On Hodge-Newton reducible local Shimura data of Hodge type

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ABSTRACT

Rapoport-Zink spaces are formal moduli spaces of p-divisible groups which give rise to local analogues of certain Shimura varieties. In particular, one can construct them from purely group theoretic data called *local Shimura data*.

The primary purpose of this dissertation is to study Rapoport-Zink spaces whose underlying local Shimura datum is of Hodge type and Hodge-Newton reducible. Our study consists of two main parts: the study of the l-adic cohomology of Rapoport-Zink spaces in relation to the local Langlands correspondence and the study of deformation spaces of p-divisible groups via the local geometry of Rapoport-Zink spaces.

The main result of the first part is a proof of the Harris-Viehmann conjecture in our setting; in particular, we prove that the l-adic cohomology of Rapoport-Zink spaces contains no supercuspidal representations under our assumptions. In the second part, we obtain a generalization of Serre-Tate deformation theory for Shimura varieties of Hodge type.

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Chapter 1

INTRODUCTION

1.1 Motivation: the local Langlands correspondence and local Shimura varieties

Let G be a connected reductive group over a nonarchimedean local field F. The local Langlands correspondence asserts that there should be a natural surjection with finite fibers from the set $\Pi(G)$ of irreducible admissible representations of $G(F)$ to the set $\Phi(G)$ of *Langlands parameters*, which are certain analogues of Galois representations. For $G = \mathrm{GL}_n$ and F a p-adic field, the correspondence is constructed by Harris and Taylor in [**HT01**] and Henniart in [**Hen00**].

The study of the local Langlands correspondence motivated the theory of *local Shimura varieties*, originally formulated by Rapoport and Viehmann in [**RV14**]. The expectation is that the l-adic cohomology of local Shimura varieties should realize many cases of the local Langlands correspondence. For example, in the work of Harris and Taylor [**HT01**] the correspondence for GL_n over a p-adic field is constructed via the l-adic cohomology of the Lubin-Tate space, which can be regarded as a local Shimura variety for GL_n. In fact, there are two main conjectures, namely the Kottwitz conjecture and the Harris-Viehmann conjecture, which relates the l-adic cohomology of local Shimura varieties to the local Langland correspondence.

The main motivation of this work is to study certain local Shimura varieties with emphasis on their relation to the local Langlands correspondence.

1.2 Overview of the results

We fix a prime $p > 2$, and write $\overline{\mathbb{F}}_p$ and $\overline{\mathbb{Q}}_p$ respectively for a fixed algebraic closure of $\mathbb{F}_p$ and $\mathbb{Q}_p$. For a p-adic local field F, we denote by $\breve{F}$ the p-adic completion of the maximal unramified extension F^{un}, and by $\sigma \in \mathrm{Gal}(\breve{F}/F)$ the relative Frobenius automorphism. We also write $\mathbb{C}_p$ for the p-adic completion of $\overline{\mathbb{Q}}_p$ and, and $\breve{\mathbb{Z}}_p$ for the ring of integers of $\breve{\mathbb{Q}}_p$.

Given a connected reductive group G over $\mathbb{Q}_p$, we say that two elements b and b' of $G(\breve{\mathbb{Q}}_p)$ are *σ-conjugate* if $b' = gb\sigma(g)^{-1}$ for some $g \in G(\breve{\mathbb{Q}}_p)$. To an element

$b \in G(\breve{\mathbb{Q}}_p)$ we associate an algebraic group J_b over $\mathbb{Q}_p$ with functor of points

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) : gb\sigma(g)^{-1} = b\}$$

for any $\mathbb{Q}_p$-algebra R. The isomorphism class of J_b depends only on the σ-conjugacy class of b.

A *local Shimura datum* is a triple $(G, [b], \{\mu\})$ consisting of a connected reductive group G over $\mathbb{Q}_p$, a σ-conjugacy class $[b]$ of $G(\breve{\mathbb{Q}}_p)$, and a geometric conjugacy class $\{\mu\}$ of cocharacters of $G_{\overline{\mathbb{Q}_p}}$ satisfying certain axioms (see 2.4.1 and [**RV14**], §5 for details). Let E denote the field of definition of $\{\mu\}$, referred to as the *local reflex field*, which is an unramified finite extension of $\mathbb{Q}_p$. The local Shimura variety associated to the datum $(G, [b], \{\mu\})$ is a tower of rigid analytic spaces over $\breve{E}$

$$\mathbb{M}(G, [b], \{\mu\}) = \{\mathbb{M}^{K_p}\},$$

where K_p ranges over all open compact subgroups of $G(\mathbb{Q}_p)$, with the following properties:

(i) each space $\mathbb{M}^{K_p}$ is equipped with an action of $J_b(\mathbb{Q}_p)$,

(ii) the group $G(\mathbb{Q}_p)$ acts on the tower as a group of Hecke correspondences,

(iii) the tower is equipped with a Weil descent datum down to E.

The l-adic cohomology groups

$$H^i(\mathbb{M}^{K_p}) := H^i_c\big(\mathbb{M}^{K_p} \otimes_{\breve{E}} \mathbb{C}_p, \mathbb{Q}_l(\dim \mathbb{M}^{K_p})\big) \quad \text{for } i > 0$$

fit into a tower $\{H^i(\mathbb{M}^{K_p})\}$ with a natural action of $G(\mathbb{Q}_p) \times \mathcal{W}_E \times J_b(\mathbb{Q}_p)$, where $\mathcal{W}_E$ is the Weil group of E. For an l-adic admissible representation ρ of $J_b(\mathbb{Q}_p)$, we define a virtual representation of $G(\mathbb{Q}_p) \times \mathcal{W}_E$

$$H^\bullet\big(\mathbb{M}(G, [b], \{\mu\})\big)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} \varinjlim_{K_p} \operatorname{Ext}^j_{J_b(\mathbb{Q}_p)}\big(H^i(\mathbb{M}^{K_p}), \rho\big).$$

This is the object of our interest for the relation between the l-adic cohomology of local Shimura varieties and the local Langlands correspondence.

In this thesis, we study local Shimura varieties and their l-adic cohomology under the following assumptions on the underlying local Shimura datum $(G, [b], \{\mu\})$:

(A1) the datum $(G, [b], \{\mu\})$ is *unramified* and *of Hodge type*.

(A2) the datum $(G, [b], \{\mu\})$ is *Hodge-Newton reducible* with respect to a parabolic subgroup P of G with Levi factor L.

The first assumption means that the datum $(G, [b], \{\mu\})$ satisfy the following two conditions:

(i) the group G admits a reductive model over $\mathbb{Z}_p$,

(ii) there is an embedding of local Shimura data

$$(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b]_{\mathrm{GL}_n}, \{\mu\}_{\mathrm{GL}_n}).$$

Under these conditions, the associated local Shimura variety $\mathbb{M}(G, [b], \{\mu\})$ arises from a certain moduli space of p-divisible groups, known as a *Rapoport-Zink space*, constructed by Rapoport and Zink [**RZ96**] (for local Shimura data of EL/PEL type) and Kim [**Kim13**] (for unramified local Shimura data of Hodge type). We will write write $\mathrm{RZ}_{G,[b],\{\mu\}}$ for the Rapoport-Zink space associated to the datum $(G, [b], \{\mu\})$, and often write $\mathrm{RZ}^{\infty}_{G,[b],\{\mu\}} = \{\mathrm{RZ}^{K_p}_{G,[b],\{\mu\}}\}$ in lieu of the corresponding local Shimura variety $\mathbb{M}(G, [b], \{\mu\})$. Any $\overline{\mathbb{F}}_p$-valued point $x \in \mathrm{RZ}_{G,[b],\{\mu\}}(\overline{\mathbb{F}}_p)$ represents an isomorphism class of p-divisible groups over $\overline{\mathbb{F}}_p$ with some additional structures induced by the underlying local Shimura datum $(G, [b], \{\mu\})$. Let us denote this isomorphism class by $\underline{X}_x$ and its underlying p-divisible group by X_x. When $G = \mathrm{GL}_n$, the class $\underline{X}_x$ simply represents the isomorphism class of X_x since local Shimura data for GL_n induce no additional structures.

The second assumption roughly means that the datum $(G, [b], \{\mu\})$ naturally reduces to a local Shimura datum $(L, [b]_L, \{\mu\}_L)$ for the specified Levi subgroup L of G. A precise definition of Hodge-Newton reducibility is given in terms of two invariants, namely the *Newton point* and the *σ-invariant Hodge point*, attached to the datum $(G, [b], \{\mu\})$. When $G = \mathrm{GL}_n$, the Newton point and the σ-invariant Hodge point of $(\mathrm{GL}_n, [b]_{\mathrm{GL}_n}, \{\mu\}_{\mathrm{GL}_n})$ can be identified with the Newton polygon and the Hodge polygon of the p-divisible group X_x for any $x \in \mathrm{RZ}_{\mathrm{GL}_n,[b]_{\mathrm{GL}_n},\{\mu\}_{\mathrm{GL}_n}}(\overline{\mathbb{F}}_p)$. We refer the readers to 4.1.4 for a precise definition of Hodge-Newton reducibility.

Our first main result verifies the Harris-Viehmann conjecture under the assumptions (A1) and (A2). The Harris-Viehmann conjecture gives a parabolic inductive formula for $H^\bullet\big(\mathbb{M}(G, [b], \{\mu\})\big)_\rho$ when the underlying local Shimura datum $(G, [b], \{\mu\})$ is not *basic*. An important implication of the conjecture is that the virtual representation $H^\bullet\big(\mathbb{M}(G, [b], \{\mu\})\big)_\rho$ contains no supercuspidal representations of $G(\mathbb{Q}_p)$ if the datum $(G, [b], \{\mu\})$ is not basic. When the datum $(G, [b], \{\mu\})$ is basic, the Kottwitz conjecture predicts how the virtual representation $H^\bullet(\mathbb{M}(G, [b], \{\mu\}))_\rho$ should realize supercuspidal representations. The readers can find a precise statement of these conjectures in [**RV14**], Conjecture 7.3 and Conjecture 8.4.

In our setting, we verify the Harris-Viehmann conjecture as follows:

Theorem 1. *Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of Hodge type which is Hodge-Newton reducible with respect to a parabolic subgroup P of G with Levi factor L. For any admissible $\overline{\mathbb{Q}}_l$-representation ρ of $J_b(\mathbb{Q}_p)$, we have the following equality of virtual representations of $G(\mathbb{Q}_p) \times \mathcal{W}_E$:*

$$H^\bullet\big(RZ^\infty_{G,[b],\{\mu\}}\big)_\rho = Ind_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H^\bullet\big(RZ^\infty_{L,[b]_L,\{\mu\}_L}\big)_\rho.$$

In particular, the virtual representation $H^\bullet\big(RZ^\infty_{G,[b],\{\mu\}}\big)_\rho$ contains no supercuspidal representations of $G(\mathbb{Q}_p)$.

The earliest result of this form is Boyer's work in [**Boy99**] for Drinfeld's modular varieties. For Rapoport-Zink spaces of PEL type, Mantovan in [**Man08**] and Shen in [**Sh13**] verified the conjecture assuming Hodge-Newton reducibility. Hansen in [**Han16**] gave another proof of the conjecture for $G = \mathrm{GL}_n$, also under the Hodge-Newton reducibility assumption, using the general construction of local Shimura varieties by Scholze in his Berkely lectures [**Sch14**].

Our second main result establishes a generalization of Serre-Tate deformation theory for local Shimura data of Hodge type. The classical Serre-Tate deformation theory states that the formal deformation space of an ordinary p-divisible group X over $\overline{\mathbb{F}}_p$ has a canonical structure of a formal torus over $\breve{\mathbb{Z}}_p$, whose identity section corresponds to a unique deformation $\mathscr{X}$ with the property that all endomorphisms of X lift to endomorphisms of $\mathscr{X}$. The deformation $\mathscr{X}$ is referred to as the *canonical deformation* of X. The theory is based on the fact that an ordinary p-divisible group over $\overline{\mathbb{F}}_p$ admits a canonical filtration, called the *slope filtration*, which can be uniquely lifted to $\breve{\mathbb{Z}}_p$. These results first appeared in the Woods Hole reports of Lubin, Serre and Tate [**LST64**].

For our generalization of Serre-Tate deformation theory, we consider the case when the local Shimura datum $(G, [b], \{\mu\})$ is *μ-ordinary*. This case corresponds to a special case of Hodge-Newton reducibility, and is characterized by the property that the Newton point and the σ-invariant Hodge point coincide. When $G = \mathrm{GL}_n$, this condition precisely means that the Newton polygon and the Hodge polygon of the

p-divisible group X_x are equal for all $x \in \mathrm{RZ}_{\mathrm{GL}_n,[b]_{\mathrm{GL}_n},\{\mu\}_{\mathrm{GL}_n}}(\overline{\mathbb{F}}_p)$; in other words, the p-divisible group X_x is ordinary for all $x \in \mathrm{RZ}_{\mathrm{GL}_n,[b]_{\mathrm{GL}_n},\{\mu\}_{\mathrm{GL}_n}}(\overline{\mathbb{F}}_p)$.

For any $x \in \mathrm{RZ}_{G,[b],\{\mu\}}(\overline{\mathbb{F}}_p)$, we denote by $\mathrm{Def}_{\underline{X}_x}$ the formal deformation space of $\underline{X}_x$, which classifies deformations of X_x that lift the additional structures on X_x. We can naturally identify the space $\mathrm{Def}_{\underline{X}_x}$ with the the formal completion of $\mathrm{RZ}_{G,[b],\{\mu\}}$ at x. When $G = \mathrm{GL}_n$, this space is the classical deformation space of X_x.

In this setting, our generalization of Serre-Tate deformation theory states that the formal deformation space $\mathrm{Def}_{\underline{X}_x}$ for any $x \in \mathrm{RZ}_{G,[b],\{\mu\}}(\overline{\mathbb{F}}_p)$ has an explicit natural "group-like" structure with an identity element corresponding to the canonical deformation $\underline{\mathscr{X}}_x$ of $\underline{X}_x$. Moreover, as in the classical setting, the p-divisible group $\underline{X}_x$ with additional structures for any $x \in \mathrm{RZ}_{G,[b],\{\mu\}}(\overline{\mathbb{F}}_p)$ admits a "slope filtration" that can be uniquely lifted to $\breve{\mathbb{Z}}_p$. When the slope filtration has two steps, the formal deformation space indeed has an explicit natural structure of a formal group as stated in the following theorem:

Theorem 2. *Let $\underline{X}$ be a p-divisible group over $\overline{\mathbb{F}}_p$ with additional structures that arises from a μ-ordinary local Shimura datum of Hodge type. If $\underline{X}$ has two steps in the slope filtration, the formal deformation space $Def_{\underline{X}}$ of $\underline{X}$ has an explicit natural structure of a p-divisible group over $\breve{\mathbb{Z}}_p$. More precisely, there exist two positive integers h and d (which can be explicitly computed) such that*

$$Def_{\underline{X}} \cong \mathscr{Y}_h^d$$

as p-divisible groups over $\breve{\mathbb{Z}}_p$, where $\mathscr{Y}_h$ is the Lubin-Tate formal group of height h. Moreover, the identity element corresponds to the canonical deformation $\underline{\mathscr{X}}$ of $\underline{X}$.

When $G = \mathrm{GL}_n$, this theorem recovers the classical Serre-Tate deformation theory. For local Shimura data of EL/PEL type, this theorem agrees with the result of Moonen in [**Mo04**].

1.3 The strategy: EL realization of Hodge-Newton reducibility

We retain the assumptions (A1) and (A2) on the datum $(G, [b], \{\mu\})$.

Our main strategy is to study the datum $(G, [b], \{\mu\})$ by embedding it into another Hodge-Newton reducible local Shimura datum of EL type, for which most of our results are previously known. We embody this strategy in the following lemma:

Lemma 3. *Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of Hodge type which is Hodge-Newton reducible with respect to a parabolic subgroup P of G with Levi factor L. Then there exists a group $\widetilde{G}$ of EL type with the following properties:*

(i) the embedding $G \hookrightarrow GL_n$ factors through $\widetilde{G}$,

(ii) the local Shimura datum $(\widetilde{G}, [b]_{\widetilde{G}}, \{\mu\}_{\widetilde{G}})$ is Hodge-Newton reducible with respect to a parabolic subgroup $\widetilde{P} \subsetneq \widetilde{G}$ and its Levi factor $\widetilde{L}$ such that $P = \widetilde{P} \cap G$ and $L = \widetilde{L} \cap G$.

The datum $(\widetilde{G}, [b]_{\widetilde{G}}, \{\mu\}_{\widetilde{G}})$ is referred to as an *EL realization* of the datum $(G, [b], \{\mu\})$. An important consequence of Lemma 3 is that, by functoriality of Rapoport-Zink spaces, we have a closed embedding

$$\mathrm{RZ}_{G,[b],\{\mu\}} \longhookrightarrow \mathrm{RZ}_{\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}}}$$

where both spaces come from Hodge-Newton reducible local Shimura data.

We remark that, if you consider the embedding

$$(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b]_{\mathrm{GL}_n}, \{\mu\}_{\mathrm{GL}_n})$$

that comes from the assumption (A1), the datum $(\mathrm{GL}_n, [b]_{\mathrm{GL}_n}, \{\mu\}_{\mathrm{GL}_n})$ is not Hodge-Newton reducible unless G is split.

For a local Shimura datum of EL type, the Hodge-Newton reducibility condition is relatively easy to study as the condition has an alternative simple characterization. The key fact is that the Newton point and the σ-invariant Hodge point for such a datum can be identified with convex polygons, called the *Newton polygon* and the *σ-invariant Hodge polygon*. These polygons have the same endpoints and satisfy a relation that the Newton polygon lies above the σ-invariant Hodge polygon. The points at which the Newton polygon changes slope are called its *break points*.

For the EL realization $(\widetilde{G}, [b]_{\widetilde{G}}, \{\mu\}_{\widetilde{G}})$, the Hodge-Newton reducibility condition means that the σ-invariant Hodge polygon passes through some break points of the Newton polygon which are specified by the Levi subgroup $\widetilde{L}$ of $\widetilde{G}$. These contact points divide the Newton polygon and the σ-invariant Hodge polygon into subpolygons, which we denote by $\nu_1, \nu_2, \cdots, \nu_r$ and $\mu_1, \mu_2, \cdots, \mu_r$. Here we choose our numbering so that the slopes of ν_i are less than the slopes of ν_{i+1}.

Let $\widetilde{X}$ be a p-divisible group over $\overline{\mathbb{F}}_p$ with additional structures that arise from the datum $(\widetilde{G}, [b]_{\widetilde{G}}, \{\mu\}_{\widetilde{G}})$; in other words, $\widetilde{X} = \underline{X}_x$ for some $x \in \mathrm{RZ}_{\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}}}(\overline{\mathbb{F}}_p)$. A *Hodge-Newton decomposition* of $\widetilde{X}$ is a decomposition of the form

$$\widetilde{X} = \widetilde{X}_1 \times \widetilde{X}_2 \times \cdots \times \widetilde{X}_r,$$

where each $\widetilde{X}_i$ is a p-divisible group over $\overline{\mathbb{F}}_p$ with additional structures that arise from some local Shimura datum $(\widetilde{G}_i, [b_i], \{\mu_i\})$ of EL type with the Newton polygon ν_i and the σ-invariant Hodge polygon μ_i. Note that we require the decomposition to be compatible with additional structures on $\widetilde{X}$ and the factors $\widetilde{X}_i$ in an appropriate sense. Given such a decomposition, we set $X^{(i)} := X_i \times \cdots \times X_r$ and get a filtration of the underlying p-divisible group

$$0 \subset X^{(r)} \subset X^{(r-1)} \subset \cdots \subset X^{(1)} = X$$

such that each quotient $X^{(i)}/X^{(i+1)} \simeq X_i$ is equipped with additional structures that arise from the datum $(\widetilde{G}_i, [b_i], \{\mu_i\})$. We refer to this filtration as a *Hodge-Newton filtration* of $\widetilde{X}$. By the work of Katz [**Ka79**], Mantovan and Viehmann [**MV10**] and Shen [**Sh13**], we have the following facts:

(1) There exists a Hodge-Newton decomposition of $\widetilde{X}$.

(2) Every deformation of $\widetilde{X}$ admits a unique filtration that lifts the Hodge-Newton filtration of $\widetilde{X}$.

Lemma 3 allows us to extend these facts to the datum $(G, [b], \{\mu\})$. For simplicity, we may assume that $\widetilde{G} = \mathrm{Res}_{K|\mathbb{Q}_p}\mathrm{GL}_n$ for some finite unramified extension K of $\mathbb{Q}_p$. The Levi subgroup $\widetilde{L} \subsetneq \widetilde{G}$ is of the form

$$\widetilde{L} = \mathrm{Res}_{K|\mathbb{Q}_p}\mathrm{GL}_{n_{n_1}} \times \cdots \times \mathrm{Res}_{K|\mathbb{Q}_p}\mathrm{GL}_{n_{n_r}}.$$

For $i = 1, 2, \cdots, r$, we denote by $\widetilde{L}_i$ the i-th factor in the above decomposition, and by L_i the image of $L = \widetilde{L} \cap G$ under the projection $\widetilde{L} \twoheadrightarrow \widetilde{L}_i$. The datum $(G, [b], \{\mu\})$ induces local Shimura data $(L_i, [b_i], \{\mu_i\})$ via the projections $L \twoheadrightarrow L_i$. Then for any p-divisible group $\underline{X}$ over $\overline{\mathbb{F}}_p$ with additional structures that arise from the datum $(G, [b], \{\mu\})$, we prove the following facts (Theorem 3.2.2 and Theorem 3.2.4):

(1) There exists a Hodge-Newton decomposition

$$\underline{X} = \underline{X}_1 \times \cdots \times \underline{X}_r$$

where $\underline{X}_i$ is a p-divisible group over $\overline{\mathbb{F}}_p$ with additional structures that arises from the datum $(L_i, [b_i], \{\mu_i\})$.

