of y on]-∞, ∞[. Basically here we have precluded the possibility of Gibbs phenomena at y = h in our series expansions, thereby ensuring the necessary decay in the coefficients.

Integrating (2. 8); on exchanging $\frac{-2h a_n'(p)}{n\pi}$ for $a_n(p)$, $\frac{2h b_n'(p)}{n\pi}$ for $b_n(p)$, $\bar{u}^a(0, h, p)$ for u(p), and invoking the symmetry criteria (1. 6); generates

$$\left.\begin{aligned} \bar{u}^a(0, y, p) &= u(p) + \sum_{n=1,3,5,\ldots}^{\infty} a_n(p) \cos \frac{n\pi y}{2h}, \\ \bar{v}^a(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} b_n(p) \sin \frac{n\pi y}{2h}, \end{aligned}\right\} \quad (2.9)$$

for $0 \le y \le h$, $p > 0$. $u(p)$ is the transformed *corner displacement* in the x direction.

Observe that (2. 9) automatically establishes the validity of term-by-term differentiation of the series there. Notice also, that, as a consequence of the large n behaviour secured for $a_n'(p)$, $b_n'(p)$, the coefficients in (2. 9) must obey the order conditions

$$a_n(p) = o\left(\frac{1}{n^2}\right), \qquad b_n(p) = o\left(\frac{1}{n^2}\right), \text{ as } n \to \infty. \quad (2.10)$$

To guarantee the satisfaction of the relevant edge conditions for $\frac{\partial \bar{u}^a}{\partial y}(0, y, p)$, $\frac{\partial \bar{v}^a}{\partial y}(0, y, p)$, $\frac{\partial \bar{u}^a}{\partial x}(0, y, p)$ and $\frac{\partial \bar{v}^a}{\partial x}(0, y, p)$; to wit, the second of (2. 1), and (2. 6); the last pair of these terms are chosen as below

$$\left.\begin{aligned} \frac{\partial \bar{u}^a}{\partial x}(0, y, p) &= -\left(\frac{k^2-2}{k^2}\right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi b_n(p)}{2h} \cos \frac{n\pi y}{2h}, \\ \frac{\partial \bar{v}^a}{\partial x}(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi a_n(p)}{2h} \sin \frac{n\pi y}{2h}, \end{aligned}\right\} \quad (2.11)$$

for $0 \le y \le h$, $p > 0$.

Now adding the contributions from the elementary theory given in (2. 5) (with an integration of the last result there to provide

$'\bar{v}^e(0, y, p)'$) and the corresponding terms from (2.9) and (2.11) gives

$$\left.\begin{aligned}
\bar{u}(0, y, p) &= \frac{\sigma_A c_p}{4\mu p^2}\left(\frac{k^2}{k^2-1}\right) + u(p) + \sum_{n=1,3,5,\ldots}^{\infty} a_n(p)\cos\frac{n\pi y}{2h}, \\
\bar{v}(0, y, p) &= \frac{\sigma_A y}{4\mu p}\left(\frac{k^2-2}{k^2-1}\right) + \sum_{n=1,3,5,\ldots}^{\infty} b_n(p)\sin\frac{n\pi y}{2h}, \\
\frac{\partial\bar{u}}{\partial x}(0, y, p) &= \frac{-\sigma_A}{4\mu p}\left(\frac{k^2}{k^2-1}\right) - \left(\frac{k^2-2}{k^2}\right)\sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi b_n(p)}{2h}\cos\frac{n\pi y}{2h}, \\
\frac{\partial\bar{v}}{\partial x}(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi a_n(p)}{2h}\sin\frac{n\pi y}{2h},
\end{aligned}\right\} \quad (2.12)$$

for $0 \leq y \leq h$, $p > 0$. (2.12) concludes the postulation of the edge unknowns in Problem A since it will yield, after differentiation of the first two expressions therein coupled with the use of (1.15), the apposite terms for substitution into the boundedness condition, (1.24).

On performing this substitution, and the subsequent simple integrations, one procures

$$U(s_j, p)u(p) + \sum_{n=1,3,5,\ldots}^{\infty} [A_n(s_j, p)a_n(p) + B_n(s_j, p)b_n(p)] = Q_A(s_j, p), \quad (2.13)$$

where

$$
\left.
\begin{aligned}
U(s_j, p) &= \frac{k_d^2}{\alpha_j^2}(k^2-2)Y(s_j, p), \\
A_n(s_j, p) &= \frac{(-1)^{\frac{n-1}{2}} \frac{n\pi s_j}{h}\left(\frac{k^2-1}{k^2}\right)\left[\frac{n^2\pi^2}{2h^2}+k_s^2\right]}{\left[\left(\frac{n\pi}{2h}\right)^2+\alpha_j^2\right]\left[\left(\frac{n\pi}{2h}\right)^2+\beta_j^2\right]}, \\
B_n(s_j, p) &= \frac{(-1)^{\frac{n-1}{2}}}{\left[\left(\frac{n\pi}{2h}\right)^2+\beta_j^2\right]}\left\{\frac{\frac{n^2\pi^2 s_j^2}{h^2}\left(\frac{k^2-1}{k^2}\right)}{\left[\left(\frac{n\pi}{2h}\right)^2+\alpha_j^2\right]} - k_s^2\right\},
\end{aligned}
\right\} \qquad (2.14)
$$

$$
Q_A(s_j, p) = Q_c(s_j, p) + Q'_A(s_j, p), \qquad (2.15)
$$

$$
Q_c(s_j, p) = \frac{\sigma_A c_p}{4\mu\alpha_j^2 c_s^2}\left(\frac{k^2-2}{k^2-1}\right)Y(s_j, p), \qquad (2.16)
$$

$$
Q'_A(s_j, p) = \frac{-\sigma_A k_s^2}{2\mu p\alpha_j^2\beta_j^2}\left(\frac{k^2-2}{k^2}\right)\left[s_j Y(s_j, p) - \frac{k^2\alpha_j^2 h}{2(k^2-1)}\right], \qquad (2.17)
$$

and in which $Y(s_j, p)$ is as in (1.23). Equations (2.13) through (2.17) comprise an infinite set of algebraic equations for $u(p)$ and the Fourier coefficients $a_n(p)$, $b_n(p)$, p being small. To solve (2.13) for $p \to 0$ we first consider the $s_j(p)$ complying with (1.21) for small p.

Asymptotics on $R(s, p)$ of (1.13) give

$$
R(s, p) = -is^2 k_s^2\left(\frac{k^2-1}{k^2}\right)(\sin 2sh + 2sh) + O(p^4), \qquad (2.18)
$$

as $p \to 0$. Hence the $s_j(p)$ for $p \to 0$ are selected from the complex zeros of

$$r(s) = \sin 2sh + 2sh = 0, \tag{2.19}$$

as those roots having a positive real part. In fact, only those roots in the first quadrant are required in the present analysis because the elements in (2.13) are symmetric about the real s-axis, that is, $U^*(s_j, p) = U(s_j^*, p)$, $A_n^*(s_j, p) = A_n(s_j^*, p)$, $B_n^*(s_j, p) = B_n(s_j^*, p) = Q_A^*(s_j, p) = Q_A(s_j^*, p)$, the * referring to the complex conjugate of the quantity beneath it. Robbins and Smith in [5] list the first 10 (in order of increasing real part) non-zero values of 2sh satisfying (2.19) in this quadrant. They also cite asymptotic expressions for large roots.

The behaviour of the slopes of $s_j(p)$ as $p \to 0$ can be determined by recognizing that along the branches of $R(s_j(p), p) = 0$,

$$dR(s_j(p), p) = \frac{\partial R}{\partial s}(s_j(p), p)ds_j(p) + \frac{\partial R}{\partial p}(s_j(p), p)dp = 0,$$

and therefore, from (2.18), (2.19), and an application of l'Hôpital's rule,

$$\lim_{\substack{p\to 0\\ p>0}} \left\{ \frac{ds_j(p)}{dp} \right\} = \lim_{\substack{p\to 0\\ p>0}} \left\{ \frac{-\frac{\partial R}{\partial p}(s_j(p), p)}{\frac{\partial R}{\partial s}(s_j(p), p)} \right\}$$

$$= \lim_{\substack{p\to 0\\ p>0}} \left\{ \frac{-2s_j(p)r(s_j(p))}{p[2r(s_j(p)) + s_j(p)\frac{dr}{ds}(s_j(p))]} \right\}$$

$$= \lim_{\substack{p\to 0\\ p>0}} \left\{ \frac{-2[r(s_j(p)) + s_j(p)\frac{dr}{ds}(s_j(p))]\frac{ds_j(p)}{dp}}{2r(s_j(p))+s_j(p)\frac{dr}{ds}(s_j(p))+p[3\frac{dr}{ds}(s_j(p))+s_j(p)\frac{d^2r}{ds^2}(s_j(p))]} \right\},$$

whence, since $\lim_{\substack{p \to 0 \\ p > 0}} r(s_j(p)) = 0$ and from (2.19), $\lim_{\substack{p \to 0 \\ p > 0}} \frac{dr}{ds}(s_j(p)) \neq 0$,

$$\lim_{\substack{p \to 0 \\ p > 0}} \left\{ \frac{ds_j(p)}{dp} \right\} = -2 \lim_{\substack{p \to 0 \\ p > 0}} \left\{ \frac{ds_j(p)}{dp} \right\},$$

i.e.

$$\lim_{\substack{p \to 0 \\ p > 0}} \left\{ \frac{ds_j(p)}{dp} \right\} = 0 .$$

It follows, as remarked in [1], that the zeros of $r(s)$ in (2.19) are a good approximation to the $s_j(p)$ for a range of p small but greater than zero.

Continuing with our solution of (2.13) for small p, we expand α_j, β_j and $\Upsilon(s_j, p)$ as $p \to 0$ to arrive at

$$\left.\begin{aligned} \alpha_j &= \alpha_j(s_j, p) = is_j\left[1 - \frac{k_d^2}{2s_j^2} + O(p^4)\right], \\ \beta_j &= \beta_j(s_j, p) = is_j\left[1 - \frac{k_s^2}{2s_j^2} + O(p^4)\right], \\ \Upsilon(s_j, p) &= -\tan s_j h + O(p^2) . \end{aligned}\right\} \qquad (2.20)$$

(2.14), (2.16), (2.17) and (2.20) then imply

$$\left.\begin{aligned}
U(s_j, p) &= \frac{(k^2-2)k_d^2}{s_j^2}\tan s_j h + \bigcirc(p^4), \\
A_n(s_j, p) &= \frac{n^3\pi^3 s_j}{2h^3 D_n(s_j)^2}\left(\frac{k^2-1}{k^2}\right)(-1)^{\frac{n-1}{2}} + \bigcirc(p^2), \\
B_n(s_j, p) &= \frac{n^2\pi^2 s_j^2}{h^2 D_n(s_j)^2}\left(\frac{k^2-2}{k^2-1}\right)(-1)^{\frac{n-1}{2}} + \bigcirc(p^2), \\
Q_c(s_j, p) &= \frac{\sigma_A c_p}{4\mu s_j^2 c_s^2}\left(\frac{k^2-2}{k^2-1}\right)\tan s_j h + \bigcirc(p^2), \\
\mathcal{Q}_A(s_j, p) &= \frac{\sigma_A k_d^2}{2\mu p s_j^3}(k^2-2)\left[\tan s_j h - \frac{s_j h k^2}{2(k^2-1)}\right] + \bigcirc(p^3),
\end{aligned}\right\} \quad (2.21)$$

as $p \to 0$, $D_n(s_j) = (\frac{n\pi}{2n})^2 - s_j^2$.

