Definability and classification of equivalence relations and logical theories

Thesis by
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In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

Caltech

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2018
Defended May 9, 2018

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Alexander Kechris, for his kind, patient, and wise guidance over the years. I owe to his mentorship a large part of who I am as a mathematician.

I would like to thank Anush Tserunyan for inviting me to the UIUC logic seminar and for supporting my academic career. I would like to thank Jindřich Zapletal for inviting me to the SEALS conference in Florida. I would like to thank Martino Lupini and Aristotelis Panagiotopoulos for teaching me about various weird and wonderful areas of logic, some of which have directly inspired my own research.

I would like to thank my colleagues Connor Meehan and William Chan for insightful discussions about math, politics, peas, and miscellanea. Finally, I would like to thank my parents for their unwavering support of my pursuit of my dreams.

My Ph.D. study was partially funded by a scholarship from the Natural Sciences and Engineering Research Council of Canada.

ABSTRACT

This thesis consists of four independent papers.

In the first paper, joint with Kechris, we study the global aspects of structurability in the theory of countable Borel equivalence relations. For a class $\mathcal{K}$ of countable relational structures, a countable Borel equivalence relation E is said to be $\mathcal{K}$-structurable if there is a Borel way to put a structure in $\mathcal{K}$ on each E-equivalence class. We show that $\mathcal{K}$-structurability interacts well with various preorders commonly used in the classification of countable Borel equivalence relations. We consider the poset of classes of $\mathcal{K}$-structurable equivalence relations for various $\mathcal{K}$, under inclusion, and show that it is a distributive lattice. Finally, we consider the effect on $\mathcal{K}$-structurability of various model-theoretic properties of $\mathcal{K}$; in particular, we characterize the $\mathcal{K}$ such that every $\mathcal{K}$-structurable equivalence relation is smooth.

In the second paper, we consider the classes of $\mathcal{K}_n$-structurable equivalence relations, where $\mathcal{K}_n$ is the class of n-dimensional contractible simplicial complexes. We show that every $\mathcal{K}_n$-structurable equivalence relation Borel embeds into one structurable by complexes in $\mathcal{K}_n$ with the further property that each vertex belongs to at most $M_n := 2^{n-1}(n^2 + 3n + 2) - 2$ edges; this generalizes a result of Jackson-Kechris-Louveau in the case $n = 1$.

In the third paper, we consider the amalgamation property from model theory in an abstract categorical context. A category is said to have the amalgamation property if every pushout diagram has a cocone. We characterize the finitely generated categories $\mathbf{I}$ such that in every category with the amalgamation property, every $\mathbf{I}$-shaped diagram has a cocone.

In the fourth paper, we prove a strong conceptual completeness theorem (in the sense of Makkai) for the infinitary logic $\mathcal{L}_{\omega_1\omega}$: every countable $\mathcal{L}_{\omega_1\omega}$-theory can be canonically recovered from its standard Borel groupoid of countable models, up to a suitable syntactical notion of equivalence. This implies that given two theories $(\mathcal{L}, \mathcal{T})$ and $(\mathcal{L}', \mathcal{T}')$, every Borel functor $\mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ between the respective groupoids of countable models is Borel naturally isomorphic to the functor induced by some $\mathcal{L}'_{\omega_1\omega}$-interpretation of $\mathcal{T}$ in $\mathcal{T}'$, which generalizes a recent result of Harrison-Trainor–Miller–Montalbán in the case where $\mathcal{T}, \mathcal{T}'$ are $\aleph_0$-categorical.

PUBLISHED CONTENT AND CONTRIBUTIONS

[CK] R. Chen and A. S. Kechris, *Structurable equivalence relations*, to appear in Fund. Math., 2018. DOI: 10.4064/fm428-7-2017. URL: https://www.impan.pl/en/publishing-house/journals-and-series/fundamenta-mathematicae/online/92249/structurable-equivalence-relations.

Both authors contributed equally to this work.

[C2] R. Chen, *Borel structurability by locally finite simplicial complexes*, to appear in Proc. Amer. Math. Soc., 2018. DOI: 10.1090/proc/13957. URL: https://www.ams.org/journals/proc/0000-000-00/S0002-9939-2018-13957-X/.

I am the sole author of this work.

[C1] R. Chen, *Amalgamable diagram shapes*, submitted, 2016. URL: https://arxiv.org/abs/1606.06777.

I am the sole author of this work.

[C3] R. Chen, *Borel functors, interpretations, and strong conceptual completeness for $\mathcal{L}_{\omega_1\omega}$*, submitted, 2018. URL: https://arxiv.org/abs/1710.02246.

I am the sole author of this work.

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Chapter 1

INTRODUCTION

This thesis consists of four independent research papers, loosely centered around the theme of classifying naturally occurring objects in mathematical logic using tools from descriptive set theory and category theory.

1.1 Structurable equivalence relations

In the paper [CK], joint with Alexander Kechris, we systematically study the framework of structurability used for classifying countable Borel equivalence relations.

A **standard Borel space** is a measurable space $X = (X, \mathcal{B})$, consisting of an underlying set X and a σ-algebra $\mathcal{B}$ on X, such that $\mathcal{B}$ is the Borel σ-algebra of a separable completely metrizable topology on X. In descriptive set theory, a standard Borel space X is thought of as a simple model of a "definable" set, where the Borel subsets $B \subseteq X$ are the "definable" subsets. Examples of standard Borel spaces include $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, any separable Banach space, and the space of all group structures (or graphs, or linear orders, etc.) on $\mathbb{N}$.

It is a basic result of classical descriptive set theory that up to Borel isomorphism, the only standard Borel spaces are $0, 1, 2, \dots, \mathbb{N}, \mathbb{R}$ (where we adopt the usual set-theoretic convention that $n \in \mathbb{N}$ is identified with the set $\{0, \dots, n-1\}$). Thus, the classification problem for standard Borel spaces is rather trivial. Such is no longer the case, however, once we pass to "definable" quotient spaces. A **countable Borel equivalence relation** E on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ whose equivalence classes are all countable. We may think of the pair (X, E) as a "presentation" of the "definable" quotient space X/E. Many naturally occurring classification problems in mathematics can be encoded in this way, allowing their complexity to be quantified in precise terms. For example, there is a countable Borel equivalence relation (X, E) whose quotient X/E is in canonical bijection with isomorphism types of finitely generated groups; by a result of Thomas and Velickovic [TV], this quotient is strictly more complicated than the space of isomorphism types of finitely generated *abelian* groups (which is itself Borel isomorphic to $\mathbb{N}$).

The most important means of comparing the complexities of two countable Borel equivalence relations (X, E) and (Y, F) is that of a **Borel reduction**

$$f : E \leq_B F,$$

which is a Borel map $f : X \to Y$ which descends to an injection $X/E \hookrightarrow Y/F$ between the quotient spaces. We think of a Borel reduction $f : E \leq_B F$ as "definably" reducing the classification problem encoded by E to that encoded by F. If a Borel reduction $f : E \leq_B F$ exists, we write $E \leq_B F$ and say that E **Borel reduces** to F. Thus, for example, the aforementioned Thomas–Velickovic result implies that the isomorphism relation between finitely generated groups does *not* Borel reduce to that between finitely generated abelian groups. The Borel reducibility preordering $\leq_B$ on the class of countable Borel equivalence relations is extremely complicated, as shown by Adams–Kechris [AK]:

Theorem 1.1.1 (Adams–Kechris)**.** *There is an order-embedding from the poset of Borel subsets of $\mathbb{R}$ under inclusion to the preorder $\leq_B$ on the class of countable Borel equivalence relations.*

There are various other, related, preorderings on the class of countable Borel equivalence relations. Given two countable Borel equivalence relations (X, E), (Y, F), a **Borel embedding** $f : E \sqsubseteq_B F$ is a Borel reduction which is moreover injective (as a function $X \to Y$); we write $E \sqsubseteq_B F$ if such f exists. Such f is an **invariant Borel embedding**, denoted $f : E \sqsubseteq_B^i F$, if furthermore the image $f(X) \subseteq Y$ is F-invariant; in that case, we write $E \sqsubseteq_B^i F$. A **Borel homomorphism** $f : E \to_B F$ is a Borel map $f : X \to Y$ which descends to a (not necessarily injective) map $X/E \to Y/F$, i.e., for all $x_1, x_2 \in X$, we have $x_1 \mathrel{E} x_2 \implies f(x_1) \mathrel{F} f(x_2)$. A Borel homomorphism $f : E \to_B F$ is **class-injective**, respectively **class-bijective**, if for each $x \in X$, the restriction $f|[x]_E : [x]_E \to [f(x)]_F$ is injective (respectively bijective); these are denoted $E \to_B^{ci} F$ and $E \to_B^{cb} F$ respectively. The relationship between these notions is depicted as follows, where a line means that the lower preordering is stronger (finer) than the upper one:

$$\begin{tikzcd}
 & \to_B \arrow[dl, dash] \arrow[dr, dash] & & \\
\leq_B \arrow[dr, dash] & & \to_B^{ci} \arrow[dl, dash] \arrow[dr, dash] & \\
 & \sqsubseteq_B \arrow[dr, dash] & & \to_B^{cb} \arrow[dl, dash] \\
 & & \sqsubseteq_B^i &
\end{tikzcd}$$

The framework of **structurability** (see [JKL, Section 2.5]) provides an *a priori* different means of quantifying the complexity of countable Borel equivalence relations. Given a class $\mathcal{K}$ of (first-order) structures, such as a class of graphs or linear orders, a countable Borel equivalence relation (X, E) is said to be $\mathcal{K}$**-structurable** if there is a "Borel assignment" of a structure in $\mathcal{K}$ to each E-class; formally, this means that there is a Borel structure on X whose restriction to each E-class is in $\mathcal{K}$. For example, the class of **treeable** equivalence relations ($\mathcal{K}$ = {connected acyclic graphs}) has been well-studied in ergodic theory (see e.g., [Ada]). For each $\mathcal{K}$, put

$$\mathcal{E}_{\mathcal{K}} := \{\mathcal{K}\text{-structurable countable Borel equivalence relations}\}.$$

We restrict attention to classes of structures $\mathcal{K}$ which are **Borel**, meaning that they can be axiomatized by a sentence in the countably infinitary logic $\mathcal{L}_{\omega_1\omega}$ (see Section 1.4 below). For Borel $\mathcal{K}$, we call $\mathcal{E}_{\mathcal{K}}$ an **elementary class** of countable Borel equivalence relations. Intuitively, each $\mathcal{E}_{\mathcal{K}}$ is a class of "sufficiently simple" equivalence relations, where the meaning of "sufficiently simple" depends on $\mathcal{K}$.

Our first goal in [CK] is to determine the precise relationship between structurability and the aforementioned preorders:

Theorem 1.1.2 (C.-Kechris; see Theorems 2.1.2 to 2.1.5)**.**

(i) Every countable Borel equivalence relation E is contained in a smallest elementary class, a smallest elementary class downward-closed under $\sqsubseteq_B$, and a smallest elementary class downward-closed under $\leq_B$.

(ii) A class C of countable Borel equivalence relations is elementary iff it is downward-closed under $\rightarrow_B^{cb}$ and contains a $\sqsubseteq_B^i$-greatest element.

There are also analogous characterizations of elementary classes downward-closed under $\sqsubseteq_B$ or $\leq_B$.

We next study the poset of elementary classes under inclusion. We show that it is order-theoretically well-behaved, yet quite rich (by adapting the proof of Theorem 1.1.1 [AK]):

Theorem 1.1.3 (C.-Kechris; see Theorems 2.1.7 to 2.1.9)**.** *The posets of elementary classes and of elementary classes closed under $\leq_B$ (both under inclusion) form countably complete distributive lattices, and admit order-embeddings from the poset of Borel subsets of $\mathbb{R}$ under inclusion.*

Together with the preceding result, this implies a positive answer to a question of Kechris–Macdonald, who had asked whether there exist $\leq_B$-incomparable countable Borel equivalence relations with a $\leq_B$-greatest lower bound (see Section 2.6.2).

An important aspect of the theory of countable Borel equivalence relations lies in connections with countable group actions and ergodic theory. In particular, one is often interested in properties of orbit equivalence relations of free Borel actions of a countable group Γ. For each Γ, let $\mathcal{E}_\Gamma$ denote the class of such equivalence relations. Then $\mathcal{E}_\Gamma = \mathcal{E}_\mathcal{K}$ for $\mathcal{K}$ the class of free transitive actions of Γ, whence $\mathcal{E}_\Gamma$ is an elementary class.

We consider the question of when $\mathcal{E}_\Gamma$ is closed under $\leq_B$. The answer is never, for the trivial reason that every orbit of a free action of Γ must have the same cardinality as Γ. To sidestep this technicality, let $\mathcal{E}_\Gamma^* \supseteq \mathcal{E}_\Gamma$ denote the elementary class of countable Borel equivalence relations which are induced by a free Borel action of Γ except on the finite equivalence classes. Then we have the following characterization in terms of the well-known notion of **amenable group** (a group that admits a left-invariant finitely additive probability measure):

Theorem 1.1.4 (C.-Kechris; see Theorem 2.1.6)**.** *Let Γ be a countably infinite group. Then $\mathcal{E}_\Gamma^*$ is closed under $\leq_B$ iff Γ is amenable.*

Finally in [CK], we begin the study of the relationship between structurability and model theory. A countable Borel equivalence relation (X, E) is **smooth** if the quotient space X/E is standard Borel. Intuitively, this means that the classification problem represented by E admits "complete invariants"; an example is the (aforementioned) classification of finitely generated abelian groups, with the invariant given by the tuple of exponents of each $\mathbb{Z}, \mathbb{Z}/p^n\mathbb{Z}$. Thus, it is of interest to determine conditions on equivalence relations which imply smoothness. The following completely characterizes all such conditions which are instances of structurability, and answers a question of Marks:

Theorem 1.1.5 (C.-Kechris; see Theorem 2.1.10)**.** *Let $\mathcal{E}_\mathcal{K}$ be an elementary class. The following are equivalent:*

(i) Every equivalence relation in $\mathcal{E}_\mathcal{K}$ is smooth.

(ii) There is an $\mathcal{L}_{\omega_1\omega}$-formula $\phi(x)$ which defines a finite nonempty subset in every structure in $\mathcal{K}$.

1.2 Borel structurability by locally finite simplicial complexes

The paper [C2] is concerned with the following elementary classes which generalize the class of treeable equivalence relations to higher dimensions.

A(n **abstract) simplicial complex** S on a set X consists of a family of nonempty finite subsets $s \subseteq X$, containing all singletons $\{x\}$ and closed under nonempty subsets. Each $s \in S$ of cardinality $n + 1$ is thought of as representing the n-simplex spanned by its elements. Thus, we say that S is n**-dimensional** if $|s| \leq n + 1$ for all $s \in S$. A simplicial complex S has a **geometric realization** $|S|$, which is a topological space formed by gluing together standard Euclidean simplices according to the data in S. We say that S is **contractible** if $|S|$ is. For example, a 1-dimensional contractible simplicial complex is the same thing as a tree.

For each $n \geq 1$, let $\mathcal{K}_n$ denote the class of n-dimensional contractible simplicial complexes. It is straightforward to encode a simplicial complex as a countable structure; thus we may view each $\mathcal{K}_n$ as a class of countable structures. We are interested in the elementary classes

$$\mathcal{E}_{\mathcal{K}_n} = \{\mathcal{K}_n\text{-structurable countable Borel equivalence relations}\}.$$

Thus, $\mathcal{E}_{\mathcal{K}_1}$ is the class of treeable equivalence relations. The study of $\mathcal{K}_n$-structurability for general n was initiated by Gaboriau [Gab], who showed using the theory of ℓ^2-Betti numbers that we have a strictly increasing hierarchy

$$\mathcal{E}_{\mathcal{K}_1} \subsetneq \mathcal{E}_{\mathcal{K}_2} \subsetneq \mathcal{E}_{\mathcal{K}_3} \subsetneq \cdots .$$

The class of treeable equivalence relations is known to have many nice properties; see [JKL, §3]. One property is the following [JKL, 3.10]:

Theorem 1.2.1 (Jackson–Kechris–Louveau)**.** *Let E be a treeable countable Borel equivalence relation. Then E Borel embeds into some countable Borel equivalence relation F which is treeable by trees of vertex degree ≤ 3.*

The main result of [C2] is the following generalization:

Theorem 1.2.2 (C.; see Theorem 3.1.1)**.** *Let $n \geq 1$, and let E be a $\mathcal{K}_n$-structurable countable Borel equivalence relation. Then E Borel embeds into some countable Borel equivalence relation F which is structurable by simplicial complexes in $\mathcal{K}_n$ with the further property that each vertex belongs to at most $M_n := 2^{n-1}(n^2 + 3n + 2) - 2$ edges.*

1.3 Amalgamable diagram shapes

The paper [C1] was inspired by some problems considered in [CK], but is not otherwise related to the two previous papers.

A recurring idea in model theory is that of **amalgamation**, which can be phrased in an abstract category-theoretic setting as follows: given a diagram

$$\begin{tikzcd} B & & C \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{$*$}$$

in some category, extend it to a commutative diagram

$$\begin{tikzcd} & D & \\ B \arrow[ur, dashed] & & C. \arrow[ul, dashed] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{$\dagger$}$$

In standard categorical terminology, this is the problem of finding a **cocone** over the diagram $(*)$.

We say that a category has the **amalgamation property** if amalgamation is always possible, i.e., every diagram $(*)$ has a cocone $(\dagger)$. The amalgamation property implies various "generalized amalgamation properties", where $(*)$ is replaced by a more complicated diagram, e.g.,

$$\begin{tikzcd} & & F & & \\ C \arrow[urr, dashed] & & D \arrow[u, dashed] & & E \arrow[ull, dashed] \\ & A \arrow[ul] \arrow[ur] & & B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

via two applications of the amalgamation property. However, the amalgamation property is not enough to imply that the following diagram has a cocone:

$$\begin{tikzcd} C & D \\ A \arrow[u] \arrow[ur] & B \arrow[u] \arrow[ul] \end{tikzcd} \tag{$\ddagger$}$$

This suggests the question of characterizing the diagrams over which a cocone may always be found via repeated use of the amalgamation property. In other words, which "generalized amalgamation properties" are implied by the amalgamation property?

A category C is **simply-connected** if the **fundamental groupoid** $\pi_1(\mathsf{C})$, obtained from C by freely adjoining an inverse for every morphism, has at most one morphism between any two objects. For example, the diagram shape (‡) is not simply-connected, because the "loop" around the diagram gives a morphism $A \to A$ in $\pi_1(\mathsf{C})$ which is not the identity. There is also a topological definition of simply-connectedness, via a certain canonical simplicial complex (the **nerve**) associated to C; see Section 4.2.1. The main result of [C1] is

Theorem 1.3.1 (C.; see Theorem 4.1.1)**.** *Let P be a finite connected poset. The following are equivalent:*

(i) Every P-shaped diagram in every category with the amalgamation property has a cocone.

(ii) Every P-shaped diagram in the category of finite sets and injections has a cocone.

(iii) Every upward-closed subset of P is simply-connected.

(iv) P can be constructed inductively according to some simple rules (see Definition 4.4.3).

Furthermore, it is decidable whether P satisfies these conditions.

These conditions can also be generalized to the case where P is replaced with an arbitrary finitely generated category.

1.4 Borel functors, interpretations, and strong conceptual completeness for $\mathcal{L}_{\omega_1\omega}$

In the paper [C3], I apply the framework of strong conceptual completeness in categorical logic to the study of the countably infinitary logic $\mathcal{L}_{\omega_1\omega}$.

In mathematical logic, "completeness" is the name given to various results relating syntax with semantics. Usually, "the completeness theorem" for a logical system refers to the following result:

"A statement ϕ is syntactically provable iff it is true under all possible semantics."

For example, consider the simple case of **propositional logic**, where atomic statements ("formulas") are combined using connectives such as $\wedge$ ("and"), $\vee$ ("or"), and $\neg$ ("not"). The formula $\phi = A \vee ((B \vee \neg B) \wedge \neg A)$ is true regardless of the truth values of A, B; thus by the completeness theorem, there is a syntactic proof of ϕ.

A "strong conceptual completeness theorem" (a term due to Makkai [M88], who proved such a theorem for finitary first-order logic $\mathcal{L}_{\omega\omega}$) includes the completeness theorem in the above sense, but is much more general:

> "There is a complete correspondence between syntax and semantics."

In particular, the correspondence includes not only the truth value of statements, but also the *statements themselves*. For example, again in propositional logic with two atomic formulas A, B, we have a semantic property Φ depending on the truth values of A, B, given by the truth table (treating 1 = true, 0 = false)

A	B	Φ
0	0	1
0	1	1
1	0	0
1	1	1

The strong conceptual completeness theorem would then imply that Φ must be given by evaluation of some syntactic formula ϕ (in this case, $\phi = \neg A \vee B$ works).

First-order logic deals not just with "absolute" statements as in propositional logic, but also with statements about elements of mathematical structures. A **first-order language** $\mathcal{L}$ is a set of **function symbols** and **relation symbols** of various **arities** $n \in \mathbb{N}$. An **$\mathcal{L}$-structure** $\mathcal{M}$ consists of an underlying set M, together with interpretations of the symbols in $\mathcal{L}$ as actual functions $f^{\mathcal{M}} : M^n \to M$ or relations $R^{\mathcal{M}} \subseteq M^n$ of the specified arities n. For example, a poset P is a structure over the language $\mathcal{L} = \{\leq\}$ where $\leq$ is a binary relation symbol.

Given a first-order language $\mathcal{L}$, the **$\mathcal{L}_{\omega_1\omega}$-formulas** are built inductively starting with atomic formulas (e.g., "$R(f(x), y)$" for a binary relation $R \in \mathcal{L}$ and unary function $f \in \mathcal{L}$) and then applying $\neg$, countably infinite $\bigwedge$ and $\bigvee$, and quantifiers $\exists x$ and $\forall x$. An $\mathcal{L}_{\omega_1\omega}$-formula $\phi(x_1, \dots, x_n)$ with free variables $x_1, \dots, x_n$ may be evaluated at some tuple $\vec{a} = (a_1, \dots, a_n)$ in an $\mathcal{L}$-structure $\mathcal{M}$ to yield a truth value $\phi^{\mathcal{M}}(\vec{a})$; thus, ϕ defines an n-ary relation $\phi^{\mathcal{M}} \subseteq M^n$. An **$\mathcal{L}_{\omega_1\omega}$-theory** $\mathcal{T}$ is a set of $\mathcal{L}_{\omega_1\omega}$-sentences (formulas without free variables). A **model** of a theory $\mathcal{T}$ is an $\mathcal{L}$-structure such that every $\phi \in \mathcal{T}$ evaluates to true in $\mathcal{M}$.

The first component of the strong conceptual completeness theorem for $\mathcal{L}_{\omega_1\omega}$ is the completeness theorem in the usual sense, due to Lopez-Escobar [Lop]:

Theorem 1.4.1 (Lopez-Escobar)**.** *Let $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$-theory. An $\mathcal{L}_{\omega_1\omega}$-sentence ϕ is provable from $\mathcal{T}$ iff it is true in all countable models of $\mathcal{T}$.*

There is an intimate connection between $\mathcal{L}_{\omega_1\omega}$ and descriptive set theory (see [Gao, Chapter 11]), which is manifested in the other two components of the strong conceptual completeness theorem. Given a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, say with no finite models for simplicity, there is a canonical standard Borel **space of countable models of** $\mathcal{T}$ with underlying set $\mathbb{N}$, which we denote by $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$; see Section 5.5. This space is the space of objects of the standard Borel **groupoid of countable models of** $\mathcal{T}$, denoted $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$, whose morphisms are isomorphisms between models. For an $\mathcal{L}_{\omega_1\omega}$-formula $\phi(x_1, \dots, x_n)$, we let

$$[\![\phi]\!] := \bigsqcup_{\mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T})} \phi^{\mathcal{M}}$$

be the disjoint union of the interpretations $\phi^{\mathcal{M}} \subseteq M^n$ in all models $\mathcal{M}$. Then $[\![\phi]\!]$ is naturally a Borel subset of $\mathrm{Mod}(\mathcal{L}, \mathcal{T}) \times \mathbb{N}^n$ which is isomorphism-invariant, i.e., invariant with respect to the action of the groupoid $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$. The second component of the strong conceptual completeness theorem for $\mathcal{L}_{\omega_1\omega}$ is the converse, again due to Lopez-Escobar [Lop] and usually (when $n = 0$) known simply as "the Lopez-Escobar theorem":

Theorem 1.4.2 (Lopez-Escobar)**.** *Let $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$-theory. Then every Borel $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-invariant subset of $\mathrm{Mod}(\mathcal{L}, \mathcal{T}) \times \mathbb{N}^n$ is equal to $[\![\phi]\!]$ for some $\mathcal{L}_{\omega_1\omega}$-formula $\phi(x_1, \dots, x_n)$.*

In other words, analogously to the case of propositional logic, every Borel isomorphism-invariant semantic property of tuples in models of $\mathcal{T}$ is defined by some syntactic formula.

Unlike in propositional logic, there is a third component, dealing neither with proofs of formulas nor with formulas themselves, but with **imaginary sorts** of the theory $\mathcal{T}$, which are certain syntactic expressions denoting sets canonically defined from any model of $\mathcal{T}$. Roughly, an imaginary sort A is given by a formal quotient of a formal countable disjoint union of (sets defined by) $\mathcal{L}_{\omega_1\omega}$-formulas; see Section 5.4 for details. An imaginary sort A may be interpreted in a model $\mathcal{M}$ of $\mathcal{T}$ to obtain a countable set $A^{\mathcal{M}}$, which depends on $\mathcal{M}$ in an isomorphism-equivariant way. Moreover, there is a canonical standard Borel structure on the disjoint union

$$[\![A]\!] := \bigsqcup_{\mathcal{M}\in\mathrm{Mod}(\mathcal{L},\mathcal{T})} A^{\mathcal{M}},$$

and the action of the isomorphism groupoid $\mathsf{Mod}(\mathcal{L},\mathcal{T})$ on $[\![A]\!]$ (equipped with the fiberwise countable projection map $\pi : [\![A]\!] \to \mathrm{Mod}(\mathcal{L},\mathcal{T})$) is Borel. The core result of [C3] is the converse, which forms the third component of the strong conceptual completeness theorem for $\mathcal{L}_{\omega_1\omega}$:

Theorem 1.4.3 (C.; see Theorem 5.1.2)**.** *Let $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$-theory. Then every standard Borel space X equipped with a fiberwise countable Borel map $p : X \to \mathrm{Mod}(\mathcal{L},\mathcal{T})$ and a Borel action of $\mathsf{Mod}(\mathcal{L},\mathcal{T})$ (in short, every* ***fiberwise countable Borel $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-space****) is isomorphic to $[\![A]\!]$ for some imaginary sort A.*

In other words, every Borel isomorphism-equivariant assignment of a countable set to each model of $\mathcal{T}$ is named by some imaginary sort. (Note that in contrast to Theorem 1.4.2, the countable set assigned to each model is not *a priori* related to the model in any way.)

In order to place Theorems 1.4.1 to 1.4.3 in their proper context, and to justify the claim that they give a “complete correspondence between syntax and semantics”, we collect the imaginary sorts of $\mathcal{T}$ into a category $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$, the **syntactic Boolean ω_1-pretopos** of $\mathcal{T}$. The morphisms $f : A \to B$ between two imaginary sorts A, B are **definable functions**, which are certain syntactic expressions denoting canonically defined functions $f^{\mathcal{M}} : A^{\mathcal{M}} \to B^{\mathcal{M}}$ for models $\mathcal{M}$; again see Section 5.4. Every such $f : A \to B$ induces a Borel $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-equivariant map $[\![f]\!] : [\![A]\!] \to [\![B]\!]$, so that we get a functor

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1} \longrightarrow \{\text{fiberwise countable Borel } \mathsf{Mod}(\mathcal{L},\mathcal{T})\text{-spaces}\}.$$

Now by standard category theory, Theorems 1.4.1 to 1.4.3 are together equivalent to the following, which is the main result of [C3]:

Theorem 1.4.4 (C.; see Theorem 5.1.3)**.** *Let $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$-theory. Then the above functor $[\![-]\!]$ is an equivalence of categories.*

The syntactic Boolean ω_1-pretopos $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ belongs to a large class of similar constructions in categorical logic; see [MR]. It is the analog for $\mathcal{L}_{\omega_1\omega}$ of the classical **Lindenbaum–Tarski algebra** (of formulas modulo provable equivalence) of a propositional theory, and like the latter, captures the "logical essence" of a theory $\mathcal{T}$ while forgetting irrelevant syntactic details. Thus, Theorem 1.4.4 can be read as saying that the "logical essence" of a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ can be canonically recovered from its groupoid of models.

The proof of Theorem 1.4.3 (and hence Theorem 1.4.4) combines methods from invariant descriptive set theory, such as the Becker–Kechris theorem on topological realization of Borel actions of a Polish group, with ideas from topos theory, namely the Joyal–Tierney representation theorem for toposes in terms of localic groupoids.

Let us mention an application of Theorem 1.4.4. Given two countable $\mathcal{L}_{\omega_1\omega}$-theories $(\mathcal{L}, \mathcal{T})$ and $(\mathcal{L}', \mathcal{T}')$ (in possibly different languages $\mathcal{L}, \mathcal{L}'$), an **interpretation** $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ is a certain kind of uniform syntactic rule for turning models of $\mathcal{T}'$ into models of $\mathcal{T}$; see Section 5.1. (This is a variation of the usual model-theoretic notion of interpretation which is suitable for $\mathcal{L}_{\omega_1\omega}$.) An interpretation $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ induces a Borel functor $F^* : \mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ which implements the rule specified by F. Conversely, it is a consequence of Theorem 1.4.4 and standard category theory that

Corollary 1.4.5 (C.; see Theorem 5.1.1)**.** *Every Borel functor* $\mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ *is Borel naturally equivalent to* F^* *for some interpretation* $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$.

This generalizes a recent result of Harrison-Trainor–Miller–Montalbán [HMM], who proved the special case where $\mathcal{T}, \mathcal{T}'$ are **$\aleph_0$-categorical**, i.e., they each have a single countable model up to isomorphism.

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Chapter 2

STRUCTURABLE EQUIVALENCE RELATIONS

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2.1 Introduction

(A) A countable Borel equivalence relation on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with the property that every equivalence class $[x]_E$, $x \in X$, is countable. We denote by $\mathcal{E}$ the class of countable Borel equivalence relations. Over the last 25 years there has been an extensive study of countable Borel equivalence relations and their connection with group actions and ergodic theory. An important aspect of this work is an understanding of the kind of countable (first-order) structures that can be assigned in a uniform Borel way to each class of a given equivalence relation. This is made precise in the following definitions; see [JKL], Section 2.5.

Let $L = \{R_i \mid i \in I\}$ be a countable relational language, where R_i has arity n_i, and $\mathcal{K}$ a class of countable structures in L closed under isomorphism. Let E be a countable Borel equivalence relation on a standard Borel space X. An **L-structure** on E is a Borel structure $\mathbb{A} = (X, R_i^{\mathbb{A}})_{i \in I}$ of L with universe X (i.e., each $R_i^{\mathbb{A}} \subseteq X^{n_i}$ is Borel) such that for $i \in I$ and $x_1, \dots, x_{n_i} \in X$, $R_i^{\mathbb{A}}(x_1, \dots, x_{n_i}) \implies x_1 \mathrel{E} x_2 \mathrel{E} \cdots \mathrel{E} x_{n_i}$. Then each E-class C is the universe of the countable L-structure $\mathbb{A}|C$. If for all such C, $\mathbb{A}|C \in \mathcal{K}$, we say that $\mathbb{A}$ is a **$\mathcal{K}$-structure** on E. Finally if E admits a $\mathcal{K}$-structure, we say that E is **$\mathcal{K}$-structurable**.

Many important classes of countable Borel equivalence relations can be described as the $\mathcal{K}$-structurable relations for appropriate $\mathcal{K}$. For example, the hyperfinite equivalence relations are exactly the $\mathcal{K}$-structurable relations, where $\mathcal{K}$ is the class of linear orderings embeddable in $\mathbb{Z}$. The treeable equivalence relations are the $\mathcal{K}$-structurable relations, where $\mathcal{K}$ is the class of countable trees (connected acyclic graphs). The equivalence relations generated by a free Borel action of a countable group Γ are the $\mathcal{K}$-structurable relations, where $\mathcal{K}$ is the class of structures corresponding to free transitive Γ-actions. The equivalence relations admitting no invariant probability Borel measure are the $\mathcal{K}$-structurable relations, where $L = \{R, S\}$, R unary and S binary, and $\mathcal{K}$ consists of all countably infinite structures

$\mathbb{A} = (A, R^{\mathbb{A}}, S^{\mathbb{A}})$, with $R^{\mathbb{A}}$ an infinite, co-infinite subset of A and $S^{\mathbb{A}}$ the graph of a bijection between A and $R^{\mathbb{A}}$.

For $L = \{R_i \mid i \in I\}$ as before and countable set X, we denote by $\mathrm{Mod}_X(L)$ the standard Borel space of countable L-structures with universe X. Clearly every countable L-structure is isomorphic to some $\mathbb{A} \in \mathrm{Mod}_X(L)$, for $X \in \{1, 2, \dots, \mathbb{N}\}$. Given a class $\mathcal{K}$ of countable L-structures, closed under isomorphism, we say that $\mathcal{K}$ is Borel if $\mathcal{K} \cap \mathrm{Mod}_X(L)$ is Borel in $\mathrm{Mod}_X(L)$, for each countable set X. We are interested in Borel classes $\mathcal{K}$ in this paper. For any $L_{\omega_1\omega}$-sentence σ, the class of countable models of σ is Borel. By a classical theorem of Lopez-Escobar [LE], every Borel class $\mathcal{K}$ of L-structures is of this form, for some $L_{\omega_1\omega}$-sentence σ. We will also refer to such σ as a theory.

Adopting this model-theoretic point of view, given a theory σ and a countable Borel equivalence relation E, we put

$$E \models \sigma$$

if E is $\mathcal{K}$-structurable, where $\mathcal{K}$ is the class of countable models of σ, and we say that E is **σ-structurable** if $E \models \sigma$. We denote by $\mathcal{E}_\sigma \subseteq \mathcal{E}$ the class of σ-structurable countable Borel equivalence relations. Finally we say that a class C of countable Borel equivalence relations is **elementary** if it is of the form $\mathcal{E}_\sigma$, for some σ (which **axiomatizes** C). In some sense the main goal of this paper is to study the global structure of elementary classes.

First we characterize which classes of countable Borel equivalence relations are elementary. We need to review some standard concepts from the theory of Borel equivalence relations. Given equivalence relations E, F on standard Borel spaces X, Y, resp., a **Borel homomorphism** of E to F is a Borel map $f\colon X \to Y$ with $x \mathrel{E} y \implies f(x) \mathrel{F} f(y)$. We denote this by $f\colon E \to_B F$. If moreover f is such that all restrictions $f|[x]_E\colon [x]_E \to [f(x)]_F$ are bijective, we say that f is a **class-bijective homomorphism**, in symbols $f\colon E \to_B^{cb} F$. If such f exists we also write $E \to_B^{cb} F$. We similarly define the notion of **class-injective homomorphism**, in symbols $\to_B^{ci}$. A **Borel reduction** of E to F is a Borel map $f\colon X \to Y$ with $x \mathrel{E} y \iff f(x) \mathrel{F} f(y)$. We denote this by $f\colon E \leq_B F$. If f is also injective, it is called a **Borel embedding**, in symbols $f\colon E \sqsubseteq_B F$. If there is a Borel reduction of E to F we write $E \leq_B F$ and if there is a Borel embedding we write $E \sqsubseteq_B F$. An **invariant Borel embedding** is a Borel embedding f as above with $f(X)$ F-invariant. We use the notation $f\colon E \sqsubseteq_B^i F$ and $E \sqsubseteq_B^i F$ for these notions. By the usual Schroeder-Bernstein argument, $E \sqsubseteq_B^i F \ \& \ F \sqsubseteq_B^i E \iff E \cong_B F$, where $\cong_B$ is Borel isomorphism.

Kechris-Solecki-Todorcevic [KST, 7.1] proved a universality result for theories of graphs, which was then extended to arbitrary theories by Miller; see Corollary 2.4.4.

Theorem 2.1.1 (Kechris-Solecki-Todorcevic, Miller)**.** *For every theory σ, there is an invariantly universal σ-structurable countable Borel equivalence relation $E_{\infty\sigma}$, i.e., $E_{\infty\sigma} \models \sigma$, and $F \sqsubseteq_B^i E_{\infty\sigma}$ for any other $F \models \sigma$.*

Clearly $E_{\infty\sigma}$ is uniquely determined up to Borel isomorphism. In fact in Theorem 2.4.1 we formulate a “relative" version of this result and its proof that allows us to capture more information.

Next we note that clearly every elementary class is closed downwards under class-bijective Borel homomorphisms. We now have the following characterization of elementary classes (see Corollary 2.4.12).

Theorem 2.1.2. *A class $\mathcal{C} \subseteq \mathcal{E}$ of countable Borel equivalence relations is elementary iff it is (downwards-)closed under class-bijective Borel homomorphisms and contains an invariantly universal element $E \in \mathcal{C}$.*

Examples of non-elementary classes include the class of non-smooth countable Borel equivalence relations (a countable Borel equivalence relation is **smooth** if it admits a Borel transversal), the class of equivalence relations admitting an invariant Borel probability measure, and the class of equivalence relations generated by a free action of *some* countable group. More generally, nontrivial unions of elementary classes are never elementary (see Corollary 2.4.5).

Next we show that every $E \in \mathcal{E}$ is contained in a (unique) smallest (under inclusion) elementary class (see Corollary 2.4.10).

Theorem 2.1.3. *For every $E \in \mathcal{E}$, there is a smallest elementary class containing E, namely $\mathcal{E}_E := \{F \in \mathcal{E} \mid F \to_B^{cb} E\}$.*

Many classes of countable Borel equivalence relations that have been extensively studied, like hyperfinite or treeable ones, are closed (downwards) under Borel reduction. It turns out that every elementary class is contained in a (unique) smallest (under inclusion) elementary class closed under Borel reduction (see Theorem 2.5.2).

Theorem 2.1.4. *For every elementary class C, there is a smallest elementary class containing C and closed under Borel reducibility, namely $C^r := \{F \in \mathcal{E} \mid \exists E \in C(F \leq_B E)\}$.*

We call an elementary class closed under reduction an **elementary reducibility class**. In analogy with Theorem 2.1.2, we have the following characterization of elementary reducibility classes (see Corollary 2.5.18). Below by a **smooth Borel homomorphism** of $E \in \mathcal{E}$ into $F \in \mathcal{E}$ we mean a Borel homomorphism for which the preimage of any point is smooth for E.

Theorem 2.1.5. *A class $C \subseteq \mathcal{E}$ is an elementary reducibility class iff it is closed (downward) under smooth Borel homomorphisms and contains an invariantly universal element $E \in C$.*

We note that as a corollary of the proof of Theorem 2.1.4 every elementary reducibility class is also closed downward under class-injective Borel homomorphisms. Hjorth-Kechris [HK, D.3] proved (in our terminology and notation) that every C^r (C elementary) is closed under $\subseteq$, i.e., containment of equivalence relations on the same space. Since containment is a class-injective homomorphism (namely the identity), Theorem 2.1.4 generalizes this.

We also prove analogous results for Borel embeddability instead of Borel reducibility (see Theorem 2.5.1).

For each countably infinite group Γ denote by $\mathcal{E}_\Gamma$ the elementary class of equivalence relations induced by free Borel actions of Γ. Its invariantly universal element is the equivalence relation induced by the free part of the shift action of Γ on $\mathbb{R}^\Gamma$. For trivial reasons this is not closed under Borel reducibility, so let $\mathcal{E}_\Gamma^*$ be the elementary class of all equivalence relations whose aperiodic part is in $\mathcal{E}_\Gamma$. Then we have the following characterization (see Theorem 2.7.1).

Theorem 2.1.6. *Let Γ be a countably infinite group. Then the following are equivalent:*

(i) *$\mathcal{E}_\Gamma^*$ is an elementary reducibility class.*

(ii) *Γ is amenable.*

We call equivalence relations of the form $E_{\infty\sigma}$ **universally structurable**. Denote by $\mathcal{E}_\infty \subseteq \mathcal{E}$ the class of universally structurable equivalence relations. In view of

Theorem 2.1.1, an elementary class is uniquely determined by its invariantly universal such equivalence relation, and the poset of elementary classes under inclusion is isomorphic to the poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ of Borel isomorphism classes of universally structurable equivalence relations under invariant Borel embeddability. It turns out that this poset has desirable algebraic properties (see Theorem 2.6.2).

Theorem 2.1.7. *The poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ is an ω_1-complete, distributive lattice. Moreover, the inclusion $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i) \subseteq (\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$ preserves (countable) meets and joins.*

This has implications concerning the structure of the class of universally structurable equivalence relations under Borel reducibility. The order-theoretic structure of the poset $(\mathcal{E}/{\sim_B}, \leq_B)$ of *all* bireducibility classes under $\leq_B$ is not well-understood, apart from that it is very complicated (by [AK]). The first general study of this structure was made only recently by Kechris-Macdonald in [KMd]. In particular, they pointed out that it was even unknown whether there exists *any* pair of $\leq_B$-incomparable $E, F \in \mathcal{E}$ for which a $\leq_B$-meet exists. However it turns out that the subposet $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ behaves quite well (see Corollary 2.6.9).

Theorem 2.1.8. *The poset of universally structurable bireducibility classes, under $\leq_B$, $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ is an ω_1-complete, distributive lattice. Moreover, the inclusion into the poset $(\mathcal{E}/{\sim_B}, \leq_B)$ of all bireducibility classes, under $\leq_B$, preserves (countable) meets and joins.*

Adapting the method of Adams-Kechris [AK], we also show that this poset is quite rich (see Theorem 2.6.20).

Theorem 2.1.9. *There is an order-embedding from the poset of Borel subsets of $\mathbb{R}$ under inclusion into the poset $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$.*

The combination of Theorem 2.1.8 and Theorem 2.1.9 answers the question mentioned in the paragraph following Theorem 2.1.7 by providing a large class of $\leq_B$-incomparable countable Borel equivalence relations for which $\leq_B$-meets exist.

An important question concerning structurability is which properties of a theory σ yield properties of the corresponding elementary class $\mathcal{E}_\sigma$. The next theorem provides the first instance of such a result. Marks [M, end of Section 4.3] asked (in our terminology) for a characterization of when the elementary class $\mathcal{E}_{\sigma_\mathbb{A}}$, where $\sigma_\mathbb{A}$ is a Scott sentence of a countable structure, consists of smooth equivalence relations, or equivalently, when $E_{\infty\sigma_{\mathbb{A}}}$ is smooth. We answer this question in full generality, i.e., for an arbitrary theory σ. Although this result belongs purely in the category of Borel equivalence relations, our proof uses ideas and results from topological dynamics and ergodic theory (see Theorem 2.8.1).

Theorem 2.1.10. *Let σ be a theory. The following are equivalent:*

(i) $\mathcal{E}_\sigma$ contains only smooth equivalence relations, i.e., $E_{\infty\sigma}$ is smooth.

(ii) There is an $L_{\omega_1\omega}$-formula $\phi(x)$ which defines a finite nonempty subset in any countable model of σ.

Along these lines an interesting question is to find out what theories σ have the property that every aperiodic countable Borel equivalence relation is σ-structurable. A result that some particular σ axiomatizes all aperiodic E shows that every such $E \in \mathcal{E}$ carries a certain type of structure, which can be useful in applications. A typical example is the very useful Marker Lemma (see [BK, 4.5.3]), which shows that every aperiodic E admits a decreasing sequence of Borel complete sections $A_0 \supseteq A_1 \supseteq \cdots$ with empty intersection. This can be phrased as: every aperiodic countable Borel equivalence relation E is σ-structurable, where σ in the language $L = \{P_0, P_1, \dots\}$ asserts that each (unary) P_i defines a nonempty subset, $P_0 \supseteq P_1 \supseteq \cdots$, and $\bigcap_i P_i = \varnothing$.

A particular case is when $\sigma = \sigma_{\mathbb{A}}$ is a Scott sentence of a countable structure. For convenience we say that E is **$\mathbb{A}$-structurable** if E is $\sigma_{\mathbb{A}}$-structurable. Marks recently pointed out that the work of [AFP] implies a very general condition under which this happens (see Theorem 2.8.2).

Theorem 2.1.11 (Marks)**.** *Let $\mathbb{A}$ be a countable structure with trivial definable closure. Then every aperiodic countable Borel equivalence relation is $\mathbb{A}$-structurable.*

In particular (see Corollary 2.8.17) the following Fraïssé structures can structure every aperiodic countable Borel equivalence relation: $(\mathbb{Q}, <)$, the random graph, the random K_n-free graph (where K_n is the complete graph on n vertices), the random poset, and the rational Urysohn space.

Finally we mention two applications of the above results and ideas. The first (see Corollary 2.8.13) is a corollary of the proof of Theorem 2.1.10.

Theorem 2.1.12. *Let σ be a consistent theory in a language L such that the models of σ form a closed subspace of* $\mathrm{Mod}_{\mathbb{N}}(L)$*. Then for any countably infinite group Γ, there is a free Borel action of Γ which admits an invariant probability measure and is σ-structurable.*

The second application is to a model-theoretic question that has nothing to do with equivalence relations. The concept of amenability of a structure in the next result (see Corollary 2.8.18) can be either the one in [JKL, 2.16(iii)] or the one in [K91, 3.4]. This result was earlier proved by the authors by a different method (still using results of [AFP]) but it can also be seen as a corollary of Theorem 2.1.11.

Theorem 2.1.13. *Let $\mathbb{A}$ be a countably infinite amenable structure. Then $\mathbb{A}$ has non-trivial definable closure.*

(B) This paper is organized as follows: In Section 2.2 we review some basic background in the theory of Borel equivalence relations and model theory. In Section 2.3 we introduce the concept of structurability of equivalence relations and discuss various examples. In Section 2.4 we study the relationship between structurability and class-bijective homomorphisms, obtaining the tight correspondence given by Theorems 2.1.1 to 2.1.3; we also apply structurability to describe a product construction (class-bijective or "tensor" product) between countable Borel equivalence relations. In Section 2.5 we study the relationship between structurability and other kinds of homomorphisms, such as reductions; we also consider the relationship between reductions and compressible equivalence relations. In Section 2.6 we introduce some concepts from order theory convenient for describing the various posets of equivalence relations we are considering, and then study the poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq^i_B)$ of universally structurable equivalence relations (equivalently of elementary classes). In Section 2.7 we consider the elementary class $\mathcal{E}_\Gamma$ of equivalence relations induced by free actions of a countable group Γ. In Section 2.8 we consider relationships between model-theoretic properties of a theory σ and the corresponding elementary class $\mathcal{E}_\sigma$. Finally, in Section 2.9 we list several open problems related to structurability.

In the appendix, we introduce fiber spaces (previously considered in [G] and [HK]), which provide a slightly more general context for several concepts appearing in the body of this paper. We discuss the relationship between fiber spaces and equivalence relations, as well as the appropriate generalizations of structurability and the various kinds of homomorphisms.

Remark 2.1.14. In a preprint of this paper uploaded to the arXiv, we included two further appendices, with some miscellaneous concepts/results which are tangential to the main subject of this paper. The first of these concerns a categorical structure on the class of all theories which interacts well with structurability. The second contains a lattice-theoretic result which can be applied to the lattice $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ considered in Section 2.6.2.

Acknowledgments. We would like to thank Andrew Marks for many valuable suggestions and for allowing us to include Theorem 2.1.11 in this paper. We are also grateful to Anush Tserunyan for extensive comments and suggestions, including spotting and correcting an error in the original version of Lemma 2.8.3.

2.2 Preliminaries

For general model theory, see [Hod]. For general classical descriptive set theory, see [K95].

2.2.1 Theories and structures

By a **language**, we will always mean a countable first-order relational language, i.e., a countable set $L = \{R_i \mid i \in I\}$ of relation symbols, where each R_i has an associated arity $n_i \geq 1$. The only logic we will consider is the infinitary logic $L_{\omega_1\omega}$. We use letters like ϕ, ψ for formulas, and σ, τ for sentences. By a **theory**, we mean a pair (L, σ) where L is a language and σ is an $L_{\omega_1\omega}$-sentence. When L is clear from context, we will often write σ instead of (L, σ).

Let L be a language. By an L**-structure**, we mean in the usual sense of first-order logic, i.e., a tuple $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ where X is a set and for each n-ary relation symbol $R \in L$, $R^{\mathbb{A}} \subseteq X^n$ is an n-ary relation on X. Then as usual, for each formula $\phi(x_1, \dots, x_n) \in L_{\omega_1\omega}$ with n free variables, we have an interpretation $\phi^{\mathbb{A}} \subseteq X^n$ as an n-ary relation on X.

We write $\mathrm{Mod}_X(L)$ for the set of L-structures with universe X. More generally, for a theory (L, σ), we write $\mathrm{Mod}_X(\sigma)$ for the set of models of σ with universe X. When X is countable, we equip $\mathrm{Mod}_X(\sigma)$ with its usual standard Borel structure (see e.g., [K95, 16.C]).

If $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ is an L-structure and $f : X \to Y$ is a bijection, then we write $f(\mathbb{A})$ for the **pushforward structure**, with universe Y and

$$R^{f(\mathbb{A})}(\overline{y}) \iff R^{\mathbb{A}}(f^{-1}(\overline{y}))$$

for n-ary R and $\overline{y} \in Y^n$. When $X = Y$, this defines the **logic action** of S_X (the group of bijections of X) on $\mathrm{Mod}_X(L, \sigma)$.

If $f : Y \to X$ is any function, then $f^{-1}(\mathbb{A})$ is the **pullback structure**, with universe Y and

$$R^{f^{-1}(\mathbb{A})}(\overline{y}) \iff R^{\mathbb{A}}(f(\overline{y})).$$

When f is the inclusion of a subset $Y \subseteq X$, we also write $\mathbb{A}|Y$ for $f^{-1}(\mathbb{A})$.

Every countable L-structure $\mathbb{A}$ has a **Scott sentence** $\sigma_{\mathbb{A}}$, which is an $L_{\omega_1\omega}$-sentence whose countable models are exactly the isomorphic copies of $\mathbb{A}$; e.g., see [Bar, §VII.6].

A **Borel class of L-structures** is a class $\mathcal{K}$ of *countable* L-structures which is closed under isomorphism and such that $\mathcal{K} \cap \mathrm{Mod}_X(L)$ is a Borel subset of $\mathrm{Mod}_X(L)$ for every countable set X (equivalently, for $X \in \{1, 2, \dots, \mathbb{N}\}$). For example, for any $L_{\omega_1\omega}$-sentence σ, the class of countable models of σ is Borel. By a classical theorem of Lopez-Escobar [LE], every Borel class $\mathcal{K}$ of L-structures is of this form, for some σ. (While Lopez-Escobar's theorem is usually stated only for $\mathrm{Mod}_{\mathbb{N}}(L)$, it is easily seen to hold also for $\mathrm{Mod}_X(L)$ with X finite.)

2.2.2 Countable Borel equivalence relations

A **Borel equivalence relation** E on a standard Borel space X is an equivalence relation which is Borel as a subset of X^2; the equivalence relation E is **countable** if each of its classes is. We will also refer to the pair (X, E) as an equivalence relation.

Throughout this paper, we use $\mathcal{E}$ to denote the class of countable Borel equivalence relations (X, E).

If Γ is a group acting on a set X, then we let $E_\Gamma^X \subseteq X^2$ be the **orbit equivalence relation**:

$$x \, E_\Gamma^X \, y \iff \exists \gamma \in \Gamma \, (\gamma \cdot x = y).$$

If Γ is countable, X is standard Borel, and the action is Borel, then E_Γ^X is a countable Borel equivalence relation. Conversely, by the Feldman-Moore Theorem [FM], every countable Borel equivalence relation on a standard Borel space X is E_Γ^X for some countable group Γ with some Borel action on X.

If Γ is a group and X is a set, the **(right) shift action** of Γ on X^Γ is given by

$$(\gamma \cdot \overline{x})(\delta) := \overline{x}(\delta\gamma)$$

for $\gamma \in \Gamma$, $\overline{x} \in X^\Gamma$, and $\delta \in \Gamma$. We let $E(\Gamma, X) := E_\Gamma^{X^\Gamma} \subseteq (X^\Gamma)^2$ denote the orbit equivalence of the shift action. If Γ is countable and X is standard Borel, then $E(\Gamma, X)$ is a countable Borel equivalence relation. If Γ already acts on X, then that action embeds into the shift action, via

$$\begin{aligned} X &\longrightarrow X^\Gamma \\ x &\longmapsto (\gamma \mapsto \gamma \cdot x). \end{aligned}$$

In particular, any action of Γ on a standard Borel space embeds into the shift action of Γ on $\mathbb{R}^\Gamma$.

The **free part** of a group action of Γ on X is

$$\{x \in X \mid \forall 1 \neq \gamma \in \Gamma\, (\gamma \cdot x \neq x)\};$$

the action is **free** if the free part is all of X. We let $F(\Gamma, X)$ denote the orbit equivalence of the free part of the shift action of Γ on X^Γ.

An **invariant measure** for a Borel group action of Γ on X is a nonzero σ-finite Borel measure μ on X such that $\gamma_* \mu = \mu$ for all $\gamma \in \Gamma$ (where $\gamma_* \mu$ is the pushforward). An **invariant measure** on a countable Borel equivalence relation (X, E) is an invariant measure for some Borel action of a countable group Γ on X which generates E, or equivalently for any such action (see [KM, 2.1]). An invariant measure μ on (X, E) is **ergodic** if for any E-invariant Borel set $A \subseteq X$, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

2.2.3 Homomorphisms

Let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations, and let $f : X \to Y$ be a Borel map (we write $f : X \to_B Y$ to denote that f is Borel). We say that f is:

- a **homomorphism**, written $f : (X, E) \to_B (Y, F)$, if

  $$\forall x, y \in X\, (x \mathrel{E} y \implies f(x) \mathrel{F} f(y)),$$

  i.e., f induces a map on the quotient spaces $X/E \to Y/F$;

- a **reduction**, written $f : (X, E) \leq_B (Y, F)$, if f is a homomorphism and

  $$\forall x, y \in X\, (f(x) \mathrel{F} f(y) \implies x \mathrel{E} y),$$

  i.e., f induces an injection on the quotient spaces;

- a **class-injective homomorphism** (respectively, **class-surjective**, **class-bijective**), written $f : (X, E) \to_B^{ci} (Y, F)$ (respectively $f : (X, E) \to_B^{cs} (Y, F)$, $f : (X, E) \to_B^{cb} (Y, F)$), if f is a homomorphism such that for each $x \in X$, the restriction $f|[x]_E : [x]_E \to [f(x)]_F$ to the equivalence class of x is injective (respectively, surjective, bijective);

- an **embedding**, written $f : (X, E) \sqsubseteq_B (Y, F)$, if f is an injective (or equivalently, class-injective) reduction;

- an **invariant embedding**, written $f : (X, E) \sqsubseteq_B^i (Y, F)$, if f is a class-bijective reduction, or equivalently an embedding such that the image $f(X) \subseteq Y$ is F-invariant.

Among these various kinds of homomorphisms, the reductions have received the most attention in the literature, while the class-bijective ones are most closely related to the notion of structurability. Here is a picture of the containments between these classes of homomorphisms, with the more restrictive classes at the bottom:

$$\begin{tikzcd}
 & & \to_B \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
\leq_B \arrow[dr, dash] & & \to_B^{ci} \arrow[dl, dash] \arrow[dr, dash] & & \to_B^{cs} \arrow[dl, dash] \\
 & \sqsubseteq_B \arrow[dr, dash] & & \to_B^{cb} \arrow[dl, dash] & \\
 & & \sqsubseteq_B^i & &
\end{tikzcd}$$

We say that (X, E) **(Borel) reduces** to (Y, F), written $(X, E) \leq_B (Y, F)$ (or simply $E \leq_B F$), if there is a Borel reduction $f : (X, E) \leq_B (Y, F)$. Similarly for the other kinds of homomorphisms, e.g., E **embeds** into F, written $E \sqsubseteq_B F$, if there is some $f : E \sqsubseteq_B F$, etc. We also write:

- $E \sim_B F$ (E is **bireducible** to F) if $E \leq_B F$ and $F \leq_B E$;

- $E <_B F$ if $E \leq_B F$ and $F \not\leq_B E$, and similarly for $\sqsubset_B$ and $\sqsubset_B^i$;

- $E \leftrightarrow_B^{cb} F$ (E is **class-bijectively equivalent** to F) if $E \to_B^{cb} F$ and $F \to_B^{cb} E$;

- $E \cong_B F$ if E is Borel isomorphic to F, or equivalently (by the Borel Schröder-Bernstein theorem) $E \sqsubseteq_B^i F$ and $F \sqsubseteq_B^i E$.

Clearly $\le_B$, $\sqsubseteq_B$, $\to_B^{cb}$, etc., are preorders on the class $\mathcal{E}$, and $\sim_B$, $\leftrightarrow_B^{cb}$, $\cong_B$ are equivalence relations on $\mathcal{E}$. The $\sim_B$-equivalence classes are called **bireducibility classes**, etc.

2.2.4 Basic operations

We have the following basic operations on Borel equivalence relations. Let (X, E), (Y, F) be Borel equivalence relations.

Their **disjoint sum** is $(X, E) \oplus (Y, F) = (X \oplus Y, E \oplus F)$ where $X \oplus Y$ is the disjoint union of X, Y, and $E \oplus F$ relates elements of X according to E and elements of Y according to F and does not relate elements of X with elements of Y. The canonical injections $\iota_1 : X \to_B X \oplus Y$ and $\iota_2 : Y \to_B X \oplus Y$ are then invariant embeddings $E, F \sqsubseteq_B^i E \oplus F$. Clearly the disjoint sum of countable equivalence relations is countable. We have obvious generalizations to disjoint sums of any countable family of equivalence relations.

Their **cross product** is $(X, E) \times (Y, F) = (X \times Y, E \times F)$, where

$$(x, y)\ (E \times F)\ (x', y') \iff x\ E\ x' \ \&\ y\ F\ y'.$$

(The "cross" adjective is to disambiguate from the *tensor products* to be introduced in Section 2.4.4.) The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are class-surjective homomorphisms $E \times F \to_B^{cs} E, F$. Cross products also generalize to countably many factors; but note that only *finite* cross products of countable equivalence relations are countable.

2.2.5 Special equivalence relations

Recall that an equivalence relation (X, E) is **countable** if each E-class is countable; similarly, it is **finite** if each E-class is finite, and **aperiodic countable** if each E-class is countably infinite. A countable Borel equivalence relation is always the disjoint sum of a finite Borel equivalence relation and an aperiodic countable Borel equivalence relation. Since many of our results become trivial when all classes are finite, we will often assume that our equivalence relations are aperiodic.

For any set X, the **indiscrete** equivalence relation on X is $I_X := X \times X$.

A Borel equivalence relation (X, E) is **smooth** if $E \le_B \Delta_Y$ where Δ_Y is the equality relation on some standard Borel space Y. When E is countable, this is equivalent to E having a Borel **transversal**, i.e., a Borel set $A \subseteq X$ meeting each E-class exactly once, or a Borel **selector**, i.e., a Borel map $f : X \to_B X$ such that $x\ E\ f(x)$ and

$x \; E \; y \implies f(x) = f(y)$ for all $x, y \in X$. Any finite Borel equivalence relation is smooth. Up to bireducibility, the smooth Borel equivalence relations consist exactly of

$$\Delta_0 <_B \Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_{\mathbb{N}} <_B \Delta_{\mathbb{R}};$$

and these form an initial segment of the preorder $(\mathcal{E}, \leq_B)$ (Silver's dichotomy; see [MK, 9.1.1]).

We will sometimes use the standard fact that a countable Borel equivalence relation (X, E) is smooth iff every ergodic invariant (σ-finite Borel) measure on E is atomic. (For the converse direction, use e.g., that if E is not smooth, then $E_t \sqsubseteq_B^i E$ (see Theorem 2.2.1 below), and E_t is isomorphic to the orbit equivalence of the translation action of $\mathbb{Q}$ on $\mathbb{R}$, which admits Lebesgue measure as an ergodic invariant nonatomic σ-finite measure.)

If $f : X \to Y$ is any function between sets, the **kernel** of f is the equivalence relation $\ker f$ on X given by $x \; (\ker f) \; y \iff f(x) = f(y)$. So a Borel equivalence relation is smooth iff it is the kernel of some Borel map.