(2) Given a deformation $\underline{\mathscr{X}}$ of $\underline{X}$ over a ring of the form $R = \breve{\mathbb{Z}}_p[[u_1, \cdots, u_N]]$ or $R = \breve{\mathbb{Z}}_p[[u_1, \cdots, u_N]]/(p^m)$, there exists a unique filtration

$$0 \subset \mathscr{X}^{(r)} \subset \mathscr{X}^{(r-1)} \subset \cdots \subset \mathscr{X}^{(1)} = \mathscr{X}$$

with the following properties:

(i) each $\mathscr{X}^{(i)}$ is a deformation of $X^{(i)} := X_i \times \cdots \times X_r$ over R,

(ii) each $\mathscr{X}^{(i)}/\mathscr{X}^{(i+1)}$ is a deformation of X_i over R, which carries additional structures that lift the additional structures on X_i.

Let us now briefly explain how these technical results are used in the proof of our main results.

For the proof of Theorem 1, we show that the rigid analytic generic fiber $\mathrm{RZ}^{\mathrm{rig}}_{G,[b],\{\mu\}}$ of the space $\mathrm{RZ}_{G,[b],\{\mu\}}$ is "parabolically induced" from the rigid analytic generic fiber $\mathrm{RZ}^{\mathrm{rig}}_{L,[b]_L,\{\mu\}_L}$ of $\mathrm{RZ}_{L,[b]_L,\{\mu\}_L}$, as stated in the following lemma:

Lemma 4. *There exists an analogue of Rapoport-Zink space $RZ_{P,[b]_P,\{\mu\}_P}$, associated to the parabolic subgroup P, with a diagram of the rigid analytic generic fibers*

$$\begin{tikzcd}
& RZ^{rig}_{P,[b]_P,\{\mu\}_P} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
RZ^{rig}_{L,[b]_L,\{\mu\}_L} \arrow[ur, bend left, "s"] & & RZ^{rig}_{G,[b],\{\mu\}}
\end{tikzcd}$$

with the following properties:

(i) s is a closed immersion,

(ii) π_1 is a fibration in balls,

(iii) π_2 is an isomorphism.

Over the space $\mathrm{RZ}_{\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}}}$, we can construct the space $\mathrm{RZ}_{\widetilde{P},[b]_{\widetilde{P}},\{\mu\}_{\widetilde{P}}}$ associated to $\widetilde{P}$ following Mantovan in [**Man08**]. Then we construct the desired space $\mathrm{RZ}_{P,[b]_P,\{\mu\}_P}$ by taking the pull-back of $\mathrm{RZ}_{\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}}}$ via the closed embedding

$$\mathrm{RZ}_{G,[b],\{\mu\}} \hookrightarrow \mathrm{RZ}_{\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}}}$$

induced by the embedding $(G,[b],\{\mu\}) \hookrightarrow (\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}})$. To construct the diagram in the lemma, we use existence of a Hodge-Newton decomposition and a unique lifting of the Hodge-Newton filtration for the datum $(G,[b],\{\mu\})$.

By construction, the space $\mathrm{RZ}_{P,[b]_P,\{\mu\}_P}$ comes with a tower of étale coverings $\mathrm{RZ}^{\infty}_{P,[b]_P,\{\mu\}_P} := \{\mathrm{RZ}^{K'_p}_{P,[b]_P,\{\mu\}_P}\}$ which is an analogue of the local Shimura varieties $\mathrm{RZ}^{\infty}_{G,[b],\{\mu\}}$ and $\mathrm{RZ}^{\infty}_{L,[b]_L,\{\mu\}_L}$ corresponding to the spaces $\mathrm{RZ}_{G,[b],\{\mu\}}$ and $\mathrm{RZ}_{L,[b]_L,\{\mu\}_L}$. Our proof of Theorem 1 is based on comparing the cohomology of the three towers $\mathrm{RZ}^{\infty}_{G,[b],\{\mu\}}$, $\mathrm{RZ}^{\infty}_{P,[b]_P,\{\mu\}_P}$ and $\mathrm{RZ}^{\infty}_{L,[b]_L,\{\mu\}_L}$ using the diagram in Lemma 4.

For our second result, we take $x \in \mathrm{RZ}_{G,[b],\{\mu\}}(\overline{\mathbb{F}}_p)$ and write $\widetilde{X}_x$ for the p-divisible group over $\overline{\mathbb{F}}_p$ with additional structures corresponding to x considered as an $\overline{\mathbb{F}}_p$-valued point of $\mathrm{RZ}_{\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}}}$ via the embedding

$$\mathrm{RZ}_{G,[b],\{\mu\}} \hookrightarrow \mathrm{RZ}_{\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}}}.$$

Taking the formal completions at x yields a closed embedding of deformation spaces

$$\mathrm{Def}_{\underline{X}_x} \hookrightarrow \mathrm{Def}_{\widetilde{X}_x}.$$

Our key observation is that the EL realization $(\widetilde{G},[b]_{\widetilde{G}},\{\mu\}_{\widetilde{G}})$ is μ-ordinary. Hence we know the structure of the latter space as studied by Moonen in [**Mo04**]. For our generalization of Serre-Tate deformation theory, it sufficecs to prove that the space $\mathrm{Def}_{\underline{X}_x}$ is closed under (some of) the structures of the space $\mathrm{Def}_{\widetilde{X}_x}$. For this, we use existence of the slope decomposition and a unique lifting of the slope filtration for $\underline{X}_x$, which follows from a special case of our general results on the Hodge-Newton decomposition and the Hodge-Newton filtration.

1.4 Structure of the thesis

In section 2, we introduce general notations and recall some preliminaries, such as the definition of local Shimura data and the construction of Rapoport-Zink spaces. Section 3 serves as the framework of the thesis where we develop the notion of EL realiation and prove related technical results. In section 4 and 5 we prove our main results.

Chapter 2

PRELIMINARIES

2.1 General notations

2.1.1. Throughout this paper, k is a perfect field of positive characteristic p. We write $W(k)$ for the ring of Witt vectors over k, and $K_0(k)$ for its quotient field. We will often write $W = W(k)$ and $K_0 = K_0(k)$. We generally denote by σ the Frobenius automorphism over k, and also its lift to $W(k)$ and $K_0(k)$.

We also fix the following standard notations:

- $\overline{\mathbb{F}}_p$ is a fixed algebraic closure of $\mathbb{F}_p$;
- $\overline{\mathbb{Q}}_p$ is a fixed algebraic closure of $\mathbb{Q}_p$;
- $\mathbb{C}_p$ is the p-adic completion of $\overline{\mathbb{Q}}_p$;
- $\breve{\mathbb{Q}}_p$ is the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$;
- $\breve{\mathbb{Z}}_p$ is the ring of integers of $\breve{\mathbb{Q}}_p$.

We remark that $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$ and $\breve{\mathbb{Q}}_p = K_0(\overline{\mathbb{F}}_p)$.

2.1.2. Given a Noetherian ring R and a free R-module Λ, we denote by $\Lambda^{\otimes}$ the direct sum of all the R-modules which can be formed from Λ using the operations of taking duals, tensor products, symmetric powers and exterior powers. An element of $\Lambda^{\otimes}$ is called a *tensor* over Λ. Note that there is a natural identification $\Lambda^{\otimes} \simeq (\Lambda^*)^{\otimes}$, where Λ^* is the dual R-module of Λ. Any isomorphism $\Lambda \xrightarrow{\sim} \Lambda'$ of free R-modules of finite rank naturally induces an isomorphism $\Lambda^{\otimes} \xrightarrow{\sim} (\Lambda')^{\otimes}$.

For a p-divisible group X over a $\mathbb{Z}_p$-scheme S, we write $\mathbb{D}(X)$ for its (contravariant) Dieudonné module and $\mathrm{Fil}^1(\mathbb{D}(X)) \subset \mathbb{D}(X)_S$ for its Hodge filtration. We generally denote by F the Frobenius map on $\mathbb{D}(X)$.

2.2 Group theoretic preliminaries

2.2.1. Let Λ be a finitely generated free module over $\mathbb{Z}_p$. Then σ acts on $\Lambda_W = \Lambda \otimes_{\mathbb{Z}_p} W$ and on $\mathrm{GL}(\Lambda_W) = \mathrm{GL}(\Lambda) \otimes_{\mathbb{Z}_p} W$ via $1 \otimes \sigma$. Alternatively, we may write this action as $\sigma(g) = (1 \otimes \sigma) \circ g \circ (1 \otimes \sigma^{-1})$ for $g \in \mathrm{GL}(\Lambda_W)$. We also have an induced action of σ on the group of cocharacters $\mathrm{Hom}_W(\mathbb{G}_m, \mathrm{GL}(\Lambda_W))$ defined by $\sigma(\mu)(a) = \sigma(\mu(a))$.

For two $\mathbb{Z}_p$-algebras $R \subseteq R'$, we will denote by $\mathrm{Res}_{R'|R}\mathrm{GL}_n$ the Weil restriction of $\mathrm{GL}_n \otimes_R R'$. If $\mathscr{O}$ is a finite unramified extension of $\mathbb{Z}_p$, a choice of σ-invariant basis of $\mathscr{O}$ over $\mathbb{Z}_p$ determines an embedding of affine $\mathbb{Z}_p$-groups

$$\mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{mn},$$

where $m = |\mathscr{O} : \mathbb{Z}_p|$. If Λ is a free module over $\mathscr{O}$ of rank n, then there is a natural identification $\mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}(\Lambda) \otimes_{\mathbb{Z}_p} W \cong \mathrm{GL}_{\mathscr{O}\otimes_{\mathbb{Z}_p} W}(\Lambda_W)$ where the latter is identified with a product of m copies of $\mathrm{GL}_n \otimes_{\mathbb{Z}_p} W$ after choosing a σ-invariant basis of $\mathscr{O}$ over $\mathbb{Z}_p$.

2.2.2. Let G be a connected reductive group over $\mathbb{Q}_p$ with a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We will write $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ for the associated root datum, and Ω for the associated Weyl group. The choice of B determines a set of positive roots $\Phi^+ \subseteq \Phi$ and a set of positive coroots $\Phi^{\vee +} \subseteq \Phi^\vee$. The group Ω naturally acts on $X_*(T)$ (resp. $X^*(T)$), and the dominant cocharacters (resp. dominant characters) form a full set of representatives for the orbits in $X_*(T)/\Omega$ (resp. $X^*(T)/\Omega$).

Except for 2.3, we will always assume that G is *unramified*. This means that G satisfies the following equivalent conditions:

(i) G is quasi-split and split over a finite unramified extension of $\mathbb{Q}_p$.

(ii) G admits a reductive model over $\mathbb{Z}_p$.

When G is unramified, we fix a reductive model $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, and will often write $G = G_{\mathbb{Z}_p}$ if there is no risk of confusion. We also fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$ which are both defined over $\mathbb{Z}_p$. We use the standard notation $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G)$) to denote the category of finite rank G-representations of over $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$).

2.2.3. For any local, strictly Henselian $\mathbb{Z}_p$-algebra R and a cocharacter $\lambda : \mathbb{G}_{m,R} \to G_R$, we denote the $G(R)$-conjugacy class of λ by $\{\lambda\}_G$, or usually by $\{\lambda\}$ if there is no risk of confusion. We have identifications $X_*(T) \cong \mathrm{Hom}_R(\mathbb{G}_m, T_R)$ and $\Omega \cong N_G(T)(R)/T(R)$, which induce a bijection between $X_*(T)/\Omega$ and the set of $G(R)$-conjugacy classes of cocharacters for G_R. We will be mostly interested in the case $R = W(k)$ for some algebraically closed k, where we also have a bijection

$$\mathrm{Hom}_W(\mathbb{G}_m, G_W)/G(W) \cong \mathrm{Hom}_{K_0}(\mathbb{G}_m, G_{K_0})/G(K_0) \xrightarrow{\sim} G(W)\backslash G(K_0)/G(W)$$

induced by $\{\lambda\} \mapsto G(W)\lambda(p)G(W)$; indeed, the first bijection follows from the fact that G is split over W, while the second bijection is the Cartan decomposition.

Let $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ be a faithful G-representation over $\mathbb{Z}_p$. By [**Ki10**], Proposition 1.3.2, we can choose a finite family of tensors $(s_i)_{i\in I}$ on Λ such that G is the pointwise stabilizer of the s_i; i.e., for any $\mathbb{Z}_p$-algebra R we have

$$G(R) = \{g \in \mathrm{GL}(\Lambda \otimes_{\mathbb{Z}_p} R) : g(s_i \otimes 1) = s_i \otimes 1 \text{ for all } i \in I\}.$$

We say that a grading $\mathrm{gr}^\bullet(\Lambda_R)$ is *induced by* λ if the following conditions are satisfied:

(i) the $\mathbb{G}_m$-action on Λ_R via λ leaves each grading stable,

(ii) the resulting $\mathbb{G}_m$-action on $\mathrm{gr}^i(\Lambda_R)$ is given by

$$\mathbb{G}_m \xrightarrow{z\mapsto z^{-i}} \mathbb{G}_m \xrightarrow{z\mapsto z\cdot \mathrm{id}} \mathrm{GL}(\mathrm{gr}^i(\Lambda_R)).$$

Let S be an R-scheme, and $\mathscr{E}$ a vector bundle on S. For a finite family of global sections (t_i) of $\mathscr{E}^{\otimes}$, we define the following scheme over S

$$\mathcal{P}_S := \mathbf{Isom}_{O_S}\Big([\mathscr{E}, (t_i)], [\Lambda \otimes_R O_S, (s_i \otimes 1)]\Big).$$

In other words, $\mathcal{P}_S$ classifies isomorphisms of vector bundles $\mathscr{E} \cong \Lambda \otimes_R O_S$ which match (t_i) and $(s_i \otimes 1)$.

Let $\mathrm{Fil}^\bullet(\mathscr{E})$ be a filtration of $\mathscr{E}$. When $\mathcal{P}_S$ is a trivial G-torsor, we say that $\mathrm{Fil}^\bullet(\mathscr{E})$ is a $\{\lambda\}$-*filtration* with respect to (t_i) if there exists an isomorphism $\mathscr{E} \cong \Lambda \otimes_R O_S$, matching (t_i) and $(1 \otimes s_i)$, which takes $\mathrm{Fil}^\bullet(\mathscr{E})$ to a filtration of $\Lambda \otimes_R O_S$ induced by $g\lambda g^{-1}$ for some $g \in G(R)$. More generally, when $\mathcal{P}_S$ a G-torsor, we say that $\mathrm{Fil}^\bullet(\mathscr{E})$ is a $\{\lambda\}$-filtration with respect to (t_i) if it is étale-locally a $\{\lambda\}$-filtration.

2.3 F-isocrystals with G-structure

We review the theory of F-isocrystals with G-structure due to R. Kottwitz in [**Ko85**] and [**Ko97**]. We do not assume that G is unramified for this section.

2.3.1. Let k be a perfect field of positive characteristic p. An F-isocrystal over k is a vector space V over $K_0(k)$ with an isomorphism $F : \sigma^* V \xrightarrow{\sim} V$. The dimension of V is called the *height* of the isocrystal. Let $F\text{-Isoc}(k)$ denote the category of F-isocrystals over k. For a connected reductive group G over $\mathbb{Q}_p$, we define an *F-isocrystal over k with G-structure* as an exact faithful tensor functor

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to F\text{-Isoc}(k).$$

Example 2.3.2. (i) An F-isocrystal with GL_n-structure is an F-isocrystal of height n.

(ii) If $G = \mathrm{Res}_{E|\mathbb{Q}_p} \mathrm{GL}_n$ where $E|\mathbb{Q}_p$ is a finite extension of degree m, an F-isocrystal with G-structure is an F-isocrystal V of height mn together with a $\mathbb{Q}_p$-homomorphism $\iota : E \to \mathrm{End}_k(V)$.

(iii) If $G = \mathrm{GSp}_{2n}$, an F-isocrystal with G-structure is an F-isocrystal V of height $2n$ together with a non-degenerate alternating pairing $V \otimes V \to \mathbf{1}$, where $\mathbf{1}$ is the unit object of the tensor category $F\text{-Isoc}(k)$.

2.3.3. Let us now assume that k is algebraically closed. We say that $b, b' \in G(K_0)$ are σ-conjugate if there exists $g \in G(K_0)$ such that $b' = gb\sigma(g)^{-1}$. We denote by $B(G)$ the set of all σ-conjugacy classes in $G(K_0)$. The definition of $B(G)$ is independent of k in the sense that any inclusion $k \hookrightarrow k'$ into another algebraically closed field of characteristic p induces a bijection between the σ-conjugacy classes of $G(K_0(k))$ and those of $G(K_0(k'))$. We will write $[b]_G$, or simply $[b]$ when there is no risk of confusion, for the σ-conjugacy class of $b \in G(K_0)$.

The set $B(G)$ classifies the F-isocrystals over k with G-structure up to isomorphism. We describe this classification as explained in [**RR96**], 3.4. Given $b \in G(K_0)$ and a G-representation (V, ρ) over $\mathbb{Q}_p$, set $N_b(\rho)$ to be $V \otimes_{\mathbb{Q}_p} K_0$ with a σ-linear automorphism $F = \rho(b) \circ (1 \otimes \sigma)$. Then $N_b : \mathrm{Rep}_{\mathbb{Q}_p}(G) \to F\text{-Isoc}(k)$ is an exact faithful tensor functor. It is evident that two elements $b_1, b_2 \in G(K_0)$ give an isomorphic functor if and only if they are σ-conjugate. One can also prove that any

F-isocrystal on k with G-structure is isomorphic to a functor N_b for some $b \in G(K_0)$. Thus the association $b \mapsto N_b$ induces the desired classification.

2.3.4. Let $\mathbb{D}$ be the pro-algebraic torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. We introduce the set

$$\mathcal{N}(G) := (\mathrm{Int}\, G(K_0)\backslash \mathrm{Hom}_{K_0}(\mathbb{D}, G_{K_0}))^{\langle\sigma\rangle}.$$

If we fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$, we can also write

$$\mathcal{N}(G) = (X_*(T)_{\mathbb{Q}}/\Omega)^{\langle\sigma\rangle}.$$

We can define a partial order $\preceq$ on $\mathcal{N}(G)$ as follows. Let $\bar{C}$ be the closed Weyl chamber. First we define a partial order $\preceq_1$ on $X_*(T)_{\mathbb{R}}$ by declaring that $\alpha \preceq_1 \alpha'$ if and only if $\alpha' - \alpha$ is a nonnegative linear combination of positive coroots. Each orbit in $X_*(T)_{\mathbb{R}}/\Omega$ is represented by a unique element in $\bar{C}$, so the restriction of $\preceq_1$ to $\bar{C}$ induces a partial order $\preceq_2$ on $X_*(T)_{\mathbb{R}}/\Omega$. Then we take $\preceq$ to be the restriction of $\preceq_2$ to $(X_*(T)_{\mathbb{Q}}/\Omega)^{\langle\sigma\rangle}$.

Remark. A closed embedding $G_1 \hookrightarrow G_2$ of connected reductive algebraic groups over $\mathbb{Q}_p$ induces an order-preserving map $\mathcal{N}(G_1) \to \mathcal{N}(G_2)$, which is not necessarily injective.

2.3.5. Kottwitz studied the set $B(G)$ by introducing two maps

$$\nu_G : B(G) \to \mathcal{N}(G), \quad \kappa_G : B(G) \to \pi_1(G)_{\langle\sigma\rangle}$$

called the Newton map and the Kottwitz map of G. We refer the readers to [**Ko85**], §4 or [**RR96**], §1 for definition of the Newton map, and [**Ko97**], §4 and §7 for definition of the Kottwitz map. Both maps are functorial in G; more precisely, they induce natural transformations of set-valued functors on the category of connected reductive groups

$$\nu : B(\cdot) \to \mathcal{N}(\cdot), \quad \kappa : B(\cdot) \to \pi_1(\cdot)_{\langle\sigma\rangle}.$$

Given $[b] \in B(G)$ (and its corresponding F-isocrystal with G-structure), we will often refer to two invariants $\nu_G([b])$ and $\kappa_G([b])$ respectively as the *Newton point* and the *Kottwitz point* of $[b]$. Kottwitz proved that a σ-conjugacy class is determined by its Newton point and Kottwitz point; in other words, the map

$$\nu_G \times \kappa_G : B(G) \to \mathcal{N}(G) \times \pi_1(G)_{\langle\sigma\rangle}$$

is injective ([**Ko97**], 4.13).

Example 2.3.6. We describe the Newton map for $G = GL_n$. Let T be the diagonal torus contained in the Borel subgroup of lower triangular matrices. Then using the identification $X_*(T) \cong \mathbb{Z}^n$ we can write

$$\mathcal{N}(\mathrm{GL}_n) = \{(r_1, r_2, \cdots, r_n) \in \mathbb{Q}^n : r_1 \leq r_2 \leq \cdots \leq r_n\},$$

which can be identified with the set of convex polygons with rational slopes. We have $(r_i) \preceq (s_i)$ if and only if $\sum_{i=1}^{l}(r_i - s_i) \geq 0$ for all $l \in \{1, 2, \cdots, n\}$, so the ordering $\preceq$ coincides with the usual “lying above" order for convex polygons.

If V is an F-isocrystal V of height n associated to $[b] \in B(\mathrm{GL}_n)$, its Newton point $\nu_{\mathrm{GL}_n}([b])$ is the same as its classical Newton polygon. In this case, the Kottwitz point $\kappa_{\mathrm{GL}_n}([b])$ is determined by the Newton point $\nu_{\mathrm{GL}_n}([b])$. Hence V and $[b]$ are determined by the Newton point $\nu_{\mathrm{GL}_n}([b])$, and we recover Manin's classification of F-isocrystals by their Newton polygons in [**Ma63**].