Now, on the basis of the premise that our elementary theory will in fact describe the nature of the dominant time variation in the very long-time, we require $\text{ord}\,\{u(p)\} \le \text{ord}\,\{\frac{1}{p^2}\}$, $\text{ord}\{a_n(p)\} \le \text{ord}\{\frac{1}{p}\}$, and $\text{ord}\,\{b_n(p)\} \le \text{ord}\,\{\frac{1}{p}\}$, for $p \to 0$. Moreover, for these terms to have significant contributions to the long-time solution we need the orders of all three quantities to be greater than one. In toto then, we seek $u(p)$, $a_n(p)$, $b_n(p)$ such that

$$\left.\begin{aligned}
&\text{ord}\,\{1\} < \text{ord}\,\{u(p)\} \le \text{ord}\,\{\frac{1}{p^2}\}, \\
&\text{ord}\,\{1\} < \left\{\begin{matrix}\text{ord}\,\{a_n(p)\} \\ \text{ord}\,\{b_n(p)\}\end{matrix}\right\} \le \text{ord}\,\{\frac{1}{p}\},
\end{aligned}\right\} \quad (2.22)$$

as $p \to 0$.

Considering the leading terms of the real and imaginary parts of the quantities in (2.21) one sees that

$$\left.\begin{aligned} &\mathrm{ord}\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\{U(s_j,p)\} = \mathrm{ord}\,\{p^2\},\ \ \mathrm{ord}\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\{\Omega_c(s_j,p)\} = \mathrm{ord}\,\{1\},\\ &\mathrm{ord}\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\{A_n(s_j,p)\} = \mathrm{ord}\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\{B_n(s_j,p)\} = \mathrm{ord}\,\{1\}, \end{aligned}\right\} \tag{2.23}$$

whilst

$$\mathrm{ord}\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\{\mathcal{Q}_A(s_j,p)\} = \mathrm{ord}\,\{p\}, \tag{2.24}$$

for $p \to 0$. Thus, and as a consequence of (2.22), the integrated boundedness condition (2.13), on using (2.15), (2.21), reduces to

$$\sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{n^2 s_j}{D_n(s_j)^2}\left[\frac{n\pi}{2h} a_n(p) + s_j b_n(p)\right] = 0, \tag{2.25}$$

for all s_j satisfying (1.21).

Concerning (2.25), we first remark that the boundedness condition has now been freed of u(p). Hence u(p) endures as an unknown term at this juncture.† This indeterminancy may reasonably be attributed to the near-field, long-time domain 'asymptotically approaching' a second boundary-value problem in elastostatics

† u(p)'s determination awaits a later application of the boundedness condition, in its original form (1.4), during the course of the inversion process.

(stresses prescribed), since this type of static problem admits an arbitrary, infinitesimal, rigid displacement field. †

In this connection observe also that, as u(p) can be ord $\{\frac{1}{p^2}\}$ as $p \to 0$, we are in essence allowing the possibility of some other plate wave speed, c_p', say. Though we expect $c_p' = c_p$, we have not yet shown this.

As a final remark on (2.25), we note the solution

$$a_n(p) = b_n(p) = 0, \qquad n = 1, 3, 5, \ldots . \ ‡ \tag{2.26}$$

(2.26) insists that, for the present small p approximation, the only supplementary terms to the edge unknown estimates derived from the elementary theory - be such terms decreasing in t, non-decreasing in t, or increasing in t at no faster than a linear rate - are confined to u(p).

From (2.12), (2.26), we have, for the six edge unknowns,

† Cf. the elastodynamic problem of the second type which has no such arbitrary displacement field.

‡ It might appear, since $a_n(p)$, $b_n(p)$ of (2.26) fulfill the order requirement of (2.10) trivially, that the care in the selection of the series representations was unwarranted. Certainly, in the present instance this is so; however, in Problem B of Section 3 our efforts are vindicated.

$$
\left.\begin{aligned}
\bar{u}(0, y, p) &= \frac{\sigma_A c_p}{4\mu p^2}\left(\frac{k^2}{k^2-1}\right) + u(p), \\
\bar{v}(0, y, p) &= \frac{\sigma_A y}{4\mu p}\left(\frac{k^2-2}{k^2-1}\right), \\
\frac{\partial \bar{u}}{\partial x}(0, y, p) &= -\left(\frac{k^2}{k^2-2}\right)\frac{\partial \bar{v}}{\partial y}(0, y, p) = \frac{-\sigma_A}{4\mu p}\left(\frac{k^2}{k^2-1}\right), \\
\frac{\partial \bar{u}}{\partial y}(0, y, p) &= \frac{\partial \bar{v}}{\partial x}(0, y, p) = 0,
\end{aligned}\right\} \quad (2.27)
$$

$0 \le y \le h$, p small.

We now undertake the inversion process for Problem A. Introducing (2.27) into (1.18) by means of (1.15), carrying out the elementary integrations encountered thereupon then substituting the resulting forms into (1.17), yields, on using (1.13), (1.19), (1.20) and simplifying,

$$
\left.\begin{aligned}
\tilde{\bar{u}}(s, y, p) = {} & \frac{-\sigma_A}{4\mu p\alpha^2\beta^2(k^2-1)}\Bigg[\left\{\frac{c_p}{p} + \frac{4\mu p\, u(p)}{\sigma_A}\left(\frac{k^2-1}{k^2}\right)\right\} s k^2\beta^2 \\
& - 3k_s^2 + 2k_d^2 + k^2 s^2 - \frac{k_d^2(k^2-2)}{R(s, p)}\Big\{k^2\alpha^2\beta\, h\phi^u(s, y, p) \\
& - \left(\frac{c_p s}{p} k^2\beta^2 + \frac{4\mu p\, u(p)}{\sigma_A} s(k^2-1)\beta^2 - 2s^2(k^2-1)\right)\psi^u(s, y, p)\Big\}\Bigg], \\
\tilde{\bar{v}}(s, y, p) = {} & \frac{-\sigma_A}{4\mu p\alpha\beta^2}\left(\frac{k^2-2}{k^2-1}\right)\Bigg[\alpha s y - \frac{k_d^2}{R(s, p)}\Big\{k^2\alpha s h\phi^v(s, y, p) \\
& - \left(\frac{c_p}{p} k^2\beta^2 + \frac{4\mu p u(p)}{\sigma_A}(k^2-1)\beta^2 - 2s(k^2-1)\right)\psi^v(s, y, p)\Big\}\Bigg].
\end{aligned}\right\} \quad (2.28)
$$

(2.28) holds for small p. In tackling its inversion we treat three ranges of s separately:

$\frac{s}{p} \to \infty$ as $p \to 0$;

$\frac{s}{p} = c$ as $p \to 0$, c a constant ($\neq 0$);

and $\frac{s}{p} \to 0$ as $p \to 0$.

The first range corresponds to the near-field. For $p \to 0$, s relatively large, R(s, p), α and β are given by (2.18), (2.20). Using (2.20) in (1.20) one sees that

$$\left.\begin{aligned} \psi^u(s,y,p) &= O(p^2), & \phi^u(s,y,p) &= O(p^2), \\ \psi^v(s,y,p) &= O(p^2), & \phi^v(s,y,p) &= O(p^2), \end{aligned}\right\} \quad (2.29)$$

as $p \to 0$. Thus, under this limiting process, (2.28) becomes

$$\tilde{\tilde{u}}(s,y,p) = \frac{\sigma_A}{4\mu ps}\left(\frac{k^2}{k^2-1}\right)\left[\frac{c_p}{p} - \frac{1}{s}\right] + \frac{u(p)}{s} + O(1),$$

$$\tilde{\tilde{v}}(s,y,p) = \frac{\sigma_A y}{4\mu ps}\left(\frac{k^2}{k^2-1}\right) + O(1),$$

as $p \to 0$, whence, on recalling the order conditions for u(p) (see (2.22)),

$$\left.\begin{aligned} u(x,y,t) &\sim \frac{\sigma_A}{4\mu}\left(\frac{k^2}{k^2-1}\right)[c_p t - x] + u_c t^{\gamma}, \\ v(x,y,t) &\sim \frac{\sigma_A y}{4\mu}\left(\frac{k^2-2}{k^2-1}\right), \end{aligned}\right\} \quad (2.30)$$

for $0 < x \le X_n$, $0 \le y \le h$, and $t \to \infty$, where u_c is a constant having the dimensions of length; $\gamma \le 1$; and X_n demarks the extent of the near-field. In view of the forms found for the edge unknowns (see (2.27)), the region of validity for (2.30) may be expanded to include $x = 0$. Due to the uniformity of the asymptotics on p in s and y, for this case, (2.30) may be differentiated with respect to x and y to produce the near-field, long-time strains.

Now for s and $p \to 0$ concurrently,

$$R(s, p) = -4(k^2 - 1)\beta h k_d^2 \left[s^2 - \frac{p^2}{c_p^2} \right] + O(p^7). \tag{2.31}$$

(2.31) shows that the zeros of R(s, p) agree asymptotically with the low-frequency, long-wave approximation recorded earlier in (2.4). (1.20), for s and $p \to 0$ simultaneously, reduces to

$$\left.\begin{aligned}
\psi^u(s, y, p) &= -k_d^2 \beta h(k^2 - 2) + O(p^5), \\
\phi^u(s, y, p) &= k_s^2 + O(p^4), \\
\psi^v(s, y, p) &= -k_s^2 \alpha\beta h y + O(p^6), \\
\phi^v(s, y, p) &= -k_d^2 \beta y(k^2 - 2) + O(p^5).
\end{aligned}\right\} \tag{2.32}$$

Accordingly, (2.28), subject to this limiting procedure, gives

$$\left.\begin{aligned}\tilde{\tilde{u}}(s, y, p) &= \frac{\sigma_A c_p}{4\mu p^2 (s + \frac{p}{c_p})}\left[\left(\frac{k^2}{k^2-1}\right) + O(p^2)\right] + \frac{su(p)}{(s^2 - \frac{p^2}{c_p^2})},\\ \tilde{\tilde{v}}(s, y, p) &= \frac{\sigma_A y}{4\mu p (s + \frac{p}{c_p})}\left[\left(\frac{k^2-2}{k^2-1}\right) + O(p^2)\right] - \frac{yk_s^2 u(p)}{4(s^2 - \frac{p^2}{c_p^2})}\left(\frac{k^2-2}{k^2-1}\right).\end{aligned}\right\} \quad (2.33)$$

Both u(p) components in the above have poles in the right-half, s-plane at $s = p/c_p$. These poles are not admissible in view of the condition at infinity (1.4) (recall the argument to this effect in Section 1). We therefore set u(p) = 0; which in turn implies $u_c = 0$ in (2.30). Inversion of the surviving terms in (2.33) then produces

$$\left.\begin{aligned}u(x, y, t) &\sim \frac{\sigma_A}{4\mu}\left(\frac{k^2}{k^2-1}\right)[c_p t-x]\, U(c_p t-x),\\ v(x, y, t) &\sim \frac{\sigma_A y}{4\mu}\left(\frac{k^2-2}{k^2-1}\right) U(c_p t-x),\end{aligned}\right\} \quad (2.34)$$

as $t \to \infty$. Comparison of (2.34) with the reduced version of (2.30) ($u_c = 0$) demonstrates that the former applies on $0 \le x$, $0 \le y \le h$, for t large.

The analogous asymptotics for $s\tilde{\tilde{u}}(s, y, p)$ and $\frac{\partial \tilde{\tilde{v}}}{\partial y}(s, y, p)$, with u(p) set equal to zero, furnish

$$
\left.\begin{aligned}
s\tilde{\tilde{u}}(s, y, p) &= \frac{\sigma_A c_p s}{4\mu p^2 (s + \frac{p}{c_p})}\left[\left(\frac{k^2}{k^2-1}\right) + O(p^2)\right], \\
\frac{\partial \tilde{\tilde{v}}}{\partial y}(s, y, p) &= \frac{\sigma_A}{4\mu p (s + \frac{p}{c_p})}\left[\left(\frac{k^2-2}{k^2-1}\right) + O(p^2)\right],
\end{aligned}\right\} \quad (2.35)
$$

whence,

$$
\left.\begin{aligned}
\frac{\partial u}{\partial x}(x, y, t) &\sim \frac{-\sigma_A}{4\mu}\left(\frac{k^2}{k^2-1}\right) U(c_p t - x), \\
\frac{\partial v}{\partial y}(x, y, t) &\sim \frac{\sigma_A}{4\mu}\left(\frac{k^2-2}{k^2-1}\right) U(c_p t - x),
\end{aligned}\right\} \quad (2.36)
$$

as $t \to \infty$, for $0 \le x$, $0 \le y \le h$; extension up to and including $x=0$ again being legitimate on account of the matching with corresponding near-field values. (2.36) is our first far-field approximation for Problem A and exhibits waves that are non-decaying in space and time. Interesting is the fact that (2.34), (2.36) could have been obtained from the elementary theory, though this is *not* what we have done in the present treatment. On the contrary, we have shown that the elementary solution, and the elementary solution alone, is obtained from the first long-time approximation of the exact theory for Problem A.