A countable Borel equivalence relation E is **universal** if E is $\leq_B$-greatest in $\mathcal{E}$, i.e., for any other countable Borel equivalence relation F, we have $F \leq_B E$. An example is $E(\mathbb{F}_2, 2)$ (where $\mathbb{F}_2$ is the free group on 2 generators) [DJK, 1.8]. Note that by [MSS, 3.6], E is universal iff it is $\sqsubseteq_B$-greatest in $\mathcal{E}$, i.e., for any other $F \in \mathcal{E}$, we have $F \sqsubseteq_B E$.

A countable Borel equivalence relation E is **invariantly universal** if E is $\sqsubseteq_B^i$-greatest in $\mathcal{E}$, i.e., for any other countable Borel equivalence relation F, we have $F \sqsubseteq_B^i E$. We denote by E_∞ any such E; in light of the Borel Schröder-Bernstein theorem, E_∞ is unique up to isomorphism. Clearly E_∞ is also $\leq_B$-universal. (Note: in the literature, E_∞ is commonly used to denote any $\leq_B$-universal countable Borel equivalence relation (which is determined only up to bireducibility).) One realization of E_∞ is $E(\mathbb{F}_\omega, \mathbb{R})$. (This follows from the Feldman-Moore Theorem.)

A (countable) Borel equivalence relation (X, E) is **hyperfinite** if E is the increasing union of a sequence of finite Borel equivalence relations on X. We will use the following facts (see [DJK, 5.1, 7.2, 9.3]):

Theorem 2.2.1. *Let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations.*

(a) E is hyperfinite iff $E = E_{\mathbb{Z}}^X$ for some action of $\mathbb{Z}$ on X.

(b) *E is hyperfinite iff there is a Borel binary relation $<$ on X such that on each E-class, $<$ is a linear order embeddable in $(\mathbb{Z}, <)$.*

(c) *If E, F are both hyperfinite and non-smooth, then $E \sqsubseteq_B F$. Thus there is a unique bireducibility (in fact biembeddability) class of non-smooth hyperfinite Borel equivalence relations.*

(d) *Let E_0, E_t be the equivalence relations on $2^{\mathbb{N}}$ given by*

$$\begin{aligned} x \; E_0 \; y &\iff \exists i \in \mathbb{N} \, \forall j \in \mathbb{N} \, (x(i+j) = y(i+j)), \\ x \; E_t \; y &\iff \exists i, j \in \mathbb{N} \, \forall k \in \mathbb{N} \, (x(i+k) = y(j+k)). \end{aligned}$$

Up to isomorphism, the non-smooth, aperiodic, hyperfinite Borel equivalence relations are

$$E_t \sqsubset_B^i E_0 \sqsubset_B^i 2 \cdot E_0 \sqsubset_B^i 3 \cdot E_0 \sqsubset_B^i \cdots \sqsubset_B^i \aleph_0 \cdot E_0 \sqsubset_B^i 2^{\aleph_0} \cdot E_0,$$

where $n \cdot E_0 := \Delta_n \times E_0$. Each $n \cdot E_0$ has exactly n ergodic invariant probability measures.

(e) *(Glimm-Effros dichotomy) E is not smooth iff $E_t \sqsubseteq_B^i E$.*

A countable Borel equivalence relation (X, E) is **compressible** if there is a $f : E \sqsubseteq_B E$ such that $f(C) \subsetneq C$ for every E-class $C \in X/E$. The basic example is $I_{\mathbb{N}}$; another example is E_t. A fundamental theorem of Nadkarni [N] asserts that E is compressible iff it does not admit an invariant probability measure. For more on compressibility, see [DJK, Section 2]; we will use the results therein extensively in Section 2.5.4.

A countable Borel equivalence relation (X, E) is **treeable** if E is generated by an acyclic Borel graph on X. For properties of treeability which we use later on, see [JKL, Section 3].

2.2.6 Fiber products

Let $(X, E), (Y, F), (Z, G)$ be Borel equivalence relations, and let $f : (Y, F) \to_B (X, E)$ and $g : (Z, G) \to_B (X, E)$ be homomorphisms. The **fiber product** of F and G (with respect to f and g) is $(Y, F) \times_{(X,E)} (Z, G) = (Y \times_X Z, F \times_E G)$, where

$$Y \times_X Z := \{(y, z) \in Y \times Z \mid f(y) = g(z)\}, \qquad F \times_E G := (F \times G)|(Y \times_X Z).$$

(Note that the notations $Y \times_X Z$, $F \times_E G$ are slight abuses of notation in that they hide the dependence on the maps f, g.) The projections $\pi_1 : F \times_E G \to F$ and

$\pi_2 : F \times_E G \to G$ fit into a commutative diagram:

$$\begin{tikzcd} F \times_E G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G \arrow[d, "g"] \\ F \arrow[r, "f"'] & E \end{tikzcd}$$

It is easily verified that if g is class-injective, class-surjective, or a reduction, then so is π_1.

2.2.7 Some categorical remarks

For each of the several kinds of homomorphisms mentioned in Section 2.2.3, we have a corresponding category of countable Borel equivalence relations and homomorphisms of that kind. We use, e.g., $(\mathcal{E}, \to_B^{cb})$ to denote the category of countable Borel equivalence relations and class-bijective homomorphisms, etc.

(Depending on context, we also use $(\mathcal{E}, \to_B^{cb})$ to denote the preorder $\to_B^{cb}$ on $\mathcal{E}$, i.e., the preorder gotten by collapsing all morphisms in the category $(\mathcal{E}, \to_B^{cb})$ between the same two objects.)

From a categorical standpoint, among these categories, the two most well-behaved ones seem to be $(\mathcal{E}, \to_B)$ and $(\mathcal{E}, \to_B^{cb})$. The latter will be treated in Sections 2.4.4 and 2.4.5. As for $(\mathcal{E}, \to_B)$, we note that (countable) disjoint sums, (finite) cross products, and fiber products give respectively coproducts, products, and pullbacks in that category. It follows that $(\mathcal{E}, \to_B)$ is finitely complete, i.e., has all finite categorical limits (see e.g., [ML, V.2, Exercise III.4.10]).

Remark 2.2.2. However, $(\mathcal{E}, \to_B)$ does not have coequalizers. Let E_0 on $2^{\mathbb{N}}$ be generated by a Borel automorphism $T : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Then it is easy to see that $T : (2^{\mathbb{N}}, \Delta_{2^{\mathbb{N}}}) \to_B (2^{\mathbb{N}}, \Delta_{2^{\mathbb{N}}})$ and the identity map do not have a coequalizer.

For later reference, let us note that the category of (not necessarily countable) Borel equivalence relations and Borel homomorphisms has inverse limits of countable chains. That is, for each $n \in \mathbb{N}$, let (X_n, E_n) be a Borel equivalence relation, and $f_n : E_{n+1} \to_B E_n$ be a Borel homomorphism. Then the **inverse limit** of the system is $\varprojlim_n (X_n, E_n) = (\varprojlim_n X_n, \varprojlim_n E_n)$, where

$$\varprojlim_n X_n := \{\overline{x} = (x_0, x_1, \dots) \in \prod_n X_n \mid \forall n\, (x_n = f_n(x_{n+1}))\},$$
$$\varprojlim_n E_n := \prod_n E_n | \varprojlim_n X_n.$$

It is easily seen that $\varprojlim_n E_n$ together with the projections $\pi_m : \varprojlim_n E_n \to_B E_m$ has the universal property of an inverse limit, i.e., for any other Borel equivalence relation (Y, F) and homomorphisms $g_m : F \to_B E_m$ such that $g_m = f_m \circ g_{m+1}$ for each m, there is a unique homomorphism $\widetilde{g} : F \to_B \varprojlim_n E_n$ such that $\pi_m \circ \widetilde{g} = g_m$ for each m. This is depicted in the following commutative diagram:

$$\begin{tikzcd}
F \arrow[r, dashed, "\widetilde{g}"] \arrow[dr, "g_2"'] \arrow[drr, "g_1"] \arrow[drrr, "g_0"] & \varprojlim_n E_n \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1"] \arrow[drr, "\pi_0"] & & \\
\cdots \arrow[r, "f_2"'] & E_2 \arrow[r, "f_1"'] & E_1 \arrow[r, "f_0"'] & E_0
\end{tikzcd}$$

It follows that the category of Borel equivalence relations and Borel homomorphisms is countably complete, i.e., has all limits of countable diagrams (again see [ML, V.2, Exercise III.4.10]).

2.3 Structures on equivalence relations

We now define the central notion of this paper.

Let L be a language and X be a standard Borel space. We say that an L-structure $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ with universe X is **Borel** if $R^{\mathbb{A}} \subseteq X^n$ is Borel for each n-ary $R \in L$.

Now let (X, E) be a countable Borel equivalence relation. We say that a Borel L-structure $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ is a **Borel L-structure on** E if for each n-ary $R \in L$, $R^{\mathbb{A}}$ only relates elements within the same E-class, i.e.,

$$R^{\mathbb{A}}(x_1, \dots, x_n) \implies x_1 \mathrel{E} x_2 \mathrel{E} \cdots \mathrel{E} x_n.$$

For an $L_{\omega_1\omega}$-sentence σ, we say that $\mathbb{A}$ is a **Borel σ-structure on** E, written

$$\mathbb{A} : E \models \sigma,$$

if for each E-class $C \in X/E$, the structure $\mathbb{A}|C$ satisfies σ. We say that E is **σ-structurable**, written

$$E \models \sigma,$$

if there is some Borel σ-structure on E. Similarly, if $\mathcal{K}$ is a Borel class of L-structures, we say that $\mathbb{A}$ is a **Borel $\mathcal{K}$-structure on** E if $\mathbb{A}|C \in \mathcal{K}$ for each $C \in X/E$, and that E is **$\mathcal{K}$-structurable** if there is some Borel $\mathcal{K}$-structure on E. Note that E is $\mathcal{K}$-structurable iff it is σ-structurable, for any $L_{\omega_1\omega}$-sentence σ axiomatizing $\mathcal{K}$.

We let

$$\mathcal{E}_\sigma \subseteq \mathcal{E}, \qquad\qquad \mathcal{E}_\mathcal{K} \subseteq \mathcal{E}$$

denote respectively the classes of σ-structurable and $\mathcal{K}$-structurable countable Borel equivalence relations. For any class $C \subseteq \mathcal{E}$ of countable Borel equivalence relations, we say that C is **elementary** if $C = \mathcal{E}_\sigma$ for some theory (L, σ), in which case we say that (L, σ) **axiomatizes** C.

2.3.1 Examples of elementary classes

Several notions of "sufficiently simple" countable Borel equivalence relations which have been considered in the literature are given by an elementary class.

For example, a countable Borel equivalence relation E is smooth iff E is structurable by pointed sets (i.e., sets with a distinguished element). By Theorem 2.2.1, E is hyperfinite iff E is structurable by linear orders that embed in $\mathbb{Z}$. Hyperfiniteness can also be axiomatized by the sentence in the language $L = \{R_0, R_1, R_2, \dots\}$ which asserts that each R_i is a finite equivalence relation and $R_0 \subseteq R_1 \subseteq \cdots$ with union the indiscrete equivalence relation. Similarly, it is straightforward to verify that for each $\alpha < \omega_1$, α-Fréchet-amenability (see [JKL, 2.11–12]) is axiomatizable. Also, E is compressible iff it is structurable via structures in the language $L = \{R\}$ where R is the graph of a non-surjective injection.

For some trivial examples: every E is σ-structurable for logically valid σ, or for the (non-valid) sentence σ in the language $L = \{R_0, R_1, \dots\}$ asserting that the R_i's form a separating family of unary predicates (i.e., $\forall x, y\, (\bigwedge_i (R_i(x) \leftrightarrow R_i(y)) \leftrightarrow x = y)$); thus $\mathcal{E}$ is elementary. The class of aperiodic countable Borel equivalence is axiomatized by the theory of infinite sets, etc.

Let $\mathcal{T}_1$ denote the class of trees (i.e., acyclic connected graphs), and more generally, $\mathcal{T}_n$ denote the class of contractible n-dimensional (abstract) simplicial complexes. Then E is $\mathcal{T}_1$-structurable iff E is treeable. Gaboriau [G] has shown that $\mathcal{E}_{\mathcal{T}_1} \subsetneq \mathcal{E}_{\mathcal{T}_2} \subsetneq \cdots$.

For any language L and countable L-structure $\mathbb{A}$, if $\sigma_\mathbb{A}$ denotes the Scott sentence of $\mathbb{A}$, then E is $\sigma_\mathbb{A}$-structurable iff it is structurable via isomorphic copies of $\mathbb{A}$. For example, if $L = \{<\}$ and $(X, \mathbb{A}) = (\mathbb{Z}, <)$, then E is $\sigma_\mathbb{A}$-structurable iff it is aperiodic hyperfinite. We write

$$\mathcal{E}_\mathbb{A} := \mathcal{E}_{\sigma_\mathbb{A}}$$

for the class of $\mathbb{A}$**-structurable** countable Borel equivalence relations.

Let Γ be a countable group, and regard Γ as a structure in the language $L_\Gamma = \{R_\gamma \mid \gamma \in \Gamma\}$, where R_γ^Γ is the graph of the map $\delta \mapsto \gamma \cdot \delta$. Then a model of σ_Γ is a Γ-action isomorphic to Γ, i.e., a free transitive Γ-action. Thus a countable Borel equivalence relation E is Γ-structurable (i.e., σ_Γ-structurable) iff it is generated by a free Borel action of Γ.

Finally, we note that several important classes of countable Borel equivalence relations are *not* elementary. This includes all classes of "sufficiently complex" equivalence relations, such as (invariantly) universal equivalence relations, non-smooth equivalence relations, and equivalence relations admitting an invariant probability measure; these classes are not elementary by Proposition 2.3.1. Another example of a different flavor is the class of equivalence relations generated by a free action of *some* countable group; more generally, nontrivial unions of elementary classes are never elementary (see Corollary 2.4.5).

2.3.2 Classwise pullback structures

Let (X, E), (Y, F) be countable Borel equivalence relations and $f : E \to_B^{cb} F$ be a class-bijective homomorphism. For an L-structure $\mathbb{A}$ on F, recall that the pullback structure of $\mathbb{A}$ along f, denoted $f^{-1}(\mathbb{A})$, is the L-structure with universe X given by

$$R^{f^{-1}(\mathbb{A})}(\overline{x}) \iff R^{\mathbb{A}}(f(\overline{x}))$$

for each n-ary $R \in L$ and $\overline{x} \in X^n$. Let $f_E^{-1}(\mathbb{A})$ denote the **classwise pullback structure**, given by

$$R^{f_E^{-1}(\mathbb{A})}(\overline{x}) \iff R^{\mathbb{A}}(f(\overline{x})) \;\&\; x_1 \mathrel{E} \cdots \mathrel{E} x_n.$$

Then $f_E^{-1}(\mathbb{A})$ is a Borel L-structure on E, such that for each E-class $C \in X/E$, the restriction $f|C : C \to f(C)$ is an isomorphism between $f_E^{-1}(\mathbb{A})|C$ and $\mathbb{A}|f(C)$. In particular, if $\mathbb{A}$ is a σ-structure for some $L_{\omega_1\omega}$-sentence σ, then so is $f_E^{-1}(\mathbb{A})$. We record the consequence of this simple observation for structurability:

Proposition 2.3.1. *Every elementary class $\mathcal{E}_\sigma \subseteq \mathcal{E}$ is (downwards-)closed under class-bijective homomorphisms, i.e., if $E \to_B^{cb} F$ and $F \in \mathcal{E}_\sigma$, then $E \in \mathcal{E}_\sigma$.*

This connection between structurability and class-bijective homomorphisms will be significantly strengthened in the next section.

2.4 Basic universal constructions

In this section we present the two main constructions relating structures on equivalence relations to class-bijective homomorphisms. Both are "universal" constructions: the first turns any theory (L, σ) into a universal equivalence relation with a σ-structure, while the second turns any equivalence relation into a universal theory.

2.4.1 The universal σ-structured equivalence relation

Kechris-Solecki-Todorcevic [KST, 7.1] proved a universality result for graphs, which was then extended to arbitrary Borel classes of structures by Miller. Here, we formulate a version of this result and its proof that allows us to capture more information.

Theorem 2.4.1. *Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation and (L, σ) be a theory. Then there is a "universal σ-structured equivalence relation lying over E", i.e., a triple $(E \ltimes \sigma, \pi, \mathbb{E})$ where*

$$E \ltimes \sigma \in \mathcal{E}, \qquad \pi : E \ltimes \sigma \to_B^{cb} E, \qquad \mathbb{E} : E \ltimes \sigma \models \sigma,$$

such that for any other $F \in \mathcal{E}$ with $f : F \to_B^{cb} E$ and $\mathbb{A} : F \models \sigma$, there is a unique *class-bijective homomorphism $\widetilde{f} : F \to_B^{cb} E \ltimes \sigma$ such that $f = \pi \circ \widetilde{f}$ and $\mathbb{A} = \widetilde{f}_F^{-1}(\mathbb{E})$. This is illustrated by the following "commutative" diagram:*

$$\begin{tikzcd}
F \arrow[dr, dashed, "\widetilde{f}"'] \arrow[drr, equal, "\mathbb{A}"] \arrow[ddr, bend right, "f"'] & & \\
& E \ltimes \sigma \arrow[r, equal, "\mathbb{E}"'] \arrow[d, "\pi"] & \sigma \\
& E &
\end{tikzcd}$$

Proof. First we describe $E \ltimes \sigma$ while ignoring all questions of Borelness, then we verify that the construction can be made Borel.

Ignoring Borelness, $E \ltimes \sigma$ will live on a set Z and will have the following form: for each E-class $C \in X/E$, and each σ-structure $\mathbb{B}$ on the universe C, there will be one $(E \ltimes \sigma)$-class lying over C (i.e., projecting to C via π), which will have the σ-structure given by pulling back $\mathbb{B}$. Thus we put

$$Z := \{(x, \mathbb{B}) \mid x \in X,\ \mathbb{B} \in \mathrm{Mod}_{[x]_E}(\sigma)\},$$

$$(x, \mathbb{B})\ (E \ltimes \sigma)\ (x', \mathbb{B}') \iff x\ E\ x' \ \&\ \mathbb{B} = \mathbb{B}',$$

$$\pi(x, \mathbb{B}) := x,$$

with the σ-structure $\mathbb{E}$ on $E \ltimes \sigma$ given by

$$R^{\mathbb{E}}((x_1, \mathbb{B}), \ldots, (x_n, \mathbb{B})) \iff R^{\mathbb{B}}(x_1, \ldots, x_n)$$

for n-ary $R \in L$, $x_1 \ E \cdots E \ x_n$, and $\mathbb{B} \in \mathrm{Mod}_{[x_1]_E}(\sigma)$. It is immediate that π is class-bijective and that $\mathbb{E}$ satisfies σ. The universal property is also straightforward: given (Y, F), f, $\mathbb{A}$ as above, the map $\widetilde{f}$ is given by

$$\widetilde{f}(y) := (f(y), f(\mathbb{A}|[y]_F)) \in Z,$$

and this choice is easily seen to be unique by the requirements $f = \pi \circ \widetilde{f}$ and $\mathbb{A} = \widetilde{f}_F^{-1}(\mathbb{E})$.

Now we indicate how to make this construction Borel. The only obstruction is the use of $\mathrm{Mod}_{[x]_E}(\sigma)$ which depends on x in the definition of Z above. We restrict to the case where E is aperiodic; in general, we may split E into its finite part and aperiodic part, and it will be clear that the finite case can be handled similarly. In the aperiodic case, the idea is to replace $\mathrm{Mod}_{[x]_E}(\sigma)$ with $\mathrm{Mod}_{\mathbb{N}}(\sigma)$, where $[x]_E$ is identified with $\mathbb{N}$ but in a manner which varies depending on x.

Let $T : X \to X^{\mathbb{N}}$ be a Borel map such that each $T(x)$ is a bijection $\mathbb{N} \to [x]_E$ (the existence of such T is easily seen from Lusin-Novikov uniformization), and replace $\mathrm{Mod}_{[x]_E}(\sigma)$ with $\mathrm{Mod}_{\mathbb{N}}(\sigma)$ while inserting $T(x)$ into the appropriate places in the above definitions:

$$\begin{aligned}
Z &:= \{(x, \mathbb{B}) \mid x \in X,\ \mathbb{B} \in \mathrm{Mod}_{\mathbb{N}}(\sigma)\} = X \times \mathrm{Mod}_{\mathbb{N}}(\sigma), \\
(x, \mathbb{B}) \ (E \ltimes \sigma) \ (x', \mathbb{B}') &\iff x \ E \ x' \ \ \& \ \ T(x)(\mathbb{B}) = T(x')(\mathbb{B}'), \\
R^{\mathbb{E}}((x_1, \mathbb{B}_1), \ldots, (x_n, \mathbb{B}_n)) &\iff R^{T(x_1)(\mathbb{B}_1)}(x_1, \ldots, x_n), \\
\widetilde{f}(y) &:= (f(y), T(f(y))^{-1}(f(\mathbb{A}|[y]_F))).
\end{aligned}$$

These are easily seen to be Borel and still satisfy the requirements of the theorem. □

Remark 2.4.2. It is clear that $E \ltimes \sigma$ satisfies a universal property in the formal sense of category theory. This in particular means that $(E \ltimes \sigma, \pi, \mathbb{E})$ is unique up to *unique* (Borel) isomorphism.

Remark 2.4.3. The construction of $E \ltimes \sigma$ for aperiodic E in the proof of Theorem 2.4.1 can be seen as an instance of the following general notion (see e.g., [K10, 10.E]):

Let (X, E) be a Borel equivalence relation, and let Γ be a (Borel) group. Recall that a **Borel cocycle** $\alpha : E \to \Gamma$ is a Borel map satisfying $\alpha(y, z)\alpha(x, y) = \alpha(x, z)$ for all $x, y, z \in X$, $x \mathrel{E} y \mathrel{E} z$. Given a cocycle α and a Borel action of Γ on a standard Borel space Y, the **skew product** $E \ltimes_\alpha Y$ is the Borel equivalence relation on $X \times Y$ given by

$$(x, y) \ (E \ltimes_\alpha Y) \ (x', y') \iff x \mathrel{E} x' \ \& \ \alpha(x, x') \cdot y = y'.$$

Note that for such a skew product, the first projection $\pi_1 : X \times Y \to X$ is always a class-bijective homomorphism $E \ltimes_\alpha Y \to_B^{cb} E$.

Now given a family $T : X \to X^{\mathbb{N}}$ of bijections $\mathbb{N} \cong_B [x]_E$, as in the proof of Theorem 2.4.1, we call $\alpha_T : E \to S_\infty$ given by $\alpha_T(x, x') := T(x')^{-1} \circ T(x)$ the **cocycle induced by** T. Then the construction of $E \ltimes \sigma$ for aperiodic E can be seen as the skew product $E \ltimes_{\alpha_T} \mathrm{Mod}_{\mathbb{N}}(\sigma)$ (with the logic action of S_∞ on $\mathrm{Mod}_{\mathbb{N}}(\sigma)$). (However, the structure $\mathbb{E}$ on $E \ltimes_{\alpha_T} \mathrm{Mod}_{\mathbb{N}}(\sigma)$ depends on T, not just on α_T.)

Theorem 2.4.1 has the following consequence:

Corollary 2.4.4 (Kechris-Solecki-Todorcevic, Miller)**.** *For every theory (L, σ), there is an invariantly universal σ-structurable countable Borel equivalence relation $E_{\infty\sigma}$, i.e., $E_{\infty\sigma} \models \sigma$, and $F \sqsubseteq_B^i E_{\infty\sigma}$ for any other $F \models \sigma$.*

Proof. Put $E_{\infty\sigma} := E_\infty \ltimes \sigma$. For any $F \models \sigma$, we have an invariant embedding $f : F \sqsubseteq_B^i E_\infty$, whence there is $\widetilde{f} : F \to_B^{cb} E_\infty \ltimes \sigma = E_{\infty\sigma}$ such that $f = \pi \circ \widetilde{f}$; since f is injective, so is $\widetilde{f}$. □

In other words, every elementary class $\mathcal{E}_\sigma$ of countable Borel equivalence relations has an invariantly universal element $E_{\infty\sigma}$ (which is unique up to isomorphism). For a Borel class of structures $\mathcal{K}$, we denote the invariantly universal $\mathcal{K}$-structurable equivalence relation by $E_{\infty\mathcal{K}}$. For an L-structure $\mathbb{A}$, we denote the invariantly universal $\mathbb{A}$-structurable equivalence relation by $E_{\infty\mathbb{A}}$.

As a basic application, we can now rule out the elementarity of a class of equivalence relations mentioned in Section 2.3.1:

Corollary 2.4.5. *If $(C_i)_{i\in I}$ is a collection of elementary classes of countable Borel equivalence relations, then $\bigcup_i C_i$ is* not *elementary, unless there is some j such that $\bigcup_i C_i = C_j$.*

In particular, the class of equivalence relations generated by a free Borel action of some countable group is not *elementary.*

Proof. If $\bigcup_i \mathcal{C}_i$ is elementary, then it has an invariantly universal element E, which is in some $\mathcal{C}_j$; then for every i and $F \in \mathcal{C}_i$, we have $F \sqsubseteq_B^i E \in \mathcal{C}_j$, whence $F \in \mathcal{C}_j$ since $\mathcal{C}_j$ is elementary.

For the second statement, the class in question is $\bigcup_\Gamma \mathcal{E}_\Gamma$ where Γ ranges over countable groups (and $\mathcal{E}_\Gamma$ is the class of equivalence relations generated by a free Borel action of Γ); and there cannot be a single $\mathcal{E}_\Gamma$ which contains all others, since if Γ is amenable then $\mathcal{E}_\Gamma$ does not contain $F(\mathbb{F}_2, 2)$ (see [HK, A4.1]), while if Γ is not amenable then $\mathcal{E}_\Gamma$ does not contain E_0 (see [K91, 2.3]). □

We also have

Proposition 2.4.6. *Let $\mathcal{C}$ denote the class of countable increasing unions of equivalence relations generated by free Borel actions of (possibly different) countable groups. Then $\mathcal{C}$ does not have a $\leq_B$-universal element, hence is not elementary.*

Proof. Let $E = \bigcup_n E_n \in \mathcal{C}$ be the countable increasing union of countable Borel equivalence relations $E_0 \subseteq E_1 \subseteq \cdots$ on X, where each E_n is generated by a free Borel action of a countable group Γ_n. Since there are uncountably many finitely generated groups, there is a finitely generated group L such that L does not embed in any Γ_n. Put $\Delta := SL_3(\mathbb{Z}) \times (L * \mathbb{Z})$, and let $F(\Delta, 2)$ live on $Y \subseteq 2^\Delta$ (the free part of the shift action), with its usual product probability measure μ. By [T2, 3.6] (see also 3.7–9 of that paper), $F(\Delta, 2)|Z \not\leq_B E_n$ for each n and $Z \subseteq Y$ of μ-measure 1.

If E were $\leq_B$-universal in $\mathcal{C}$, then we would have some $f : F(\Delta, 2) \leq_B E$. Let $F_n := f^{-1}(E_n)$, so that $F(\Delta, 2) = \bigcup_n F_n$. By [GT, 1.1] (using that $SL_3(\mathbb{Z})$ acts strongly ergodically [HK, A4.1]), there is an n and a Borel $A \subseteq Y$ with $\mu(A) > 0$ such that $F(\Delta, 2)|A = F_n|A$. By ergodicity of μ, $Z := [A]_{F(\Delta,2)}$ has μ-measure 1; but $F(\Delta, 2)|Z \sim_B F(\Delta, 2)|A = F_n|A \leq_B E_n$, a contradiction. □

We conclude this section by explicitly describing the invariantly universal equivalence relation in several elementary classes:

- The $\sqsubseteq_B^i$-universal finite Borel equivalence relation is $\bigoplus_{1 \leq n \in \mathbb{N}} (\Delta_\mathbb{R} \times I_n)$.
- The $\sqsubseteq_B^i$-universal aperiodic smooth countable Borel equivalence relation is $\Delta_\mathbb{R} \times I_\mathbb{N}$.

- The $\sqsubseteq_B^i$-universal aperiodic hyperfinite Borel equivalence relation is $2^{\aleph_0} \cdot E_0 = \Delta_{\mathbb{R}} \times E_0$, and the $\sqsubseteq_B^i$-universal compressible hyperfinite Borel equivalence relation is E_t (see Theorem 2.2.1).

- The $\sqsubseteq_B^i$-universal countable Borel equivalence relation is E_∞, and the $\sqsubseteq_B^i$-universal compressible Borel equivalence relation is $E_\infty \times I_{\mathbb{N}}$ (see Section 2.5.4).

- For a countable group Γ, the $\sqsubseteq_B^i$-universal equivalence relation $E_{\infty\Gamma}$ generated by a free Borel action of Γ is $F(\Gamma, \mathbb{R})$.

2.4.2 The "Scott sentence" of an equivalence relation

We now associate to every $E \in \mathcal{E}$ a "Scott sentence" σ_E. Just as the Scott sentence $\sigma_{\mathbb{A}}$ of an ordinary first-order structure $\mathbb{A}$ axiomatizes structures isomorphic to $\mathbb{A}$, the "Scott sentence" σ_E will axiomatize equivalence relations class-bijectively mapping to E.

Theorem 2.4.7. *Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation. Then there is a sentence σ_E (in some fixed language not depending on E) and a σ_E-structure $\mathbb{H} : E \models \sigma_E$, such that for any $F \in \mathcal{E}$ and $\mathbb{A} : F \models \sigma_E$, there is a unique class-bijective homomorphism $f : F \to_B^{cb} E$ such that $\mathbb{A} = f_F^{-1}(\mathbb{H})$. This is illustrated by the following diagram:*

$$\begin{tikzcd}
F \arrow[d, dashed, "f"'] \arrow[dr, "\mathbb{A}", "\models"'] & \\
E \arrow[r, "\models", "\mathbb{H}"'] & \sigma_E
\end{tikzcd}$$

Proof. We may assume that X is a Borel subspace of $2^{\mathbb{N}}$. Let $L = \{R_0, R_1, \ldots\}$ where each R_i is unary. The idea is that a Borel L-structure will code a Borel map to $X \subseteq 2^{\mathbb{N}}$. Note that since L is unary, there is no distinction between Borel L-structures on X and Borel L-structures on E, or between pullback L-structures and classwise pullback L-structures.

Let $\mathbb{H}'$ be the Borel L-structure on $2^{\mathbb{N}}$ given by

$$R_i^{\mathbb{H}'}(x) \iff x(i) = 1.$$

It is clear that for any standard Borel space Y, we have a bijection

$$\begin{aligned}
\{\text{Borel maps } Y \to_B 2^{\mathbb{N}}\} &\longleftrightarrow \{\text{Borel } L\text{-structures on } Y\} \\
f &\longmapsto f^{-1}(\mathbb{H}') \qquad\qquad (*) \\
(y \mapsto (i \mapsto R_i^{\mathbb{A}}(y))) &\longleftarrow\!\shortmid \mathbb{A}.
\end{aligned}$$

It will suffice to find an $L_{\omega_1\omega}$-sentence σ_E such that for all $(Y, F) \in \mathcal{E}$ and $f : Y \to_B 2^{\mathbb{N}}$,

$$f^{-1}(\mathbb{H}') : F \models \sigma_E \iff f(Y) \subseteq X \;\&\; f : F \to_B^{cb} E. \tag{$**$}$$

Indeed, we may then put $\mathbb{H} := \mathbb{H}'|X$, and $(*)$ will restrict to a bijection between class-bijective homomorphisms $F \to_B^{cb} E$ and σ_E-structures on F, as claimed in the theorem.

Now we find σ_E satisfying $(**)$. The conditions $f(Y) \subseteq X$ and $f : F \to_B^{cb} E$ can be rephrased as: for each F-class $D \in Y/F$, the restriction $f|D : D \to 2^{\mathbb{N}}$ is a bijection between D and some E-class. Using $(*)$, this is equivalent to: for each F-class $D \in Y/F$, the structure $\mathbb{B} := f^{-1}(\mathbb{H}')|D$ on D is such that

$$y \mapsto (i \mapsto R_i^{\mathbb{B}}(y)) \text{ is a bijection from the universe of } \mathbb{B} \text{ to some } E\text{-class.} \tag{$***$}$$

So it suffices to show that the class $\mathcal{K}$ of L-structures $\mathbb{B}$ satisfying $(***)$ is Borel (so we may let σ_E be any $L_{\omega_1\omega}$-sentence axiomatizing $\mathcal{K}$), i.e., that for any $I = 1, 2, \dots, \mathbb{N}$, $\mathcal{K} \cap \mathrm{Mod}_I(L) \subseteq \mathrm{Mod}_I(L)$ is Borel. Using $(*)$ again, $\mathcal{K} \cap \mathrm{Mod}_I(L)$ is the image of the Borel injection

$$\begin{aligned} \{\text{bijections } I \to (\text{some } E\text{-class})\} &\longrightarrow \mathrm{Mod}_I(L) \\ f &\longmapsto f^{-1}(\mathbb{H}'). \end{aligned}$$

The domain of this injection is clearly a Borel subset of X^I, whence its image is Borel. □

In the rest of this section, we give an alternative, more "explicit" construction of σ_E (rather than obtaining it from Lopez-Escobar's definability theorem as in the above proof). Using the same notations as in the proof, we want to find σ_E satisfying $(**)$.

By Lusin-Novikov uniformization, write $E = \bigcup_i G_i$ where $G_0, G_1, \dots \subseteq X^2$ are graphs of (total) Borel functions. For each i, let $\phi_i(x, y)$ be a quantifier-free $L_{\omega_1\omega}$-formula whose interpretation in the structure $\mathbb{H}'$ is $\phi_i^{\mathbb{H}'} = G_i \subseteq (2^{\mathbb{N}})^2$. (Such a formula can be obtained from a Borel definition of $G_i \subseteq (2^{\mathbb{N}})^2$ in terms of the basic rectangles $R_j^{\mathbb{H}'} \times R_k^{\mathbb{H}'}$, by replacing each $R_j^{\mathbb{H}'} \times R_k^{\mathbb{H}'}$ with $R_j(x) \wedge R_k(y)$.) Define the $L_{\omega_1\omega}$-sentences:

$$\begin{aligned} \sigma_E^h &:= \forall x\, \forall y\, \textstyle\bigvee_i \phi_i(x, y), \\ \sigma_E^{ci} &:= \forall x\, \forall y\, (\textstyle\bigwedge_i (R_i(x) \leftrightarrow R_i(y)) \to x = y), \\ \sigma_E^{cs} &:= \forall x\, \textstyle\bigwedge_i \exists y\, \phi_i(x, y). \end{aligned}$$

Lemma 2.4.8. *In the notation of the proof of Theorem 2.4.7,*

$$\begin{aligned}
f^{-1}(\mathbb{H}') : F \models \sigma_E^h &\iff f(Y) \subseteq X \ \&\ f : F \to_B E,\\
f^{-1}(\mathbb{H}') : F \models \sigma_E^{ci} &\iff f|D : D \to 2^{\mathbb{N}} \text{ is injective } \forall D \in Y/F,\\
f^{-1}(\mathbb{H}') : F \models \sigma_E^{cs} &\iff f(Y) \subseteq X \ \&\ f(D) \text{ is } E\text{-invariant } \forall D \in Y/F.
\end{aligned}$$

Proof. $f^{-1}(\mathbb{H}') : F \models \sigma_E^h$ iff for all $(y, y') \in F$, there is some i such that $\phi_i^{f^{-1}(\mathbb{H}')}(y, y')$; $\phi_i^{f^{-1}(\mathbb{H}')}(y, y')$ is equivalent to $\phi_i^{\mathbb{H}'}(f(y), f(y'))$, i.e., $f(y)\ G_i\ f(y')$, so we get that $f^{-1}(\mathbb{H}') : F \models \sigma_E^h$ iff for all $(y, y') \in F$, we have $f(y)\ E\ f(y')$. (Taking $y = y'$ yields $f(y) \in X$.)

$f^{-1}(\mathbb{H}') : F \models \sigma_E^{ci}$ iff for all $(y, y') \in F$ with $y \neq y'$, there is some i such that $R_i^{f^{-1}(\mathbb{H}')}(y) \not\Longleftrightarrow R_i^{f^{-1}(\mathbb{H}')}(y)$, i.e., $R_i^{\mathbb{H}'}(f(y)) \not\Longleftrightarrow R_i^{\mathbb{H}'}(f(y'))$, i.e., $f(y) \neq f(y')$.

$f^{-1}(\mathbb{H}') : F \models \sigma_E^{cs}$ iff for all $y \in Y$ and all $i \in \mathbb{N}$, there is some $y'\ F\ y$ such that $\phi_i^{f^{-1}(\mathbb{H}')}(y, y')$, i.e., $\phi_i^{\mathbb{H}'}(f(y), f(y'))$, i.e., $f(y)\ G_i\ f(y')$; from the definition of the G_i, this is equivalent to: for all $y \in Y$, we have $f(y) \in X$, and for every $x'\ E\ f(y)$ there is some $y'\ F\ y$ such that $f(y') = x'$. □

So defining $\sigma_E := \sigma_E^h \wedge \sigma_E^{ci} \wedge \sigma_E^{cs}$, we have that $f^{-1}(\mathbb{H}') : F \models \sigma_E$ iff $f : F \to_B^{cb} E$, as desired. Moreover, by modifying these sentences, we may obtain theories for which structures on F correspond to other kinds of homomorphisms $F \to E$. We will take advantage of this later, in Sections 2.5.1 and 2.5.2.

2.4.3 Structurability and class-bijective homomorphisms

The combination of Theorems 2.4.1 and 2.4.7 gives the following (closely related) corollaries, which imply a tight connection between structurability and class-bijective homomorphisms.

Corollary 2.4.9. *For $E, F \in \mathcal{E}$, we have $F \models \sigma_E$ iff $F \to_B^{cb} E$.*

Proof. By Theorem 2.4.7 and Proposition 2.3.1. □

Corollary 2.4.10. *For every $E \in \mathcal{E}$, there is a smallest elementary class containing E, namely $\mathcal{E}_{\sigma_E} = \{F \in \mathcal{E} \mid F \to_B^{cb} E\}$.*

Proof. By Proposition 2.3.1, $\mathcal{E}_{\sigma_E}$ is contained in every elementary class containing E. □

We define $\mathcal{E}_E := \mathcal{E}_{\sigma_E} = \{F \in \mathcal{E} \mid F \to_B^{cb} E\}$, and call it the **elementary class of** E.

Remark 2.4.11. E is not necessarily $\sqsubseteq_B^i$-universal in $\mathcal{E}_E$: for example, E_0 is not invariantly universal in $\mathcal{E}_{E_0} = \{\text{aperiodic hyperfinite}\}$ (see Theorem 2.2.1).

Corollary 2.4.12. *A class $C \subseteq \mathcal{E}$ of countable Borel equivalence relations is elementary iff it is (downwards-)closed under class-bijective homomorphisms and contains an invariantly universal element $E \in C$, in which case $C = \mathcal{E}_E$.*

Proof. One implication is Proposition 2.3.1 and Corollary 2.4.4. Conversely, if C is closed under $\to_B^{cb}$ and $E \in C$ is invariantly universal, then clearly $C = \{F \mid F \to_B^{cb} E\} = \mathcal{E}_E$. ◻

So every elementary class C is determined by a canonical isomorphism class contained in C, namely the invariantly universal elements of C. We now characterize the class of equivalence relations which are invariantly universal in some elementary class.

Corollary 2.4.13. *Let $E \in \mathcal{E}$. The following are equivalent:*

(i) $E \cong_B E_{\infty\sigma_E}$, *i.e., E is invariantly universal in $\mathcal{E}_E$.*

(ii) $E \cong_B E_{\infty\sigma}$ *for some σ, i.e., E is invariantly universal in some elementary class.*

(iii) For every $F \in \mathcal{E}$, $F \to_B^{cb} E$ iff $F \sqsubseteq_B^i E$.

Proof. Clearly (i) $\implies$ (ii) $\implies$ (iii), and if (iii) holds, then $\mathcal{E}_E = \{F \mid F \to_B^{cb} E\} = \{F \mid F \sqsubseteq_B^i E\}$ so E is invariantly universal in $\mathcal{E}_E$. ◻

Remark 2.4.14. The awkward notation $E_{\infty\sigma_E}$ will be replaced in the next section (with $E_\infty \otimes E$).

We say that $E \in \mathcal{E}$ is **universally structurable** if the equivalent conditions in Corollary 2.4.13 hold. We let $\mathcal{E}_\infty \subseteq \mathcal{E}$ denote the class of universally structurable countable Borel equivalence relations. The following summarizes the relationship between $\mathcal{E}_\infty$ and elementary classes:

Corollary 2.4.15. *We have an order-isomorphism of posets*

$$\begin{aligned}(\{\textit{elementary classes}\}, \subseteq) &\longleftrightarrow (\mathcal{E}_\infty/\cong_B, \sqsubseteq^i_B) = (\mathcal{E}_\infty/\leftrightarrow^{cb}_B, \to^{cb}_B)\\ C &\longmapsto \{\sqsubseteq^i_B\textit{-universal elements of } C\}\\ \mathcal{E}_E &\longleftarrow\!\!\mathrel{\mid} E.\end{aligned}$$

We will study the purely order-theoretical aspects of the poset $(\mathcal{E}_\infty/\cong_B, \sqsubseteq^i_B)$ (equivalently, the poset of elementary classes) in Section 2.6.

We conclude this section by pointing out the following consequence of universal structurability:

Corollary 2.4.16. *If $E \in \mathcal{E}$ is universally structurable, then $E \cong_B \Delta_\mathbb{R} \times E$. In particular, E has either none or continuum many ergodic invariant probability measures.*

Proof. Clearly $E \sqsubseteq^i_B \Delta_\mathbb{R} \times E$, and $\Delta_\mathbb{R} \times E \to^{cb}_B E$, so $\Delta_\mathbb{R} \times E \sqsubseteq^i_B E$. □

2.4.4 Class-bijective products

In this section and the next, we use the theory of the preceding sections to obtain some structural results about the category $(\mathcal{E}, \to^{cb}_B)$ of countable Borel equivalence relations and class-bijective homomorphisms. For categorical background, see [ML].

This section concerns a certain product construction between countable Borel equivalence relations, which, unlike the cross product $E \times F$, is well-behaved with respect to class-bijective homomorphisms.

Proposition 2.4.17. *Let $E, F \in \mathcal{E}$ be countable Borel equivalence relations. There is a countable Borel equivalence relation, which we denote by $E \otimes F$ and call the **class-bijective product** (or **tensor product**) of E and F, which is the categorical product of E and F in the category $(\mathcal{E}, \to^{cb}_B)$. In other words, there are canonical class-bijective projections*

$$\pi_1 : E \otimes F \to^{cb}_B E, \qquad\qquad \pi_2 : E \otimes F \to^{cb}_B F,$$

such that the triple $(E \otimes F, \pi_1, \pi_2)$ is universal in the following sense: for any other $G \in \mathcal{E}$ with $f : G \to^{cb}_B E$ and $g : G \to^{cb}_B F$, there is a unique *class-bijective homomorphism $\langle f, g\rangle : G \to^{cb}_B E \otimes F$ such that $f = \pi_1 \circ \langle f, g\rangle$ and $g = \pi_2 \circ \langle f, g\rangle$.*

This is illustrated by the following commutative diagram:

$$\begin{tikzcd}
 & G \arrow[ld, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[rd, "g"] & \\
E & E \otimes F \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & F
\end{tikzcd}$$

Proof. Put $E \otimes F := E \ltimes \sigma_F$. The rest follows from chasing through the universal properties in Theorem 2.4.1 and Theorem 2.4.7 (or equivalently, the Yoneda lemma). For the sake of completeness, we give the details.

From Theorem 2.4.1, we have a canonical projection $\pi_1 : E \otimes F \to_B^{cb} E$. We also have a canonical σ_F-structure on $E \otimes F$, namely $\mathbb{E} : E \otimes F = E \ltimes \sigma_F \models \sigma_F$. This structure corresponds to a unique class-bijective map $\pi_2 : E \otimes F \to_B^{cb} F$ such that $\mathbb{E} = (\pi_2)^{-1}_{E\otimes F}(\mathbb{H})$, where $\mathbb{H} : F \models \sigma_F$ is the canonical structure from Theorem 2.4.7.

Now given G, f, g as above, the map $\langle f, g\rangle$ is produced as follows. We have the classwise pullback structure $g_G^{-1}(\mathbb{H}) : G \models \sigma_F$, which, together with $f : G \to_B^{cb} E$, yields (by Theorem 2.4.1) a unique map $\langle f, g\rangle : G \to_B^{cb} E \ltimes \sigma_F = E \otimes F$ such that $f = \pi_1 \circ \langle f, g\rangle$ and $g_G^{-1}(\mathbb{H}) = \langle f, g\rangle_G^{-1}(\mathbb{E})$. Since $\mathbb{E} = (\pi_2)^{-1}_{E\otimes F}(\mathbb{H})$, we get $g_G^{-1}(\mathbb{H}) = \langle f, g\rangle_G^{-1}((\pi_2)^{-1}_{E\otimes F}(\mathbb{H})) = (\pi_2 \circ \langle f, g\rangle)_G^{-1}(\mathbb{H})$; since (by Theorem 2.4.7) g is the *unique* map $h : G \to_B^{cb} F$ such that $g_G^{-1}(\mathbb{H}) = h_G^{-1}(\mathbb{H})$, we get $g = \pi_2 \circ \langle f, g\rangle$, as desired. It remains to check uniqueness of $\langle f, g\rangle$. If $h : G \to_B^{cb} E \otimes F$ is such that $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$, then (reversing the above steps) we have $g_G^{-1}(\mathbb{H}) = h_G^{-1}(\mathbb{E})$; since $\langle f, g\rangle$ was unique with these properties, we get $h = \langle f, g\rangle$, as desired. $\square$

Remark 2.4.18. It follows immediately from the definitions that $\mathcal{E}_{E\otimes F} = \mathcal{E}_E \cap \mathcal{E}_F$.

Remark 2.4.19. As with all categorical products, $\otimes$ is unique up to unique (Borel) isomorphism, as well as associative and commutative up to (Borel) isomorphism. Note that the two latter properties are not immediately obvious from the definition $E \otimes F := E \ltimes \sigma_F$.

Remark 2.4.20. However, by unravelling the proofs of Theorems 2.4.1 and 2.4.7, we may explicitly describe $E \otimes F$ in a way that makes associativity and commutativity more obvious. Since this explicit description also sheds some light on the structure of $E \otimes F$, we briefly give it here.

Let E live on X, F live on Y, and $E \otimes F$ live on Z. We have one $(E \otimes F)$-class for each E-class C, F-class D, and bijection $b : C \cong D$; the elements of the $(E \otimes F)$-class corresponding to (C, D, b) are the elements of C, or equivalently via the bijection b, the elements of D. Thus, ignoring Borelness, we put

$$Z := \{(x, y, b) \mid x \in X,\ y \in Y,\ b : [x]_E \cong [y]_F,\ b(x) = y\},$$
$$(x, y, b)\ (E \otimes F)\ (x', y', b') \iff x\ E\ x' \ \&\ y\ F\ y' \ \&\ b = b',$$
$$\pi_1(x, y, b) := x, \qquad\qquad \pi_2(x, y, b) := y.$$

Given G, f, g as in Proposition 2.4.17 (G living on W, say), the map $\langle f, g\rangle : G \to_B^{cb} E \otimes F$ is given by $\langle f, g\rangle(w) = (f(w), g(w), (g|[w]_G) \circ (f|[w]_G)^{-1})$, where $(g|[w]_G) \circ (f|[w]_G)^{-1}) : [f(w)]_E \cong [g(w)]_F$ since f, g are class-bijective.

To make this construction Borel, we assume that E, F are aperiodic, and replace Z in the above with a subspace of $X \times Y \times S_\infty$, where bijections $b : [x]_E \cong [y]_F$ are transported to bijections $\mathbb{N} \cong \mathbb{N}$ via Borel enumerations of the E-classes and F-classes, as with the map $T : X \to X^{\mathbb{N}}$ in the proof of Theorem 2.4.1. (If E, F are not aperiodic, then we split them into the parts consisting of classes with each cardinality $n \in \{1, 2, \dots, \aleph_0\}$; then there will be no $(E \otimes F)$-classes lying over an E-class and an F-class with different cardinalities.)

The tensor product $\otimes$ and the cross product $\times$ are related as follows: we have a canonical homomorphism $(\pi_1, \pi_2) : E \otimes F \to_B^{ci} E \times F$, where $(\pi_1, \pi_2)(z) = (\pi_1(z), \pi_2(z))$ (where $\pi_1 : E \otimes F \to_B^{cb} E$ and $\pi_2 : E \otimes F \to_B^{cb} F$ are the projections from the tensor product), which is class-injective because $\pi_1' \circ (\pi_1, \pi_2) = \pi_1$ is class-injective (where $\pi_1' : E \times F \to_B E$ is the projection from the *cross* product).

When we regard $E \otimes F$ as in Remark 2.4.20, (π_1, π_2) is the obvious projection from Z to $X \times Y$. This in particular shows that

Proposition 2.4.21. *(a) $(\pi_1, \pi_2) : E \otimes F \to_B^{ci} E \times F$ is surjective iff E and F have all classes of the same cardinality (in particular, if both are aperiodic);*

(b) $(\pi_1, \pi_2) : E \otimes F \to_B^{ci} E \times F$ is an isomorphism if $E = \Delta_X$ and $F = \Delta_Y$.

We now list some formal properties of $\otimes$:

Proposition 2.4.22. *Let $E, E_i, F, G \in \mathcal{E}$ for $i < n \le \mathbb{N}$ and let (L, σ) be a theory.*

(a) If $E \models \sigma$ then $E \otimes F \models \sigma$.

(b) If $f : E \sqsubseteq_B^i F$ and $g : E \to_B^{cb} G$, then $\langle f, g\rangle : E \sqsubseteq_B^i F \otimes G$.

(c) If E is universally structurable, then so is $E \otimes F$.

(d) $(E \otimes F) \ltimes \sigma \cong_B E \otimes (F \ltimes \sigma)$ (and the isomorphism is natural in E, F).

(e) $\bigoplus_i (E_i \otimes F) \cong_B (\bigoplus_i E_i) \otimes F$ (and the isomorphism is natural in E_i, F).

Proof. (a): follows from $\pi_1 : E \otimes F \to_B^{cb} E$.

(b): since $f = \pi_1 \circ \langle f, g \rangle$ is class-injective, so is $\langle f, g \rangle$.

(c): if $f : G \to_B^{cb} E \otimes F$, then $\pi_1 \circ f : G \to_B^{cb} E$, whence there is some $g : G \sqsubseteq_B^i E$ since E is universally structurable, whence $\langle g, \pi_2 \circ f \rangle : G \sqsubseteq_B^i E \otimes F$ (by (b)).

(d): follows from a chase through the universal properties of $\otimes$ and $\ltimes$ (or the Yoneda lemma). (A class-bijective homomorphism $G \to_B^{cb} (E \otimes F) \ltimes \sigma$ is the same thing as pair of class-bijective homomorphisms $G \to_B^{cb} E$ and $G \to_B^{cb} F$ together with a σ-structure on G, which is the same thing as a class-bijective homomorphism $G \to_B^{cb} E \otimes (F \ltimes \sigma)$.)

(e): this is an instance of the following more general fact, which follows easily from the construction of $E \ltimes \sigma$ in Theorem 2.4.1 (and which could have been noted earlier, in Section 2.4.1):

Proposition 2.4.23. $\bigoplus_i (E_i \ltimes \sigma) \cong_B (\bigoplus_i E_i) \ltimes \sigma$.

Moreover, the isomorphism can be taken to be the map $d : \bigoplus_i (E_i \ltimes \sigma) \to_B^{cb} (\bigoplus_i E_i) \ltimes \sigma$ such that for each i, the restriction $d|(E_i \ltimes \sigma) : E_i \ltimes \sigma \to_B^{cb} (\bigoplus_i E_i) \ltimes \sigma$ is the canonical such map induced by the inclusion $E_i \sqsubseteq_B^i \bigoplus_i E_i$.

(In other words, the functor $E \mapsto E \ltimes \sigma$ preserves countable coproducts.)

To get from this to (e), simply put $\sigma := \sigma_F$. Naturality is straightforward. □

Remark 2.4.24. The analog of Proposition 2.4.22(a) for cross products is false: the class of treeable countable Borel equivalence relations is *not* closed under cross products (see e.g., [JKL, 3.28]).

We note that for any $E \in \mathcal{E}$, the equivalence relation $E_{\infty\sigma_E}$ (i.e., the invariantly universal element of $\mathcal{E}_E$) can also be written as the less awkward $E_\infty \otimes E$, which is therefore how we will write it from now on.

Here are some sample computations of class-bijective products:

- $\Delta_m \otimes \Delta_n \cong_B \Delta_m \times \Delta_n = \Delta_{m \times n}$ for $m, n \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ (by Proposition 2.4.21(b)).
- $I_\mathbb{N} \otimes I_\mathbb{N} \cong_B \Delta_\mathbb{R} \times I_\mathbb{N}$, since $I_\mathbb{N} \otimes I_\mathbb{N}$ is aperiodic smooth (Proposition 2.4.22(a)) and there are continuum many bijections $\mathbb{N} \cong \mathbb{N}$.
- $E_\infty \otimes E_\infty \cong_B E_\infty$, since $E_\infty \sqsubseteq_B^i E_\infty \otimes E_\infty$ (Proposition 2.4.22(b)).
- If E is universally structurable and $E \to_B^{cb} F$, then $E \otimes F \cong_B E$, since $E \sqsubseteq_B^i E \otimes F$ (Proposition 2.4.22(b)), and $\pi_1 : E \otimes F \to_B^{cb} E$ so $E \otimes F \sqsubseteq_B^i E$.
- $E_0 \otimes E_0 \cong_B \Delta_\mathbb{R} \times E_0$, since $E_0 \otimes E_0$ is aperiodic hyperfinite (Proposition 2.4.22(a)), and there are $2^{\aleph_0}$ pairwise disjoint copies of E_0 in $E_0 \otimes E_0$. This last fact can be seen by taking a family $(f_r)_{r \in \mathbb{R}}$ of Borel automorphisms $f_r : E_0 \cong_B E_0$ such that f_r, f_s disagree on every E_0-class whenever $r \neq s$, and then considering the embeddings $\langle 1_{E_0}, f_r \rangle : E_0 \sqsubseteq_B^i E_0 \otimes E_0$, which will have pairwise disjoint images (by Remark 2.4.20).

  (The existence of the family $(f_r)_r$ is standard; one construction is by regarding E_0 as the orbit equivalence of the translation action of $\mathbb{Z}$ on $\mathbb{Z}_2$, the 2-adic integers, then taking $f_r : \mathbb{Z}_2 \to \mathbb{Z}_2$ for $r \in \mathbb{Z}_2$ to be translation by r.)

Note that the last example can be used to compute $\otimes$ of all hyperfinite equivalence relations.

We conclude by noting that we do not currently have a clear picture of tensor products of general countable Borel equivalence relations. For instance, the examples above suggest that perhaps $E \otimes E$ is universally structurable (equivalently, $E \otimes E \cong_B E_\infty \otimes E$, since $E_\infty \otimes E \otimes E \cong_B E_\infty \otimes E$) for all aperiodic E; but we do not know if this is true.

2.4.5 Categorical limits in $(\mathcal{E}, \to_B^{cb})$

This short section concerns general categorical limits in the category $(\mathcal{E}, \to_B^{cb})$ of countable Borel equivalence relations and class-bijective homomorphisms. Throughout this section, we use categorical terminology, e.g., “product” means categorical product (i.e., class-bijective product), “pullback” means categorical pullback (i.e., fiber product), etc. For definitions, see [ML, III.3–4, V].

We have shown that $(\mathcal{E}, \to_B^{cb})$ contains binary products. By iterating (or by generalizing the construction outlined in Remark 2.4.20), we may obtain all finite (nontrivial) products. The category $(\mathcal{E}, \to_B^{cb})$ also contains pullbacks (see Section 2.2.6). It follows that it contains all finite nonempty limits, i.e., limits of all diagrams $F : J \to (\mathcal{E}, \to_B^{cb})$ where the indexing category J is finite and nonempty (see [ML, V.2, Exercise III.4.9]).

Remark 2.4.25. $(\mathcal{E}, \to_B^{cb})$ does *not* contain a terminal object, i.e., a limit of the empty diagram. This would be a countable Borel equivalence relation E such that any other countable Borel equivalence relation F has a *unique* class-bijective map $F \to_B^{cb} E$; clearly such E does not exist.

We now verify that $(\mathcal{E}, \to_B^{cb})$ has inverse limits of countable chains, and that these coincide with the same limits in the category of all Borel equivalence relations and Borel homomorphisms (Section 2.2.7):

Proposition 2.4.26. *Let $(X_n, E_n)_{n\in\mathbb{N}}$ be countable Borel equivalence relations, and $(f_n : E_{n+1} \to_B^{cb} E_n)_n$ be class-bijective homomorphisms. Then the inverse limit $\varprojlim_n (X_n, E_n)$ in the category of Borel equivalence relations and Borel homomorphisms, as defined in Section 2.2.7, is also the inverse limit in $(\mathcal{E}, \to_B^{cb})$. More explicitly,*

- *(a) the projections $\pi_m : \varprojlim_n E_n \to_B E_m$ are class-bijective (so in particular $\varprojlim_n E_n$ is countable);*
- *(b) if $(Y, F) \in \mathcal{E}$ is a countable Borel equivalence relation with class-bijective homomorphisms $g_n : F \to_B^{cb} E_n$ such that $g_n = f_n \circ g_{n+1}$, then the unique homomorphism $\widetilde{g} : F \to_B \varprojlim_n E_n$ such that $\pi_n \circ \widetilde{g} = g_n$ for each n (namely $\widetilde{g}(y) = (g_n(y))_n$) is class-bijective.*

Proof. For (a), note that since $\pi_m = f_m \circ \pi_{m+1}$ and the f_m are class-bijective, it suffices to check that π_0 is class-bijective, since we may then inductively get that $\pi_1, \pi_2, \ldots$ are class-bijective. Let $\overline{x} = (x_n)_n \in \varprojlim_n X_n$ and $x_0 = \pi_0(\overline{x})\ E_0\ x_0'$; we must find a unique $\overline{x}'\ (\varprojlim_n E_n)\ \overline{x}$ such that $x_0' = \pi_0(\overline{x}')$. For the coordinate $x_1' = \pi_1(\overline{x}')$, we must have $x_0' = f_0(x_1')$ (in order to have $\overline{x}' \in \varprojlim_n X_n$) and $x_1'\ E_1\ x_1$ (in order to have $\overline{x}'\ (\varprojlim_n E_n)\ \overline{x}$); since $x_0'\ E_0\ x_0$, by class-bijectivity of f_0, there is a unique such x_1'. Continuing inductively, we see that there is a unique choice of $x_n' = \pi_n(\overline{x}')$ for each $n > 0$. Then $\overline{x}' := (x_n')_n$ is the desired element.

For (b), simply note that since $\pi_n \circ \widetilde{g} = g_n$ and π_n, g_n are class-bijective, so must be $\widetilde{g}$. $\square$

Corollary 2.4.27. $(\mathcal{E}, \to_B^{cb})$ *has all countable (nontrivial) products.*

Proof. To compute the product $\bigotimes_i E_i$ of $E_0, E_1, E_2, \ldots \in \mathcal{E}$, take the inverse limit of the chain $\cdots \to_B^{cb} E_0 \otimes E_1 \otimes E_2 \to_B^{cb} E_0 \otimes E_1 \to_B^{cb} E_0$ (where the maps are the projections). □

Remark 2.4.28. Countable products can also be obtained by generalizing Remark 2.4.20.

Corollary 2.4.29. $(\mathcal{E}, \to_B^{cb})$ *has all countable nonempty limits, i.e., limits of all diagrams* $F : J \to (\mathcal{E}, \to_B^{cb})$*, where the indexing category* J *is countable and nonempty.*

Proof. Follows from countable products and pullbacks; again see [ML, V.2, Exercise III.4.9]. □

2.5 Structurability and reducibility

This section has two parts: the first part (Sections 2.5.1 to 2.5.3) relates structurability to various classes of homomorphisms, in the spirit of Sections 2.4.1 to 2.4.3; while the second part (Section 2.5.4) relates reductions to compressibility, using results from the first part and from [DJK, Section 2].