2.3.7. Let $\mu \in X_*(T)$ be a dominant cocharacter. Then μ represents a unique conjugacy class of cocharacters of $G(K_0)$ which we denote by $\{\mu\}$. We identify μ with its image in $X_*(T)/\Omega$, and define

$$\bar{\mu} = \frac{1}{m}\sum_{i=0}^{m-1}\sigma^i(\mu) \in \mathcal{N}(G),$$

where m is some integer such that $\sigma^m(\mu) = \mu$. Note that our definition of $\bar{\mu}$ does not depend on the choice of m. We also let $\mu^\natural \in \pi_1(G)_{\langle\sigma\rangle}$ be the image of μ under the natural projection $X_*(T) \longrightarrow \pi_1(G)_{\langle\sigma\rangle} = (X_*(T)/\langle\alpha^\vee : \alpha^\vee \in \Phi^\vee\rangle)_{\langle\sigma\rangle}$. The characterization of the Newton map in [**Ko85**], 4.3 shows that $\bar{\mu}$ is the image of $[\mu(p)]$ under ν_G. It also follows directly from the definition of κ_G that $\mu^\natural$ is the image of $[\mu(p)]$ under κ_G.

Let us now define the set

$$B(G, \{\mu\}) := \{[b] \in B(G) : \kappa_G([b]) = \mu^\natural, \nu_G([b]) \preceq \bar{\mu}\}.$$

This set is known to be finite (see [**RR96**], 2.4.). It is also non-empty since we have $[\mu(p)] \in B(G, \{\mu\})$ by the discussion in the previous paragraph.

Since the Newton map is injective on $B(G, \{\mu\})$ (see 2.3.5), the partial order $\preceq$ on $\mathcal{N}(G)$ induces a partial order on $B(G, \{\mu\})$. We will also use the symbol $\preceq$ to denote this induced partial order. Note that $[\mu(p)]$ is a unique maximal element in $B(G, \{\mu\})$ as the inequality $[b] \leq [\mu(p)]$ clearly holds for all $[b] \in B(G, \{\mu\})$.

We refer to the σ-conjugacy class $[\mu(p)]$ as the *μ-ordinary* element of $B(G, \{\mu\})$. We say that an F-isocrystal over k with G-structure is μ-ordinary if it corresponds to $[\mu(p)]$ in the sense of 2.3.3. Note that a σ-conjugacy class $[b] \in B(G, \{\mu\})$ is μ-ordinary if and only if $\nu_G([b]) = \bar{\mu}$.

2.4 Unramified local Shimura data of Hodge type

In this subsection, we review the notion of unramified local Shimura data of Hodge type and describe F-crystals with additional structures that arise from such data.

2.4.1. Assume that k is algebraically closed. By an *unramified (integral) local Shimura datum of Hodge type*, we mean a tuple $(G, [b], \{\mu\})$ where

- G is an unramified connected reductive group over $\mathbb{Q}_p$;
- $[b]$ is a σ-conjugacy class of $G(K_0)$;
- $\{\mu\}$ is a $G(W)$-conjugacy class of cocharacters of G,

which satisfy the following two conditions:

(i) $[b] \in B(G, \{\mu\})$,

(ii) there exists a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ (with its dual Λ^*) such that, for all $b \in [b]$ and $\mu \in \{\mu\}$ satisfying $b \in G(W)\mu(p)G(W)$, the W-lattice

$$M := \Lambda^* \otimes_{\mathbb{Z}_p} W \subset N_b(\Lambda^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

satisfies the property $pM \subset FM \subset M$ (where F is defined from b as explained in 2.3.3).

Here $N_b : \mathrm{Rep}_{\mathbb{Q}_p}(G) \to F\text{-Isoc}(k)$ is the functor defined in 2.3.3 which is uniquely determined by $[b]$. The set $G(W)\mu(p)G(W)$ is independent of the choice $\mu \in \{\mu\}$ as explained in 2.2.2. The property $pM \subset FM \subset M$ means that M is an F-crystal over k (with a σ-linear endomorphism F). The requirement $b \in G(W)\mu(p)G(W)$ ensures that the Hodge filtration of M is induced by $\sigma^{-1}(\mu)$.

In practice when one tries to check that a given tuple $(G, [b], \{\mu\})$ is an unramified local Shimura datum, it is often more convenient to work with the following equivalent conditions of (i) and (ii):

(i') $[b] \cap G(W)\mu(p)G(W)$ is not empty for some (and hence for all) $\mu \in \{\mu\}$,

(ii') there exists a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ (with its dual Λ^*) such that, for *some* $b \in [b]$ and $\mu \in \{\mu\}$ satisfying $b \in G(W)\mu(p)G(W)$, the W-lattice

$$M := \Lambda^* \otimes_{\mathbb{Z}_p} W \subset N_b(\Lambda^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

satisfies the property $pM \subset FM \subset M$.

The equivalence of (i) and (i') is due to work of several authors, including Kottwitz-Rapoport [**KR03**], Lucarelli [**Lu04**] and Gashi [**Ga10**]. Note that (i') ensures that the condition (ii) is never vacuously satisfied. The equivalence of (ii) and (ii'), one observes that both conditions are equivalent to the condition that the linearization of F has an integer matrix representation after taking some σ-conjugate, which depends only on $[b]$.

Remark. When $\{\mu\}$ is minuscule, an unramified local Shimura datum of Hodge type as defined above is a local Shimura datum as defined by Rapoport and Viehmann in [**RV14**], Definition 5.1. In fact, since G is split over W, we may view geometric conjugacy classes of cocharacters as $G(W)$-conjugacy classes of cocharacters.

Using the conditions (i') and (ii') one easily verifies the following functorial properties of unramified local Shimura data of Hodge type:

Lemma 2.4.2. *Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of Hodge type.*

(1) If $(G', [b'], \{\mu'\})$ is another unramified local Shimura datum of Hodge type, the tuple $(G \times G', [b, b'], \{\mu, \mu'\})$ is also an unramified local Shimura datum of Hodge type.

(2) For any homomorphism $f : G \longrightarrow G'$ of unramified connected reductive group defined over $\mathbb{Z}_p$, the tuple $(G', [f(b)], \{f \circ \mu\})$ is an unramified local Shimura datum of Hodge type.

2.4.3. For the rest of this section, we fix our unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$ and also a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ in the condition (ii) of 2.4.1. By Lemma 2.4.2, we obtain a morphism of unramified local Shimura data of Hodge type

$$(G, [b], \{\mu\}) \longrightarrow (\mathrm{GL}(\Lambda), [b]_{\mathrm{GL}(\Lambda)}, \{\mu\}_{\mathrm{GL}(\Lambda)}).$$

Let us now choose an element $b \in [b] \cap G(W)\mu(p)G(W)$ and take $M := \Lambda^* \otimes_{\mathbb{Z}_p} W$ as in the condition (ii) of 2.4.1. We also choose a finite family of tensors $(s_i)_{i\in I}$ on Λ as in 2.2.3. Then $M = \Lambda^* \otimes_{\mathbb{Z}_p} W$ is equipped with tensors $(t_i) := (s_i \otimes 1)$, which are F-invariant since the linearization of F on $M[1/p] = N_b(\Lambda^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is given by an element $b \in G(K_0)$ in the conjugacy class $[b]$. We may regard the tensors (t_i) as additional structures on M induced by the group G. Following the terminology of 2.3, we will often refer to these additional structures as *G-structure*. We will also write $\underline{M} := (M, (t_i))$, which will often be referred to as an *F-crystal with G-structure* (induced by b).

When $\{\mu\}$ is minuscule, we have a unique p-divisible group X over k with $\mathbb{D}(X) = M$. The Hodge filtration $\mathrm{Fil}^1(\mathbb{D}(X)) \subset \mathbb{D}(X)$ is a $\{\sigma^{-1}(\mu^{-1})\}$-filtration with respect to (t_i), as explained in [**Kim13**], Lemma 2.5.7 and Remark 2.5.8. In this case, we will often write $\underline{X} := (X, (t_i))$ and refer to it as a *p-divisible group with G-structure* (induced by b). We will sometimes use the phrase "tensors on $\underline{X}$" to indicate the tensors (t_i), although strictly speaking they are tensors on the Dieudonne module $\mathbb{D}(X) = M$.

2.4.4. For the datum $(G, [b], \{\mu\})$, we can define its Newton point and Kottwitz point by $\nu_G([b])$ and $\kappa_G([b])$. Taking a unique dominant representative μ of $\{\mu\}$, we can also define $\bar{\mu}$ as in 2.3.7, which we call the *σ-invariant Hodge point* of $(G, [b], \{\mu\})$. We say that $(G, [b], \{\mu\})$ is μ-ordinary if $[b]$ is μ-ordinary.

For the F-crystal with G-structure $\underline{M}$, we define its Newton point, Kottwitz point and σ-invariant Hodge point to be the corresponding invariants for $(G, [b], \{\mu\})$. We say that $\underline{M}$ is ordinary if $(G, [b], \{\mu\})$ is ordinary. When $\{\mu\}$ is minuscule, these definitions obviously extend to the corresponding p-divisible group with G-structure $\underline{X}$.

Remark. We can further extend most of the notions defined in this section to the case when k is not algebraically closed. For example, we may define an F-crystal over k with G-structure as an F-crystal M over k equipped with tensors (t_i) such that the pair $(M \otimes_{W(k)} W(\bar{k}), (t_i \otimes 1))$ is an F-crystal over $\bar{k}$ with G-structure as defined in 2.4.3. Then we have natural notions of the Newton point, Kottwitz point, σ-invariant Hodge point, and μ-ordinariness induced by the corresponding notions for $(M \otimes_{W(k)} W(\bar{k}), (t_i \otimes 1))$. This explains why we may safely focus our study on the case when k is algebraically closed.

Example 2.4.5. As a concrete example, let us consider the case $G = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n$, where $\mathscr{O}$ is the ring of integers of some finite unramified extension E of $\mathbb{Q}_p$.

Choosing a family of tensors (s_i) on Λ whose pointwise stabilizer is G amounts to choosing a $\mathbb{Z}_p$-basis of $\mathscr{O}$. Hence $\underline{M} = (M, (t_i))$ can be identified with an F-crystal M with an action of $\mathscr{O}$ (cf. Example 2.3.2.(ii)). Following Moonen in [**Mo04**], we will often say *$\mathscr{O}$-module structure* in lieu of G-structure.

We now take $\mathscr{I} := \mathrm{Hom}(\mathscr{O}, W(k))$ and $m := |E : \mathbb{Q}_p|$. Note that $\mathscr{I}$ has m elements. For convenience, we will write $i + s := \sigma^s \circ i$ for any $i \in \mathscr{I}$ and $s \in \mathbb{Z}$. Then M, being a module over $\mathscr{O} \otimes_{\mathbb{Z}_p} W(k) = \prod_{i \in \mathscr{I}} W(k)$, decomposes into character spaces

$$M = \bigoplus_{i \in \mathscr{I}} M_i \qquad \text{where } M_i = \{x \in M : a \cdot x = i(a)x\}. \tag{2.4.5.1}$$

For each $i \in \mathscr{I}$, the Frobenius map F restricts to a σ-linear map $F_i : M_i \to M_{i+1}$. Then the map F^m restricts to a σ^m-linear endomorphism ϕ_i of M_i, thereby yielding a σ^m-F-crystal (M_i, ϕ_i) over k. By construction, F_i induces an isogeny from $\sigma^*(M_i, \phi_i)$ to (M_{i+1}, ϕ_{i+1}). This implies that the rank and the Newton polygon of (M_i, ϕ_i) is independent of $i \in \mathscr{I}$. We will write d for the rank of (M_i, ϕ_i).

The decomposition (2.4.5.1) yields a decomposition

$$M/FM = \bigoplus_{i \in \mathscr{I}} M_i/F_{i-1}M_i.$$

Define a function $\mathfrak{f} : \mathscr{I} \to \mathbb{Z}$ by setting $\mathfrak{f}(i)$ to be the rank of $M_i/F_{i-1}M_i$. We refer to the datum $(d, \mathfrak{f})$ as the *type* of $\underline{M}$.

Let us describe the Newton point in this setting. Using the identifications $G_W \cong \prod_{i \in \mathscr{I}} \mathrm{GL}(M_i)$ and $X_*(T) \cong \mathbb{Z}^{md}$ we can write

$$X_*(T)_{\mathbb{Q}}/\Omega = \{(x_1, \cdots, x_{md}) \in \mathbb{Q}^{md} : x_{ds+1} \leq \cdots \leq x_{d(s+1)} \text{ for } s = 0, 1, \cdots, m-1\}.$$

For $\mu = (x_1, \cdots, x_{md}) \in X_*(T)_{\mathbb{Q}}/\Omega$ the action of σ is given by $\sigma(\mu) = (y_1, \cdots, y_{md})$ where $y_t = x_{t+d}$. Therefore we obtain an identification

$$\mathcal{N}(G) = \{(r_1, r_2, \cdots, r_d) \in \mathbb{Q}^d : r_1 \leq r_2 \leq \cdots \leq r_d\}. \tag{2.4.5.2}$$

Under this identification, the Newton point $\nu_G([b])$ of $\underline{M}$ coincides with the Newton polygon of (M_i, ϕ_i) which was already seen to be independent of $i \in \mathscr{I}$. We will refer to this polygon as the *Newton polygon* of $\underline{M}$. The polygon $\nu_G([b])$ is closely related with the Newton polygon of M (without $\mathscr{O}$-module structure) as follows: a rational number λ appears with multiplicity α in $\nu_G([b])$ (viewed as a d-tuple) if and only if it appears with multiplicity $m\alpha$ in the Newton polygon of M (viewed as an md-tuple).

We can also regard the σ-invariant Hodge point $\bar{\mu}$ as a polygon under the identification (2.4.5.2). We will refer to this polygon as the *σ-invariant Hodge polygon* of $\underline{M}$. The inequality $\nu_G([b]) \preceq \bar{\mu}$ serves as a generalized Mazur's inequality, which says that the Newton polygon $\nu_G([b])$ lies above the σ-invariant Hodge polygon $\bar{\mu}$. Note that $\underline{M}$ is μ-ordinary if and only if the two polygons coincide.

When $\{\mu\}$ is minuscule, we also identify $\underline{X} = (X, (t_i))$ with a p-divisible group X with an action of $\mathscr{O}$. All of the discussions above evidently apply to $\underline{X}$. Namely, we can define the type, the Newton polygon and the σ-invariant Hodge polygon of $\underline{X}$. In addition, when $\{\mu\}$ is minuscule we have the following facts:

(1) The σ-invariant Hodge polygon $\bar{\mu}$ of $\underline{X}$ is determined by the type $(d, \mathfrak{f})$ as follows: if we write $\bar{\mu} = (a_1, a_2, \cdots, a_d)$, the slopes a_j are given by

$$a_j = \#\{i \in \mathscr{I} : \mathfrak{f}(i) > d - j\}$$

(see [**Mo04**], 1.2.5.).

(2) There exists a unique isomorphism class of μ-ordinary p-divisible groups with $\mathscr{O}$-module structure of a fixed type $(d, \mathfrak{f})$ (see [**Mo04**], Theorem 1.3.7.).

Remark. As seen in 2.2.1, we have an embedding $G_W = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n \otimes_{\mathbb{Z}_p} W \hookrightarrow \mathrm{GL}(M)$ where the image is identified with a product of m copies of $\mathrm{GL}_n \otimes_{\mathbb{Z}_p} W$. The decomposition (2.4.5.1) shows that these copies are given by $\mathrm{GL}(M_i)$. In particular, we have $n = d$.

2.4.6. The isomorphism class of $\underline{M} = (M, (t_i))$ depends on the choice $b \in [b]$, even though $M[1/p] \simeq N_b(\Lambda^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is independent of this choice. To see this, let $\underline{M'} = (M', (t_i'))$ be the F-crystal over k with G-structure that arises from another choice $b' = gb\sigma(g)^{-1} \in [b] \cap G(W)\mu(p)G(W)$ for some $g \in G(K_0)$. Then g gives an isomorphism

$$M[1/p] \simeq N_b(\Lambda^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \xrightarrow{\sim} N_{b'}(\Lambda^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \simeq M'[1/p],$$

which also matches (t_i) with (t'_i) since $g \in G(K_0)$. However, this isomorphism does not induce an isomorphism between M and M' unless $g \in G(W)$.

The above discussion motivates us to consider the set

$$X^G_{\{\mu\}}([b]) := \{g \in G(K_0)/G(W) | gb\sigma(g)^{-1} \in G(W)\mu(p)G(W)\}.$$

This set is clearly independent of our choice of $b \in [b]$ up to bijection. It is also independent of the choice of $\mu \in \{\mu\}$ as we already noted that the set $G(W)\mu(p)G(W)$ only depends on the conjugacy class of μ. The set $X^G_{\{\mu\}}([b])$ is called the *affine Deligne-Lusztig set* associated to $(G, [b], \{\mu\})$.

Proposition 2.4.7. *Fix an element $b \in [b]$, and let $\underline{M} = (M, (t_i))$ denote the F-crystal with G-structure induced by b (as defined in 2.4.3). Then the affine Deligne-Lusztig set $X^G_{\{\mu\}}([b])$ classifies isomorphism classes of tuples $(M', (t'_i), \iota)$ where*

- *$(M', (t'_i))$ is an F-crystal over k with G-structure;*
- *$\iota : M'[1/p] \xrightarrow{\sim} M[1/p]$ is an isomorphism which matches (t'_i) with (t_i).*

When $\{\mu\}$ is minuscule, take X to be the p-divisible group with $\mathbb{D}(X) = M$. Then the set $X^G_{\{\mu\}}([b])$ also classifies isomorphism classes of tuples $(X', (t'_i), \iota)$ where

- *$(X', (t'_i))$ is a p-divisible group over k with G-structure;*
- *$\iota : X \to X'$ is a quasi-isogeny such that the induced isomorphism $\mathbb{D}(X')[1/p] \xrightarrow{\sim} \mathbb{D}(X)[1/p]$ matches (t'_i) with (t_i).*

Proof. The second part follows immediately from the first part using Dieudonné theory, so we need only prove the first part.

Let g be a representative of $gG(W) \in X^G_{\{\mu\}}([b])$. Then as discussed in 2.4.6, the element $b' := g^{-1}b\sigma(g)$ gives rise to an F-crystal over k with G-structure $(M', (t'_i))$ and an isomorphism $\iota : M'[1/p] \xrightarrow{\sim} M[1/p]$ which matches (t'_i) with (t_i). It is clear that the isomorphism class of $(M', (t'_i), \iota)$ does not depend on the choice of the representative g.

Conversely, let $(M', (t_i'), \iota)$ be a tuple as in the statement. Let $b' \in G(K_0)$ be the linearization of the Frobenius map on $M'[1/p]$. Then the isomorphism $\iota : M'[1/p] \xrightarrow{\sim} M[1/p]$ determines an element $g \in G(K_0)$ such that $b' = gb\sigma(g)^{-1}$. Moreover, we have $b' \in G(W)\mu(p)G(W)$ since $(M', (t_i'))$ is an F-crystal over k with G-structure. Changing $(M', (t_i'), \iota)$ to an isomorphic tuple will change g to gh for some $h \in G(W)$, so we get a well-defined element $gG(W) \in X^G_{\{\mu\}}([b])$.

These associations are clearly inverse to each other, so we complete the proof. $\square$

We now describe some functorial properties of affine Deligne-Lusztig sets which are compatible with the functorial properties of unramified local Shimura data of Hodge type described in Lemma 2.4.2.

Lemma 2.4.8. *Let G' be an unramified connected reductive group over $\mathbb{Q}_p$.*

(1) If $(G', [b'], \{\mu'\})$ is an unramified local Shimura datum of Hodge type, we have an isomorphism

$$X^{G\times G'}_{\{\mu,\mu'\}}([b, b']) \xrightarrow{\sim} X^G_{\{\mu\}}([b]) \times X^{G'}_{\{\mu'\}}([b'])$$

induced by the natural projections.

(2) For any homomorphism $f : G \longrightarrow G'$ defined over $\mathbb{Z}_p$, we have a natural map

$$X^G_{\{\mu\}}([b]) \longrightarrow X^{G'}_{\{f\circ\mu\}}([f(b)])$$

induced by $gG(W) \mapsto f(g)G'(W)$, which is injective if f is a closed immersion.

Proof. The only possibly non-trivial assertion is the injectivity of the natural map $X^G_{\{\mu\}}([b]) \longrightarrow X^{G'}_{\{f\circ\mu\}}([f(b)])$ in (2) when f is a closed immersion. To see this, one may assume that $G' = \mathrm{GL}_n$ by embedding G' into some GL_n. Then the assertion follows from the fact that the map

$$G(K_0)/G(W) \longrightarrow GL_n(K_0)/\mathrm{GL}_n(W)$$

is injective (see [**HP17**], 2.4.4.). $\square$

2.5 Deformation Spaces of p-divisible groups with Tate tensors

In this subsection, we review Faltings's construction of a "universal" deformation of p-divisible groups with Tate tensors, given in [**Fal99**], §7. We refer readers to [**Mo98**], §4 for a more detailed discussion of these results.

2.5.1. Let R be a ring of the form $R = W[[u_1, \cdots, u_N]]$ or $R = W[[u_1, \cdots, u_N]]/(p^m)$. We can define a lift of the Frobenius map on R, which we also denote by σ, by setting $\sigma(u_i) = u_i^p$.

We define a *filtered crystalline Dieudonné module* over R to be a 4-tuple $(\mathscr{M}, \mathrm{Fil}^1(\mathscr{M}), \nabla, F)$ where

- $\mathscr{M}$ is a free R-module of finite rank;
- $\mathrm{Fil}^1(\mathscr{M}) \subset \mathscr{M}$ is a direct summand;
- $\nabla : \mathscr{M} \to \mathscr{M} \otimes \widehat{\Omega}_{R/W}$ is an integrable, topologically quasi-nilpotent connection;
- $F_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is a σ-linear horizontal endomorphism,

which satisfy the following conditions:

(i) $F_{\mathscr{M}}$ induces an isomorphism $\left(\mathscr{M} + p^{-1}\mathrm{Fil}^1(\mathscr{M})\right) \otimes_{R,\sigma} R \xrightarrow{\sim} \mathscr{M}$, and

(ii) $\mathrm{Fil}^1(\mathscr{M}) \otimes_R (R/p) = \mathrm{Ker}\big(F \otimes \sigma_{R/p} : \mathscr{M} \otimes_R (R/p) \to \mathscr{M} \otimes_R (R/p)\big)$.