To facilitate the illustration of results found here and subsequently, we introduce the notation

$$\left.\begin{aligned}
\hat{x} &= \frac{x}{h}, \qquad \hat{y} = \frac{y}{h}, \qquad \hat{t} = \frac{c_p t}{h}, \\
\hat{u} &= \hat{u}(\hat{x}, \hat{y}, \hat{t}) = u(\hat{x}, \hat{y}, \hat{t})/h, \\
\hat{v} &= \hat{v}(\hat{x}, \hat{y}, \hat{t}) = v(\hat{x}, \hat{y}, \hat{t})/h.
\end{aligned}\right\} \qquad (2.37)$$

With this nomenclature, the dimensionless equivalent of (2.36) is

$$\frac{\partial \hat{u}}{\partial \hat{x}} \Big/ \hat{\sigma}_A \sim -\left(\frac{k^2}{k^2-2}\right) \frac{\partial \hat{v}}{\partial \hat{y}} \Big/ \hat{\sigma}_A \sim -U(\hat{t}-\hat{x}), \qquad (2.38)$$

as $\hat{t} \to \infty$, for $0 \le \hat{x}$, $0 \le \hat{y} \le 1$. Here

$$\hat{\sigma}_A = \frac{\sigma_A}{4\mu}\left(\frac{k^2}{k^2-1}\right). \qquad (2.39)$$

(2.38) attests the long-time *Poisson's ratio coupling* of the *longitudinal* and *thickness strains*.

A higher order approximation than (2.36), based on a second approximation to R(s, p) rather than the first in (2.31), is available for s and p tending to zero together. † This approximation, which gives more of the wave features than (2.36), closely represents the physical nature of a longitudinal pulse in the far-field of an end-loaded rod or plate. Accordingly such an approximation will be derived here.

† The derivation of this approximation is now well-known; see [1] for a selection of pertinent references.

The residues associated with the inner integral of (1.8) at

$$s = \frac{-p}{c_p}\left[1 - \frac{k'h^2p^2}{c_p^2}\right], \quad k' = \frac{1}{6}\left(\frac{k^2-2}{k^2}\right)^2, \dagger$$

in view of (2.35), are given by

$$\left\{\begin{matrix}\text{Residue} \\ \text{at } s = \\ \frac{-p}{c_p}\left[1 - \frac{k'h^2p^2}{c_p^2}\right]\end{matrix}\right\}\left\{\begin{matrix} s\tilde{\bar{u}}(s, y, p) \\ \\ \frac{\partial \tilde{\bar{v}}}{\partial y}(s, y, p)\end{matrix}\right\} = \frac{-\sigma_A}{4\mu p}\left(\frac{k^2}{k^2-1}\right)e^{-\frac{px}{c_p}\left[1 - \frac{k'h^2p^2}{c_p^2}\right]}\left\{\begin{matrix} 1 \\ \\ -\left(\frac{k^2-2}{k^2}\right)\end{matrix}\right\} + O(p).$$

Consequently, (1.8), (2.39) imply

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(x, y, t) &\sim \frac{-\hat{\sigma}_A}{2\pi i}\int_{Br_p} e^{p\left\{t - \frac{x}{c_p}\left[1 - \frac{k'h^2p^2}{c_p^2}\right]\right\}}\frac{dp}{p}, \\ \frac{\partial v}{\partial y}(x, y, t) &\sim -\left(\frac{k^2-2}{k^2}\right)\frac{\partial u}{\partial x}(x, y, t), \end{aligned}\right\} \quad (2.40)$$

as x, $t \to \infty$. The second of these reiterates the Poisson's ratio coupling of (2.38) for the present approximation. The sole singularity of the integrand in (2.40) is a simple pole at p = 0. Therefore, Br_p may be chosen along the imaginary p-axis provided we indent the contour near p = 0. Collapsing this indentation - thereby collecting the residue contribution at the origin - setting $p = i\omega$ and combining the resulting integrals from $-\infty$ to 0 and from 0 to ∞, leads to

† Cf. (2.4)

$$\frac{\partial u}{\partial x}(x, y, t)/\hat{\sigma}_A \sim \frac{1}{\pi}\int_0^\infty \sin\left(\frac{\omega x}{c_p}\left[1 + \frac{k' h^2 \omega^2}{c_p^2}\right] - \omega t\right)\frac{d\omega}{\omega} - \frac{1}{2},$$

for x, $t \to \infty$. Based on stationary phase arguments and certain transformations - see for instance, Skalak, [6] - this result may be reduced to, on introducing our dimensionless variables,

$$\frac{\partial \hat{u}}{\partial \hat{x}}\Big/\hat{\sigma}_A \sim -\left[\int_0^\eta Ai(-\eta')d\eta' + \frac{1}{3}\right], \tag{2.41}$$

for x, $t \to \infty$. Here $\eta = \dfrac{\hat{t}-\hat{x}}{3\sqrt{3k'\hat{x}}}$; $Ai(-\eta) = \int_0^\infty \cos(\omega^3 - \eta\omega)d\omega$, the Airy function. $\dfrac{\partial \hat{v}}{\partial \hat{y}}$ follows from (2.40).

This higher order approximation was first found by Skalak, [6], in a mixed, edge condition, problem. In the closely related elastic rod problem it has been shown to be in close agreement with experimental results - see discussion in [1].

Note that our second approximation is also non-decaying in space and time, but, unlike our first approximation, exhibits dispersion. To compare the two approximations, Fig. 3 shows a plot of them both.

To conclude the inversion of the small p, formal solution for Problem A, we consider the third s range, namely; p small, $s \to 0$. For this limiting procedure (2.28), with u(p) = 0, gives

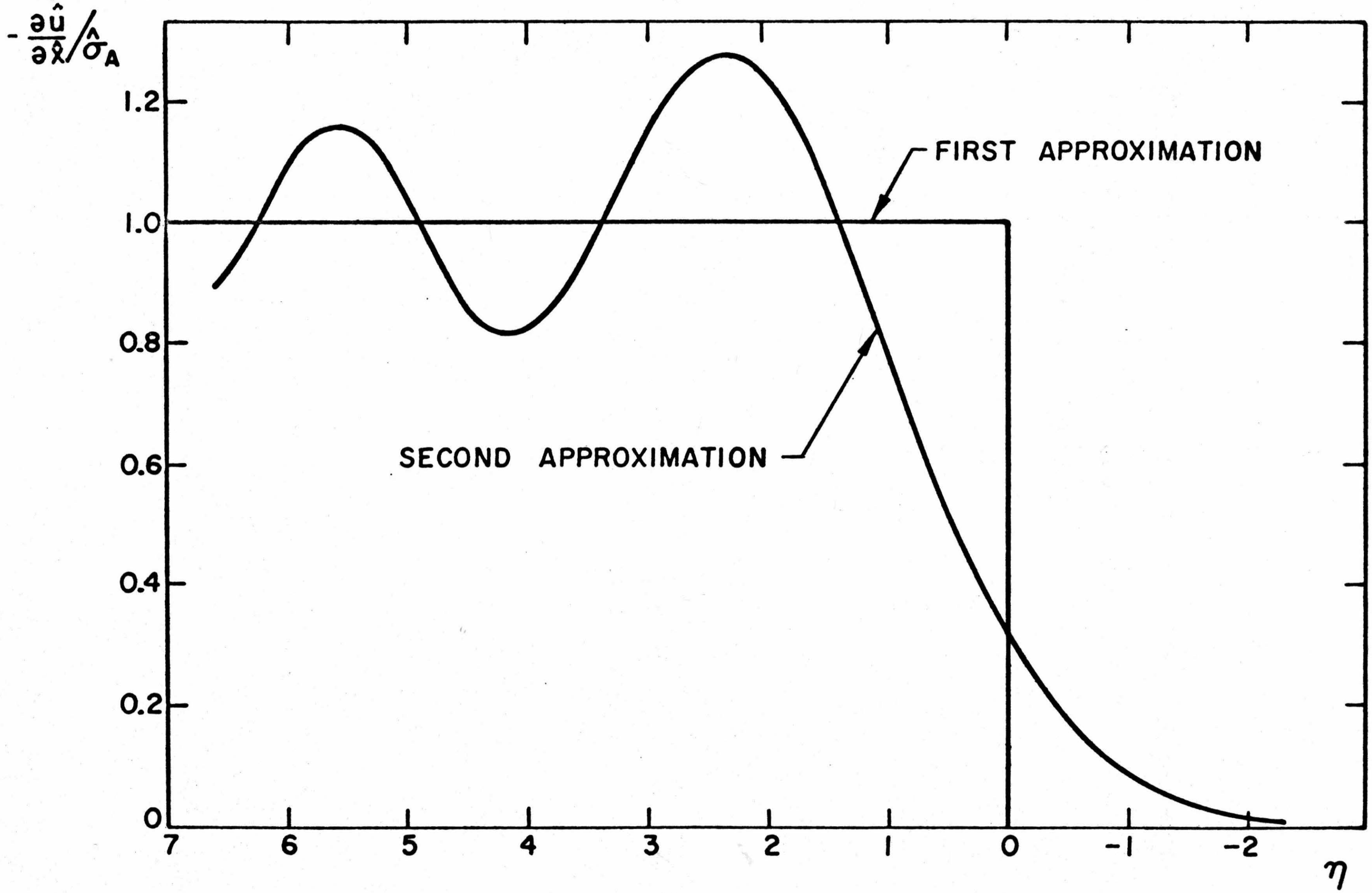

Fig. 3. Long-time, far-field response of the longitudinal strain

$$\tilde{\tilde{u}}(s, y, p) = O(1), \qquad \tilde{\tilde{v}}(s, y, p) = O(1),$$

as $s \to 0$ and hence we have no contributions to the displacements for t large, $x \to \infty$. Similarly it may be shown that the strains are zero as $x \to \infty$ and thus our solution is in accord with the conditions at infinity, (1.4).

3. PROBLEM B: LINE-LOAD

We now parallel the procedures adopted in Section 2 to procure long-time information for a line-load impact on the end of our wave-guide (Fig. 4). Thus σ_B is a positive quantity with the dimensions of force per unit length; $\delta(y)$ is the *symmetric delta*, a generalized function defined by

$$\delta(y) = \lim_{\Delta \to 0} \delta(y;\Delta), \tag{3.1}$$

where

$$\delta(y;\Delta) = \frac{1}{2\Delta}\left[U(y+\Delta) - U(y-\Delta)\right], \tag{3.2}$$

with $\Delta > 0$.