We describe here the various classes of homomorphisms that we will be considering. These fit into the following table:

Table 2.5.1: global and local classes of homomorphisms

Global	Local
$\sqsubseteq_B^i$	$\to_B^{cb}$
$\sqsubseteq_B$	$\to_B^{ci}$
$\leq_B$	$\to_B^{sm}$

The last entry in the table denotes the following notion: we say that a Borel homomorphism $f : (X, E) \to_B (Y, F)$ is **smooth**, written $f : E \to_B^{sm} F$, if the f-preimage of every smooth set is smooth (where by a smooth subset of Y (resp., X) we mean a subset to which the restriction of F (resp., E) is smooth). This notion was previously considered by Clemens-Conley-Miller [CCM], under the name **smooth-to-one homomorphism** (because of Proposition 2.5.8(ii)). See Proposition 2.5.8 for basic properties of smooth homomorphisms.

Let $(\mathcal{G}, \mathcal{L})$ be a row in Table 2.5.1. We say that $\mathcal{G}$ is a "global" class of homomorphisms, while $\mathcal{L}$ is the corresponding "local" class. Note that $\mathcal{G} \subseteq \mathcal{L}$. The idea is that $\mathcal{G}$ is a condition on homomorphisms requiring injectivity between classes (i.e., $\mathcal{G}$ consists only of reductions), while $\mathcal{L}$ is an analogous "classwise" condition which can be captured by structurability.

Our main results in this section state the following: for any elementary class $C \subseteq \mathcal{E}$, the downward closure of C under $\mathcal{G}$ is equal to the downward closure under $\mathcal{L}$, and is elementary. In particular, when $C = \mathcal{E}_E$, this implies that the downward closure of $\{E\}$ under $\mathcal{L}$ is elementary. In the case $(\mathcal{G}, \mathcal{L}) = (\sqsubseteq^i_B, \to^{cb}_B)$, these follow from Section 2.4.3; thus, our results here generalize our results therein to the other classes of homomorphisms appearing in Table 2.5.1.

Theorem 2.5.1. *Let $C \subseteq \mathcal{E}$ be an elementary class. Then the downward closures of C under $\sqsubseteq_B$ and $\to^{ci}_B$, namely*

$$C^e := \{F \in \mathcal{E} \mid \exists E \in C\, (F \sqsubseteq_B E)\},$$
$$C^{cih} := \{F \in \mathcal{E} \mid \exists E \in C\, (F \to^{ci}_B E)\},$$

are equal and elementary.

In particular, when $C = \mathcal{E}_E$, we get that

$$\mathcal{E}^e_E = \mathcal{E}^{cih}_E = \{F \in \mathcal{E} \mid F \to^{ci}_B E\}$$
$$(= \{F \in \mathcal{E} \mid F \sqsubseteq_B E\}, \text{ if } E \text{ is universally structurable})$$

is the smallest elementary class containing E and closed under $\sqsubseteq_B$.

Theorem 2.5.2. *Let $C \subseteq \mathcal{E}$ be an elementary class. Then the downward closures of C under $\leq_B$ and $\to^{sm}_B$, namely*

$$C^r := \{F \in \mathcal{E} \mid \exists E \in C\, (F \leq_B E)\},$$
$$C^{smh} := \{F \in \mathcal{E} \mid \exists E \in C\, (F \to^{sm}_B E)\},$$

are equal and elementary.

In particular, when $C = \mathcal{E}_E$, we get that

$$\mathcal{E}^r_E = \mathcal{E}^{smh}_E = \{F \in \mathcal{E} \mid F \to^{sm}_B E\}$$
$$(= \{F \in \mathcal{E} \mid F \leq_B E\}, \text{ if } E \text{ is universally structurable})$$

is the smallest elementary class containing E and closed under $\leq_B$.

Our proof strategy is as follows. For each $(\mathcal{G}, \mathcal{L})$ $(= (\sqsubseteq_B, \to_B^{ci})$ or $(\leq_B, \to_B^{sm}))$, we prove a "factorization lemma" which states that $\mathcal{L}$ consists precisely of composites of homomorphisms in $\mathcal{G}$ followed by class-bijective homomorphisms (in that order). This yields that the closures of C under $\mathcal{G}$ and $\mathcal{L}$ are equal, since C is already closed under class-bijective homomorphisms. We then prove that for any $E \in \mathcal{E}$, a variation of the "Scott sentence" from Theorem 2.4.7 can be used to code $\mathcal{L}$-homomorphisms to E. This yields that for any $E \in \mathcal{E}$, the $\mathcal{L}$-downward closure of $\{E\}$ is elementary, which completes the proof.

2.5.1 Embeddings and class-injective homomorphisms

We begin with embeddings and class-injective homomorphisms, for which we have the following factorization lemma:

Proposition 2.5.3. *Let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations and $f : E \to_B^{ci} F$ be a class-injective homomorphism. Then there is a countable Borel equivalence relation $(Z, G) \in \mathcal{E}$, an embedding $g : E \sqsubseteq_B G$, and a class-bijective homomorphism $h : G \to_B^{cb} F$, such that $f = h \circ g$:*

$$\begin{tikzcd}
(X,E) \arrow[rr, "f"] \arrow[dr, dashed, "g"'] & & (Y,F) \\
& (Z,G) \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

Furthermore, g can be taken to be a complete section embedding, i.e., $[g(X)]_G = Z$.

Proof. Consider the equivalence relation (W, D) where

$$W := \{(x, y) \in X \times Y \mid f(x) \; F \; y\},$$
$$(x, y) \; D \; (x', y') \iff x \; E \; x' \;\; \& \;\; y = y'.$$

Then (W, D) is a countable Borel equivalence relation. We claim that it is smooth. Indeed, by Lusin-Novikov uniformization, write $F = \bigcup_i G_i$ where $G_i \subseteq Y^2$ for $i \in \mathbb{N}$ are graphs of Borel functions $g_i : Y \to Y$. Then a Borel selector for D is found by sending $(x, y) \in W$ to $((f|[x]_E)^{-1}(g_i(y)), y)$ for the least i such that $g_i(y)$ is in the image of $f|[x]_E$. (Here we are using that f is class-injective.)

Now put $Z := W/D$, and let G be the equivalence relation on Z given by

$$[(x, y)]_D \; G \; [(x', y')]_D \iff x \; E \; x' \;\; \& \;\; y \; F \; y'.$$

Then (Z, G) is a countable Borel equivalence relation. Let $g : X \to Z$ and $h : Z \to Y$ be given by

$$g(x) := [(x, f(x))]_D, \qquad h([(x, y)]_D) := y.$$

It is easily seen that $g : E \sqsubseteq_B G$ is a complete section embedding, $h : G \to_B^{cb} F$, and $f = h \circ g$, as desired. ◻

Remark 2.5.4. It is easy to see that the factorization produced by Proposition 2.5.3 (with the requirement that g be a complete section embedding) is unique up to unique Borel isomorphism. In other words, if $(Z', G') \in \mathcal{E}$, $g' : E \sqsubseteq_B G'$, and $h' : G' \to_B^{cb} F$ with $f = h' \circ g'$ are another factorization, with g' a complete section embedding, then there is a unique Borel isomorphism $i : G \cong_B G'$ such that $i \circ g = g'$ and $h = h' \circ i$.

Corollary 2.5.5. *If $E \in \mathcal{E}$ is universally structurable, then $F \sqsubseteq_B E \iff F \to_B^{ci} E$, for all $F \in \mathcal{E}$. Similarly, if C is an elementary class, then $C^e = C^{cih}$.*

Proof. If E is universally structurable and $F \to_B^{ci} E$, then by Proposition 2.5.3, $F \sqsubseteq_B G \to_B^{cb} E$ for some G; then $G \sqsubseteq_B^i E$, whence $F \sqsubseteq_B E$. The second statement is similar. ◻

We now have the following analog of Theorem 2.4.7 for class-injective homomorphisms, which we state in the simpler but slightly weaker form of Corollary 2.4.9 since that is all we will need:

Proposition 2.5.6. *Let $E \in \mathcal{E}$ be a countable Borel equivalence relation. Then there is a sentence σ_E^{cih} (in some fixed language) such that for all $F \in \mathcal{E}$, we have $F \models \sigma_E^{cih}$ iff $F \to_B^{ci} E$.*

Proof. We may either modify the proof of Theorem 2.4.7 (by considering "injections $I \to$ (some E-class)" instead of bijections in the last few lines of the proof), or take $\sigma_E^{cih} := \sigma_E^h \wedge \sigma_E^{ci}$ where σ_E^h and σ_E^{ci} are as in Lemma 2.4.8. ◻

Corollary 2.5.7. *If $C = \mathcal{E}_E$ is an elementary class, then so is $C^e = C^{cih} = \{F \in \mathcal{E} \mid F \to_B^{ci} E\}$.*

This completes the proof of Theorem 2.5.1.

2.5.2 Reductions, smooth homomorphisms, and class-surjectivity

Recall that a Borel homomorphism $f : E \to_B F$ between countable Borel equivalence relations E, F is **smooth** if the preimage of every smooth set is smooth. We have the following equivalent characterizations of smooth homomorphisms, parts of which are implicit in [CCM, 2.1–2.3]:

Proposition 2.5.8. *Let $(X, E), (Y, F) \in \mathcal{E}$ and $f : E \to_B F$. The following are equivalent:*

(i) f is smooth.

(ii) For every $y \in Y$, $f^{-1}(y)$ is smooth (i.e., $E|f^{-1}(y)$ is smooth).

(iii) $E \cap \ker f$ is smooth (as a countable Borel equivalence relation on X).

(iv) f can be factored into a surjective reduction $g : E \le_B G$, followed by a complete section embedding $h : G \sqsubseteq_B H$, followed by a class-bijective homomorphism $k : H \to_B^{cb} F$, for some $G, H \in \mathcal{E}$:

$$\begin{tikzcd} E \arrow[rrr, "f"] \arrow[dr, dashed, "g"'] & & & F \\ & G \arrow[r, dashed, "h"'] & H \arrow[ur, dashed, "k"'] & \end{tikzcd}$$

(In particular, f can be factored into a reduction $h \circ g$ (with image a complete section) followed by a class-bijective homomorphism k, or a surjective reduction g followed by a class-injective homomorphism $k \circ h$.)

(v) f belongs to the smallest class of Borel homomorphisms between countable Borel equivalence relations which is closed under composition and contains all reductions and class-injective homomorphisms.

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii): [CCM, 2.2] If $E \cap \ker f$ is not smooth, then it has an ergodic invariant σ-finite non-atomic measure μ. The pushforward $f_*\mu$ is then a Δ_Y-ergodic (because μ is $(\ker f)$-ergodic) measure on Y, hence concentrates at some $y \in Y$, i.e., $\mu(f^{-1}(y)) > 0$, whence $f^{-1}(y)$ is not smooth.

(iii) $\Longrightarrow$ (iv): Letting $g : X \to X/(E \cap \ker f)$ be the projection and G be the equivalence relation on $X/(E \cap \ker f)$ induced by E, we have that $g : E \le_B G$ is a surjective reduction, and f descends along g to a class-injective homomorphism

$f' : G \to_B^{ci} F$. By Proposition 2.5.3, f' factors as a complete section embedding $h : G \sqsubseteq_B H$ followed by a class-bijective homomorphism $k : H \to_B^{cb} F$, for some $H \in \mathcal{E}$.

(iv) $\Longrightarrow$ (v) is obvious.

(v) $\Longrightarrow$ (i): Clearly reductions are smooth, as are class-bijective homomorphisms; it follows that so are class-injective homomorphisms, by Proposition 2.5.3 (see also [CCM, 2.3]). □

Similarly to before we now have

Corollary 2.5.9. *If $E \in \mathcal{E}$ is universally structurable, then $F \leq_B E \iff F \to_B^{sm} E$, for all $F \in \mathcal{E}$. Similarly, if C is an elementary class, then $C^r = C^{smh}$.*

Proposition 2.5.10. *Let $E \in \mathcal{E}$ be a countable Borel equivalence relation. Then there is a sentence σ_E^{smh} (in some fixed language) such that for all $F \in \mathcal{E}$, we have $F \models \sigma_E^{smh}$ iff $F \to_B^{sm} E$.*

Proof. The language is $L = \{R_0, R_1, \dots\} \cup \{P\}$ where R_i, P are unary predicates, and the sentence is $\sigma_E^{smh} := \sigma_E^h \wedge \sigma_E^{sm}$, where σ_E^h is as in Lemma 2.4.8, and

$$\sigma_E^{sm} := \forall x\, \exists! y\, (P(y) \wedge \textstyle\bigwedge_i (R_i(x) \leftrightarrow R_i(y))).$$

It is easily seen that for any L-structure $\mathbb{A}$ on F, we will have $\mathbb{A} : F \models \sigma_E^{sm}$ iff the interpretation $P^{\mathbb{A}}$ is a Borel transversal of $F \cap \ker f$, where f is the Borel map to E coded by $\mathbb{A}$. □

Corollary 2.5.11. *If $C = \mathcal{E}_E$ is an elementary class, then so is $C^r = C^{smh} = \{F \in \mathcal{E} \mid F \to_B^{sm} E\}$.*

This completes the proof of Theorem 2.5.2.

Remark 2.5.12. Theorem 2.5.2 generalizes [CCM, 2.3], which shows that some particular classes of the form C^r, C elementary, are closed under $\to_B^{sm}$.

Hjorth-Kechris [HK, D.3] proved that every C^r (C elementary) is closed under $\subseteq$, i.e., containment of equivalence relations on the same space. Since containment is a class-injective homomorphism (namely the identity), Theorem 2.5.2 also generalizes this.

See Section 2.A.5 for more on the relation between [HK, Appendix D] and the above.

We end this section by pointing out that exactly analogous proofs work for yet another pair of ("global" resp. "local") classes of homomorphisms (which we did not include in Table 2.5.1), which forms a natural counterpart to $(\sqsubseteq_B, \to_B^{ci})$. We write $\leq_B^{cs}$ to denote a **(Borel) class-surjective reduction**, and $\to_B^{cssm}$ to denote a **class-surjective smooth homomorphism**. Then we have

Theorem 2.5.13. *Let $C = \mathcal{E}_E$ be an elementary class. Then*

$$C^{csr} := \{F \in \mathcal{E} \mid \exists E \in C\, (F \leq_B^{cs} E)\},$$
$$C^{cssmh} := \{F \in \mathcal{E} \mid \exists E \in C\, (F \to_B^{cssm} E)\}$$

are equal and elementary, and $C^{csr} = \{F \in \mathcal{E} \mid F \to_B^{cssm} E\}$.

Proof. Exactly as before, we have the following chain of results:

Proposition 2.5.14. *Let $(X, E), (Y, F) \in \mathcal{E}$ and $f : E \to_B^{cssm} F$. Then there is a $(Z, G) \in \mathcal{E}$, a surjective reduction $g : E \leq_B G$, and a class-bijective homomorphism $h : G \to_B^{cb} F$, such that $f = h \circ g$.*

Proof. By Proposition 2.5.8, f can be factored into a surjective reduction g followed by a class-injective homomorphism h; since $h \circ g = f$ is class-surjective and g is surjective, h must be class-surjective, i.e., class-bijective. $\square$

Corollary 2.5.15. *If $E \in \mathcal{E}$ is universally structurable, then $F \leq_B^{cs} E \iff F \to_B^{cssm} E$, for all $F \in \mathcal{E}$. Similarly, if C is an elementary class, then $C^{csr} = C^{cssmh}$.*

Proposition 2.5.16. *Let $E \in \mathcal{E}$. Then there is a sentence σ_E^{cssmh} (in some fixed language) such that for all $F \in \mathcal{E}$, we have $F \models \sigma_E^{cssmh}$ iff $F \to_B^{cssm} E$.*

Proof. Like Proposition 2.5.10, but put $\sigma_E^{cssmh} := \sigma_E^h \wedge \sigma_E^{cs} \wedge \sigma_E^{sm}$. $\square$

It follows that $C^{csr} = C^{cssmh} = \{F \in \mathcal{E} \mid F \to_B^{cssm} E\}$ is elementary. $\square$

Remark 2.5.17. Since any reduction $f : E \leq_B F$ can be factored into a surjective reduction onto its image followed by an embedding, we could have alternatively proved that C^r is elementary (for elementary C) by combining Theorem 2.5.1 with Theorem 2.5.13.

2.5.3 Elementary reducibility classes

We say that an elementary class $C \subseteq \mathcal{E}$ is an **elementary reducibility class** if it is closed under reductions. The following elementary classes mentioned in Section 2.3.1 are elementary reducibility classes: smooth equivalence relations, hyperfinite equivalence relations, treeable equivalence relations [JKL, 3.3], $\mathcal{E}$. The following classes are *not* elementary reducibility classes: finite equivalence relations, aperiodic equivalence relations, compressible equivalence relations, compressible hyperfinite equivalence relations. In Section 2.7, we will prove that for a countably infinite group Γ, $\mathcal{E}_\Gamma^*$ is an elementary reducibility class iff Γ is amenable, where $\mathcal{E}_\Gamma^*$ consists of equivalence relations whose aperiodic part is generated by a free action of Γ.

By Theorem 2.5.2, for every $E \in \mathcal{E}$, $\mathcal{E}_E^r$ is the smallest elementary reducibility class containing E; this is analogous to Corollary 2.4.10. We also have the following analog of Corollary 2.4.12:

Corollary 2.5.18. *A class $C \subseteq \mathcal{E}$ is an elementary reducibility class iff it is closed under smooth homomorphisms and contains an invariantly universal element $E \in C$, in which case $C = \mathcal{E}_E^r$.*

As well, there is the analog of Corollary 2.4.13:

Corollary 2.5.19. *Let $E \in \mathcal{E}$. The following are equivalent:*

(i) E is invariantly universal in $\mathcal{E}_E^r$.

(ii) E is invariantly universal in some elementary reducibility class.

(iii) For every $F \in \mathcal{E}$, $F \to_B^{sm} E$ iff $F \sqsubseteq_B^i E$.

We call $E \in \mathcal{E}$ **stably universally structurable** if these equivalent conditions hold. We write $\mathcal{E}_\infty^r \subseteq \mathcal{E}$ for the class of stably universally structurable countable Borel equivalence relations. For any $E \in \mathcal{E}$, we write $E_{\infty E}^r := E_{\infty\sigma_E^{smh}}$ for the $\sqsubseteq_B^i$-universal element of $\mathcal{E}_E^r$.

As a simple example illustrating these notions, consider the equivalence relation E_0. Its elementary class $\mathcal{E}_{E_0}$ is the class of all aperiodic hyperfinite equivalence relations: since E_0 is aperiodic hyperfinite, so is every $F \in \mathcal{E}_{E_0}$, and conversely every aperiodic hyperfinite F admits a class-bijective homomorphism to E_0 by the Dougherty-Jackson-Kechris classification (Theorem 2.2.1). Thus, $\mathcal{E}_{E_0}$ is not an elementary reducibility class. Its closure $\mathcal{E}^r_{E_0}$ under reduction is the class of all hyperfinite equivalence relations, whose $\sqsubseteq^i_B$-universal element is $E^r_{\infty E_0} \cong \bigoplus_{1 \le n \in \mathbb{N}} (\Delta_{\mathbb{R}} \times I_n) \oplus (\Delta_{\mathbb{R}} \times E_0)$.

Remark 2.5.20. We emphasize that being stably universally structurable is a *stronger* notion than being universally structurable ($\mathcal{E}^r_\infty$ is a transversal of $\leftrightarrow^{sm}_B$, which is a coarser equivalence relation than $\leftrightarrow^{cb}_B$). In particular, "stably universally structurable" is not the same as "$\le_B$-universal in some elementary class" (which would be a weaker notion).

Remark 2.5.21. By Proposition 2.5.8, the preorder $\to^{sm}_B$ on $\mathcal{E}$ is the composite $(\to^{cb}_B) \circ (\le_B)$ of the two preorders $\le_B$ and $\to^{cb}_B$ on $\mathcal{E}$, hence also the join of $\le_B$ and $\to^{cb}_B$ in the complete lattice of all preorders on $\mathcal{E}$ (that are $\cong_B$-invariant, say), i.e., $\to^{sm}_B$ is the finest preorder on $\mathcal{E}$ coarser than both $\le_B$ and $\to^{cb}_B$. Similarly, $\leftrightarrow^{sm}_B$ is the join of $\sim_B$ and $\leftrightarrow^{cb}_B$ in the lattice of equivalence relations on $\mathcal{E}$; this follows from noting that $E \leftrightarrow^{cb}_B E_\infty \otimes E \sim_B E^r_{\infty E}$.

One may ask what is the meet of the preorders $\le_B$ and $\to^{cb}_B$. We do not know of a simple answer. Note that the meet is strictly coarser than $\sqsubseteq^i_B$; indeed, $2 \cdot E_0 \le_B E_0$ and $2 \cdot E_0 \to^{cb}_B E_0$, but $2 \cdot E_0 \not\sqsubseteq^i_B E_0$. (Similarly, the meet of $\sim_B$ and $\leftrightarrow^{cb}_B$ is strictly coarser than $\cong_B$.)

Remark 2.5.22. Clearly one can define similar notions of "elementary embeddability class" and "elementary class-surjective reducibility class".

2.5.4 Reductions and compressibility

Dougherty-Jackson-Kechris proved several results relating Borel reducibility to compressibility [DJK, 2.3, 2.5, 2.6], which we state here in a form suited for our purposes.

Proposition 2.5.23 (Dougherty-Jackson-Kechris)**.** *Let E, F be countable Borel equivalence relations.*

(a) E is compressible iff $E \cong_B E \times I_{\mathbb{N}}$ (and the latter is always compressible).

(b) If E is compressible and $E \sqsubseteq_B F$, then $E \sqsubseteq^i_B F$.

(c) If F is compressible and $E \le^{cs}_B F$, then $E \sqsubseteq^i_B F$.

(d) If E, F are compressible and $E \le_B F$, then $E \sqsubseteq^i_B F$.

(e) $E \le_B F$ iff $E \times I_{\mathbb{N}} \sqsubseteq^i_B F \times I_{\mathbb{N}}$.

Proof. While these were all proved at some point in [DJK], not all of them were stated in this form. For (a), see [DJK, 2.5]. For (b), see [DJK, 2.3]. Clearly (e) follows from (a) and (d) (and that $E \sim_B E \times I_{\mathbb{N}}$). We now sketch (c) and (d), which are implicit in the proof of [DJK, 2.6].

For (c), take $f : E \leq_B^{cs} F$, and let $G \sqsubseteq_B^i F$ be the image of f. Then f is a surjective reduction $E \leq_B G$, hence we can find a $g : G \sqsubseteq_B E$ such that $f \circ g = 1_G$; in particular, g is a complete section embedding. Now G is compressible, so applying [DJK, 2.2], we get $G \cong_B E$, whence $E \cong_B G \sqsubseteq_B^i F$.

For (d), take $f : E \leq_B F$, and let $G \sqsubseteq_B F$ be the image of f. Then $f : E \leq_B^{cs} G$ and $G \sqsubseteq_B F$, whence $E \times I_{\mathbb{N}} \leq_B^{cs} G \times I_{\mathbb{N}} \sqsubseteq_B F \times I_{\mathbb{N}}$. By (a–c), $E \cong_B E \times I_{\mathbb{N}} \sqsubseteq_B^i G \times I_{\mathbb{N}} \sqsubseteq_B^i F \times I_{\mathbb{N}} \cong_B F$. □

Remark 2.5.24. In passing, we note that Proposition 2.5.23(b) and Proposition 2.5.3 together give the following: if E is compressible and $E \to_B^{ci} F$, then $E \to_B^{cb} F$.

It follows from Proposition 2.5.23 that the compressible equivalence relations (up to isomorphism) form a transversal of bireducibility, with corresponding selector $E \mapsto E \times I_{\mathbb{N}}$, which is moreover compatible with the reducibility ordering. We summarize this as follows. Let $\mathcal{E}_c \subseteq \mathcal{E}$ denote the compressible countable Borel equivalence relations.

Corollary 2.5.25. *We have an order-isomorphism of posets*

$$\begin{aligned}(\mathcal{E}/{\sim_B}, \leq_B) &\longleftrightarrow (\mathcal{E}_c/{\cong_B}, \sqsubseteq_B^i)\\ E &\longmapsto E \times I_{\mathbb{N}}.\end{aligned}$$

Remark 2.5.26. Unlike the selector $E \mapsto E_\infty \otimes E$ for $\leftrightarrow_B^{cb}$, the selector $E \mapsto E \times I_{\mathbb{N}}$ for $\sim_B$ does not take E to the $\sqsubseteq_B^i$-greatest element of its $\sim_B$-class (e.g., $E_0 \times I_{\mathbb{N}} \cong_B E_t \sqsubset_B^i E_0$). Nor does it always take E to the $\sqsubseteq_B^i$-least element of its $\sim_B$-class, or even to an element $\sqsubseteq_B^i$-less than E: for finite E clearly $E \times I_{\mathbb{N}} \not\sqsubseteq_B^i E$, while for aperiodic E, a result of Thomas [T] (see also [HK, 3.9]) states that there are aperiodic E such that $E \times I_2 \not\sqsubseteq_B E$.

We now relate compressibility to structurability. Let $E_{\infty c}$ denote the invariantly universal compressible countable Borel equivalence relation, i.e., the $\sqsubseteq_B^i$-universal element of $\mathcal{E}_c$. Aside from $E \mapsto E \times I_{\mathbb{N}}$, we have another canonical way of turning any E into a compressible equivalence relation, namely $E \mapsto E_{\infty c} \otimes E$. These two maps are related as follows:

Proposition 2.5.27. *Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation.*

(a) $E_{\infty c} \otimes E \to_B^{cb} E \times I_{\mathbb{N}}$.

(b) Suppose E is universally structurable. Then:

(i) $E \times I_{\mathbb{N}}$ *is universally structurable;*

(ii) $E_{\infty c} \otimes E \sqsubseteq_B^i E \times I_{\mathbb{N}}$*;*

(iii) $E \times I_{\mathbb{N}} \sqsubseteq_B E$ *iff* $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c} \otimes E$ *(iff* $E \times I_{\mathbb{N}} \cong_B E_{\infty c} \otimes E$*).*

Proof. For (a), we have $E_{\infty c} \otimes E \to_B^{cb} E$, whence $E_{\infty c} \otimes E \cong_B (E_{\infty c} \otimes E) \times I_{\mathbb{N}} \to_B^{cb} E \times I_{\mathbb{N}}$.

For (i), let $f : F \to_B^{cb} E \times I_{\mathbb{N}}$; we need to show that $F \sqsubseteq_B^i E \times I_{\mathbb{N}}$. Letting $F_0 := F|f^{-1}(X \times \{0\})$, it is easily seen that $F \cong F_0 \times I_{\mathbb{N}}$. We have $f|f^{-1}(X \times \{0\}) : F_0 \to_B^{cb} (E \times I_{\mathbb{N}})|(X \times \{0\}) \cong E$, so $F_0 \sqsubseteq_B^i E$ by universal structurability of E, whence $F \cong F_0 \times I_{\mathbb{N}} \sqsubseteq_B^i E \times I_{\mathbb{N}}$. (This argument is due to Anush Tserunyan, and is simpler than our original argument.)

(ii) follows from (a) and (i).

For (iii), if $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c} \otimes E$, then $E_{\infty c} \otimes E \sqsubseteq_B^i E$ gives $E \times I_{\mathbb{N}} \sqsubseteq_B^i E$. Conversely, if $E \times I_{\mathbb{N}} \sqsubseteq_B E$, then since $E \times I_{\mathbb{N}}$ is compressible, $E \times I_{\mathbb{N}} \sqsubseteq_B^i E$, and also $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c}$, whence $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c} \otimes E$. □

Remark 2.5.28. We do not know if there is an aperiodic universally structurable E with $E \times I_{\mathbb{N}} \not\sqsubseteq_B E$. The example of Thomas [T] mentioned above is far from universally structurable, since it has a unique ergodic invariant probability measure.

We call a bireducibility class $C \subseteq \mathcal{E}$ **universally structurable** if it contains a universally structurable element. In this case, by Theorem 2.5.2, C contains an invariantly universal (stably universally structurable) element, namely $E^r_{\infty E}$ for any $E \in C$; and by Proposition 2.5.27, it also contains a compressible universally structurable element, namely $E \times I_{\mathbb{N}}$ for any $E \in C$. Between these two (in the ordering $\sqsubseteq_B^i$) lie all those universally structurable $E \in C$ such that $E \times I_{\mathbb{N}} \sqsubseteq_B E$.

Let $\mathcal{E}_{\infty c} := \mathcal{E}_\infty \cap \mathcal{E}_c$ denote the class of compressible universally structurable equivalence relations. Since $\mathcal{E}_\infty$ forms a transversal (up to isomorphism) of the equivalence relation $\leftrightarrow_B^{cb}$, while $\mathcal{E}_c$ forms a transversal of $\sim_B$, we would expect $\mathcal{E}_{\infty c}$ to form a transversal of $\leftrightarrow_B^{sm}$, the join of $\leftrightarrow_B^{cb}$ and $\sim_B$. That this is the case follows from the fact that the two corresponding selectors $E \mapsto E_\infty \otimes E$ (for $\leftrightarrow_B$) and $E \mapsto E \times I_{\mathbb{N}}$ (for $\sim_B$) commute:

Proposition 2.5.29. *For any $E \in \mathcal{E}$, $(E_\infty \otimes E) \times I_{\mathbb{N}} \cong_B E_\infty \otimes (E \times I_{\mathbb{N}})$.*

Proof. We have $E_\infty \otimes E \to_B^{cb} E$, whence $(E_\infty \otimes E) \times I_{\mathbb{N}} \to_B^{cb} E \times I_{\mathbb{N}}$, and so $(E_\infty \otimes E) \times I_{\mathbb{N}} \sqsubseteq_B^i E_\infty \otimes (E \times I_{\mathbb{N}})$. Conversely, we have $E \sqsubseteq_B^i E_\infty \otimes E$, whence $E \times I_{\mathbb{N}} \sqsubseteq_B^i (E_\infty \otimes E) \times I_{\mathbb{N}}$, and so $E_\infty \otimes (E \times I_{\mathbb{N}}) \sqsubseteq_B^i (E_\infty \otimes E) \times I_{\mathbb{N}}$, since the latter is universally structurable by Proposition 2.5.27. □

2.6 The poset of elementary classes

In this section, we consider the order-theoretic structure of the poset of elementary classes under inclusion (equivalently the poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$), as well as the poset of elementary reducibility classes under inclusion (equivalently $(\mathcal{E}_\infty^r/{\cong_B}, \sqsubseteq_B^i)$, or $(\mathcal{E}_{\infty c}/{\cong_B}, \sqsubseteq_B^i)$, or $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$).

In Section 2.6.1, we introduce some concepts from order theory which give us a convenient way of concisely stating several results from previous sections. In Section 2.6.2, we discuss meets and joins in the poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$. In Section 2.6.4, we extend a well-known result of Adams-Kechris [AK] to show that $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ is quite complicated, by embedding the poset of Borel subsets of reals.

We remark that we always consider the empty equivalence relation $\varnothing$ on the empty set to be a countable Borel equivalence relation; this is particularly important in this section. Note that $\varnothing$ is (vacuously) σ-structurable for any σ, hence is the $\sqsubseteq_B^i$-universal $\perp$-structurable equivalence relation, where $\perp$ denotes an inconsistent theory.

2.6.1 Projections and closures

Among the various posets (or preordered sets) of equivalence relations we have considered so far (e.g., $(\mathcal{E}, \to_B^{cb})$, $(\mathcal{E}_\infty, \sqsubseteq_B^i)$, $(\mathcal{E}_c, \sqsubseteq_B^i)$), there is one which is both the finest and the most inclusive, namely $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$. Several of the other posets and preorders may be viewed as derived from $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$ via the following general order-theoretic notion.

Let $(P, \leq)$ be a poset. A **projection operator** on P is an idempotent order-preserving map $e : P \to P$, i.e.,

$$\forall x, y \in P\,(x \leq y \implies e(x) \leq e(y)), \qquad\qquad e \circ e = e.$$

The image $e(P)$ of a projection operator e is a **retract** of P, i.e., the inclusion $i : e(P) \to P$ has a one-sided (order-preserving) inverse $e : P \to e(P)$, such that $e \circ i = 1_{e(P)}$. A projection operator e also gives rise to an **induced preorder** $\lesssim$ on P, namely the pullback of $\leq$ along e, i.e.,

$$x \lesssim y \iff e(x) \leq e(y).$$

Letting $\sim := \ker e$, which is also the equivalence relation associated with $\lesssim$, we thus have two posets derived from $(P, \leq)$ associated with each projection operator e, namely the quotient poset $(P/{\sim}, \lesssim)$ and the subposet $(e(P), \leq)$. These are related by an order-isomorphism:

$$\begin{aligned} (P/{\sim}, \lesssim) &\longleftrightarrow (e(P), \leq) = (e(P), \lesssim) \\ [x]_\sim &\longmapsto e(x) \\ e^{-1}(y) = [y]_\sim &\longleftarrow\!\shortmid\; y. \end{aligned}$$

(There is the following analogy with equivalence relations: set $\leftrightarrow$ poset, equivalence relation $\leftrightarrow$ preorder, selector $\leftrightarrow$ projection, and transversal $\leftrightarrow$ retract.)

Summarizing previous results, we list here several projection operators on $(\mathcal{E}/{\cong_B}, \sqsubseteq^i_B)$ that we have encountered, together with their images and induced preorders.

- $E \mapsto E_\infty \otimes E$, which has image $\mathcal{E}_\infty/{\cong_B}$ (the universally structurable equivalence relations) and induces the preorder $\to^{cb}_B$ (Section 2.4.3);
- $E \mapsto E \times I_{\mathbb{N}}$, which has image $\mathcal{E}_c/{\cong_B}$ (the compressible equivalence relations) and induces the preorder $\leq_B$ (Proposition 2.5.23);
- $E \mapsto (E_\infty \otimes E) \times I_{\mathbb{N}} \cong_B E_\infty \otimes (E \times I_{\mathbb{N}})$ (Proposition 2.5.29), which has image $\mathcal{E}_{\infty c}/{\cong_B}$ (the compressible universally structurable equivalence relations) and induces the preorder $\to^{sm}_B$;
- $E \mapsto E^r_{\infty E}$ (the $\sqsubseteq^i_B$-universal element of $\mathcal{E}^r_E$), which has image $\mathcal{E}^r_\infty/{\cong_B}$ (the stably universally structurable equivalence relations) and also induces the preorder $\to^{sm}_B$;
- similarly, $E \mapsto$ the $\sqsubseteq^i_B$-universal element of $\mathcal{E}^e_E$, which induces $\to^{ci}_B$.

Also note that some of these projection operators can be restricted to the images of others; e.g., the restriction of $E \mapsto E \times I_{\mathbb{N}}$ to $\mathcal{E}_\infty$ is a projection operator on $\mathcal{E}_\infty/{\cong_B}$ (by Proposition 2.5.27), with image $\mathcal{E}_{\infty c}/{\cong_B}$.

Again let $(P, \leq)$ be a poset, and let $e : P \to P$ be a projection operator. We say that e is a **closure operator** if

$$\forall x \in P\, (x \leq e(x)).$$

In other words, each $e(x)$ is the ($\leq$-)greatest element of its $\sim$-class. In that case, the induced preorder $\lesssim$ satisfies

$$x \lesssim y \iff x \leq e(y) \quad (\iff e(x) \leq e(y)).$$

Among the projection operators on $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$ listed above, three are closure operators, namely $E \mapsto$ the $\sqsubseteq_B^i$-universal element of $\mathcal{E}_E$, $\mathcal{E}_E^r$, or $\mathcal{E}_E^e$ (the first of these being $E \mapsto E_\infty \otimes E$).

For another example, let us say that a countable Borel equivalence relation $E \in \mathcal{E}$ is **idempotent** if $E \cong_B E \oplus E$. This is easily seen to be equivalent to $E \cong_B \aleph_0 \cdot E$; hence, the idempotent elements of $\mathcal{E}$ form the image of the closure operator $E \mapsto \aleph_0 \cdot E$ on $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$. Note that all universally structurable equivalence relations are idempotent (Corollary 2.4.16).

2.6.2 The lattice structure

We now discuss the lattice structure of the poset of elementary classes under inclusion, equivalently the poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ of universally structurable isomorphism classes under $\sqsubseteq_B^i$.

Let us first introduce the following notation. For theories (L, σ) and (L', τ), we write

$$(L, \sigma) \Rightarrow^* (L', \tau) \qquad (\text{or } \sigma \Rightarrow^* \tau)$$

to mean that $\mathcal{E}_\sigma \subseteq \mathcal{E}_\tau$, i.e., for every $E \in \mathcal{E}$, if $E \models \sigma$, then $E \models \tau$. Thus $\Rightarrow^*$ is a preorder on the class of theories which is equivalent to the poset of elementary classes (via $\sigma \mapsto \mathcal{E}_\sigma$), and hence also to the poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ (via $\sigma \mapsto E_{\infty\sigma}$). We denote the associated equivalence relation by $\Leftrightarrow^*$.

Remark 2.6.1. We stress that in the notation $\sigma \Rightarrow^* \tau$, σ and τ may belong to *different* languages. Of course, if they happen to belong to the same language and σ logically implies τ, then also $\sigma \Rightarrow^* \tau$; but the latter is in general a weaker condition.

Let $(P, \leq)$ be a poset. We say that P is an ω_1**-complete lattice** if every countable subset $A \subseteq P$ has a meet (i.e., greatest lower bound) $\bigwedge A$, as well as a join (i.e., least upper bound) $\bigvee A$. We say that P is an ω_1-**distributive lattice** if it is an ω_1-complete lattice which satisfies the ω_1-**distributive laws**

$$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i), \qquad\qquad x \vee \bigwedge_i y_i = \bigwedge_i (x \vee y_i),$$

where i runs over a countable index set.

Theorem 2.6.2. *The poset $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ is an ω_1-distributive lattice, in which joins are given by $\bigoplus$, nonempty meets are given by $\bigotimes$, the greatest element is E_∞, and the least element is $\varnothing$.*

Moreover, the inclusion $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i) \subseteq (\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$ preserves (countable) meets and joins. In other words, if $E_0, E_1, \ldots \in \mathcal{E}_\infty$ are universally structurable equivalence relations, then $\bigotimes_i E_i$ (respectively $\bigoplus_i E_i$) is their meet (respectively join) in $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$ as well as in $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$.

Before giving the proof of Theorem 2.6.2, we discuss the operations on theories which correspond to the operations $\bigotimes$ and $\bigoplus$. That is, let $((L_i, \sigma_i))_i$ be a countable family of theories; we want to find theories (L', σ') and (L'', σ'') such that $\bigotimes_i E_{\infty\sigma_i} \cong_B E_{\infty\sigma'}$ and $\bigoplus_i E_{\infty\sigma_i} \cong_B E_{\infty\sigma''}$.

Proposition 2.6.3. *Let $\bigotimes_i (L_i, \sigma_i) = (\bigsqcup_i L_i, \bigotimes_i \sigma_i)$ be the theory where $\bigsqcup_i L_i$ is the disjoint union of the L_i, and $\bigotimes_i \sigma_i$ is the conjunction of the σ_i's regarded as being in the language $\bigsqcup_i L_i$ (so that the different σ_i's have disjoint languages). Then $\bigotimes_i E_{\infty\sigma_i} \cong_B E_{\infty \bigotimes_i \sigma_i}$.*

Proof. For each i, the $(\bigotimes_j \sigma_j)$-structure on $E_{\infty \bigotimes_j \sigma_j}$ has a reduct which is a σ_i-structure, so $E_{\infty \bigotimes_j \sigma_j} \models \sigma_i$, i.e., $E_{\infty \bigotimes_j \sigma_j} \sqsubseteq_B^i E_{\infty\sigma_i}$; hence $E_{\infty \bigotimes_j \sigma_j} \sqsubseteq_B^i \bigotimes_i E_{\infty\sigma_i}$. Conversely, for each j we have $\bigotimes_i E_{\infty\sigma_i} \to_B^{cb} E_{\infty\sigma_j} \models \sigma_j$ so $\bigotimes_i E_{\infty\sigma_i} \models \sigma_j$; combining these σ_j-structures yields a $\bigotimes_j \sigma_j$-structure, so $\bigotimes_i E_{\infty\sigma_i} \sqsubseteq_B^i E_{\infty \bigotimes_j \sigma_j}$. $\square$

While we can similarly prove that $\bigoplus_i E_{\infty\sigma_i}$ corresponds to the theory given by the disjunction of the σ_i's, we prefer to work with the following variant, which is slightly better behaved with respect to structurability. Let $\bigoplus_i (L_i, \sigma_i) = (\bigoplus_i L_i, \bigoplus_i \sigma_i)$ be the theory where $\bigoplus_i L_i := \bigsqcup_i (L_i \sqcup \{P_i\})$ where each is P_i is a unary relation symbol, and

$$\bigoplus_i \sigma_i := \bigvee_i ((\forall x\, P_i(x)) \wedge \sigma_i \wedge \bigwedge_{j \neq i} \bigwedge_{R \in L_i \sqcup \{P_i\}} \forall \overline{x}\, \neg R(\overline{x}))$$

(where on the right-hand side, σ_i is regarded as having language $\bigoplus_i L_i$). In other words, $\bigoplus_i \sigma_i$ asserts that for some (unique) i, P_i holds for all elements, and we have a σ_i-structure; and for all $j \neq i$, P_j and all relations in L_j hold for no elements. Then for a countable Borel equivalence relation $(X, E) \in \mathcal{E}$, a $\bigoplus_i \sigma_i$-structure $\mathbb{A} : E \models \bigoplus_i \sigma_i$ is the same thing as a Borel E-invariant partition $(P_i^{\mathbb{A}})_i$ of X, together with a σ_i-structure $\mathbb{A}|P_i^{\mathbb{A}} : E|P_i^{\mathbb{A}} \models \sigma_i$ for each i.

Proposition 2.6.4. $\bigoplus_i E_{\infty\sigma_i} \cong_B E_{\infty \bigoplus_i \sigma_i}$.

Proof. The σ_i-structure on each $E_{\infty\sigma_i}$ yields a $\bigoplus_j \sigma_j$-structure (with $P_i^{\mathbb{A}}$ = everything, and $P_j^{\mathbb{A}} = \varnothing$ for $j \neq i$); so $\bigoplus_i E_{\infty\sigma_i} \sqsubseteq_B^i E_{\infty \bigoplus_j \sigma_j}$. Conversely, letting $\mathbb{A} : E_{\infty \bigoplus_j \sigma_j} \models \bigoplus_j \sigma_j$, we have $E_{\infty \bigoplus_j \sigma_j} = \bigoplus_i E_{\infty \bigoplus_j \sigma_j}|P_i^{\mathbb{A}}$ and $\mathbb{A}|P_i^{\mathbb{A}} : E_{\infty \bigoplus_j \sigma_j}|P_i^{\mathbb{A}} \models \sigma_i$ for each i, whence $E_{\infty \bigoplus_i \sigma_i} \sqsubseteq_B^i \bigoplus_i E_{\infty\sigma_i}$. ◻

As noted in [KMd, 2.C], the next lemma follows from abstract properties of the poset $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$; for the convenience of the reader, we include a direct proof.

Lemma 2.6.5. *Let $E_0, E_1, \ldots \in \mathcal{E}$ be countably many idempotent countable Borel equivalence relations. Then $\bigoplus_i E_i$ is their join in the preorder $(\mathcal{E}, \sqsubseteq_B^i)$.*

Proof. Clearly each $E_j \sqsubseteq_B^i \bigoplus_i E_i$. Let $F \in \mathcal{E}$ and $E_i \sqsubseteq_B^i F$ for each i; we must show that $\bigoplus_i E_i \sqsubseteq_B^i F$. Since each $E_i \sqsubseteq_B^i F$, we have $F \cong_B E_i \oplus F_i$ for some F_i; since $E_i \cong_B E_i \oplus E_i$, we have $F \cong_B E_i \oplus E_i \oplus F_i \cong_B E_i \oplus F$. So an invariant embedding $\bigoplus_i E_i \sqsubseteq_B^i F$ is built by invariantly embedding E_0 into $E_0 \oplus F \cong_B F$ so that the remainder (complement of the image) is isomorphic to F, then similarly embedding E_1 into the remainder, etc. ◻

Proof of Theorem 2.6.2. By Propositions 2.6.3 and 2.6.4, we may freely switch between the operations $\bigotimes$ and $\bigoplus$ on universally structurable equivalence relations, and the same operations on theories.

First we check that $\bigotimes$ is meet and $\bigoplus$ is join. Let $(L_0, \sigma_0), (L_1, \sigma_1), \ldots$ be theories; it suffices to show that $\bigotimes_i \sigma_i$, resp., $\bigoplus_i \sigma_i$, is their meet, resp., join, in the preorder $\Rightarrow^*$. For $\bigotimes$ this is clear, since a $(\bigotimes_i \sigma_i)$-structure on $E \in \mathcal{E}$ is the same thing as a σ_i-structure for each i. For $\bigoplus$, a σ_i-structure on E for any i yields a $(\bigoplus_j \sigma_j)$-structure (corresponding to the partition of E where the ith piece is everything); thus $\sigma_i \Rightarrow^* \bigoplus_j \sigma_j$ for each i. And if (L', τ) is another theory with $\sigma_i \Rightarrow^* \tau$ for each i, then given a $(\bigoplus_i \sigma_i)$-structure on E, we have a partition of E into pieces which are

σ_i-structured for each i, so by $\sigma_i \Rightarrow^* \tau$ we can τ-structure each piece of the partition; thus $\bigoplus_i \sigma_i \Rightarrow^* \tau$.

That the inclusion $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i) \to (\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$ preserves (*all* existing) meets follows from the fact that $\mathcal{E}_\infty/{\cong_B} \subseteq \mathcal{E}/{\cong_B}$ is the image of the closure operator $E \mapsto E_\infty \otimes E$. That it preserves countable joins follows from Lemma 2.6.5.

Now we check the ω_1-distributive laws. Distributivity of $\otimes$ over $\bigoplus$ follows from Proposition 2.4.22(e). To check distributivity of $\oplus$ over $\bigotimes$, we again work with theories. Let $\sigma, \tau_0, \tau_1, \ldots$ be theories; we need to show that $\sigma \oplus \bigotimes_i \tau_i \Leftrightarrow^* \bigotimes_i (\sigma \oplus \tau_i)$. The $\Rightarrow^*$ inequality, as in any lattice, is trivial. For the converse inequality, let $(X, E) \in \mathcal{E}$ and $\mathbb{A} : E \models \bigotimes_i (\sigma \oplus \tau_i)$, which amounts to a $\mathbb{A}_i : E \models \sigma \oplus \tau_i$ for each i. Then for each i, we have a Borel E-invariant partition $X = A_i \cup B_i$ such that $\mathbb{A}_i|A_i : E|A_i \models \sigma$ and $\mathbb{A}_i|B_i : E|B_i \models \tau_i$. By combining the various $\mathbb{A}_i$, we get $E|\bigcup_i A_i \models \sigma$ and $G|\bigcap_i B_i \models \bigotimes_i \tau_i$; and so the partition $X = (\bigcup_i A_i) \cup (\bigcap_i B_i)$ witnesses that $E \models \sigma \oplus \bigotimes_i \tau_i$. $\square$

Remark 2.6.6. It is not true that $\bigoplus$ is join in $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$, since there exist $E \in \mathcal{E}$ such that $E \ncong_B E \oplus E$ (e.g., $E = E_0$). Similarly, it is not true that $\bigotimes$ is meet in $(\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$, since there are E with $E \ncong_B E \otimes E$ (see examples near the end of Section 2.4.4).

Remark 2.6.7. That the inclusion $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i) \to (\mathcal{E}/{\cong_B}, \sqsubseteq_B^i)$ preserves countable joins suggests that perhaps $\mathcal{E}_\infty/{\cong_B} \subseteq \mathcal{E}/{\cong_B}$ is also the image of an "interior operator". This would mean that every countable Borel equivalence relation $E \in \mathcal{E}$ contains (in the sense of $\sqsubseteq_B^i$) a greatest universally structurable equivalence relation. We do not know if this is true.

By restricting Theorem 2.6.2 to the class $\mathcal{E}_c$ of compressible equivalence relations, which is downward-closed under $\sqsubseteq_B^i$, closed under $\bigoplus$, and has greatest element $E_{\infty c}$, we immediately obtain

Corollary 2.6.8. *The poset $(\mathcal{E}_{\infty c}/{\cong_B}, \sqsubseteq_B^i)$ is an ω_1-distributive lattice, in which joins are given by $\bigoplus$, nonempty meets are given by $\bigotimes$, the greatest element is $E_{\infty c}$, and the least element is $\varnothing$. Moreover, the inclusion $(\mathcal{E}_{\infty c}/{\cong_B}, \sqsubseteq_B^i) \subseteq (\mathcal{E}_c/{\cong_B}, \sqsubseteq_B^i)$ preserves (countable) meets and joins.*

Now using that $(\mathcal{E}_c/{\cong_B}, \sqsubseteq_B^i)$ is isomorphic to $(\mathcal{E}/{\sim_B}, \leq_B)$, we may rephrase this as

Corollary 2.6.9. *The poset of universally structurable bireducibility classes under $\le_B$ is an ω_1-distributive lattice. Moreover, the inclusion into the poset $(\mathcal{E}/{\sim_B}, \le_B)$ of all bireducibility classes under $\le_B$ preserves (countable) meets and joins.*

Remark 2.6.10. We stress that the $\le_B$-meets in Corollary 2.6.9 must be computed using the *compressible* elements of bireducibility classes. That is, if E, F are universally structurable, then their $\le_B$-meet is $(E \times I_{\mathbb{N}}) \otimes (F \times I_{\mathbb{N}})$, but *not necessarily* $E \otimes F$. For example, if E is invariantly universal finite and F is invariantly universal aperiodic, then $E \otimes F = \varnothing$ is clearly not the $\le_B$-meet of E, F. Also, if there is an aperiodic universally structurable E with $E \times I_{\mathbb{N}} \not\sqsubseteq_B E$, then (by Proposition 2.5.27) $E \otimes E_{\infty c}$ is not the $\le_B$-meet of E and $E_{\infty c} \sim_B E_\infty$.

The order-theoretic structure of the poset $(\mathcal{E}/{\sim_B}, \le_B)$ of *all* bireducibility classes under $\le_B$ is not well-understood, apart from that it is very complicated (by [AK]). The first study of this structure was made by Kechris-Macdonald in [KMd]. In particular, they raised the question of whether there exists any pair of $\le_B$-incomparable $E, F \in \mathcal{E}$ for which a $\le_B$-meet exists. Corollary 2.6.9, together with the existence of many $\le_B$-incomparable universally structurable bireducibility classes (Theorem 2.6.20), answers this question by providing a large class of bireducibility classes for which $\le_B$-meets always exist.

There are some natural order-theoretic questions one could ask about the posets $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ and $(\mathcal{E}_\infty/{\sim}, \le_B)$, which we do not know how to answer. For example, is either a complete lattice? If so, is it completely distributive? Is it a "zero-dimensional" ω_1-complete lattice, in that it embeds (preserving all countable meets and joins) into 2^X for some set X? (See Corollary 2.6.15 below for some partial results concerning this last question.)

Remark 2.6.11. It can be shown that every ω_1-distributive lattice is a *quotient* of a sublattice of 2^X for some set X (see the arXiv version of this paper). In particular, this implies that the set of algebraic identities involving $\bigotimes$ and $\bigoplus$ which hold in $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ is "completely understood", in that it consists of exactly those identities which hold in $2 = \{0 < 1\}$.

2.6.3 Closure under independent joins

We mention here some connections with recent work of Marks [M].

Let $E_0, E_1, \ldots$ be countably many countable Borel equivalence relations on the same standard Borel space X. We say that the E_i are **independent** if there is no sequence $x_0, x_1, \ldots, x_n$ of distinct elements of X, where $n \geq 1$, such that $x_0 \ E_{i_0} \ x_1 \ E_{i_1} \cdots E_{i_{n-1}} \ x_n \ E_{i_n} \ x_0$ for some $i_0, \ldots, i_n$ with $i_j \neq i_{j+1}$ for each j. In that case, their **independent join** is the smallest equivalence relation on X containing each E_i. For example, the independent join of treeable equivalence relations is still treeable. Marks proves the following for elementary classes closed under independent joins [M, 4.15, 4.16]:

Theorem 2.6.12 (Marks)**.** *If $\mathcal{E}_\sigma$ is an elementary class of aperiodic equivalence relations closed under binary independent joins, then for any Borel homomorphism $p : E_{\infty\sigma} \to_B \Delta_X$ (where X is any standard Borel space), there is some $x \in X$ such that $E_{\infty\sigma} \sim_B E_{\infty\sigma}|p^{-1}(x)$.*

Theorem 2.6.13 (Marks)**.** *If $\mathcal{E}_\sigma$ is an elementary class of aperiodic equivalence relations closed under countable independent joins, then for any $E \in \mathcal{E}$, if $E_{\infty\sigma} \leq_B E$, then $E_{\infty\sigma} \sqsubseteq_B E$.*

Remark 2.6.14. Clearly the aperiodicity condition in Theorems 2.6.12 and 2.6.13 can be loosened to the condition that $I_{\mathbb{N}} \in \mathcal{E}_\sigma$ (so that restricting $\mathcal{E}_\sigma$ to the aperiodic elements does not change $E_{\infty\sigma}$ up to biembeddability).

Above we asked whether the ω_1-distributive lattice $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ is zero-dimensional, i.e., embeds into 2^X for some set X. This is equivalent to asking whether there are enough **ω_1-prime filters** (i.e., filters closed under countable meets whose complements are closed under countable joins) in $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ to separate points. Theorem 2.6.12 gives some examples of ω_1-prime filters:

Corollary 2.6.15. *If $\mathcal{E}_\sigma$ contains $I_{\mathbb{N}}$ and is closed under binary independent joins, then*

$$\{E \in \mathcal{E}_\infty \mid E_{\infty\sigma} \leq_B E\}$$

is an ω_1-prime filter in $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$.

Proof. If $g : E_{\infty\sigma} \leq_B \bigoplus_i E_i$ then we have a homomorphism $E_{\infty\sigma} \to_B \Delta_{\mathbb{N}}$ sending the g-preimage of E_i to i; by Theorem 2.6.12, it follows that $E_{\infty\sigma} \leq_B E_i$ for some i. ∎

We also have the following simple consequence of Theorem 2.6.13:

Corollary 2.6.16. *If $\mathcal{E}_\sigma$ is an elementary class such that $\mathcal{E}_\sigma^r$ is closed under countable independent joins, then $\mathcal{E}_\sigma^e = \mathcal{E}_\sigma^r$.*

Proof. The $\sqsubseteq_B^i$-universal element of $\mathcal{E}_\sigma^r$ reduces to $E_{\infty\sigma}$, whence by Theorem 2.6.13 it embeds into $E_{\infty\sigma}$, i.e., belongs to $\mathcal{E}_\sigma^e$. □

Remark 2.6.17. Although the conclusions of Theorems 2.6.12 and 2.6.13 are invariant with respect to bireducibility (respectively biembeddability), Marks has pointed out that the notion of being closed under independent joins is not similarly invariant: there are $E_{\infty\sigma} \sim_B E_{\infty\tau}$ such that $\mathcal{E}_\sigma$ is closed under independent joins but $\mathcal{E}_\tau$ is not. In particular, if σ axiomatizes trees while τ axiomatizes trees of degree ≤ 3, then $\mathcal{E}_{\infty\sigma} \sim_B E_{\infty\tau}$ by [JKL, 3.10]; but it is easy to see (using an argument like that in Proposition 2.6.18 below) that independent joins of τ-structurable equivalence relations can have arbitrarily high cost, so are not all τ-structurable.

Clearly if $\mathcal{E}_\sigma, \mathcal{E}_\tau$ are closed under independent joins, then so is $\mathcal{E}_{\sigma\otimes\tau} = \mathcal{E}_\sigma \cap \mathcal{E}_\tau$. In particular, the class $\mathcal{E}_c$ of compressible equivalence relations is closed under arbitrary (countable) joins, since the join of compressible equivalence relations contains a compressible equivalence relation; thus the class of compressible treeable equivalence relations is closed under independent joins. We note that this is the smallest nontrivial elementary class to which Theorems 2.6.12 and 2.6.13 apply:

Proposition 2.6.18. *If $\mathcal{E}_\sigma$ is an elementary class containing $I_\mathbb{N}$ and closed under binary independent joins, then $\mathcal{E}_\sigma$ contains all compressible treeable equivalence relations.*

Proof. Since $\mathcal{E}_\sigma$ is elementary and contains $I_\mathbb{N}$, it contains all aperiodic smooth countable Borel equivalence relations. Now let $(X, E) \in \mathcal{E}$ be compressible treeable. By [JKL, 3.11], there is a Borel treeing $T \subseteq E$ with degree ≤ 3. By [KST, 4.6] (see also remarks following [KST, 4.10]), there is a Borel edge coloring $c : T \to 5$. Then E is the independent join of the equivalence relations $E_i := c^{-1}(i) \cup \Delta_X$ for $i = 0, 1, 2, 3, 4$. Since the E_i are not aperiodic, consider the following modification. Let

$$X' := X \sqcup (X \times 5 \times \mathbb{N}),$$

let T' be the tree on X' consisting of T on X and the edges $(x, (x, i, 0))$ and $((x, i, n), (x, i, n + 1))$ for $x \in X$, $i \in 5$, and $n \in \mathbb{N}$, and let $c' : T' \to 5$ extend c with

$c'(x,(x,i,0)) = c'((x,i,n),(x,i,n+1)) = i$ (note that c' is not an edge coloring). Then the inclusion $X \to X'$ is a complete section embedding of each E_i into the equivalence relation E'_i generated by $c'^{-1}(i)$, and of E into the equivalence relation E' generated by T'. It follows that each E'_i is (aperiodic) smooth (because E_i is), hence in $\mathcal{E}_\sigma$, while $E \cong_B E'$ (because E is compressible). But it is easily seen that E' is the independent join of the E'_i, whence $E \in \mathcal{E}_\sigma$. □

2.6.4 Embedding the poset of Borel sets

Adams-Kechris [AK] proved the following result showing that the poset $(\mathcal{E}/{\sim_B}, \leq_B)$ is extremely complicated:

Theorem 2.6.19 (Adams-Kechris)**.** *There is an order-embedding from the poset of Borel subsets of $\mathbb{R}$ under inclusion into the poset $(\mathcal{E}/{\sim_B}, \leq_B)$.*

In this short section, we show that their proof may be strengthened to yield

Theorem 2.6.20. *There is an order-embedding from the poset of Borel subsets of $\mathbb{R}$ under inclusion into the poset $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$.*

Proof. By [AK, 4.2], there is a countable Borel equivalence relation (X,E), a Borel homomorphism $p : (X,E) \to_B (\mathbb{R}, \Delta_\mathbb{R})$, and a Borel map $x \mapsto \mu_x$ taking each $x \in \mathbb{R}$ to a Borel probability measure μ_x on X, such that, putting $E_x := E|p^{-1}(x)$, we have

(i) for each $x \in \mathbb{R}$, μ_x is nonatomic, concentrated on $p^{-1}(x)$, E_x-invariant, and E_x-ergodic;

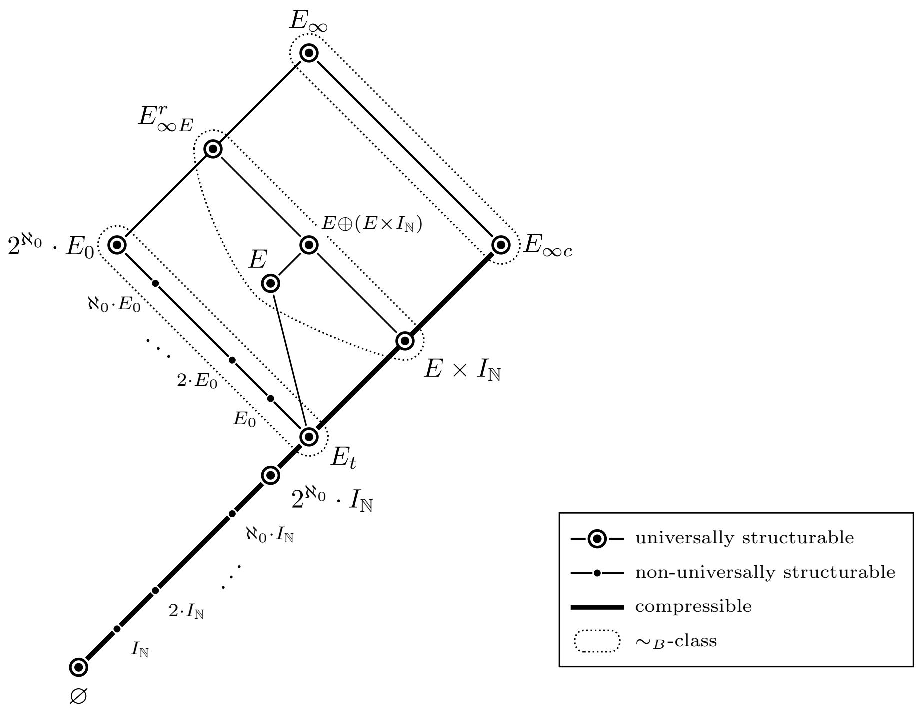
(ii) if $x, y \in \mathbb{R}$ with $x \neq y$, then every Borel homomorphism $f : E_x \to_B E_y$ maps a Borel E_x-invariant set $M \subseteq p^{-1}(x)$ of μ_x-measure 1 to a single E_y-class.