Combining the work of de Jong in [**dJ95**] and Grothendieck-Messing theory, we obtain an equivalence between the category of filtered crystalline Dieudonné modules over R and the (opposite) category of p-divisible groups over R (see also [**Mo98**], 4.1.).

2.5.2. Let X be a p-divisible group over k. We write $\mathbf{C}_W$ for the category of artinian local W-algebras with residue field k. By a *deformation* or *lifting* of X over $R \in \mathbf{C}_W$, we mean a p-divisible group $\mathscr{X}$ over R with an isomorphism $\alpha : \mathscr{X} \otimes_R k \cong X$. We define a functor $\mathrm{Def}_X : \mathbf{C}_W \to \mathbf{Sets}$ by setting $\mathrm{Def}_X(R)$ to be the set of isomorphism classes of deformations of X over R.

We take $M := \mathbb{D}(X)$, the contravariant Dieudonné module of X, and write F for the Frobenius map and $\mathrm{Fil}^1(M) \subset M$ for its Hodge filtration. We choose a cocharacter $\mu : \mathbb{G}_m \to \mathrm{GL}_W(M)$ such that $\sigma^{-1}(\mu)$ induces this filtration; for instance, we take μ to be the dominant cocharacter that represents the Hodge polygon of X under the identification of the Newton set $\mathcal{N}(\mathrm{GL}_n)$ in Example 2.3.6. The stabilizer of the complement of $\mathrm{Fil}^1(M)$ is a parabolic subgroup of $\mathrm{GL}_W(M)$. We let U^μ be its unipotent radical, and take the formal completion $\widehat{U}^\mu = \mathrm{Spf} R^\mu_{\mathrm{GL}}$ of U^μ at the identity section. Then R^μ_{GL} is a formal power series ring over W, so we can define a lift of Frobenius map on R^μ_{GL}.

Proposition 2.5.3 ([**Fal99**], §7)**.** *Let $u_t \in \widehat{U}^\mu(R^\mu_{GL})$ be the tautological point. Define*

$$\mathscr{M} := M \otimes_W R^\mu_{GL}, \quad Fil^1(\mathscr{M}) := Fil^1(M) \otimes_W R^\mu_{GL}, \quad F_{\mathscr{M}} := u_t \circ (F \otimes_W \sigma).$$

(1) There exists a unique topologically quasi-nilpotent connection $\nabla : \mathscr{M} \to \mathscr{M} \otimes \widehat{\Omega}_{R^\mu_{GL}/W}$ that commutes with $F_{\mathscr{M}}$, and this connection is integrable.

(2) If $p > 2$, the filtered crystalline Dieudonné module $(\mathscr{M}, Fil^1(\mathscr{M}), \nabla, F_{\mathscr{M}})$ corresponds to the universal deformation of X via the equivalence described in 2.5.1.

In particular, (2) implies that we have an identification $\mathrm{Def}_X \cong \mathrm{Spf} R^\mu_{\mathrm{GL}}$. We will write $\mathscr{X}^\mu_{\mathrm{GL}}$ for the universal deformation of X.

2.5.4. We now consider deformations of p-divisible groups with G-structure. We fix an unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$ with minuscule $\{\mu\}$. We also fix a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ in the condition (ii) of 2.4.1, and choose $b \in [b]$ and $\mu \in \{\mu\}$ such that $b \in G(W)\mu(p)G(W)$. Then we obtain an F-crystal with G-structure $\underline{M} = (M, (t_i))$ as explained in 2.4.3, which gives rise to a p-divisible group with G-structure $\underline{X} = (X, (t_i))$ since $\{\mu\}$ is minuscule. The condition $b \in G(W)\mu(p)G(W)$ ensures that the Hodge filtration $\mathrm{Fil}^1(M) \subset M$ is induced by $\sigma^{-1}(\mu)$, so all the constructions from 2.5.2 and Proposition 2.5.3 are valid for X.

Let $U^\mu_G := U^\mu \cap G_W$, which is a smooth unipotent subgroup of G_W. Take $\widehat{U}^\mu_G = \mathrm{Spf} R^\mu_G$ to be its formal completion at the identity section. Then R^μ_G is a formal power series ring over W, so we get a lift of Frobenius map to R^μ_G. Alternatively, we get this lift from the lift on R^μ_{GL} via the surjection $R^\mu_{\mathrm{GL}} \twoheadrightarrow R^\mu_G$ induced by the embedding $\widehat{U}^\mu_G \hookrightarrow \widehat{U}^\mu$.

Let $u_{t,G} \in \widehat{U}^\mu_G(R^\mu_G)$ be the tautological point. Define

$$\mathscr{M}_G := M \otimes_W R^\mu_G, \quad \mathrm{Fil}^1(\mathscr{M}_G) := \mathrm{Fil}^1(M) \otimes_W R^\mu_G, \quad F_{\mathscr{M}_G} := u_{t,G} \circ (F \otimes_W \sigma).$$

Then we have an integrable, topologically quasi-nilpotent connection $\nabla_G : \mathscr{M}_G \to \mathscr{M}_G \otimes \widehat{\Omega}_{R^\mu_G/W}$ induced by $\nabla : \mathscr{M} \to \mathscr{M} \otimes \widehat{\Omega}_{R^\mu_{\mathrm{GL}}/W}$ from Proposition 2.5.3. In addition, ∇_G clearly commutes with $F_{\mathscr{M}_G}$ by construction. Hence we have a filtered crystalline Dieudonné module $(\mathscr{M}_G, \mathrm{Fil}^1(\mathscr{M}_G), \nabla_G, F_{\mathscr{M}_G})$.

Note that $\mathscr{M}_G$ is equipped with tensors $(\mathbf{t}_i^{\mathrm{univ}}) := (t_i \otimes 1)$, which are evidently $F_{\mathscr{M}_G}$-invariant by construction. If $p > 2$, one can prove that these tensors lie in the 0th filtration (see [**Kim13**], Lemma 2.2.7 and Proposition 2.5.9.).

Let $\mathscr{X}_G^\mu$ be the p-divisible group over R_G^μ corresponding to $(\mathscr{M}_G, \mathrm{Fil}^1(\mathscr{M}_G), \nabla_G, F_{\mathscr{M}_G})$ via the equivalence described in 2.5.1. Alternatively, one can get $\mathscr{X}_G^\mu$ by simply pulling back $\mathscr{X}_{\mathrm{GL}}^\mu$ over R_G^μ. Then $\mathscr{X}_G^\mu$ is the "universal deformation" of $(X, (t_i))$ in the following sense:

Proposition 2.5.5 ([**Fal99**], §7)**.** *Assume that $p > 2$. Let R be a ring of the form $R = W[[u_1, \cdots, u_N]]$ or $R = W[[u_1, \cdots, u_N]]/(p^m)$. Choose a deformation $\mathscr{X}$ of X over R, and let $f : R_{GL}^\mu \to R$ be the morphism induced by $\mathscr{X}$ via $SpfR_{GL}^\mu \cong Def_X$. Then f factors through R_G^μ if and only if the tensors (t_i) can be lifted to tensors $(\boldsymbol{t}_i) \in \mathbb{D}(\mathscr{X})^\otimes$ which are Frobenius-invariant and lie in the 0th filtration with respect to the Hodge filtration. If this holds, then we necessarily have $(f^*\boldsymbol{t}_i^{univ}) = (\boldsymbol{t}_i)$.*

We define $\mathrm{Def}_{X,G}$ to be the image of the closed immersion $\mathrm{Spf}R_G^\mu \hookrightarrow \mathrm{Spf}R_{\mathrm{GL}}^\mu \cong \mathrm{Def}_X$. Then $\mathrm{Def}_{X,G}$ classifies deformations of $(X, (t_i))$ over formal power series rings over W or $W/(p^m)$ in the sense of Proposition 2.5.5. Note that our definition of $\mathrm{Def}_{X,G}$ is independent of the choice of (t_i) and $\mu \in \{\mu\}$; indeed, the independence of the choice of (t_i) is clear by construction, and the independence of the choice of μ follows from the universal property.

We close this section with some functorial properties of deformation spaces, which are compatible with the functorial properties of unramified local Shimura data of Hodge type described in Lemma 2.4.2. The proof is straightforward and thus omitted.

Lemma 2.5.6. *Let $(G', [b'], \{\mu'\})$ be another unramified local Shimura datum of Hodge type. Choose $b' \in [b']$ and $\mu' \in \{\mu'\}$ such that $b' \in G'(W)\mu'(p)G'(W)$, and let $(X', (t_i'))$ be a p-divisible group with G'-structure that arises from this choice.*

(1) The natural morphism $Def_X \times Def_{X'} \longrightarrow Def_{X \times X'}$, defined by taking the product of deformations, induces an isomorphism

$$Def_{X,G} \times Def_{X',G'} \xrightarrow{\sim} Def_{X \times X', G \times G'}.$$

(2) For any homomorphism $f : G \to G'$ defined over $\mathbb{Z}_p$ such that $f(b) = b'$, we have a natural morphism

$$Def_{X,G} \to Def_{X',G'}$$

induced by the map $\widehat{U}_G^{\mu} \to \widehat{U}_{G'}^{f\circ\mu}$.

Remark. With some additional work, one can show that the natural morphism $\mathrm{Def}_{X,G} \to \mathrm{Def}_{X',G'}$ in (2) is independent of the choice of $\mu \in \{\mu\}$. See [**Kim13**], Proposition 3.7.2 for details.

2.6 Rapoport-Zink spaces of Hodge type

In this section, we discuss the construction and key properties of Rapoport-Zink spaces of Hodge type, following [**Kim13**].

2.6.1. Let us fix some notations for this section. We set $k = \overline{\mathbb{F}}_p$ so that $W = \breve{\mathbb{Z}}_p$ and $K_0 = \breve{\mathbb{Q}}_p$. We fix an unraified local Shimura datum $(G, [b], \{\mu\})$ such that $\{\mu\}$ is minuscule. We also choose $b \in [b] \cap G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)$ and take $\underline{X} := (X, (t_i))$ as in 2.4.3.

Let $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ denote the category of $\breve{\mathbb{Z}}_p$-algebra where p is nilpotent. For any $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ we set $\mathrm{RZ}_b(R)$ to be the set of isomorphism classes of pairs $(\mathcal{X}, \iota)$ where

- $\mathcal{X}$ is a p-divisible group over R;
- $\iota : X_{R/p} \longrightarrow \mathcal{X}_{R/p}$ is a quasi-isogeny, i.e., an invertible global section of $\mathrm{Hom}(X_{R/p}, \mathcal{X}_{R/p}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Then RZ_b defines a covariant set-valued functor on $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$, which does not depend on the choice of $b \in [b] \cap G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)$ up to isomorphism. Rapoport and Zink in [**RZ96**] proved that the functor RZ_b is represented by a formal scheme which is locally formally of finite type and formally smooth over $\breve{\mathbb{Z}}_p$. We write RZ_b also for the representing formal scheme, and $\mathcal{X}_{\mathrm{GL},b}$ for the universal p-divisible group over RZ_b.

2.6.2. Given a pair $(\mathcal{X}, \iota) \in \mathrm{RZ}_b(R)$ with $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$, we have an isomorphism

$$\mathbb{D}(\iota) : \mathbb{D}(\mathcal{X}_{R/p})[1/p] \xrightarrow{\sim} \mathbb{D}(X_{R/p})[1/p]$$

induced by ι. We write $(t_{\mathcal{X},i})$ for the inverse image of the tensors $(t_i)_R$ under this isomorphism.

Let $\mathrm{Nilp}^{\mathrm{sm}}_{\breve{\mathbb{Z}}_p}$ denote the full subcategory of $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ consisting of formally smooth and formally finitely generated algebra over $\breve{\mathbb{Z}}_p/p^m$ for some positive integer m. For any $R \in \mathrm{Nilp}^{\mathrm{sm}}_{\breve{\mathbb{Z}}_p}$, we define the set $\mathrm{RZ}^{(s_i)}_{G,b}(R) \subset \mathrm{Hom}_{\breve{\mathbb{Z}}_p}(\mathrm{Spf}(R), \mathrm{RZ}_b)$ as follows: for a morphism $f : \mathrm{Spf}(R) \to \mathrm{RZ}_b$ and a p-divisible group $\mathcal{X}$ over $\mathrm{Spec}(R)$ which pulls back to $f^*\mathcal{X}_{\mathrm{GL},b}$ over $\mathrm{Spf}(R)$, we have $f \in \mathrm{RZ}^{(s_i)}_{G,b}(R)$ if and only if there exists a (unique) family of tensors $(\mathbf{t}_i)$ on $\mathbb{D}(\mathcal{X})$ with the following properties:

(i) for some ideal of definition J of R containing p, the pull-back of $(\mathbf{t}_i)$ over R/J agrees with the pull-back of $(t_{\mathcal{X},i})$ over R/J,

(ii) for a p-adic lift $\mathcal{R}$ of R which is formally smooth over $\breve{\mathbb{Z}}_p$, the $\mathcal{R}$-scheme
$$\mathcal{P}_{\mathcal{R}} := \mathbf{Isom}_{\mathcal{R}}\Big([\mathbb{D}(\mathcal{X})_{\mathcal{R}}, (\mathbf{t}_i)_{\mathcal{R}}], [\Lambda^* \otimes_{\mathbb{Z}_p} \mathcal{R}, (s_i \otimes 1)]\Big)$$
defined as in 2.2.3 is a G-torsor,

(iii) the Hodge filtration of $\mathcal{X}$ is a $\{\sigma^{-1}(\mu^{-1})\}$-filtration with respect to $(\mathbf{t}_i)$.

Then $\mathrm{RZ}^{(s_i)}_{G,b}$ defines a set-valued functor on $\mathrm{Nilp}^{\mathrm{sm}}_{\breve{\mathbb{Z}}_p}$.

Proposition 2.6.3 ([**Kim13**], Theorem 4.9.1.)**.** *Assume that $p > 2$. Then there exists a closed formal subscheme $RZ_{G,b} \subset RZ_b$, which is formally smooth over $\breve{\mathbb{Z}}_p$ and represents the functor $RZ^{(s_i)}_{G,b}$ for any choice of the tensors (s_i) in 2.4.3. Moreover, the isomorphism class of the formal scheme $RZ_{G,b}$ depends only on the datum $(G, [b], \{\mu\})$.*

We let $\mathcal{X}_{G,b}$ denote the "universal p-divisible group" over $\mathrm{RZ}_{G,b}$, obtained by taking the pull-back of $\mathcal{X}_{\mathrm{GL},b}$. Then we obtain a family of "universal tensors" $(\mathbf{t}^{\mathrm{univ}}_i)$ on $\mathbb{D}(\mathcal{X}_{G,b})$ by applying the universal property to an open affine covering of $\mathrm{RZ}_{G,b}$.

For the rest of this section, we assume that $p > 2$ and take $\mathrm{RZ}_{G,b}$ as in Proposition 2.6.3.

Example 2.6.4. Consider the case $G = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n$ where $\mathscr{O}$ is the ring of integers of some finite unramified extension of $\mathbb{Q}_p$. As explained in Example 2.4.5, we can regard $\underline{X} = (X, (t_i))$ as a p-divisible group X with an action of $\mathscr{O}$. In this setting, the construction of $\mathrm{RZ}_{G,b}$ agrees with the construction of Rapoport-Zink spaces of EL type in [**RZ96**] (see [**Kim13**], Proposition 4.7.1.). In other words, for any $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ the set $\mathrm{RZ}_{G,b}(R)$ classifies the isomorphism classes of pairs $(\mathcal{X}, \iota)$ where

- $\mathcal{X}$ is a p-divisible group over R, endowed with an action of $\mathcal{O}$ such that

$$\det{}_R(a, \mathrm{Lie}(\mathcal{X})) = \det(a, \mathrm{Fil}^0(\mathbb{D}(X))_{\breve{\mathbb{Q}}_p}) \quad \text{for all } a \in \mathcal{O},$$

- $\iota : X_{R/p} \to \mathcal{X}_{R/p}$ is a quasi-isogeny which commutes with the action of $\mathcal{O}$.

2.6.5. Let us give a concrete description of the set $\mathrm{RZ}_{G,b}(\overline{\mathbb{F}}_p)$. Consider a pair $(\mathcal{X}, \iota) \in \mathrm{RZ}_b(\overline{\mathbb{F}}_p)$ with a family of tensors $(\mathbf{t}_i)$ on $\mathbb{D}(\mathcal{X})$. Then $(\mathbf{t}_i)$ has the property (i) of 2.6.2 if and only if it is matched with the family (t_i) under the isomorphism

$$\mathbb{D}(\iota) : \mathbb{D}(\mathcal{X})[1/p] \xrightarrow{\sim} \mathbb{D}(X)[1/p]$$

induced by ι. In addition, it satisfies the properties (ii) and (iii) of 2.6.2 if and only if $(\mathcal{X}, (\mathbf{t}_i))$ is a p-divisible group with G-structure that arises from the datum $(G, [b], \{\mu\})$. Hence the set $\mathrm{RZ}_{G,b}(\overline{\mathbb{F}}_p)$ classifies the isomorphism classes of tuples $(\mathcal{X}, (\mathbf{t}_i), \iota)$ where

- $(\mathcal{X}, (\mathbf{t}_i))$ is a p-divisible group over $\overline{\mathbb{F}}_p$ with G-structure;
- $\iota : X \longrightarrow \mathcal{X}$ is a quasi-isogeny such that the induced isomorphism $\mathbb{D}(\mathcal{X})[1/p] \xrightarrow{\sim} \mathbb{D}(X)[1/p]$ matches $(\mathbf{t}_i)$ with (t_i).

By Proposition 2.4.7, we have a natural bijection

$$X^G_{\{\mu\}}([b]) \xrightarrow{\sim} \mathrm{RZ}_{G,b}(\overline{\mathbb{F}}_p).$$

Now we consider an $\overline{\mathbb{F}}_p$-valued point $x \in \mathrm{RZ}_{G,b}(\overline{\mathbb{F}}_p)$. Let us write $(X_x, (t_{x,i}), \iota_x)$ for the corresponding tuple under the above classification, and $\widehat{(\mathrm{RZ}_{G,b})_x}$ for the formal completion of $\mathrm{RZ}_{G,b}$ at x. Then we have a natural isomorphism

$$\mathrm{Def}_{X_x,G} \simeq \widehat{(\mathrm{RZ}_{G,b})_x}$$

as explained in [**Kim13**], 4.8.

Proposition 2.6.6 ([**Kim13**], Theorem 4.9.1.)**.** *Let $(G', [b'], \{\mu'\})$ be another unramified local Shimura datum of Hodge type, and choose $b' \in [b'] \cap G(\breve{\mathbb{Z}}_p)\mu'(p)G(\breve{\mathbb{Z}}_p)$ that gives rise to a p-divisible group over $\overline{\mathbb{F}}_p$ with G'-structure as in 2.4.3.*

(1) The natural morphism $RZ_b \times_{Spf(\breve{\mathbb{Z}}_p)} RZ_{b'} \longrightarrow RZ_{(b,b')}$, defined by the product of p-divisible groups with quasi-isogeny, induces an isomorphism

$$RZ_{G,b} \times_{Spf(\breve{\mathbb{Z}}_p)} RZ_{G',b'} \xrightarrow{\sim} RZ_{G\times G',(b,b')}$$

(2) For any homomorphism $f : G \longrightarrow G'$ with $f(b) = b'$, there exists an induced morphism

$$RZ_{G,b} \longrightarrow RZ_{G',b'},$$

which is a closed embedding if f is a closed embedding.

Moreover, via the natural bijections given in 2.6.5, the functorial properties (1) and (2) induce the functorial properties of affine Deligne-Lusztig sets and the formal deformation spaces described in Lemma 2.4.8 and Lemma 2.5.6.

2.6.7. We define a group valued functor J_b on the category of $\mathbb{Q}_p$-algebras by setting for any $\mathbb{Q}_p$-algebra R

$$J_b(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) : gb\sigma(g)^{-1} = b\}.$$

This functor is represented by an algebraic group over $\mathbb{Q}_p$ which is an inner form of some Levi subgroup of $G_{\mathbb{Q}_p}$ (see [**RZ96**], Corollary 1.14.). The isomorphism class of J_b does not depend on the choice $b \in [b]$ since any $g \in G(\breve{\mathbb{Q}}_p)$ induces an isomorphism $J_b \cong J_{gb\sigma(g)^{-1}}$ via conjugation. Note that $J_b(\mathbb{Q}_p)$ can be identified with the group of quasi-isogenies $\gamma : X \longrightarrow X$ that preserve the tensors (t_i). One can show that $\mathrm{RZ}_{G,b}$ carries a natural left $J_b(\mathbb{Q}_p)$-action defined by

$$\gamma(\mathcal{X}, \iota) = (\mathcal{X}, \iota \circ \gamma^{-1})$$

for any $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}, (\mathcal{X}, \iota) \in \mathrm{RZ}_{G,b}(R)$ and $\gamma \in J_b(\mathbb{Q}_p)$ (see [**Kim13**], 7.2.).

2.6.8. Let E be the field of definition of the $G(\breve{\mathbb{Q}}_p)$-conjugacy class of μ, and let $\mathscr{O}_E$ denote its ring of integers. Note that E is a finite unramified extension of $\mathbb{Q}_p$ since $G_{\mathbb{Q}_p}$ is split over a finite unramified extension of $\mathbb{Q}_p$. Let d be the degree of the extension, and write τ for the Frobenius automorphism of $\breve{\mathbb{Q}}_p$ relative to E.