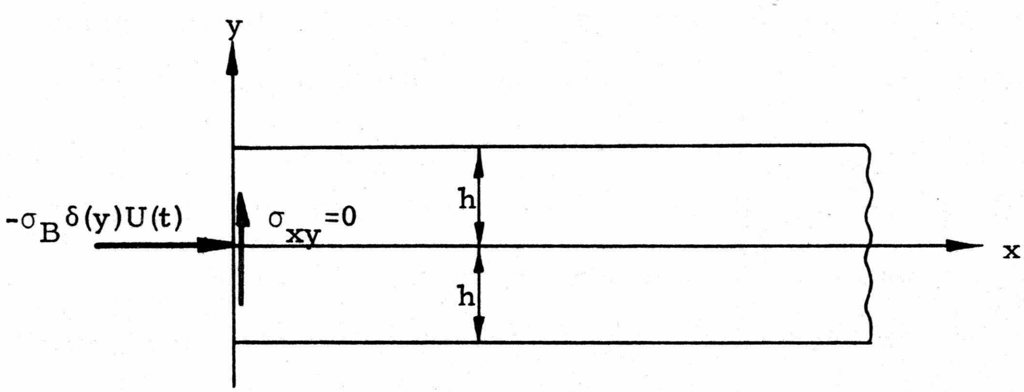

Fig. 4. Edge conditions for Problem B

Operations on $\delta(y)$ will be defined, in general, as the result of performing the equivalent operation on $\delta(y;\Delta)$ then taking the limit $\Delta \to 0$. In view of this, the edge conditions for Problem B - which may be expressed through (1.2) by

$$\left.\begin{aligned} \bar{\sigma}_x(0,y,p) &= \mu\left[k^2 \frac{\partial \bar{u}}{\partial x}(0,y,p) + (k^2-2)\frac{\partial \bar{v}}{\partial y}(0,y,p)\right] = -\frac{\sigma_B \delta(y)}{p}, \\ \bar{\sigma}_{xy}(0,y,p) &= \mu\left[\frac{\partial \bar{u}}{\partial y}(0,y,p) + \frac{\partial \bar{v}}{\partial x}(0,y,p)\right] = 0, \end{aligned}\right\} \quad (3.3)$$

for $0 \le y < h$, $p > 0$ - can be regarded as the conditions satisfied by the problem arrived at as the limit, $\Delta \to 0$, of the sequence of normal loadings, $\sigma_x(0,y,t) = -\sigma_B \delta(y;\Delta)U(t)$, acting on the edge of our plate.

As in Section 2, we must now postulate forms for the edge unknowns in order to 'open up' the boundedness condition (1.24). In undertaking this postulation, the only differences encountered, between the pattern established for Problem A and that required for Problem B, will be occasioned by the singular nature of the latter problem. Hence, to facilitate the appraisal of such singular nature, we resolve Problem B, for the long-time, into three problems (Fig. 5):

Problem B1...the line-load on the elastostatic half-space, sometimes referred to as the Flamant problem. Problem B1 will provide the long-time *singular parts of the edge unknowns.*

Problem B2...the residual associated elastostatic problem. The stresses applied to the plate-faces here, σ_y', $\pm\sigma_{xy}'$ ($\sigma_y' > 0$, $\sigma_{xh}' > 0$), are of the same magnitude as those acting on the corresponding

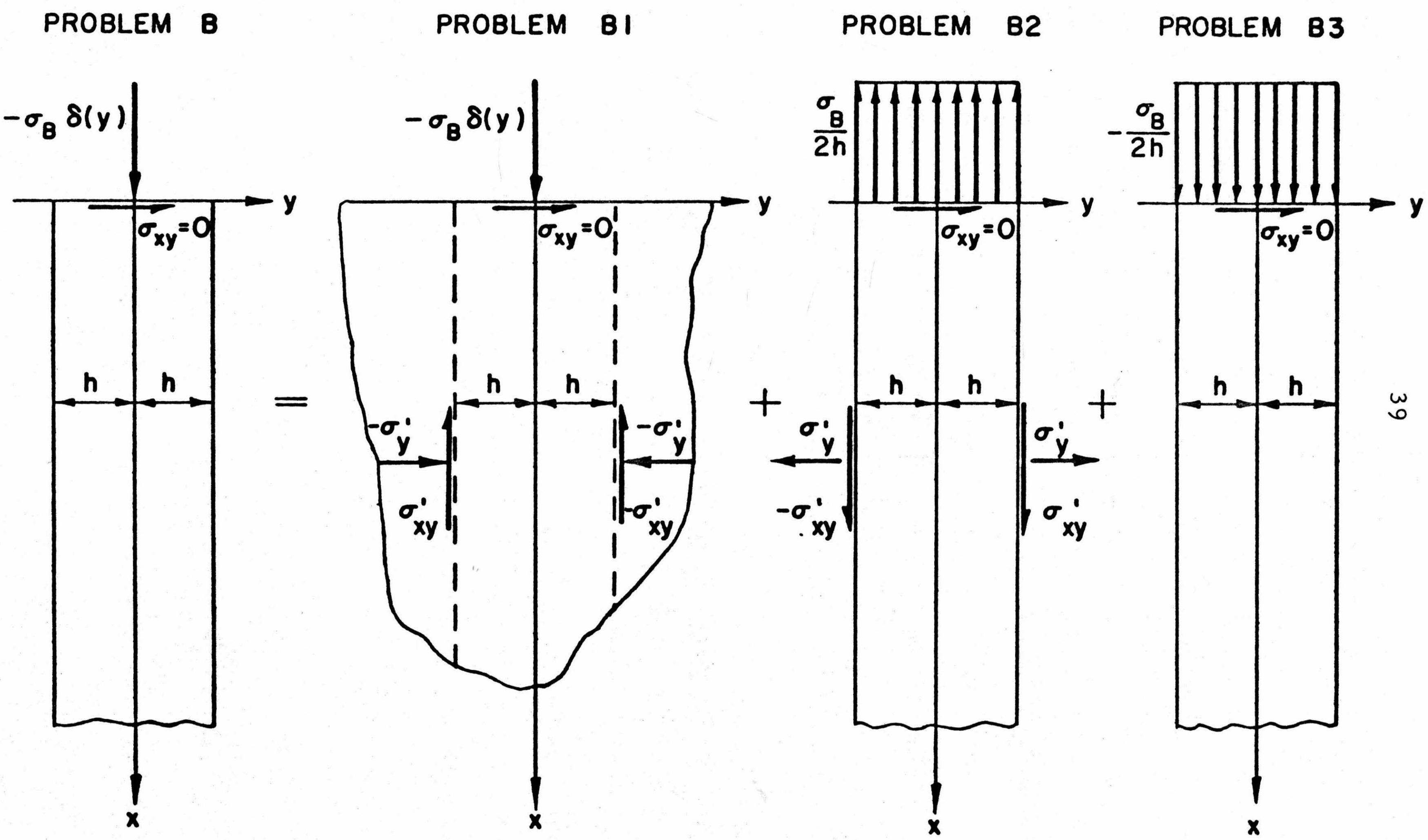

Fig. 5. Decomposition of Problem B

sections of the half-space in Problem B1, but opposite in sign.

Problem B2 is made self-equilibrating by the introduction of a uniform normal stress, $\frac{\sigma_B}{2h}$, acting on the plate edge. The attendant edge values for this problem will contribute to the *regular parts of the edge unknowns.*

Problem B3... the uniform normal load applied to our waveguide, recognizable as Problem A with a modified stress input. Problem B3 will furnish the dominant, long-time, time dependence of the edge quantities, thereby completing the selection of the representations for the regular parts of the edge unknowns.

We now consider each problem in this decomposition individually; first, the singular problem.

It is apparent, from the edge unknown involvement in the boundedness condition, that to perform the requisite integrations in (1.24) we will need to determine the values at the origin of $\bar{u}^s(0, y, p)$, $\bar{v}^s(0, y, p)$, $\frac{\partial \bar{u}^s}{\partial y}(0, y, p)$, $\frac{\partial \bar{v}^s}{\partial y}(0, y, p)$, $\frac{\partial \bar{u}^s}{\partial x}(0, y, p)$, $\frac{\partial \bar{v}^s}{\partial y}(0, y, p)$ - the superscript s here denoting the respective quantity's singular source. In order to obtain this additional information on the well-known Flamant problem, we solve Problem B1 as the limit, $\Delta \to 0$, of the sequence of *kernel problems* indicated by $\delta(y, \Delta)$ of (3.2).

A kernel problem will thus constitute a second boundary-value problem in elastostatics with:

$\sigma_x(0, y; \Delta) = -\sigma_B\, \delta(y; \Delta)$, $\sigma_{xy}(0, y; \Delta) = 0$, for $-\infty < y < \infty$;

and $\sigma_x(x, y; \Delta) = o(1)$, $\sigma_y(x, y; \Delta) = o(1)$, $\sigma_{xy}(x, y; \Delta) = o(1)$ as $r \to \infty$, $r = \sqrt{x^2+y^2}$.

This problem is tractable to Fourier transformation on y as a means of deriving its solution.† In what follows, we confine our attention to the results of such a derivation.

Consider the displacement field, for the half-space,

$$\left.\begin{aligned} u(x,y;\Delta) &= \frac{\sigma_B}{4\Delta\mu\pi(k^2-1)}\left\{x\left[\arctan\left(\frac{y-\Delta}{x}\right)-\arctan\left(\frac{y+\Delta}{x}\right)\right]\right. \\ &\quad\left.+k^2\left[(y-\Delta)\ln\sqrt{x^2+(y-\Delta)^2}-(y+\Delta)\ln\sqrt{x^2+(y+\Delta)^2}\right]\right\}, \\ v(x,y;\Delta) &= \frac{\sigma_B}{4\Delta\mu\pi(k^2-1)}\left\{(y-\Delta)\arctan\left(\frac{y-\Delta}{x}\right)-(y+\Delta)\arctan\left(\frac{y+\Delta}{x}\right)\right. \\ &\quad\left.+k^2x\left[\ln\sqrt{x^2+(y+\Delta)^2}-\ln\sqrt{x^2+(y-\Delta)^2}\right]\right\}, \end{aligned}\right\} \quad (3.4)$$

wherein the arctan ranges from $-\pi/2$ to $\pi/2$.‡ Differentiating (3.4) and substituting the terms found thereon in the elastostatic analogue of (1.2), gives

$$\left.\begin{aligned} \sigma_x(x,y;\Delta) &= \frac{-\sigma_B}{2\Delta\pi}\left\{\arctan\left(\frac{2\Delta x}{x^2+y^2-\Delta^2}\right)+\frac{2\Delta x(x^2-y^2+\Delta^2)}{[x^2+(y+\Delta)^2][x^2+(y-\Delta)^2]}\right\}, \\ \sigma_y(x,y;\Delta) &= \frac{-\sigma_B}{2\Delta\pi}\left\{\arctan\left(\frac{2\Delta x}{x^2+y^2-\Delta^2}\right)-\frac{2\Delta x(x^2-y^2+\Delta^2)}{[x^2+(y+\Delta)^2][x^2+(y-\Delta)^2]}\right\}, \\ \sigma_{xy}(x,y;\Delta) &= \frac{-2\sigma_B x^2 y}{\pi[x^2+(y+\Delta)^2][x^2+(y-\Delta)^2]}. \end{aligned}\right\} \quad (3.5)$$

† See Sneddon, [7], Art. 45.

‡ Throughout our treatment of Problem B1, results will apply at discontinuous and singular points only in some 'integrable sense'. Nevertheless, since providing values for the integration of (1.24) is the ultimate purpose of the treatment, existence in such a sense suffices.