For Borel $A \subseteq \mathbb{R}$, put $E_A := E|p^{-1}(A)$ and $F_A := E_\infty \otimes E_A$. We claim that $A \mapsto F_A$ gives the desired order-embedding. It is clearly order-preserving. Now suppose $A, B \subseteq \mathbb{R}$ with $A \not\subseteq B$ but $F_A \leq_B F_B$. By taking $x \in A \setminus B$, we get $x \notin B$ but $E_x \sqsubseteq^i_B E_\infty \otimes E_x = F_{\{x\}} \leq_B F_A \leq_B F_B$. Let $f : E_x \leq_B F_B = E_\infty \otimes E_B$, and let $\pi_2 : E_\infty \otimes E_B \to^{cb}_B E_B$ be the second projection. Then $p \circ \pi_2 \circ f : E_x \to_B \Delta_B$, whence by E_x-ergodicity of μ_x, there is a $y \in B$ and an E_x-invariant $M \subseteq p^{-1}(x)$ of μ_x-measure 1 such that $(p \circ \pi_2 \circ f)(M) = \{y\}$, i.e., $(\pi_2 \circ f)(M) \subseteq p^{-1}(y)$. By (ii) above, there is a further E_x-invariant $N \subseteq M$ of μ_x-measure 1 such that $(\pi_2 \circ f)(N)$ is contained in a single E_y-class. But since π_2 is class-bijective and f is a reduction, this implies that $E|N$ is smooth, a contradiction. □

Remark 2.6.21. If in Theorem 2.6.20 we replace $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ with $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq^i_B)$ (thus weakening the result), then a simpler proof may be given, using groups of different costs (see [KM, 36.4]) instead of [AK].

2.6.5 A global picture

The picture below is a simple visualization of the poset $(\mathcal{E}/{\cong_B}, \sqsubseteq^i_B)$. For the sake of clarity, among the hyperfinite equivalence relations, only the aperiodic ones are shown.



Six landmark universally structurable equivalence relations are shown (circled dots): $\varnothing$, $2^{\aleph_0} \cdot I_\mathbb{N}$ ($\sqsubseteq^i_B$-universal aperiodic smooth), E_t ($\sqsubseteq^i_B$-universal compressible hyperfinite), $2^{\aleph_0} \cdot E_0$ ($\sqsubseteq^i_B$-universal aperiodic hyperfinite), $E_{\infty c}$ ($\sqsubseteq^i_B$-universal compressible), and E_∞ ($\sqsubseteq^i_B$-universal).

Also shown is the "backbone" of compressible equivalence relations (bold line), which contains one element from each bireducibility class (dotted loops).

The middle of the picture shows a "generic" universally structurable E and its relations to some canonical elements of its bireducibility class: the $\sqsubseteq^i_B$-universal element $E^r_{\infty E}$ and the compressible element $E \times I_\mathbb{N}$. Note that $E \times I_\mathbb{N}$ is not depicted as being below E, in accordance with Remark 2.5.28. Note also that for non-smooth E, the $\sqsubseteq^i_B$-universal element $E^r_{\infty E}$ of its bireducibility class would indeed be above

$2^{\aleph_0} \cdot E_0$, as shown: $E_0 \leq_B E$ implies $2^{\aleph_0} \cdot E_0 \sqsubseteq_B^i E_{\infty E}^r$ since $E_{\infty E}^r$ is stably universally structurable.

Finally, note that the picture is somewhat misleading in a few ways. It is not intended to suggest that the compressibles form a linear order. Nor is it intended that any of the pairs $E \sqsubset_B^i F$ do not have anything strictly in between them (except of course for the things below $2^{\aleph_0} \cdot E_0$, which are exactly as shown).

2.7 Free actions of a group

Let Γ be a countably infinite group. Recall (from Section 2.3.1) that we regard Γ as a structure in the language $L_\Gamma = \{R_\gamma \mid \gamma \in \Gamma\}$, where $R_\gamma^\Gamma \subseteq \Gamma^2$ is the graph of the left multiplication of γ on Γ. Thus, $\mathcal{E}_\Gamma = \mathcal{E}_{\sigma_\Gamma}$ (where σ_Γ is the Scott sentence of Γ in L_Γ) is the class of Borel equivalence relations generated by a free Borel action of Γ. Our main goal in this section is to characterize when $\mathcal{E}_\Gamma$ is an elementary reducibility class.

Actually, to deal with a technicality, we need to consider the following variant of $\mathcal{E}_\Gamma$. Let $\mathcal{E}_\Gamma^* := \mathcal{E}_{\sigma_\Gamma \oplus \sigma_f}$, where σ_f is a sentence axiomatizing the finite equivalence relations. Thus $\mathcal{E}_\Gamma^*$ consists of countable Borel equivalence relations whose aperiodic part is generated by a free Borel action of Γ. This is needed because every equivalence relation in $\mathcal{E}_\Gamma$ must have all classes of the same cardinality as Γ.

Theorem 2.7.1. *Let Γ be a countably infinite group. The following are equivalent:*

(i) Γ is amenable.

(ii) $\mathcal{E}_\Gamma^*$ is closed under $\sqsubseteq_B$.

(iii) $\mathcal{E}_\Gamma^*$ is closed under $\leq_B$, i.e., $\mathcal{E}_\Gamma^*$ is an elementary reducibility class.

To motivate Theorem 2.7.1, consider the following examples. By Theorem 2.2.1, $\mathcal{E}_\mathbb{Z}^*$ is the class of all hyperfinite equivalence relations, which is closed under $\leq_B$. On the other hand, for every $2 \leq n \leq \aleph_0$, the free group $\mathbb{F}_n$ on n generators is such that $(\mathcal{E}_{\mathbb{F}_n}^*)^r$ is the class of treeable equivalence relations, by [JKL, 3.17]; but $\mathcal{E}_{\mathbb{F}_n}^*$ is not itself the class of all treeables, since every $E \in \mathcal{E}_{\mathbb{F}_n}^*$ with a nonatomic invariant probability measure has cost n (see [KM, 36.2]).

Recall that the $\sqsubseteq_B^i$-universal element of $\mathcal{E}_\Gamma$ is $F(\Gamma, \mathbb{R})$, the orbit equivalence of the free part of the shift action of Γ on $\mathbb{R}^\Gamma$. Thus the $\sqsubseteq_B^i$-universal element of $\mathcal{E}_\Gamma^*$ is

$F(\Gamma, \mathbb{R}) \oplus E_{\infty f}$, where $E_{\infty f}$ is the $\sqsubseteq^i_B$-universal finite equivalence relation (given by $E_{\infty f} = \bigoplus_{1 \le n \in \mathbb{N}} 2^{\aleph_0} \cdot I_n$).

Remark 2.7.2. Seward and Tucker-Drob [ST] have shown that for countably infinite Γ, every free Borel action of Γ admits an equivariant class-bijective map into $F(\Gamma, 2)$ (clearly the same holds for finite Γ). It follows that $F(\Gamma, 2)$ is $\to^{cb}_B$-universal in $\mathcal{E}_\Gamma$.

A well-known open problem asks whether every orbit equivalence of a Borel action of a countable amenable group Γ is hyperfinite. In the purely Borel context, the best known general result is the following [SS]:

Theorem 2.7.3 (Schneider-Seward)**.** *If Γ is a countable locally nilpotent group, i.e., every finitely generated subgroup of Γ is nilpotent, then every orbit equivalence E^X_Γ of a Borel action of Γ is hyperfinite.*

Remark 2.7.4. Recently Conley, Jackson, Marks, Seward, and Tucker-Drob have found examples of solvable but not locally nilpotent countable groups for which the conclusion of Theorem 2.7.3 still holds.

If Theorem 2.7.3 generalizes to arbitrary countable amenable Γ, then it would follow that $\mathcal{E}^*_\Gamma$ is the class of all hyperfinite equivalence relations (since it contains $F(\Gamma, \mathbb{R})$ which admits an invariant probability measure); then the main implication (i) $\Longrightarrow$ (iii) in Theorem 2.7.1 would trivialize.

In the measure-theoretic context, a classical result of Ornstein-Weiss [OW] states that the orbit equivalence of a Borel action of an amenable group Γ is hyperfinite almost everywhere with respect to every probability measure. We will need a version of this result which is uniform in the measure, which we now state. For a standard Borel space X, we let $P(X)$ denote the **space of probability Borel measures on** X (see [K95, 17.E]).

Lemma 2.7.5. *Let X, Y be standard Borel spaces, $E = E^X_\Gamma$ be the orbit equivalence of a Borel action of a countable amenable group Γ on X, and $m : Y \to_B P(X)$. Then there is a Borel set $A \subseteq Y \times X$, with $\pi_1(A) = Y$ (where $\pi_1 : Y \times X \to_B Y$ is the first projection), such that*

(i) for each $y \in Y$, $A_y := \{x \in X \mid (y, x) \in A\}$ has $m(y)$-measure 1 and is E-invariant;

(ii) $(\Delta_Y \times E)|A$ is hyperfinite.

Proof. This follows from verifying that the proofs of [KM, 9.2, 10.1] can be made uniform. We omit the details, which are tedious but straightforward. □

We now have the following, which forms the core of Theorem 2.7.1:

Proposition 2.7.6. *Let Γ be a countable amenable group, and let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations. If $E \leq_B F$ and $F = E_\Gamma^Y$ for some Borel action of Γ on Y, then E is the disjoint sum of a hyperfinite equivalence relation and a compressible equivalence relation.*

Proof. If E is compressible, then we are done. Otherwise, E has an invariant probability measure. Consider the ergodic decomposition of E; see e.g., [KM, 3.3]. This gives a Borel homomorphism $p : E \to_B \Delta_{P(X)}$ such that

(i) p is a surjection onto the Borel set $P_e(E) \subseteq P(X)$ of ergodic invariant probability measures on E;

(ii) for each $\mu \in P_e(E)$, we have $\mu(p^{-1}(\mu)) = 1$.

Let $f : E \leq_B F$, and apply Lemma 2.7.5 to F and $f_* : P_e(E) \to_B P(Y)$, where f_* is the pushforward of measures. This gives Borel $A \subseteq P_e(E) \times Y$ such that

(iii) for each $\mu \in P_e(E)$, $\mu(f^{-1}(A_\mu)) = (f_*\mu)(A_\mu) = 1$, and $A_\mu \subseteq Y$ is F-invariant (so A is $(\Delta_{P_e(E)} \times F)$-invariant);

(iv) $(\Delta_{P_e(E)} \times F)|A$ is hyperfinite.

Now consider the homomorphism $g := (p, f) : E \to_B \Delta_{P_e(E)} \times F$, i.e., $g(x) = (p(x), f(x))$. Then g is a reduction because f is. It follows that $B := g^{-1}(A)$ is E-invariant and $E|B$ is hyperfinite. It now suffices to note that $E|(X \setminus B)$ is compressible. Indeed, otherwise it would have an ergodic invariant probability measure, i.e., there would be some $\mu \in P_e(E)$ such that $\mu(X \setminus B) = 1$. But then $\mu(p^{-1}(\mu) \cap f^{-1}(A_\mu)) = 1$, while $p^{-1}(\mu) \cap f^{-1}(A_\mu) \subseteq B$, a contradiction. □

Proof of Theorem 2.7.1. Clearly (iii) $\implies$ (ii). If (ii) holds, then by the Glimm-Effros dichotomy, $E_0 \sqsubseteq_B F(\Gamma, \mathbb{R})$, so (ii) implies $E_0 \to_B^{cb} F(\Gamma, \mathbb{R})$, i.e., E_0 is generated by a free action of Γ, and so since E_0 is hyperfinite and has an invariant probability measure, Γ is amenable (see [JKL, 2.5(ii)]). So it remains to prove (i) $\implies$ (iii).

Let $E \leq_B F \in \mathcal{E}_\Gamma^*$. Then E splits into a smooth part, which is clearly in $\mathcal{E}_\Gamma^*$, and a part which reduces to some $F' \in \mathcal{E}_\Gamma$; so we may assume $F \in \mathcal{E}_\Gamma$. Factor the reduction $E \leq_B F$ into a surjective reduction $f : E \leq_B G$ (onto the image) followed by an embedding $G \sqsubseteq_B F$. By Proposition 2.7.6, $G = G' \oplus G''$, where G' is hyperfinite and G'' is compressible. Then $E = f^{-1}(G') \oplus f^{-1}(G'')$. Since $f^{-1}(G'') \leq_B^{cs} G'' \sqsubseteq_B F$ and G'' is compressible, we have $f^{-1}(G'') \sqsubseteq_B^i F$ (Proposition 2.5.23) and so $f^{-1}(G'') \in \mathcal{E}_\Gamma$. Finally, we have $f^{-1}(G') \in \mathcal{E}_\Gamma^*$, since $\mathcal{E}_\Gamma^*$ contains all hyperfinite equivalence relations (because $E_0 \sqsubseteq_B^i F(\Gamma, \mathbb{R})$, by Ornstein-Weiss's theorem and Theorem 2.2.1). $\square$

2.8 Structurability and model theory

In the previous sections, we have studied the relationship between structurability and common notions from the theory of countable Borel equivalence relations. This section, by contrast, concerns the other side of the $\models$ relation, i.e., logic. In particular, we are interested in model-theoretic properties of theories (L, σ) which are reflected in the elementary class $\mathcal{E}_\sigma$ that they axiomatize.

A general question one could ask is when two theories $(L, \sigma), (L', \tau)$ axiomatize the same elementary class, i.e., in the notation of Section 2.6.2, when does $\sigma \Leftrightarrow^* \tau$. Our main result here answers one instance of this question. Let σ_{sm} denote any sentence axiomatizing the smooth countable Borel equivalence relations.

Theorem 2.8.1. *Let (L, σ) be a theory. The following are equivalent:*

(i) There is an $L_{\omega_1\omega}$-formula $\phi(x)$ which defines a finite nonempty subset in any countable model of σ.

(ii) $\sigma \Rightarrow^* \sigma_{sm}$, i.e., any σ-structurable equivalence relation is smooth, or equivalently $E_{\infty\sigma}$ is smooth.

(iii) For any countably infinite group Γ, we have $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$, i.e., any σ-structurable equivalence relation generated by a free Borel action of Γ is smooth.

(iv) There is a countably infinite group Γ such that $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$.

In particular, this answers a question of Marks [M, end of Section 4.3], who asked for a characterization of when $E_{\infty\sigma_{\mathbb{A}}}$ ($\sigma_{\mathbb{A}}$ a Scott sentence) is smooth. The proof uses ideas from topological dynamics and ergodic theory.

Marks observed that recent work of Ackerman-Freer-Patel [AFP] implies the following sufficient condition for a structure $\mathbb{A}$ to structure every aperiodic countable Borel equivalence relation. In Section 2.8.2, we present his proof of this result, as well as several corollaries and related results. The result refers to the model-theoretic notion of **trivial definable closure**; see Section 2.8.2 for details. Let σ_a denote any sentence axiomatizing the aperiodic countable Borel equivalence relations.

Theorem 2.8.2 (Marks)**.** *Let L be a language and $\mathbb{A}$ be a countable L-structure with trivial definable closure. Then $\sigma_a \Rightarrow^* \sigma_{\mathbb{A}}$, i.e., every aperiodic countable Borel equivalence relation is $\mathbb{A}$-structurable.*

In Section 2.8.3 we discuss the problem of when an elementary class can be axiomatized by a Scott sentence.

2.8.1 Smoothness of $E_{\infty\sigma}$

We now begin the proof of Theorem 2.8.1. The implication (i) $\Longrightarrow$ (ii) is easy: given a formula ϕ as in (i), ϕ may be used to uniformly pick out a finite nonempty subset of each $E_{\infty\sigma}$-class, thus $E_{\infty\sigma}$ is smooth. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious. So let Γ be as in (iv).

Consider the logic action of S_Γ on $\mathrm{Mod}_\Gamma(L)$, the space of L-structures with universe Γ. Recall that this is given as follows: for $f \in S_\Gamma$, $\overline{\delta} \in \Gamma^n$, n-ary $R \in L$, and $\mathbb{A} \in \mathrm{Mod}_\Gamma(L)$, we have

$$R^{f(\mathbb{A})}(\overline{\delta}) \iff R^{\mathbb{A}}(f^{-1}(\overline{\delta})).$$

We regard Γ as a subgroup of S_Γ via the left multiplication action, so that Γ acts on $\mathrm{Mod}_\Gamma(L)$.

In an earlier version of this paper, we had stated the following lemma without the condition on finite stabilizers; only the $\Longrightarrow$ direction (without the condition) is used in what follows. Anush Tserunyan pointed out to us that the $\Longleftarrow$ direction was wrong, and gave the corrected version below together with the necessary additions to its proof.

Lemma 2.8.3. *Let Γ be a countably infinite group. Then $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$ iff $E_\Gamma^{\mathrm{Mod}_\Gamma(\sigma)}$ is smooth and the action of Γ on $\mathrm{Mod}_\Gamma(\sigma)$ has finite stabilizers.*

Proof. The proof is largely based on that of [KM, 29.5].

$\Longleftarrow$: Suppose (X, E) is generated by a free Borel action of Γ and $\mathbb{A} : E \models \sigma$. Define $f : X \to \mathrm{Mod}_\Gamma(\sigma)$ by

$$R^{f(x)}(\gamma_1, \ldots, \gamma_n) \iff R^{\mathbb{A}}(\gamma_1^{-1} \cdot x, \ldots, \gamma_n^{-1} \cdot x).$$

Then f is Γ-equivariant, so since $\Gamma \curvearrowright \mathrm{Mod}_\Gamma(\sigma)$ has finite stabilizers, f is finite-to-one on every E-class. Thus f is a smooth homomorphism, and so since $E_\Gamma^{\mathrm{Mod}_\Gamma(\sigma)}$ is smooth, so is E.

$\Longrightarrow$: First, suppose $E_\Gamma^{\mathrm{Mod}_\Gamma(\sigma)}$ is not smooth. Let ν be an ergodic non-atomic invariant σ-finite measure on $E_\Gamma^{\mathrm{Mod}_\Gamma(\sigma)}$. Consider the free part $Y \subseteq 2^\Gamma$ of the shift action of Γ on 2^Γ, with orbit equivalence $F = F(\Gamma, 2)$. The usual product measure ρ on 2^Γ concentrates on Y, and is invariant and mixing with respect to the action of Γ on Y (see [KM, 3.1]). Then consider the product action of Γ on $Y \times \mathrm{Mod}_\Gamma(\sigma)$, which is free since Γ acts freely on Y. By [SW, 2.3, 2.5], this product action admits $\rho \times \nu$ as an ergodic non-atomic invariant σ-finite measure. Thus $E_\Gamma^{Y \times \mathrm{Mod}_\Gamma(\sigma)}$ is not smooth.

Observe that $E_\Gamma^{Y \times \mathrm{Mod}_\Gamma(\sigma)}$ is the skew product $F \ltimes \mathrm{Mod}_\Gamma(\sigma)$ with respect to the cocycle $\alpha : F \to \Gamma$ associated to the free action of Γ on Y; and that α, when regarded as a cocycle $F \to S_\Gamma$, is induced, in the sense of Remark 2.4.3, by $T : Y \to Y^\Gamma$ where $T(y)(\gamma) := \gamma^{-1} \cdot y$. So (as in the proof of Theorem 2.4.1) $E_\Gamma^{Y \times \mathrm{Mod}_\Gamma(\sigma)}$ is σ-structurable, hence witnesses that $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$.

Now, suppose that the stabilizer $\Gamma_{\mathbb{A}}$ of some $\mathbb{A} \in \mathrm{Mod}_\Gamma(\sigma)$ is infinite. Again, we let $Y \subseteq 2^\Gamma$ be the free part of the shift action, and consider the product action of Γ on $Y \times [\mathbb{A}]_\Gamma$, which is σ-structurable as above. The action of $\Gamma_{\mathbb{A}}$ on $Y \times [\mathbb{A}]_\Gamma$ is not smooth because it contains the action on $Y \times \{\mathbb{A}\} \cong Y$ which in turn contains the free part of the shift on $2^{\Gamma_{\mathbb{A}}} \cong 2^{\Gamma_{\mathbb{A}}} \times \{0\}^{\Gamma \setminus \Gamma_{\mathbb{A}}} \subseteq 2^\Gamma$. Since $E_{\Gamma_{\mathbb{A}}}^{Y \times [\mathbb{A}]_\Gamma} \subseteq E_\Gamma^{Y \times [\mathbb{A}]_\Gamma}$, it follows that $E_\Gamma^{Y \times [\mathbb{A}]_\Gamma}$ is not smooth, hence witnesses that $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$. $\square$

So we have converted (iv) in Theorem 2.8.1 into a property of the action of Γ on $\mathrm{Mod}_\Gamma(\sigma)$. Our next step requires some preparation.

Let L be a language and $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ be a countable L-structure. We say that $\mathbb{A}$ has the **weak duplication property (WDP)** if for any finite sublanguage $L' \subseteq L$ and finite subset $F \subseteq X$, there is a finite subset $G \subseteq X$ disjoint from F such that $(\mathbb{A}|L')|F \cong (\mathbb{A}|L')|G$ (here $\mathbb{A}|L'$ denotes the reduct in the sublanguage L').

Remark 2.8.4. If we define the **duplication property (DP)** for $\mathbb{A}$ by replacing L' in the above by L, then clearly the DP is equivalent to the **strong joint embedding**

property (SJEP) for the age of $\mathbb{A}$: for any $\mathbb{F}, \mathbb{G} \in \mathrm{Age}(\mathbb{A})$, there is $\mathbb{H} \in \mathrm{Age}(\mathbb{A})$ and embeddings $\mathbb{F} \to \mathbb{H}$ and $\mathbb{G} \to \mathbb{H}$ with disjoint images. (Recall that $\mathrm{Age}(\mathbb{A})$ is the class of finite L-structures embeddable in $\mathbb{A}$.)

For a countable group Γ acting continuously on a topological space X, we say that a point $x \in X$ is **recurrent** if x is not isolated in the orbit $\Gamma \cdot x$ (with the subspace topology). When X is a Polish space, a basic fact is that E_Γ^X is smooth iff it does not have a recurrent point; see e.g., [K10, 22.3].

Thus far, we have only regarded the space $\mathrm{Mod}_X(L)$ of L-structures on a countable set X as a standard Borel space. Below we will also need to consider the topological structure on $\mathrm{Mod}_X(L)$. See e.g., [K95, 16.C]. In particular, we will use the system of basic clopen sets consisting of

$$N_{\mathbb{F}} := \{\mathbb{A} \in \mathrm{Mod}_X(L) \mid (\mathbb{A}|L')|F = \mathbb{F}\}$$

where $L' \subseteq L$ is a finite sublanguage and $\mathbb{F} = (F, R^{\mathbb{F}})_{R \in L'}$ is an L'-structure on a finite nonempty subset $F \subseteq X$.

The next lemma, which translates between the dynamics of $\mathrm{Mod}_\Gamma(\sigma)$ and a model-theoretic property of σ, is the heart of the proof of (iv) $\Longrightarrow$ (i) in Theorem 2.8.1:

Lemma 2.8.5. *Let Γ be a countably infinite group. Let L be a language, and let σ be an $L_{\omega_1\omega}$-sentence such that $\mathrm{Mod}_\Gamma(\sigma) \subseteq \mathrm{Mod}_\Gamma(L)$ is a G_δ subspace. Suppose there is a countable model $\mathbb{A} \models \sigma$ with the WDP, such that the interpretation $R_0^{\mathbb{A}}$ of some $R_0 \in L$ is not definable (without parameters) from equality. Then the action of Γ on $\mathrm{Mod}_\Gamma(\sigma)$ has a recurrent point, thus $E_\Gamma^{\mathrm{Mod}_\Gamma(\sigma)}$ is not smooth.*

Proof. We claim that it suffices to show that

- $(*)$ every basic clopen set $N_{\mathbb{F}} \subseteq \mathrm{Mod}_\Gamma(L)$ containing some isomorphic copy of $\mathbb{A}$ also contains two distinct isomorphic copies of $\mathbb{A}$ from the same Γ-orbit, i.e., there is $\mathbb{B} \in N_{\mathbb{F}}$ and $\gamma \in \Gamma$ such that $\mathbb{B} \cong \mathbb{A}$ and $\gamma \cdot \mathbb{B} \neq \mathbb{B}$.

Suppose this has been shown; we complete the proof. Note that since $\mathbb{A}$ has WDP, $\mathbb{A}$ must be infinite. Let $\overline{\mathrm{Mod}_\Gamma(\sigma_{\mathbb{A}})}$ denote the closure in $\mathrm{Mod}_\Gamma(\sigma)$ of $\mathrm{Mod}_\Gamma(\sigma_{\mathbb{A}})$ (where $\sigma_{\mathbb{A}}$ is the Scott sentence of $\mathbb{A}$). Since $\mathrm{Mod}_\Gamma(\sigma) \subseteq \mathrm{Mod}_\Gamma(L)$ is G_δ, $\overline{\mathrm{Mod}_\Gamma(\sigma_{\mathbb{A}})}$ is a Polish space, which is nonempty because it contains an isomorphic copy of $\mathbb{A}$. For each basic clopen set $N_{\mathbb{F}} \subseteq \mathrm{Mod}_\Gamma(L)$, the set of $\mathbb{B} \in \overline{\mathrm{Mod}_\Gamma(\sigma_{\mathbb{A}})}$ such that

$$\mathbb{B} \in N_{\mathbb{F}} \implies \exists \gamma \in \Gamma\, (\mathbb{B} \neq \gamma \cdot \mathbb{B} \in N_{\mathbb{F}}) \tag{$**$}$$

is clearly open; and by $(*)$, it is also dense. Thus the set of recurrent points in $\overline{\mathrm{Mod}_\Gamma(\sigma_{\mathbb{A}})}$, i.e., the set of $\mathbb{B} \in \overline{\mathrm{Mod}_\Gamma(\sigma_{\mathbb{A}})}$ for which $(**)$ holds for *every* $N_{\mathbb{F}}$, is comeager.

So it remains to prove $(*)$. Let $\mathbb{F}$ be such that $N_{\mathbb{F}}$ contains an isomorphic copy of $\mathbb{A}$. Let $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$, and let $\mathbb{F} = (F, R^{\mathbb{F}})_{R \in L'}$ where $F \subseteq \Gamma$ is finite nonempty and $L' \subseteq L$ is finite. We may assume $R_0 \in L'$.

Since $N_{\mathbb{F}}$ contains an isomorphic copy of $\mathbb{A}$, there is a map $f : F \to X$ which is an embedding $\mathbb{F} \to \mathbb{A}$. We will extend f to a bijection $\Gamma \to X$, and then define $\mathbb{B} := f^{-1}(\mathbb{A})$, thus ensuring that $\mathbb{B} \in N_{\mathbb{F}}$; we need to choose f appropriately so that there is $\gamma \in \Gamma$ with $\mathbb{B} \neq \gamma \cdot \mathbb{B} \in N_{\mathbb{F}}$.

Put $G := f(F) \subseteq X$. By WDP, there is a $G' \subseteq X$ disjoint from G such that $(\mathbb{A}|L')|G \cong (\mathbb{A}|L')|G'$, say via $g : G \to G'$. By the hypothesis that $R_0^{\mathbb{A}}$ is not definable from equality, there are $\overline{x} = (x_1, \dots, x_n) \in X^n$ and $\overline{x}' = (x'_1, \dots, x'_n) \in X^n$, where n is the arity of R_0, such that $\overline{x} \in R_0^{\mathbb{A}}$, $\overline{x}' \notin R_0^{\mathbb{A}}$, and $\overline{x}, \overline{x}'$ have the same equality type, i.e., we have a bijection $\{x_1, \dots, x_n\} \to \{x'_1, \dots, x'_n\}$ sending x_i to x'_i. Again by WDP, we may find $\overline{x}, \overline{x}'$ disjoint from G, G', and each other.

Now pick $\overline{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$ disjoint from F and with the same equality type as $\overline{x}$, and pick $\gamma \in \Gamma$ such that $\gamma^{-1}F$ and $\gamma^{-1}\overline{\delta}$ are disjoint from F and $\overline{\delta}$. Extend $f : F \to X$ to a bijection $f : \Gamma \to X$ such that

$$f|\gamma^{-1}F = g \circ (f|F) \circ \gamma : \gamma^{-1}F \to G', \qquad f(\overline{\delta}) = \overline{x}, \qquad f(\gamma^{-1}\overline{\delta}) = \overline{x}'.$$

Then putting $\mathbb{B} := f^{-1}(\mathbb{A})$, it is easily verified that $(\gamma \cdot \mathbb{B}|L')|F = \mathbb{F}$, i.e., $\gamma \cdot \mathbb{B} \in N_{\mathbb{F}}$; but $R_0^{\gamma \cdot \mathbb{B}}(\overline{\delta}) \iff R_0^{\mathbb{B}}(\gamma^{-1}\overline{\delta}) \iff R_0^{\mathbb{A}}(\overline{x}') \iff \neg R_0^{\mathbb{A}}(\overline{x}) \iff \neg R_0^{\mathbb{B}}(\overline{\delta})$, so $\gamma \cdot \mathbb{B} \neq \mathbb{B}$. ◻

Corollary 2.8.6. *Let Γ be a countably infinite group. Let L be a language, and let σ be an $L_{\omega_1\omega}$-sentence such that $\mathrm{Mod}_\Gamma(\sigma) \subseteq \mathrm{Mod}_\Gamma(L)$ is a G_δ subspace. If $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$, then no countable model of σ has the WDP.*

Proof. Suppose a countable (infinite) $\mathbb{A} \models \sigma$ has the WDP. If for some $R_0 \in L$, the interpretation $R_0^{\mathbb{A}}$ is not definable (without parameters) from equality, then $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$ by Lemma 2.8.3 and Lemma 2.8.5. Otherwise, clearly any aperiodic countable Borel equivalence relation is $\sigma_{\mathbb{A}}$-structurable, hence σ-structurable; taking $F(\Gamma, 2)$ then yields that $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$. ◻

Working towards (i) in Theorem 2.8.1, which asserts the existence of a formula with certain properties, we now encode the WDP into formulas, using the following combinatorial notion.

Let X be a set and $1 \leq n \in \mathbb{N}$. An n-**ary intersecting family** on X is a nonempty collection $\mathcal{F}$ of subsets of X of size n such that every pair $A, B \in \mathcal{F}$ has $A \cap B \neq \emptyset$.

Lemma 2.8.7. *Let L be a language. There are $L_{\omega_1\omega}$-formulas $\phi_n(x_0, \ldots, x_{n-1})$ for each $1 \leq n \in \mathbb{N}$, such that for any countable L-structure $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ without the WDP, there is some n such that*

$$\{\{x_0, \ldots, x_{n-1}\} \mid \phi_n^{\mathbb{A}}(x_0, \ldots, x_{n-1})\}$$

is an n-ary intersecting family on X.

Proof. Let $((L_k, n_k, \mathbb{F}_k))_k$ enumerate all countably many triples where $L_k \subseteq L$ is a finite sublanguage, $1 \leq n_k \in \mathbb{N}$, and $\mathbb{F}_k \in \mathrm{Mod}_{n_k}(L_k)$ is an L_k-structure with universe n_k $(= \{0, \ldots, n_k - 1\})$. For each k, let $\psi_k(x_0, \ldots, x_{n_k-1})$ be an L_k-formula asserting that $x_0, \ldots, x_{n_k-1}$ (are pairwise distinct and) form an L_k-substructure isomorphic to $\mathbb{F}_k$, which is not disjoint from any other such substructure. Thus $\mathbb{A}$ does not have the WDP iff some ψ_k holds for some tuple in $\mathbb{A}$; and in that case, the collection of all tuples (regarded as sets) for which ψ_k holds will form an n_k-ary intersecting family. Finally put

$$\phi_n(\overline{x}) := \bigvee_{n_k = n}(\psi_k(\overline{x}) \wedge \neg \bigvee_{k' < k} \exists \overline{y}\, \psi_{k'}(\overline{y})),$$

so that ϕ_{n_k} is equivalent to ψ_k for the least k which holds for some tuple. $\square$

Recall that (i) in Theorem 2.8.1 asserts the existence of a single formula defining a finite nonempty set. The following lemma, due to Clemens-Conley-Miller [CCM, 4.3], gives a way of uniformly defining a finite nonempty set from an intersecting family. For the convenience of the reader, we include its proof here.

Lemma 2.8.8 (Clemens-Conley-Miller)**.** *Let $\mathcal{F}$ be an n-ary intersecting family on X. For $1 \leq m < n$, define*

$$\mathcal{F}^{(m)} := \{A \subseteq X \mid |A| = m \ \& \ |\{B \in \mathcal{F} \mid A \subseteq B\}| \geq \aleph_0\}.$$

Then there exist $m_k < m_{k-1} < \cdots < m_1 < n$ such that $\mathcal{F}^{(m_1)}, \mathcal{F}^{(m_1)(m_2)}, \ldots$ are (respectively m_1-ary, m_2-ary, etc.) intersecting families, and $\mathcal{F}^{(m_1)\cdots(m_k)}$ is finite.

Proof. It suffices to show that if $\mathcal{F}$ is infinite, then there is some $1 \leq m < n$ such that $\mathcal{F}^{(m)}$ is an m-ary intersecting family. Indeed, having shown this, we may find the desired $m_1, m_2, \ldots$ inductively; the process must terminate since a 1-ary intersecting family is necessarily a singleton.

So assume $\mathcal{F}$ is infinite, and let $m < n$ be greatest so that $\mathcal{F}^{(m)}$ is nonempty. Let $A, B \in \mathcal{F}^{(m)}$. For each $x \in B \setminus A$, by our choice of m, there are only finitely many $C \in \mathcal{F}$ such that $A \cup \{x\} \subseteq C$. Thus by definition of $\mathcal{F}^{(m)}$, there is $C \in \mathcal{F}$ such that $A \subseteq C$ and $(B \setminus A) \cap C = \varnothing$. Similarly, there is $D \in \mathcal{F}$ such that $B \subseteq D$ and $(C \setminus B) \cap D = \varnothing$. Then $A \cap B = C \cap B = C \cap D \neq \varnothing$, as desired. □

Corollary 2.8.9. *Let L be a language. There is an $L_{\omega_1\omega}$-formula $\phi(x)$ such that for any countable L-structure $\mathbb{A}$ without the WDP, $\phi^{\mathbb{A}}$ is a finite nonempty subset.*

Proof. This follows from Lemma 2.8.7 and Lemma 2.8.8, by a straightforward encoding of the operation $\mathcal{F} \mapsto \mathcal{F}^{(m)}$ in $L_{\omega_1\omega}$.

In more detail, for each $L_{\omega_1\omega}$-formula $\psi(x_0, \ldots, x_{n-1})$ and $m < n$, let $\psi^{(m)}(x_0, \ldots, x_{m-1})$ be a formula asserting that $x_0, \ldots, x_{m-1}$ are pairwise distinct and there are infinitely many extensions $(x_m, \ldots, x_{n-1})$ such that $\psi(x_0, \ldots, x_{n-1})$ holds, so that if ψ defines (in the sense of Lemma 2.8.7) a family $\mathcal{F}$ of subsets of size n, then $\psi^{(m)}$ defines $\mathcal{F}^{(m)}$. Let ϕ_n for $1 \leq n \in \mathbb{N}$ be given by Lemma 2.8.7. For each finite tuple $t = (n, m_1, \ldots, m_k)$ such that $n > m_1 > \cdots > m_k \geq 1$, let τ_t be a sentence asserting that $\phi_n^{(m_1)\cdots(m_k)}$ holds for at least one but only finitely many tuples. Then, letting $(t^l = (n^l, m_1^l, \ldots, m_{k^l}^l))_{l \in \mathbb{N}}$ enumerate all such tuples, the desired formula ϕ can be given by

$$\phi(x) = \bigvee_l \left(\tau_{t^l} \wedge \exists \overline{x} \left(\phi_{n^l}^{(m_1^l)\cdots(m_{k^l}^l)}(\overline{x}) \wedge \bigvee_i (x = x_i) \right) \wedge \neg \bigvee_{l' < l} \tau_{t^{l'}} \right).$$

By Lemmas 2.8.7 and 2.8.8, in any countable L-structure $\mathbb{A}$ without the WDP, $\phi^{\mathbb{A}}$ will be the union of the finitely many sets in some intersecting family. □

Corollary 2.8.10. *Let Γ be a countably infinite group. Let L be a language, and let σ be an $L_{\omega_1\omega}$-sentence such that* $\mathrm{Mod}_\Gamma(\sigma) \subseteq \mathrm{Mod}_\Gamma(L)$ *is a G_δ subspace. If $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$, then there is an $L_{\omega_1\omega}$-formula $\phi(x)$ which defines a finite nonempty subset in any countable model of σ.*

Proof. By Corollary 2.8.6 and Corollary 2.8.9. □

To complete the proof of Theorem 2.8.1, we need to remove the assumption that $\mathrm{Mod}_\Gamma(\sigma) \subseteq \mathrm{Mod}_\Gamma(L)$ is G_δ from Corollary 2.8.10. This can be done using the standard trick of **Morleyization**, as described for example in [Hod, Section 2.6] for finitary first-order logic, or [AFP, 2.5] for $L_{\omega_1\omega}$. Given any language L and $L_{\omega_1\omega}$-sentence σ, by adding relation symbols for each formula in a countable fragment of $L_{\omega_1\omega}$ containing the sentence σ, we obtain a new (countable) language L' and an $L'_{\omega_1\omega}$-sentence σ' such that

- the L-reduct of every countable model of σ' is a model of σ;
- every countable model of σ has a unique expansion to a model of σ';
- σ' is (logically equivalent to a formula) of the form

$$\bigwedge_i \forall \overline{x}\, \exists y\, \bigvee_j \phi_{i,j}(\overline{x}, y),$$

  where each $\phi_{i,j}$ is a quantifier-free finitary L'-formula, whence $\mathrm{Mod}_\Gamma(\sigma') \subseteq \mathrm{Mod}_\Gamma(L')$ is G_δ.

It follows that the conditions (i) and (iv) in Theorem 2.8.1 for (L, σ) are equivalent to the same conditions for (L', σ'). So Corollary 2.8.10 holds also without the assumption that $\mathrm{Mod}_\Gamma(\sigma) \subseteq \mathrm{Mod}_\Gamma(L)$ is G_δ, which completes the proof of Theorem 2.8.1.

We conclude this section by pointing out the following analog of Lemma 2.8.3:

Lemma 2.8.11. *Let Γ be a countably infinite group and (L, σ) be a theory. Then $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_c$ iff $E_\Gamma^{\mathrm{Mod}_\Gamma(\sigma)}$ is compressible.*

Proof. For $\Longrightarrow$, the proof is exactly the same as the first part of the proof of $\Longrightarrow$ in Lemma 2.8.3, but using probability measures instead of non-atomic σ-finite measures. Similarly, for $\Longleftarrow$, let (X, E) and $f : X \to \mathrm{Mod}_\Gamma(\sigma)$ be as in the proof of $\Longleftarrow$ in Lemma 2.8.3; if E were not compressible, then it would have an invariant probability measure μ, whence $f_*\mu$ would be an invariant probability measure on $E_\Gamma^{\mathrm{Mod}_\Gamma(\sigma)}$, contradicting compressibility of the latter. □

This has the following corollaries. The first strengthens [AFP, Section 6.1.10]:

Corollary 2.8.12. *Let $\mathcal{T}_1$ denote the class of trees, and more generally, let $\mathcal{T}_n$ denote the class of contractible n-dimensional simplicial complexes. Then for each n,* *there is some countably infinite group* Γ *such that* $\mathrm{Mod}_\Gamma(\mathcal{T}_n)$ *admits no* Γ*-invariant measure (and thus no* S_Γ*-invariant measure).*

Proof. For each n, let σ_n be a sentence axiomatizing $\mathcal{T}_n$. For $n = 1$, take Γ to be any infinite Kazhdan group. By [AS], no free Borel action of Γ admitting an invariant probability measure is treeable, i.e., $\sigma_1 \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_c$; thus $\mathrm{Mod}_\Gamma(\mathcal{T}_n)$ admits no Γ-invariant measure by Lemma 2.8.11. For $n > 1$, take $\Gamma := \mathbb{F}_2^n \times \mathbb{Z}$. By a result of Gaboriau (see, e.g., [HK, p. 59]), no free Borel action of Γ admitting an invariant probability measure can be $\mathcal{T}_n$-structurable. $\square$

Corollary 2.8.13. *Let* L *be a language, and let* σ *be an* $L_{\omega_1\omega}$*-sentence such that* $\mathrm{Mod}_\mathbb{N}(\sigma) \subseteq \mathrm{Mod}_\mathbb{N}(L)$ *is a closed subspace. Then for any countably infinite group* Γ*, there is a free Borel action of* Γ *which admits an invariant probability measure and is* σ*-structurable.*

Proof. Since $\mathrm{Mod}_\Gamma(\sigma) \subseteq \mathrm{Mod}_\Gamma(L)$ is closed, it is compact, so since S_Γ is amenable, $\mathrm{Mod}_\Gamma(\sigma)$ admits a S_Γ-invariant probability measure, and thus a Γ-invariant probability measure; then apply Lemma 2.8.11. $\square$

2.8.2 Universality of $E_{\infty\sigma}$

Several theories (L, σ) are known to axiomatize $\mathcal{E}$, the class of all countable Borel equivalence relations. For example, by [JKL, (proof of) 3.12], every $E \in \mathcal{E}$ is structurable via locally finite graphs. More generally, one can consider σ such that every aperiodic or compressible countable Borel equivalence relation is σ-structurable. For example, it is folklore that every aperiodic countable Borel equivalence relation can be structured via dense linear orders (this will also follow from Theorem 2.8.2), while the proof of [JKL, 3.10] shows that every compressible $E \in \mathcal{E}$ is structurable via graphs of vertex degree ≤ 3.

A result that some particular σ axiomatizes $\mathcal{E}$ (or all aperiodic E) shows that every (aperiodic) $E \in \mathcal{E}$ carries a certain type of structure, which can be useful in applications. A typical example is the very useful Marker Lemma (see [BK, 4.5.3]), which shows that every aperiodic E admits a decreasing sequence of Borel complete sections $A_0 \supseteq A_1 \supseteq \cdots$ with empty intersection. This can be phrased as follows: every aperiodic countable Borel equivalence relation E is σ-structurable, where σ in the language $L = \{P_0, P_1, \dots\}$ asserts that each (unary) P_i defines a nonempty subset, $P_0 \supseteq P_1 \supseteq \cdots$, and $\bigcap_i P_i = \varnothing$.

We now give the proof of Theorem 2.8.2, which provides a large class of examples of such theories. To do so, we first review the main result from [AFP].

Let L be a language and $\mathbb{A} = (X, R^{\mathbb{A}})_{R\in L}$ be a countable L-structure. For a subset $F \subseteq X$, let $\mathrm{Aut}_F(\mathbb{A}) \subseteq \mathrm{Aut}(\mathbb{A})$ denote the pointwise stabilizer of F, i.e., the set of all automorphisms $f \in \mathrm{Aut}(\mathbb{A})$ fixing every $x \in F$. We say that $\mathbb{A}$ has **trivial definable closure (TDC)** if the following equivalent conditions hold (see [AFP, 2.12–15], [Hod, 4.1.3]):

- for every finite $F \subseteq X$, $\mathrm{Aut}_F(\mathbb{A}) \curvearrowright X$ fixes no element of $X \setminus F$;
- for every finite $F \subseteq X$, $\mathrm{Aut}_F(\mathbb{A}) \curvearrowright X$ has infinite orbits on $X \setminus F$ (**trivial algebraic closure**);
- for every finite $F \subseteq X$ and $L_{\omega_1\omega}$-formula $\phi(x)$ with parameters in F, if there is a unique $x \in X$ such that $\phi^{\mathbb{A}}(x)$ holds, then $x \in F$;
- for every finite $F \subseteq X$ and $L_{\omega_1\omega}$-formula $\phi(x)$ with parameters in F, if there are only finitely many $x_1, \ldots, x_n \in X$ such that $\phi^{\mathbb{A}}(x_i)$ holds, then $x_i \in F$ for each i.

Remark 2.8.14. If $\mathbb{A}$ is a Fraïssé structure, then TDC is further equivalent to the **strong amalgamation property (SAP)** for the age of $\mathbb{A}$: for any $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathrm{Age}(\mathbb{A})$ living on F, G, H respectively and embeddings $f : \mathbb{H} \to \mathbb{F}$ and $g : \mathbb{H} \to \mathbb{G}$, there is $\mathbb{K} \in \mathrm{Age}(\mathbb{A})$ and embeddings $f' : \mathbb{F} \to \mathbb{K}$ and $g' : \mathbb{G} \to \mathbb{K}$ with $f' \circ f = g' \circ g$ and $f'(F) \cap g'(G) = (f' \circ f)(H)$.

Theorem 2.8.15 (Ackerman-Freer-Patel [AFP, 1.1])**.** *Let L be a language and $\mathbb{A} = (X, R^{\mathbb{A}})_{R\in L}$ be a countably infinite L-structure. The following are equivalent:*

(i) The logic action of S_X on $\mathrm{Mod}_X(\sigma_{\mathbb{A}})$ ($\sigma_{\mathbb{A}}$ the Scott sentence of $\mathbb{A}$) admits an invariant probability measure.

(ii) $\mathbb{A}$ has TDC.

We will in fact need the following construction from Ackerman-Freer-Patel's proof of Theorem 2.8.15. Starting with a countable L-structure $\mathbb{A}$ with TDC, they consider the Morleyization $(L', \sigma'_{\mathbb{A}})$ of the Scott sentence $\sigma_{\mathbb{A}}$ of $\mathbb{A}$, where

$$\sigma'_{\mathbb{A}} = \bigwedge_i \forall \overline{x}\, \exists y\, \psi_i(\overline{x}, y)$$

with each ψ_i quantifier-free, as described following Corollary 2.8.10. They then produce (see [AFP, Section 3.4]) a Borel L'-structure $\mathbb{A}' \models \sigma'_{\mathbb{A}}$ with universe $\mathbb{R}$ such that for each i and $\overline{x} \in \mathbb{R}$, the corresponding subformula $\exists y\, \psi_i(\overline{x}, y)$ in $\sigma'_{\mathbb{A}}$ is witnessed either by some y in the tuple $\overline{x}$, or by all y in some nonempty open interval. Clearly then the restriction of $\mathbb{A}'$ to any countable dense set of reals still satisfies $\sigma'_{\mathbb{A}}$, hence (its L-reduct) is isomorphic to $\mathbb{A}$. This shows:

Corollary 2.8.16 (of proof of [AFP, 1.1])**.** *Let L be a language and $\mathbb{A}$ be a countable L-structure with TDC. Then there is a Borel L-structure $\mathbb{A}'$ with universe $\mathbb{R}$ such that for any countable dense set $A \subseteq \mathbb{R}$, $\mathbb{A}'|A \cong \mathbb{A}$.*

Proof of Theorem 2.8.2. (Marks) If E is smooth, then clearly it is $\mathbb{A}$-structurable. So we may assume $X = 2^{\mathbb{N}}$. Let $N_s = \{x \in 2^{\mathbb{N}} \mid s \subseteq x\}$ for $s \in 2^{<\mathbb{N}}$ denote the basic clopen sets in $2^{\mathbb{N}}$. Note that the set

$$X_1 := \{x \in X \mid \exists s \in 2^{<\mathbb{N}}\, (|[x]_E \cap N_s| = 1)\}$$

of points whose class contains an isolated point is Borel, and $E|X_1$ is smooth (with a selector given by $x \mapsto$ the unique element of $[x]_E \cap N_s$ for the least s such that $|[x]_E \cap N_s| = 1$), hence $\mathbb{A}$-structurable. For $x \in X \setminus X_1$, the closure $\overline{[x]_E}$ has no isolated points, hence is homeomorphic to $2^{\mathbb{N}}$. For each such x, define $f_x(t)$ inductively for $t \in 2^{<\mathbb{N}}$ by

$$\begin{aligned} f_x(\varnothing) &:= \varnothing, \\ f_x(t^\frown i) &:= s^\frown i \text{ for the unique } s \supseteq f_x(t) \text{ such that } [x]_E \cap N_{f_x(t)} \subseteq N_s \\ &\qquad\qquad \text{but } [x]_E \cap N_{f_x(t)} \not\subseteq N_{s^\frown 0}, N_{s^\frown 1} \end{aligned}$$

(for $i = 0, 1$), so that $f_x : 2^{\mathbb{N}} \to \overline{[x]_E}$, $f_x(y) := \bigcup_{t \subseteq y} f_x(t)$ is a homeomorphism, such that $x\; E\; x' \implies f_x = f_{x'}$. It is easy to see that $(x, y) \mapsto f_x(y)$ is Borel.

Now let the structure $\mathbb{A}'$ on $\mathbb{R}$ be given by Corollary 2.8.16. Let $Z = \{z_0, z_1, \dots\} \subseteq 2^{\mathbb{N}}$ be a countable set so that there is a continuous bijection $g : 2^{\mathbb{N}} \setminus Z \to \mathbb{R}$. Let

$$X_2 := \{x \in X \setminus X_1 \mid \exists x' \in [x]_E\, (f_x^{-1}(x') \in Z)\}.$$

Then $E|X_2$ is smooth (with selector $x \mapsto x' \in [x]_E$ such that $f_x^{-1}(x') = z_j$ with j minimal), hence $\mathbb{A}$-structurable. Finally, $E|(X \setminus (X_1 \cup X_2))$ is $\mathbb{A}$-structurable: for each $x \in X \setminus (X_1 \cup X_2)$, we have that $f_x^{-1}([x]_E) \subseteq 2^{\mathbb{N}} \setminus Z$ is dense, so $g \circ f_x^{-1}$ gives a bijection between $[x]_E$ and a dense subset of $\mathbb{R}$, along which we may pull back $\mathbb{A}'$ to get a structure on $[x]_E$ isomorphic to $\mathbb{A}$. ◻

Theorem 2.8.2 has the following immediate corollary.

Corollary 2.8.17. *The following Fraïssé structures can structure every aperiodic countable Borel equivalence relation:* $(\mathbb{Q}, <)$*, the random graph, the random* K_n*-free graph (where* K_n *is the complete graph on* n *vertices), the random poset, and the rational Urysohn space.*

The concept of amenability of a structure in the next result can be either the one in [JKL, 2.6(iii)] or the one in [K91, 3.4]. This result was first proved by the authors by a different method but it can also be seen as a corollary of Theorem 2.8.2.

Corollary 2.8.18. *Let* $\mathbb{A}$ *be a countably infinite amenable structure. Then* $\mathbb{A}$ *fails TDC.*

Proof. Since $\mathbb{A}$ is amenable, every $\mathbb{A}$-structurable equivalence relation is amenable (see [JKL, 2.18] or [K91, 2.6]), thus it is not true that $\mathbb{A}$ structures every aperiodic countable Borel equivalence relation, and so $\mathbb{A}$ fails TDC by Theorem 2.8.2. □

We do not know of a counterexample to the converse of Theorem 2.8.2, i.e., of a single structure $\mathbb{A}$ without TDC such that every aperiodic $E \in \mathcal{E}$ is $\mathbb{A}$-structurable. There do exist structures without TDC which structure every compressible E, as the following simple example shows:

Proposition 2.8.19. *For any countable linear order* $(Y, <)$*, every compressible* $(X, E) \in \mathcal{E}_c$ *is structurable via linear orders isomorphic to* $\mathbb{Q} \times (Y, <)$ *with the lexicographical order.*

In particular, $\mathbb{Q} \times \mathbb{Z}$ *structures every compressible equivalence relation.*

Proof. By Theorem 2.8.2, E is structurable via linear orders isomorphic to $\mathbb{Q}$. Take the lexicographical order on $E \times I_Y$ and apply Proposition 2.5.23(a). □

Concerning classes of structures (or theories) which can structure every (compressible) equivalence relation, we can provide the following examples. Below a **graphing** of an equivalence relation E is a $\mathcal{K}$-structuring, where $\mathcal{K}$ is the class of connected graphs.

Proposition 2.8.20. *Every* $(X, E) \in \mathcal{E}$ *is structurable via connected bipartite graphs.*

Proof. The finite part of E can be treed, so assume E is aperiodic. Then we may partition $X = Y \cup Z$ where Y, Z are complete sections (this is standard; see e.g., [BK, 4.5.4]). Then the graph $G \subseteq E$ which connects each $y \in Y$ and $z \in Z$ (and with no other edges) works. $\square$

Proposition 2.8.21. *For every $k \geq 1$, every compressible $(X, E) \in \mathcal{E}_c$ is structurable via connected graphs in which all cycles have lengths divisible by k.*

Proof. Let $<$ be a Borel linear order on X, and let $G \subseteq E$ be any Borel graphing, e.g., $G = E \setminus \Delta_X$. Let

$$X' := X \sqcup (G \times \{1, \ldots, k-1\}),$$

let E' be the equivalence relation on X' generated by E and $x \mathrel{E'} (y, z, i)$ for $x \mathrel{E} y \mathrel{E} z$, $(y, z) \in G$, and $1 \leq i < k$, and let $G' \subseteq E'$ be the graph generated by, for each $(x, y) \in G$ with $x < y$,

$$x \mathrel{G'} (x, y, 1) \mathrel{G'} (x, y, 2) \mathrel{G'} \cdots \mathrel{G'} (x, y, k-1) \mathrel{G'} y$$

(and no other edges). That is, every edge in G has been replaced by a k-length path with the same endpoints. It is clear that G' graphs E' and every cycle in G' has length divisible by k. Now since E is compressible, and the inclusion $X \subseteq X'$ is a complete section embedding $E \sqsubseteq_B E'$, we have $E \cong_B E'$, thus E is structurable via a graph isomorphic to G'. $\square$

A similar example is provided by

Theorem 2.8.22 (Kechris-Miller [Mi, 3.2])**.** *Let E be a countable Borel equivalence relation and $n \in \mathbb{N}$. Then every graphing of E admits a spanning subgraphing with no cycles of length $\leq n$.*

Thus in contrast to the fact that not every countable Borel equivalence relation is treeable, we have the following result, using also [JKL, proof of 3.12].

Corollary 2.8.23. *Every countable Borel equivalence relation has locally finite graphings of arbitrarily large girth.*

2.8.3 Classes axiomatizable by a Scott sentence

Let us say that an elementary class $\mathcal{C} \subseteq \mathcal{E}$ is **Scott axiomatizable** if it is axiomatizable by a Scott sentence $\sigma_{\mathbb{A}}$ of some structure $\mathbb{A}$, or equivalently by some sentence σ which is countably categorical (i.e., it has exactly one countable model up to isomorphism). Several elementary classes we have considered are naturally Scott axiomatizable: e.g., aperiodic, aperiodic smooth, aperiodic hyperfinite (by $\sigma_{\mathbb{Z}}$), free actions of a group Γ (by σ_Γ), and compressible (by the sentence in the language $\{R\}$ asserting that R is the graph of an injective function with infinite and coinfinite image and with no fixed points).

It is an open problem to characterize the elementary classes which are Scott axiomatizable. In fact, we do not even know if every elementary class of aperiodic equivalence relations is Scott axiomatizable. Here we describe a general construction which can be used to show that certain compressible elementary classes are Scott axiomatizable.

Let $(L, \sigma), (M, \tau)$ be theories. Let $\sigma \times \tau$ be a sentence in the language $L \sqcup M \sqcup \{R_1, R_2\}$ asserting

(i) R_1, R_2 are equivalence relations such that the quotient maps $X \to X/R_1$ and $X \to X/R_2$ (where X is the universe) exhibit a bijection between X and $X/R_1 \times X/R_2$; and

(ii) the L-reduct (respectively M-reduct) is an R_1-invariant (resp., R_2-invariant) structure which induces a model of σ (resp., τ) on the quotient X/R_1 (resp., X/R_2).

Thus, a countable $(\sigma \times \tau)$-structure $\mathbb{A}$ on a set X is essentially the same thing as a σ-structure $\mathbb{B}$ on a set Y and a τ-structure $\mathbb{C}$ on a set Z, together with a bijection $X \cong Y \times Z$. The following are clear:

Proposition 2.8.24. *$E_{\infty\sigma} \times E_{\infty\tau} \models \sigma \times \tau$ (equivalently, $E_{\infty\sigma} \times E_{\infty\tau} \sqsubseteq^i_B E_{\infty(\sigma\times\tau)}$).*

Remark 2.8.25. It is not true in general that $E_{\infty\sigma} \times E_{\infty\tau} \cong_B E_{\infty(\sigma\times\tau)}$. For example, if $\sigma = \sigma_\Gamma$ and $\tau = \sigma_\Delta$ axiomatize free actions of countable groups Γ, Δ, then it is easy to see that $\sigma_\Gamma \times \sigma_\Delta$ axiomatizes free actions of $\Gamma \times \Delta$; taking $\Gamma = \Delta = \mathbb{F}_2$, we have that $E_{\infty(\sigma\times\tau)}$ is the universal orbit equivalence of a free action of $\mathbb{F}_2 \times \mathbb{F}_2$, which does not reduce to a product of two treeables (such as $E_{\infty\sigma} \times E_{\infty\tau}$) by [HK, 8.1(iii)].

Proposition 2.8.26. *If σ, τ are countably categorical, then so is $\sigma \times \tau$.*

Now consider the case where τ in the language $\{P_0, P_1, \dots\}$ asserts that the P_i are disjoint singleton subsets which enumerate the universe. Then clearly τ axiomatizes the aperiodic smooth countable Borel equivalence relations, i.e., $E_{\infty\tau} = \Delta_{\mathbb{R}} \times I_{\mathbb{N}}$, whence $E_{\infty\sigma} \times E_{\infty\tau} \cong_B E_{\infty\sigma} \times I_{\mathbb{N}}$.

Proposition 2.8.27. *For this choice of τ, $E_{\infty(\sigma\times\tau)} \cong_B E_{\infty\sigma} \times E_{\infty\tau} \cong_B E_{\infty\sigma} \times I_{\mathbb{N}}$.*

Proof. Let $E_{\infty(\sigma\times\tau)}$ live on X and let $\mathbb{E} : E_{\infty(\sigma\times\tau)} \models \sigma \times \tau$. Then from the definition of $\sigma \times \tau$, we have that (the reduct to the language of σ of) $\mathbb{E}|P_0^{\mathbb{E}} : E_{\infty(\sigma\times\tau)}|P_0^{\mathbb{E}} \models \sigma$ (where P_i is from the language of τ as above). Let $f : E_{\infty(\sigma\times\tau)}|P_0^{\mathbb{E}} \sqsubseteq_B^i E_{\infty\sigma}$. Then it is easy to see that $g : E_{\infty(\sigma\times\tau)} \sqsubseteq_B^i E_{\infty\sigma} \times I_{\mathbb{N}}$, where $g(x) := (f(x), i)$ for the unique i such that $x \in P_i^{\mathbb{E}}$. □

Since τ is clearly countably categorical, this yields

Corollary 2.8.28. *If an elementary class $\mathcal{E}_E$ is Scott axiomatizable, then so is $\mathcal{E}_{E\times I_{\mathbb{N}}}$.*

In particular, if an elementary class C is Scott axiomatizable and closed under $E \mapsto E \times I_{\mathbb{N}}$, then $C \cap \mathcal{E}_c$ (i.e., the compressible elements of C) is Scott axiomatizable.

Proof. If $\mathcal{E}_E = \mathcal{E}_\sigma$ where σ is countably categorical, then $E_\infty \otimes (E \times I_{\mathbb{N}}) = (E_\infty \otimes E) \times I_{\mathbb{N}} = E_{\infty\sigma} \times I_{\mathbb{N}} = E_{\infty(\sigma\times\tau)}$ (using Proposition 2.5.29), whence $\mathcal{E}_{E\times I_{\mathbb{N}}} = \mathcal{E}_{\sigma\times\tau}$.