For any formal scheme S over $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$, we write $S^\tau := S \times_{\mathrm{Spf}(\breve{\mathbb{Z}}_p),\tau} \mathrm{Spf}(\breve{\mathbb{Z}}_p)$. By a *Weil descent datum* on S over $\mathscr{O}_E$, we mean an isomorphism $S \xrightarrow{\sim} S^\tau$. If $S \cong S_0 \times_{\mathrm{Spf}(\mathscr{O}_E)} \mathrm{Spf}(\breve{\mathbb{Z}}_p)$ for some formal scheme S_0 over $\mathrm{Spf}(\mathscr{O}_E)$, then there exists a natural Weil descent datum on S over $\mathscr{O}_E$, called an *effective* Weil descent datum.

For any $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$, we define R^τ to be R viewed as a $\breve{\mathbb{Z}}_p$-algebra via τ. Note that we have a natural identification $\mathrm{RZ}_b^\tau(R) = \mathrm{RZ}_b(R^\tau)$. Following Rapoport and Zink in [**RZ96**], 3.48, we define a Weil descent datum Φ on RZ_b over $\mathscr{O}_E$ by sending $(\mathcal{X}, \iota) \in \mathrm{RZ}_b(R)$ with $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ to $(\mathcal{X}^\Phi, \iota^\Phi) \in \mathrm{RZ}_b(R^\tau)$, where

- $\mathcal{X}^\Phi$ is $\mathcal{X}$ viewed as a p-divisible group over R^τ;
- ι^Φ is the quasi-isogeny

$$\iota^\Phi : X_{R^\tau/p} = (\tau^* X)_{R/p} \xrightarrow{\mathrm{Frob}^{-d}} X_{R/p}, \xrightarrow{\iota} \mathcal{X}_{R/p} = \mathcal{X}^\Phi_{R/p}$$

where $\mathrm{Frob}^d : X \to \tau^* X$ is the relative q-Frobenius with $q = p^d$.

One can check that Φ restricts to a Weil descent datum Φ_G on $\mathrm{RZ}_{G,b}$ over $\mathcal{O}_E$ by looking at $\overline{\mathbb{F}}_p$-points and the formal completions thereof. The Weil descent datum Φ_G clearly commutes with the $J_b(\mathbb{Q}_p)$-action defined in 2.6.7.

2.6.9. Since $\mathrm{RZ}_{G,b}$ is locally formally of finite type over $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$, it admits a rigid analytic generic fiber which we denote by $\mathrm{RZ}^{\mathrm{rig}}_{G,b}$ (see [**Ber96**].). The $J_b(\mathbb{Q}_p)$-action and the Weil descent datum Φ_G on $\mathrm{RZ}_{G,b}$ induce an action of $J_b(\mathbb{Q}_p)$ on $\mathrm{RZ}^{\mathrm{rig}}_{G,b}$ and an Weil descent datum $\Phi_G : \mathrm{RZ}^{\mathrm{rig}}_{G,b} \xrightarrow{\sim} (\mathrm{RZ}^{\mathrm{rig}}_{G,b})^\tau$ over E.

Recall that we have a universal p-divisible group $\mathcal{X}_{G,b}$ over $\mathrm{RZ}_{G,b}$ and a family of universal tensors $(\mathbf{t}_i^{\mathrm{univ}})$ on $\mathbb{D}(\mathcal{X}_{G,b})$. In addition, the family $(\mathbf{t}_i^{\mathrm{univ}})$ has a "étale realization" $(\mathbf{t}_{i,\text{ét}}^{\mathrm{univ}})$ on the Tate module $T_p(\mathcal{X}_{G,b})$ (see [**Kim13**], Theorem 7.1.6.).

For any open compact subgroup K_p of $G(\mathbb{Z}_p)$, we define the following rigid analytic étale cover of $\mathrm{RZ}^{\mathrm{rig}}_{G,b}$:

$$\mathrm{RZ}^{K_p}_{G,b} := \mathbf{Isom}_{\mathrm{RZ}^{\mathrm{rig}}_{G,b}}\Big([\Lambda, (s_i)], [T_p(\mathcal{X}_{G,b}), (\mathbf{t}_{i,\text{ét}}^{\mathrm{univ}})]\Big)\Big/ K_p.$$

The $J_b(\mathbb{Q}_p)$-action and the Weil descent datum over E on $\mathrm{RZ}^{\mathrm{rig}}_{G,b}$ pull back to $\mathrm{RZ}^{K_p}_{G,b}$. As the level K_p varies, these covers form a tower $\{\mathrm{RZ}^{K_p}_{G,b}\}$ with Galois group $G(\mathbb{Z}_p)$. We denote this tower by $\mathrm{RZ}^\infty_{G,b}$.

By [**Kim13**], Proposition 7.4.8, there exists a right $G(\mathbb{Q}_p)$-action on the tower $\mathrm{RZ}^\infty_{G,b}$ extending the Galois action of $G(\mathbb{Z}_p)$, which commutes with the natural $J_b(\mathbb{Q}_p)$-action and the Weil descent datum over E. In addition, there is a well-defined period map on $\mathrm{RZ}^{\mathrm{rig}}_{G,b}$ as explained in [**Kim13**], 7.5. Hence the tower $\mathrm{RZ}^\infty_{G,b}$ is a local Shimura variety in the sense of Rapoport and Viehmann in [**RV14**], 5.1.

2.6.10. We fix a prime $l \neq p$, and let $\mathcal{W}_E$ denote the Weil group of E. For any level $K_p \subset G(\mathbb{Z}_p)$, we consider the cohomology groups

$$H^i(\mathrm{RZ}^{K_p}_{G,b}) = H^i_c(\mathrm{RZ}^{K_p}_{G,b} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \mathbb{Q}_l(\dim \mathrm{RZ}^{K_p}_{G,b})).$$

As the level K_p varies, these cohomology groups form a tower $\{H^i(\mathrm{RZ}_{G,b}^{K_p})\}$ for each i, endowed with a natural action of $G(\mathbb{Q}_p) \times \mathcal{W}_E \times J_b(\mathbb{Q}_p)$.

Let ρ be an admissible l-adic representation of $J_b(\mathbb{Q}_p)$. The groups

$$H^{i,j}(\mathrm{RZ}_{G,b}^{\infty})_\rho := \varinjlim_{K_p} \mathrm{Ext}^j_{J_b(\mathbb{Q}_p)}(H^i(\mathrm{RZ}_{G,b}^{K_p}), \rho)$$

satisfy the following properties (see [**RV14**], Proposition 6.1 and [**Man08**], Theorem 8):

(1) The groups $H^{i,j}(\mathrm{RZ}_{G,b}^{\infty})_\rho$ vanish for almost all i, j.

(2) There is a natural action of $G(\mathbb{Q}_p) \times \mathcal{W}_E$ on each $H^{i,j}(\mathrm{RZ}_{G,b}^{\infty})_\rho$.

(3) The representations $H^{i,j}(\mathrm{RZ}_{G,b}^{\infty})_\rho$ are admissible.

Hence we can define a virtual representation of $G(\mathbb{Q}_p) \times \mathcal{W}_E$

$$H^{\bullet}(\mathrm{RZ}_{G,b}^{\infty})_\rho := \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathrm{RZ}_{G,b}^{\infty})_\rho.$$

Chapter 3

THE HODGE-NEWTON FILTRATION FOR HODGE-NEWTON REDUCIBLE LOCAL SHIMURA DATA

In this chapter, we state and prove our results on the Hodge-Newton decomposition and the Hodge-Newton filtration in the setting of unramified local Shimura data of Hodge type.

Throughout this chapter, k is assumed to be algebraically closed.

3.1 EL realization of Hodge-Newton reducibility

3.1.1. Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of Hodge type. Choose a maximal torus $T \subseteq G$ and a Borel subgroup $B \subseteq G$ containing T, both defined over $\mathbb{Z}_p$. Let P be a proper standard parabolic subgroup of G with Levi factor L and unipotent radical U. We say that $(G, [b], \{\mu\})$ is *Hodge-Newton reducible* (with respect to P and L) if there exist $\mu \in \{\mu\}$ which factors through L and an element $b \in [b] \cap L(K_0)$ which satisfy the following conditions:

(i) $[b]_L \in B(L, \{\mu\}_L)$,

(ii) in the action of μ and $\nu_G([b])$ on $\mathrm{Lie}(U) \otimes_{\mathbb{Q}_p} K_0$, only non-negative characters occur.

Since G is unramified, one can give an alternative definition in terms of some specific choice of $b \in [b] \cap L(K_0)$ and $\mu \in \{\mu\}$ (see [**RV14**], Remark 4.25.).

Example 3.1.2. Consider the case $G = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n$, where $\mathscr{O}$ is the ring of integers of some finite unramified extension of $\mathbb{Q}_p$. Then L is of the form

$$L = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_1} \times \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_2} \times \cdots \times \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_r}.$$

Recall from Example 2.4.5 that we have an identification

$$\mathcal{N}(G) = \{(r_1, r_2, \cdots, r_n) \in \mathbb{Q}^d : r_1 \leq r_2 \leq \cdots \leq r_n\}.$$

Using this, we may write $\nu_G([b]) = (\nu_1, \nu_2, \cdots, \nu_n)$ and $\bar{\mu} = (\mu_1, \mu_2, \cdots, \mu_r)$. In this setting, the conditions (i) and (ii) in 3.1.1 are equivalent to the following conditions:

(i') $\nu_1 + \nu_2 + \cdots + \nu_{j_k} = \mu_1 + \mu_2 + \cdots + \mu_{j_k}$ for each $k = 1, 2, \cdots, r$,

(ii') $\nu_{j_k} < \nu_{j_k+1}$ for each $k = 1, 2, \cdots, r$.

In other words, $(G, [b], \{\mu\})$ is Hodge-Newton reducible (with respect to P and L) if and only if the Newton polygon $\nu_G([b])$ and the σ-invariant Hodge polygon $\bar{\mu}$ have contact points which are break points of $\nu_G([b])$ specified by L. We refer the readers to [**MV10**], §3 for more details.

3.1.3. For the rest of this chapter, we fix an unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$ which is Hodge-Newton reducible with respect to P and L. Let us also fix a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ in the condition (ii) of 2.4.1 and choose a finite family of tensors (s_i) on Λ as in 2.4.3. Our strategy is to study $(G, [b], \{\mu\})$ by embedding G into another group $\widetilde{G}$ of EL type such that the datum $(\widetilde{G}, [b], \{\mu\})$ is also Hodge-Newton reducible.

Note that if G is not split, the datum $(\mathrm{GL}(\Lambda), [b], \{\mu\})$ may not be Hodge-Newton reducible in general. In fact, the map on the Newton sets $\mathcal{N}(G) \longrightarrow \mathcal{N}(\mathrm{GL}(\Lambda))$ induced by the embedding $G \hookrightarrow \mathrm{GL}(\Lambda)$ does not map $\bar{\mu}_G$ to the Hodge polygon $\mu_{\mathrm{GL}(\Lambda)}$ since it does not respect the action of σ.

Lemma 3.1.4. *There exists a group $\widetilde{G}$ of EL type with the following properties:*

(i) the embedding $G \hookrightarrow GL(\Lambda)$ factors through $\widetilde{G}$.

(ii) the datum $(\widetilde{G}, [b], \{\mu\})$ is Hodge-Newton reducible with respect to a parabolic subgroup $\widetilde{P} \subsetneq \widetilde{G}$ and its Levi factor $\widetilde{L}$ such that $P = \widetilde{P} \cap G$ and $L = \widetilde{L} \cap G$.

Proof. Write $V := \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}^{\mathrm{un}}$ where $\mathbb{Q}^{\mathrm{un}}$ is the maximal unramified extension of $\mathbb{Q}_p$ in a fixed algebraic closure. We know that G is split over $\mathbb{Q}^{\mathrm{un}}$ for being unramified over $\mathbb{Q}_p$. Hence V admits a decomposition into character spaces

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi \tag{3.1.4.1}$$

with the property that $\sigma(V_\chi) = V_{\sigma\chi}$.

For each $\chi \in X^*(T)$, let $\langle\chi\rangle$ denote the Ω-conjugacy class of χ and write $V_{\langle\chi\rangle} := \oplus_{\omega\in\Omega} V_{\omega\cdot\chi}$. Since V is a G-representation, we can rewrite the decomposition (3.1.4.1) as

$$V = \bigoplus_{\langle\chi\rangle\in X^*(T)/\Omega} V_{\langle\chi\rangle},$$

where $V_{\langle\chi\rangle}$'s are sub G-representations (see [**Se68**], Theorem 4.) with the property that $V_{\langle\sigma\chi\rangle} = \sigma(V_{\langle\chi\rangle})$. If a Ω-conjugacy class $\langle\chi\rangle \in X^*(T)/\Omega$ is in an orbit of size m under the action of σ, the G-representation

$$\bigoplus_{i=0}^{m-1} V_{\langle\sigma^i\chi\rangle}$$

is also a $\mathrm{Res}_{E|\mathbb{Q}_p}\mathrm{GL}_n$-representation where E is the field of definition of $\langle\chi\rangle$, which is a degree m unramified extension of $\mathbb{Q}_p$ (cf. (2.4.5.1) in Example 2.4.5). Hence the embedding $G_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(\Lambda_{\mathbb{Q}_p})$ factors through a group of the form $\prod \mathrm{Res}_{E_j|\mathbb{Q}_p}\mathrm{GL}_{n_j}$ where each E_j is the field of definition of an orbit in $X^*(T)/\Omega$. Then by [**Se68**], Theorem 5, we can take the pull-back of this embedding over $\mathbb{Z}_p$ to obtain

$$G \longhookrightarrow \prod \mathrm{Res}_{\mathscr{O}_j|\mathbb{Z}_p}\mathrm{GL}_{n_j} \longhookrightarrow \mathrm{GL}(\Lambda)$$

where $\mathscr{O}_j$ is the ring of integers of E_j.

We take

$$\widetilde{G} := \prod \mathrm{Res}_{\mathscr{O}_j|\mathbb{Z}_p}\mathrm{GL}_{n_j}.$$

Choose a Borel pair $(\widetilde{B}, \widetilde{T})$ of $\widetilde{G}$ such that $B \subseteq \widetilde{B}$ and $T \subseteq \widetilde{T}$. Then we get a proper standard parabolic subgroup $\widetilde{P} \subsetneq \widetilde{G}$ with Levi factor $\widetilde{L}$ such that $P = \widetilde{P} \cap G$ and $L = \widetilde{L} \cap G$ (e.g. by using [**SGA3**], Exp. XXVI, Cor. 6.10.).

It is evident from the construction that the embedding $G \longhookrightarrow \widetilde{G}$ respects the action of σ on cocharacters. Hence the induced map on the Newton sets $\mathcal{N}(G) \longrightarrow \mathcal{N}(\widetilde{G})$ maps $\bar{\mu}_G$ to the σ-invariant Hodge polygon $\bar{\mu}_{\widetilde{G}}$. Combining this fact with the functoriality of the Kottwitz map and the Newton map, we verify that the datum $(\widetilde{G}, [b], \{\mu\})$ is Hodge-Newton reducible with respect to $\widetilde{P}$ and $\widetilde{L}$. $\square$

We will refer to the datum $(\widetilde{G}, [b], \{\mu\})$ in Lemma 3.1.4 as an *EL realization* of the Hodge-Newton reducible datum $(G, [b], \{\mu\})$.

Remark. If G is split, the construction in the proof above yields $\widetilde{G} = \mathrm{GL}(\Lambda)$.

3.2 The Hodge-Newton decomposition and the Hodge-Newton filtration

3.2.1. Fix an EL realization $(\widetilde{G}, [b], \{\mu\})$ of our datum $(G, [b], \{\mu\})$, and take $\widetilde{P}$ and $\widetilde{L}$ as in Lemma 3.1.4. In a view of the functorial properties in Lemma 2.4.2, Lemma 2.4.8, Lemma 2.5.6 and Proposition 2.6.6, we will always assume for simplicity that $\widetilde{G}$ is of the form

$$\widetilde{G} := \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n,$$

where $\mathscr{O}$ is the ring of integers of some finite unramified extension E of $\mathbb{Q}_p$. Then $\widetilde{L}$ is of the form

$$\widetilde{L} = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_1} \times \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_2} \times \cdots \times \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_r}. \tag{3.2.1.1}$$

Let us now choose $b \in [b] \cap L(K_0)$ and $\mu \in \{\mu\}$ as in (i) of 3.1.1. After taking σ-conjugate in $L(K_0)$ if necessary, we may assume that $b \in L(W)\mu(p)L(W)$. Let $\underline{M} = (M, (t_i))$ be the corresponding F-crystal over k with G-structure (in the sense of 2.4.3). If $\{\mu\}$ is minuscule, we let $\underline{X} = (X, (t_i))$ denote the corresponding p-divisible group over k with G-structure.

Note that the tuple $(L, [b]_L, \{\mu\}_L)$ is an unramified local Shimura datum of Hodge type; indeed, with our choice of $b \in [b]_L$ and $\mu \in \{\mu\}_L$ one immediately verifies the conditions (i') and (ii') of 2.4.1.

Theorem 3.2.2. *Notations as above. In addition, we set the following notations:*

- $\widetilde{L}_j$ *denotes the j-th factor in* (3.2.1.1)*,*
- L_j *is the image of L under the projection $\widetilde{L} \twoheadrightarrow \widetilde{L}_j$,*
- b_j *is the image of b under the projection $L \twoheadrightarrow L_j$,*
- μ_j *is the cocharacter of L_j obtained by composing μ with the projection $L \twoheadrightarrow L_j$.*

Then $\underline{M}$ can be naturally regarded as an F-crystal with $L_1 \times L_2 \times \cdots \times L_r$-structure, and admits a decomposition

$$\underline{M} = \underline{M}_1 \times \underline{M}_2 \times \cdots \times \underline{M}_r, \tag{3.2.2.1}$$

where $\underline{M}_j$ is an F-crystal with L_j-structure that arises from an unramified local Shimura datum of Hodge type $(L_j, [b_j], \{\mu_j\})$.

When $\{\mu\}$ is minuscule, we also have a decomposition

$$\underline{X} = \underline{X}_1 \times \underline{X}_2 \times \cdots \times \underline{X}_r, \tag{3.2.2.2}$$

where $\underline{X}_j$ is a p-divisible group with L_j-structure corresponding to $\underline{M}_j$.

Proof. We need only prove the first part, as the second part follows immediately from the first part via Dieudonné theory.

We first note that $\underline{M}$ has a natural $L_1 \times L_2 \times \cdots \times L_r$-structure as follows: our choice of $b \in [b]_L$ and $\mu \in \{\mu\}_L$ gives rise to an L-structure on M, which can be regarded as an $L_1 \times L_2 \times \cdots \times L_r$-structure via the embedding $L \hookrightarrow L_1 \times L_2 \times \cdots \times L_r$.

Now considering b as an element of $[b]_{\widetilde{G}}$, we get an F-crystal over k with $\widetilde{G}$-structure $\widetilde{M}$ from an unramified local Shimura datum of Hodge type $(\widetilde{G}, [b], \{\mu\})$. As explained in Example 2.4.5, we can regard the $\widetilde{G}$-structure as an action of $\mathscr{O}$ which we refer to as $\mathscr{O}$-module structure. Since $(\widetilde{G}, [b], \{\mu\})$ is Hodge-Newton reducible, [**MV10**], Corollary 7 yields a decomposition

$$\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2 \times \cdots \times \widetilde{M}_r, \tag{3.2.2.3}$$

where $\widetilde{M}_j$ is an F-crystal over k with $\mathscr{O}$-module structure which arises from an unramified local Shimura datum of Hodge type $(\widetilde{L}_j, [b_j], \{\mu_j\})$. In fact, $\widetilde{M}_j$ corresponds to the choice $b_j \in [b_j]$ (and $\mu_j \in \{\mu_j\}$).

A priori, it is not clear that the tuple $(\widetilde{L}_j, [b_j], \{\mu_j\})$ is an unramified local Shimura datum of Hodge type. This is indeed implied in the statement and the proof of [**MV10**], Corollary 7.

We check that the tuple $(L_j, [b_j], \{\mu_j\})$ is an unramified local Shimura datum of Hodge type by verifying the conditions (i') and (ii') of 2.4.1. For (i'), we simply observe that $b_j \in L_j(W)\mu(p)L_j(W)$, which follows from our assumption that $b \in L(W)\mu(p)L(W)$ using the decomposition $\widetilde{L} = \widetilde{L}_1 \times \widetilde{L}_2 \times \cdots \times \widetilde{L}_r$. Then the condition (ii') immediately follows since we already know that M_j gives the desired W-lattice for b_j and μ_j.

Since $(L_j, [b_j], \{\mu_j\})$ is an unramified local Shimura datum of Hodge type, we can equip each M_j with an L_j-structure corresponding to the choice $b_j \in [b_j]$ (and $\mu_j \in \{\mu_j\}$). We thus get the desired decomposition (3.2.2.1) from the decomposition (3.2.2.3). □

Remark. We give an alternative proof of Theorem 3.2.2 using affine Deligne-Lusztig sets. After proving that the tuples $(L_j, [b_j], \{\mu_j\})$ are unramified local Shimura data of Hodge type, we find the following maps of affine Deligne-Lusztig sets:

$$X^G_{\{\mu\}}([b]) \xrightarrow{\sim} X^L_{\{\mu\}}([b]) \hookrightarrow X^{L_1}_{\{\mu_1\}}([b_1]) \times X^{L_2}_{\{\mu_2\}}([b_2]) \times \cdots \times X^{L_r}_{\{\mu_r\}}([b_r]).$$

Here the first isomorphism is given by [**MV10**], Theorem 6, whereas the second map is induced by the embedding $L \hookrightarrow L_1 \times L_2 \times \cdots \times L_r$ as in Lemma 2.4.8. Now the desired decomposition follows from the composition of these two maps via the moduli interpretation of affine Deligne-Lusztig sets given in Proposition 2.4.7.