Here the arctan varies between 0 and π. Clearly the stresses in (3. 5) conform with our boundary and order conditions for a kernel problem. A second differentiation of (3. 4) and substitution into the time-independent counterpart of (1. 1) reveals that $u(x, y;\Delta)$, $v(x, y;\Delta)$ satisfy the displacement equations of equilibrium. Consequently (3. 4), (3. 5) comprise the appropriate values for a kernel problem. In particular, for $x = 0$, (3. 4) yields

$$\left.\begin{aligned}
u(0, y;\Delta) &= \frac{\sigma_B k^2}{4\Delta\mu\pi(k^2-1)} \{(y-\Delta)\ln|y-\Delta| - (y+\Delta)\ln|y+\Delta|\}, \\
v(0, y;\Delta) &= \frac{-\sigma_B}{4\mu(k^2-1)} \operatorname{sgn}(y;\Delta), \\
\frac{\partial u}{\partial x}(0, y;\Delta) &= \frac{\partial v}{\partial y}(0, y;\Delta) = \frac{-\sigma_B}{2\mu(k^2-1)} \delta(y;\Delta), \\
\frac{\partial u}{\partial y}(0, y;\Delta) &= -\frac{\partial v}{\partial x}(0, y;\Delta) = \frac{\sigma_B k^2}{4\Delta\mu\pi(k^2-1)} \ln\left|\frac{y-\Delta}{y+\Delta}\right|,
\end{aligned}\right\} \qquad (3.6)$$

in which

$$\operatorname{sgn}(y;\Delta) = (1 - \frac{y}{\Delta})U(y-\Delta) + (1 + \frac{y}{\Delta})U(y+\Delta) - 1. \qquad (3.7)$$

Proceeding to the limit $\Delta \to 0$ in (3. 4), (3. 5) produces displacements and stresses in agreement with the usual forms, valid away from the origin, for the Flamant problem. The same limiting process in (3. 6) implies that we take, as forms for the singular components of the edge unknowns for Problem B:

$$\left.\begin{aligned} \bar{u}^s(0, y, p) &= \frac{-\sigma_B k^2}{2\mu\pi p(k^2 - 1)} \, \ell n\left(\frac{y}{h}\right), \\ \frac{\partial \bar{u}^s}{\partial y}(0, y, p) &= -\frac{\partial \bar{v}^s}{\partial x}(0, y, p) = \frac{-\sigma_B k^2}{2\mu\pi p(k^2 - 1)} \left(\frac{1}{y}\right), \end{aligned}\right\} \quad (3.8)$$

for $0 < y \leq h$, $p > 0$;

$$\left.\begin{aligned} \bar{v}^s(0, y, p) &= \frac{-\sigma_B}{4\mu p (k^2 - 1)} \, \mathrm{sgn}(y), \\ \frac{\partial \bar{u}^s}{\partial x}(0, y, p) &= \frac{\partial \bar{v}}{\partial y}(0, y, p) = \frac{-\sigma_B}{2\mu p(k^2 - 1)} \, \delta(y) \end{aligned}\right\} \quad (3.9)$$

on $0 \leq y \leq h$ as integrable quantities, † for $p > 0$; where $\mathrm{sgn}(y) = 2U(y) - 1$, is the signum function, with $\mathrm{sgn}(0)$ defined to be zero. Here $\bar{u}^s(0, y, p)$ is determined to within an arbitrary, infinitesimal, rigid body displacement; such indefiniteness is consistent with an elastostatic, second boundary-value problem and will be incorporated in a $u(p)$ term in the same manner as in Section 2.

The forms in (3.8), (3.9) fulfill the edge conditions (3.3). Indeed, the use of (3.3) in conjunction with the standard results for the Flamant problem and the definition of the derivative of the signum function by

$$\frac{d}{dy}\{\mathrm{sgn}(y)\} = \lim_{\Delta \to 0} \frac{d}{dy}\{\mathrm{sgn}(y;\Delta\},$$

† As intimated earlier, the integral from zero to some upper limit of an integrand containing $\delta(y)$ will be defined to be equal to the limit, $\Delta \to 0$, of the equivalent integral entailing $\delta(y;\Delta)$.

$sgn(y;\Delta)$ as in (3.7); viz. $\frac{d}{dy}\{sgn(y)\} = 2\delta(y)$; will furnish (3.8), (3.9).

One should bear in mind that currently we are merely postulating the forms for the singular parts of the edge unknowns. Accordingly (3.8), (3.9) are only reasonable guesses as to what these terms might be, based on the thesis that the long-time singular nature, in the near-field, of a problem involving, exclusively, outward propagating disturbances, is the same as the singular behaviour of the corresponding elastostatic problem. Should it transpire that this is not the case for Problem B, the convergence of the Fourier series representations that we subsequently select will, at best, be attained in a generalized sense.

For use in the next problem we set y = h in the limiting values ($\Delta \to 0$) of $\sigma_y(x, y;\Delta)$, $\sigma_{xy}(x, y;\Delta)$ of (3.5) to obtain

$$\left.\begin{aligned} \sigma_y(x, h) &= -\frac{2\sigma_B}{\pi h}\frac{(x/h)}{[1 + (x/h)^2]^2}, \\ \sigma_{xy}(x, h) &= -\frac{2\sigma_B}{\pi h}\frac{(x/h)^2}{[1 + (x/h)^2]^2}, \end{aligned}\right\} \quad (3.10)$$

$x \geq 0$.

In Problem B2 we continue our theme that the analysis of corresponding elastostatic problems will provide long-time, near-field information; putting $\sigma_y' = -\sigma_y(x, h)$, $\sigma_{xy}' = -\sigma_{xy}(x, h)$, with $\sigma_y(x, h)$, $\sigma_{xy}(x,h)$ as in (3.10). Integration of these surface stresses, σ_y', σ_{xy}', then shows this problem to be self-equilibrated. Hence it is amenable to a finite-element approach. Such an approach is outlined in Appendix 1.

In choosing edge forms for Problem B2 we observe the similarity of the prescribed stresses at x = 0 with those in Problem A, suggesting as suitable representations of the edge unknowns for the present problem the expressions in (2.12) with σ_A therein replaced by $\frac{-\sigma_B}{2h}$. However, Problem B2 has, in addition to edge stresses, the equilibrating plate-face stresses, σ'_y, σ'_{xy}. This implies that we modify these forms by omitting the c_p/p^2 term.

Now turning to Problem B3, we note its complete equivalence to Problem A subject to $\frac{\sigma_B}{2h}$ being exchanged for σ_A. It follows that the appropriate forms for Problem B3 are contained in (2.27) on swapping σ_A there for $\frac{\sigma_B}{2h}$.

Combining the edge representations for these last two problems will cancel the elements in $\bar{v}(0, y, p)$, $\frac{\partial \bar{u}}{\partial x}(0, y, p)$ which give rise to $\pm \frac{\sigma_B}{2h}$, and thus produce the compact forms

$$\left.\begin{aligned}
\bar{u}^r(0, y, p) &= \frac{\sigma_B c_p}{8\mu h p^2}\left(\frac{k^2}{k^2-1}\right) + u(p) + \sum_{n=1,3,5,\ldots}^{\infty} a_n(p)\cos\frac{n\pi y}{2h}, \\
\bar{v}^r(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} b_n(p)\sin\frac{n\pi y}{2h}, \\
\frac{\partial \bar{u}^r}{\partial x}(0, y, p) &= -\left(\frac{k^2-2}{k^2}\right)\sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi b_n(p)}{2h}\cos\frac{n\pi y}{2h}, \\
\frac{\partial \bar{v}^r}{\partial x}(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi a_n(p)}{2h}\sin\frac{n\pi y}{2h},
\end{aligned}\right\} \quad (3.11)$$

for $0 \leq y \leq h$, $p > 0$. Here, the r atop quantities signifies their regular nature (which is assumed at this juncture); the u(p) is the amalgamation of the two 'u(p)' in (2.12), (2.27).

Contingent upon the validity of the thesis that the forms in (3.8), (3.9) will describe all the singular contributions to the edge unknowns, the terms in (3.11) will, in fact, be regular and, consequently, the $a_n(p)$, $b_n(p)$ there will be subject to the large n, order condition, (2.10).† Further, such regularity then guarantees that term-by-term differentiation of the displacement representations in (3.11) is legitimate. Thus, adjoining (3.8), (3.9) to (3.11) and the terms, $\frac{\partial \bar{u}^r}{\partial y}(0, y, p)$, $\frac{\partial \bar{v}^r}{\partial y}(0, y, p)$ obtained therefrom, we have

$$\left.\begin{aligned}
\bar{u}(0, y, p) &= \frac{\sigma_B}{2\mu p}\left(\frac{k^2}{k^2-1}\right)\left[\frac{cp}{4hp} - \frac{1}{\pi}\ln\left(\frac{y}{h}\right)\right] + u(p) + \sum_{n=1,3,5,\ldots}^{\infty} a_n(p)\cos\frac{n\pi y}{2h}, \\
\bar{v}(0, y, p) &= \frac{-\sigma_B}{4\mu p(k^2-1)}\,\mathrm{sgn}(y) + \sum_{n=1,3,5,\ldots}^{\infty} b_n(p)\sin\frac{n\pi y}{2h}, \\
\frac{\partial \bar{u}}{\partial y}(0, h, p) = -\frac{\partial \bar{v}}{\partial x}(0, y, p) &= \frac{-\sigma_B}{2\mu\pi p}\left(\frac{k^2}{k^2-1}\right)\left(\frac{1}{y}\right) - \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi a_n(p)}{2h}\sin\frac{n\pi y}{2h}, \\
\frac{\partial \bar{u}}{\partial x}(0, y, p) &= \frac{-\sigma_B}{2\mu p(k^2-1)}\,\delta(y) - \left(\frac{k^2-2}{k^2}\right)\sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi b_n(p)}{2h}\cos\frac{n\pi y}{2h}, \\
\frac{\partial \bar{v}}{\partial y}(0, y, p) &= \frac{-\sigma_B}{2\mu p(k^2-1)}\,\delta(y) + \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi b_n(p)}{2h}\cos\frac{n\pi y}{2h},
\end{aligned}\right\} \quad (3.12)$$

for $0 < y \le h$, $p > 0$ - with extension to include $y = 0$ in an integrable sense whenever this is asked for by the boundedness condition, (1.24). (3.12) ends our postulation of the edge unknowns for Problem B.

† To see this, recall the argument on quarter-range Fourier series in Section 2.

We note that: the $u(p)$, $a_n(p)$, $b_n(p)$ terms in (3.12) are the same as in (2.12); the c_p terms would be the same if $\frac{\sigma_B}{2h}$ in (3.12) was replaced by σ_A. Hence, on using the integral definitions

$$\mathrm{Shi}(y) = \int_0^y \frac{\sinh y'}{y'}\,dy' \text{ (the sinh integral)},$$

$$\int_0^h \delta(y')\cosh\{{}^{\alpha}_{\beta}\}y'dy' = \lim_{\Delta\to 0}\int_0^h \delta(y';\Delta)\cosh\{{}^{\alpha}_{\beta}\}y'dy' \; (=\tfrac{1}{2}, \text{ by } (3.2)),$$

(1.24) and (3.12) together yield

$$U(s_j,p)u(p) + \sum_{n=1,3,5,\ldots}^{\infty}[A_n(s_j,p)a_n(p) + B_n(s_j,p)b_n(p)] = Q_B(s_j,p), \tag{3.13}$$

where

$$\left.\begin{aligned}
Q_B(s_j,p) &= Q'_c(s_j,p) + Q'_B(s_j,p),\\
Q'_c(s_j,p) &= \frac{\sigma_B c_p}{8\mu h\alpha_j^2 c_s^2}\left(\frac{k^2-2}{k^2-1}\right)Y(s_j,p),\\
Q'_B(s_j,p) &= \frac{\sigma_B}{2\mu\pi p k_s^2}\left(\frac{k^2}{k^2-1}\right)\left\{\frac{2s_j}{\alpha_j\cosh\alpha_j h}[\pi s_j\alpha_j - \mathrm{Shi}(\alpha_j h)(k_s^2-2k_d^2+2s_j^2)]\right.\\
&\quad\left. + \frac{(2s_j^2-k_s^2)}{2\beta_j^2\cosh\beta_j h}[\pi(2s_j^2-2k_s^2+k_d^2)+4\mathrm{Shi}(\beta_j h)s_j\beta_j] - \frac{\pi k_d^2 k_s^2}{2\beta_j^2}\right\},
\end{aligned}\right\}\tag{3.14}$$

and $Y(s_j,p)$, $U(s_j,p)$, $A_n(s_j,p)$, $B_n(s_j,p)$ are given in (1.23), (2.14).