For the second statement, if $C = \mathcal{E}_E$ where E is universally structurable, then $C \cap \mathcal{E}_c = \mathcal{E}_{E\times I_{\mathbb{N}}}$ (Proposition 2.5.27). □

Corollary 2.8.29. *The following elementary classes are Scott axiomatizable: compressible hyperfinite, compressible treeable.*

Proof. For the compressible treeables, use that $E_{\infty\mathbb{F}_2}$ (i.e., the $\sqsubseteq_B^i$-universal orbit equivalence of a free action of $\mathbb{F}_2$) is $\sqsubseteq_B$-universal treeable [JKL, 3.17]; it follows that $\mathcal{E}_{\mathbb{F}_2}$ is closed under $E \mapsto E \times I_{\mathbb{N}}$, and also that $\mathcal{E}_{\mathbb{F}_2} \cap \mathcal{E}_c$ is the class of compressible treeables. □

However, we do not know if the elementary class of aperiodic treeable equivalence relations is Scott axiomatizable.

2.9 Some open problems

2.9.1 General questions

At the end of Section 2.4.4 we asked:

Problem 2.9.1. Is $E \otimes E$ universally structurable (or equivalently, isomorphic to $E_\infty \otimes E$) for every aperiodic E?

The following question (Remark 2.5.28) concerns the structure of universally structurable $\sim_B$-classes:

Problem 2.9.2. Is $E \times I_{\mathbb{N}} \sqsubseteq_B E$ for every aperiodic universally structurable E? Equivalently, is the compressible element of every universally structurable $\sim_B$-class the $\sqsubseteq_B^i$-least of the aperiodic elements?

By Theorem 2.6.20, we know that there are many incomparable elementary reducibility classes, or equivalently, many $\leq_B$-incomparable universally structurable E. However, these were produced using the results in [AK], which use rigidity theory for measure-preserving group actions. One hope for the theory of structurability is the possibility of producing $\leq_B$-incomparable equivalence relations using other methods, e.g., using model theory.

Problem 2.9.3. Show that there are $\leq_B$-incomparable $E_{\infty\sigma}, E_{\infty\tau}$ without using ergodic theory.

2.9.2 Order-theoretic questions

We turn now to the order-theoretic structure of the lattice $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ (equivalently, the poset of elementary classes) and the lattice $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ (equivalently, the poset of elementary reducibility classes). The following questions, posed in Section 2.6.2 (near end), are natural from an abstract order-theoretic perspective, though perhaps not so approachable:

Problem 2.9.4. Is either $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ or $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ a complete lattice? If so, is it completely distributive?

Problem 2.9.5. Is either $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq_B^i)$ or $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ a zero-dimensional ω_1-complete lattice, in that it embeds into 2^X for some set X?

We noted above (Corollary 2.6.15) that the recent work of Marks [M] gives some examples of ω_1-prime filters on $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$, and also (Proposition 2.6.18) that these filters cannot separate elements of $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$ below the universal treeable equivalence relation $E_{\infty T}$.

Regarding Problem 2.9.4, a natural attempt at a negative answer would be to show that some "sufficiently complicated" collection of universally structurable equivalence relations does not have a join. For example, one could try to find the join of a strictly increasing ω_1-sequence.

Problem 2.9.6. Is there an "explicit" strictly increasing ω_1-sequence in $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq^i_B)$? Similarly for $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$.

Note that by Theorem 2.6.20, such a sequence does exist, abstractly; the problem is thus to find a sequence which is in some sense "definable", preferably corresponding to some "natural" hierarchy of countable structures. For example, a long-standing open problem (implicit in e.g., [JKL, Section 2.4]) asks whether the sequence of elementary classes $(\mathcal{E}_\alpha)_{\alpha<\omega_1}$, where

$$\begin{aligned} \mathcal{E}_0 &:= \{\text{hyperfinite}\}, \\ \mathcal{E}_\alpha &:= \{\text{countable increasing union of } E \in \mathcal{E}_\beta \text{ for } \beta < \alpha\}, \end{aligned}$$

stabilizes (or indeed is constant); a negative answer would constitute a positive solution to Problem 2.9.6.

One possible approach to defining an ω_1-sequence would be by iterating a "jump" operation, $E \mapsto E'$, that sends any non-universal $E \in \mathcal{E}_\infty$ to some non-universal $E' \in \mathcal{E}_\infty$ such that $E <_B E'$.

Problem 2.9.7. Is there an "explicit" jump operation on the non-universal elements of $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$?

On the other hand, this would not be possible if there were a greatest non-universal element:

Problem 2.9.8. Is there a greatest element among the non-universal elements of $(\mathcal{E}_\infty/{\cong_B}, \sqsubseteq^i_B)$, or of $(\mathcal{E}_\infty/{\sim_B}, \leq_B)$? If so, do the non-universal equivalence relations form an elementary class, i.e., are they downward-closed under $\to^{cb}_B$?

2.9.3 Model-theoretic questions

The general model-theoretic question concerning structurability is which properties of a theory (L, σ) (or a Borel class of structures $\mathcal{K}$) yield properties of the corresponding elementary class $\mathcal{E}_\sigma$ (or $\mathcal{E}_\mathcal{K}$). Theorem 2.8.1 fits into this mold, by characterizing the σ which yield smoothness. One could seek similar results for other properties of countable Borel equivalence relations.

Problem 2.9.9. Find a model-theoretic characterization of the σ such that $\mathcal{E}_\sigma$ consists of only hyperfinite equivalence relations, i.e., such that $\sigma \Rightarrow \sigma_{hf}$, for any sentence σ_{hf} axiomatizing hyperfiniteness.

Less ambitiously, one might look for "natural" examples of such σ, for specific classes of structures. For example, for the Borel class of locally finite graphs, we have:

- If E is structurable via locally finite trees with one end, then E is hyperfinite [DJK, 8.2].
- If E is structurable via locally finite graphs with two ends, then E is hyperfinite [Mi, 5.1].

Remark 2.9.10. If E is structurable via locally finite graphs with at least 3 but finitely many ends, then E is smooth; this follows from [Mi, 6.2], or simply by observing that in any such graph, a finite nonempty subset may be defined as the set of all vertices around which the removal of a ball of minimal radius leaves ≥ 3 infinite connected components.

Problem 2.9.11. Find "natural" examples of σ such that $\mathcal{E}_\sigma$ consists of only Fréchet-amenable equivalence relations (see [JKL, 2.12]).

For example, every E structurable via countable scattered linear orders is Fréchet-amenable [JKL, 2.19] (recall that a countable linear order is **scattered** if it does not embed the rationals); note however that the scattered linear orders do not form a Borel class of structures.

Problem 2.9.12. Find "natural" examples of σ such that $\mathcal{E}_\sigma$ consists of only compressible equivalence relations.

For example, by [Mi2], the class of E structurable via locally finite graphs whose space of ends is not perfect but has cardinality at least 3 is exactly $\mathcal{E}_c$.

There is also the converse problem of determining for which σ is every equivalence relation of a certain form (e.g., compressible) σ-structurable. Theorem 2.8.2 fits into this mold, by giving a sufficient condition for a single structure to structure every aperiodic equivalence relation.

Problem 2.9.13. Is there a structure $\mathbb{A}$ without TDC which structures every aperiodic countable Borel equivalence relation? That is, does the converse of Theorem 2.8.2 hold?

In particular, does $\mathbb{Q} \times \mathbb{Z}$ structure every aperiodic equivalence relation? We noted above that it structures every compressible equivalence relation, thus the analogous question for the compressibles has a negative answer.

Problem 2.9.14. Find a model-theoretic characterization of the structures $\mathbb{A}$ such that every compressible equivalence relation is $\mathbb{A}$-structurable, i.e., $\mathcal{E}_c \subseteq \mathcal{E}_{\mathbb{A}}$.

We also have the corresponding questions for theories (or classes of structures):

Problem 2.9.15. Find a model-theoretic characterization of the σ such that $\mathcal{E}_\sigma = \mathcal{E}$, or more generally, $\mathcal{E}_c \subseteq \mathcal{E}_\sigma$.

We gave several examples in Section 2.8.2. Another example is the following [Mi, 4.1]: every $E \in \mathcal{E}$ is structurable via locally finite graphs with at most one end.

For a different sort of property that $\mathcal{E}_\sigma$ may or may not have, recall (Section 2.5.3) that $\mathcal{E}_\sigma$ is an elementary reducibility class, i.e., closed under $\leq_B$, when σ axiomatizes linear orders embeddable in $\mathbb{Z}$, or when σ axiomatizes trees.

Problem 2.9.16. Find a model-theoretic characterization of the σ such that $\mathcal{E}_\sigma$ is closed under $\leq_B$.

We considered in Section 2.8.3 the question of which elementary classes are Scott axiomatizable, i.e., axiomatizable by a Scott sentence.

Problem 2.9.17. Find other "natural" examples of Scott axiomatizable elementary classes.

We showed above (Corollary 2.8.29) that the class of compressible treeable equivalence relations is Scott axiomatizable.

Problem 2.9.18. Is the class of aperiodic treeable (countable Borel) equivalence relations Scott axiomatizable?

Remark 2.9.19. The class of aperiodic treeables cannot be axiomatized by the Scott sentence of a single countable tree T. Indeed, since E_0 would have to be treeable by T, by a result of Adams (see [KM, 22.3]), T can have at most 2 ends; but then by [DJK, 8.2] and [Mi, 5.1], every E treeable by T is hyperfinite.

Problem 2.9.20. Find a model-theoretic characterization of the σ such that $\mathcal{E}_\sigma$ is axiomatizable by a Scott sentence (possibly in some other language). In particular, is every elementary class of aperiodic, or compressible, equivalence relations axiomatizable by a Scott sentence?

We conclude by stating two very general (and ambitious) questions concerning the relationship between structurability and model theory. For the first, note that by (i) $\iff$ (ii) in Theorem 2.8.1, the condition $\sigma \Rightarrow^* \sigma_{sm}$ is equivalent to the existence of a formula(s) in the language of σ with some definable properties which are logically implied by σ. Our question is whether a similar equivalence continues to hold when σ_{sm} is replaced by an arbitrary sentence τ.

Problem 2.9.21. Is there, for any τ, a sentence $\tau'(R_1, R_2, \ldots)$ in a language consisting of relation symbols $R_1, R_2, \ldots$ (thought of as "predicate variables"), such that for any σ, we have $\sigma \Rightarrow^* \tau$ iff there are formulas $\phi_1, \phi_2, \ldots$ in the language of σ such that σ logically implies $\tau'(\phi_1, \phi_2, \ldots)$ (the result of "substituting" ϕ_i for R_i in τ')?

Finally, there is the question of completely characterizing containment between elementary classes:

Problem 2.9.22. Find a model-theoretic characterization of the pairs (σ, τ) such that $\sigma \Rightarrow^* \tau$.

2.A Appendix: Fiber spaces

In this appendix, we discuss fiber spaces on countable Borel equivalence relations, which provide a more general context for structurability and related notions. The application of fiber spaces to structurability was previously considered in [G] and [HK, Appendix D].

In both this appendix and the next, we will use categorical terminology somewhat more liberally than in the body of this paper.

2.A.1 Fiber spaces

Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation. A **fiber space over** E consists of a countable Borel equivalence relation (U, P), together with a surjective countable-to-1 class-bijective homomorphism $p : P \to_B^{cb} E$. We refer to the fiber space by (U, P, p), by (P, p, E), by (P, p), or (ambiguously) by P. We call (U, P) the **total space**, (X, E) the **base space**, and p the **projection**. For $x \in X$, the **fiber** over x is the set $p^{-1}(x) \subseteq U$. For $x, x' \in X$ such that $x \mathrel{E} x'$, we let

$$p^{-1}(x, x') : p^{-1}(x) \to p^{-1}(x')$$

denote the **fiber transport map**, where for $u \in p^{-1}(x)$, $p^{-1}(x, x')(y)$ is the unique $u' \in p^{-1}(x')$ such that $u \mathrel{P} u'$.

For two fiber spaces (U, P, p), (V, Q, q) over (X, E), a **fiberwise map** between them **over** E, denoted $\widetilde{f} : (P, p) \to_E (Q, q)$ (we use letters like $\widetilde{f}, \widetilde{g}$ for maps between total spaces), is a homomorphism $\widetilde{f} : P \to_B Q$ such that $p = q \circ \widetilde{f}$ (note that this implies that $\widetilde{f}$ is class-bijective):

$$\begin{tikzcd}
(U, P) \arrow[rr, "\widetilde{f}"] \arrow[dr, "p"'] & & (V, Q) \arrow[dl, "q"] \\
& (X, E) &
\end{tikzcd}$$

For a fiber space (U, P, p) over (X, E) and a fiber space (V, Q, q) over (Y, F), a **fiber space homomorphism** from (P, p, E) to (Q, q, F), denoted $f : (P, p, E) \to (Q, q, F)$, consists of two homomorphisms $f : E \to_B F$ and $\widetilde{f} : P \to_B Q$ such that $f \circ p = q \circ \widetilde{f}$:

$$\begin{tikzcd}
(U, P) \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & (V, Q) \arrow[d, "q"] \\
(X, E) \arrow[r, "f"'] & (Y, F)
\end{tikzcd}$$

We sometimes refer to $\widetilde{f}$ as the fiber space homomorphism; note that f is determined by $\widetilde{f}$ (since p is surjective). We say that $\widetilde{f}$ is a fiber space homomorphism **over** f. Note that a fiberwise map over E is the same thing as a fiber space homomorphism over the identity function on E.

A fiber space homomorphism $f : (P, p, E) \to (Q, q, F)$ is **fiber-bijective** if $\widetilde{f}|p^{-1}(x) : p^{-1}(x) \to q^{-1}(f(x))$ is a bijection for each $x \in X$ (where E lives on X); **fiber-injective**, **fiber-surjective** are defined similarly.

Let (U, P, p) be a fiber space over (X, E), and let $(Y, F) \in \mathcal{E}$ be a countable Borel equivalence relation with a homomorphism $f : F \to_B E$. Recall (Section 2.2.6) that we have the fiber product equivalence relation $(Y \times_X U, F \times_E P)$ with respect to f and p, which comes equipped with the canonical projections $\pi_1 : F \times_E P \to_B F$ and $\pi_2 : F \times_E P \to_B P$ obeying $f \circ \pi_1 = p \circ \pi_2$. It is easy to check that π_1 is class-bijective, surjective, and countable-to-1 (because p is). In this situation, we also use the notation

$$(f^{-1}(U), f^{-1}(P), f^{-1}(p)) = f^{-1}(U, P, p) := (Y \times_X U, F \times_E P, \pi_1).$$

Note that $\widetilde{f} := \pi_2 : f^{-1}(P) \to_B P$ is then a fiber space homomorphism over f. We refer to $f^{-1}(U, P, p)$ as the **pullback of** (U, P, p) **along** f. Here is a diagram:

$$\begin{tikzcd}
f^{-1}(U,P) \arrow[r, "\widetilde{f}"] \arrow[d, "f^{-1}(p)"'] & (U,P) \arrow[d, "p"] \\
(Y,F) \arrow[r, "f"'] & (X,E)
\end{tikzcd}$$

Let $\mathbf{Fib}(E)$ denote the category of fiber spaces and fiberwise maps over E, and let $\int_{\mathcal{E}} \mathbf{Fib}$ denote the category of fiber spaces and fiber space homomorphisms. For a homomorphism $f : E \to_B F$, pullback along f gives a functor $f^{-1} : \mathbf{Fib}(F) \to \mathbf{Fib}(E)$ (with the obvious action on fiberwise maps). The assignment $f \mapsto f^{-1}$ is itself functorial, and turns $\mathbf{Fib}$ into a contravariant functor from the category $(\mathcal{E}, \to_B)$ to the category of (essentially small) categories. (Technically $f \mapsto f^{-1}$ is only *pseudo*functorial, i.e., $f^{-1}(g^{-1}(P))$ is naturally isomorphic, not equal, to $(g \circ f)^{-1}(P)$; we will not bother to make this distinction.)

2.A.2 Fiber spaces and cocycles

Let (U, P, p) be a fiber space over $(X, E) \in \mathcal{E}$. By Lusin-Novikov uniformization, we may Borel partition X according to the cardinalities of the fibers. Suppose for simplicity that each fiber is countably infinite. Again by Lusin-Novikov uniformization, there is a Borel map $T : X \to U^{\mathbb{N}}$ such that each $T(x)$ is a bijection $\mathbb{N} \to p^{-1}(x)$. Let $\alpha_T : E \to S_\infty$ be the cocycle given by

$$\alpha_T(x, x') := T(x')^{-1} \circ p^{-1}(x, x') \circ T(x)$$

(where $p^{-1}(x, x')$ is the fiber transport map; compare Remark 2.4.3). We then have a (fiberwise) isomorphism of fiber spaces over E, between (U, P, p) and the skew product $E \ltimes_{\alpha_T} \mathbb{N}$ (with its canonical projection $q : E \ltimes_{\alpha_T} \mathbb{N} \to_B^{cb} E$):

$$\begin{aligned}(U, P, p) &\longleftrightarrow (X \times \mathbb{N}, E \ltimes_{\alpha_T} \mathbb{N}, q)\\ u &\longmapsto (p(u), T(p(u))^{-1}(u))\\ T(x)(n) &\longleftarrow\!\shortmid (x, n).\end{aligned}$$

Recall that two cocycles $\alpha, \beta : E \to S_\infty$ are **cohomologous** if there is a Borel map $\phi : X \to S_\infty$ such that $\phi(x')\alpha(x, x') = \beta(x, x')\phi(x)$, for all $(x, x') \in E$. It is easy to see that in the above, changing the map $T : X \to U^{\mathbb{N}}$ results in a cohomologous cocycle $\alpha_T : E \to S_\infty$; so we get a well-defined map from (isomorphism classes of) fiber spaces over E with countably infinite fibers to S_∞-valued cohomology classes on E. Conversely, given any cocycle $\alpha : E \to S_\infty$, the skew product $E \ltimes_\alpha \mathbb{N}$ yields a fiber space over E with countably infinite fibers. These two operations are inverse to each other, so we have a bijection

$$\begin{aligned}&\{\text{isomorphism classes of fiber spaces over } E \text{ with } \aleph_0\text{-sized fibers}\}\\ &\qquad\qquad \cong \{S_\infty\text{-valued cohomology classes on } E\}.\end{aligned}$$

Remark 2.A.1. In fact, we have the following more refined correspondence, which also smoothly handles the case with finite fibers. Let $\mathbf{C}$ denote the category whose objects are $1, 2, \dots, \mathbb{N}$ and morphisms are maps between them (where as usual, $n = \{0, \dots, n-1\}$ for $n \in \mathbb{N}$). Then $\mathbf{C}$ is a "standard Borel category". Regarding E as the groupoid on X with a single morphism between any two related points, we have a **Borel functor category** $\mathbf{C}_B^E$, whose objects are Borel functors $E \to \mathbf{C}$ and morphisms are Borel natural transformations. We then have a functor

$$\mathbf{C}_B^E \longrightarrow \mathbf{Fib}(E)$$

which takes a Borel functor $\alpha : E \to \mathbf{C}$ to the obvious generalization of the skew product of E with respect to α (but where the fibers are no longer uniformly $\mathbb{N}$, but vary from point to point according to α); and this functor is an equivalence of categories. We leave the details to the reader.

Using this correspondence between fiber spaces and cocycles, we obtain

Proposition 2.A.2. *There is a fiber space $(U_\infty, P_\infty, p_\infty)$ over E_∞, which is universal with respect to fiber-bijective invariant embeddings: for any other fiber space (U, P, p) over E, there is a fiber-bijective homomorphism $\widetilde{f} : P \to P_\infty$ over an invariant embedding $f : E \sqsubseteq_B^i E_\infty$.*

Proof. For simplicity, we restrict again to the case where P has countably infinite fibers. Let σ be a sentence over the language $L = \{R_{ij}\}_{i,j\in\mathbb{N}}$, where each R_{ij} is binary, asserting that

$$\alpha(x, y)(i) = j \iff R_{ij}(x, y)$$

defines a cocycle $\alpha : I_X \to S_\infty$, where X is the universe of the structure (and I_X is the indiscrete equivalence relation on X). Then the canonical σ-structure on $E_{\infty\sigma}$ corresponds to a cocycle $\alpha_\infty : E_{\infty\sigma} \to S_\infty$. We will in fact define the universal fiber space P_∞ over $E_{\infty\sigma}$, since clearly $E_\infty \sqsubseteq^i_B E_{\infty\sigma}$ (by giving E_∞ the trivial cocycle). Let $P_\infty := E_{\infty\sigma} \ltimes_{\alpha_\infty} \mathbb{N}$, with $p_\infty : P_\infty \to^{cb}_B E_{\infty\sigma}$ the canonical projection. For another fiber space (U, P, p) over E with countably infinite fibers, by the above remarks, P is isomorphic (over E) to a skew product $E \ltimes_\alpha \mathbb{N}$, for some cocycle $\alpha : E \to S_\infty$. This α corresponds to a σ-structure on E, which yields an invariant embedding $f : E \sqsubseteq^i_B E_{\infty\sigma}$ such that α is the restriction of α_∞ along f, giving the desired fiber-bijective homomorphism $\widetilde{f} := f \times \mathbb{N} : E \ltimes_\alpha \mathbb{N} \to E_{\infty\sigma} \ltimes_{\alpha_\infty} \mathbb{N}$ over f. □

There is a different kind of universality one could ask for, which we do not know how to obtain. Namely, for each $E \in \mathcal{E}$, is there a fiber space $(U_\infty, P_\infty, p_\infty)$ over E which is universal with respect to fiberwise injective maps?

2.A.3 Equivalence relations as fiber spaces

Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation. The **tautological fiber space** over E is $(E, \widehat{E}, \pi_1)$, where $\widehat{E}$ is the equivalence relation on the set $E \subseteq X^2$ given by

$$(x, x') \mathrel{\widehat{E}} (y, y') \iff x' = y',$$

and $\pi_1 : (E, \widehat{E}) \to^{cb}_B (X, E)$ is the first coordinate projection (i.e., $\pi(x, x') = x$). In other words, the $\widehat{E}$-fiber over each E-class $C \in X/E$ consists of the elements of C.

Note that $\widehat{E}$ is the kernel of the second coordinate projection $\pi_2 : E \to X$; thus $\widehat{E}$ is smooth, and in fact $E/\widehat{E}$ is isomorphic to X (via π_2). Now let (U, P, p) be any smooth fiber space over (X, E), and let F be the (countable Borel) equivalence relation on $Y := U/P$ given by

$$[u]_P \mathrel{F} [u']_P \iff p(u) \mathrel{E} p(u').$$

By Lusin-Novikov uniformization, there is a Borel map $X \to U$ which is a section of p, which when composed with the projection $U \to Y$ gives a reduction $f : (X, E) \leq_B (Y, F)$ whose image is a complete section.

Let us say that a **presentation** of the quotient space X/E consists of a countable Borel equivalence relation $(Y, F) \in \mathcal{E}$ together with a bijection $X/E \cong Y/F$ which admits a Borel lifting $X \to_B Y$ (which is then a reduction $E \leq_B F$ with image a complete section). By the above, every smooth fiber space over E gives rise to a presentation of X/E. Conversely, given any presentation (Y, F) of X/E, letting $f : E \leq_B F$ with image a complete section, the pullback $f^{-1}(\widehat{F})$ is a fiber space over E, which is smooth (because $f^{-1}(\widehat{F})$ reduces to $\widehat{F}$, via the map $\widetilde{f}$ coming from the pullback). It is easily seen that the two operations we have just described are mutually inverse up to isomorphism, yielding a bijection

$$\{\text{iso. classes of smooth fiber spaces over } E\} \cong \{\text{iso. classes of presentations of } X/E\}.$$

Remark 2.A.3. This correspondence between smooth fiber spaces and presentations of the same quotient space is essentially the proof of [HK, D.1].

We now describe the correspondence between homomorphisms of equivalence relations and fiber space homomorphisms. Let $(X, E), (Y, F) \in \mathcal{E}$. A homomorphism $f : E \to_B F$ induces a fiber space homomorphism $\widehat{f} : \widehat{E} \to \widehat{F}$ over f, given by

$$\widehat{f}(x, x') := (f(x), f(x')).$$

Conversely, let $\widetilde{g} : \widehat{E} \to \widehat{F}$ be any fiber space homomorphism over $g : E \to_B F$. Then $\widetilde{g}$ must be given by $\widetilde{g}(x, x') = (g(x), f(x'))$ for some $f : X \to Y$ such that $g(x) \; F \; f(x)$ for each $x \in X$; in particular, f is a homomorphism $E \to_B F$.

Let us say that two homomorphisms $f, g : E \to_B F$ are **equivalent**, denoted $f \simeq g$, if $f(x) \; F \; g(x)$ for each $x \in X$; equivalently, they induce the same map on the quotient spaces $X/E \to Y/F$. The above yield mutually inverse bijections

$$\{\text{homomorphisms } E \to_B F\} \cong \{\simeq\text{-classes of fiber space homomorphisms } \widehat{E} \to \widehat{F}\}.$$

Class-injectivity on the left translates to fiber-injectivity on the right, etc.

2.A.4 Countable Borel quotient spaces

We discuss here an alternative point of view on fiber spaces and equivalence relations. The idea is that the tautological fiber space $\widehat{E}$ over an equivalence relation (X, E) allows a clean distinction to be made between the quotient space X/E and the presentation (X, E).

A **countable Borel quotient space** is, formally, the same thing as a countable Borel equivalence relation (X, E), except that we denote it by X/E. A **Borel map** between countable Borel quotient spaces X/E and Y/F, denoted $f : X/E \to_B Y/F$, is a map which admits a Borel lifting $X \to Y$, or equivalently an $\simeq$-class of Borel homomorphisms $E \to_B F$. Let $(\mathcal{Q}, \to_B)$ denote the category of countable Borel quotient spaces and Borel maps. (Note that $X/E, Y/F$ are isomorphic in $(\mathcal{Q}, \to_B)$ iff they are *bireducible* as countable Borel equivalence relations.)

Let $\mathcal{B}$ denote the class of standard Borel spaces. By identifying $X \in \mathcal{B}$ with $X/\Delta_X \in \mathcal{Q}$, we regard $(\mathcal{B}, \to_B)$ as a full subcategory of $(\mathcal{Q}, \to_B)$. By regarding Borel maps in $\mathcal{Q}$ as $\simeq$-classes of homomorphisms, we have that $(\mathcal{Q}, \to_B)$ is the quotient category of $(\mathcal{E}, \to_B)$ (with the same objects) by the congruence $\simeq$.

A **(quotient) fiber space** over a quotient space $X/E \in \mathcal{Q}$ is a quotient space $U/P \in \mathcal{Q}$ together with a countable-to-1 surjection $p : U/P \to_B X/E$. This definition agrees with the previous notion of fiber space over (X, E), in that fiber spaces over X/E are in natural bijection with fiber spaces over (X, E), up to isomorphism. Indeed, by Proposition 2.5.8, we may factor any lifting $(U, P) \to_B (X, E)$ of p into a reduction with image a complete section, followed by a class-bijective homomorphism; the former map becomes an isomorphism when we pass to the quotient, so U/P is isomorphic to a fiber space with class-bijective projection.

We have obvious versions of the notions of **fiberwise map over** X/E, **fiber space homomorphism**, and **fiber-bijective homomorphism** for quotient fiber spaces. Let $\mathbf{Fib}(X/E)$ denote the category of fiber spaces over X/E; in light of the above remarks, $\mathbf{Fib}(X/E)$ is equivalent to $\mathbf{Fib}(E)$. Let $\int_{\mathcal{Q}} \mathbf{Fib}$ denote the category of quotient fiber spaces and homomorphisms ($\int_{\mathcal{Q}} \mathbf{Fib}$ is then the quotient of $\int_{\mathcal{E}} \mathbf{Fib}$ by $\simeq$). We now have a full embedding

$$(\mathcal{E}, \to_B) \longrightarrow \int_{\mathcal{Q}} \mathbf{Fib},$$

that sends an equivalence relation (X, E) to its tautological fiber space $(E, \widehat{E})$ but regarded as the quotient fiber space $\widehat{E}/E \cong X$ over X/E, and sends a homomorphism f to the corresponding fiber space homomorphism $\widehat{f}$ given above. Thus, we may regard equivalence relations as special cases of fiber spaces over quotient spaces.

To summarize, here is a (non-commuting) diagram of several relevant categories and functors:

$$\begin{tikzcd}
\int_{\mathcal{E}} \mathbf{Fib} \arrow[d] \arrow[dr, two heads] & \\
(\mathcal{E}, \to_B) \arrow[r, hook] \arrow[dr, two heads] & \int_{Q} \mathbf{Fib} \arrow[d] \\
(\mathcal{B}, \to_B) \arrow[r, hook] & (\mathcal{Q}, \to_B)
\end{tikzcd}$$

The horizontal arrows are full embeddings, the diagonal arrows are quotients by $\simeq$, and the vertical arrows are forgetful functors that send a fiber space to its base space.

2.A.5 Factorizations of fiber space homomorphisms

Let (U, P, p), (V, Q, q) be fiber spaces over (X, E), (Y, F) respectively, and $f : E \to_B F$. A fiber space homomorphism $\widetilde{f} : P \to Q$ over f corresponds, via the universal property of the pullback $f^{-1}(V, Q, q)$, to a fiberwise map $\widetilde{f}' : P \to_E f^{-1}(Q)$ over E:

$$\begin{tikzcd}
P \arrow[dr, "\widetilde{f}'"'] \arrow[drr, "\widetilde{f}"] \arrow[ddr, bend right, "p"'] & & \\
& f^{-1}(Q) \arrow[r] \arrow[d, "f^{-1}(q)"] & Q \arrow[d, "q"] \\
& E \arrow[r, "f"'] & F
\end{tikzcd}$$

Note that $\widetilde{f}$ is fiber-bijective iff $\widetilde{f}'$ is an isomorphism. In general, since $\widetilde{f}'$ is countable-to-1, we may further factor it into the surjection onto its image $(\widetilde{f}'(U), \widetilde{f}'(P))$ followed by an inclusion:

$$\begin{tikzcd}
P \arrow[dr, two heads, "\widetilde{f}'"'] \arrow[drrr, bend left, "\widetilde{f}"] \arrow[ddrr, bend right, "p"'] & & & \\
& \widetilde{f}'(P) \arrow[r, hook] \arrow[dr] & f^{-1}(Q) \arrow[r] \arrow[d, "f^{-1}(q)"] & Q \arrow[d, "q"] \\
& & E \arrow[r, "f"'] & F
\end{tikzcd}$$

So we have a canonical factorization of any fiber space homomorphism $\widetilde{f}$ into a fiberwise surjection over E, followed by a fiberwise injection over E, followed by a fiber-bijective homomorphism.

In the case where $P = \widehat{E}$, $Q = \widehat{F}$, and $\widetilde{f} = \widehat{f} : \widehat{E} \to \widehat{F}$ is the fiber space homomorphism induced by f, the fiber space $f^{-1}(V, Q, q) = f^{-1}(F, \widehat{F}, \pi_1)$ is given by

$$f^{-1}(F) = \{(x, (y_1, y_2)) \in F \mid f(x) = y_1\} \cong \{(x, y) \in F \mid f(x) \; F \; y\},$$
$$(x, y) \; f^{-1}(\widehat{F}) \; (x', y') \iff y = y',$$
$$f^{-1}(\pi_1)(x, y) = x,$$

while the map $\widetilde{f'} : (E, \widehat{E}) = (U, P) \to f^{-1}(V, Q) = f^{-1}(F, \widehat{F})$ is given by

$$\widetilde{f'}(x, x') = (x, f(x')).$$

Comparing with the proofs of Propositions 2.5.3 and 2.5.8 reveals that when $f : E \to_B F$ is smooth, the above factorization of $\widehat{f}$ corresponds (via the correspondence between smooth fiber spaces and presentations from Section 2.A.3) to the factorization of f produced by Proposition 2.5.8. In particular, we obtain a characterization of smooth homomorphisms in terms of fiber spaces:

Proposition 2.A.4. *$f : E \to_B F$ is smooth iff the fiber space $f^{-1}(\widehat{F})$ over E is smooth.*

Remark 2.A.5. In fact, the proof of Proposition 2.5.3 is essentially just the above correspondence, plus the observation that $\widetilde{f'}(\widehat{E}) \to_B^{ci} \widehat{F}$ and smoothness of $\widehat{F}$ imply that $\widetilde{f'}(\widehat{E})$ is smooth (compare also [HK, D.2]).

2.A.6 Structures on fiber spaces

Let L be a language and (U, P, p) be a fiber space over $(X, E) \in \mathcal{E}$. A **Borel L-structure on** (U, P, p) is a Borel L-structure $\mathbb{A} = (U, R^{\mathbb{A}})_{R \in L}$ with universe U which only relates elements within the same fiber, i.e.,

$$R^{\mathbb{A}}(u_1, \dots, u_n) \implies p(u_1) = \cdots = p(u_n),$$

such that structures on fibers over the same E-class are related via fiber transport, i.e.,

$$x \; E \; x' \implies p^{-1}(x, x')(\mathbb{A}|p^{-1}(x)) = \mathbb{A}|p^{-1}(x').$$

For an $L_{\omega_1\omega}$-sentence σ, we say that $\mathbb{A}$ is a **Borel σ-structure on** (U, P, p), denoted

$$\mathbb{A} : (U, P, p) \models \sigma,$$

if $\mathbb{A}|p^{-1}(x)$ satisfies σ for each $x \in X$.

For $(X,E) \in \mathcal{E}$, σ-structures on E are in bijection with σ-structures on the tautological fiber space $(\widehat{E}, \pi_1)$ over E, where $\mathbb{A} : E \models \sigma$ corresponds to $\widehat{\mathbb{A}} : (\widehat{E}, \pi_1) \models \sigma$ given by

$$R^{\widehat{\mathbb{A}}}((x,x_1),\dots,(x,x_n)) \iff R^{\mathbb{A}}(x_1,\dots,x_n).$$

In other words, for each $x \in X$, $\mathbb{A}|[x]_E$ and $\widehat{\mathbb{A}}|\pi_1^{-1}(x)$ are isomorphic via the canonical bijection $x' \mapsto (x,x')$ between $[x]_E$ and $\pi_1^{-1}(x)$.

For a fiber space homomorphism $f : (P,p,E) \to (Q,q,F)$ and a σ-structure $\mathbb{A} : (Q,q) \models \sigma$, the **fiberwise pullback structure** $f^{-1}_{(P,p)}(\mathbb{A}) : (P,p) \models \sigma$ is defined in the obvious way, i.e.,

$$R^{f^{-1}_{(P,p)}(\mathbb{A})}(u_1,\dots,u_n) \iff R^{\mathbb{A}}(\widetilde{f}(u_1),\dots,\widetilde{f}(u_n)) \;\&\; p(u_1) = \cdots = p(u_n).$$

We have the following generalization of Theorem 2.4.1:

Proposition 2.A.6. *Let (U,P,p) be a fiber space over $(X,E) \in \mathcal{E}$ and (L,σ) be a theory. There is a fiber space $(U,P,p) \ltimes \sigma = (U \ltimes_p \sigma, P \ltimes_p \sigma, p \ltimes \sigma)$ over an equivalence relation $E \ltimes_p \sigma \in \mathcal{E}$, together with a fiber-bijective homomorphism $\pi : (P,p,E) \ltimes \sigma \to (P,p,E)$ and a σ-structure $\mathbb{E} : (P,p) \ltimes \sigma \models \sigma$, such that the triple $((U,P,p) \ltimes \sigma, \pi, \mathbb{E})$ is universal: for any other fiber space (V,Q,q) over $(Y,F) \in \mathcal{E}$ with a fiber-bijective homomorphism $f : (Q,q,F) \to (P,p,E)$ and a structure $\mathbb{A} : (Q,q) \models \sigma$, there is a unique fiber-bijective $g : (Q,q,F) \to (P,p,E) \ltimes \sigma$ such that $f = \pi \circ g$ and $\mathbb{A} = g^{-1}_{(Q,q)}(\mathbb{E})$.*

Proof sketch. This is a straightforward generalization of Theorem 2.4.1 (despite the excessive notation). The equivalence relation $E \ltimes_p \sigma$ lives on

$$\{(x,\mathbb{B}) \mid x \in X,\ \mathbb{B} \in \mathrm{Mod}_{p^{-1}(x)}(\sigma)\},$$

and is given by

$$(x,\mathbb{B})\ (E \ltimes_p \sigma)\ (x',\mathbb{B}') \iff x\ E\ x' \;\&\; p^{-1}(x,x^{-1})(\mathbb{B}) = \mathbb{B}'.$$

As usual, the Borel structure on $E \ltimes_p \sigma$ is given by uniformly enumerating each $p^{-1}(x)$. The base space part of π is given by $\pi(x,\mathbb{B}) := x$, the fiber space $(U,P,p) \ltimes \sigma$ is given by the pullback $\pi^{-1}(U,P,p)$, and the structure $\mathbb{E}$ is given by

$$R^{\mathbb{E}}((x,\mathbb{B},u_1),\dots,(x,\mathbb{B},u_n)) \iff R^{\mathbb{B}}(u_1,\dots,u_n)$$

for $x \in X$, $\mathbb{B} \in \mathrm{Mod}_{p^{-1}(x)}(\sigma)$, and $u_1,\dots,u_n \in p^{-1}(x)$. The universal property is straightforward. □

Remark 2.A.7. However, our other basic universal construction for structuring equivalence relations, the "Scott sentence" (Theorem 2.4.7), fails to generalize in a straightforward fashion to fiber spaces; this is essentially because we require languages to be countable, whereas the invariant Borel σ-algebra of a nonsmooth fiber space is not countably generated.

Remark 2.A.8. Nonetheless, we may define the **fiber-bijective product** $(P, p, E) \otimes (Q, q, F)$ of two fiber spaces $(U, P, p), (V, Q, q)$ over $(X, E), (Y, F) \in \mathcal{E}$ respectively, by generalizing Remark 2.4.20, yielding their categorical product in the category of fiber spaces and fiber-bijective homomorphisms; we leave the details to the reader. In particular, by taking (Q, q, F) to be the universal fiber space $(P_\infty, p_\infty, E_\infty)$ from Proposition 2.A.2, we obtain

Proposition 2.A.9. *For every fiber space (P, p) over $E \in \mathcal{E}$, there is a fiber space $(P_\infty, p_\infty, E_\infty) \otimes (P, p, E)$ admitting a fiber-bijective homomorphism to (P, p, E) and which is universal among such fiber spaces with respect to fiber-bijective embeddings.*

We conclude by noting that restricting attention to smooth fiber spaces and applying the correspondence with presentations gives a different perspective on some results from Section 2.5:

- [HK, D.1] If $(X, E) \in \mathcal{E}$ admits a smooth fiber space (P, p), and $\mathbb{A} : (P, p) \models \sigma$, then (P, p) corresponds to a presentation (Y, F) of X/E, and $\mathbb{A}$ corresponds to a structure $(\widehat{F}, \pi_1) \models \sigma$, i.e., a structure $F \models \sigma$; hence E is bireducible with a σ-structurable equivalence relation.
- In particular, if $f : E \to_B^{sm} F$ and $\mathbb{A} : F \models \sigma$, then pulling back $\widehat{\mathbb{A}} : (\widehat{F}, \pi_1) \models \sigma$ along f gives a smooth σ-structured fiber space (namely $f^{-1}(\widehat{F})$) over E, whence E is bireducible with a σ-structurable equivalence relation. So $\mathcal{E}_\sigma^r$ is closed under $\to_B^{sm}$ (Theorem 2.5.2).
- If $f : E \to_B^{ci} F$, then the induced $\widehat{f} : \widehat{E} \to \widehat{F}$ is fiber-injective, which yields a fiberwise injection $\widehat{E} \to_E f^{-1}(\widehat{F})$ over E, whence E embeds into the σ-structurable presentation corresponding to $f^{-1}(\widehat{F})$; this similarly re-proves part of Theorem 2.5.1.

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Chapter 3

BOREL STRUCTURABILITY BY LOCALLY FINITE SIMPLICIAL COMPLEXES

Ruiyuan Chen

3.1 Introduction

A **countable Borel equivalence relation** E on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ for which each equivalence class is countable. The class of **treeable** countable Borel equivalence relations, for which there is a Borel way to put a tree (acyclic connected graph) on each equivalence class, has been studied extensively by many authors, especially in relation to ergodic theory; see e.g., [Ada], [Ga1], [JKL], [KM], [HK], [Hjo]. It is a basic result, due to Jackson-Kechris-Louveau [JKL, 3.10], that every treeable equivalence relation embeds into one treeable by trees in which each vertex has degree at most 3. The purpose of this paper is to present a generalization of this result to higher dimensions.

Recall that a **simplicial complex** on a set X is a collection S of finite nonempty subsets of X which contains all singletons and is closed under nonempty subsets. A simplicial complex S has a **geometric realization** $|S|$, which is a topological space formed by gluing together Euclidean simplices according to S (see Section 3.2 for the precise definition); S is **contractible** if $|S|$ is. Given a distinguished class $\mathcal{K}$ of simplicial complexes (e.g., the contractible ones) and a countable Borel equivalence relation (X, E), a **(Borel) structuring of E by simplicial complexes in $\mathcal{K}$** is, informally (see Section 3.2), a Borel assignment of a simplicial complex $S_C \in \mathcal{K}$ on each equivalence class $C \in X/E$. If such a structuring exists, we say that E is **structurable by complexes in $\mathcal{K}$**. We are interested here mainly in $\mathcal{K} = n$-dimensional contractible simplicial complexes; when $n = 1$, we recover the notion of treeability. The study of equivalence relations structurable by n-dimensional contractible simplicial complexes was initiated by Gaboriau [Ga2], who proved (among other things) that for $n = 1, 2, 3, \dots$ these classes of countable Borel equivalence relations form a strictly increasing hierarchy under $\subseteq$.

Recall also the notion of a **Borel embedding** $f : E \to F$ between countable Borel equivalence relations (X, E) and (Y, F), which is an injective Borel map $f : X \to Y$

such that $x \; E \; x' \iff f(x) \; F \; f(x')$ for all $x, x' \in X$.

Theorem 3.1.1. *Let $n \geq 1$, and let (X, E) be a countable Borel equivalence relation structurable by n-dimensional contractible simplicial complexes. Then E Borel embeds into a countable Borel equivalence relation (Y, F) structurable by n-dimensional contractible simplicial complexes in which each vertex belongs to at most (or even exactly) $M_n := 2^{n-1}(n^2 + 3n + 2) - 2$ edges.*

In particular, every E structurable by n-dimensional contractible simplicial complexes Borel embeds into an F structurable by locally finite such complexes, where a simplicial complex is **locally finite** if each vertex is contained in finitely many edges (or equivalently finitely many simplices). The constant M_n above is not optimal: for $n = 1$ we have $M_1 = 4$, whereas by the aforementioned result of Jackson-Kechris-Louveau we may take $M_1 = 3$ instead, which is optimal; for $n = 2$ we have $M_2 = 22$, whereas by a construction different from the one below we are able to get $M_2 = 10$. We do not know what the optimal M_n is for $n > 1$; however, the result of Gaboriau mentioned above implies that the optimal M_n is at least $n + 1$.

The referee has pointed out that by an easy argument, one may strengthen "at most" to "exactly" in Theorem 3.1.1 (as well as in the following reformulations).

We may reformulate Theorem 3.1.1 in terms of **compressible** countable Borel equivalence relations, which are those admitting no invariant probability Borel measure (see e.g., [DJK] for various equivalent definitions of compressibility):

Corollary 3.1.2. *Let $n \geq 1$, and let (X, E) be a compressible countable Borel equivalence relation structurable by n-dimensional contractible simplicial complexes. Then E is structurable by n-dimensional contractible simplicial complexes in which each vertex belongs to at most (or even exactly) M_n edges.*

Note that by the theory of cost (see [Ga1], [KM]), Corollary 3.1.2 cannot be true of non-compressible equivalence relations, i.e., there cannot be a uniform bound M_n on the number of edges containing each vertex.

Theorem 3.1.1 fits into a general framework for classifying countable Borel equivalence relations according to the (first-order) structures one may assign in a Borel way to each equivalence class; see [JKL], [Mks], [CK]. As with most such results, the "underlying" result is that there is a procedure for turning every structure of the kind we are starting with (n-dimensional contractible simplicial complexes) into a structure of the kind we want (n-dimensional contractible simplicial complexes satisfying the additional condition), which is "uniform" enough that it may be performed simultaneously on all equivalence classes in a Borel way. We state this as follows. We say that a simplicial complex is **locally countable** if each vertex is contained in countably many edges (or equivalently countably many simplices).

Theorem 3.1.3. *There is a procedure for turning a locally countable simplicial complex (X, S) into a locally finite simplicial complex (Y, T), such that*

(i) T is homotopy equivalent to S;

(ii) if S is n-dimensional, then T can be chosen to be n-dimensional and with each vertex in at most (or even exactly) M_n edges.

Furthermore, given a countable Borel equivalence relation (X, E) and a structuring S of E by simplicial complexes, this procedure may be performed simultaneously (in a Borel way) on all E-classes, yielding a countable Borel equivalence relation (Y, F) with a structuring T by simplicial complexes and a Borel embedding $f : E \to F$ such that applying the above procedure to the complex $S_{[x]_E}$ on an E-class $[x]_E$ yields the complex $T_{[f(x)]_F}$ on the corresponding F-class $[f(x)]_F$.

The theorem in this form also yields the following (easy) corollary:

Corollary 3.1.4. *Every countable Borel equivalence relation (X, E) embeds into a countable Borel equivalence relation (Y, F) structurable by locally finite contractible simplicial complexes.*

Again, this may be reformulated as

Corollary 3.1.5. *Every compressible countable Borel equivalence relation (X, E) is structurable by locally finite contractible simplicial complexes.*

The proof of Theorem 3.1.3 is based on a classical theorem of Whitehead on CW-complexes [Wh, Theorem 13], which states that every locally countable CW-complex is homotopy equivalent to a locally finite CW-complex of the same dimension. While the statement of this theorem is useless for Theorem 3.1.3 (every contractible complex is homotopy equivalent to a point, but one cannot replace every class of a non-smooth equivalence relation with a point), its proof may be adapted to our setting, with the help of some lemmas from descriptive set theory.

We review some definitions and standard lemmas in Section 3.2, then give the proofs of the above results in Section 3.3; the proofs are structured so that it should be possible to read the combinatorial/homotopy-theoretic argument without the descriptive set theory, and vice-versa. In Section 3.4 we list some other properties of treeable equivalence relations which we do not currently know how to generalize to higher dimensions.

Acknowledgments. We would like to thank Alexander Kechris, Damien Gaboriau, and the anonymous referee for providing some comments on drafts of this paper.

3.2 Preliminaries

We begin by reviewing some notions related to simplicial complexes; see e.g., [Spa].

A **simplicial complex** on a set X is a set S of finite nonempty subsets of X such that $\{x\} \in S$ for all $x \in X$ and every nonempty subset of an element of S is in S. The elements $s \in S$ are called **simplices**. The **dimension** $\dim(s)$ of $s \in S$ is $|s| - 1$; if $\dim(s) = n$, we call s an **n-simplex**. We let $S^{(n)} := \{s \in S \mid \dim(s) = n\}$ be the n-simplices, and call S **n-dimensional** if $S^{(m)} = \varnothing$ for $m > n$. (To avoid confusion, we will sometimes call a simplicial complex with an n-simplex containing all other simplices a **standard n-simplex**.)

A **subcomplex** of (X, S) is a simplicial complex (Y, T) such that $Y \subseteq X$ and $T \subseteq S$. For a simplicial complex (X, S) and a subset $Y \subseteq X$, the **induced subcomplex** on Y is $S|Y := \{s \in S \mid s \subseteq Y\}$. A **simplicial map** $f : S \to T$ between complexes (X, S) and (Y, T) is a map $f : X \to Y$ such that $f(s) \in T$ for all $s \in S$.

The **geometric realization** of a simplicial complex (X, S) is the topological space $|S|$ formed by gluing together standard Euclidean n-simplices Δ^n for each $s \in S^{(n)}$, according to the subset relation. Explicitly, $|S|$ can be defined as the set $\bigcup_{s \in S} |s|_S \subseteq [0, 1]^X$, where $|s|_S := \{(a_x)_{x \in X} \mid \sum_{x \in X} a_x = 1, \forall x \notin s\, (a_x = 0)\}$ is (thought of as) the set of formal convex combinations of elements of X supported on s, equipped with the topology where a subset of $|S|$ is open iff its intersection with each $|s|_S$ is open in the Euclidean topology on $|s|_S$. We say that S is **contractible** if $|S|$ is. Likewise, a simplicial map $f : S \to T$ induces a continuous map $|f| : |S| \to |T|$ in the obvious way; we say that f is a **homotopy equivalence** if $|f|$ is.

We also need the more refined notion of an **ordered simplicial complex**, which is a simplicial complex S on a poset X such that every simplex $s \in S$ is a chain $\{x_0 < \cdots < x_n\}$ in X. The **product** of ordered simplicial complexes (X, S) and

(Y, T) is the complex $(X \times Y, S \times T)$ where $X \times Y$ is the usual product poset and

$$\{(x_0, y_0) \leq \cdots \leq (x_n, y_n)\} \in S \times T \iff \{x_0 \leq \cdots \leq x_n\} \in S \;\&\; \{y_0 \leq \cdots \leq y_n\} \in T.$$

It is standard that $|S \times T|$ is canonically homeomorphic to $|S| \times |T|$ with the CW-product topology (which coincides with the product topology if S, T are locally countable).

In order to prove contractibility/homotopy equivalence, we use the following standard results from homotopy theory.

Lemma 3.2.1. *Let S, T be simplicial complexes which are the unions of subcomplexes $S = \bigcup_{i \in I} S_i$ and $T = \bigcup_{i \in I} T_i$ over the same index set I, and let $f : S \to T$ be a simplicial map such that $f(S_i) \subseteq T_i$ for each i. If for each finite family of indices $i_1, \ldots, i_n \in I$, the restriction $f : S_{i_1} \cap \cdots \cap S_{i_n} \to T_{i_1} \cap \cdots \cap T_{i_n}$ is a homotopy equivalence, then $f : S \to T$ is a homotopy equivalence.*

Proof. See e.g., [Hat, 4K.2]. □

Corollary 3.2.2. *Let S be a simplicial complex which is the union of subcomplexes $U, V \subseteq S$. If the inclusion $U \cap V \to U$ is a homotopy equivalence, then so is the inclusion $V \to S$. In particular, if U, V, and $U \cap V$ are contractible, then so is S.*

Proof. Apply Lemma 3.2.1 to the inclusion from $V = (U \cap V) \cup V$ into $S = U \cup V$. □

Corollary 3.2.3. *Let $S = \bigcup_{i \in I} S_i$ and $T = \bigcup_{i \in I} T_i$ be simplicial complexes which are directed unions of subcomplexes (over the same directed poset), and let $f : S \to T$ be a simplicial map such that $f(S_i) \subseteq T_i$ for each i. If each restriction $f|S_i : S_i \to T_i$ is a homotopy equivalence, then so is f.*

In particular, if S_i is contractible for each i, then (taking $T = T_i =$ a point) S is contractible.

Proof. In the case where I is a well-ordered set, this is immediate from Lemma 3.2.1; the two places below where we use this result both follow from this case. (To deduce the general form of the result, one can appeal to Iwamura's lemma from order theory which reduces an arbitrary directed union to iterated well-ordered unions; see e.g., [Mky].) □

We say that a simplicial map $f : S \to T$ is a **trivial pseudofibration** if for each $t \in T$, the subcomplex $S|f^{-1}(t) \subseteq S$ is contractible.

Corollary 3.2.4. *A trivial pseudofibration is a homotopy equivalence.*

Proof. Apply Lemma 3.2.1 to $S = \bigcup_{t \in T} S|f^{-1}(t)$ and $T = \bigcup_{t \in T} T|t$. □

Finally, we come to the notion of Borel structurability. Let (X, E) be a countable Borel equivalence relation. We say that a simplicial complex S on X is **Borel** if for each n the $(n+1)$-ary relation "$\{x_0, \ldots, x_n\} \in S$" is Borel, or equivalently S is Borel as a subset of the standard Borel space of finite subsets of X. A Borel simplicial complex S on X is a **Borel structuring of E by simplicial complexes** if in addition each simplex $s \in S$ is contained in a single E-class; such an S represents the "Borel assignment" $C \mapsto S_C := S|C$ of the (countable) complex S_C to each E-class $C \in X/E$. More generally, for a class $\mathcal{K}$ of simplicial complexes (e.g., the contractible ones), S is a **structuring of E by complexes in $\mathcal{K}$** if $S_C \in \mathcal{K}$ for each $C \in X/E$; if such a structuring exists, we say that E is **structurable by complexes in $\mathcal{K}$**.

3.3 Proofs

3.3.1 Some lemmas

Let $N = \{\{i\}, \{i, i+1\} \mid i \in \mathbb{N}\}$ denote the ordered simplicial complex on $\mathbb{N} = \{0 < 1 < 2 < \ldots\}$ with an edge between $i, i+1$ for each i, whose geometric realization is a ray.

For a simplicial complex (X, S), a set Y, and a map $f : X \to Y$, define the **image complex**

$$f(S) := \{f(s) \mid s \in S\},$$

which is a simplicial complex on $f(X)$; we write $f(X, S)$ for $(f(X), f(S))$. If (X, S) is an ordered simplicial complex, Y is a poset, and f is monotone, then $(f(X), f(S))$ is also ordered.

Let X be a poset and T be an ordered simplicial complex on $X \times \mathbb{N}^n$, for some $n \in \mathbb{N}$. We define the **telescope** $\mathcal{T}_n(T)$, an ordered simplicial complex on $X \times \mathbb{N}^n$, by induction on n as follows:

$$\begin{aligned}
\mathcal{T}_0(T) &:= T, \\
\mathcal{T}_n(T) &:= (p_1(T) \times N) \cup (\mathcal{T}_{n-1}(p_1(T)) \times \{0\}) \qquad \text{for } n \geq 1,
\end{aligned}$$

where $p_i : X \times \mathbb{N}^n \to X \times \mathbb{N}^{n-i}$ is the projection onto all but the last i factors. Explicitly, we have

$$\mathcal{T}_n(T) = (p_1(T) \times N) \cup (p_2(T) \times N \times \{0\}) \cup \cdots \cup (p_n(T) \times N \times \{0\}^{n-1}) \cup (p_n(T) \times \{0\}^n)$$

(the last term $p_n(T) \times \{0\}^n$ is redundant unless $n = 0$). Here are some simple properties of $\mathcal{T}_n(T)$:

Lemma 3.3.1. *(a)* $T \subseteq \mathcal{T}_n(T)$.

(b) The projection $p_n : \mathcal{T}_n(T) \to p_n(T)$ *is a homotopy equivalence (with homotopy inverse the inclusion* $p_n(T) \cong p_n(T) \times \{0\}^n \subseteq \mathcal{T}_n(T)$*).*

(c) For a subset $Z \subseteq X$*, we have* $\mathcal{T}_n(T)|(Z \times \mathbb{N}^n) = \mathcal{T}_n(T|(Z \times \mathbb{N}^n))$.

(d) If T *is (at most)* k*-dimensional, then* $\mathcal{T}_n(T)$ *is (at most)* $(k+1)$*-dimensional.*

Proof. (a), (c), and (d) are straightforward. For $n \geq 1$, it is easily seen that $|\mathcal{T}_n(T)|$ deformation retracts onto $|\mathcal{T}_{n-1}(p_1(T)) \times \{0\}| \cong |\mathcal{T}_{n-1}(p_1(T))|$; a simple induction then yields (b). □

We need one more (straightforward) lemma:

Lemma 3.3.2. *A trivial pseudofibration* $f : S \to T$ *is surjective on simplices.*

Proof. Let $t \in T$. Put $S' := \{s \in S \mid f(s) \subsetneq t\} = S|f^{-1}(t) \setminus \{s \in S \mid f(s) = t\}$. Since f is a trivial pseudofibration, for every $t' \subsetneq t$, $S'|f^{-1}(t') = S|f^{-1}(t')$ is contractible; thus $f : S' \to T|t \setminus \{t\}$ is a homotopy equivalence. But $T|t \setminus \{t\}$ is the boundary of the simplex t, hence not contractible; thus for $S|f^{-1}(t)$ to be contractible, there must be $s \in S$ with $f(s) = t$. □

3.3.2 The main construction

We now give the main construction in the proof of Theorem 3.1.3. Let (X, S) be a locally countable simplicial complex, which we may assume to be ordered by taking any linear order on X. By local countability, for each n we may find a function $c_n : S^{(n)} \to \mathbb{N}$ which colors the intersection graph on the n-simplices $S^{(n)}$, which means that for $s, t \in S^{(n)}$ with $s \neq t$ and $s \cap t \neq \varnothing$ we have $c_n(s) \neq c_n(t)$. The idea is that for each n, we will multiply the complex by the ray N and then attach each n-simplex $s \in S^{(n)}$ at position $c_n(s)$ along the ray, so that distinct simplices have non-overlapping boundaries.

Let $S_n := \bigcup_{m\le n} S^{(m)} = \{s \in S \mid \dim(s) \le n\}$, the n-skeleton of S. We will inductively define ordered simplicial complexes T_n on $X \times \mathbb{N}^n$ and for $n \ge 1$, T'_n on $X \times \mathbb{N}^n$ such that

$$T_n \subseteq S_n \times N^n, \qquad T'_{n+1} \subseteq S_n \times N^{n+1}, \qquad T_n \times N \subseteq T'_{n+1} \subseteq T_{n+1},$$

fitting into the following commutative diagram of monotone simplicial maps:

$$\begin{tikzcd}
 & & & & T_2 \times N \arrow[r, hook] \arrow[d, two heads, "p_1"', "\simeq"] & T'_3 \arrow[r, hook] \arrow[dddl, two heads, "p_3", "\simeq"'] & \cdots \\
 & & T_1 \times N \arrow[r, hook] \arrow[d, two heads, "p_1"', "\simeq"] & T'_2 \arrow[r, hook] \arrow[ddl, two heads, "p_2", "\simeq"'] & T_2 \arrow[dd, two heads, "p_2"', "\simeq"] & & \\
T_0 \times N \arrow[r, hook] \arrow[d, two heads, "p_1"', "\simeq"] & T'_1 \arrow[r, hook] \arrow[dl, two heads, "p_1", "\simeq"'] & T_1 \arrow[d, two heads, "p_1"', "\simeq"] & & & & \\
T_0 = S_0 \arrow[rr, hook] & & S_1 \arrow[rr, hook] & & S_2 \arrow[rr, hook] & & \cdots
\end{tikzcd} \tag{$*$}$$

The horizontal maps are the inclusions, while the vertical/diagonal maps are the projections $p_i : X\times\mathbb{N}^n \to X\times\mathbb{N}^{n-i}$ onto all but the last i factors as before; furthermore each vertical/diagonal map will be a trivial pseudofibration between the respective complexes.

Start with $T_0 := S_0$. Given T_n such that $p_n : T_n \to S_n$ is a trivial pseudofibration, put

$$T'_{n+1} := (T_n \times N) \cup \textstyle\bigcup_{s\in S^{(n+1)}} (\mathcal{T}_n(T_n|(s \times \mathbb{N}^n)) \times \{c_{n+1}(s)\}).$$

Clearly this is an ordered simplicial complex on $X \times \mathbb{N}^{n+1}$.