3.2.3. We will refer to the decomposition (3.2.2.1) in Theorem 3.2.2 as the *Hodge-Newton decomposition* of $\underline{M}$ (associated to P and L). For $1 \leq a \leq b \leq r$, we define

$$M_{a,b} := \prod_{s=a}^{b} M_s.$$

Then we obtain a filtration

$$0 \subset M_{1,1} \subset M_{1,2} \subset \cdots \subset M_{1,r} = M \tag{3.2.3.1}$$

such that each quotient $M_{1,s}/M_{1,s-1} \simeq M_s$ carries an L_s-structure. We call this filtration the *Hodge-Newton filtration* of $\underline{M}$ (associated to P and L).

When $\{\mu\}$ is minuscule, we will refer to the decomposition (3.2.2.2) in Theorem 3.2.2 as the *Hodge-Newton decomposition* of $\underline{X}$ (associated to P and L). For $1 \leq a \leq b \leq r$, we define

$$X_{a,b} := \prod_{s=a}^{b} X_s.$$

Then via (contravariant) Dieudonné theory, the filtration (3.2.3.1) yields a filtration

$$0 \subset X_{r,r} \subset X_{r-1,r} \subset \cdots \subset X_{1,r} = X, \tag{3.2.3.2}$$

where each quotient $X_{s,r}/X_{s+1,r} \simeq X_s$ carries an L_s-structure. We call this filtration the *Hodge-Newton filtration* of $\underline{X}$ (associated to P and L).

Theorem 3.2.4. *Assume that $p > 2$ and $\{\mu\}$ is minuscule. Let R to be a ring of the form $R = W[[u_1, \cdots, u_N]]$ or $R = W[[u_1, \cdots, u_N]]/(p^m)$. Let $\underline{\mathscr{X}}$ be a deformation of $\underline{X}$ over R with an isomorphism $\alpha : \underline{\mathscr{X}} \otimes_R k \cong \underline{X}$. Then there exists a unique filtration of $\mathscr{X}$*

$$0 \subset \mathscr{X}_{r,r} \subset \mathscr{X}_{r-1,r} \subset \cdots \subset \mathscr{X}_{1,r} = \mathscr{X}$$

which lifts the Hodge-Newton filtration (3.2.3.2) *in the sense that α induces isomorphisms $\mathscr{X}_{s,r} \otimes_R k \cong X_{s,r}$ and $\underline{\mathscr{X}_{s,r}/\mathscr{X}_{s+1,r}} \otimes_R k \cong \underline{X}_s$ for $s = 1, 2, \cdots, r$.*

Note that we require each quotient $\underline{\mathscr{X}_{s,r}/\mathscr{X}_{s+1,r}}$ to carry tensors that lift those on $\underline{X}_s$.

Proof. We will only consider the case $r = 2$ as the argument easily extends to the general case.

Take unramified local Shimura data of Hodge type $(L_j, [b_j], \{\mu\}_j)$ and $(\widetilde{L}_j, [b_j], \{\mu_j\})$ as in Theorem 3.2.2. In addition, let $\widetilde{X}$ be the p-divisible group over k with $\mathscr{O}$-module structure that arises from the datum $(\widetilde{G}, [b], \{\mu\})$ with the choice $b \in [b]$, and let $\widetilde{X}_j$ be the p-divisible group over k with $\mathscr{O}$-module structure that arises from the datum $(\widetilde{L}_j, [b_j], \{\mu_j\})$ with the choice $b_j \in [b_j]$. Then the filtration

$$0 \subseteq \widetilde{X}_2 \subseteq \widetilde{X}$$

is the Hodge-Newton filtration of $\widetilde{X}$.

By the functorial properties of deformation spaces in Lemma 2.5.6, the closed embedding $G \hookrightarrow \widetilde{G}$ induces a closed embedding

$$\mathrm{Def}_{X,G} \hookrightarrow \mathrm{Def}_{X,\widetilde{G}}.$$

Thus $\underline{\mathscr{X}}$ yields a deformation $\widetilde{\mathscr{X}}$ of $\widetilde{X}$ over R. Then by [**Sh13**], Theorem 5.4, $\widetilde{\mathscr{X}}$ admits a (unique) filtration

$$0 \subseteq \mathscr{X}_2 \subseteq \mathscr{X}$$

such that α induces isomorphisms $\alpha_1 : \widetilde{\mathscr{X}/\mathscr{X}_2} \otimes_R k \cong \widetilde{X}_1$ and $\alpha_2 : \widetilde{\mathscr{X}_2} \otimes_R k \cong \widetilde{X}_2$.

It remains to show that $\mathscr{X}/\mathscr{X}_2$ and $\mathscr{X}_2$ are equipped with tensors which lift the tensors of $\underline{X}_1$ and $\underline{X}_2$ respectively in the sense of Proposition 2.5.5. Note that we have isomorphisms of Dieudonné modules

$$\beta : \mathbb{D}(\mathscr{X} \otimes_R k) \cong \mathbb{D}(X), \quad \beta_1 : \mathbb{D}((\mathscr{X}/\mathscr{X}_2) \otimes_R k) \cong \mathbb{D}(X_1), \quad \beta_2 : \mathbb{D}(\mathscr{X}_2 \otimes_R k) \cong \mathbb{D}(X_2)$$

corresponding to the isomorphisms α, α_1 and α_2. We may regard β as an element of $G(W)$ by identifying both modules with $\Lambda^* \otimes_{\mathbb{Z}_p} W$. Similarly, we may regard each β_j as an element of $\widetilde{L}_j(W)$. Then β_j should be in the image of $\widetilde{L}(W) \cap G(W) = L(W)$ under the projection $\widetilde{L} \twoheadrightarrow \widetilde{L}_2$ since it is induced by β. Hence we have $\beta_j \in L_j(W)$ for each $j = 1, 2$. This implies that $\mathscr{X}/\mathscr{X}_2$ and $\mathscr{X}_2$ respectively lift the tensors of $\underline{X}_1$ and $\underline{X}_2$ via α_1 and α_2, completing the proof. $\square$

Chapter 4

HARRIS-VIEHMANN CONJECTURE FOR HODGE-NEWTON REDUCIBLE RAPOPORT-ZINK SPACES

4.1 Harris-Viehmann conjecture: statement

4.1.1. Throughout this chapter, we fix a prime $p > 2$ and set $k = \overline{\mathbb{F}}_p$ so that $W = \breve{\mathbb{Z}}_p$ and $K_0 = \breve{\mathbb{Q}}_p$. We also fix an unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$ such that $\{\mu\}$ is minuscule. We choose a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ in the condition (ii) of 2.4.1 and a finite family of tensors (s_i) on Λ as in 2.4.3. In addition, we fix a maximal torus $T \subseteq G$ and a Borel subgroup $B \subseteq G$ containing T, both defined over $\mathbb{Z}_p$.

Let P be a proper standard parabolic subgroup of G with Levi factor L and unipotent radical U. For any element $b \in [b] \cap L(\breve{\mathbb{Q}}_p)$, we define $I_{b,\{\mu\},L}$ to be the set of $L(\breve{\mathbb{Z}}_p)$-conjugacy classes of cocharacters of L with a representative μ' such that

(i) $\mu' \in \{\mu\}_G$,

(ii) $[b]_L \cap L(\breve{\mathbb{Z}}_p)\mu'(p)L(\breve{\mathbb{Z}}_p)$ is not empty.

Then $I_{b,\{\mu\},L}$ is finite and nonempty (see [**RV14**], Lemma 8.1.).

Lemma 4.1.2. *For any $\{\mu'\}_L \in I_{b,\{\mu\},L}$, the tuple $(L, [b]_L, \{\mu'\}_L)$ is an unramified local Shimura datum of Hodge type.*

Proof. By construction, the tuple $(L, [b]_L, \{\mu'\}_L)$ satisfies the conditions (i') of 2.4.1. After taking σ-conjugate in $L(\breve{\mathbb{Q}}_p)$ if necessary, we may assume that $b \in L(\breve{\mathbb{Z}}_p)\mu'(p)L(\breve{\mathbb{Z}}_p)$. Then we have $b \in G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)$ since $\mu' \in \{\mu\}_G$. Now we verify the condition (ii') with b since $(G, [b], \{\mu\})$ is an unramified local Shimura datum of Hodge type. □

We can now state the Harris-Viehmann conjecture in the setting of Rapoport-Zink spaces of Hodge type.

Conjecture 4.1.3 ([**RV14**], Conjecture 8.4.)**.** *Let P be a parabolic subgroup of G with Levi factor L. Assume that $b \in [b] \cap G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)$ satisfies the following properties:*

(i) $[b] \cap L(\breve{\mathbb{Q}}_p)$ *is not empty,*

(ii) J_b *is an inner form of a Levi subgroup of* G *contained in* L*.*

Choose representatives $\mu_1, \mu_2, \cdots, \mu_s$ *of the* $L(\breve{\mathbb{Z}}_p)$*-conjugacy classes of cocharacters in* $I_{b,\{\mu\},L}$*, and also choose* $b_k \in [b]_L \cap L(\breve{\mathbb{Z}}_p)\mu_k(p)L(\breve{\mathbb{Z}}_p)$ *for each* $k = 1, 2, \cdots, s$*. Then for any admissible* $\overline{\mathbb{Q}}_l$*-representation* ρ *of* $J_b(\mathbb{Q}_p)$*, we have an equality of virtual representations of* $G(\mathbb{Q}_p) \times \mathcal{W}_E$

$$H^\bullet(RZ^\infty_{G,b})_\rho = \bigoplus_{k=1}^{s} Ind_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H^\bullet(RZ^\infty_{L,b_k})_\rho.$$

In particular, the virtual representation $H^\bullet(RZ^\infty_{G,b})_\rho$ *contains no supercuspidal representations of* $G(\mathbb{Q}_p)$*.*

Here we consider the groups $H^\bullet(\mathrm{RZ}^\infty_{L,b_k})_\rho$ as a virtual representation of $P(\mathbb{Q}_p) \times \mathcal{W}_E$ by letting the unipotent radical of $P(\mathbb{Q}_p)$ act trivially. Note that the choice of b_k's (or μ_k's) is unimportant since the isomorphism class of the spaces $\mathrm{RZ}^\infty_{L,b_k}$ only depend on the tuples $(L, [b]_L, \{\mu_k\}_L)$.

4.1.4. For the rest of this chapter, we assume that the datum $(G, [b], \{\mu\})$ is Hodge-Newton reducible with respect to P and L. We also choose $\mu \in \{\mu\}$ and $b \in L(\breve{\mathbb{Z}}_p)\mu(p)L(\breve{\mathbb{Z}}_p)$ as in 3.2.1, and denote by $\underline{X} = (X, (t_i))$ the corresponding p-divisible group over $\overline{\mathbb{F}}_p$ with G-structure.

Let us interpret the statement of Conjecture 4.1.3 under our assumption. Note that b and L clearly satisfy the condition (i) of Conjecture 4.1.3. One can also check that b and L satisfy the condition (ii) of Conjecture 4.1.3 (see [**RV14**], Remark 8.9.). Moreover, under our assumption the set $I_{b,\{\mu\},L}$ consists of a single element, namely $\{\mu\}_L$ (see [**RV14**], Theorem 8.8.).

We now state the main result for this chapter, which proves Conjecture 4.1.3 under our assumption.

Theorem 4.1.5. *Assume that* $(G, [b], \{\mu\})$ *is Hodge-Newton reducible with respect to a standard parabolic subgroup* P *with Levi factor* L*. Choose* $\mu \in \{\mu\}$ *and* $b \in L(\breve{\mathbb{Z}}_p)\mu(p)L(\breve{\mathbb{Z}}_p)$ *as in 3.2.1. Then for any admissible* $\overline{\mathbb{Q}}_l$*-representation* ρ *of* $J_b(\mathbb{Q}_p)$*, we have an equality of virtual representations of* $G(\mathbb{Q}_p) \times \mathcal{W}_E$

$$H^\bullet(RZ^\infty_{G,b})_\rho = Ind_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H^\bullet(RZ^\infty_{L,b})_\rho.$$

In particular, the virtual representation $H^\bullet(RZ^\infty_{G,b})_\rho$ contains no supercuspidal representations of $G(\mathbb{Q}_p)$.

4.2 Rigid analytic tower associated to the parabolic subgroup

For our proof of Theorem 4.1.5, we construct an intermediate tower of rigid analytic spaces associated to the parabolic subgroup P.

4.2.1. For the rest of this chapter, we fix an EL realization $(\widetilde{G}, [b], \{\mu\})$ of our datum $(G, [b], \{\mu\})$ and take $\widetilde{P}$ and $\widetilde{L}$ as in Lemma 3.1.4. We continue to assume that $\widetilde{G}$ is of the form

$$\widetilde{G} = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n,$$

where $\mathscr{O}$ is the ring of integers of some finite unramified extension of $\mathbb{Q}_p$. We also take $\widetilde{L}_j, L_j, b_j, \mu_j$ for $j = 1, 2, \cdots, r$ as in Theorem 3.2.2. Then Theorem 3.2.2 gives a Hodge-Newton decomposition of $\underline{X}$

$$\underline{X} = \underline{X}_1 \times \underline{X}_2 \times \cdots \times \underline{X}_r \tag{4.2.1.1}$$

and the induced Hodge-Newton filtration of $\underline{X}$

$$0 \subset X^{(r)} \subset X^{(r-1)} \subset \cdots \subset X^{(1)} = X, \tag{4.2.1.2}$$

where each quotient $X^{(j)}/X^{(j+1)} \simeq X_j$ carries L_j-structure that arises from the datum $(L_j, [b_j], \{\mu_j\})$ with the choice $b_j \in [b_j]$.

4.2.2. Following Mantovan in [**Man08**], Definition 9, we define a set-valued functor $\mathrm{RZ}_{\widetilde{P},b}$ on $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ as follows: for any $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$, we set $\mathrm{RZ}_{\widetilde{P},b}(R)$ to be the set of isomorphism classes of triples $(\mathcal{X}, \mathcal{X}^\bullet, \iota)$ where

- $\mathcal{X}$ is a p-divisible group over R with an action of $\mathscr{O}$ (see Example 2.4.5);
- $\mathcal{X}^\bullet$ is a filtration of p-divisible groups over R

$$0 \subset \mathcal{X}^{(r)} \subset \mathcal{X}^{(r-1)} \subset \cdots \subset \mathcal{X}^{(1)} = \mathcal{X},$$

  which is preserved by the action of $\mathscr{O}$ such that the quotients $\mathcal{X}^{(j)}/\mathcal{X}^{(j+1)}$ are p-divisible groups (with the induced action of $\mathscr{O}$);

- $\iota : X_{R/p} \to \mathcal{X}_{R/p}$ is a quasi-isogeny which is compatible with the action of $\mathcal{O}$ and induces quasi-isogenies $\iota^{(j)} : X^{(j)}_{R/p} \longrightarrow \mathcal{X}^{(j)}_{R/p}$ for $j = 1, 2, \cdots, r$,

such that for all $a \in \mathcal{O}$ and $j = 1, 2, \cdots, r$,

$$\det{}_R(a, \mathrm{Lie}(\mathcal{X}^{(j)})) = \det(a, \mathrm{Fil}^0(X^{(j)})_{\breve{\mathbb{Q}}_p}).$$

Mantovan in [**Man08**], Proposition 11 proved that the functor $\mathrm{RZ}_{\widetilde{P},b}$ is represented by a formal scheme which is formally smooth and locally formally of finite type over $\breve{\mathbb{Z}}_p$. We write $\mathrm{RZ}_{\widetilde{P},b}$ also for this representing formal scheme, and $\mathrm{RZ}^{\mathrm{rig}}_{\widetilde{P},b}$ for its rigid analytic generic fiber. In addition, we write $\mathcal{X}_{\widetilde{P},b}$ and $\mathcal{X}^{\bullet}_{\widetilde{P},b}$ respectively for the universal filtered p-divisible group over $\mathrm{RZ}_{\widetilde{P},b}$ and the associated "universal filtration".

Remark. As in [**Man08**], Definition 10, we can also define a tower of étale covers $\mathrm{RZ}^{\infty}_{\widetilde{P},b} = \{\mathrm{RZ}^{\widetilde{K_p}'}_{X,\widetilde{P}}\}$ over $\mathrm{RZ}^{\mathrm{rig}}_{\widetilde{P},b}$ with a natural action of $\widetilde{P}(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ and a Weil descent datum over E, where $\widetilde{K_p}'$ runs over open and compact subgroups of $\widetilde{P}(\mathbb{Z}_p)$.

4.2.3. By the functoriality of Rapoport-Zink spaces described in Proposition 2.6.6, the embedding $G \hookrightarrow \widetilde{G}$ induces a closed embedding

$$\mathrm{RZ}_{G,b} \hookrightarrow \mathrm{RZ}_{\widetilde{G},b}.$$

In addition, we have a natural map

$$\widetilde{\pi}_2 : \mathrm{RZ}_{\widetilde{P},b} \longrightarrow \mathrm{RZ}_{\widetilde{G},b}$$

defined by $(\mathcal{X}, \mathcal{X}^{\bullet}, \iota) \mapsto (\mathcal{X}, \iota)$ on the points. We define $\mathrm{RZ}_{P,b} := \mathrm{RZ}_{\widetilde{P},b} \times_{\mathrm{RZ}_{\widetilde{G},b}} \mathrm{RZ}_{G,b}$. Then we have the following Cartesian diagram:

$$\begin{tikzcd}
\mathrm{RZ}_{P,b} \arrow[r, "\pi_2"] \arrow[d, hook] & \mathrm{RZ}_{G,b} \arrow[d, hook] \\
\mathrm{RZ}_{\widetilde{P},b} \arrow[r, "\widetilde{\pi}_2"] & \mathrm{RZ}_{\widetilde{G},b}
\end{tikzcd}$$

Moreover, π_2 is a local isomorphism which gives an isomorphism on the rigid analytic generic fiber since $\widetilde{\pi}_2$ has the same property (see [**Man08**], Theorem 36 and [**Sh13**], Proposition 6.3.).

We want to describe the universal property of the closed embedding $\mathrm{RZ}_{P,b} \hookrightarrow \mathrm{RZ}_{\widetilde{P},b}$ in an analogous way to the universal property of $\mathrm{RZ}_{G,b} \subset \mathrm{RZ}_b$ described in 2.6.2. For this, we choose a decomposition of Λ

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_r$$

corresponding to the decomposition of $\widetilde{L}$ in (3.2.1.1). We set $\Lambda^{(j)} = \Lambda_1 \oplus \cdots \oplus \Lambda_j$ for $j = 1, 2, \cdots, r$, and denote by $\Lambda^\bullet$ the filtration

$$0 \subset \Lambda^{(1)} \subset \cdots \subset \Lambda^{(r)} = \Lambda.$$

Then for any $\mathbb{Z}_p$-algebra R we have

$$P(R) = \{g \in G(R) : g(\Lambda^\bullet_R) = \Lambda^\bullet_R\}.$$

Now consider a morphism $f : \mathrm{Spf}(R) \to \mathrm{RZ}_{\widetilde{P},b}$ for some $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$. Let $(\mathcal{X}, \mathcal{X}^\bullet)$ be a p-divisible group over $\mathrm{Spec}(R)$ with a filtration which pulls back to $(f^*\mathcal{X}_{\widetilde{P},b}, f^*\mathcal{X}^\bullet_{\widetilde{P},b})$ over $\mathrm{Spf}(R)$. We denote by $\mathbb{D}(\mathcal{X}^\bullet)$ the filtration of Dieudonné modules

$$0 = \mathbb{D}(\mathcal{X}/\mathcal{X}^{(1)}) \subset \mathbb{D}(\mathcal{X}/\mathcal{X}^{(2)}) \subset \cdots \subset \mathbb{D}(\mathcal{X}/\mathcal{X}^{(r)}) \subset \mathbb{D}(\mathcal{X})$$

induced by $\mathcal{X}^\bullet$ via (contravariant) Dieudonne theory. We choose tensors $(\hat{t}_i)$ on $\mathbb{D}(\mathcal{X})[1/p]$ as in 2.6.2. Then f factors through $\mathrm{RZ}_{P,b}$ if and only if $\widetilde{\pi}_2 \circ f$ factors through $\mathrm{RZ}_{G,b} \hookrightarrow \mathrm{RZ}_{\widetilde{G},b}$, which is equivalent to existence of a (unique) family of tensors $(\mathbf{t}_i)$ on $\mathbb{D}(\mathcal{X})$ such that

(i) for some ideal of definition J of R containing p, the pull-back of $(\mathbf{t}_i)$ over R/J agrees with the pull-back of $(\hat{t}_i)$ over R/J,

(ii) for a p-adic lift $\mathcal{R}$ of R which is formally smooth over $\breve{\mathbb{Z}}_p$, the $\mathcal{R}$-scheme

$$\mathcal{P}_{\mathcal{R}} := \mathbf{Isom}_{\mathcal{R}}\Big([\mathbb{D}(\mathcal{X})_{\mathcal{R}}, (\mathbf{t}_i)_{\mathcal{R}}], [\Lambda^* \otimes_{\mathbb{Z}_p} \mathcal{R}, (s_i \otimes 1)]\Big)$$

defined in 2.6.2 is a G-torsor, and consequently the $\mathcal{R}$-scheme

$$\mathcal{P}'_{\mathcal{R}} := \mathbf{Isom}_{\mathcal{R}}\Big([\mathbb{D}(\mathcal{X}^\bullet)_{\mathcal{R}}, (\mathbf{t}_i)_{\mathcal{R}}], [(\Lambda^\bullet)^* \otimes_{\mathbb{Z}_p} \mathcal{R}, (s_i \otimes 1)]\Big)$$

is a P-torsor,

(iii) the Hodge filtration of $\mathcal{X}$ is a $\{\mu\}$-filtration with respect to $(\mathbf{t}_i)$.