We must next solve this infinite system of linear equations in the infinite set of unknowns comprised of u(p), $a_n(p)$, $b_n(p)$; for small p. $U(s_j, p)$, $A_n(s_j, p)$, $B_n(s_j, p)$, for p small, are as in (2.21). (2.16), (2.21), (3.14) imply

$$Q'_c(s_j, p) = \frac{\sigma_B c_p}{8\mu s_j^2 c_s^2}\left(\frac{k^2-2}{k^2-1}\right)\tan s_j h + O(p^2), \text{ as } p \to 0. \tag{3.15}$$

$\mathrm{Shi}(\alpha_j h)$, $\mathrm{Shi}(\beta_j h)$, for small p, may be found by using (2.20) and the series expansions for Shi(y),† sinhy, and are given by

$$\mathrm{Shi}\begin{Bmatrix}\alpha_j y\\ \beta_j y\end{Bmatrix} = i\left[\mathrm{Si}(s_j y) - \frac{1}{2s_j^2}\begin{Bmatrix}k_d^2\\ k_s^2\end{Bmatrix}\sin s_j y + O(p^4)\right], \text{ as } p \to 0, \tag{3.16}$$

in which $\mathrm{Si}(y) = \int_0^y \frac{\sin y'}{y'}\,dy'$, the sine integral. (2.20), (3.16) then imply

$$Q_B(s_j, p) = \frac{\sigma_B}{2\mu\pi p \cos s_j h}\{[\pi - 2\mathrm{Si}(s_j h)][1 + s_j h \tan s_j h] - 2\sin s_j h + O(p^2)\} \tag{3.17}$$

as $p \to 0$.

Now, on the basis of the premise that Problem B3 will in fact describe the nature of the dominant time variation in the very long-time, we require u(p), $a_n(p)$, $b_n(p)$ to comply with the same small p order condition as we had in Problem A, to wit (2.22). Considering the leading terms of the real and imaginary parts of the quantities in (2.21), (3.15), (3.17), one sees that (2.23) holds with $Q_c(s_j, p)$ therein

† See [8], equation (5.2.17).

exchanged for $Q'_C(s_j, p)$, and

$$\text{ord}\begin{Bmatrix}\text{Re}\\ \text{Im}\end{Bmatrix}\{Q'_B(s_j, p)\} = \text{ord}\left\{\frac{1}{p}\right\}.$$

Thus, as a consequence of these order stipulations, the integrated boundedness condition (3.13), on using (2.21), (3.14), (3.15), (3.17), reduces to

$$\sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{4n^2\pi^2 z_j}{[z_j^2-n^2\pi^2]^2}\{n\pi\hat{a}_n + z_j\hat{b}_n\} = \{[\pi-2\text{Si}(\frac{z_j}{2})][1+\frac{z_j}{2}\tan\frac{z_j}{2}] - 2\sin\frac{z_j}{2}\}/\cos\frac{z_j}{2}, \tag{3.18}$$

wherein all quantities have been rendered dimensionless by the introduction of

$$\hat{a}_n = \frac{pa_n(p)}{\hat{\sigma}_B h}, \qquad \hat{b}_n = \frac{pb_n(p)}{\hat{\sigma}_B h}, \qquad z_j = 2s_j h, \tag{3.19}$$

where

$$\hat{\sigma}_B = \frac{\sigma_B}{2\mu h\pi}\left(\frac{k^2}{k^2-1}\right). \tag{3.20}$$

(3.18) holds for all z_j satisfying $\sin z_j + z_j = 0$. Accordingly we employ the *method of reduction* whereby:

one takes the first N non-zero roots, z_j, (in order of increasing real part), with Re $z_j > 0$, Im $z_j > 0$,† and solves the 2N×2N finite system,

† This set of roots suffices since (3.18) holds for z_j^* if it holds for z_j.

that results on decomposing (3.18) into real and imaginary parts, for $\hat{a}_1, \hat{a}_3, \ldots \hat{a}_{2N-1}, \hat{b}_1, \hat{b}_3, \ldots, \hat{b}_{2N-1}$; then increases the number of roots taken to N+1, solving the enlarged finite set of linear equations for the first N+1 $\hat{a}_n$s and $\hat{b}_n$s; and continues enlarging the system until the $\hat{a}_n$, $\hat{b}_n$ estimates have attained *stability*, i.e. the differences in the $\hat{a}_n$, $\hat{b}_n$ estimates from successive finite matrices is negligible.

The z_j and $Si(z_j/2)$ needed to solve these finite systems are tabulated in Appendix 2 (for the first 24 zeros). The method used to compute z_j is the same as in [5]; the method used to compute $Si(z_j/2)$ is outlined in Appendix 2.

The results for the first ten $\hat{a}_n$, $\hat{b}_n$, found using 24 roots, are displayed in Table 1.†

Table 1. Fourier coefficient estimates

n	$\hat{a}_n$	$\hat{b}_n$
1	-0.1129	0.2891
3	-0.0242	0.0036
5	0.0077	-0.0026
7	-0.0030	0.0011
9	0.0014	-0.0005
11	-0.0007	0.0003
13	0.0004	-0.0001
15	-0.0003	0.0001
17	0.0002	-0.0001
19	-0.0001	0.0000

† These results are independent of elastic constants. This follows from their non-dimensioning in (3.19), (3.20) and the form of (3.18).

We define $\Delta^N\{\hat{a}_n\}$, $\Delta^N\{\hat{b}_n\}$, as measures of stability, by

$$\Delta^N\{\hat{a}_n\} = |\hat{a}_n(N) - \hat{a}_n(N-2)|,$$

where $\hat{a}_n(N)$ are the $\hat{a}_n$ calculated using N roots, z_j, with a similar definition for $\Delta^N\{\hat{b}_n\}$. For the values in Table 1, it is found that $\Delta^{24}\{\hat{a}_n\} < 3\times10^{-6}$, $\Delta^{24}\{\hat{b}_n\} < 10^{-6}$, with $\Delta^{24}\{\hat{a}_n\}$ uniformly (in n) less than $\Delta^{22}\{\hat{a}_n\}$, $\Delta^{24}\{\hat{b}_n\}$ uniformly less than $\Delta^{22}\{\hat{b}_n\}$. Hence it would appear that our coefficient estimates are sufficiently stable, viz. the values listed are probably correct to the 4 decimal places quoted.

The numerical decay of the $\hat{a}_n$, $\hat{b}_n$ estimates is faster than $1/n^2$ for $n \geq 5$, in agreement with our large n order condition, (2.10). Indeed, for $n \geq 7$, these values decrease faster than $1/n^3$. Such numerical convergence supports our thesis that the long-time, near-field, singular nature of Problem B is the same as the singular behaviour of the corresponding elastostatic problem.

Using $\hat{a}_n$, $\hat{b}_n$ of Table 1, the edge displacements associated with Problem B2 can be evaluated, thus enabling comparison with the finite-element analysis of this problem (Appendix 1). The displacement sets so computed agree to within 1% of the finite-element values, for all y.† Appendix 1 contains the relevant plots for this comparison.

Turning to our forms for the edge unknowns for Problem B, (3.12), and carrying out the simple inversion on p, allows us to define

† Equivalent to within 0.1% of u, v in (3.21).

the *static edge displacements* for Problem B, u and v, by

$$\left.\begin{aligned} u &= \left[u(0,y,t) - \frac{\hat{\sigma}_B \pi c_p}{4} t\right] \Big/ \hat{\sigma}_B h \\ &= -\ell n \hat{y} + \hat{u}_c + \sum_{n=1,3,5\ldots}^{\infty} \hat{a}_n \cos\frac{n\pi\hat{y}}{2}, \\ v &= v(0,y,t)/\hat{\sigma}_B h \\ &= -\frac{\pi}{2k^2}\,\mathrm{sgn}(\hat{y}) + \sum_{n=1,3,5,\ldots}^{\infty} \hat{b}_n \sin\frac{n\pi\hat{y}}{2}, \end{aligned}\right\} \qquad (3.21)$$

for $0 < \hat{y} \le 1$. Here the dimensionless terms in (2.37), (3.19), (3.20) have been introduced, and we have anticipated a result yet to be established, namely

$$u(p) = \frac{\hat{\sigma}_B h \hat{u}_c}{p}. \qquad (3.22)$$

(3.22) will be a by-product of the inversion process (cf. u(p) for Problem A). Note that u, v of (3.21) are independent of time - hence the name static edge displacements.

The attendant displacement gradients, u_y and v_y, arising from (2.37), (3.12), (3.19), (3.20), are

$$\left.\begin{aligned} u_y &= -\left[\frac{1}{\hat{y}} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi\hat{a}_n}{2} \sin\frac{n\pi\hat{y}}{2}\right], \\ v_y &= \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi\hat{b}_n}{2} \cos\frac{n\pi\hat{y}}{2}, \end{aligned}\right\} \qquad (3.23)$$

for $0 < \hat{y} \leq 1$. v_y has a symmetric delta at $y = 0$. The numbers in Table 1 afford a means of calculating u, v, u_y, v_y, which are then plotted for $k^2 = 7/2$ (Fig. 6). In Fig. 6 we include the value of $\hat{u}_c$, 0.935, which is actually determined in the course of the inversion process.

In proceeding to the formal solution of Problem B, for small p, one substitutes (3.12) into (1.18) by means of (1.15) and performs the integrations encountered thereupon, using, in particular,

$$\text{Cinh}(y) = \int_0^y \frac{\cosh y' - 1}{y'} dy' \text{ (the cosh integral).}$$

Substitution of the forms that eventuate into (1.17), and simplifying using (1.13), (1.19), (1.20), yields the counterpart of (2.28) in Section 2. The algebra here is straightforward but tedious, and the forms so produced, lengthy. For the sake of brevity, we suppress these forms here.

We next outline the inversion process, focusing on the results found, and closely following the methods in Section 2.

For the near-field inversion, we require; in addition to (2.18), (2.20), (2.29); (3.16) and the companion relation for the cosh integral (derived similarly)

$$\text{Cinh}\begin{Bmatrix} \alpha y \\ \beta y \end{Bmatrix} = -\left[\text{Cin}(sy) + \frac{1}{2s^2}\begin{Bmatrix} k_d^2 \\ k_s^2 \end{Bmatrix}(\cos sy - 1) + O(p^4)\right],$$

as $p \to 0$, wherein $\text{Cin}(y) = \int_0^y \frac{1-\cos y'}{y'} dy'$. These expressions coupled