Claim. *$p_{n+1} : (X \times \mathbb{N}^{n+1}, T'_{n+1}) \to (X, S_n)$ is a trivial pseudofibration.*

Proof. Let $t \in S_n$; we must check that $T'_{n+1}|p_{n+1}^{-1}(t) = T'_{n+1}|(t \times \mathbb{N}^{n+1})$ is contractible. We have

$$\begin{aligned}
T'_{n+1}|(t \times \mathbb{N}^{n+1}) &= (T_n|(t \times \mathbb{N}^n) \times N) \cup \textstyle\bigcup_{s\in S^{(n+1)}} (\mathcal{T}_n(T_n|((s \cap t) \times \mathbb{N}^n)) \times \{c_{n+1}(s)\}) \\
&= \underbrace{(T_n|p_n^{-1}(t) \times N)}_{A} \cup \textstyle\bigcup_{s\in S^{(n+1)}} \underbrace{(\mathcal{T}_n(T_n|p_n^{-1}(s \cap t)) \times \{c_{n+1}(s)\})}_{B_s}
\end{aligned}$$

(using Lemma 3.3.1(c)); let A, B_s be as shown. The subcomplex A is contractible since $p_n : T_n \to S_n$ is a trivial pseudofibration by the induction hypothesis whence $T_n|p_n^{-1}(t)$ is contractible. For each $s \in S^{(n+1)}$ such that $s \cap t \neq \varnothing$ (otherwise B_s is empty), the subcomplex B_s is contractible since the telescope $\mathcal{T}_n(T_n|p_n^{-1}(s \cap t))$ is homotopy equivalent (by Lemma 3.3.1(b)) to the projection $p_n(T_n|p_n^{-1}(s \cap t)) = p_n(T_n)|(s \cap t) = S_n|(s \cap t)$ which is a standard simplex; and also $A \cap B_s$ is contractible since

$$\begin{aligned} A \cap B_s &= (T_n|(t \times \mathbb{N}^n) \cap \mathcal{T}_n(T_n|((s \cap t) \times \mathbb{N}^n))) \times \{c_{n+1}(s)\} \\ &= (T_n|((s \cap t) \times \mathbb{N}^n) \cap \mathcal{T}_n(T_n|((s \cap t) \times \mathbb{N}^n))) \times \{c_{n+1}(s)\} \\ &= T_n|((s \cap t) \times \mathbb{N}^n) \times \{c_{n+1}(s)\} \\ &= T_n|p_n^{-1}(s \cap t) \times \{c_{n+1}(s)\} \end{aligned}$$

(the second equality since the telescope is a complex on $(s \cap t) \times \mathbb{N}^n$, the third equality by Lemma 3.3.1(a)), which is contractible because again p_n is a trivial pseudofibration. For two distinct $s, s' \in S^{(n+1)}$, we have $B_s \cap B_{s'} = \emptyset$: either $c_{n+1}(s) \neq c_{n+1}(s')$ in which case clearly $B_s \cap B_{s'} = \emptyset$, or $c_{n+1}(s) = c_{n+1}(s')$ whence by the coloring property of c_{n+1} we have $s \cap s' = \emptyset$. Now by repeated use of Corollary 3.2.2, we get that $A \cup B_{s_1} \cup \cdots \cup B_{s_m}$ is contractible for every finite collection of $s_1, \ldots, s_m \in S^{(n+1)}$, whence by Corollary 3.2.3, $T'_{n+1}|(t \times \mathbb{N}^{n+1})$ is contractible. □

Now put

$$T_{n+1} := T'_{n+1} \cup \{s \times \{0\}^n \times \{c_{n+1}(s)\} \mid s \in S^{(n+1)}\}.$$

Claim. *T_{n+1} is an ordered simplicial complex on $X \times \mathbb{N}^{n+1}$.*

Proof. The only thing that needs to be checked is that for each $s \in S^{(n+1)}$, a nonempty subset $s' \times \{0\}^n \times \{c_{n+1}(s)\}$ of $s \times \{0\}^n \times \{c_{n+1}(s)\}$ is still in T_{n+1}. We may assume $s' \subsetneq s$. Then $s' \in S_n$, so since $p_n : T_n \to S_n$ is a trivial pseudofibration, hence surjective on simplices, we have $s' \in p_n(T_n|(s \times \mathbb{N}^n))$, whence $s' \times \{0\}^n \times \{c_{n+1}(s)\} \in p_n(T_n|(s \times \mathbb{N}^n)) \times \{0\}^n \times \{c_{n+1}(s)\} \subseteq \mathcal{T}_n(T_n|(s \times \mathbb{N}^n)) \times \{c_{n+1}(s)\} \subseteq T'_{n+1} \subseteq T_{n+1}$. □

Claim. *$p_{n+1} : (X \times \mathbb{N}^{n+1}, T_{n+1}) \to (X, S_{n+1})$ is a trivial pseudofibration.*

Proof. Let $s \in S_{n+1}$; we must check that $T_{n+1}|p_{n+1}^{-1}(s)$ is contractible. If $s \in S_n$ then clearly $T_{n+1}|p_{n+1}^{-1}(s) = T'_{n+1}|p_{n+1}^{-1}(s)$ so this follows from the previous claim that $p_{n+1} : T'_{n+1} \to S_n$ is a trivial pseudofibration. So we may assume that $s \in S^{(n+1)}$, in which case

$$T_{n+1}|p_{n+1}^{-1}(s) = T'_{n+1}|p_{n+1}^{-1}(s) \cup \{s \times \{0\}^n \times \{c_{n+1}(s)\}\}.$$

Since $p_{n+1} : T'_{n+1} \to S_n$ is a trivial pseudofibration, so is the restriction $p_{n+1} : T'_{n+1}|p_{n+1}^{-1}(s) \to S_n|s$; but this restriction has one-sided inverse the inclusion $S_n|s \cong S_n|s \times \{0\}^n \times \{c_{n+1}(s)\} \subseteq \mathcal{T}_n(T_n|(s \times \mathbb{N}^n)) \times \{c_{n+1}(s)\} \subseteq T'_{n+1}|p_{n+1}^{-1}(s)$, which is therefore a homotopy equivalence. Now applying Corollary 3.2.2 to

$$T_{n+1}|p_{n+1}^{-1}(s) = T'_{n+1}|p_{n+1}^{-1}(s) \cup (S|s \times \{0\}^n \times \{c_{n+1}(s)\}),$$

where the two subcomplexes on the right-hand side have intersection $S_n|s \times \{0\}^n \times \{c_{n+1}(s)\}$, yields that the inclusion $S|s \times \{0\}^n \times \{c_{n+1}(s)\} \subseteq T_{n+1}|p_{n+1}^{-1}(s)$ is a homotopy equivalence; but $S|s$ is a standard simplex, hence contractible, whence $T_{n+1}|p_{n+1}^{-1}(s)$ is contractible. □

This completes the definition of the complexes T_n, T'_n and the verification that $p_n : T_n \to S_n$ is a homotopy equivalence for each n. Note that from the definition and Lemma 3.3.1(d), it is clear that each T_n is n-dimensional.

3.3.3 The constant bound

We next bound the number of edges containing a point in T_n. To do so, we will define for each $n \geq 1$ a constant K_n such that for each $y \in X \times \mathbb{N}^n$ there are at most K_n distinct $y' \in X \times \mathbb{N}^n$ with $y \leq y'$ and $\{y, y'\} \in T_n$, and also the same holds with $y' \leq y$.

For $n = 1$, we have $T'_1 = T_0 \times N = S_0 \times N$, while $T_1 = T'_1 \cup \{s \times \{c_1(s)\} \mid s \in S^{(1)}\}$. Thus

$$K_1 := 3$$

works: for $t = \{y \leq y'\} \in T_1$, either $t \in T'_1$, in which case we have $y = (x, i)$ and $y' \in \{(x, i), (x, i+1)\}$ for some $(x, i) \in X \times \mathbb{N}$, or $t = s \times \{c_1(s)\}$ for some $s \in S^{(1)}$, in which case $y = (x, c_1(s))$ and $y' = (x', c_1(s))$ for some $s = \{x < x'\} \in S^{(1)}$, which is uniquely determined by y by the coloring property of c_1; and similarly for $y' \leq y$.

Now suppose for $n \geq 1$ that we are given K_n; we find K_{n+1} by a similar argument. Let $t = \{y \leq y'\} \in T_{n+1}$. Since $n + 1 \geq 2$, T_{n+1} adds no 0- or 1-simplices to T'_{n+1}, so $t \in T'_{n+1}$. If $t \in T_n \times N$, then we have $y = (z, i)$ and $y' = (z', i')$ for some $\{z \leq z'\} \in T_n$ and $\{i \leq i'\} \in N$, i.e., $i' \in \{i, i+1\}$; there are thus $\leq 2K_n$ choices for y' given y in this case. Otherwise, we have $t \in \mathcal{T}_n(T_n|(s \times \mathbb{N}^n)) \times \{c_{n+1}(s)\} \subseteq S|s \times N^n \times \{c_{n+1}(s)\}$ for some $s \in S^{(n+1)}$, whence $y = (x, i_1, \dots, i_n, c_{n+1}(s))$ and $y' = (x', i'_1, \dots, i'_n, c_{n+1}(s))$ where $x, x' \in s$ and each $i'_j \in \{i_j, i_j + 1\}$; by the coloring property of $c_{n+1}(s)$, s is uniquely determined by y, so there are at most $|s| = n + 2$ choices for x' and so at most $(n + 2)2^n$ choices for y' given y. In total, there are thus at most

$$K_{n+1} := 2K_n + (n + 2)2^n$$

choices for $y' \geq y$; similarly for $y' \leq y$.

Solving this recurrence yields

$$K_n = 2^{n-2}(n^2 + 3n + 2).$$

So, for each $n \geq 1$ and $y \in X \times \mathbb{N}^n$, there are at most $2(K_n - 1)$ distinct edges $\{y < y'\}$ or $\{y' < y\}$ in T_n; that is, there are at most

$$M_n := 2(K_n - 1) = 2^{n-1}(n^2 + 3n + 2) - 2$$

edges in T_n containing y. When $S = S_n$ is n-dimensional, truncating the above inductive construction at T_n and taking $T := T_n$ proves the combinatorial part of Theorem 3.1.3 (with the weaker condition "at most M_n" in (ii)) in this case.

3.3.4 Growing edges

Still in the n-dimensional case, in order to modify T_n so that each vertex is contained in exactly M_n edges, we use the following simple construction. Put $T_{n,0} := T_n$. Given $T_{n,k}$, let $T_{n,k+1}$ be $T_{n,k}$ together with, for each vertex y of T_n with fewer than M_n edges, a new vertex y' and an edge $\{y, y'\}$. Then clearly

$$T_n^* := \bigcup_{k \in \mathbb{N}} T_{n,k}$$

is still n-dimensional and has each vertex contained in exactly M_n edges. Also, clearly $T_{n,k+1}$ deformation retracts onto $T_{n,k}$; thus (by Corollary 3.2.3) the inclusion $T_n = T_{n,0} \subseteq T_n^*$ is a homotopy equivalence. So we may replace T_n with T_n^* to get the stronger form of Theorem 3.1.3(ii).

3.3.5 The infinite-dimensional case

Next we handle the case where S is infinite-dimensional. Let $i_n : (X \times \mathbb{N}^n, T_n) \hookrightarrow (X \times \mathbb{N}^{n+1}, T_{n+1})$ be the composite

$$i_n : T_n \cong T_n \times \{0\} \subseteq T_n \times N \subseteq T'_{n+1} \subseteq T_{n+1}.$$

From the above diagram $(*)$, we get a commutative diagram

$$\begin{tikzcd} T_0 \arrow[r, hook, "i_0"] \arrow[d, two heads, "p_0"', "\simeq"] & T_1 \arrow[r, hook, "i_1"] \arrow[d, two heads, "p_1"', "\simeq"] & T_2 \arrow[r, hook, "i_2"] \arrow[d, two heads, "p_2"', "\simeq"] & \cdots \\ S_0 \arrow[r, hook] & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & \cdots \end{tikzcd} \tag{†}$$

We would like to let T be the direct limit of the top row of this diagram, but that might not be locally finite. Instead, we take the mapping telescope of the top row, which can be defined explicitly as follows.

Let $\mathbb{N}^\infty$ be the direct limit of $\mathbb{N} \cong \mathbb{N} \times \{0\} \subseteq \mathbb{N}^2 \cong \mathbb{N}^2 \times \{0\} \subseteq \mathbb{N}^3 \subseteq \cdots$; explicitly, $\mathbb{N}^\infty$ can be taken as the subset of $\mathbb{N}^\mathbb{N}$ consisting of the eventually zero sequences. Then $X \times \mathbb{N}^\infty$ is the direct limit of the sequence $X \times \mathbb{N}^0 \xrightarrow{i_0} X \times \mathbb{N}^1 \xrightarrow{i_1} \cdots$, with injections

$$i^n : X \times \mathbb{N}^n \cong X \times \mathbb{N}^n \times \{0\}^\infty \subseteq X \times \mathbb{N}^\infty;$$

and so the direct limit of the top row of (†) can be taken explicitly as the ordered simplicial complex $\bigcup_{n\in\mathbb{N}} i^n(T_n)$ on $X \times \mathbb{N}^\infty$.

The **mapping telescope** of the top row of (†) is the complex (Y, T) where

$$\begin{aligned} Y &:= \textstyle\bigcup_{n\in\mathbb{N}}(X \times \mathbb{N}^n \times \{0\}^\infty \times \{n, n+1\}) \subseteq X \times \mathbb{N}^\infty \times \mathbb{N}, \\ T &:= \textstyle\bigcup_{n\in\mathbb{N}}(i^n(T_n) \times N|\{n, n+1\}). \end{aligned}$$

For each n, let

$$\widetilde{T}_n := \textstyle\bigcup_{m\le n}(i^m(T_m) \times N|\{m, m+1\}).$$

It is easy to see that the projection $p_1 : X \times \mathbb{N}^\infty \times \mathbb{N} \to X \times \mathbb{N}^\infty$ restricts to simplicial maps $\widetilde{T}_n \to i^n(T_n)$ for each n, yielding a commutative diagram

$$\begin{tikzcd} \widetilde{T}_0 \arrow[r, hook] \arrow[d, "p_1"', "\simeq"] & \widetilde{T}_1 \arrow[r, hook] \arrow[d, "p_1"', "\simeq"] & \widetilde{T}_2 \arrow[r, hook] \arrow[d, "p_1"', "\simeq"] & \cdots \\ i^0(T_0) \arrow[r, hook] & i^1(T_1) \arrow[r, hook] & i^2(T_2) \arrow[r, hook] & \cdots \end{tikzcd} \tag{‡}$$

in which the horizontal maps are inclusions and the vertical maps are homotopy equivalences by the usual argument: the (geometric realization of the) first cylinder $i^0(T_0) \times N|\{0, 1\}$ in $\widetilde{T}_n$ deformation retracts onto its base $i^0(T_0) \times \{1\}$, which is contained in the second cylinder $i^1(T_1) \times N|\{1, 2\}$, which deformation retracts onto its base $i^1(T_1) \times \{2\}$, etc. Since, as noted above, the bottom row of (‡) may be identified with the top row of (†), combining the two diagrams and applying Corollary 3.2.3 yields that $T = \bigcup_n \widetilde{T}_n$ is homotopy equivalent to $S = \bigcup_n S_n$ (via the restriction of the projection $X \times \mathbb{N}^\infty \times \mathbb{N} \to X$).

Since, clearly, each T_n being locally finite implies that T is locally finite, this proves the combinatorial part of Theorem 3.1.3 in the infinite-dimensional case.

3.3.6 The Borel case

Finally, suppose we start with a Borel structuring S of a countable Borel equivalence relation (X, E) by simplicial complexes. Recall that this means S is a simplicial complex on X with simplices contained in E-classes and such that S is Borel in the standard Borel space of finite subsets of X. We may then simply apply the above construction to the locally countable simplicial complex (X, S), while observing that each step is Borel. To do so, we first pick a Borel linear order on X to turn (X, S) into an ordered simplicial complex, and then pick the coloring functions $c_n : S^{(n)} \to \mathbb{N}$ to be Borel (in fact restrictions of a single $c : S \to \mathbb{N}$) using the following standard lemma:

Lemma 3.3.3 (Kechris-Miller [KM, 7.3])**.** *Let (X, E) be a countable Borel equivalence relation, and let $[E]^{<\infty}$ be the standard Borel space of finite subsets of X which are contained in some E-class. Then there is a Borel $\mathbb{N}$-coloring of the intersection graph on $[E]^{<\infty}$, i.e., a Borel map $c : [E]^{<\infty} \to \mathbb{N}$ such that if $A, B \in [E]^{<\infty}$ with $A \neq B$ and $A \cap B \neq \varnothing$ then $c(A) \neq c(B)$.*

It is now straightforward to check that the definitions of T_n, T'_n are Borel; in the definition of T'_{n+1}, note that the union over $s \in S^{(n+1)}$ is disjoint, by the coloring property of c_{n+1}. In the n-dimensional case, we end up with an ordered Borel simplicial complex $(X \times \mathbb{N}^n, T_n)$ such that the projection $p_n : X \times \mathbb{N}^n \to X$ is a homotopy equivalence $T_n \to S_n = S$. Defining the countable Borel equivalence relation F on $Y := X \times \mathbb{N}^n$ by

$$(x, i_1, \dots, i_n)\ F\ (x', i'_1, \dots, i'_n) \iff x\ E\ x',$$

we get that $T := T_n$ is a Borel structuring of (Y, F); and we have a Borel embedding $f : (X, E) \to (Y, F)$ given by $f(x) := (x, 0, \dots, 0)$ such that $S|[x]_E$ is homotopy equivalent to $T|[f(x)]_F$ (via the map $p_n|([x]_E \times \mathbb{N}^n) = p_n|[f(x)]_F : T|[f(x)]_F \to S|[x]_E$) for each $x \in X$.

For the stronger condition that each vertex is contained in exactly M_n edges, it is straightforward that the definition of T^*_n above can be taken to be a Borel simplicial complex on a standard Borel space $Y^* \supseteq Y$; letting $F^* \supseteq F$ be the obvious equivalence relation on Y^* (so that each newly added edge in T^*_n lies in one F^*-class), T^*_n is a Borel structuring of (Y^*, F^*) such that the composite $(X, E) \xrightarrow{f} (Y, F) \subseteq (Y^*, F^*)$ is a homotopy equivalence on each class. So we may replace (Y, F, T_n) by (Y^*, F^*, T^*_n).

Similarly, in the infinite-dimensional case, it is straightforward that the definition of the mapping telescope T on $Y \subseteq X \times \mathbb{N}^\infty \times \mathbb{N}$ is Borel; so the same definitions of

F, f as in the finite-dimensional case work (note that $(x, 0, \ldots, 0) \in Y$ for all $x \in X$). This completes the proof of Theorem 3.1.3, which implies Theorem 3.1.1.

To prove Corollary 3.1.2, apply Theorem 3.1.1 to get (Y, F) with structuring T and an embedding $f : (X, E) \to (Y, F)$; since E is compressible, f may be modified so that its image is F-invariant (see [DJK, 2.3]), whence we get the desired structuring of E by restricting T.

To prove Corollary 3.1.4, let S be the trivial structuring of E given by $\{x_0, \ldots, x_n\} \in S \iff x_0 \ E \cdots E \ x_n$; this is obviously contractible on each E-class, so by Theorem 3.1.3 E Borel embeds into some F structurable by locally finite contractible complexes. As before, this implies Corollary 3.1.5.

3.3.7 Some remarks

In the dimension $n = 1$ case, the construction of T_1 above can be seen as a slight variant of the proof of Jackson-Kechris-Louveau [JKL, 3.10]. Thus the general case of our construction can be seen as a generalization of their proof to higher dimensions.

As mentioned in the Introduction, our construction is based on the proof of Whitehead [Wh, Theorem 13] that every countable CW-complex is homotopy equivalent to a locally finite complex of the same dimension. That proof uses the same idea of "spreading out" cells along a ray to make their boundaries disjoint, but uses more abstract tools from homotopy theory in place of our explicit "telescope" construction $\mathcal{T}_n$. While it should be possible to give a more direct combinatorial transcription of Whitehead's proof, using (for example) simplicial sets, it does not seem that such an approach would yield a uniform bound M_n on the number of edges containing a vertex in the n-dimensional case.

3.4 Problems

There are several other nice properties of treeable countable Borel equivalence relations, for which we do not know if they generalize to higher dimensions. Each of the following is known to be true in the case $n = 1$; see [JKL, 3.3, 3.12, 3.17].

Problem 3.4.1. Let E, F be countable Borel equivalence relations such that E Borel embeds into F. If F is structurable by n-dimensional contractible simplicial complexes, then must E be also?

Problem 3.4.2. Let E be a countable Borel equivalence relation. If E is structurable by n-dimensional contractible simplicial complexes, then is E necessarily structurable by n-dimensional locally finite contractible simplicial complexes? (As noted in the Introduction, there cannot be a uniform bound on the number of edges containing each vertex.)

Problem 3.4.3. Is there a single countably infinite n-dimensional contractible simplicial complex S_n, such that every countable Borel equivalence relation E structurable by n-dimensional contractible simplicial complexes Borel embeds into an F structurable by isomorphic copies of S_n?

Problem 3.4.4. Is there a countable group Γ_n with an n-dimensional Eilenberg-MacLane complex $K(\Gamma_n, 1)$, such that every countable Borel equivalence relation E structurable by n-dimensional contractible simplicial complexes Borel embeds into the orbit equivalence relation of a free Borel action of Γ_n?

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Chapter 4

AMALGAMABLE DIAGRAM SHAPES

Ruiyuan Chen

4.1 Introduction

This paper concerns a category-theoretic question which arose in a model-theoretic context. In model theory, specifically in the construction of Fraïssé limits (see [H, §7.1]), one considers a **Fraïssé class**, i.e., a class $\mathcal{K}$ of structures (in some first-order language) with the following properties:

- the **joint embedding property (JEP)**: ($\mathcal{K}$ is nonempty and) for every two structures $A, B \in \mathcal{K}$, there are embeddings $f : A \to X$ and $g : B \to X$ into some $X \in \mathcal{K}$:

$$\begin{tikzcd} & X & \\ A \arrow[ur, dashed, hook, "f"] & & B \arrow[ul, dashed, hook, "g"'] \end{tikzcd};$$

- the **amalgamation property (AP)**: every diagram of embeddings

$$\begin{tikzcd} B & & C \\ & A \arrow[ul, hook, "f"] \arrow[ur, hook, "g"'] & \end{tikzcd}$$

  between structures $A, B, C \in \mathcal{K}$ can be completed into a commutative diagram

$$\begin{tikzcd} & X & \\ B \arrow[ur, dashed, hook, "h"] & & C \arrow[ul, dashed, hook, "k"'] \\ & A \arrow[ul, hook, "f"] \arrow[ur, hook, "g"'] & \end{tikzcd}$$

  for some structure $X \in \mathcal{K}$ and embeddings $h : B \to X$ and $k : C \to X$.

Common examples of classes $\mathcal{K}$ with these properties include: finitely generated groups, posets, nontrivial Boolean algebras, and finite fields of fixed characteristic p.

From the AP (and optionally the JEP), one has various "generalized amalgamation properties", whereby more complicated diagrams (of embeddings) can be completed into commutative diagrams (of embeddings), e.g., the following diagram by two uses of AP:

$$\begin{tikzcd}
& & Y \\
& X \arrow[ur, dashed] \\
A \arrow[ur, dashed] & & C \arrow[ul, dashed] & & E \arrow[uull, dashed] \\
& B \arrow[ul] \arrow[ur] & & D \arrow[ul] \arrow[ur]
\end{tikzcd} .$$

However, the following diagram cannot be completed using just the AP (and the JEP):

$$\begin{tikzcd}
A & B \\
C \arrow[u, "f"] \arrow[ur, "h"] & D \arrow[u, "k"'] \arrow[ul, "g"']
\end{tikzcd} . \tag{4.1}$$

For example, take $\mathcal{K}$ = the class of finite sets, $A = C = D = 1$, $B = 2$, and $h \neq k$. This leads to the following

Question. Can we characterize the shapes of the diagrams which can always be completed using the AP, i.e., the "generalized amalgamation properties" which are implied by the AP? If so, is such a characterization decidable?

This question concerns only abstract properties of diagrams and arrows, hence is naturally phrased in the language of category theory. Let $\mathbf{C}$ be a category. Recall that a **cocone** over a diagram in $\mathbf{C}$ consists of an object $X \in \mathbf{C}$, together with morphisms $f_A : A \to X$ in $\mathbf{C}$ for each object A in the diagram, such that the morphisms f_A commute with the morphisms in the diagram; this is formally what it means to "complete" a diagram. Recall also that a **colimit** of a diagram is a universal cocone, i.e., one which admits a unique morphism to any other cocone. (See Section 4.2 for the precise definitions.)

We say that $\mathbf{C}$ has the AP if every pushout diagram (i.e., diagram of shape $\bullet \leftarrow \bullet \rightarrow \bullet$) in $\mathbf{C}$ has a cocone, and that $\mathbf{C}$ has the JEP[1] if every diagram in $\mathbf{C}$ consisting of finitely many objects (without any arrows) has a cocone. When $\mathbf{C}$ is the category of structures

[1] We borrow this terminology from model theory, even when not assuming that morphisms are "embeddings" in any sense; in particular, we do not assume that every morphism in $\mathbf{C}$ is monic (although see Section 4.5).

in a class $\mathcal{K}$ and embeddings between them, this recovers the model-theoretic notions defined above. Category-theoretic questions arising in Fraïssé theory have been considered previously in the literature; see e.g., [K], [C].

The possibility of answering the above question in the generality of an arbitrary category is suggested by an analogous result of Paré [P] (see Theorem 4.2.3 below), which characterizes the diagram shapes over which a colimit may be built by pushouts (i.e., colimits of pushout diagrams). There, the crucial condition is that the diagram shape must be **simply-connected** (see Definition 4.2.1); failure to be simply-connected is witnessed by the **fundamental groupoid** of the diagram shape, whose morphisms are "paths up to homotopy". For example, the fundamental groupoid of the shape of (4.1) is equivalent to $\mathbb{Z}$, with generator given by the "loop" $A \leftarrow D \rightarrow B \leftarrow C \rightarrow A$.

However, simply-connectedness of a diagram's shape does not guarantee that a cocone over it may be built using only the AP (see Example 4.2.4 below). Intuitively, the discrepancy with Paré's result is because the universal property of a pushout allows it to be used in more ways to build further cocones. Simply-connectedness nonetheless plays a role in the following characterization, which is the main result of this paper:

Theorem 4.1.1. *Let $\mathbf{I}$ be a finitely generated category. The following are equivalent:*

(i) Every $\mathbf{I}$-shaped diagram in a category with the AP and the JEP has a cocone.

(ii) Every $\mathbf{I}$-shaped diagram in the category of sets and injections has a cocone. (When $\mathbf{I}$ is finite, it suffices to consider finite sets.)

(iii) $\mathbf{I}$ is upward-simply-connected (see Definition 4.3.2).

When $\mathbf{I}$ is a finite poset, these are further equivalent to:

(iv) Every upward-closed subset of $\mathbf{I}$ is simply-connected.

(v) $\mathbf{I}$ is forest-like (see Definition 4.4.3; this means $\mathbf{I}$ is built via some simple inductive rules).

Similarly, every $\mathbf{I}$-shaped diagram in a category with the AP has a cocone, iff $\mathbf{I}$ is connected and any/all of (ii), (iii) (also (iv), (v) if $\mathbf{I}$ is a poset) hold.

A corollary of our proof yields a simple decision procedure for these conditions (for finite $\mathbf{I}$). This is somewhat surprising, because Paré's result (Theorem 4.2.3) implies that the analogous question of whether every $\mathbf{I}$-shaped diagram has a colimit in a category with pushouts is *un*decidable.

This paper is organized as follows. In Section 4.2, we fix notations and review some categorical concepts. In Section 4.3, we introduce an invariant $\mathcal{L}(\mathbf{I})$, similar to the fundamental groupoid, and use it to prove the equivalence of (ii) and (iii) in Theorem 4.1.1 for arbitrary small (not necessarily finitely generated) $\mathbf{I}$. In Section 4.4, we analyze upward-simply-connected posets in more detail, deriving the conditions (iv) and (v) equivalent to (iii) and proving that they imply (i) when $\mathbf{I}$ is a finite poset. In Section 4.5, we remove this restriction on $\mathbf{I}$ and complete the proof. Finally, in Section 4.6, we discuss decidability of the equivalent conditions in Theorem 4.1.1 and of the analogous conditions in Paré's result.

Acknowledgments. We would like to thank Alexander Kechris for providing some feedback on an earlier draft of this paper.

4.2 Preliminaries

We begin by fixing notations and terminology for some basic categorical notions; see [ML].

For a category $\mathbf{C}$ and objects $X, Y \in \mathbf{C}$, we denote a morphism between them by $f : X \underset{\mathbf{C}}{\longrightarrow} Y$. We use the terms **morphism** and **arrow** interchangeably.

We use **Set** to denote the category of sets and functions, **Inj** to denote the category of sets and injections, and **PInj** to denote the category of sets and partial injections. We use **Cat**, **Gpd** to denote the categories of small categories, resp., small groupoids.[2]

We regard a preordered set $(\mathbf{I}, \leq)$ as a category where there is a unique arrow $I \underset{\mathbf{I}}{\longrightarrow} J$ iff $I \leq J$.

We say that a category $\mathbf{C}$ is **monic** if every morphism $f : X \underset{\mathbf{C}}{\longrightarrow} Y$ in it is monic (i.e., if $f \circ g = f \circ h$ then $g = h$, for all $g, h : Z \underset{\mathbf{C}}{\longrightarrow} X$). Similarly, $\mathbf{C}$ is **idempotent** if every endomorphism $f : X \underset{\mathbf{C}}{\longrightarrow} X$ is idempotent (i.e., $f \circ f = f$).

A category $\mathbf{I}$ is **finitely generated** if there are finitely many arrows in $\mathbf{I}$ whose closure under composition is all arrows in $\mathbf{I}$. Note that such $\mathbf{I}$ necessarily has finitely many objects, and that a preorder is finitely generated iff it is finite.

[2] We will generally ignore size issues; it is straightforward to check that except where smallness is explicitly assumed, the following definitions and results work equally well for large categories.

For a category $\mathbf{C}$ and a small category $\mathbf{I}$, a **diagram** of shape $\mathbf{I}$ in $\mathbf{C}$ is simply a functor $F : \mathbf{I} \to \mathbf{C}$. A **cocone** $(X, \overline{f})$ over a diagram F consists of an object $X \in \mathbf{C}$ together with a family of morphisms $\overline{f} = (f_I : F(I) \underset{\mathbf{C}}{\to} X)_{I \in \mathbf{I}}$, such that for each $i : I \underset{\mathbf{I}}{\to} J$, we have $f_I = f_J \circ F(i)$. A **morphism between cocones** $(X, \overline{f})$ and $(Y, \overline{g})$ over the diagram F is a morphism $h : X \underset{\mathbf{C}}{\to} Y$ such that for each object $I \in \mathbf{I}$, we have $h \circ f_I = g_I$. A cocone $(X, \overline{f})$ over F is a **colimit** of F if it is initial in the category of cocones over F, i.e., for any other cocone $(Y, \overline{g})$ there is a unique cocone morphism $h : (X, \overline{f}) \to (Y, \overline{g})$; in this case we write $X = \varinjlim F$, and usually use a letter like ι_I for the cocone maps f_I.

$$\begin{tikzcd} F(I) \arrow[r, "F(i)"] \arrow[dr, "f_I"'] \arrow[drr, "g_I"] & F(J) \arrow[d, "f_J"] \arrow[dr, "g_J"] & \\ & X = \varinjlim F \arrow[r, dashed, "h"'] & Y \end{tikzcd}$$

As mentioned above, a category $\mathbf{C}$ has the **amalgamation property (AP)** if every **pushout diagram** (i.e., diagram of shape $\bullet \leftarrow \bullet \rightarrow \bullet$) in $\mathbf{C}$ has a cocone (colimits of such diagrams are called **pushouts**), and $\mathbf{C}$ has the **joint embedding property (JEP)** (regardless of whether $\mathbf{C}$ is monic) if every finite coproduct diagram (i.e., diagram of finite discrete shape) in $\mathbf{C}$ has a cocone. (So the empty category does not have the JEP.)

A category $\mathbf{C}$ is **connected** if it has exactly one **connected component**, where $X, Y \in \mathbf{C}$ are in the same connected component if they are joined by a zigzag of morphisms

$$\begin{tikzcd} & X_1 & & X_3 & & Y \\ X \arrow[ur] & & X_2 \arrow[ul] \arrow[ur] & & \cdots & \end{tikzcd}$$

(So the empty category is not connected.) We use $\pi_0(\mathbf{C})$ to denote the set (or class, if $\mathbf{C}$ is large) of connected components of $\mathbf{C}$. Note that in the presence of the AP, connectedness is equivalent to the JEP, since the AP may be used to turn "troughs" into "peaks" in a zigzag.

4.2.1 Simply-connected categories

Definition 4.2.1. The **fundamental groupoid** of a category $\mathbf{I}$, denoted $\pi_1(\mathbf{I})$, is the groupoid freely generated by $\mathbf{I}$ (as a category). Thus $\pi_1(\mathbf{I})$ has the same objects as $\mathbf{I}$, while its morphisms are words made up of the morphisms in **I** together with their formal inverses, modulo the relations which hold in **I** (and the relations which say that the formal inverses are inverses).

We say that **I** is **simply-connected** if $\pi_1(\mathbf{I})$ is an equivalence relation, i.e., has at most one morphism between any two objects.

Remark 4.2.2. There is also a topological definition: $\pi_1(\mathbf{I})$ is the same as the fundamental groupoid of the (simplicial) nerve of **I**; see [Q, §1] for the general case. When **I** is a poset, the **nerve** of **I** can be defined as the (abstract) simplicial complex whose n-simplices are the chains of cardinality $n + 1$ in **I**; see [B, 1.4.4, §2.4] (in which the nerve is called the *order complex*).

We now state Paré's result [P], mentioned in the introduction, characterizing colimits which can be built using pushouts:

Theorem 4.2.3 (Paré)**.** *Let* **I** *be a finitely generated category. The following are equivalent:*

(i) Every **I***-shaped diagram in a category with pushouts has a colimit.*

(ii) **I** *is simply-connected.*

Example 4.2.4. Let **I** be the shape of the diagram (4.1) in the Introduction. As mentioned there, $\pi_1(\mathbf{I})$ is equivalent to $\mathbb{Z}$ (i.e., it is connected and its automorphism group at each object is $\mathbb{Z}$). Now let $\mathbf{J} \supseteq \mathbf{I}$ be the shape of the (commuting) diagram

$$\begin{tikzcd}
A & & B \\
C \arrow[u, "f"] \arrow[urr, "h"] & & D \arrow[u, "k"'] \arrow[ull, "g"'] \\
& E \arrow[ul, "u"] \arrow[ur, "v"'] &
\end{tikzcd} .$$

Unlike **I**, **J** is simply-connected. Thus by Theorem 4.2.3, a colimit of a **J**-shaped diagram can be constructed out of pushouts. However, a cocone over a **J**-shaped diagram cannot necessarily be constructed from the AP: take $A = C = D = 1$, $B = 2$, and $h \neq k$ in **Inj** as in the Introduction, and $E = \varnothing$. Note that there is no contradiction, since **Inj** does *not* have pushouts.

The "reason" that **J** is simply-connected even though **I** is not is that the generating "loop" $f \circ h^{-1} \circ k \circ g^{-1}$ in $\pi_1(\mathbf{I})$ becomes trivial in $\pi_1(\mathbf{J})$:

$$\begin{aligned} f \circ h^{-1} \circ k \circ g^{-1} &= (f \circ u) \circ (u^{-1} \circ h^{-1}) \circ k \circ g^{-1} \\ &= (g \circ v) \circ (v^{-1} \circ k^{-1}) \circ k \circ g^{-1} = 1_A. \end{aligned}$$

This suggests that to characterize when a **J**-shaped diagram has a cocone in any category with the AP, we need a finer invariant than π_1, which does not allow the use of u, v to simplify the loop $f \circ h^{-1} \circ k \circ g^{-1}$ above.

4.2.2 Inverse categories

Definition 4.2.5. An **inverse category** is a category **C** such that every morphism $f : X \underset{\mathbf{C}}{\longrightarrow} Y$ has a unique **pseudoinverse** $f^{-1} : Y \underset{\mathbf{C}}{\longrightarrow} X$ obeying $f \circ f^{-1} \circ f = f$ and $f^{-1} \circ f \circ f^{-1} = f^{-1}$.

We write **InvCat** for the category of small inverse categories.

For basic properties of inverse categories, see e.g., [Li, §2]; the one-object case of *inverse monoids* is well-known in semigroup theory [La]. Here are some elementary facts about inverse categories we will use without mention:

Lemma 4.2.6. *Let* **C** *be an inverse category.*

- $f \mapsto f^{-1}$ *is an involutive functor* $\mathbf{C}^{\mathrm{op}} \to \mathbf{C}$.
- *Idempotents in* **C** *commute.*
- $f : X \underset{\mathbf{C}}{\longrightarrow} Y$ *is monic iff it is split monic iff* $f^{-1} \circ f = 1_X$.

The archetypical example of an inverse category is **PInj**, the category of sets and partial injections (where f^{-1} is given by the partial inverse of f). In fact, the axioms of an inverse category capture precisely the algebraic properties of **PInj**, in the sense that we have the following representation theorem (see [Li, 2.5], [CL, 3.8]), generalizing the Wagner-Preston representation theorem for inverse semigroups and the Yoneda lemma:

Theorem 4.2.7. *Let* $\mathbf{C}$ *be a small inverse category. We have an embedding functor*

$$
\begin{aligned}
\Psi_{\mathbf{C}} : \mathbf{C} &\longrightarrow \mathbf{PInj} \\
X &\longmapsto \textstyle\sum_{Z \in \mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(Z, X) \\
(X \xrightarrow[\mathbf{C}]{f} Y) &\longmapsto \left(\begin{array}{rcl} \sum_Z f^{-1} \circ f \circ \mathrm{Hom}_{\mathbf{C}}(Z, X) & \xrightarrow{\sim} & \sum_Z f \circ f^{-1} \circ \mathrm{Hom}_{\mathbf{C}}(Z, Y) \\ g & \longmapsto & f \circ g \end{array} \right).
\end{aligned}
$$

Here $\sum$ *denotes disjoint union, and* $f^{-1} \circ f \circ \mathrm{Hom}_{\mathbf{C}}(Z, X)$ *denotes the set of all composites* $f^{-1} \circ f \circ g$ *for* $g : Z \xrightarrow[\mathbf{C}]{} X$*, equivalently the set of all* $g : Z \xrightarrow[\mathbf{C}]{} X$ *such that* $g = f^{-1} \circ f \circ g$ *(and similarly for* $f \circ f^{-1} \circ \mathrm{Hom}_{\mathbf{C}}(Z, Y)$*).*

4.3 Amalgamating sets

In this section, we characterize the small categories $\mathbf{I}$ such that every $\mathbf{I}$-shaped diagram in $\mathbf{Inj}$ has a cocone. We begin with the following easy observation:

Lemma 4.3.1. *A diagram* $F : \mathbf{I} \to \mathbf{Inj}$ *has a cocone iff the colimit of* F *in* $\mathbf{Set}$ *is such that the canonical maps* $\iota_I : F(I) \to \varinjlim F$ *are injective, for all* $I \in \mathbf{I}$.

Proof. If the ι_I are injective, then $(\varinjlim F, (\iota_I)_{I \in \mathbf{I}})$ is a cocone in $\mathbf{Inj}$. Conversely, if F has a cocone $(X, (f_I)_{I \in \mathbf{I}})$ in $\mathbf{Inj}$, then the unique cocone morphism $g : \varinjlim F \to F$ is such that $g \circ \iota_I = f_I$ is injective for each I, hence ι_I is injective for each I. □

Now recall that for a diagram $F : \mathbf{I} \to \mathbf{Inj}$, the standard construction of $\varinjlim F$ in $\mathbf{Set}$ is as the quotient of the disjoint sum:

$$\varinjlim F := (\textstyle\sum_{I \in \mathbf{I}} F(I)) / \{(x, F(i)(x)) \mid i : I \xrightarrow[\mathbf{I}]{} J,\ x \in F(I)\}.$$

Two elements $x, y \in F(I)$ are thus identified iff they are connected by a zigzag

$$
\begin{tikzcd}[column sep=small]
 & x_1 \in F(I_1) & & x_3 \in F(I_3) & & x_{2n-1} \in F(I_{2n-1}) & \\
x \in F(I) \arrow[ur, mapsto, "F(i_1)"] & & x_2 \in F(I_2) \arrow[ul, mapsto, "F(i_2)"'] \arrow[ur, mapsto, "F(i_3)"] & & \cdots & & y \in F(I). \arrow[ul, mapsto, "F(i_{2n})"']
\end{tikzcd}
$$

Since the $F(i_k)$ are injective, the endpoint y of this zigzag is determined by x together with the "path" $I \xrightarrow[\mathbf{I}]{i_1} I_1 \xleftarrow[\mathbf{I}]{i_2} I_2 \xrightarrow[\mathbf{I}]{i_2} \cdots \xleftarrow[\mathbf{I}]{i_{2n}} I$ in I; in other words, F induces an action of such "paths" via partial injections between sets. This motivates defining the category of such "paths", while keeping in mind that they will be acting via partial injections:

Definition 4.3.2. The **left fundamental inverse category** of a category $\mathbf{I}$, denoted $\mathcal{L}(\mathbf{I})$, is the inverse category freely generated by $\mathbf{I}$ such that every morphism in $\mathbf{I}$ becomes monic in $\mathcal{L}(\mathbf{I})$.

Thus, we have a functor $\eta = \eta_{\mathbf{I}} : \mathbf{I} \to \mathcal{L}(\mathbf{I})$, such that $\mathcal{L}(\mathbf{I})$ is generated by the morphisms $\eta(i), \eta(i)^{-1}$ for $i : I \underset{\mathbf{I}}{\to} J$, such that $\eta(i)^{-1} \circ \eta(i) = 1_I$ for each such i, and such that any other functor $F : \mathbf{I} \to \mathbf{C}$ into an inverse category with $F(i)^{-1} \circ F(i) = 1_{F(I)}$ for each i factors uniquely through η. (We write i for $\eta(i)$ when there is no risk of confusion.)

We say that $\mathbf{I}$ is **upward-simply-connected** if $\mathcal{L}(\mathbf{I})$ is idempotent. (See Corollary 4.4.8 below for equivalent conditions when $\mathbf{I}$ is a poset.)

Thus $\mathcal{L}$ extends in an obvious manner to a functor $\mathbf{Cat} \to \mathbf{InvCat}$, which is left adjoint to the functor $\mathcal{S} : \mathbf{InvCat} \to \mathbf{Cat}$ taking an inverse category to its subcategory of monomorphisms. To see that $\mathcal{L}(\mathbf{I})$ is a finer invariant than $\pi_1(\mathbf{I})$, note that the forgetful functor $\mathbf{Gpd} \to \mathbf{Cat}$ (to which π_1 is left adjoint) factors through $\mathcal{S}$; indeed, a groupoid is precisely an inverse category in which every morphism is monic. Thus $\pi_1(\mathbf{I})$ is a quotient of $\mathcal{L}(\mathbf{I})$ (by the least congruence which makes every arrow monic). Since idempotents in a groupoid are identities, we get

Corollary 4.3.3. *If a category $\mathbf{I}$ is upward-simply-connected, then it is simply-connected.* □

Proposition 4.3.4. *Let $\mathbf{I}$ be a small category. The following are equivalent:*

(i) Every diagram $F : \mathbf{I} \to \mathbf{Inj}$ has a cocone.

(ii) The diagram $\Psi_{\mathcal{L}(\mathbf{I})} \circ \eta_{\mathbf{I}} : \mathbf{I} \to \mathbf{Inj}$ has a cocone, where $\Psi_{\mathcal{L}(\mathbf{I})} : \mathcal{L}(\mathbf{I}) \to \mathbf{PInj}$ is the embedding from Theorem 4.2.7 (whose restriction along $\eta_{\mathbf{I}} : \mathbf{I} \to \mathcal{L}(\mathbf{I})$ lands in the subcategory $\mathbf{Inj}$).

(iii) $\mathbf{I}$ is upward-simply-connected.

Proof. First we remark on why $\Psi_{\mathcal{L}(\mathbf{I})} \circ \eta_{\mathbf{I}}$ in (ii) lands in $\mathbf{Inj}$. This is because every morphism i in $\mathbf{I}$ becomes monic in $\mathcal{L}(\mathbf{I})$, hence in $\mathbf{PInj}$; and the monomorphisms in $\mathbf{PInj}$ are precisely the total injections.

(i) $\implies$ (ii) is obvious.

(ii) $\implies$ (iii): Let $f : I \xrightarrow[\mathcal{L}(\mathbf{I})]{} I$ be an endomorphism. To show that f is idempotent, it suffices to show that $f \circ f^{-1} \circ f = f^{-1} \circ f$, since then $f \circ f = f \circ (f \circ f^{-1} \circ f) = f \circ (f^{-1} \circ f) = f$. Since $\mathcal{L}(\mathbf{I})$ is generated by the morphisms in $\mathbf{I}$ and their pseudoinverses, we have

$$f = i_{2n}^{-1} \circ i_{2n-1} \circ \cdots \circ i_3 \circ i_2^{-1} \circ i_1$$

for some zigzag "path" in $\mathbf{I}$

$$\begin{tikzcd}[column sep=small] & I_1 & & I_3 & & \cdots & I_{2n-1} & \\ I = I_0 \arrow[ur, "i_1"] & & I_2 \arrow[ul, "i_2"'] \arrow[ur, "i_3"] & & & & & I_{2n} = I. \arrow[ul, "i_{2n}"'] \end{tikzcd}$$

This yields a zigzag (with some obvious abbreviations for clarity)

$$\begin{tikzcd}[column sep=tiny] & i_1 f^{-1} f \in \Psi(I_1) & & i_3 i_2^{-1} i_1 f^{-1} f \in \Psi(I_3) & \cdots & i_{2n-1} \cdots i_2^{-1} i_1 f^{-1} f \in \Psi(I_{2n-1}) & \\ f^{-1} f \in \Psi(I) \arrow[ur, mapsto, "i_1"] & & i_2^{-1} i_1 f^{-1} f \in \Psi(I_2) \arrow[ul, mapsto, "i_2"'] \arrow[ur, mapsto, "i_3"] & & & & f f^{-1} f \in \Psi(I) \arrow[ul, mapsto, "i_{2n}"'] \end{tikzcd}$$

where the even-numbered mappings are by the following calculation:

$$\begin{aligned} i_{2k}(i_{2k}^{-1} i_{2k-1} \cdots i_2^{-1} i_1 f^{-1} f) &= (i_{2k} i_{2k}^{-1})(i_{2k-1} \cdots i_1 \overbrace{i_1^{-1} \cdots i_{2k-1}^{-1}) i_{2k} \cdots i_{2n}}^{f^{-1}} f \\ &= (i_{2k-1} \cdots i_1 i_1^{-1} \cdots i_{2k-1}^{-1})(i_{2k} i_{2k}^{-1}) i_{2k} \cdots i_{2n} f \\ &= (i_{2k-1} \cdots i_1 i_1^{-1} \cdots i_{2k-1}^{-1}) i_{2k} \cdots i_{2n} f \\ &= i_{2k-1} \cdots i_1 f^{-1} f. \end{aligned}$$

By the discussion following Lemma 4.3.1, this last zigzag implies that $f^{-1} f = f f^{-1} f$ in $\varinjlim(\Psi \circ \eta)$, whence by (ii) and Lemma 4.3.1, the same equality holds in $\Psi(I)$ and hence in $\mathcal{L}(\mathbf{I})$, as desired.

(iii) $\implies$ (i): Let $F : \mathbf{I} \to \mathbf{Inj}$ be a diagram. By the universal property of $\mathcal{L}(\mathbf{I})$, F extends along η to a functor $\widetilde{F} : \mathcal{L}(\mathbf{I}) \to \mathbf{PInj}$ such that $\widetilde{F} \circ \eta = F : \mathbf{I} \to \mathbf{PInj}$. This functor $\widetilde{F}$ takes a pseudoinverse $i^{-1} : J \xrightarrow[\mathcal{L}(\mathbf{I})]{} I$, for $i : I \xrightarrow[\mathbf{I}]{} J$, to the (partial) inverse of $F(i)$, hence takes a "path" $f = i_{2n}^{-1} \circ i_{2n-1} \circ \cdots \circ i_1 : I \xrightarrow[\mathcal{L}(\mathbf{I})]{} I$ to the partial injection $\widetilde{F}(f) : F(I) \xrightarrow[\mathbf{PInj}]{} F(I)$ mapping $x \in F(I)$ to the endpoint y of the zigzag in the remarks following Lemma 4.3.1. Since every $f : I \xrightarrow[\mathcal{L}(\mathbf{I})]{} I$ is idempotent, so is every $\widetilde{F}(f)$. Since the idempotents in $\mathbf{PInj}$ are precisely the partial identity functions, it follows (by the remarks following Lemma 4.3.1) that the canonical maps $\iota_I : F(I) \to \varinjlim F$ are injective. $\square$

This proves (ii) $\iff$ (iii) in Theorem 4.1.1, for arbitrary small $\mathbf{I}$; the parenthetical in (ii) follows from

Lemma 4.3.5. *Let* $\mathbf{I}$ *be a finite category. If a diagram* $F : \mathbf{I} \to \mathbf{Inj}$ *does not have a cocone, then some diagram* F' *which is a pointwise restriction of* F *to finite subsets also does not have a cocone.*

Proof. If F does not have a cocone, then there is some $I \in \mathbf{I}$ and $x \neq y \in F(I)$ which are identified in $\varinjlim F$. Take a zigzag $x = x_0 \mapsto x_1 \mapsfrom x_2 \mapsto \cdots \mapsfrom x_{2n} = y$ witnessing that x, y are identified in $\varinjlim F$, as in the remarks following Lemma 4.3.1, where $x_k \in F(I_k)$. Put $F'(J) := \{F(i)(x_k) \mid i : I_k \underset{\mathbf{I}}{\to} J\}$. □

4.4 Posetal diagrams

In this section, we prove that (iii) $\implies$ (i) in Theorem 4.1.1 for finite posets $\mathbf{I}$. To do so, we first examine the structure of upward-simply-connected posets; this will lead to the conditions (iv) and (v) in Theorem 4.1.1.

Lemma 4.4.1. *Let* $\mathbf{I}$ *be any category and* $\mathbf{J} \subseteq \mathbf{I}$ *be an upward-closed subcategory (or* ***cosieve****), i.e., a full subcategory such that if* $I \in \mathbf{J}$ *and* $i : I \underset{\mathbf{I}}{\to} J$ *then* $i, J \in \mathbf{J}$*. Then the canonical induced functor* $\mathcal{L}(\mathbf{J}) \to \mathcal{L}(\mathbf{I})$ *is faithful.*

Proof. Let $\mathbf{K}$ be the inverse category obtained by taking $\mathcal{L}(\mathbf{J})$, adding the objects in $\mathbf{I} \setminus \mathbf{J}$, and freely adjoining zero morphisms between every pair of objects (taking the zero morphism from an object in $\mathbf{I} \setminus \mathbf{J}$ to itself as the identity). Let $F : \mathbf{I} \to \mathbf{K}$ send morphisms i in $\mathbf{J}$ to $\eta_{\mathbf{J}}(i)$ and all other morphisms to 0. Then F sends all morphisms to monomorphisms, hence extends along $\eta_{\mathbf{I}} : \mathbf{I} \to \mathcal{L}(\mathbf{I})$ to $\widetilde{F} : \mathcal{L}(\mathbf{I}) \to \mathbf{K}$.

$$\begin{tikzcd}
\mathbf{J} \arrow[r, hook] \arrow[d, "\eta_{\mathbf{J}}"'] & \mathbf{I} \arrow[d, "\eta_{\mathbf{I}}"'] \arrow[ddr, "F"] & \\
\mathcal{L}(\mathbf{J}) \arrow[r] \arrow[drr, hook, bend right] & \mathcal{L}(\mathbf{I}) \arrow[dr, dashed, "\widetilde{F}"'] & \\
 & & \mathbf{K}
\end{tikzcd}$$

The composite $\mathcal{L}(\mathbf{J}) \to \mathcal{L}(\mathbf{I}) \xrightarrow{\widetilde{F}} \mathbf{K}$ is equal to the inclusion $\mathcal{L}(\mathbf{J}) \subseteq \mathbf{K}$, because it takes morphisms $\eta_{\mathbf{J}}(i)$ for i in $\mathbf{J}$ to $F(i) = \eta_{\mathbf{J}}(i)$. It follows that $\mathcal{L}(\mathbf{J}) \to \mathcal{L}(\mathbf{I})$ must be faithful, as desired. □

Corollary 4.4.2. *If* $\mathbf{I}$ *is upward-simply-connected, then every upward-closed subcategory* $\mathbf{J} \subseteq \mathbf{I}$ *is simply-connected.*

Proof. By Lemma 4.4.1 and Corollary 4.3.3. $\square$

We will show that for finite posets $\mathbf{I}$, the two conditions in Corollary 4.4.2 are equivalent to each other and to the following combinatorial notion:

Definition 4.4.3. The class of finite **tree-like** posets is defined inductively by the following rule:

($*$) if $\mathbf{K}_1, \ldots, \mathbf{K}_n$ are finite tree-like posets, and $\mathbf{U}_k \subseteq \mathbf{K}_k$ is a connected upward-closed subset for each k, then a new finite tree-like poset is formed by taking the disjoint union of the $\mathbf{K}_k$'s and adjoining a single point which lies below each $\mathbf{U}_k$.

A finite **forest-like** poset is a finite disjoint union of tree-like posets.

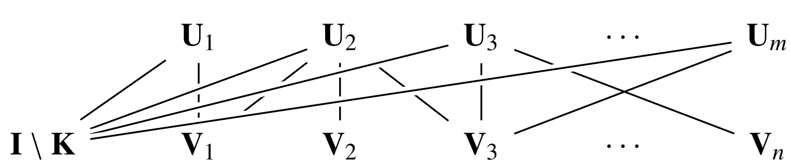
Proposition 4.4.4. *Let* $\mathbf{I}$ *be a finite connected simply-connected poset and* $I \in \mathbf{I}$ *be minimal. Then for each connected component* $\mathbf{K} \in \pi_0(\mathbf{I} \setminus \{I\})$*, the subposet* $\mathbf{K} \cap {\uparrow}I$ *(where* ${\uparrow}I := \{J \in \mathbf{I} \mid J \geq I\}$*) is connected.*

Proof. This is straightforward to show using topological arguments, by considering the nerve of $\mathbf{I}$ as in Remark 4.2.2; however, in the interest of keeping this paper self-contained, we will give a more elementary proof.

Let $\pi_0(\mathbf{K} \cap {\uparrow}I) = \{\mathbf{U}_1, \ldots, \mathbf{U}_m\}$ and $\pi_0(\mathbf{K} \setminus {\uparrow}I) = \{\mathbf{V}_1, \ldots, \mathbf{V}_n\}$. Note that we must have $m \geq 1$, since $\mathbf{K} \neq \varnothing$ and $\mathbf{I}$ is connected. Consider the partition of connected subposets

$$\mathcal{P} := \{\mathbf{I} \setminus \mathbf{K}\} \cup \pi_0(\mathbf{K} \cap {\uparrow}I) \cup \pi_0(\mathbf{K} \setminus {\uparrow}I)$$

of $\mathbf{I}$, ordered by $\mathbf{A} \leq \mathbf{B} \iff \exists A \in \mathbf{A}, B \in \mathbf{B}.\, A \leq B$ for $\mathbf{A}, \mathbf{B} \in \mathcal{P}$. Then $\mathcal{P}$ is connected (because $\mathbf{I}$ is), $\pi_0(\mathbf{K} \cap {\uparrow}I) \cup \pi_0(\mathbf{K} \setminus {\uparrow}I) \subseteq \mathcal{P}$ is connected (because $\mathbf{K}$ is), and we have two antichains $\{\mathbf{I} \setminus \mathbf{K}\} \cup \pi_0(\mathbf{K} \setminus {\uparrow}I)$ and $\pi_0(\mathbf{K} \cap {\uparrow}I)$, with only elements of the former below elements of the latter (and $\mathbf{I} \setminus \mathbf{K}$ below every element of the latter). So the Hasse diagram of $\mathcal{P}$ looks like



Using that each element of $\mathcal{P}$ is connected, it is easy to check that the functor $\pi_1(\mathbf{I}) \to \pi_1(\mathcal{P})$ induced by the quotient map $\mathbf{I} \to \mathcal{P}$ is full. So since $\mathbf{I}$ is simply-connected, so must be $\mathcal{P}$. But since $\mathcal{P}$ has no chains of cardinality > 2, $\pi_1(\mathcal{P})$ is just the graph-theoretic fundamental groupoid of its Hasse diagram (depicted above). So for $\mathcal{P}$ to be simply-connected, its Hasse diagram must be acyclic, which clearly implies $|\pi_0(\mathbf{K} \cap \uparrow I)| = m = 1$, as desired. ◻

Corollary 4.4.5. *Let* $\mathbf{I}$ *be a finite (connected) poset such that every upward-closed subset is simply-connected. Then* $\mathbf{I}$ *is forest-like (tree-like).* ◻

Proposition 4.4.6. *Let* $\mathbf{I}$ *be a finite tree-like poset. Then for every category* $\mathbf{C}$ *with the AP, every diagram* $F : \mathbf{I} \to \mathbf{C}$ *has a cocone.*

Proof. By induction on the construction of $\mathbf{I}$, there is $I \in \mathbf{I}$ minimal such that $\mathbf{I} \setminus \{I\}$ is the disjoint union of tree-like posets $\mathbf{K}_1, \dots, \mathbf{K}_n$ and $\mathbf{U}_k := \mathbf{K}_k \cap \uparrow I$ is connected for each k. For each k, there is some $U_k \in \mathbf{U}_k$ since $\mathbf{U}_k$ is connected, and there is a cocone $(X_k, (f^k_K)_{K \in \mathbf{K}_k})$ over $F|\mathbf{K}_k$ by the induction hypothesis. By the AP in $\mathbf{C}$, there is a cocone $(Y, (g_I, g_k)_k)$ over the pushout diagram consisting of the composite maps

$$F(I) \xrightarrow[\mathbf{C}]{F(I,U_k)} F(U_k) \xrightarrow[\mathbf{C}]{f^k_{U_k}} X_k$$

(where $F(I, U_k)$ denotes F applied to the unique morphism $I \underset{\mathbf{I}}{\to} U_k$) for all k, where $g_I : F(I) \underset{\mathbf{C}}{\to} Y$ and $g_k : X_k \underset{\mathbf{C}}{\to} Y$ with

$$g_k \circ f^k_{U_k} \circ F(I, U_k) = g_I.$$

For $K \in \mathbf{K}_k$, let $g_K : F(K) \to Y$ be the composite

$$F(K) \xrightarrow{f^k_K} X_k \xrightarrow{g_k} Y.$$

We claim that $(Y, (g_J)_{J \in \mathbf{I}})$ is a cocone over F. For $K < K'$ where $K \in \mathbf{K}_k$ for some k, also $K' \in \mathbf{K}_k$, whence $g_K = g_k \circ f^k_K = g_k \circ f^k_{K'} \circ F(K, K') = g_{K'} \circ F(K, K')$ because $(X_k, (f^k_K)_{K \in \mathbf{K}_k})$ is a cocone over $F|\mathbf{K}_k$. So we only need to check that for $I < K \in \mathbf{K}_k$, i.e., $K \in \mathbf{U}_k$, we have $g_I = g_K \circ F(I, K)$. By connectedness of $\mathbf{U}_k$, there is a path $K = K_0 \leq K_1 \geq K_2 \leq \cdots \geq K_{2n} = U_k$ in $\mathbf{U}_k$, whence

$$\begin{aligned} g_K \circ F(I, K) &= g_k \circ f^k_{K_0} \circ F(I, K_0) \\ &= g_k \circ f^k_{K_1} \circ F(K_0, K_1) \circ F(I, K_0) = g_k \circ f^k_{K_1} \circ F(I, K_1) \\ &= g_k \circ f^k_{K_1} \circ F(K_2, K_1) \circ F(I, K_2) = g_k \circ f^k_{K_2} \circ F(I, K_2) \\ &= \cdots \\ &= g_k \circ f^k_{K_{2n}} \circ F(I, K_{2n}) = g_k \circ f^k_{U_k} \circ F(I, U_k) = g_I. \end{aligned}$$ □

Corollary 4.4.7. *Let $\mathbf{I}$ be a finite forest-like poset. Then for every category $\mathbf{C}$ with the AP and the JEP, every diagram $F : \mathbf{I} \to \mathbf{C}$ has a cocone.*

Proof. Find cocones over F restricted to each connected component of $\mathbf{I}$, then apply JEP. □

Corollary 4.4.8. *A finite poset $\mathbf{I}$ is upward-simply-connected, iff every upward-closed subset $\mathbf{J} \subseteq \mathbf{I}$ is simply-connected, iff $\mathbf{I}$ is forest-like.*

Proof. By Corollary 4.4.2, Corollary 4.4.5, Corollary 4.4.7, and Proposition 4.3.4. □

This proves (iii) $\iff$ (iv) $\iff$ (v) $\implies$ (i) in Theorem 4.1.1 for finite posets $\mathbf{I}$.