Here the scheme $\mathcal{P}'_{\mathcal{R}}$ in (ii) classifies the isomorphisms $\mathbb{D}(\mathcal{X})_{\mathcal{R}} \cong \Lambda^*_{\mathcal{R}}$ which map the tensors $(\mathbf{t}_i)$ to $(s_i \otimes 1)$ and the filtration $\mathbb{D}(\mathcal{X}^\bullet)_{\mathcal{R}}$ to $(\Lambda^\bullet)^* \otimes_{\mathbb{Z}_p} \mathcal{R}$.

We obtain the "universal p-divisible group" $\mathcal{X}_{P,b}$ over $\mathrm{RZ}_{P,b}$ with the associated "universal filtration" $\mathcal{X}^\bullet_{P,b}$ by taking the pull-back of $\mathcal{X}_{\widetilde{P},b}$ and $\mathcal{X}^\bullet_{\widetilde{P},b}$ over $\mathrm{RZ}_{P,b}$. We also obtain a family of "universal tensors" $(\mathbf{t}_i^{\mathrm{univ},P})$ on $\mathbb{D}(\mathcal{X}_{P,b})$ by applying the universal property to an open affine covering of $\mathrm{RZ}_{P,b}$. Moreover, this family has a "étale realization" $(\mathbf{t}_{i,\text{ét}}^{\mathrm{univ},P})$ on the Tate module $T_p(\mathcal{X}_{P,b})$ (see [**Kim13**], Theorem 7.1.6.).

4.2.4. The formal scheme $\mathrm{RZ}_{P,b}$ is formally smooth and locally formally of finite type over $\breve{\mathbb{Z}}_p$ by construction. Hence it admits a rigid analytic generic fiber, which we denote by $\mathrm{RZ}^{\mathrm{rig}}_{P,b}$. Moreover, since π_2 gives an isomorphism on the rigid analytic generic fiber, we have a $J_b(\mathbb{Q}_p)$-action and a Weil descent datum over E on $\mathrm{RZ}^{\mathrm{rig}}_{P,b}$ induced by the corresponding structures on $\mathrm{RZ}^{\mathrm{rig}}_{G,b}$.

For any open compact subgroup $K_p{}'$ of $P(\mathbb{Z}_p)$, we define the following rigid analytic étale cover of $\mathrm{RZ}^{\mathrm{rig}}_{P,b}$:

$$\mathrm{RZ}^{K_p{}'}_{P,b} := \mathbf{Isom}_{\mathrm{RZ}^{\mathrm{rig}}_{P,b}}\Big([\Lambda^\bullet, (s_i)], [T_p(\mathcal{X}^\bullet_{P,b}), (\mathbf{t}_{i,\text{ét}}^{\mathrm{univ},P})]\Big)\Big/ K_p{}'.$$

The $J_b(\mathbb{Q}_p)$-action and the Weil descent datum over E on $\mathrm{RZ}^{\mathrm{rig}}_{P,b}$ pull back to $\mathrm{RZ}^{K_p{}'}_{P,b}$. We denote by $\mathrm{RZ}^{\infty}_{P,b} := \{\mathrm{RZ}^{K_p{}'}_{P,b}\}$ the tower of these covers with Galois group $P(\mathbb{Z}_p)$. The Galois action on this tower gives rise to a natural $P(\mathbb{Q}_p)$-action which commutes with the $J_b(\mathbb{Q}_p)$-action and the Weil descent datum over E (cf. [**Kim13**], Proposition 7.4.8.). Hence the cohomology groups

$$H^i(\mathrm{RZ}^{K_p{}'}_{P,b}) = H^i_c(\mathrm{RZ}^{K_p}_{P,b} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \mathbb{Q}_l(\dim \mathrm{RZ}^{K_p{}'}_{P,b}))$$

form a tower $\{H^i(\mathrm{RZ}^{K_p{}'}_{P,b})\}$ for each i, which are endowed with a natural action of $P(\mathbb{Q}_p) \times \mathcal{W}_E \times J_b(\mathbb{Q}_p)$. Moreover, for any admissible l-adic representation ρ of $J_b(\mathbb{Q}_p)$, the groups

$$H^{i,j}(\mathrm{RZ}^{\infty}_{P,b})_\rho := \varinjlim_{K_p{}'} \mathrm{Ext}^j_{J_b(\mathbb{Q}_p)}(H^i(\mathrm{RZ}^{K_p{}'}_{P,b}), \rho)$$

satisfy the following properties (cf. 2.6.10):

(1) The groups $H^{i,j}(\mathrm{RZ}^{\infty}_{P,b})_\rho$ vanish for almost all i, j.

(2) There is a natural action of $P(\mathbb{Q}_p) \times \mathcal{W}_E$ on each $H^{i,j}(\mathrm{RZ}^{\infty}_{P,b})_\rho$.

(3) The representations $H^{i,j}(\mathrm{RZ}^{\infty}_{P,b})_\rho$ are admissible.

We can thus define a virtual representation of $P(\mathbb{Q}_p) \times \mathcal{W}_E$

$$H^{\bullet}(\mathrm{RZ}^{\infty}_{P,b})_\rho := \sum_{i,j\geq 0} (-1)^{i+j} H^{i,j}(\mathrm{RZ}^{\infty}_{P,b})_\rho.$$

Remark. Alternatively, we can obtain the tower $\mathrm{RZ}^{\infty}_{P,b}$ as the pull-back of the tower $\mathrm{RZ}^{\infty}_{\widetilde{P},b}$ over $\mathrm{RZ}^{\mathrm{rig}}_{P,b}$.

4.3 Harris-Viehmann conjecture: proof

Let us now present our proof of Theorem 4.1.5. We retain all the notations from 4.2.

Lemma 4.3.1. *There exists a diagram*

$$\begin{tikzcd} & RZ^{rig}_{P,b} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ RZ^{rig}_{L,b} \arrow[ur, bend left, "s"] & & RZ^{rig}_{G,b} \end{tikzcd}$$

such that

(i) s is a closed immersion,

(ii) π_1 is a fibration in balls,

(iii) π_2 is an isomorphism.

Proof. For notational simplicity, we assume that $r = 2$, i.e., the decomposition of $\widetilde{L}$ in (3.2.1.1) has two factors. Our argument will naturally extend to the general case.

Note that we have already constructed π_2 and proved (iii) in 4.2.3.

Let us now construct s and prove (i). From the decomposition $\widetilde{L} = \widetilde{L}_1 \times \widetilde{L}_2$ we obtain a natural isomorphism $\mathrm{RZ}_{\widetilde{L},b} \simeq \mathrm{RZ}_{\widetilde{L}_1,b_1} \times \mathrm{RZ}_{\widetilde{L}_2,b_2}$ by Proposition 2.6.6. Consider the map

$$\widetilde{s} : \mathrm{RZ}_{\widetilde{L},b} \simeq \mathrm{RZ}_{\widetilde{L}_1,b_1} \times \mathrm{RZ}_{\widetilde{L}_2,b_2} \longrightarrow \mathrm{RZ}_{\widetilde{P},b},$$

where the second arrow is defined by $(\mathcal{X}_1, \iota_1, \mathcal{X}_2, \iota_2) \mapsto (\mathcal{X}_1 \times \mathcal{X}_2, 0 \subset \mathcal{X}_2 \subset \mathcal{X}_1 \times \mathcal{X}_2, \iota_1 \times \iota_2)$ on the points. Then $\widetilde{s}$ gives a closed immersion on the rigid analytic generic fibers by [**Man08**], Proposition 14. We define s to be the restriction of $\widetilde{s}$ on $\mathrm{RZ}_{L,b}$. Since s also gives a closed immersion on the rigid analytic generic fibers by construction, it suffices to show that s factors through the embedding $\mathrm{RZ}_{P,b} \hookrightarrow \mathrm{RZ}_{\widetilde{P},b}$, which amounts to proving that $\widetilde{\pi}_2 \circ s$ factors through $\mathrm{RZ}_{G,b}$. In fact, $\widetilde{\pi}_2 \circ s$ is the natural closed embedding $\mathrm{RZ}_{L,b} \hookrightarrow \mathrm{RZ}_{\widetilde{G},b}$ which is functorially induced by the embedding $L \hookrightarrow \widetilde{G}$ in the sense of Proposition 2.6.6. Hence $\widetilde{\pi}_2 \circ s$ factors through $\mathrm{RZ}_{G,b}$ as the embedding $L \hookrightarrow \widetilde{G}$ factors through G.

It remains to construct π_1 and prove (ii). Note that we have a natural embedding

$$\mathrm{RZ}_{L,b} \hookrightarrow \mathrm{RZ}_{\widetilde{L},b}$$

which is functorially induced by the embedding $L \hookrightarrow \widetilde{L}$ in the sense of Proposition 2.6.6. Consider the map

$$\widetilde{\pi}_1 : \mathrm{RZ}_{\widetilde{P},b} \longrightarrow \mathrm{RZ}_{\widetilde{L}_1,b_1} \times \mathrm{RZ}_{\widetilde{L}_2,b_2} \xrightarrow{\sim} \mathrm{RZ}_{\widetilde{L},b}$$

defined by $(\mathcal{X}, \mathcal{X}^\bullet, \iota) \mapsto (\mathcal{X}/\mathcal{X}^{(2)}, \iota/\iota^{(2)}, \mathcal{X}^{(2)}, \iota^{(2)}) \mapsto ((\mathcal{X}/\mathcal{X}^{(2)}) \times \mathcal{X}^{(2)}, (\iota/\iota^{(2)}) \times \iota^{(2)})$ on the points, where $\iota/\iota^{(2)} : (X_1)_{R/p} = (X/X^{(2)})_{R/p} \longrightarrow (\mathcal{X}/\mathcal{X}^{(2)})_{R/p}$ denotes the quasi-isogeny induced by ι and $\iota^{(2)}$. We define π_1 be the restriction of $\widetilde{\pi}_1$ on $\mathrm{RZ}_{P,b}$.

We claim that π_1 factor through the embedding $\mathrm{RZ}_{L,b} \hookrightarrow \mathrm{RZ}_{\widetilde{L},b}$. It suffices to show that (locally) the map $\pi_2^{-1} \circ \pi_1$ factors through $\mathrm{RZ}_{L,b} \hookrightarrow \mathrm{RZ}_{\widetilde{L},b}$. Moreover, we only need to check this on the set of $\overline{\mathbb{F}}_p$-valued points and the completions thereof.

Recall that we have natural identifications

$$X^G_{\{\mu\}}([b]) \xrightarrow{\sim} \mathrm{RZ}_{G,b}(\overline{\mathbb{F}}_p), \quad X^L_{\{\mu\}}([b]) \xrightarrow{\sim} \mathrm{RZ}_{L,b}(\overline{\mathbb{F}}_p)$$

as explained in 2.6.5. Under these identifications, $\pi_2^{-1} \circ \pi_1$ on the $\overline{\mathbb{F}}_p$-valued points coincides with the map

$$X^G_{\{\mu\}}([b]) \xrightarrow{\sim} X^L_{\{\mu\}}([b]) \hookrightarrow X^{L_1}_{\{\mu_1\}}([b_1]) \times X^{L_2}_{\{\mu_2\}}([b_2])$$

induced by the Hodge-Newton decompositon of $\underline{X}$ as explained in the remark following Theorem 3.2.2. Hence we see that $\pi_2^{-1} \circ \pi_1$ factors through $\mathrm{RZ}_{L,b}$ on the set of $\overline{\mathbb{F}}_p$-valued points.

We now take an $\overline{\mathbb{F}}_p$-valued point $x \in \mathrm{RZ}_{G,b}(\overline{\mathbb{F}}_p)$ and consider the map induced by $\pi_2^{-1} \circ \pi_1$ on the formal completion $\widehat{(\mathrm{RZ}_{G,b})_x}$ at x. We denote by $(X_x, (t_{x,i}), \iota_x)$ the tuple corresponding to x under the description of $\mathrm{RZ}_{G,b}(\overline{\mathbb{F}}_p)$ in 2.6.5. Then $\underline{X}_x := \big(X_x, (t_{x,i})\big)$ admits a Hodge-Newton decomposition

$$\underline{X}_x = \underline{X}_{x,1} \times \underline{X}_{x,2}$$

and the induced Hodge-Newton filtration

$$0 \subset X_x^{(2)} \subset X_x^{(1)} = X_x.$$

Take R to be a ring of the form $R = \breve{\mathbb{Z}}_p[[u_1, \cdots, u_N]]$ or $R = \breve{\mathbb{Z}}_p[[u_1, \cdots, u_N]]/(p^m)$. Then for any $\underline{\mathscr{X}} \in \mathrm{Def}_{X_x,G}(R)$ with an isomorphism $\alpha : \underline{\mathscr{X}} \otimes_R k \cong \underline{X}_x$, Theorem 3.2.4 gives a filtration

$$0 \subset \mathscr{X}^{(2)} \subset \mathscr{X}^{(1)} = \mathscr{X}$$

with isomorphisms $\alpha_1 : \underline{\mathscr{X}_x/\mathscr{X}_x^{(2)}} \otimes_R \overline{\mathbb{F}}_p \cong \underline{X}_{x,1}$ and $\alpha^{(2)} : \mathscr{X}_x^{(2)} \otimes_R \overline{\mathbb{F}}_p \cong X_x^{(2)}$.

Under the identifications

$$\mathrm{Def}_{X_x,G} \simeq \widehat{(\mathrm{RZ}_{G,b})_x}, \quad \mathrm{Def}_{X_x,L} \simeq \widehat{(\mathrm{RZ}_{L,b})_x}$$

described in 2.6.5, the map $\pi_2^{-1} \circ \pi_1$ gives rise to a map

$$\mathrm{Def}_{X_x,G} \longrightarrow \mathrm{Def}_{X_{x_1},\widetilde{L}_1} \times \mathrm{Def}_{X_{x_2},\widetilde{L}_2} \simeq \mathrm{Def}_{X_{x_1}\times X_{x_2},\widetilde{L}_1\times\widetilde{L}_2} = \mathrm{Def}_{X_x,\widetilde{L}}$$

induced by the association $\mathscr{X}_x \mapsto (\mathscr{X}_x/\mathscr{X}_x^{(2)}) \times \mathscr{X}_x^{(2)}$. Note that $(\mathscr{X}_x/\mathscr{X}_x^{(2)}) \times \mathscr{X}_x^{(2)}$ is a deformation of X_x via the isomorphism $\alpha_1 \times \alpha^{(2)}$. Since this isomorphism is induced by α, we see that $(\mathscr{X}_x/\mathscr{X}_x^{(2)}) \times \mathscr{X}_x^{(2)}$ lifts the tensors that define G-structure on X_x. Hence the image of this map must lie in $\mathrm{Def}_{X_x,\widetilde{L}} \cap \mathrm{Def}_{X_x,G} = \mathrm{Def}_{X_x,L}$.

Finally, we easily see that π_1 is a fibration in balls. In fact, for any point $x \in \mathrm{RZ}_{L,b}(\overline{\mathbb{F}}_p)$ the completion of $\mathrm{RZ}_{P,b}$ at $s(x)$ is isomorphic to $\mathrm{Def}_{X_x,G}$, which is isomorphic to a formal spectrum of a power series ring over $\breve{\mathbb{Z}}_p$ by Proposition 2.5.5. $\square$

Proposition 4.3.2. *For any admissible l-adic representation ρ of $J_b(\mathbb{Q}_p)$, we have*

$$H^\bullet(RZ^\infty_{L,b})_\rho = H^\bullet(RZ^\infty_{P,b})_\rho$$

as virtual representations of $P(\mathbb{Q}_P) \times \mathcal{W}_E$.

Proof. For any open compact subgroups $K_p{}' \subseteq P(\mathbb{Z}_p)$, we get morphisms of rigid analytic spaces

$$s_{K_p{}'} : \mathrm{RZ}_{L,b}^{K_p{}' \cap L(\mathbb{Q}_p)} \longrightarrow \mathrm{RZ}_{P,b}^{K_p{}'} \quad \text{and} \quad \pi_{1,K_p{}'} : \mathrm{RZ}_{P,b}^{K_p{}'} \longrightarrow \mathrm{RZ}_{L,b}^{K_p{}' \cap L(\mathbb{Q}_p)}$$

which are $P(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-equivariant and compatible with the Weil descent datum. Moreover, $s_{K_p{}'}$'s are closed immersions and satisfy $\pi_{1,K_p{}'} \circ s_{K_p{}'} = \mathrm{id}_{\mathrm{RZ}_{L,b}^{K_p{}' \cap L(\mathbb{Q}_p)}}$.

Recall that we have a universal p-divisible group $\mathcal{X}_{\widetilde{P},b}$ over $\mathrm{RZ}_{\widetilde{P},b}$ with the associated filtration $\mathcal{X}^{\bullet}_{\widetilde{P},b}$. By [**Man08**], Proposition 30, we have a formal scheme $\mathrm{RZ}^{(m)}_{\widetilde{P},b} \longrightarrow \mathrm{RZ}_{\widetilde{P},b}$ for each integer $m > 0$ with the following properties:

(i) a morphism $f : \mathrm{Spf}(R) \longrightarrow \mathrm{RZ}_{\widetilde{P},b}$ for some $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ factors through $\mathrm{RZ}^{(m)}_{\widetilde{P},b}$ if and only if the filtration $f^*\mathcal{X}^{\bullet}_{\widetilde{P},b}[p^m]$ is split,

(ii) the formal schemes $\mathrm{RZ}^{(m)}_{\widetilde{P},b}$ and $\mathrm{RZ}_{\widetilde{P},b}$ become isomorphic when considered as formal schemes over $\mathrm{RZ}_{\widetilde{L},b}$ via the map $\widetilde{\pi}_1 : \mathrm{RZ}_{\widetilde{P},b} \longrightarrow \mathrm{RZ}_{\widetilde{L},b}$.

Taking the pull back of $\mathrm{RZ}^{(m)}_{\widetilde{P},b}$ over $\mathrm{RZ}_{P,b}$, we obtain a formal scheme $\mathrm{RZ}^{(m)}_{P,b} \longrightarrow \mathrm{RZ}_{P,b}$ for each integer $m > 0$ with analogous properties. We write $\mathrm{RZ}^{(m),\mathrm{rig}}_{P,b}$ for the rigid analytic generic fiber of $\mathrm{RZ}^{(m)}_{P,b}$.

For each integer $m > 0$, we set $K_p{}'^{(m)} := \ker\big(P(\mathbb{Z}_p) \twoheadrightarrow P(\mathbb{Z}_p/p^m\mathbb{Z}_p)\big)$ and define two distinct covers $\mathcal{P}_m \longrightarrow \mathrm{RZ}^{(m)}_{P,b}$ and $\mathcal{P}'_m \longrightarrow \mathrm{RZ}^{(m)}_{P,b}$ by the following Cartesian diagrams:

$$\begin{tikzcd}
\mathcal{P}_m \arrow[r] \arrow[d] & \mathrm{RZ}^{(m),\mathrm{rig}}_{P,b} \arrow[d] \\
\mathrm{RZ}^{K_p{}'^{(m)}}_{P,b} \arrow[r] & \mathrm{RZ}^{\mathrm{rig}}_{P,b}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}'_m \arrow[r] \arrow[d] & \mathrm{RZ}^{(m),\mathrm{rig}}_{P,b} \arrow[d, "\pi_1"] \\
\mathrm{RZ}^{K_p{}'^{(m)}}_{L,b} \arrow[r] & \mathrm{RZ}^{\mathrm{rig}}_{L,b}
\end{tikzcd}$$

Since π_1 is a fibration in balls, we obtain quasi-isomorphisms

$$R\Gamma_c(\mathcal{P}'_m \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_l) \cong R\Gamma_c(\mathrm{RZ}^{K_p{}'^{(m)}}_{L,b} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_l(-D))[-2D] \quad \text{for all } m > 0,$$

where $D = \dim \mathrm{RZ}_{P,b} - \dim \mathrm{RZ}_{L,b}$. Moreover, we can argue as in [**Man08**], Lemma 31 and Proposition 32 to deduce quasi-isomorphisms

$$R\Gamma_c(\mathrm{RZ}^{K_p{}'^{(m)}}_{P,b} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_l) \cong R\Gamma_c(\mathcal{P}'_m \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_l) \quad \text{for all } m > 0.$$

Thus we have quasi-isomorphisms

$$R\Gamma_c(\mathrm{RZ}_{P,b}^{K_p{}'^{(m)}} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_l) \cong R\Gamma_c(\mathrm{RZ}_{L,b}^{K_p{}'^{(m)}} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_l(-D))[-2D] \quad \text{for all } m > 0,$$

which yield the desired equality. □

Proposition 4.3.3. *For any admissible l-adic representation ρ of $J_b(\mathbb{Q}_p)$, we have*

$$H^\bullet(RZ_{G,b}^\infty)_\rho = Ind_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H^\bullet(RZ_{P,b}^\infty)_\rho$$

as virtual representations of $P(\mathbb{Q}_P) \times \mathcal{W}_E$.

Proof. For any open compact subgroup $K_p \subseteq G(\mathbb{Z}_p)$, we have natural morphisms of rigid analytic spaces

$$\pi_{2,K_p} : \mathrm{RZ}_{P,b}^{K_p \cap P(\mathbb{Q}_p)} \longrightarrow \mathrm{RZ}_{G,b}^{K_p}$$

which are $P(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-equivariant and compatible with the Weil descent datum. Moreover, these maps are evidently closed immersions. Hence we have isomorphisms

$$\mathrm{RZ}_{G,b}^{K_p} \cong \mathrm{RZ}_{G,b}^{K_p} \times_{\mathrm{RZ}_{G,b}^{\mathrm{rig}}} \mathrm{RZ}_{P,b}^{\mathrm{rig}} \cong \coprod_{K_p \backslash G(\mathbb{Q}_p)/P(\mathbb{Q}_p)} \mathrm{RZ}_{P,b}^{K_p \cap P(\mathbb{Q}_p)} \quad \text{for all } K_p \subseteq G(\mathbb{Z}_p),$$

thereby obtaining the desired identity. □

Proposition 4.3.2 and 4.3.3 together imply Theorem 4.1.5.