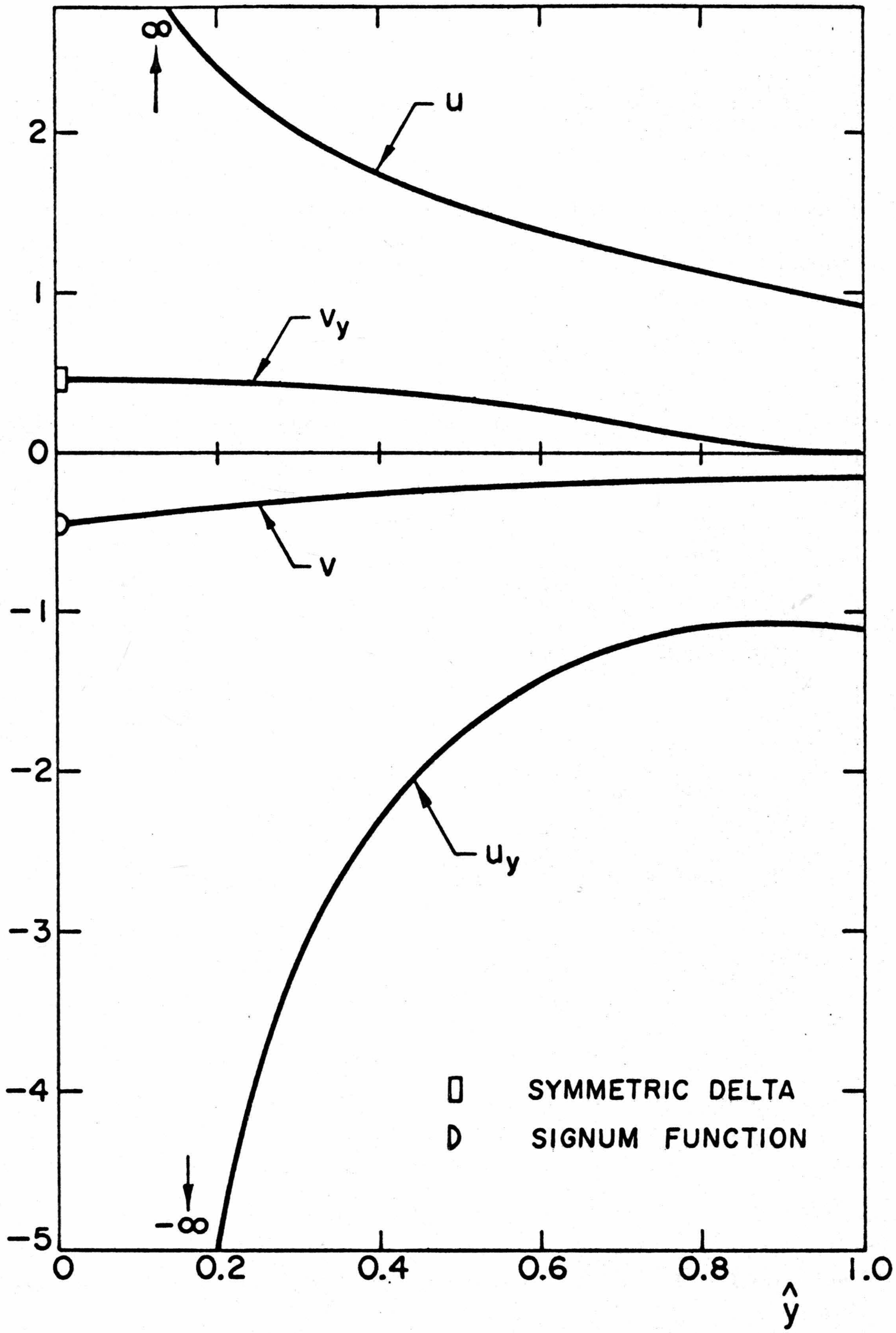

Fig. 6. Static edge displacements and derivatives thereof for Problem B

with the pertinent asymptotic forms from [8], Chapter 5, allow us to recover the edge values on taking the limit, $s \to \infty$; to wit (3.12) on inversion with respect to p.

For s, $p \to 0$, concurrently, (2.31), (2.32), the series expansions of Shi(y), Cinh(y), and the small p formal solution of Problem B, imply

$$\left.\begin{aligned}
\tilde{\tilde{u}}(s,y,p) &= \frac{\sigma_B c_p}{8\mu h p^2 (s+\frac{p}{c_p})}\left[\left(\frac{k^2}{k^2-1}\right)+O(p^2)\right] \\
&\quad + \frac{s}{(s^2-\frac{p^2}{c_p^2})}\left[u(p)-\frac{\sigma_B k^2}{2\mu\pi p(k^2-1)}\left\{1+\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\frac{2\hat{a}_n}{n\pi}\right\}\right], \\
\tilde{\tilde{v}}(s,y,p) &= \frac{\sigma_B y}{8\mu h p(s+\frac{p}{c_p})}\left[\left(\frac{k^2-1}{k^2-1}\right)+O(p^2)\right] \\
&\quad - \frac{sy}{(s^2-\frac{p^2}{c_p^2})}\left(\frac{k^2-2}{k^2}\right)\left[su(p)-\frac{\sigma_B k^2}{2\mu\pi c_p(k^2-1)}\left\{1+\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\frac{2\hat{a}_n}{n\pi}\right\}\right].
\end{aligned}\right\} \quad (3.24)$$

The inadmissible pole at $s = \frac{p}{c_p}$ is removed from both expressions in (3.24) by taking

$$u(p) = \frac{\hat{\sigma}_B h}{p}\left\{1+\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\frac{2\hat{a}_n}{n\pi}\right\}. \quad (3.25)$$

This is the form that we previously anticipated in (3.22). Using the

$\hat{a}_n$ of Table 1, (3.22), (3.25) imply

$$\hat{u}_c = \left\{ 1 + \sum_{n=1,3,5,\ldots} (-1)^{\frac{n-1}{2}} \frac{2\hat{a}_n}{n\pi} \right\} = 0.935,$$

the value employed in Fig. 6.

Inversion of the surviving terms in (3.24) now produces

$$\left.\begin{aligned} u(x,y,t) &\sim \frac{\sigma_B}{8\mu h}\left(\frac{k^2}{k^2-1}\right)\left[c_p t - x\right] U(c_p t - x), \\ v(x,y,t) &\sim \frac{\sigma_B y}{8\mu h}\left(\frac{k^2-2}{k^2-1}\right) U(c_p t - x), \end{aligned}\right\} \quad (3.26)$$

as x, $t \to \infty$, for $0 \le y \le h$. Analogous asymptotics for $s\tilde{\bar{u}}(s,y,p)$ and $\frac{\partial \tilde{\bar{v}}}{\partial y}(s,y,p)$, with u(p) as in (3.25), furnish

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(x,y,t) &\sim \frac{-\sigma_B}{8\mu h}\left(\frac{k^2}{k^2-1}\right) U(c_p t - x), \\ \frac{\partial v}{\partial y}(x,y,t) &\sim \frac{\sigma_B}{8\mu h}\left(\frac{k^2-2}{k^2-1}\right) U(c_p t - x), \end{aligned}\right\} \quad (3.27)$$

x, t large, $0 \le y \le h$. These last two equations, (3.26) and (3.27), are our first far-field approximation for Problem B. Derivation of the second far-field approximation, using the dimensionless forms in (2.37), (3.20), gives

$$\frac{\partial \hat{u}}{\partial \hat{x}} \Big/ \hat{\sigma}_B \sim -\frac{\pi}{4}\left[\int_0^{\eta} Ai(-\eta')d\eta' + \frac{1}{3}\right], \quad (3.28)$$

$$\frac{\partial \hat{v}}{\partial \hat{x}} \sim -\left(\frac{k^2-2}{k^2}\right)\frac{\partial \hat{u}}{\partial \hat{x}}, \quad (3.29)$$

as x, $t \to \infty$. Comparison of (2.35), (2.36), (2.41), (2.40), in that order, with (3.26), (3.27), (3.28), (3.29), respectively, demonstrates that the long-time, far-field approximations for Problems A and B are the same if equal normal forces act on the edge, $x = 0$, in both problems; that is, if $\sigma_A = \frac{\sigma_B}{2h}$.

Inversion for p small, $s \to 0$, shows the formal solution for Problem B to comply with the infinity condition (1.4).

REFERENCES

[1] J. Miklowitz, 'Analysis of elastic waveguides involving an edge,' in Wave Propagation in Solids. ASME Publications, N. Y. (1969).

[2] J. P. Benthem, ' A Laplace transform method for the solution of semi-infinite and finite strip problems in stress analysis.' Quart. J. Mech. Appl. Math. 16, 413 (1963).

[3] J. Miklowitz, 'On solving elastic waveguide problems involving non-mixed edge conditions.' ONR Technical Report No. 5, Contract Nonr-220(57), California Institute of Technology (Aug. 1967).

[4] L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis. P. Noordhoff Ltd. (1958).

[5] C. I. Robbins and R. C. T. Smith, 'A table of roots of sinz = -z.' Phil. Mag. (7) 39, 1004 (1948).

[6] R. Skalak, 'Longitudinal impact of a semi-infinite circular elastic bar.' J. Appl. Mech. 24, 59 (1957).

[7] I. N. Sneddon, Fourier Transforms. McGraw-Hill (1951).

[8] Handbk. Mathematical Functions, edited by M. Abramowitz and I. A. Stegun. National Bureau of Standards (1964).

APPENDIX 1. FINITE-ELEMENT ANALYSIS

The problem analysed here corresponds to Problem B2 of Section 2 (Fig. 7). Thus $\hat{x}$, $\hat{y}$ are given in (2.37); $\hat{\sigma}_x = \frac{1}{2}$ from Fig. 5; and $\hat{\sigma}_y$, $\hat{\sigma}_{xy}$ are implied by (3.10), viz.

$$\hat{\sigma}_y = \frac{h\sigma'_y}{\sigma_B} = \frac{2}{\pi}\,\frac{\hat{x}}{[1+\hat{x}^2]^2}, \quad \hat{\sigma}_{xy} = \frac{h\sigma'_{xy}}{\sigma_B} = \frac{2}{\pi}\,\frac{\hat{x}^2}{[1+\hat{x}^2]^2}.$$

The analysis employs a first-order rectangular element with the displacement field (which varies linearly on the element boundaries)

$$\hat{u}(\hat{x}, \hat{y}) = \alpha_1 + \alpha_2\hat{x} + \alpha_3\hat{y} + \alpha_4\hat{x}\hat{y},$$

$$\hat{v}(\hat{x}, \hat{y}) = \beta_1 + \beta_2\hat{x} + \beta_3\hat{y} + \beta_4\hat{x}\hat{y},$$

where α_i, β_i (i = 1, 2, 3, 4) are constants; $\hat{u}$, $\hat{v}$ are the dimensionless displacements inferred by (2.37). Calculations are made, with $k^2 = 7/2$, for two non-uniform meshes: a coarse mesh (156 nodes), illustrated in Fig. 7; and a fine mesh (561 nodes) created by quartering the coarse mesh elements.

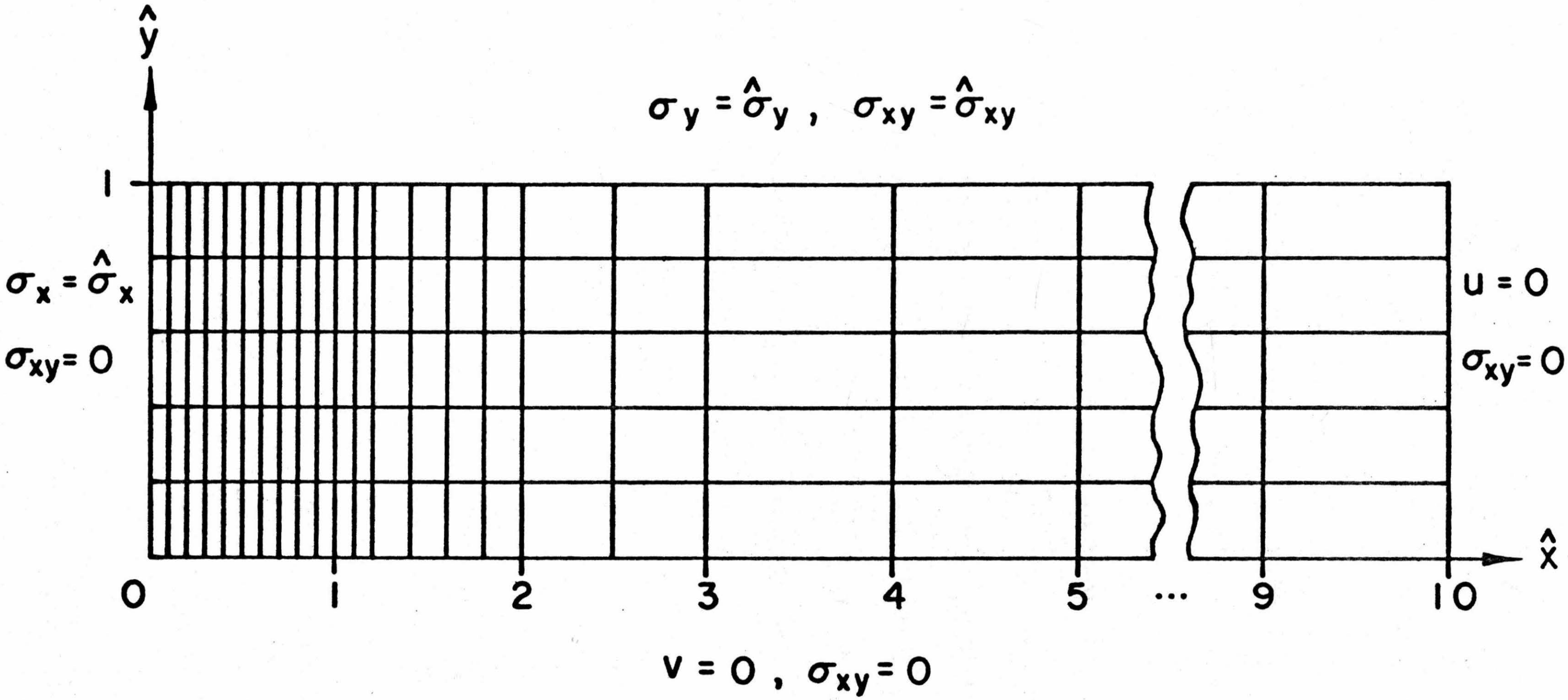

Fig. 7. Coarse mesh and boundary-values for finite-element method

Results for the edge ($\hat{x} = 0$) displacements, obtained at the nodes there, are plotted in Fig. 8; wherein

$$u = \frac{2\mu\pi}{\sigma_B}\left(\frac{k^2-1}{k^2}\right)[\hat{u}(0,\hat{y})-\hat{u}(0,1)], \quad v = \frac{2\mu\pi}{\sigma_B}\left(\frac{k^2-1}{k^2}\right)\hat{v}(0,\hat{y}).$$

In Fig. 8, the broken line denotes the coarse mesh values whilst the solid lines denote the fine mesh values. The coarse and fine mesh results for v are virtually indistinguishable on the scale of this drawing. Consequently, only the fine mesh values for v are shown.