4.5 General diagrams

We begin this section by explaining why in a context such as Fraïssé theory, where we are looking at diagrams in a monic category $\mathbf{C}$ (of embeddings in the case of Fraïssé theory), the only diagram shapes $\mathbf{I}$ worth considering are posets.

Definition 4.5.1. The **monic reflection** of a category $\mathbf{I}$ is the monic category $\mathcal{M}(\mathbf{I})$ freely generated by $\mathbf{I}$. Explicitly, $\mathcal{M}(\mathbf{I}) = \mathbf{I}/{\sim}$ for the least congruence $\sim$ on $\mathbf{I}$ such that $\mathbf{I}/{\sim}$ is monic.

Thus for a diagram $F : \mathbf{I} \to \mathbf{C}$ in a monic category $\mathbf{C}$, F factors through $\mathcal{M}(\mathbf{I})$, say as $F' : \mathcal{M}(\mathbf{I}) \to \mathbf{C}$; and pullback of cocones over F' along the projection $\pi : \mathbf{I} \to \mathcal{M}(\mathbf{I})$ is an isomorphism of categories (between the category of cocones over F' and the category of cocones over F). So in a monic category $\mathbf{C}$, we may as well only consider diagrams whose shape $\mathbf{I}$ is monic. And by the following, an upward-simply-connected monic $\mathbf{I}$ is necessarily a preorder; clearly $\mathbf{I}$ may be replaced with an equivalent poset in that case.

Lemma 4.5.2. *Let* $\mathbf{C}$ *be an idempotent inverse category. Then there is at most one monomorphism between any two objects in* $\mathbf{C}$*.*

Proof. Let $f, g : X \underset{\mathbf{C}}{\to} Y$ be two monomorphisms. Then $f \circ g^{-1} : Y \to Y$ is idempotent, whence $g^{-1} \circ f = f^{-1} \circ f \circ g^{-1} \circ f \circ g^{-1} \circ g = f^{-1} \circ f \circ g^{-1} \circ g = 1_X$. This implies that g^{-1} is a pseudoinverse of f, whence $g^{-1} = f^{-1}$, whence $g = f$. □

Corollary 4.5.3. *An upward-simply-connected small monic category* $\mathbf{I}$ *is necessarily a preorder.*

Proof. Since $\mathbf{I}$ is monic, we have an embedding

$$\begin{aligned} \mathbf{I} &\longrightarrow \mathbf{Inj} \subseteq \mathbf{PInj} \\ I &\longmapsto \textstyle\sum_{K \in \mathbf{I}} (K \underset{\mathbf{I}}{\to} I) \\ (I \underset{\mathbf{I}}{\xrightarrow{i}} J) &\longmapsto (j \mapsto i \circ j); \end{aligned}$$

this embedding factors through the canonical functor $\eta : \mathbf{I} \to \mathcal{L}(\mathbf{I})$, so η must be faithful. By Lemma 4.5.2, for $i, j : I \underset{\mathbf{I}}{\to} J$, we have $\eta(i) = \eta(j)$, whence $i = j$. □

Now we check that $\mathcal{L}(\mathbf{I})$ is invariant when passing from $\mathbf{I}$ to $\mathcal{M}(\mathbf{I})$:

Lemma 4.5.4. *For any (small) category* $\mathbf{I}$*,* $\mathcal{L}(\mathbf{I}) \cong \mathcal{L}(\mathcal{M}(\mathbf{I}))$*.*

Proof. $\mathcal{L} : \mathbf{Cat} \to \mathbf{InvCat}$ is left adjoint to the composite

$$\mathbf{InvCat} \longrightarrow \mathbf{MonCat} \longrightarrow \mathbf{Cat},$$

where **MonCat** is the category of (small) monic categories, the first functor takes an inverse category to its subcategory of monomorphisms, and the second functor is the full inclusion, which has left adjoint $\mathcal{M} : \mathbf{Cat} \to \mathbf{MonCat}$. □

Corollary 4.5.5. *Let* $\mathbf{I}$ *be a connected upward-simply-connected category with finitely many objects. Then every diagram* $F : \mathbf{I} \to \mathbf{C}$ *in a monic category* $\mathbf{C}$ *with the AP has a cocone.*

Proof. Since $\mathbf{C}$ is monic, F factors through $\mathcal{M}(\mathbf{I})$, as $F' : \mathcal{M}(\mathbf{I}) \to \mathbf{C}$, say. Since $\mathbf{I}$ is upward-simply-connected, so is $\mathcal{M}(\mathbf{I})$, whence $\mathcal{M}(\mathbf{I})$ is a finite preorder. Clearly replacing $\mathcal{M}(\mathbf{I})$ with an equivalent poset does not change whether F' has a cocone, so by Proposition 4.4.6, F' has a cocone, which induces a cocone over F. □

We now deduce the general case of (iii) $\Longrightarrow$ (i) in Theorem 4.1.1:

Lemma 4.5.6. *Let* $\mathbf{C}$ *be a category with the AP. Then*

$$f \sim g \iff \exists h \in \mathbf{C}.\, h \circ f = h \circ g$$

defines a congruence on $\mathbf{C}$*, thus* $\mathcal{M}(\mathbf{C}) = \mathbf{C}/{\sim}$*.*

Proof. Clearly $\sim$ is reflexive, symmetric, and right-compatible. To check left-compatibility, suppose $f \sim g : X \xrightarrow[\mathbf{C}]{} Y$ and $h : Y \xrightarrow[\mathbf{C}]{} Z$; we show $h \circ f \sim h \circ g$. Let $k : Y \xrightarrow[\mathbf{C}]{} W$ witness $f \sim g$, so that $k \circ f = k \circ g$. Then $h \circ f \sim h \circ g$ is witnessed by l such that the following diagram commutes:

$$\begin{tikzcd}
& V & \\
Z \arrow[ur, dashed, "l"] & & W \arrow[ul, dashed] \\
& Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

To check transitivity, suppose $f \sim g \sim h : X \xrightarrow[\mathbf{C}]{} Y$; we show $f \sim h$. Let $k : Y \xrightarrow[\mathbf{C}]{} Z$ witness $f \sim g$ and $l : Y \xrightarrow[\mathbf{C}]{} W$ witness $g \sim h$. Then $f \sim h$ is witnessed by an amalgam of k, l. $\square$

Proposition 4.5.7. *Let* $\mathbf{I}$ *be a finitely generated connected upward-simply-connected category. Then every diagram* $F : \mathbf{I} \to \mathbf{C}$ *in a category* $\mathbf{C}$ *with the AP has a cocone.*

Proof. Let $\pi : \mathbf{C} \to \mathcal{M}(\mathbf{C})$ be the monic reflection of $\mathbf{C}$. By Corollary 4.5.5, $\pi \circ F : \mathbf{I} \to \mathcal{M}(\mathbf{C})$ has a cocone $(Y, \overline{g})$, where $g_I : F(I) \xrightarrow[\mathcal{M}(\mathbf{C})]{} Y$ for each $I \in \mathbf{I}$. Pick for each $I \in \mathbf{I}$ a lift $f_I : F(I) \xrightarrow[\mathbf{C}]{} Y$ of g_I. For each $i : I \xrightarrow[\mathbf{I}]{} J$, since $(Y, \overline{g})$ is a cocone over $\pi \circ F$, we have $\pi(f_I) = g_I = g_J \circ \pi(F(i)) = \pi(f_J \circ F(i))$; by Lemma 4.5.6, this means there is some $h_i : Y \xrightarrow[\mathbf{C}]{} Z_i$ such that $h_i \circ f_I = h_i \circ f_J \circ F(i)$. Now letting $h : Y \xrightarrow[\mathbf{C}]{} Z$ be an amalgam of h_i for all arrows i in some finite generating set of arrows in $\mathbf{I}$, we have $h \circ f_I = h \circ f_J \circ F(i)$ for all $i : I \xrightarrow[\mathbf{I}]{} J$ in the generating set, so $(Z, h \circ f_I)_{I \in \mathbf{I}}$ is a cocone over F. $\square$

Corollary 4.5.8. *Let* $\mathbf{I}$ *be a finitely generated upward-simply-connected category. Then every diagram* $F : \mathbf{I} \to \mathbf{C}$ *in a category* $\mathbf{C}$ *with the AP and the JEP has a cocone.* $\square$

This proves (iii) $\Longrightarrow$ (i) in Theorem 4.1.1. Since (i) $\Longrightarrow$ (ii) is obvious, to complete the proof of the theorem it only remains to check

Lemma 4.5.9. *Let $\mathbf{I}$ be a category such that every $\mathbf{I}$-shaped diagram in a category with the AP has a cocone. Then $\mathbf{I}$ is connected.*

Proof. Consider the diagram $\mathbf{I} \to \pi_0(\mathbf{I})$ where $\pi_0(\mathbf{I})$ is regarded as a discrete category. □

4.6 Decidability

Suppose we are given a finite category $\mathbf{I}$ in some explicit form (say, a list of its morphisms and a composition table). Then our proof of Theorem 4.1.1 yields a simple procedure for testing whether a "generalized amalgamation property" holds for $\mathbf{I}$-shaped diagrams:

Corollary 4.6.1. *For a finite category $\mathbf{I}$, it is decidable whether $\mathbf{I}$ is upward-simply-connected, hence whether every $\mathbf{I}$-shaped diagram in a category with the AP (and possibly the JEP) has a cocone.*

In particular, it is decidable whether a "generalized amalgamation property" for $\mathbf{I}$-shaped diagrams holds for every Fraïssé class.

Proof. First, compute the monic reflection $\mathcal{M}(\mathbf{I})$; this can be done in finite time, since $\mathcal{M}(\mathbf{I}) = \mathbf{I}/{\sim}$ for the least congruence $\sim$ such that $\mathbf{I}/{\sim}$ is monic, and $\sim$ can be computed by taking the equality congruence and closing it under finitely many conditions, which takes finitely many steps since $\mathbf{I}$ is finite. If $\mathcal{M}(\mathbf{I})$ is not a preorder, then $\mathbf{I}$ is not upward-simply-connected by Corollary 4.5.3. If $\mathcal{M}(\mathbf{I})$ is a preorder, then replace it with an equivalent poset $\mathbf{J}$ and recursively test whether $\mathbf{J}$ is forest-like using Proposition 4.4.4, i.e., for each connected component $\mathbf{K} \in \pi_0(\mathbf{J})$, test whether $\mathbf{K}$ is tree-like by picking some minimal $K \in \mathbf{K}$ and then for each $\mathbf{L} \in \pi_0(\mathbf{K} \setminus \{K\})$ testing whether $\mathbf{L} \cap \uparrow K$ is connected and whether $\mathbf{L}$ is tree-like. □

This cannot be extended to finitely presented $\mathbf{I}$, since if $\mathbf{I}$ is a group regarded as a one-object category then $\mathcal{L}(\mathbf{I}) = \mathbf{I}$ is idempotent iff $\mathbf{I}$ is trivial, and it is undecidable whether a finite group presentation presents the trivial group (see e.g., [Mi, 3.4]).

We end by pointing out the following simple, but somewhat surprising, consequence of Paré's result (Theorem 4.2.3), which shows that the analogy between the AP and pushouts breaks down when it comes to decidability:

Corollary 4.6.2 (of Theorem 4.2.3)**.** *For a finite poset* $\mathbf{I}$*, it is undecidable whether* $\mathbf{I}$ *is simply-connected, hence whether every* $\mathbf{I}$*-shaped diagram in a category with pushouts has a colimit.*

Proof (sketch). There is a standard procedure to turn a finite presentation of a group G into a finite connected 2-dimensional simplicial complex K with fundamental group G (see [LS]); then the nerve (see Remark 4.2.2) of the face poset of K is the barycentric subdivision of K, hence the face poset of K has fundamental group(oid equivalent to) G (see [B] for details). ◻

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Chapter 5

BOREL FUNCTORS, INTERPRETATIONS, AND STRONG CONCEPTUAL COMPLETENESS FOR $\mathcal{L}_{\omega_1\omega}$

Ruiyuan Chen

5.1 Introduction

A "strong conceptual completeness" theorem for a logic, in the sense of Makkai [M88], is a strengthening of the usual completeness theorem which allows the syntax of a theory in that logic to be completely recovered from its semantics, up to a suitable notion of equivalence. In this paper, we prove such a result for the infinitary logic $\mathcal{L}_{\omega_1\omega}$.

Let $\mathcal{L}$ be a countable first-order language and $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$-theory. An $(\mathcal{L}_{\omega_1\omega}, \mathcal{T})$**-imaginary sort** A is a certain kind of syntactical name for a countable set $A^{\mathcal{M}}$ uniformly definable from each countable model $\mathcal{M}$ of $\mathcal{T}$, which is built up from $\mathcal{L}_{\omega_1\omega}$-formulas by taking (formal) countable disjoint unions and quotients by definable equivalence relations. Given two imaginary sorts A, B, an $(\mathcal{L}_{\omega_1\omega}, \mathcal{T})$**-definable function** $f : A \to B$ is a syntactical name for a uniformly definable function $f^{\mathcal{M}} : A^{\mathcal{M}} \to B^{\mathcal{M}}$ for each model $\mathcal{M}$; formally, f is given by a $\mathcal{T}$-equivalence class of (families of) $\mathcal{L}_{\omega_1\omega}$-formulas defining the graph of such a function. The notion of **definable relation** $R \subseteq A$ on a definable sort A is defined similarly. See Section 5.4 for the precise definitions.

Let $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ denote the **standard Borel groupoid of countable models of** $\mathcal{T}$, whose space of objects $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ is the standard Borel space of models of $\mathcal{T}$ whose underlying set is an initial segment of $\mathbb{N}$, and whose morphisms are isomorphisms between models.

Before stating the strong conceptual completeness theorem for $\mathcal{L}_{\omega_1\omega}$, we first state some of its consequences. Given a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, an $\mathcal{L}'_{\omega_1\omega}$-**interpretation** $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ of $\mathcal{T}$ in another countable $\mathcal{L}'_{\omega_1\omega}$-theory $\mathcal{T}'$ (in possibly a different language $\mathcal{L}'$) consists of:

- an $(\mathcal{L}'_{\omega_1\omega}, \mathcal{T}')$-imaginary sort[1] $F(\mathbb{X})$;
- for each n-ary relation symbol $R \in \mathcal{L}$, an n-ary definable relation $F(R) \subseteq F(\mathbb{X})^n$;
- for each n-ary function symbol $f \in \mathcal{L}$, an n-ary definable function $F(f) : F(\mathbb{X})^n \to F(\mathbb{X})$;
- such that "applying" F to the axioms in $\mathcal{T}$ results in $\mathcal{L}'$-sentences implied by $\mathcal{T}'$.

Given such an interpretation F, every countable model $\mathcal{M} = (M, R^{\mathcal{M}}, f^{\mathcal{M}})_{R,f \in \mathcal{L}'}$ of $\mathcal{T}'$ gives rise to a countable model $F^*(\mathcal{M}) = (F(\mathbb{X})^{\mathcal{M}}, F(R)^{\mathcal{M}}, F(f)^{\mathcal{M}})_{R,f\in\mathcal{L}}$ of $\mathcal{T}$; this yields a Borel functor

$$F^* : \mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$$

(after suitable coding to make the underlying set of $F^*(\mathcal{M})$ an initial segment of $\mathbb{N}$). Conversely,

Theorem 5.1.1. *Every Borel functor* $\mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ *is Borel naturally isomorphic to* F^* *for some interpretation* $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$.

This generalizes the Borel version of the main result of Harrison-Trainor–Miller–Montalbán [HMM, Theorem 9], which is the case where $\mathcal{T}, \mathcal{T}'$ are Scott sentences, i.e., they each have a single countable model up to isomorphism.

For an $(\mathcal{L}_{\omega_1\omega}, \mathcal{T})$-imaginary sort A, let

$$[\![A]\!] := \{(\mathcal{M}, a) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \ \& \ a \in A^{\mathcal{M}}\}$$

be the disjoint union of the countable sets $A^{\mathcal{M}}$ defined by A in all models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T})$. There is a natural standard Borel structure on $[\![A]\!]$, and we have the fiberwise countable Borel projection map $\pi : [\![A]\!] \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$. Since the set $A^{\mathcal{M}}$ is uniformly defined for each model $\mathcal{M}$, an isomorphism between models $\mathcal{M} \cong \mathcal{N}$ induces a bijection $A^{\mathcal{M}} \cong A^{\mathcal{N}}$. This gives a Borel action of the groupoid $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ on $[\![A]\!]$, turning $[\![A]\!]$ into a **fiberwise countable Borel** $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$**-space**, i.e., a standard Borel space X equipped with a fiberwise countable Borel map $p : X \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ and a Borel action of $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$. The core result of this paper is

[1] Here $\mathbb{X}$ is thought of as the $\mathcal{T}$-imaginary sort which names the underlying set of a model; see Remark 5.4.2.

Theorem 5.1.2. *Every fiberwise countable Borel* $\mathsf{Mod}(\mathcal{L},\mathcal{T})$*-space is isomorphic to* $[\![A]\!]$ *for some* $(\mathcal{L}_{\omega_1\omega},\mathcal{T})$*-imaginary sort* A*.*

In other words, every Borel isomorphism-equivariant assignment of a countable set to every countable model of $\mathcal{T}$ is named by some imaginary sort.

In order to place Theorems 5.1.1 and 5.1.2 in their proper context, we organize the $(\mathcal{L}_{\omega_1\omega},\mathcal{T})$-imaginary sorts and definable functions into a category, the **syntactic Boolean** ω_1**-pretopos** of $\mathcal{T}$, denoted

$$\overline{\langle\mathcal{L}\mid\mathcal{T}\rangle}^B_{\omega_1}.$$

The syntactic Boolean ω_1-pretopos is the categorical "Lindenbaum–Tarski algebra" of an $\mathcal{L}_{\omega_1\omega}$-theory: it "remembers" the logical structure of the theory, such as $\mathcal{T}$-equivalence classes of formulas and implications between them, while "forgetting" irrelevant syntactic details. See [MR] or [J02, D] for the general theory of syntactic pretoposes and related notions.

Let $\mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L},\mathcal{T}))$ denote the category of fiberwise countable Borel $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-spaces and Borel equivariant maps between them. Given a definable function $f : A \to B$, taking the disjoint union of the functions $f^{\mathcal{M}} : A^{\mathcal{M}} \to B^{\mathcal{M}}$ for every countable model $\mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T})$ yields a Borel $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-equivariant map $[\![f]\!] : [\![A]\!] \to [\![B]\!]$. We thus have a functor

$$[\![-]\!] : \overline{\langle\mathcal{L}\mid\mathcal{T}\rangle}^B_{\omega_1} \longrightarrow \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L},\mathcal{T})),$$

which can be thought of as taking syntax to semantics for the $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$. Our main result is

Theorem 5.1.3. *The functor* $[\![-]\!] : \overline{\langle\mathcal{L}\mid\mathcal{T}\rangle}^B_{\omega_1} \longrightarrow \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L},\mathcal{T}))$ *is an equivalence of categories.*

We say that two theories $(\mathcal{L},\mathcal{T})$, $(\mathcal{L}',\mathcal{T}')$ (in possibly different languages) are **Morita equivalent** if their syntactic Boolean ω_1-pretoposes are equivalent categories; informally, this means that they are different presentations of the "same" theory. Thus, by Theorem 5.1.3, a theory can be recovered up to Morita equivalence from its standard Borel groupoid of countable models.

To see the connection of Theorem 5.1.3 with Theorem 5.1.2, as well as the sense in which it is a strong form of the completeness theorem, note that (by general category theory) the statement that $[\![-]\!]$ is an equivalence may be broken into three parts:

(i) $\llbracket - \rrbracket$ is conservative, i.e., injective when restricted to the lattice of subobjects of each imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$. This is equivalent to the Lopez-Escobar completeness theorem for $\mathcal{L}_{\omega_1\omega}$ [Lop], provided that in the definitions above of imaginary sorts and definable functions, when we say e.g., that a formula ϕ defines the graph of a function in models of $\mathcal{T}$, we actually mean that $\mathcal{T}$ proves various $\mathcal{L}_{\omega_1\omega}$-sentences which say "$\phi$ is the graph of a function". (If we instead interpret these conditions semantically, then conservativity becomes vacuous.)

(ii) $\llbracket - \rrbracket$ is full on subobjects, i.e., surjective when restricted to subobject lattices. This is equivalent to the Lopez-Escobar definability theorem for isomorphism-invariant Borel sets [Lop].

(iii) $\llbracket - \rrbracket$ is essentially surjective. This is Theorem 5.1.2.

We will explain the equivalences in (i) and (ii) when we prove Theorem 5.1.3 in Sections 5.8 and 5.9.

Theorem 5.1.3 is the essence of a strong conceptual completeness theorem for $\mathcal{L}_{\omega_1\omega}$. There is a large family of such theorems known for various kinds of logic. Typically, these take the form of a "Stone-type duality" arising from a dualizing (or "schizophrenic") object equipped with two commuting kinds of structure; see [PT] or [J82, VI §4] for the general theory of such dualities. Here is a partial list of such dualities interpreted as strong conceptual completeness theorems:

- The original Stone duality between Boolean algebras and compact Hausdorff zero-dimensional spaces arises from equipping the set $2 = \{0, 1\}$ with both kinds of structure. When interpreted as a strong conceptual completeness theorem for (finitary) propositional logic, Stone duality says that the Lindenbaum–Tarski algebra $\langle \mathcal{L} \mid \mathcal{T} \rangle$ of a propositional theory $(\mathcal{L}, \mathcal{T})$ may be recovered as the algebra of clopen sets of its space of models.

- A version of Łoś's theorem says that ultraproducts on the category of sets commute with the structure (finite intersection, finite union, etc.) used to interpret finitary first-order logic $\mathcal{L}_{\omega\omega}$. Makkai [M87] proved that for an $\mathcal{L}_{\omega\omega}$-theory $\mathcal{T}$, its syntactic Boolean pretopos $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega}$ (defined similarly to $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ above but for $\mathcal{L}_{\omega\omega}$) may be recovered as the category of ultraproduct-preserving actions of the category of models and elementary embeddings.

- Analogous results of Gabriel–Ulmer, Lawvere, Makkai and others (see [AR], [ALR], [M90]) apply to various well-behaved fragments of $\mathcal{L}_{\omega\omega}$.
- In perhaps the closest relative to this paper, Awodey–Forssell [AF] proved that for an $\mathcal{L}_{\omega\omega}$-theory $\mathcal{T}$, $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^{B}_{\omega}$ may be recovered as certain continuous actions of the topological groupoid of models on subsets of a fixed set of large enough cardinality.

We will explain how to view Theorem 5.1.3 as a Stone-type duality theorem in Section 5.11. One benefit of doing so is that general duality theory then automatically yields Theorem 5.1.1. Indeed, the whole of Theorem 5.1.3 is equivalent to the following strengthening of Theorem 5.1.1: given two theories $(\mathcal{L}, \mathcal{T})$, $(\mathcal{L}', \mathcal{T}')$, the functor $F \mapsto F^*$ taking interpretations $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ and "definable natural isomorphisms" between them to Borel functors and Borel natural isomorphisms is an equivalence of groupoids. (See Corollary 5.11.5 for a precise statement.)

The proof of Theorem 5.1.3 is by reduction to a continuous version of the result. Recall that a **countable fragment** $\mathcal{F}$ of $\mathcal{L}_{\omega_1\omega}$ is a countable set of $\mathcal{L}_{\omega_1\omega}$-formulas containing atomic formulas and closed under subformulas, $\mathcal{L}_{\omega\omega}$-logical operations, and variable substitutions. Given a countable fragment $\mathcal{F}$ containing a countable theory $\mathcal{T}$, we define the notions of $(\mathcal{F}, \mathcal{T})$**-imaginary sort** and $(\mathcal{F}, \mathcal{T})$**-definable function** in the same way as the $(\mathcal{L}_{\omega_1\omega}, \mathcal{T})$- versions above, except that the formulas involved must be *countable disjunctions of formulas in* $\mathcal{F}$. The resulting category is called the **syntactic** ω_1**-pretopos**, denoted

$$\overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1}.$$

Let $\mathrm{Mod}(\mathcal{F}, \mathcal{T})$ denote $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ equipped with the Polish topology induced by the countable fragment $\mathcal{F}$; see [Gao, Ch. 11]. Let $\mathsf{Mod}(\mathcal{F}, \mathcal{T})$ denote the **Polish groupoid of countable models of** $\mathcal{T}$, whose space of objects is $\mathrm{Mod}(\mathcal{F}, \mathcal{T})$ and whose morphisms are isomorphisms with the usual pointwise convergence topology. We say that a topological space X equipped with a continuous map $p : X \to \mathrm{Mod}(\mathcal{F}, \mathcal{T})$ is **countable étalé over** $\mathrm{Mod}(\mathcal{F}, \mathcal{T})$ if X has a countable cover by open sets $U \subseteq X$ such that $p|U$ is an open embedding; a **countable étalé** $\mathsf{Mod}(\mathcal{F}, \mathcal{T})$**-space** is such a space equipped with a continuous action of $\mathsf{Mod}(\mathcal{F}, \mathcal{T})$. For an $(\mathcal{F}, \mathcal{T})$-imaginary sort A, the space $[\![A]\!]$ is countable étalé over $\mathrm{Mod}(\mathcal{F}, \mathcal{T})$ in a canonical way; and we get a functor

$$[\![-]\!] : \overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1} \longrightarrow \mathsf{Act}_{\omega_1}(\mathsf{Mod}(\mathcal{F}, \mathcal{T})),$$

where $\mathsf{Act}_{\omega_1}(\mathsf{Mod}(\mathcal{F}, \mathcal{T}))$ is the category of countable étalé $\mathsf{Mod}(\mathcal{F}, \mathcal{T})$-spaces and continuous equivariant maps. We now have the following continuous analog of Theorem 5.1.3:

Theorem 5.1.4. *The functor* $[\![-]\!] : \overline{\langle \mathcal{F} \mid \mathcal{T} \rangle_{\omega_1}} \longrightarrow \mathsf{Act}_{\omega_1}(\mathsf{Mod}(\mathcal{F}, \mathcal{T}))$ *is an equivalence of categories.*

The proof of Theorem 5.1.3 from Theorem 5.1.4 in Section 5.9 uses techniques from invariant descriptive set theory, in particular Vaught transforms and the Becker–Kechris method for topological realization of Borel actions (see [BK] or [Gao]), to show that every fiberwise countable Borel action can be realized as a countable étalé action by picking a large enough countable fragment $\mathcal{F}$. Along the way, we will prove the following more abstract result, which may be of independent interest:

Theorem 5.1.5. *Let* G *be an open Polish groupoid and* X *be a fiberwise countable Borel* G*-space. Then there is a finer open Polish groupoid topology on* G*, such that letting* G' *be the resulting Polish groupoid,* X *is Borel isomorphic to a countable étalé* G'*-space.*

As for Theorem 5.1.4, we will give a direct proof in Section 5.8. The proof we give is analogous to that of the duality result of Awodey–Forssell [AF] for $\mathcal{L}_{\omega\omega}$-theories mentioned above.

However, as with Awodey–Forssell's result, in some sense the proper context for Theorem 5.1.4 is the theory of groupoid representations for toposes. As such, we will sketch in Section 5.15 an alternative proof of (a generalization of) Theorem 5.1.4 using the Joyal–Tierney representation theorem [JT]. While this proof (together with its prerequisite definitions and lemmas) is admittedly much longer than the direct proof, it uses only straightforward variations of well-known concepts and arguments, thereby showing that Theorem 5.1.4 is in some sense a purely "formal" consequence of standard topos theory.

We have tried to organize this paper so as to minimize the amount of category theory needed in the earlier sections. We begin with basic definitions involving groupoids and étalé spaces in Section 5.2, followed by the proof of Theorem 5.1.5 in Section 5.3. We then give in Sections 5.4 to 5.6 the precise definitions of the syntactic (Boolean) ω_1-pretopos, the groupoid of countable models, and the functor $[\![-]\!]$. Along the way, we introduce the notion of an "ω_1-coherent theory", which generalizes that of a countable fragment. In Section 5.7, we present a version of Vaught's proof of Lopez-Escobar's (definability) theorem; this will be needed in what follows. In Section 5.8, we give the direct proof of Theorem 5.1.4, which is then used (along with the proof of Theorem 5.1.5) to prove Theorem 5.1.3 in Section 5.9.

The rest of the paper involves more heavy-duty categorical notions. In Section 5.10, we define the notion of an interpretation F between theories and the induced functor F^* between the groupoids of models; this defines a (contravariant) pseudofunctor from the 2-category of theories to the 2-category of standard Borel groupoids. In Section 5.11, we explain how Theorem 5.1.3 may be viewed as a Stone-type duality, yielding Theorem 5.1.1; we also explain how this latter result may be viewed as a generalization of the main result of [HMM]. In Sections 5.12 to 5.14, we give some prerequisite definitions and lemmas, which are used in the proof of (a generalization of) Theorem 5.1.4 from the Joyal–Tierney theorem, in Section 5.15.

Acknowledgments. We would like to thank Alexander Kechris for providing some comments on an earlier draft of this paper.

5.2 (Quasi-)Polish spaces and groupoids

In this section, we recall some basic definitions involving groupoids and their actions, étalé spaces, and quasi-Polish spaces.

For sets X, Y, Z and functions $f : X \to Z$ and $g : Y \to Z$, the **fiber product** or **pullback** is

$$X \times_Z Y := \{(x, y) \mid f(x) = g(y)\} \subseteq X \times Y.$$

The maps f, g are hidden in the notation and will be explicitly specified if not clear from context.

A (small) **groupoid** $\mathsf{G} = (G^0, G^1)$ consists of a set of *objects* G^0, a set of *morphisms* G^1, *source* and *target* maps $\partial_1, \partial_0 : G^1 \to G^0$ (note the order; usually $g \in G^1$ with $(\partial_1(g), \partial_0(g)) = (x, y)$ is denoted $g : x \to y$), a *unit* map $\iota : G^0 \to G^1$ (usually denoted $\iota(x) = 1_x$), a *multiplication* map $\mu : G^1 \times_{G^0} G^1 \to G^1$ (usually denoted $\mu(h, g) = h \cdot g$; here $\times_{G^0}$ means fiber product with respect to ∂_1 on the left and ∂_0 on the right), and an *inversion* map $\nu : G^1 \to G^1$, subject to the usual axioms ($\partial_0(h \cdot g) = \partial_0(h)$, $g^{-1} \cdot g = 1_{\partial_1(g)}$, etc.).

An **action** of a groupoid G on a set X equipped with a map $p : X \to G^0$ is a map $a : G^1 \times_{G^0} X \to X$ (usually denoted $a(g, x) = g \cdot x$; here $\times_{G^0}$ means ∂_1 on the left and p on the right) satisfying the usual axioms ($p(g \cdot x) = \partial_0(g)$, $1_{p(x)} \cdot x = x$, and $(h \cdot g) \cdot x = h \cdot (g \cdot x)$). The set X equipped with the map p and the action is also called a G-**set**. A G-**equivariant map** between two G-sets $p : X \to G^0$ and $q : Y \to G^0$ is a map $f : X \to Y$ such that $p = q \circ f$ and $f(g \cdot x) = g \cdot f(x)$.

The **trivial action** of a groupoid G on G^0 equipped with the identity $1_{G^0} : G^0 \to G^0$ is given by $g \cdot x := y$ for $g : x \to y$. Note that for any G-set $p : X \to G^0$, p is G-equivariant.

A **topological groupoid** is a groupoid $\mathsf{G} = (G^0, G^1)$ in which G^0, G^1 are topological spaces and the structure maps $\partial_1, \partial_0, \iota, \mu, \nu$ are continuous. We will usually be concerned with topological groupoids G which are **open**, meaning that ∂_1 (equivalently ∂_0, μ) is an open map. A **continuous action** of a topological groupoid G is an action $a : G^1 \times_{G^0} X \to X$ on $p : X \to G^0$ such that X is a topological space and p, a are continuous. In that case, (X, p, a) is a **continuous** G-**space**.

A continuous map $f : X \to Y$ between topological spaces is **étalé** (or a **local homeomorphism**) if X has a cover by open sets $U \subseteq X$ such that $f|U : U \to Y$ is an open embedding (U is then an **open section over** $f(U) \subseteq Y$), and **countable étalé** if such a cover can be taken to be countable. A continuous action of a topological groupoid G on $p : X \to G^0$ is **(countable) étalé** if p is (countable) étalé. We denote the category of countable étalé G-spaces and continuous G-equivariant maps by

$$\mathsf{Act}_{\omega_1}(\mathsf{G}).$$

We will need the following standard facts about (countable) étalé maps:

Lemma 5.2.1. *Let X, Y, Z be topological spaces.*

(i) Étalé maps are open.

(ii) (Countable) étalé maps are closed under composition.

(iii) If $f : X \to Y$ is continuous, $g : Y \to Z$ is (countable) étalé, and $g \circ f$ is (countable) étalé, then f is (countable) étalé.

(iv) If $f : X \to Z$ is continuous and $g : Y \to Z$ is (countable) étalé, then the pullback of g along f, i.e., the projection $p : X \times_Z Y \to X$, is (countable) étalé.

(v) If $E \subseteq X \times X$ is an equivalence relation such that either (equivalently both) of the projections $p, q : E \to X$ is open, then the quotient map $h : X \to X/E$ is open.

(vi) If $f : X \to Y$ is an open surjection, $g : Y \to Z$ is continuous, and $g \circ f$ is (countable) étalé, then g is (countable) étalé.

Proof. Most of these are proved in standard references on sheaf theory (at least without the countability restrictions); see e.g., [Ten, §2.3].

(i): If $f : X \to Y$ is étalé, with X covered by open sections U_i, then for any open $V \subseteq X$, $f(V) = \bigcup_i f(U_i \cap V)$ which is open.

(ii): If $f : X \to Y$ and $g : Y \to Z$ are étalé, with X covered by open f-sections U_i and Y covered by open g-sections V_j, then X is covered by open $(g \circ f)$-sections $U_i \cap g^{-1}(V_j)$.

(iii): If $U \subseteq X$ is an open $(g \circ f)$-section, then $f(U) \subseteq Y$ is open, since if $Y = \bigcup_j V_j$ is a cover by open g-sections, then $f(U) = \bigcup_j (V_j \cap g^{-1}((g \circ f)(U \cap f^{-1}(V_j))))$. Thus if $U \subseteq X$ is an open $(g \circ f)$-section, then U is also an open f-section.

(iv): If $V \subseteq Y$ is an open g-section, then $X \times_Z V \subseteq X \times_Z Y$ is an open p-section.

(v): If $U \subseteq X$ is open, then $h^{-1}(h(U)) = [U]_E = p(E \cap (X \times U)) \subseteq X$ is open, whence $h(U) \subseteq X/E$ is open.

(vi): If $U \subseteq X$ is an open $(g \circ f)$-section, then $f(U) \subseteq Y$ is an open g-section. $\square$

For basic descriptive set theory, see [Kec]. A **standard Borel groupoid** is a groupoid $\mathsf{G} = (G^0, G^1)$ in which G^0, G^1 are standard Borel spaces and the structure maps $\partial_1, \partial_0, \iota, \mu, \nu$ are Borel. A **Borel action** of a standard Borel groupoid G (or a **Borel G-space**) is an action a on $p : X \to G^0$ such that X is a standard Borel space and p, a are Borel maps. The action is **fiberwise countable** if p is (and we call a Borel set $A \subseteq X$ a **Borel section over** $f(A)$ if $p|A$ is injective). We denote the category of fiberwise countable Borel G-spaces and Borel G-equivariant maps by

$$\mathsf{Act}^B_{\omega_1}(\mathsf{G}).$$

We will be considering the following generalization of Polish spaces. We say that a subset of $A \subseteq X$ of a topological space X is $\mathbf{\Pi}^0_2$ if it is a countable intersection $A = \bigcap_i (U_i \cup F_i)$ where $U_i \subseteq X$ is open and $F_i \subseteq X$ is closed. Note that if every closed set in X is G_δ (a countable intersection of open sets, e.g., if X is metrizable), then a set if $\mathbf{\Pi}^0_2$ iff it is G_δ. A **quasi-Polish space** is a topological space which is homeomorphic to a $\mathbf{\Pi}^0_2$ subset of a countable power of the Sierpiński space $\mathbb{S} = \{0 < 1\}$ (with $\{0\}$ closed but not open). Quasi-Polish spaces are closed under countable products, $\mathbf{\Pi}^0_2$ subsets, and continuous open T_0 images. A space is Polish iff it is quasi-Polish and regular. Every quasi-Polish space can be made Polish by adjoining countably many closed sets to the topology, hence is in particular a standard Borel space. For these and other basic facts about quasi-Polish spaces, see [deB].

The following is a version of a more abstract "presentability" result; see Lemma 5.12.2.

Lemma 5.2.2. *Let $f : X \to Y$ be a countable étalé map with Y quasi-Polish. Then X is quasi-Polish.*

Proof. Let $\mathcal{A}$ be a countable basis of open sections in X, closed under binary intersections. Consider

$$\begin{aligned} g : X &\longrightarrow Y \times \mathbb{S}^{\mathcal{A}} \\ x &\longmapsto (f(x), \chi_A(x))_{A \in \mathcal{A}}, \end{aligned}$$

where χ_A is the indicator function of A. Clearly g is a continuous embedding. We claim that for $(y, i_A)_{A \in \mathcal{A}} \in Y \times \mathbb{S}^{\mathcal{A}}$,

$$(y, i_A)_A \in g(X) \iff \overbrace{\exists A\, (i_A = 1)}^{(1)} \;\&\; \forall A, B \left(\begin{array}{l} \overbrace{(i_A \wedge i_B = i_{A \cap B})}^{(2)} \;\& \\ \underbrace{(A \subseteq B \implies i_A = \chi_{f(A)}(y) \wedge i_B)}_{(3)} \end{array} \right).$$

$\Longrightarrow$ is straightforward. For $\Longleftarrow$, given $(y, i_A)_A$ satisfying the right-hand side, by (1), let A_0 be such that $i_{A_0} = 1$; then (2) and (3) give that for all A, $i_A = 1$ iff $i_{A \cap A_0} = 1$ iff $y \in f(A \cap A_0)$, so in particular $y \in f(A_0)$, whence letting $x \in A_0$ be (unique) such that $f(x) = y$, it is easily verified that $(y, i_A)_A = g(x)$. Clearly the right-hand side is $\mathbf{\Pi}^0_2$ in $Y \times \mathbb{S}^{\mathcal{A}}$, so X is quasi-Polish. ◻

A **(quasi-)Polish groupoid** G is a topological groupoid such that G^0, G^1 are (quasi-)Polish. See Lupini [Lup] for the basic theory of Polish groupoids (note that by *Polish groupoid*, [Lup] refers to a slight generalization of what we are calling *open Polish groupoid*). A **(quasi-)Polish G-space** is a continuous G-space $p : X \to G^0$ such that X is (quasi-)Polish. By Lemma 5.2.2, for quasi-Polish G, every countable étalé G-space is quasi-Polish.

Let G be an open Polish groupoid and $p : X \to G^0$ be a Borel G-space. For a ∂_1-fiberwise open set $U \subseteq G^1$ and a Borel set $A \subseteq X$, the **Vaught transforms** are defined by

$$A^{\triangle U} := \{x \in X \mid \exists^* g \in \partial_1^{-1}(p(x)) \cap U\,(g \cdot x \in A)\},$$
$$A^{*U} := \{x \in X \mid \forall^* g \in \partial_1^{-1}(p(x)) \cap U\,(g \cdot x \in A)\}.$$

Here $\exists^*$ and $\forall^*$ denote Baire category quantifiers; see [Kec, 8.J]. We also put

$$\begin{aligned} A^{\circledast U} &:= \{x \in X \mid \partial_1^{-1}(p(x)) \cap U \neq \varnothing \;\&\; \forall^* g \in \partial_1^{-1}(p(x)) \cap U\,(g \cdot x \in A)\} \\ &= A^{*U} \cap p^{-1}(\partial_1(U)) \\ &= A^{*U} \cap A^{\triangle U} \end{aligned}$$

(in [Lup], this is defined to be A^{*U}). We will usually be interested in the case where $U \subseteq G^1$ is open, but it is convenient to have the more general notation available. Here are some basic properties of the Vaught transforms:

Lemma 5.2.3. *(a)* $\neg A^{*U} = (\neg A)^{\triangle U}$.

(b) $A^{\triangle U}$ *preserves countable unions in each argument.*

(c) For any basis of open sets $V_i \subseteq G^1$, $A^{\triangle U} = \bigcup_{V_i \subseteq U} A^{\circledast V_i}$. *(It is enough to assume that the* V_i *form a "*∂_1*-fiberwise weak basis for* U*", i.e., every nonempty open subset of a* ∂_1*-fiber of* U *contains a nonempty* ∂_1*-fiber of some* $V_i \subseteq U$*.)*

(d) For U *open,* $(A^{*U})^{*V} = A^{*(U \cdot V)}$ *(where* $U \cdot V = \{g \cdot h \mid g \in U \;\&\; h \in V \;\&\; \partial_1(g) = \partial_0(h)\}$*).*

(e) If $\partial_0^{-1}(p(A)) \cap U = \partial_0^{-1}(p(A)) \cap V$*, then* $A^{\triangle U} = A^{\triangle V}$.

Proof. (a)–(c) are standard (see [Lup, 2.10.2]).

(e) is trivial, amounting to $g \cdot x \in A \implies g \in \partial_0^{-1}(\partial_0(g)) = \partial_0^{-1}(p(g \cdot x)) \subseteq \partial_0^{-1}(p(A))$.

For (d), we have

$$x \in (A^{*U})^{*V} \iff \forall^* h \in \partial_1^{-1}(p(x)) \cap V\, \forall^* g \in \partial_1^{-1}(\partial_0(h)) \cap U\,(g \cdot h \cdot x \in A);$$

applying the Kuratowski–Ulam theorem for open maps (see [MT, A.1]) to the projection $U \times_{G^0} (\partial_1^{-1}(p(x)) \cap V) \to \partial_1^{-1}(p(x)) \cap V$ (a pullback of $\partial_1 : U \to G^0$, hence open) yields

$$\iff \forall^*(g, h) \in U \times_{G^0} (\partial_1^{-1}(p(x)) \cap V)\,(g \cdot h \cdot x \in A);$$

applying Kuratowski–Ulam to the multiplication $\mu : U \times_{G^0} (\partial_1^{-1}(p(x)) \cap V) \to \partial_1^{-1}(p(x))$ yields

$$\begin{aligned} &\iff \forall^* k \in \partial_1^{-1}(p(x))\, \forall^*(g,h) \in \mu^{-1}(k)\, (k \cdot x \in A) \\ &\iff \forall^* k \in \partial_1^{-1}(p(x)) \cap (U \cdot V)\, (k \cdot x \in A) \\ &\iff x \in A^{*(U \cdot V)}. \end{aligned}$$

□

5.3 Étalé realizations of fiberwise countable Borel actions

In this section, we prove Theorem 5.1.5.

Let G be an open Polish groupoid, $p : X \to G^0$ be a Borel G-space, and $\mathcal{U}$ be a countable basis of open sets in G^1. By the proof of [Lup, 4.1.1] (the Becker–Kechris theorem for Polish groupoid actions), if we let $\mathcal{A}$ be a countable Boolean algebra of Borel subsets of X generating a Polish topology and closed under $A \mapsto A^{\triangle U}$ for each $U \in \mathcal{U}$, then

$$\mathcal{A}^{\triangle \mathcal{U}} := \{A^{\triangle U} \mid A \in \mathcal{A} \ \&\ U \in \mathcal{U}\}$$

generates a topology making X into a Polish G-space.

Lemma 5.3.1. *Under these hypotheses, $\mathcal{A}^{\triangle \mathcal{U}}$ forms a basis for a topology.*

Proof. Clearly $X = X^{\triangle G^1} = \bigcup_{U \in \mathcal{U}} X^{\triangle U}$ is covered by $\mathcal{A}^{\triangle \mathcal{U}}$. Let $x \in A_1^{\triangle U_1} \cap A_2^{\triangle U_2}$ where $A_1, A_2 \in \mathcal{A}$ and $U_1, U_2 \in \mathcal{U}$; we must find $A_3 \in \mathcal{A}$ and $U_3 \in \mathcal{U}$ such that $x \in A_3^{\triangle U_3} \subseteq A_1^{\triangle U_1} \cap A_2^{\triangle U_2}$. Let $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$ be open so that

$$x \in A_1^{\circledast V_1} \cap A_2^{\circledast V_2}.$$

Let $h_1 \in \partial_1^{-1}(p(x)) \cap V_1$ and $h_2 \in \partial_1^{-1}(p(x)) \cap V_2$. We have $h_2 = (h_2 \cdot h_1^{-1}) \cdot h_1$, so by continuity of multiplication, there are open $V_3 \ni h_1$ and $V_4 \ni h_2 \cdot h_1^{-1}$ such that $V_4 \cdot V_3 \subseteq V_2$. Let $U_3, U_4 \in \mathcal{U}$ be such that $h_2 \cdot h_1^{-1} \in U_4 \subseteq V_4$ and $h_1 \in U_3 \subseteq V_1 \cap V_3 \cap \partial_0^{-1}(\partial_1(U_4))$; the latter is possible since $\partial_0(h_1) = \partial_1(h_2 \cdot h_1^{-1}) \in \partial_1(U_4)$. Then $U_4 \cdot U_3 \subseteq V_4 \cdot V_3 \subseteq V_2$, so since $h_1 \in \partial_1^{-1}(p(x)) \cap U_3 \neq \varnothing$, from $x \in A_2^{\circledast V_2}$ we get

$$x \in A_2^{\circledast (U_4 \cdot U_3)} \subseteq (A_2^{*U_4})^{\circledast U_3}.$$

Since also $U_3 \subseteq V_1$, $\partial_1^{-1}(p(x)) \cap U_3 \neq \varnothing$, and $x \in A_1^{\circledast V_1}$, we have $x \in A_1^{\circledast U_3}$, so

$$x \in A_1^{\circledast U_3} \cap (A_2^{*U_4})^{\circledast U_3} = (A_1 \cap A_2^{*U_4})^{\circledast U_3} \subseteq (A_1 \cap A_2^{*U_4})^{\triangle U_3}.$$

Now suppose $y \in (A_1 \cap A_2^{*U_4})^{\triangle U_3}$. Then $y \in A_1^{\triangle U_3} \subseteq A_1^{\triangle V_1} \subseteq A_1^{\triangle U_1}$. Let $W \subseteq U_3$ be open so that

$$y \in (A_2^{*U_4})^{\circledast W}.$$

Since $W \subseteq U_3 \subseteq \partial_0^{-1}(\partial_1(U_4))$, it is easily seen that $(A_2^{*U_4})^{\circledast W} = A_2^{\circledast(U_4 \cdot W)}$, whence

$$y \in A_2^{\circledast(U_4 \cdot W)} \subseteq A_2^{\triangle(U_4 \cdot W)} \subseteq A_2^{\triangle(U_4 \cdot U_3)} \subseteq A_2^{\triangle V_2} \subseteq A_2^{\triangle U_2}.$$

Thus $(A_1 \cap A_2^{*U_4})^{\triangle U_3} \subseteq A_1^{\triangle U_1} \cap A_2^{\triangle U_2}$. Put $A_3 := A_1 \cap A_2^{*U_4}$. □

Lemma 5.3.2. *Let $A \subseteq X$ be a Borel section (i.e., $p|A : A \to G^0$ is injective) and let $U \subseteq G^1$ be ∂_1-fiberwise open. Then $A^{\circledast U}$ is also a Borel section. Furthermore, if $U, V \subseteq G^1$ are open with $V \cdot V^{-1} \subseteq U^{-1} \cdot U$, then $V^{-1} \cdot A^{\circledast U}$ (and hence $(A^{\circledast U})^{\triangle V} \subseteq V^{-1} \cdot A^{\circledast U}$) is also a Borel section.*

Proof. Let $x, y \in A^{\circledast U}$ with $p(x) = p(y)$. Then

$$\forall^* g \in \partial_1^{-1}(p(x)) \cap U = \partial_1^{-1}(p(y)) \cap U \; (g \cdot x, g \cdot y \in A).$$

Let $g \in \partial_1^{-1}(p(x)) \cap U$ such that $g \cdot x, g \cdot y \in A$. Then $p(g \cdot x) = \partial_0(g) = p(g \cdot y)$, so since A is a Borel section, $g \cdot x = g \cdot y$, whence $x = y$. So $A^{\circledast U}$ is a Borel section.

Now suppose U, V are open, and let $x, y \in V^{-1} \cdot A^{\circledast U}$ with $p(x) = p(y)$. Let $g \in \partial_1^{-1}(p(x)) \cap V$ and $h \in \partial_1^{-1}(p(y)) \cap V$ with $g \cdot x, h \cdot y \in A^{\circledast U}$. Since $h \cdot g^{-1} \in V \cdot V^{-1} \subseteq U^{-1} \cdot U$, there are $k, l \in U$ such that $h \cdot g^{-1} = l^{-1} \cdot k$, i.e., $k \cdot g = l \cdot h$. In particular, $k \cdot g = l \cdot h \in \partial_1^{-1}(p(x)) \cap (U \cdot g) \cap (U \cdot h) \neq \varnothing$, so that from $g \cdot x \in A^{\circledast U}$ we get $x \in A^{\circledast(U \cdot g)}$ and hence $x \in A^{\circledast((U \cdot g) \cap (U \cdot h))}$, and similarly $y \in A^{\circledast((U \cdot g) \cap (U \cdot h))}$. By the first claim, $x = y$. □

Lemma 5.3.3. *For Borel $A \subseteq X$ and open $W \subseteq G^1$,*

$$A^{\triangle W} = \bigcup \{ (A^{\circledast U})^{\triangle V} \mid U, V \in \mathcal{U} \;\&\; U \cdot V \subseteq W \;\&\; V \cdot V^{-1} \subseteq U^{-1} \cdot U \}.$$

Proof. For $g \in W$, since $g = g \cdot 1_{\partial_1(g)}$ and μ is continuous, there are $U, V \in \mathcal{U}$ such that $U \cdot V \subseteq W$, $g \in U$, and $1_{\partial_1(g)} \in V$; thus

$$W = \bigcup \{ U \cdot V \mid U, V \in \mathcal{U} \;\&\; U \cdot V \subseteq W \}.$$

So

$$
\begin{aligned}
A^{\triangle W} &= A^{\triangle \bigcup\{U\cdot V \mid U,V\in\mathcal{U} \,\&\, U\cdot V\subseteq W\}} \\
&= \bigcup\{A^{\triangle(U\cdot V)} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W\} \\
&= \bigcup\{(A^{\triangle U})^{\triangle V} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W\} \\
&= \bigcup\{(\bigcup\{A^{\circledast U'} \mid U \supseteq U' \in \mathcal{U}\})^{\triangle V} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W\} \\
&= \bigcup\{(A^{\circledast U'})^{\triangle V} \mid U',U,V \in \mathcal{U} \;\&\; U' \subseteq U \;\&\; U\cdot V \subseteq W\} \\
&= \bigcup\{(A^{\circledast U})^{\triangle V} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W\}.
\end{aligned}
$$

Since $\partial_0^{-1}(p(A^{\circledast U})) \cap V = \partial_0^{-1}(p(A^{\circledast U})) \cap \partial_0^{-1}(\partial_1(U)) \cap V$, we have

$$(A^{\circledast U})^{\triangle V} = (A^{\circledast U})^{\triangle(\partial_0^{-1}(\partial_1(U))\cap V)}.$$

For any open $U, V \subseteq G^1$ with $V \subseteq \partial_0^{-1}(\partial_1(U))$, for each $g \in V$, there is some $h \in U$ with $\partial_0(g) = \partial_1(h)$, whence $g \cdot g^{-1} = 1_{\partial_0(g)} = 1_{\partial_1(h)} = h^{-1} \cdot h \in U^{-1} \cdot U$, whence by continuity of μ there are open $V_1 \ni g$ and $V_2 \ni g^{-1}$ such that $V_1 \cdot V_2 \subseteq U^{-1} \cdot U$, whence letting $V' \in \mathcal{U}$ with $g \in V' \subseteq V \cap V_1 \cap V_2^{-1}$, we have $V' \cdot V'^{-1} \subseteq U^{-1} \cdot U$; thus

$$V = \bigcup\{V' \in \mathcal{U} \mid V' \subseteq V \;\&\; V' \cdot V'^{-1} \subseteq U^{-1} \cdot U\}.$$

So from above we get

$$
\begin{aligned}
A^{\triangle W} &= \bigcup\{(A^{\circledast U})^{\triangle(\partial_0^{-1}(\partial_1(U))\cap V)} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W\} \\
&= \bigcup\{(A^{\circledast U})^{\triangle \bigcup\{V'\in\mathcal{U} \mid V' \subseteq \partial_0^{-1}(\partial_1(U))\cap V \,\&\, V'\cdot V'^{-1}\subseteq U^{-1}\cdot U\}} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W\} \\
&= \bigcup\{(A^{\circledast U})^{\triangle V'} \mid U,V,V' \in \mathcal{U} \;\&\; U\cdot V \subseteq W \;\&\; V' \subseteq \partial_0^{-1}(\partial_1(U)) \cap V \\
&\qquad\qquad \& \; V' \cdot V'^{-1} \subseteq U^{-1} \cdot U\} \\
&= \bigcup\{(A^{\circledast U})^{\triangle V} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W \;\&\; V \subseteq \partial_0^{-1}(\partial_1(U)) \\
&\qquad\qquad \& \; V \cdot V^{-1} \subseteq U^{-1} \cdot U\} \\
&= \bigcup\{(A^{\circledast U})^{\triangle(\partial_0^{-1}(\partial_1(U))\cap V)} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W \;\&\; V \cdot V^{-1} \subseteq U^{-1} \cdot U\} \\
&= \bigcup\{(A^{\circledast U})^{\triangle V} \mid U,V \in \mathcal{U} \;\&\; U\cdot V \subseteq W \;\&\; V \cdot V^{-1} \subseteq U^{-1} \cdot U\}. \qquad \square
\end{aligned}
$$

Lemma 5.3.4. *Let $p : X \to G^0$ and $q : Y \to G^0$ be Borel* G*-spaces, $f : X \to Y$ be a* G*-equivariant map, and $U \subseteq G^1$ be ∂_1-fiberwise open. Then for any $B \subseteq Y$, $f^{-1}(B^{\triangle U}) = f^{-1}(B)^{\triangle U}$. If furthermore f is fiberwise countable, then for any $A \subseteq X$, $f(A^{\triangle U}) = f(A)^{\triangle U}$.*

Proof. The first claim is straightforward. For the second claim, we have $A \subseteq f^{-1}(f(A))$ whence $A^{\triangle U} \subseteq f^{-1}(f(A))^{\triangle U} = f^{-1}(f(A)^{\triangle U})$ whence $f(A^{\triangle U}) \subseteq$

$f(A)^{\triangle U}$ (regardless of fiberwise countability). Conversely, if f is fiberwise countable and $y \in f(A)^{\triangle U}$, then

$$\{g \in \partial_1^{-1}(q(y)) \cap U \mid g \cdot y \in f(A)\} \subseteq \bigcup_{x \in f^{-1}(y)} \{g \in \partial_1^{-1}(p(x)) \cap U \mid g \cdot x \in A\}$$

since given g in the left-hand side we have $g \cdot y = f(x')$ for some $x' \in A$ whence we may take $x := g^{-1} \cdot x'$; since the left-hand side is non-meager and the union on the right is countable, some term in it is non-meager, i.e., there is $x \in f^{-1}(y)$ such that $x \in A^{\triangle U}$, whence $y = f(x) \in f(A^{\triangle U})$. □

Lemma 5.3.5. *Let $p : X \to G^0$ be a Polish G-space and $\mathcal{A}$ be a basis of open sets in X. Then $\mathcal{A}^{\triangle \mathcal{U}}$ is also a basis of open sets in X.*

Proof. Let $B \subseteq X$ be open and $x \in B$. Since $1_{p(x)} \cdot x = x$ and the action is continuous, there are $1_{p(x)} \in U \in \mathcal{U}$ and $x \in A \in \mathcal{A}$ such that $U^{-1} \cdot A \subseteq B$, as well as $1_{p(x)} \in V \in \mathcal{U}$ such that $V \cdot x \subseteq A$. Then $(U \cap V) \cdot x \subseteq A$, whence $x \in A^{\triangle U} \subseteq U^{-1} \cdot A \subseteq B$. □

Proof of Theorem 5.1.5. Let $\mathcal{A}, \mathcal{B}$ be countable Boolean algebras of Borel sets in X, G^0 respectively, such that

(i) each generates a Polish topology and is closed under $(-)^{\triangle U}$ for each $U \in \mathcal{U}$ (where for $\mathcal{B}$ this refers to the trivial action of G on G^0);

(ii) $\mathcal{A}$ contains a countable cover of X by Borel sections (which exists by Lusin–Novikov uniformization; see [Kec, 18.10]);

(iii) $p(A) \in \mathcal{B}$ for every $A \in \mathcal{A}$;

(iv) $\mathcal{B}$ contains a countable basis of open sets in G^0;

(v) $p^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.

It is clear that this can be achieved via a ω-length procedure like in the proof of [BK, 5.2.1]. Let X', G'^0 be X, G^0 with the topologies generated by $\mathcal{A}^{\triangle \mathcal{U}}, \mathcal{B}^{\triangle \mathcal{U}}$ respectively. By the proof of [Lup, 4.1.1], X', G'^0 are Polish G-spaces. By Lemma 5.3.4 and (v), $p : X' \to G'^0$ is continuous. By Lemma 5.3.5 and (iv), the topology of G'^0 is finer than that of G^0. Let $\mathsf{G}' = (G'^0, G'^1)$ be the **action groupoid** of G'^0, where $G'^1 = G^1 \times_{G^0} G'^0$ is G^1 with $\partial_1^{-1}(U)$ adjoined to its topology for each open $U \subseteq G'^0$, with composition, unit, and inversion in G' as in G. Then G' is an open Polish groupoid (see [Lup, 2.7.1]), and we have a continuous functor $\mathsf{G}' \to \mathsf{G}$, namely the identity, which when composed with the action $\mathsf{G} \curvearrowright X'$ gives a continuous action $\mathsf{G}' \curvearrowright X'$. This action is countable étalé, since by Lemmas 5.3.1 to 5.3.3 and (ii) X' has a countable basis of open sets of the form $A^{\triangle U}$ which are Borel sections, and by Lemma 5.3.4 and (iii) the image of each such set is open in G'^0. □

We end this section with a general question concerning Polish groupoids:

Problem 5.3.6. Let G, H be (open) Polish groupoids and $F : \mathsf{G} \to \mathsf{H}$ be a Borel functor. Is there necessarily a finer (open) Polish groupoid topology on G which renders F continuous?

To motivate this problem, recall that every Borel homomorphism between Polish *groups* is automatically continuous (see e.g., [Kec, 9.10]). Naive form of this statement for Borel functors between Polish groupoids are false. For example, there is a Polish groupoid G and a Borel endofunctor $F : \mathsf{G} \to \mathsf{G}$ which is the identity (hence continuous) on objects but not continuous: take $2^{\aleph_0}$ many disjoint copies of a Polish group with a nontrivial automorphism, and apply that automorphism on a Borel but non-clopen set of objects. Problem 5.3.6 is a weaker analog of automatic continuity for Borel functors, meant to exclude such trivial counterexamples.

If Problem 5.3.6 has a positive solution, then that would imply Theorem 5.1.5, since every fiberwise countable Borel action of G can be encoded as a Borel functor $\mathsf{G} \to \mathsf{S}$ where S is the disjoint union of the symmetric groups $S_0, S_1, \dots, S_\infty$; see Lemma 5.11.2.

5.4 Imaginary sorts and definable functions

In this section, we review $\mathcal{L}_{\omega_1\omega}$ and the notions of countable fragments, ω_1-coherent formulas, imaginary sorts, definable functions, and the syntactic ω_1-pretopos of a theory.

Let $\mathcal{L}$ be a first-order language. For simplicity, we will only consider relational languages; functions may be coded via their graphs in the usual way. Recall that the logic $\mathcal{L}_{\omega_1\omega}$ is the extension of finitary first-order logic $\mathcal{L}_{\omega\omega}$ with countably infinite conjunctions $\bigwedge$ and disjunctions $\bigvee$; see e.g., [Gao, 11.2]. By $\mathcal{L}_{\omega_1\omega}$**-formula**, we always mean a formula with finitely many free variables.