Chapter 5

SERRE-TATE DEFORMATION THEORY FOR LOCAL SHIMURA DATA OF HODGE TYPE

Our goal for this chapter is to establish a generalization of Serre-Tate deformation theory for p-divisible groups that arise from μ-ordinary local Shimura data of Hodge type. There are two main ingredients for our theory, namely

(a) existence of a "slope filtration" which admits a unique lifting over deformation rings;

(b) existence of a "canonical deformation".

We prove (a) by applying Theorem 3.2.2 and Theorem 3.2.4 to μ-ordinary local Shimura data of Hodge type. To prove (b), we first embed our deformation space into a deformation space that arises from an EL realization of our local Shimura datum (cf. the proof of Theorem 3.2.4), then use the existence of a canonical deformation in the latter space proved by Moonen in [**Mo04**].

Throughout this section, we fix a prime $p > 2$ and assume that k is algebraically closed.

5.1 The slope filtration of μ-ordinary p-divisible groups

5.1.1. Let us first fix some notations for this chapter. We fix a μ-ordinary unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$. We assume that $\{\mu\}$ is minuscule, and take a unique dominant representative $\mu \in \{\mu\}$. Then we have $[b] = [\mu(p)]$ by definition of μ-ordinariness, so we may take $b = \mu(p)$ and write $\underline{X}$ for the p-divisible group over k with G-structure that arises from this choice $b \in [b] \cap G(W)\mu(p)G(W)$. Let m be a positive integer such that $\sigma^m(\mu) = \mu$, and take L to be the centralizer of $m \cdot \bar{\mu}$ in G which is a Levi subgroup (see [**SGA3**], Exp. XXVI, Cor. 6.10.). We set P to be a proper standard parabolic subgroup of G with Levi factor L.

5.1.2. One can check that $(G, [b], \{\mu\})$ is Hodge-Newton reducible with respect to P and L (see [**Wo13**], Proposition 7.4.). Hence Theorem 3.2.2 gives us the Hodge-Newton decomposition associated to P and L

$$\underline{X} = \underline{X}_1 \times \underline{X}_2 \times \cdots \times \underline{X}_r \tag{5.1.2.1}$$

which we call the *slope decomposition* of $\underline{X}$. If we set

$$X_{a,b} := \prod_{s=a}^{b} X_s$$

for $1 \leq a \leq b \leq r$, we obtain the induced Hodge-Newton filtration

$$0 \subset X_{r,r} \subset X_{r-1,r} \subset \cdots \subset X_{1,r} = X, \tag{5.1.2.2}$$

which we refer to as the *slope filtration* of $\underline{X}$.

Now Theorem 3.2.4 readily gives us the first main ingredient of the theory, namely the unique lifting of the slope filtration.

Proposition 5.1.3. *Let R be a W-algebra of the form $R = W[[u_1, \cdots, u_N]]$ or $R = W[[u_1, \cdots, u_N]]/(p^m)$. Let $\underline{\mathscr{X}}$ be a deformation of $\underline{X}$ over R with an isomorphism $\alpha : \underline{\mathscr{X}} \otimes_R k \cong \underline{X}$. Then there exists a unique filtration of $\mathscr{X}$*

$$0 \subset \mathscr{X}_{r,r} \subset \mathscr{X}_{r-1,r} \subset \cdots \subset \mathscr{X}_{1,r} = \mathscr{X}$$

which lifts the slope filtration (5.1.2.2) *in the sense that α induces isomorphisms $\mathscr{X}_{s,r} \otimes_R k \cong X_{s,r}$ and $\underline{\mathscr{X}_{s,r}/\mathscr{X}_{s+1,r}} \otimes_R k \cong \underline{X}_s$ for $s = 1, 2, \cdots, r$.*

Proof. This is an immediate consequence of Theorem 3.2.4. □

5.2 The canonical deformation of μ-ordinary p-divisible groups

5.2.1. We now aim to find the canonical deformation $\underline{\mathscr{X}}^{\mathrm{can}}$ of $\underline{X}$ over W, which has the property that all endomorphisms of $\underline{X}$ lifts to endomorphisms of $\underline{\mathscr{X}}^{\mathrm{can}}$. When G is of EL type, we already know existence of such a deformation thanks to the work of Moonen in [**Mo04**]. Our strategy is to deduce the existence of $\underline{\mathscr{X}}^{\mathrm{can}}$ from Moonen's result by means of an EL realization of the datum $(G, [b], \{\mu\})$.

The following lemma is crucial for our strategy.

Lemma 5.2.2. *Let $(\widetilde{G}, [b], \{\mu\})$ be an EL realization of the datum $(G, [b], \{\mu\})$. Then $(\widetilde{G}, [b], \{\mu\})$ is μ-ordinary.*

Proof. Consider the map on the Newton sets

$$\mathcal{N}(G) \longrightarrow \mathcal{N}(\widetilde{G})$$

induced by the embedding $G \hookrightarrow \widetilde{G}$. It maps $\bar{\mu}_G$ to $\bar{\mu}_{\widetilde{G}}$ by the proof of Lemma 3.1.4, and $\nu_G([b])$ to $\nu_{\widetilde{G}}([b])$ by the functoriality of the Newton map. On the other hand, we have $\nu_G([b]) = \bar{\mu}_G$ since $(G, [b], \{\mu\})$ is μ-ordinary. Hence we deduce that $\nu_{\widetilde{G}}([b]) = \bar{\mu}_{\widetilde{G}}$, which implies the assertion. □

5.2.3. Let us now fix an EL realization $(\widetilde{G}, [b], \{\mu\})$ of the datum $(G, [b], \{\mu\})$. Then $(\widetilde{G}, [b], \{\mu\})$ is Hodge-Newton reducible with respect to some parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$ with Levi factor $\widetilde{L}$ such that $P = \widetilde{P} \cap G$ and $L = \widetilde{L} \cap G$. In fact, since L is the centralizer of $m \cdot \bar{\mu}$ in G, we may take $\widetilde{P}$ such that $\widetilde{L}$ is the centralizer of $m \cdot \bar{\mu}$ in $\widetilde{G}$. As in 3.2.1, we assume for simplicity that $\widetilde{G}$ is of the form

$$\widetilde{G} := \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_n,$$

where $\mathscr{O}$ is the ring of integers of some finite unramified extension of $\mathbb{Q}_p$. Then $\widetilde{L}$ takes the form

$$\widetilde{L} = \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_1} \times \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_2} \times \cdots \times \mathrm{Res}_{\mathscr{O}|\mathbb{Z}_p}\mathrm{GL}_{j_r}. \tag{5.2.3.1}$$

We define $\widetilde{L}_j, L_j, b_j, \mu_j$ as in Theorem 3.2.2. Then by the proof of Theorem 3.2.2 we have the following facts:

(1) The tuples $(L_j, [b_j], \{\mu_j\})$ and $(\widetilde{L}_j, [b_j], \{\mu_j\})$ are unramified Shimura data of Hodge type,

(2) Each factor $\underline{X}_j$ in the slope decomposition (5.1.2.1) arises from the datum $(L_j, [b_j], \{\mu_j\})$ with the choice $b_j \in [b_j]$.

Let $\widetilde{X}$ be the p-divisible group over k with $\mathscr{O}$-module structure that arises from the datum $(\widetilde{G}, [b], \{\mu\})$ with the choice $b \in [b]$. It admits the Hodge-Newton decomposition

$$\widetilde{X} = \widetilde{X}_1 \times \widetilde{X}_2 \times \cdots \times \widetilde{X}_r, \tag{5.2.3.2}$$

which gives rise to the slope decomposition (5.1.2.1) of $\underline{X}$. By Lemma 5.2.2, the Newton polygon $\nu_{\widetilde{G}}([b])$ and the σ-invariant Hodge polygon $\bar{\mu}_{\widetilde{G}}$ of $\widetilde{X}$ coincide. Since $\widetilde{L}$ is the centralizer of $m \cdot \bar{\mu}$ in $\widetilde{G}$, each factor in the decompositions (5.2.3.1) and (5.2.3.2) corresponds to a unique slope in the polygon $\bar{\mu}_{\widetilde{G}} = \nu_{\widetilde{G}}([b])$. Hence the decomposition (5.2.3.2) is in fact the slope decomposition of $\widetilde{X}$.

Proposition 5.2.4. *Each factor $\underline{X}_j$ in the slope decomposition* (5.1.2.1) *is rigid, i.e., Def_{X_j,L_j} is pro-represented by W.*

Proof. Note that $\widetilde{X}_j$ arises from the datum $(\widetilde{L}_j, [b_j], \{\mu_j\})$ with the choice $b_j \in [b_j]$ (see the proof of Theorem 3.2.2). It corresponds to a unique slope in the polygon $\bar{\mu}_{\widetilde{G}} = \nu_{\widetilde{G}}([b])$, so it is μ-ordinary with single slope. By [**Mo04**], Corollary 2.1.5, its deformation space $\mathrm{Def}_{X_j,\widetilde{L}_j}$ is pro-represented by W. Now the assertion follows from the closed embedding of deformation spaces

$$\mathrm{Def}_{X_j,L_j} \hookrightarrow \mathrm{Def}_{X_j,\widetilde{L}_j}$$

induced by the embedding $L_j \hookrightarrow \widetilde{L}_j$ (Lemma 2.5.6). □

Let $\underline{\mathscr{X}}_j^{\mathrm{can}}$ be the universal deformation of $\underline{X}_j$ in the sense of Proposition 2.5.5. Proposition 5.2.4 says that $\underline{\mathscr{X}}_j^{\mathrm{can}}$ is defined over W. Hence for any ring R of the form $R = W[[u_1, \cdots, u_N]]$ or $R = W[[u_1, \cdots, u_N]]/(p^m)$, there exists a unique deformation of $\underline{X}_j$ over R, namely $\underline{\mathscr{X}}_j^{\mathrm{can}} \otimes_W R$.

We define the *canonical deformation* of $\underline{X}$ to be a deformation of $\underline{X}$ over W given by

$$\underline{\mathscr{X}}^{\mathrm{can}} := \underline{\mathscr{X}}_1^{\mathrm{can}} \times \underline{\mathscr{X}}_2^{\mathrm{can}} \times \cdots \times \underline{\mathscr{X}}_r^{\mathrm{can}}.$$

It is clear from this construction that all endomorphisms of $\underline{X}$ lifts to endomorphisms of $\underline{\mathscr{X}}^{\mathrm{can}} \otimes_W R$ for any ring R of the form $R = W[[u_1, \cdots, u_N]]$ or $R = W[[u_1, \cdots, u_N]]/(p^m)$.

5.3 Structure of deformation spaces

5.3.1. When $r = 1$, we have $\mathrm{Def}_{X,G} \simeq \mathrm{Spf}(W)$ by Proposition 5.2.4.

Let us now consider the case $r = 2$. Then we have the slope decompositions

$$\underline{X} = \underline{X}_1 \times \underline{X}_2 \quad \text{and} \quad \widetilde{X} = \widetilde{X}_1 \times \widetilde{X}_2.$$

Let $(d_s, \mathfrak{f}_s)$ be the type of $\widetilde{X}_s$ for $s \in \{1, 2\}$ (see Example 2.4.5 for definition). Define a function $\mathfrak{f}' : \mathscr{I} \to \{0, 1\}$ by

$$\mathfrak{f}'(i) = \begin{cases} 0 & \text{if } \mathfrak{f}_1(i) = \mathfrak{f}_2(i) = 0; \\ 0 & \text{if } \mathfrak{f}_1(i) = d_1 \text{ and } \mathfrak{f}_2(i) = d_2; \\ 1 & \text{if } \mathfrak{f}_1(i) = 0 \text{ and } \mathfrak{f}_2(i) = d_2. \end{cases}$$

As noted in Example 2.4.5 for definition, there exists a unique isomorphism class of μ-ordinary p-divisible group over k with $\mathscr{O}$-module structure of type $(1, \mathfrak{f}')$. We let $\widetilde{\mathscr{X}}^{\mathrm{can}}(1, \mathfrak{f}')$ denote its canonical lifting.

Theorem 5.3.2. *Notations above. The deformation space $Def_{X,G}$ has a natural structure of a p-divisible group over W. More precisely, we have an isomorphism*

$$Def_{X,G} \cong \widetilde{\mathscr{X}}^{can}(1, \mathfrak{f}')^{d'}$$

as p-divisible groups over W with $\mathscr{O}$-structure for some integer $d' \leq d_1 d_2$.

Proof. Consider the category $\mathbf{C}_W$ of artinian local W-algebra with residue field k. Let $\widetilde{\mathscr{X}}_j^{\mathrm{can}}$ denote the canonical deformation of $\widetilde{X}_j$ for $j = 1, 2$. We define the functor

$$\mathrm{Ext}(\widetilde{\mathscr{X}}_1^{\mathrm{can}}, \widetilde{\mathscr{X}}_2^{\mathrm{can}}) : \mathbf{C}_W \to \mathbf{Sets}$$

by setting $\mathrm{Ext}(\widetilde{\mathscr{X}}_1^{\mathrm{can}}, \widetilde{\mathscr{X}}_2^{\mathrm{can}})(R)$ to be the set of isomorphism classes of extensions of $\widetilde{\mathscr{X}}_j^{\mathrm{can}} \otimes_W R$ by $\widetilde{\mathscr{X}}_2^{\mathrm{can}} \otimes_W R$ as fppf sheaves of $\mathscr{O}$-module.

By [**Mo04**], Theorem 2.3.3, we have the following isomorphisms:

(a) $\mathrm{Def}_{X,\widetilde{G}} \cong \mathrm{Ext}(\widetilde{\mathscr{X}}_1^{\mathrm{can}}, \widetilde{\mathscr{X}}_2^{\mathrm{can}})$ as smooth formal groups over W,

(b) $\mathrm{Def}_{X,\widetilde{G}} \cong \widetilde{\mathscr{X}}^{\mathrm{can}}(1, \mathfrak{f}')^{d_1 d_2}$ as p-divisible groups over W with $\mathscr{O}$-module structure.

On the other hand, by Lemma 2.5.6 we have a closed embedding of deformation spaces

$$\mathrm{Def}_{X,G} \hookrightarrow \mathrm{Def}_{X,\widetilde{G}}. \tag{5.3.2.1}$$

Our first task is to show that $\mathrm{Def}_{X,G}$ is a subgroup of $\mathrm{Def}_{X,\widetilde{G}}$ with $\mathscr{O}$-module structure. Let R be a smooth formal W-algebra of the form $R = W[[u_1, \cdots, u_N]]$ or $R = W[[u_1, \cdots, u_N]]/(p^m)$, and take two arbitrary deformations $\underline{\mathscr{X}}$ and $\underline{\mathscr{X}'}$ of $\underline{X}$ over R. By Proposition 5.1.3, we have exact sequences

$$0 \longrightarrow \underline{\mathscr{X}}_1^{\mathrm{can}} \otimes_W R \longrightarrow \underline{\mathscr{X}} \longrightarrow \underline{\mathscr{X}}_2^{\mathrm{can}} \otimes_W R \longrightarrow 0,$$
$$0 \longrightarrow \underline{\mathscr{X}}_1^{\mathrm{can}} \otimes_W R \longrightarrow \underline{\mathscr{X}'} \longrightarrow \underline{\mathscr{X}}_2^{\mathrm{can}} \otimes_W R \longrightarrow 0.$$

We denote by $\mathscr{X} \odot \mathscr{X}'$ the underlying p-divisible group of their Baer sum taken in $\mathrm{Ext}(\widetilde{\mathscr{X}}_1^{\mathrm{can}}, \widetilde{\mathscr{X}}_2^{\mathrm{can}})(R)$.

We wish to show that $\mathscr{X} \odot \mathscr{X}' \in \mathrm{Def}_{X,G}(R)$. By the isomorphism (a), we already know that $\mathscr{X} \odot \mathscr{X}' \in \mathrm{Def}_{X,\widetilde{G}}(R)$. Hence it remains to show that we have tensors on (the Dieudonné module of) $\mathscr{X} \odot \mathscr{X}'$ which lift the tensors (t_i) on $\underline{X}$ in the sense of Proposition 2.5.3. Unfortunately, it is not easy to explicitly find these tensors in terms of the tensors on $\underline{\mathscr{X}}$ and $\underline{\mathscr{X}}'$. Instead, we start with the family of all tensors $(\mathfrak{s}_j)$ on Λ which are fixed by G. Then we have a family $(\mathfrak{t}_j) := (\mathfrak{s}_j \otimes 1)$ on $\Lambda^* \otimes_{\mathbb{Z}_p} W = M$, where M denotes the Dieudonne module of X as before. Since the formal deformation space $\mathrm{Def}_{X,G}$ is independent of the choice of tensors (t_i), we get tensors $(\bar{\mathfrak{t}}_j)$ on $\underline{\mathscr{X}}$ and $(\bar{\mathfrak{t}}'_j)$ on $\underline{\mathscr{X}}'$ which lift $(\mathfrak{t}_j)$ (in the sense of Proposition 2.5.3). Moreover, the families $(\bar{\mathfrak{t}}_j)$ and $(\bar{\mathfrak{t}}'_j)$ map to the same family of tensors on $\underline{\mathscr{X}}_2^{\mathrm{can}}$ under the surjections $\underline{\mathscr{X}} \twoheadrightarrow \underline{\mathscr{X}}_2^{\mathrm{can}}$ and $\underline{\mathscr{X}}' \twoheadrightarrow \underline{\mathscr{X}}_2^{\mathrm{can}}$. Hence the families $(\bar{\mathfrak{t}}_j)$ and $(\bar{\mathfrak{t}}'_j)$ define the same family of tensors on $\mathscr{X} \odot \mathscr{X}'$ which lift $(\mathfrak{t}_j)$. In particular, there exists a family of tensors on $\mathscr{X} \odot \mathscr{X}'$ which lift (t_i).

Since $\mathrm{Def}_{X,\widetilde{G}}$ has a finite p-torsion for being a p-divisible group, we observe from the embedding (5.3.2.1) that $\mathrm{Def}_{X,G}$ also has finite p-torsion. Using the same argument as in the proof of [**Mo04**], Theorem 2.3.3, we deduce that $\mathrm{Def}_{X,G}$ is a p-divisible group.

Hence $\mathrm{Def}_{\underline{X}}$ is a p-divisible subgroup of $\mathrm{Def}_{X,\widetilde{G}} \cong \widetilde{\mathscr{X}}^{\mathrm{can}}(1,\mathfrak{f}')^{d_1 d_2}$ with $\mathscr{O}$-module structure. Now the dimension of $\mathrm{Def}_{X,G}$ determines an integer d' such that

$$\mathrm{Def}_{X,G} \cong \widetilde{\mathscr{X}}^{\mathrm{can}}(1,\mathfrak{f}')^{d'}$$

as p-divisible groups over W with $\mathscr{O}$-module structure. $\square$

Remark. From the proof, one sees that the canonical deformation $\underline{\mathscr{X}}^{\mathrm{can}}$ corresponds to the identity element in the p-divisible group structure of $\mathrm{Def}_{X,G}$.

5.3.3. We finally consider the case $r \geq 3$. For convenience, we write $\mathrm{Def}_{\widetilde{X}_{a,b}}$ for the deformation space of $\widetilde{X}_{a,b}$. These spaces fit into a diagram

$$\begin{tikzcd}[column sep=tiny]
 & & & \mathrm{Def}_{\widetilde{X}_{1,r}} = \mathrm{Def}_{X,\widetilde{G}} \arrow[dl] \arrow[dr] & & & \\
 & & \mathrm{Def}_{\widetilde{X}_{1,r-1}} \arrow[dl] \arrow[dr] & & \mathrm{Def}_{\widetilde{X}_{2,r}} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{Def}_{\widetilde{X}_{1,r-2}} \arrow[dl] \arrow[dr] & & \mathrm{Def}_{\widetilde{X}_{2,r-1}} \arrow[dl] \arrow[dr] & & \mathrm{Def}_{\widetilde{X}_{3,r}} \arrow[dl] \arrow[dr] & \\
\cdots & & \cdots & & \cdots & & \cdots
\end{tikzcd}$$

where each map comes from the restriction of the filtration in Proposition 5.1.3 (see [**Mo04**], 2.3.6.). This diagram carries some additional structures called the *cascade structure*, as described by Moonen in loc. cit.

We denote by $\mathrm{Def}_{\underline{X}_{a,b}}$ the pull back of $\mathrm{Def}_{\widetilde{X}_{a,b}}$ over $\mathrm{Def}_{X,G}$. Then $\mathrm{Def}_{\underline{X}_{a,b}}$ classifies deformations of $X_{a,b}$ with a filtration that comes from the filtration of $\mathscr{X}$ in Proposition 5.1.3. If we pull back the above diagram over over $\mathrm{Def}_{X,G}$, we get another diagram

$$\begin{tikzcd}[column sep=small]
 & & & \mathrm{Def}_{\underline{X}_{1,r}} = \mathrm{Def}_{X,G} \arrow[dl] \arrow[dr] & & & \\
 & & \mathrm{Def}_{\underline{X}_{1,r-1}} \arrow[dl] \arrow[dr] & & \mathrm{Def}_{\underline{X}_{2,r}} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{Def}_{\underline{X}_{1,r-2}} \arrow[dl] \arrow[dr] & & \mathrm{Def}_{\underline{X}_{2,r-1}} \arrow[dl] \arrow[dr] & & \mathrm{Def}_{\underline{X}_{3,r}} \arrow[dl] \arrow[dr] & \\
\dots & & \dots & & \dots & & \dots
\end{tikzcd}$$

where each map comes from the restriction of the filtration in Proposition 5.1.3. With similar arguments as in the proof of Theorem 5.3.2, one can give a group structure on $\mathrm{Def}_{\underline{X}_{a,b}}$ over $\mathrm{Def}_{\underline{X}_{a,b-1}}$ and $\mathrm{Def}_{\underline{X}_{a+1,b}}$ (cf. [**Mo04**], 2.3.6.). However, this diagram does not carry the full cascade structure in general.

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