For comparison, Fig. 8 also exhibits the analogous displacements evaluated via the Fourier series representations, u' and v'. Differences between v and v' are only discernable on the scale used in the vicinity of $\hat{y} = 1$, and, though not demonstrated in the figure, these differences are uniformly less in magnitude for the fine mesh than for the coarse mesh.

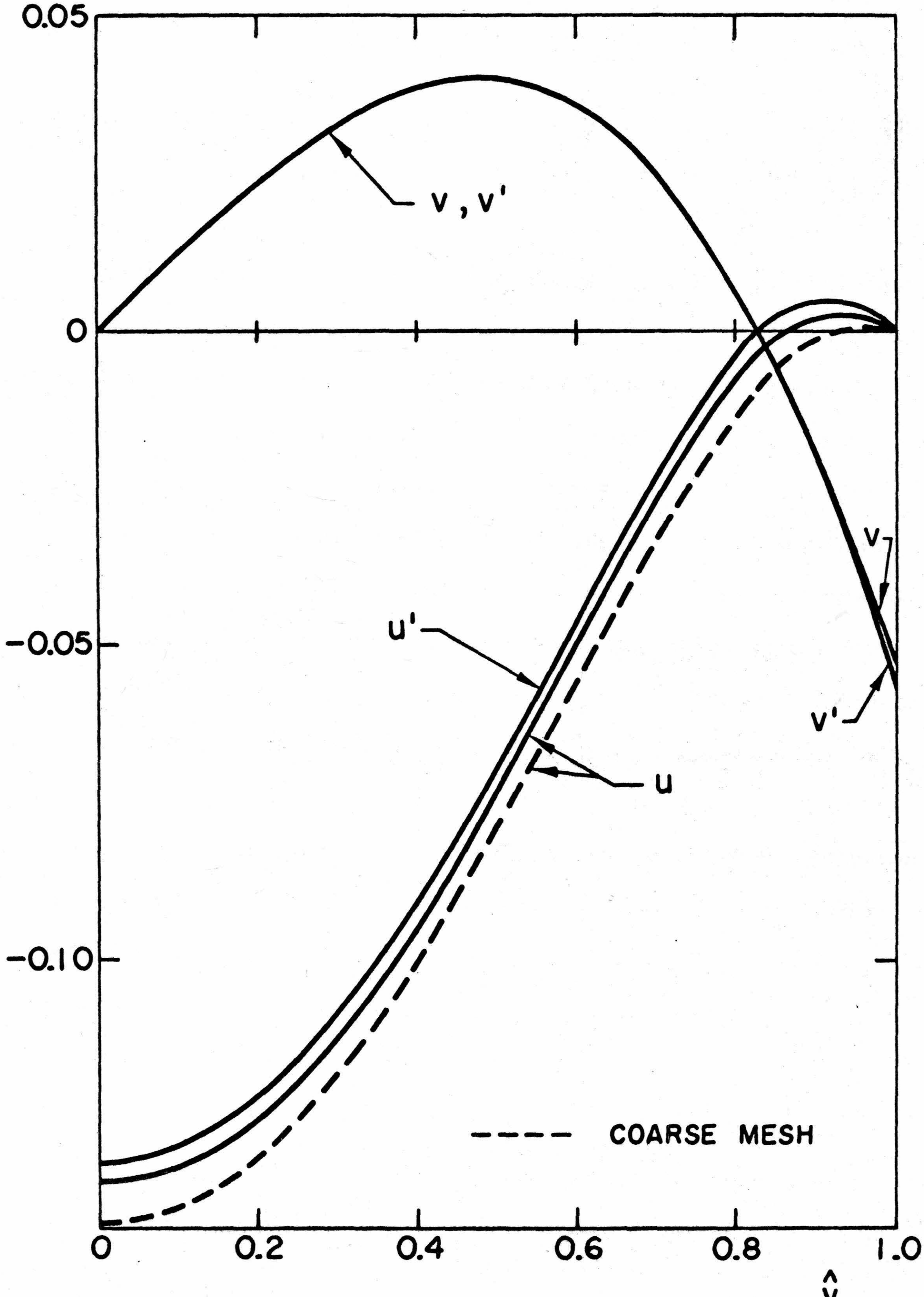

Fig. 8. Values of the edge displacements for Problem B2

APPENDIX 2. NUMERICAL RESULTS FOR THE SINE INTEGRAL

Here we give the computational method and results for $Si(z_j/2)$, z_j having real and imaginary parts greater than zero and satisfying $\sin z_j + z_j = 0$.

First we evaluate z_j using a two-dimensional Newton's method, the same method used by Robbins and Smith in [5]. From the asymptotic forms stated in [5]; viz. $\operatorname{Re} z_j \sim \frac{\pi}{2}(4j-1)$, $\operatorname{Im} z_j \sim \ln \pi(4j-1)$, $j = 1, 2, 3, \ldots,$ j large; and the form of the denominator in (3.18); namely, $[z_j^2 - n^2\pi^2]^2$; it is apparent that for $n = 2j-1$, n and j large, round-off error may become a factor in solving (3.18). To avoid this difficulty, Newton's method is iterated until the roots are accurate to 10 decimal places, a check on this accuracy being afforded on substituting the values for the roots back into $\sin z_j + z_j$. The error estimate defined on such a substitution, e, is less than 10^{-11} for all z_j. The results for the first 24 z_j are shown in Table 2, p. 66.

Now calculating $Si(z_j/2)$. For small $|z_j|$, the series expansion for the sine integral † is a rapidly converging method of computation. The pertinent forms are

$$\operatorname{Re} Si(z_j/2) \approx \sum_{n=1,2,3,\ldots}^{N} (-1)^{n-1} \frac{r_j^{2n-1} \cos(2n-1)\theta_j}{(2n-1)(2n-1)!},$$

$$\operatorname{Im} Si(z_j/2) \approx \sum_{n=1,2,3,\ldots}^{N} (-1)^{n-1} \frac{r_j^{2n-1} \sin(2n-1)\theta_j}{(2n-1)(2n-1)!},$$

† See [8], equation (5.2.14).

with

$$e_s < \frac{r_j^{2N}}{2N(2N)!}$$

wherein $z_j/2 = r_j e^{i\theta_j}$; N is the number of terms taken in the series; e_s is the truncation error, for which the stated upper bound can readily be obtained on bounding the remainders of the infinite series for Re Si$(z_j/2)$, Im Si$(z_j/2)$. These forms were used to estimate the first seven roots with $N < 36$.

For larger roots, $|z_j| > 45$, the faster converging asymptotics forms † are used. These àre

$$\mathrm{Re\,Si}(z_j/2) \sim \frac{\pi}{2} - \cos x_j \cosh y_j \,\mathrm{Re\,f}(z_j/2) - \sin x_j \sinh y_j \,\mathrm{Im\,f}(z_j/2)$$

$$- \sin x_j \cosh y_j \,\mathrm{Re\,g}(z_j/2) + \cos x_j \sinh y_j \,\mathrm{Im\,g}(z_j/2),$$

$$\mathrm{Im\,Si}(z_j/2) \sim \sin x_j \sinh y_j \,\mathrm{Re\,f}(z_j/2) - \cos x_j \cosh y_j \,\mathrm{Im\,f}(z_j/2)$$

$$- \cos x_j \sinh y_j \,\mathrm{Re\,g}(z_j/2) - \sin x_j \cosh y_j \,\mathrm{Im\,g}(z_j/2),$$

$$\mathrm{Re\,f}(z_j/2) = \sum_{n=1,2,3,\ldots}^{10} (-1)^{n-1} \frac{(2n-2)!}{r^{2n-1}} \cos(2n-1)\theta_j,$$

$$\mathrm{Im\,f}(z_j/2) = \sum_{n=1,2,3,\ldots}^{10} (-1)^{n-1} \frac{(2n-2)!}{r^{2n-1}} \sin(2n-1)\theta_j,$$

$$\mathrm{Re\,g}(z_j/2) = \sum_{n=1,2,3,\ldots}^{10} (-1)^{n-1} \frac{(2n-1)!}{r^{2n}} \cos 2n\theta_j,$$

$$\mathrm{Im\,g}(z_j/2) = \sum_{n=1,2,3,\ldots}^{10} (-1)^{n-1} \frac{(2n-1)!}{r^{2n}} \sin 2n\theta_j,$$

† See [8], equations (5.2.8), (5.2.34), (5.2.35).

in which $z_j/2 = x_j + iy_j$. These expressions can be derived by repeated integrating by parts of the sine integral, whence on bounding the residual integral, one may show that

$$e_a < \frac{19!}{x_j^{20}} (1 + \cosh y_j) \ ,$$

where e_a is the error in the asymptotic series evaluation.

Table 2 gives the first 24 values of $Si(z_j/2)$, the last 17 of these being computed via the asymptotic series. The e quoted in the table is the maximum of the associated e_s and e_a values.

Table 2. z_j satisfying $\sin z_j + z_j = 0$, $Si(z_j/2)$

Re z_j	Im z_j	Re $Si(z_j/2)$	Im $Si(z_j/2)$
4.21239 22305	2.25072 86116	1.94348	0.45761
10.71253 73973	3.10314 87458	1.27733	-0.29396
17.07336 48532	3.55108 73470	1.81179	0.23447
23.39835 52257	3.85880 89931	1.36267	-0.20112
29.70811 98253	4.09370 49248	1.75626	0.17902
36.00986 60164	4.28378 15878	1.40206	-0.16297
42.30682 67176	4.44344 58303	1.72656	0.15063
48.60068 41241	4.58110 45735	1.42546	-0.14075
54.89240 57881	4.70209 64604	1.70753	0.13260
61.18259 01968	4.81002 51375	1.44132	-0.12574
67.47162 86350	4.90743 84165	1.69406	0.11985
73.75978 83468	4.99620 44099	1.45295	-0.11472
80.04725 84359	5.07773 37322	1.68388	0.11020
86.33417 66904	5.15311 77014	1.46194	-0.10618
92.62064 60143	5.22321 79892	1.67585	0.10257
98.90674 48938	5.28872 68571	1.46917	-0.09931
105.19253 42895	5.35020 88486	1.66931	0.09635
111.47806 23079	5.40813 03964	1.47512	-0.09363
117.76336 74457	5.46288 13161	1.66385	0.09114
124.04848 08941	5.51479 07194	1.48015	-0.08883
130.33342 82072	5.56413 89982	1.65921	0.08669
136.61823 05302	5.61116 69898	1.48446	-0.08470
142.90290 55184	5.65608 30843	1.65519	0.08284
149.18746 80349	5.69906 88025	1.48822	-0.08110
$e < 10^{-11}$		$e < 10^{-9}$	