We adopt the following convention regarding formulas and free variables. A **formula-in-context** is a pair $(\vec{x}, \phi(\vec{x}))$ where $\vec{x}$ is a finite tuple of distinct variables and $\phi(\vec{x})$

is a formula with free variables among $\vec{x}$. We identify formulas-in-context up to variable renaming, i.e., $(\vec{x}, \phi(\vec{x})) = (\vec{y}, \phi(\vec{y}))$. *By abuse of terminology, henceforth by "formula" we always mean "formula-in-context"; we denote a formula-in-context* $(\vec{x}, \phi(\vec{x}))$ *simply by* ϕ.

Given a formula ϕ with n variables and an $\mathcal{L}$-structure $\mathcal{M} = (M, R^{\mathcal{M}})_{R \in \mathcal{L}}$, we write $\phi^{\mathcal{M}} \subseteq M^n$ for the **interpretation of ϕ in $\mathcal{M}$**. For an n-tuple $\vec{a} \in M$, we write $\phi^{\mathcal{M}}(\vec{a})$ or $\vec{a} \in \phi^{\mathcal{M}}$ interchangeably.

There is a Gentzen-type proof system for $\mathcal{L}_{\omega_1\omega}$, which can be found in [Lop] or (in a slightly different presentation) [J02, D1.3]. By the Lopez-Escobar completeness theorem [Lop], this proof system is complete for a *countable* theory $\mathcal{T}$: if an $\mathcal{L}_{\omega_1\omega}$-sentence ϕ is true in every countable model of $\mathcal{T}$, then ϕ is provable from $\mathcal{T}$. In the following definitions, we will often refer to provability, while keeping in mind that this is equivalent to semantic implication in the case of countable theories by soundness and completeness. In particular, when we say that two formulas are "equivalent" or "$\mathcal{T}$-equivalent", we mean that they are provably so.

It is convenient to consider the following restriction of $\mathcal{L}_{\omega_1\omega}$. An **$\omega_1$-coherent $\mathcal{L}$-formula** is an $\mathcal{L}_{\omega_1\omega}$-formula which uses only atomic formulas, finite $\wedge$ (including nullary $\top$), countable $\bigvee$, and $\exists$. Note that every ω_1-coherent formula ϕ is equivalent to one in the following **normal form**:

$$\phi(\vec{x}) = \bigvee_i \exists \vec{y}_i \, (\phi_{i1}(\vec{x}, \vec{y}_i) \wedge \cdots \wedge \phi_{ik_i}(\vec{x}, \vec{y}_i)),$$

where the ϕ_{ij} are atomic. An **ω_1-coherent $\mathcal{L}$-axiom** is an $\mathcal{L}_{\omega_1\omega}$-sentence of the form

$$\forall \vec{x} \, (\phi(\vec{x}) \Rightarrow \psi(\vec{x})),$$

where ϕ, ψ are ω_1-coherent $\mathcal{L}$-formulas. An **ω_1-coherent $\mathcal{L}$-theory** is a set of ω_1-coherent $\mathcal{L}$-axioms.

An ω_1-coherent $\mathcal{L}$-theory $\mathcal{T}$ is **decidable** if there is an ω_1-coherent $\mathcal{L}$-formula $\phi(x, y)$ with two free variables which is $\mathcal{T}$-equivalent to the formula $x \neq y$ (which is *not* ω_1-coherent). If such a formula $\phi(x, y)$ exists, we will generally denote it by $x \neq y$, it being understood that this refers to an ω_1-coherent compound formula and not the (non-ω_1-coherent) negated atomic formula.

A **fragment** $\mathcal{F}$ of $\mathcal{L}_{\omega_1\omega}$ is a set of $\mathcal{L}_{\omega_1\omega}$-formulas which contains all atomic formulas and is closed under subformulas, finitary first-order logical operations $\wedge, \vee, \neg, \exists, \forall$, and variable substitutions. An $\mathcal{F}$**-theory** is an $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ such that $\mathcal{T} \subseteq \mathcal{F}$. The **Morleyization** of a fragment $\mathcal{F}$ is the ω_1-coherent theory $\mathcal{T}'$ in the language $\mathcal{L}'$ consisting of $\mathcal{L}$ together with a new relation symbol $R_\phi(\vec{x})$ for each $\mathcal{F}$-formula $\phi(\vec{x})$, whose axioms consist of

$$\begin{array}{lll}
\forall \vec{x}\,(& R_\phi(\vec{x}) \Leftrightarrow \phi(\vec{x}) &) \quad \text{for } \phi \text{ atomic}, \\
\forall \vec{x}\,(& R_{\bigvee_i \phi_i}(\vec{x}) \Leftrightarrow \bigvee_i R_{\phi_i}(\vec{x}) &), \\
\forall \vec{x}\,(& R_{\exists y\, \phi(\vec{x},y)}(\vec{x}) \Leftrightarrow \exists y\, R_\phi(\vec{x}, y) &), \\
\forall \vec{x}\,(& R_\phi(\vec{x}) \wedge R_{\neg\phi}(\vec{x}) \Rightarrow \bot &), \\
\forall \vec{x}\,(& \top \Rightarrow R_\phi(\vec{x}) \vee R_{\neg\phi}(\vec{x}) &), \\
\forall \vec{x}\,(& R_{\bigwedge_i \phi_i}(\vec{x}) \Rightarrow R_{\phi_j}(\vec{x}) &), \\
\forall \vec{x}\,(& \top \Rightarrow R_{\bigwedge_i \phi_i}(\vec{x}) \vee \bigvee_i R_{\neg\phi_i}(\vec{x}) &), \\
\forall \vec{x}\,(& R_{\forall y\, \phi(\vec{x},y)}(\vec{x}) \Leftrightarrow R_{\neg\exists y\, \neg\phi(\vec{x},y)}(\vec{x}) &)
\end{array} \tag{5.4.Mor}$$

whenever the formulas in the subscripts belong to $\mathcal{F}$ (where the axioms with $\Leftrightarrow$ really abbreviate two ω_1-coherent axioms, $\Leftarrow$ and $\Rightarrow$). Note that if $\mathcal{F}$ is countable, then so is $\mathcal{T}'$. The **Morleyization** of an $\mathcal{F}$-theory $\mathcal{T}$ is defined in the same way, except that $\mathcal{T}'$ also includes the axiom

$$R_\phi$$

(which is a nullary relation symbol) for each axiom ϕ in $\mathcal{T}$. The Morleyization is a decidable theory, as witnessed by the formula $R_{\neq}(x, y)$. For more on Morleyization, see [Hod, §2.6] or [J02, D1.5.13]. An $\mathcal{F}$-theory is "equivalent" to its Morleyization, in the following sense:

Lemma 5.4.1. *Let $\mathcal{F}$ be a fragment of $\mathcal{L}_{\omega_1\omega}$, $\mathcal{T}$ be an $\mathcal{F}$-theory, and $(\mathcal{L}', \mathcal{T}')$ be its Morleyization.*

(i) $\mathcal{L}_{\omega_1\omega}$-formulas modulo $\mathcal{T}$-equivalence are in canonical bijection with $\mathcal{L}'_{\omega_1\omega}$-formulas modulo $\mathcal{T}'$-equivalence.

(ii) Countable disjunctions of $\mathcal{F}$-formulas modulo $\mathcal{T}$-equivalence are in canonical bijection with ω_1-coherent $\mathcal{L}'$-formulas modulo $\mathcal{T}'$-equivalence.

(iii) Models of $\mathcal{T}$ are in canonical bijection with models of $\mathcal{T}'$.

Proof. By an easy induction, $\mathcal{T}' \vdash \forall \vec{x}\,(R_\phi(\vec{x}) \Leftrightarrow \phi(\vec{x}))$ for every $\phi \in \mathcal{F}$. Since $R_\phi \in \mathcal{T}'$ for every $\phi \in \mathcal{T}$, it follows that $\mathcal{T}'$ proves every axiom in $\mathcal{T}$.

For (i), if $\mathcal{T}$ proves an $\mathcal{L}_{\omega_1\omega}$-sentence ϕ, then also $\mathcal{T}' \vdash \phi$; thus if two $\mathcal{L}_{\omega_1\omega}$-formulas ϕ, ψ are $\mathcal{T}$-equivalent, then they are also $\mathcal{T}'$-equivalent. Conversely, for an $\mathcal{L}'_{\omega_1\omega}$-formula ψ, an easy induction shows that $\mathcal{T}' \vdash \forall \vec{x}\, (\psi \Leftrightarrow \psi')$ where ψ' is ψ with every R_ϕ replaced by ϕ; thus every $\mathcal{L}'_{\omega_1\omega}$-formula is $\mathcal{T}'$-equivalent to an $\mathcal{L}_{\omega_1\omega}$-formula. Furthermore, for an $\mathcal{L}'_{\omega_1\omega}$-sentence ψ such that $\mathcal{T}' \vdash \psi$, if we take the proof of ψ from $\mathcal{T}'$ (in the proof system in [Lop] or [J02, D1.3]) and replace every R_ϕ in it with ϕ, we obtain a proof tree whose root node is ψ' and whose leaves (i.e., axioms of $\mathcal{T}'$) are all either tautologies (for one of the axioms (5.4.Mor)) or axioms in $\mathcal{T}$ (for $R_\phi \in \mathcal{T}'$ where $\phi \in \mathcal{T}$), whence $\mathcal{T} \vdash \psi'$. Thus if two $\mathcal{L}'_{\omega_1\omega}$-formulas ψ, θ are $\mathcal{T}'$-equivalent, then ψ', θ' are $\mathcal{T}$-equivalent.

For (ii), a countable disjunction $\bigvee_i \phi_i$ of $\mathcal{F}$-formulas ϕ_i is $\mathcal{T}'$-equivalent to $\bigvee_i R_{\phi_i}$, which is a ω_1-coherent $\mathcal{L}'$-formula. Conversely, an ω_1-coherent $\mathcal{L}'$-formula is equivalent to one in normal form $\psi(\vec{x}) = \bigvee_i \exists \vec{y}_i\, (\psi_{i1}(\vec{x}, \vec{y}_i) \wedge \cdots \wedge \psi_{ik_i}(\vec{x}, \vec{y}_i))$, where the ψ_{ij} are either equalities or some $R_{\phi_{ij}}$ which is $\mathcal{T}'$-equivalent to ϕ_{ij}; hence $\exists \vec{y}_i\, (\psi_{i1}(\vec{x}, \vec{y}_i) \wedge \cdots \wedge \psi_{ik_i}(\vec{x}, \vec{y}_i))$ is $\mathcal{T}'$-equivalent to an $\mathcal{F}$-formula (since $\mathcal{F}$ is closed under $\wedge, \exists$), and so ψ is $\mathcal{T}'$-equivalent to a countable disjunction of $\mathcal{F}$-formulas.

For (iii), since $\mathcal{T}' \vdash \mathcal{T}$, the $\mathcal{L}$-reduct of a model of $\mathcal{T}'$ is a model of $\mathcal{T}$; conversely, for a model $\mathcal{M}$ of $\mathcal{T}$, an easy induction shows that the unique $\mathcal{L}'$-expansion of $\mathcal{M}$ which satisfies $\mathcal{T}'$ is given by $R_\phi^{\mathcal{M}} := \phi^{\mathcal{M}}$ for each $\phi \in \mathcal{F}$. □

Let $(\mathcal{L}, \mathcal{T})$ be an ω_1-coherent theory. An **ω_1-coherent $\mathcal{T}$-imaginary sort** A is a pair $A = ((\alpha_i)_{i \in I}, (\varepsilon_{ij})_{i,j \in I})$ consisting of countable families of ω_1-coherent formulas $\alpha_i(\vec{x}_i)$ (with possibly different numbers of free variables, say $n_i := |\vec{x}_i|$) and $\varepsilon_{ij}(\vec{x}_i, \vec{x}_j)$, such that $\mathcal{T}$ proves the following sentences which say that "$\bigsqcup_{i,j} \varepsilon_{ij}$ is an equivalence relation on $\bigsqcup_i \alpha_i$":

$$\begin{aligned}
\forall \vec{x}, \vec{y}\, (\quad & \varepsilon_{ij}(\vec{x}, \vec{y}) \Rightarrow \alpha_i(\vec{x}) \wedge \alpha_j(\vec{y})), \\
\forall \vec{x}\, (\quad & \alpha_i(\vec{x}) \Rightarrow \varepsilon_{ii}(\vec{x}, \vec{x})\quad), \\
\forall \vec{x}, \vec{y}\, (\quad & \varepsilon_{ij}(\vec{x}, \vec{y}) \Rightarrow \varepsilon_{ji}(\vec{y}, \vec{x})\quad), \\
\forall \vec{x}, \vec{y}, \vec{z}\, (\varepsilon_{ij}(\vec{x}, \vec{y}) \wedge {} & \varepsilon_{jk}(\vec{y}, \vec{z}) \Rightarrow \varepsilon_{ik}(\vec{x}, \vec{z})\quad).
\end{aligned} \tag{5.4.Eqv}$$

For $(\mathcal{L}, \mathcal{T})$ countable, using the completeness theorem, this is easily seen to be equivalent to the following: in every countable model $\mathcal{M} = (M, R)_{R \in \mathcal{L}}$ of $\mathcal{T}$, the set

$$\textstyle\bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}} \subseteq \bigsqcup_{i,j} (M^{n_i} \times M^{n_j}) \cong (\bigsqcup_i M^{n_i})^2$$

is an equivalence relation on

$$\textstyle\bigsqcup_i \alpha_i^{\mathcal{M}} \subseteq \bigsqcup_i M^{n_i}.$$

The **interpretation of** A **in** $\mathcal{M}$ is the quotient set

$$A^{\mathcal{M}} := (\bigsqcup_i \alpha_i^{\mathcal{M}})/(\bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}}).$$

We will denote the imaginary sort $A = ((\alpha_i)_i, (\varepsilon_{ij})_{i,j})$ itself suggestively by

$$A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij}).$$

We identify a single formula α with the imaginary sort given by α quotiented by the equality relation (i.e., the imaginary sort $((\alpha), (\varepsilon))$ where $\varepsilon(\vec{x}, \vec{y}) = \alpha(\vec{x}) \wedge (\vec{x} = \vec{y})$). Note that the notation $\phi^{\mathcal{M}}$ means the same thing whether we regard ϕ as a formula or as an imaginary sort. Likewise, for countably many formulas α_i, we write $\bigsqcup_i \alpha_i$ for the corresponding imaginary sort where the equivalence relation is equality.

Let $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$ and $B = (\bigsqcup_k \beta_k)/(\bigsqcup_{k,l} \eta_{kl})$ be two $\mathcal{T}$-imaginary sorts, where $\alpha_i = \alpha_i(\vec{x}_i)$ and $\beta_k = \beta_k(\vec{y}_k)$, say. An ω_1**-coherent** $\mathcal{T}$**-definable function** $f : A \to B$ is a $\mathcal{T}$-equivalence class $f = [(\phi_{ik})_{i,k}]$ of families of formulas $\phi_{ik}(\vec{x}_i, \vec{y}_k)$ such that $\mathcal{T}$ proves the following sentences which say that "$\bigsqcup_{i,k} \phi_{ik} \subseteq (\bigsqcup_i \alpha_i) \times (\bigsqcup_k \beta_k)$ is the lift of the graph of a function $A \to B$":

$$\begin{aligned}
&\forall \vec{x}, \vec{y}\, (&& \phi_{ik}(\vec{x}, \vec{y}) \Rightarrow \alpha_i(\vec{x}) \wedge \beta_k(\vec{y}) \ ), \\
&\forall \vec{x}, \vec{x}', \vec{y}\, (&& \phi_{ik}(\vec{x}, \vec{y}) \wedge \varepsilon_{ij}(\vec{x}, \vec{x}') \Rightarrow \phi_{jk}(\vec{x}', \vec{y}) \ ), \\
&\forall \vec{x}, \vec{y}, \vec{y}'\, (&& \phi_{ik}(\vec{x}, \vec{y}) \wedge \eta_{kl}(\vec{y}, \vec{y}') \Rightarrow \phi_{il}(\vec{x}, \vec{y}') \ ), \\
&\forall \vec{x}, \vec{y}, \vec{y}'\, (&& \phi_{ik}(\vec{x}, \vec{y}) \wedge \phi_{il}(\vec{x}, \vec{y}') \Rightarrow \eta_{kl}(\vec{y}, \vec{y}') \ ), \\
&\forall \vec{x}\, (&& \alpha_i(\vec{x}) \Rightarrow \textstyle\bigvee_k \exists \vec{y}\, \phi_{ik}(\vec{x}, \vec{y})).
\end{aligned} \tag{5.4.Fun}$$

Again by the completeness theorem, for $(\mathcal{L}, \mathcal{T})$ countable this is equivalent to: in every countable model $\mathcal{M}$ of $\mathcal{T}$, $\bigsqcup_{i,k} \phi_{ik}^{\mathcal{M}}$ is the lift of the graph of a function

$$f^{\mathcal{M}} : A^{\mathcal{M}} \to B^{\mathcal{M}},$$

the **interpretation of** f **in** $\mathcal{M}$.

The **identity function** on $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$ is

$$1_A := [(\varepsilon_{ij})_{i,j}] : A \to A.$$

Given also $B = (\bigsqcup_k \beta_k)/(\bigsqcup_{k,l} \eta_{kl})$ and $C = (\bigsqcup_m \gamma_m)/(\bigsqcup_{m,n} \xi_{m,n})$ and definable functions $f = [(\phi_{ik})_{i,k}] : A \to B$ and $g = [(\psi_{km})_{k,m}] : B \to C$, their **composite** is $g \circ f := [(\theta_{im})_{i,m}] : A \to C$ where

$$\theta_{im}(\vec{x}, \vec{z}) := \textstyle\bigvee_k \exists \vec{y}\, (\phi_{ik}(\vec{x}, \vec{y}) \wedge \psi_{km}(\vec{y}, \vec{z})).$$

It is straightforward to verify (by explicitly writing down formal proofs) that 1_A and $g \circ f$ are definable functions and that composition is associative and unital.

The **syntactic ω_1-pretopos** of an ω_1-coherent theory $(\mathcal{L}, \mathcal{T})$ is the category of imaginary sorts and definable functions, denoted

$$\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}.$$

The categorical structure in the syntactic ω_1-pretopos encodes the logical structure of the theory:

Remark 5.4.2.

- There is an object $\mathbb{X} \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, the **home sort**, given by the true formula $\top(x)$ in one variable (quotiented by the equality relation), whose interpretation in a model $\mathcal{M} = (M, R^{\mathcal{M}})_{R \in \mathcal{L}}$ is the underlying set M.

- The categorical **product** $\mathbb{X}^n$ of n copies of the home sort $\mathbb{X}$ is given by the true formula $\top(x_0, \dots, x_{n-1})$ in n variables, with the ith projection $p_i : \mathbb{X}^n \to \mathbb{X}$ defined by the formula $\pi_i(x_0, \dots, x_{n-1}, y) = (x_i = y)$.

  More generally, given two imaginary sorts $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$ and $B = (\bigsqcup_k \beta_k)/(\bigsqcup_{k,l} \eta_{kl})$, their **product** is $A \times B = (\bigsqcup_{i,k} \gamma_{ik})/(\bigsqcup_{i,k,j,l} \xi_{ikjl})$, where $\gamma_{ik}(\vec{x}, \vec{y}) = \alpha_i(\vec{x}) \wedge \beta_k(\vec{y})$ and $\xi_{ikjl}(\vec{x}, \vec{y}, \vec{z}, \vec{w}) = \varepsilon_{ij}(\vec{x}, \vec{z}) \wedge \eta_{kl}(\vec{y}, \vec{w})$, so that $(A \times B)^{\mathcal{M}} \cong A^{\mathcal{M}} \times B^{\mathcal{M}}$.

- Recall that a **subobject** of $\mathbb{X}^n$ is an equivalence class of monomorphisms $A \hookrightarrow \mathbb{X}^n$; as usual, we will abuse terminology and also refer to single monomorphisms as subobjects. Every formula α with n variables yields a subobject $\alpha \hookrightarrow \mathbb{X}^n$ given by the identity function 1_α (as defined above, but regarded as a definable function $\alpha \to \mathbb{X}^n$), with two such subobjects α, β being equal iff $\mathcal{T} \vdash \alpha \Leftrightarrow \beta$. Conversely, every subobject of $\mathbb{X}^n$ is of this form; so there is an (order-preserving) bijection between subobjects of $\mathbb{X}^n$ and $\mathcal{T}$-equivalence classes of formulas with n variables.

  More generally, given an imaginary sort $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij}) \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, the **subobjects** of A are in bijection with "subsorts" or **definable relations** on A, i.e., families of formulas $(\beta_i)_i$ such that $\mathcal{T} \vdash \beta_i \Rightarrow \alpha_i$ and $\mathcal{T}$ proves that "$\bigsqcup_i \beta_i \subseteq \bigsqcup_i \alpha_i$ is $(\bigsqcup_{i,j} \varepsilon_{ij})$-invariant".

- Given two definable functions $f, g : A \to B$, say $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$, $B = (\bigsqcup_k \beta_k)/(\bigsqcup_{k,l} \eta_{kl})$, $f = [(\phi_{ik})_{i,k}]$, and $g = [(\psi_{ik})_{i,k}]$, their **equalizer** is the subsort $A' \subseteq A$ on which f, g are equal, given by $A' = (\bigsqcup_i \alpha'_i)/(\bigsqcup_{i,j} \varepsilon'_{ij})$ where $\alpha'_i(\vec{x}) = \bigvee_k \exists \vec{y}\, (\phi_{ik}(\vec{x}, \vec{y}) \wedge \psi_{ik}(\vec{x}, \vec{y}))$ and $\varepsilon'_{ij}(\vec{x}, \vec{x}') = \alpha'_i(\vec{x}) \wedge \alpha'_i(\vec{x}') \wedge \varepsilon_{ij}(\vec{x}, \vec{x}')$. In particular, the equalizer of the two projections $\mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is (equivalent to) the equality formula $x = y$.

- Intersection (pullback) of subobjects corresponds to taking conjunction of formulas.
- Union (join in subobject lattice) of subobjects corresponds to taking disjunction of formulas.
- For a subobject $A \hookrightarrow \mathbb{X}^n \times \mathbb{X}$ corresponding to a formula $\phi(\vec{x}, y)$, the formula $\exists y\, \phi(\vec{x}, y)$ corresponds to the subobject of $\mathbb{X}^n$ given by the image of the composite $A \hookrightarrow \mathbb{X}^n \times \mathbb{X} \to \mathbb{X}^n$, where the second map is the projection.

The proofs of these statements are straightforward syntactic calculations; see [J02, D1.4].

There is an alternative, multi-step construction of the syntactic ω_1-pretopos. First, one defines the **syntactic category** $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\omega_1}$ in the same way as $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, except that instead of imaginary sorts, one considers only single ω_1-coherent $\mathcal{L}$-formulas (representing definable subsets); see [MR, 8.1.3] or [J02, D1.4]. This category already has all of the structure encoding logical operations in the list above. To form $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, one "completes" $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\omega_1}$ by first freely adjoining countable disjoint unions of objects, and then freely adjoining quotients of equivalence relations. This may be done either directly on the categorical level (see [J02, A1.4.5, A3.3.10]), or syntactically, by considering multi-sorted ω_1-coherent theories (see [MR, 8.4.1]). We have chosen to combine these steps, for the sake of brevity.

The notations $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\omega_1}$ and $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ are meant to suggest that the syntactic category (resp., ω_1-pretopos) is the category "freely presented" by $(\mathcal{L}, \mathcal{T})$ under the categorical structures listed in Remark 5.4.2 (resp., plus countable disjoint unions and quotients of equivalence relations). For the precise sense in which this is true, see [J02, D1.4.12] or Section 5.10 below.

Two ω_1-coherent theories $(\mathcal{L}, \mathcal{T}), (\mathcal{L}', \mathcal{T}')$ are **(ω_1-coherently) Morita equivalent** if their syntactic ω_1-pretoposes are equivalent categories. Intuitively, this means that the two theories have the same logical structure, modulo different presentations.

Sometimes, it is convenient to change the definition of **imaginary sort** $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ to a $\mathcal{T}$*-equivalence class* of pairs $((\alpha_i)_i, (\varepsilon_{ij})_{i,j})$ of formulas. Doing so results in a definition of $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ which is equivalent to the original one, since if two imaginary sorts A, B (in the original sense) were $\mathcal{T}$-equivalent, then the identity function $1_A : A \to A$ is also an isomorphism $A \cong B$.

For a fragment $\mathcal{F}$ of $\mathcal{L}_{\omega_1\omega}$ and an $\mathcal{F}$-theory $\mathcal{T}$, we define its **syntactic** ω_1**-pretopos** $\overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1}$ to be that of its Morleyization. By Lemma 5.4.1, we may equivalently define $\overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1}$ in the same way as $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, but using only countable disjunctions of $\mathcal{F}$-formulas in the definitions of both "imaginary sort" and "definable function"; we call these $(\mathcal{F}, \mathcal{T})$**-imaginary sorts** (or simply $\mathcal{F}$**-imaginary sorts**) and $(\mathcal{F}, \mathcal{T})$**-definable functions**. (If we quotient by $\mathcal{T}$-equivalence in the definition of imaginary sort, the two definitions of $\overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1}$ become isomorphic and not merely equivalent.)

For an $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, we define its **syntactic Boolean** ω_1**-pretopos** $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ to be that of $\mathcal{T}$ regarded as a theory in the *uncountable* fragment of all $\mathcal{L}_{\omega_1\omega}$-formulas. By Lemma 5.4.1, this is equivalent (or isomorphic if we consider sorts modulo $\mathcal{T}$-equivalence) to taking the definition of $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ but allowing *arbitrary* $\mathcal{L}_{\omega_1\omega}$-formulas; we call the objects and morphisms $(\mathcal{L}_{\omega_1\omega}, \mathcal{T})$**-imaginary sorts** (or simply $\mathcal{L}_{\omega_1\omega}$**-imaginary sorts**) and $(\mathcal{L}_{\omega_1\omega}, \mathcal{T})$**-definable functions**. Note that since every $\mathcal{L}_{\omega_1\omega}$-formula is contained in a countable fragment, for $\mathcal{T}$ countable, $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ is the direct limit of $\overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1}$ as $\mathcal{F}$ varies over all countable fragments containing $\mathcal{T}$. Two theories $(\mathcal{L}, \mathcal{T})$ and $(\mathcal{L}', \mathcal{T}')$ are **(**$\mathcal{L}_{\omega_1\omega}$**-)Morita equivalent** if their syntactic Boolean ω_1-pretoposes are equivalent.

5.5 The groupoid of models

In this section, we define the space and groupoid of countable models of a theory. We will first consider the general case of an ω_1-coherent theory, and then specialize (via Morleyization) to the more familiar case of an $\mathcal{F}$-theory in a countable fragment $\mathcal{F}$.

Let $\mathcal{L}$ be a countable relational language. The **space of countable** $\mathcal{L}$**-structures** $\mathrm{Mod}(\mathcal{L})$ consists of countable $\mathcal{L}$-structures $\mathcal{M} = (M, R)_{R \in \mathcal{L}}$ whose underlying set M is an initial segment of $\mathbb{N}$ (i.e., one of $0, 1, 2, \dots, \mathbb{N}$, where as usual $n = \{0, \dots, n-1\}$ for $n \in \mathbb{N}$), equipped with the topology generated by the subbasic open sets denoted by the following symbols:

$$\llbracket |\mathbb{X}| \geq n \rrbracket := \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) \mid |M| \geq n\} \qquad \text{for } n \in \mathbb{N},$$

$$\llbracket R(\vec{a}) \rrbracket := \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) \mid \vec{a} \in M^n \ \ \& \ \ R^{\mathcal{M}}(\vec{a})\} \quad \text{for } n\text{-ary } R \in \mathcal{L} \text{ and } \vec{a} \in \mathbb{N}^n$$

(here "$\mathbb{X}$" is thought of as the home sort). We have a homeomorphism

$$\begin{aligned}\mathrm{Mod}(\mathcal{L}) &\cong \left\{ (x, y_R)_{R \in \mathcal{L}} \;\middle|\; \begin{array}{l} \forall a \in \mathbb{N}\,(x(a+1) = 1 \implies x(a) = 1) \;\; \& \\ \forall n\text{-ary } R \in \mathcal{L},\, \vec{a} \in \mathbb{N}^n\,(y_R(\vec{a}) = 1 \implies \forall i\,(x(a_i) = 1)) \end{array} \right\} \\ &\subseteq \mathbb{S}^{\mathbb{N}} \times \textstyle\prod_{n\text{-ary } R \in \mathcal{L}} \mathbb{S}^{\mathbb{N}^n}\end{aligned}$$

to a $\mathbf{\Pi}^0_2$ subset of a countable power of $\mathbb{S}$, whence $\mathrm{Mod}(\mathcal{L})$ is a quasi-Polish space.

For an $\mathcal{L}_{\omega_1\omega}$-formula ϕ with n variables and $\vec{a} \in \mathbb{N}^n$, we define

$$[\![\phi(\vec{a})]\!] := \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) \mid \vec{a} \in M^n \;\; \& \;\; \phi^{\mathcal{M}}(\vec{a})\}.$$

Note that the subbasic open set $[\![|\mathbb{X}| \geq n]\!]$ above can also be written as $[\![\top(0, \dots, n-1)]\!]$ (i.e., we consider the true formula $\top$ with n variables). Let us say that a **basic formula** is a finite conjunction of atomic relations $R(\vec{x})$. Thus, a countable basis of open sets in $\mathrm{Mod}(\mathcal{L})$ consists of $[\![\phi(\vec{a})]\!]$ for basic formulas ϕ.

By the usual induction on ϕ (see [Kec, 16.7]), $[\![\phi(\vec{a})]\!]$ is a Borel subset of $\mathrm{Mod}(\mathcal{L})$. Moreover if ϕ is ω_1-coherent, then it is easily seen that $[\![\phi(\vec{a})]\!]$ is open. It follows that for two ω_1-coherent formulas ϕ, ψ, $[\![\phi(\vec{a}) \Rightarrow \psi(\vec{a})]\!]$ is the union of a closed set and an open set, and hence that for an ω_1-coherent axiom ϕ, $[\![\phi]\!]$ is $\mathbf{\Pi}^0_2$. For a countable ω_1-coherent $\mathcal{L}$-theory $\mathcal{T}$, put

$$\mathrm{Mod}(\mathcal{L}, \mathcal{T}) := \textstyle\bigcap_{\phi \in \mathcal{T}} [\![\phi]\!] \subseteq \mathrm{Mod}(\mathcal{L}).$$

This is also a quasi-Polish space, the **space of countable models of $\mathcal{T}$**. We will continue to denote $[\![\phi(\vec{a})]\!] \cap \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \subseteq \mathrm{Mod}(\mathcal{L})$ by $[\![\phi(\vec{a})]\!]$; similarly with $[\![|\mathbb{X}| \geq n]\!]$.

For a countable fragment $\mathcal{F}$ of $\mathcal{L}_{\omega_1\omega}$ and a countable $\mathcal{F}$-theory $\mathcal{T}$, we define the **space of countable models of $\mathcal{T}$ with topology induced by $\mathcal{F}$** to be

$$\mathrm{Mod}(\mathcal{F}, \mathcal{T}) := \mathrm{Mod}(\mathcal{L}', \mathcal{T}')$$

where $(\mathcal{L}', \mathcal{T}')$ is the Morleyization of $\mathcal{T}$. Using Lemma 5.4.1, it is easily seen that $\mathrm{Mod}(\mathcal{F}, \mathcal{T})$ is equivalently the set of countable models of $\mathcal{T}$ on initial segments of $\mathbb{N}$, equipped with the topology generated by the sets

$$[\![\phi(\vec{a})]\!] \quad \text{for } \phi \in \mathcal{F}.$$

Since $\mathcal{F}$ is closed under $\neg$, the topology is zero-dimensional, hence regular, hence Polish. This is the usual definition of the topology induced by a countable fragment; see [Gao, 11.4].

For a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, we define its standard Borel **space of countable models** as

$$\mathrm{Mod}(\mathcal{L}, \mathcal{T}) := \mathrm{Mod}(\mathcal{F}, \mathcal{T})$$

for any countable fragment $\mathcal{F}$ containing $\mathcal{T}$. Since for two countable fragments $\mathcal{F}' \supseteq \mathcal{F} \supseteq \mathcal{T}$, the Polish topology induced by $\mathcal{F}'$ is clearly finer than that induced by $\mathcal{F}$, the standard Borel structure on $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ does not depend on which countable fragment we take. Moreover if $\mathcal{T}$ happens to be ω_1-coherent, the standard Borel structure on $\mathrm{Mod}(\mathcal{F}, \mathcal{T})$ is induced by the topology on $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$.

We now turn to isomorphisms between models. Let $\mathrm{Iso}(\mathcal{L})$ denote the set of triples $(\mathcal{N}, g, \mathcal{M})$ where $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ and $g : \mathcal{M} \cong \mathcal{N}$ is an isomorphism. Let

$$\begin{aligned}
\partial_0 : \mathrm{Iso}(\mathcal{L}) &\longrightarrow \mathrm{Mod}(\mathcal{L}) & (\mathcal{N}, g, \mathcal{M}) &\longmapsto \mathcal{N}, \\
\partial_1 : \mathrm{Iso}(\mathcal{L}) &\longrightarrow \mathrm{Mod}(\mathcal{L}) & (\mathcal{N}, g, \mathcal{M}) &\longmapsto \mathcal{M}, \\
\iota : \mathrm{Mod}(\mathcal{L}) &\longrightarrow \mathrm{Iso}(\mathcal{L}) & \mathcal{M} &\longmapsto (\mathcal{M}, 1_{\mathcal{M}}, \mathcal{M}), \\
\mu : \mathrm{Iso}(\mathcal{L}) \times_{\mathrm{Mod}(\mathcal{L})} \mathrm{Iso}(\mathcal{L}) &\longrightarrow \mathrm{Iso}(\mathcal{L}) & ((\mathcal{P}, h, \mathcal{N}), (\mathcal{N}, g, \mathcal{M})) &\longmapsto (\mathcal{P}, h \circ g, \mathcal{M}), \\
\nu : \mathrm{Iso}(\mathcal{L}) &\longrightarrow \mathrm{Iso}(\mathcal{L}) & (\mathcal{N}, g, \mathcal{M}) &\longmapsto (\mathcal{M}, g^{-1}, \mathcal{N}).
\end{aligned}$$

Equipped with these maps, $\mathsf{Mod}(\mathcal{L}) := (\mathrm{Mod}(\mathcal{L}), \mathrm{Iso}(\mathcal{L}))$ is a groupoid. We usually denote its morphisms by g instead of $(\mathcal{N}, g, \mathcal{M})$ when $\mathcal{M}, \mathcal{N}$ are clear from context. We equip $\mathrm{Iso}(\mathcal{L})$ with the topology generated by the subbasic open sets

$$\begin{aligned}
&\partial_1^{-1}(U) \quad \text{for } U \subseteq \mathrm{Mod}(\mathcal{L}) \text{ (subbasic) open}, \\
&[\![a \mapsto b]\!] := \{(\mathcal{N}, g, \mathcal{M}) \mid a, b \in M \;\&\; g(a) = b\} \quad \text{for } a, b \in \mathbb{N}.
\end{aligned}$$

It is easily verified that the maps $\partial_0, \partial_1, \iota, \mu, \nu$ are continuous, and that ∂_1 is open. Thus, $\mathsf{Mod}(\mathcal{L})$ is an open quasi-Polish groupoid, the **quasi-Polish groupoid of countable $\mathcal{L}$-structures.**

We note that the space $\mathrm{Iso}(\mathcal{L})$ can alternatively be regarded as consisting of pairs $(g, \mathcal{M})$ where $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ is a countable structure and $g \in S_M$ (the symmetric group on M) is a permutation of its underlying set. This definition of $\mathrm{Iso}(\mathcal{L})$ can be regarded as a subspace of $S_\infty \times \mathrm{Mod}(\mathcal{L})$ (consisting of the $(g, \mathcal{M}) \in S_\infty \times \mathrm{Mod}(\mathcal{L})$ such that g is the identity outside of M), with the subspace topology.

For a countable ω_1-coherent $\mathcal{L}$-theory $\mathcal{T}$, we define the **quasi-Polish groupoid of countable models of $\mathcal{T}$**

$$\mathsf{Mod}(\mathcal{L}, \mathcal{T}) = (\mathrm{Mod}(\mathcal{L}, \mathcal{T}), \mathrm{Iso}(\mathcal{L}, \mathcal{T})) \subseteq \mathsf{Mod}(\mathcal{L})$$

to be the full subgroupoid on $\mathrm{Mod}(\mathcal{L},\mathcal{T}) \subseteq \mathrm{Mod}(\mathcal{L})$; clearly it is also an open quasi-Polish groupoid. For a countable theory $\mathcal{T}$ in a countable fragment $\mathcal{F}$ of $\mathcal{L}_{\omega_1\omega}$, we define the **Polish groupoid of countable models of $\mathcal{T}$ with topology induced by $\mathcal{F}$**

$$\mathsf{Mod}(\mathcal{F},\mathcal{T}) = (\mathrm{Mod}(\mathcal{F},\mathcal{T}), \mathrm{Iso}(\mathcal{F},\mathcal{T})) := \mathsf{Mod}(\mathcal{L}',\mathcal{T}')$$

where $(\mathcal{L}',\mathcal{T}')$ is the Morleyization of $\mathcal{T}$. For a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, we define the **standard Borel groupoid of countable models of $\mathcal{T}$**

$$\mathsf{Mod}(\mathcal{L},\mathcal{T}) = (\mathrm{Mod}(\mathcal{L},\mathcal{T}), \mathrm{Iso}(\mathcal{L},\mathcal{T})) := \mathsf{Mod}(\mathcal{F},\mathcal{T})$$

for any countable fragment $\mathcal{F} \supseteq \mathcal{T}$; again, the Borel structure does not depend on the fragment $\mathcal{F}$, and is consistent with the topology in case $\mathcal{T}$ is ω_1-coherent.

5.6 Interpretations of imaginary sorts

In this section, we define the interpretation functor $[\![-]\!]$ taking imaginary sorts to $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-spaces. As before, we begin with the general case of an ω_1-coherent theory.

Let $\mathcal{L}$ be a countable relational language and $\mathcal{T}$ be a countable ω_1-coherent $\mathcal{L}$-theory. For an imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, we put

$$[\![A]\!] := \{(\mathcal{M}, a) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T}) \ \& \ a \in A^{\mathcal{M}}\},$$

the disjoint union the interpretations $A^{\mathcal{M}}$ in all models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T})$, equipped with the projection $\pi : [\![A]\!] \to \mathrm{Mod}(\mathcal{L},\mathcal{T})$; we may call $[\![A]\!]$ simply the **interpretation of** A. We have the following alternative definition of $[\![A]\!]$ which is uniform over all models, which will yield the topology on $[\![A]\!]$.

For a single ω_1-coherent $\mathcal{L}$-formula α with n variables, regarded as an imaginary sort, put

$$[\![\alpha]\!] := \{(\mathcal{M}, \vec{a}) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T}) \ \& \ \vec{a} \in M^n \ \& \ \alpha^{\mathcal{M}}(\vec{a})\}.$$

There is an obvious countable étalé (over $\mathrm{Mod}(\mathcal{L},\mathcal{T})$) topology on $[\![\alpha]\!]$, with a cover by disjoint open sections of the form

$$[\![\alpha]\!]_{\vec{a}} := \{(\mathcal{M}, \vec{a}) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T}) \ \& \ \vec{a} \in M^n \ \& \ \alpha^{\mathcal{M}}(\vec{a})\},$$

each of which is a section over $[\![\alpha(\vec{a})]\!] \subseteq \mathrm{Mod}(\mathcal{L},\mathcal{T})$. Thus, a countable basis for $[\![\alpha]\!]$ consists of sets of the form

$$[\![\alpha]\!]_{\vec{a}} \cap \pi^{-1}([\![\phi(\vec{b})]\!])$$

with $[\![\phi(\vec{b})]\!] \subseteq \mathrm{Mod}(\mathcal{L},\mathcal{T})$ a basic open set, i.e., ϕ a basic formula. Note that when $\alpha = \top$, so that α as an imaginary sort is a power $\mathbb{X}^n$ of the home sort $\mathbb{X}$,

$$[\![\mathbb{X}^n]\!] = \{(\mathcal{M},\vec{a}) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T}) \ \& \ \vec{a} \in M^n\};$$

and for general α with n variables, we have $[\![\alpha]\!] \subseteq [\![\mathbb{X}^n]\!]$.

When $n = 0$, the notation $[\![\alpha]\!]$ agrees with the notation $[\![\alpha(\vec{a})]\!] \subseteq \mathrm{Mod}(\mathcal{L},\mathcal{T})$ defined earlier (for $\vec{a}$ the empty tuple). For future use, we introduce the following common generalization of both notations: for a formula ϕ with $m+n$ variables and $\vec{a} \in \mathbb{N}^m$, put

$$[\![\phi(\vec{a},-)]\!] := \{(\mathcal{M},\vec{b}) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L},\mathcal{T}) \ \& \ \vec{a} \in M^m \ \& \ \vec{b} \in M^n \ \& \ \phi^{\mathcal{M}}(\vec{a},\vec{b})\}.$$

When $n = 0$, this reduces to $[\![\phi(\vec{a})]\!]$; when $m = 0$, this reduces to $[\![\phi]\!]$.

For countably many ω_1-coherent formulas α_i, we put

$$[\![\bigsqcup_i \alpha_i]\!] := \bigsqcup_i [\![\alpha_i]\!]$$

with the disjoint union topology. Finally, for an arbitrary imaginary sort $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$, where α_i has n_i variables, note that $[\![\bigsqcup_{i,j} \varepsilon_{ij}]\!] \subseteq \bigsqcup_{i,j}[\![\mathbb{X}^{n_i+n_j}]\!] \cong (\bigsqcup_i [\![\mathbb{X}^{n_i}]\!]) \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} (\bigsqcup_j [\![\mathbb{X}^{n_j}]\!])$ is fiberwise (over $\mathrm{Mod}(\mathcal{L},\mathcal{T})$) an equivalence relation on $[\![\bigsqcup_i \alpha_i]\!] \subseteq \bigsqcup_i [\![\mathbb{X}^{n_i}]\!]$ by (5.4.Eqv) (and soundness); we define $[\![A]\!]$ to be the corresponding quotient

$$[\![(\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})]\!] := [\![\bigsqcup_i \alpha_i]\!]/[\![\bigsqcup_{i,j} \varepsilon_{ij}]\!]$$

with the quotient topology. By Lemma 5.2.1(v,vi), the quotient of a countable étalé space by an étalé equivalence relation is countable étalé; a countable basis of open sections in $[\![\bigsqcup_i \alpha_i]\!]/[\![\bigsqcup_{i,j} \varepsilon_{ij}]\!]$ is given by the images of basic open sections in $[\![\bigsqcup_i \alpha_i]\!]$.

We let the groupoid $\mathsf{Mod}(\mathcal{L},\mathcal{T})$ act on $[\![A]\!]$ in the obvious way, via application. That is, for a single formula α, we put

$$g \cdot (\mathcal{M},\vec{a}) := (\mathcal{N}, g(\vec{a}))$$

for $(\mathcal{M},\vec{a}) \in [\![\alpha]\!]$ and $g : \mathcal{M} \cong \mathcal{N}$. This action is continuous, since

$$\begin{aligned} g \cdot (\mathcal{M},\vec{a}) \in [\![\alpha]\!]_{\vec{b}} &\iff g(\vec{a}) = \vec{b} \ \& \ \alpha^{\mathcal{N}}(\vec{b}) \\ &\iff \exists \vec{a}' \, (g \in \bigcap_i [\![a'_i \mapsto b_i]\!] \ \& \ (\mathcal{M},\vec{a}) \in [\![\alpha]\!]_{\vec{a}'}). \end{aligned}$$

For countably many formulas α_i, equip $[\![\bigsqcup_i \alpha_i]\!]$ with the disjoint union of the actions. For a general imaginary sort $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$, equip $[\![A]\!]$ with the quotient action. Thus for every imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, we have defined a countable étalé $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-space $[\![A]\!]$.

For a definable function $f : A \to B$, we let

$$[\![f]\!] : [\![A]\!] \to [\![B]\!]$$

be given fiberwise by $f^{\mathcal{M}} : A^{\mathcal{M}} \to B^{\mathcal{M}}$ for each $\mathcal{M}$. Again there is a uniform definition, which shows that $[\![f]\!]$ is continuous. Let $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$, $B = (\bigsqcup_k \beta_k)/(\bigsqcup_{k,l} \eta_{kl})$, and $f = [(\phi_{ik})_{i,k}]$. By (5.4.Fun) (and soundness), the sub-countable étalé space $[\![\bigsqcup_{i,k} \phi_{ik}]\!] \subseteq [\![\bigsqcup_i \alpha_i]\!] \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} [\![\bigsqcup_k \beta_k]\!]$ is invariant with respect to the equivalence relation $[\![\bigsqcup_{i,j} \varepsilon_{ij}]\!] \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} [\![\bigsqcup_{k,l} \eta_{kl}]\!]$, and its image in the quotient $[\![A]\!] \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} [\![B]\!]$ is fiberwise the graph of a function $[\![f]\!] : [\![A]\!] \to [\![B]\!]$, which is continuous because its fiberwise graph is open. It is clear that $[\![f]\!]$ is $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-equivariant, and that $[\![-]\!]$ preserves identity and composition, so that we have defined a functor

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1} \longrightarrow \mathsf{Act}_{\omega_1}(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$$

(recall from Section 5.2 that $\mathsf{Act}_{\omega_1}(\mathsf{G})$ denotes the category of countable étalé G-spaces).

For a countable fragment $\mathcal{F}$ of $\mathcal{L}_{\omega_1\omega}$ and a countable $\mathcal{F}$-theory $\mathcal{T}$, we define

$$[\![-]\!] : \overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1} \longrightarrow \mathsf{Act}_{\omega_1}(\mathsf{Mod}(\mathcal{F}, \mathcal{T}))$$

by taking the Morleyization. Note that since every countable étalé action is Borel, we have $\mathsf{Act}_{\omega_1}(\mathsf{Mod}(\mathcal{F}, \mathcal{T})) \subseteq \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{F}, \mathcal{T})) = \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$. For a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, we define

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1} \longrightarrow \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$$

by regarding $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ as the direct limit $\varinjlim_{\mathcal{F}} \overline{\langle \mathcal{F} \mid \mathcal{T} \rangle}_{\omega_1}$ over countable fragments $\mathcal{F} \supseteq \mathcal{T}$, i.e., $[\![A]\!]$ for an imaginary sort A is defined as above for any countable fragment $\mathcal{F}$ containing all of the formulas in A, and similarly for definable functions. Both of these definitions are the same as if we had repeated the definition in the ω_1-coherent case, except that we do not have to re-check that the actions are continuous/Borel.

5.7 The Lopez-Escobar theorem

In this section, we present what is essentially Vaught's proof [Vau, 3.1] of Lopez-Escobar's theorem (see also [Kec, 16.9] or [Gao, 11.3.5]). We do so for the sake of completeness, since we are working in a slightly more general context (we allow finite models), and since later we will need precise statements of some intermediate parts of the proof.

Let $\mathcal{L}$ be a countable relational language and $\mathcal{T}$ be a countable *decidable* ω_1-coherent $\mathcal{L}$-theory. Recall (Section 5.4) that this means that the formula $x \neq y$ is $\mathcal{T}$-equivalent to an ω_1-coherent formula, which by abuse of notation we also write as $x \neq y$.

Recall also the subbasic open sets $[\![a \mapsto b]\!] \subseteq \mathrm{Iso}(\mathcal{L}, \mathcal{T})$ from Section 5.5, consisting of isomorphisms taking a to b. We say that two tuples $\vec{a}, \vec{b} \in \mathbb{N}^n$ have the same **equality type**, written $\vec{a} \equiv \vec{b}$, if $a_i = a_j \iff b_i = b_j$ for all i, j. For two such tuples, put $[\![\vec{a} \mapsto \vec{b}]\!] := \bigcap_i [\![a_i \mapsto b_i]\!]$.

Let $n \in \mathbb{N}$, $\vec{a} \in \mathbb{N}^n$, and $\vec{x}$ be an n-tuple of variables. We introduce the following notational abbreviations for certain ω_1-coherent formulas we will use repeatedly:

$$\begin{aligned}
(|\mathbb{X}| \geq n) &:= \exists y_0, \dots, y_{n-1} \ \textstyle\bigwedge_{i \neq j} (y_i \neq y_j), \\
(\vec{a} \in \mathbb{X}^n) &:= (|\mathbb{X}| \geq \max_i (a_i + 1)), \\
(\vec{x} \equiv \vec{a}) &:= \textstyle\bigwedge_{a_i = a_j} (x_i = x_j) \wedge \bigwedge_{a_i \neq a_j} (x_i \neq x_j), \\
(S_{\mathbb{X}} \cdot \vec{x} \ni \vec{a}) &:= (\vec{a} \in \mathbb{X}^n) \wedge (\vec{x} \equiv \vec{a}).
\end{aligned}$$

These have the expected interpretations in models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T})$: for $\vec{b} \in M^n$,

$$\begin{aligned}
(|\mathbb{X}| \geq n)^{\mathcal{M}} &\iff |M| \geq n, \\
(\vec{a} \in \mathbb{X}^n)^{\mathcal{M}} &\iff \vec{a} \in M^n, \\
(\vec{b} \equiv \vec{a})^{\mathcal{M}} &\iff \vec{b} \equiv \vec{a}, \\
(S_{\mathbb{X}} \cdot \vec{b} \ni \vec{a})^{\mathcal{M}} &\iff S_M \cdot \vec{b} \ni \vec{a}
\end{aligned}$$

(where S_M is the symmetric group on M). In particular, note that $[\![|\mathbb{X}| \geq n]\!] \subseteq \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ is as defined in Section 5.5.

Lemma 5.7.1. *Let $\vec{b} \in \mathbb{N}^k$ and let $U \subseteq [\![\mathbb{X}^n]\!]$ be open. Then there is an ω_1-coherent formula ϕ with $k + n$ variables such that for all $\vec{a} \equiv \vec{b}$,*

$$[\![\vec{a} \mapsto \vec{b}]\!]^{-1} \cdot U = [\![\phi(\vec{a}, -)]\!].$$

Proof. We may assume that U is a basic open set, i.e., $U = [\![\mathbb{X}^n]\!]_{\vec{d}} \cap \pi^{-1}([\![\psi(\vec{f})]\!])$ for some basic formula ψ (say with l variables) and tuples $\vec{d} \in \mathbb{N}^n$ and $\vec{f} \in \mathbb{N}^l$. So

$$\begin{aligned} U &= \{(\mathcal{M}, \vec{d}) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \;\;\&\;\; \vec{d} \in M^n \;\;\&\;\; \psi^{\mathcal{M}}(\vec{f})\}, \\ [\![\vec{a} \mapsto \vec{b}]\!]^{-1} \cdot U &= \{(\mathcal{M}, \vec{c}) \in [\![\mathbb{X}^n]\!] \mid \exists g \in S_M\, (g(\vec{a}, \vec{c}) = (\vec{b}, \vec{d}) \;\;\&\;\; \psi^{\mathcal{M}}(g^{-1}(\vec{f})))\} \\ &= \{(\mathcal{M}, \vec{c}) \in [\![\mathbb{X}^n]\!] \mid \exists \vec{e} \in M^l\, (S_M \cdot (\vec{a}, \vec{c}, \vec{e}) \ni (\vec{b}, \vec{d}, \vec{f}) \;\;\&\;\; \psi^{\mathcal{M}}(\vec{e}))\} \\ &= [\![\phi(\vec{a}, -)]\!], \end{aligned}$$

where ϕ is the formula

$$\begin{aligned} &\phi(x_0, \dots, x_{k+n-1}) \\ &= \exists x_{k+n}, \dots, x_{k+n+l-1}\, ((S_{\mathbb{X}} \cdot \vec{x} \ni (\vec{b}, \vec{d}, \vec{f})) \wedge \psi(x_{k+n}, \dots, x_{k+n+l-1})). \qquad \square \end{aligned}$$

Corollary 5.7.2. *If $U \subseteq [\![\mathbb{X}^n]\!]$ is open and $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-invariant, then there is an ω_1-coherent formula ϕ with n variables such that $U = [\![\phi]\!]$.* $\square$

Now replace $\mathcal{T}$ above with the Morleyization of a countable theory $\mathcal{T}$ in a countable fragment $\mathcal{F}$, so that $\mathsf{Mod}(\mathcal{F}, \mathcal{T})$ is an open Polish groupoid and so we may talk about Vaught transforms.

Lemma 5.7.3. *Let $\vec{b} \in \mathbb{N}^k$ and let $B \subseteq [\![\mathbb{X}^n]\!]$ be Borel. Then there is an $\mathcal{L}_{\omega_1\omega}$-formula ϕ with $k + n$ variables such that for all $\vec{a} \equiv \vec{b}$,*

$$B^{\triangle[\![\vec{a} \mapsto \vec{b}]\!]} = [\![\phi(\vec{a}, -)]\!].$$

Proof. By induction on the complexity of B. For B open, this is Lemma 5.7.1. For a countable union $B = \bigcup_i B_i$, let for each i the corresponding formula for B_i be ϕ_i, then put $\phi := \bigvee_i \phi_i$. For a complement $B = \neg C$, using the "∂_1-fiberwise weak basis for $[\![\vec{a} \mapsto \vec{b}]\!]$" (see Lemma 5.2.3) consisting of $[\![\vec{c} \mapsto \vec{d}]\!]$ for $\vec{a} \subseteq \vec{c} \equiv \vec{d} \supseteq \vec{b}$, we have

$$\begin{aligned} B^{\triangle[\![\vec{a} \mapsto \vec{b}]\!]} &= \textstyle\bigcup_{\vec{a} \subseteq \vec{c} \equiv \vec{d} \supseteq \vec{b}} B^{\circledast[\![\vec{c} \mapsto \vec{d}]\!]} \\ &= \textstyle\bigcup_{\vec{a} \subseteq \vec{c} \equiv \vec{d} \supseteq \vec{b}} (B^{*[\![\vec{c} \mapsto \vec{d}]\!]} \cap p^{-1}(\partial_1([\![\vec{c} \mapsto \vec{d}]\!]))) \\ &= \textstyle\bigcup_{\vec{a} \subseteq \vec{c} \equiv \vec{d} \supseteq \vec{b}} (\neg C^{\triangle[\![\vec{c} \mapsto \vec{d}]\!]} \cap p^{-1}([\![\vec{c} \in \mathbb{X}^{|\vec{c}|}]\!] \cap [\![\vec{d} \in \mathbb{X}^{|\vec{d}|}]\!])); \end{aligned}$$

let for each $\vec{d} \supseteq \vec{b}$ the formula $\psi_{\vec{d}}$ be such that $C^{\triangle[\![\vec{c} \mapsto \vec{d}]\!]} = [\![\psi_{\vec{d}}(\vec{c}, -)]\!]$ for all $\vec{c} \equiv \vec{d}$, whence

$$\begin{aligned} B^{\triangle[\![\vec{a} \mapsto \vec{b}]\!]} &= \textstyle\bigcup_{\vec{a} \subseteq \vec{c} \equiv \vec{d} \supseteq \vec{b}} (\neg [\![\psi_{\vec{d}}(\vec{c}, -)]\!] \cap p^{-1}([\![\vec{c} \in \mathbb{X}^{|\vec{c}|}]\!] \cap [\![\vec{d} \in \mathbb{X}^{|\vec{d}|}]\!])) \\ &= \{(\mathcal{M}, \vec{e}) \mid \vec{e} \in M^n \;\;\&\;\; \exists \vec{a} \subseteq \vec{c} \equiv \vec{d} \supseteq \vec{b}\, (\vec{c}, \vec{d} \in M^{|\vec{d}|} \;\;\&\;\; \neg\psi_{\vec{d}}^{\mathcal{M}}(\vec{c}, \vec{e}))\} \\ &= [\![\phi(\vec{a}, -)]\!], \end{aligned}$$

where

$$\phi(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{n-1}) = \bigvee_{\vec{d} \supseteq \vec{b}} \exists x_k, \ldots, x_{|\vec{d}|-1} ((S_{\mathbb{X}} \cdot \vec{x} \ni \vec{d}) \wedge \neg \psi_{\vec{d}}(\vec{x}, \vec{y})). \square$$

Corollary 5.7.4. *If $B \subseteq [\![\mathbb{X}^n]\!]$ is Borel and $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-invariant, then there is an $\mathcal{L}_{\omega_1\omega}$-formula ϕ with n variables such that $B = [\![\phi]\!]$.* $\square$

The usual statement of Lopez-Escobar's theorem is the case $n = 0$.

5.8 Naming countable étalé actions

In this section, we give a direct proof of the following generalization of Theorem 5.1.4 (which follows from it via Morleyization). The proof is analogous to that of a similar result of Awodey–Forssell [AF, §1.4] for $\mathcal{L}_{\omega\omega}$-theories. Later in Section 5.15 we will give a more abstract proof of this result, by extracting it from the Joyal–Tierney representation theorem.

Theorem 5.8.1. *Let $\mathcal{L}$ be a countable relational language and $\mathcal{T}$ be a countable decidable ω_1-coherent $\mathcal{L}$-theory. Then the interpretation functor*

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1} \longrightarrow \mathsf{Act}_{\omega_1}(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$$

is an equivalence of categories.

Proof. We will prove that the functor is conservative, full on subobjects, and essentially surjective. This is enough, since by standard category theory, a finite limit-preserving functor between categories with finite limits is an equivalence iff it is conservative, full on subobjects, and essentially surjective (see e.g., [J02, D3.5.6]), and $[\![-]\!]$ is easily seen to preserve finite limits (using the explicit constructions of finite products and equalizers of imaginary sorts in Remark 5.4.2).

Conservative means that given an imaginary sort $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij}) \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ and a subsort (i.e., definable relation) $B \subseteq A$, if $[\![B]\!] = [\![A]\!]$, then B and A are provably equivalent. Recall (Remark 5.4.2) that B is given by a family of formulas $(\beta_i)_i$ such that $\mathcal{T}$ proves that "$\bigsqcup_i \beta_i \subseteq \bigsqcup_i \alpha_i$ is $(\bigsqcup_{i,j} \varepsilon_{ij})$-invariant". If $[\![B]\!] = [\![A]\!]$, then clearly $[\![\beta_i]\!] = [\![\alpha_i]\!]$ for every i, i.e., $\beta_i^{\mathcal{M}} = \alpha_i^{\mathcal{M}}$ for every countable model $\mathcal{M}$ of $\mathcal{T}$. By the completeness theorem, $\mathcal{T} \vdash \beta_i \Leftrightarrow \alpha_i$, as desired.

Full on subobjects means that given an imaginary sort $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij}) \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ and a sub-countable étalé $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-space $Y \subseteq [\![A]\!]$, there is a subsort $B \subseteq A$ such that $[\![B]\!] = Y$. Since Y is étalé, $Y \subseteq [\![A]\!]$ is open. Let $q : [\![\bigsqcup_i \alpha_i]\!] \to [\![A]\!]$

be the quotient map. Then for each i, $[\![\alpha_i]\!] \cap q^{-1}(Y) \subseteq [\![\alpha_i]\!]$ is open and $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-invariant, hence by Corollary 5.7.2 equal to $[\![\beta_i]\!]$ for some ω_1-coherent formula β_i. Since $[\![\beta_i]\!] \subseteq [\![\alpha_i]\!]$, by the completeness theorem (or conservativity as above applied to $\alpha_i \wedge \beta_i \subseteq \beta_i$), $\beta_i \subseteq \alpha_i$. Similarly, since $q^{-1}(Y) = \bigsqcup_i [\![\beta_i]\!] \subseteq [\![\bigsqcup_i \alpha_i]\!]$ is $[\![\bigsqcup_{i,j} \varepsilon_{ij}]\!]$-invariant, $\mathcal{T}$ proves that "$\bigsqcup_i \beta_i \subseteq \bigsqcup_i \alpha_i$ is $(\bigsqcup_{i,j} \varepsilon_{ij})$-invariant". So the desired subsort B is given by $(\beta_i)_i$.

We now come to the heart of the proof: essentially surjective means that given a countable étalé $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-space $p : X \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$, there is an imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ such that $X \cong [\![A]\!]$. Since X is countable étalé, it has a countable basis $\mathcal{U}$ of open sections $U \subseteq X$, which we may assume to be over basic open sets $[\![\phi(\vec{a})]\!] \subseteq \mathrm{Mod}(\mathcal{L}, \mathcal{T})$. We claim that we may choose these so that

$$[\![\vec{a} \mapsto \vec{a}]\!] \cdot U \subseteq U. \tag{$*$}$$

Proof of claim. Let $V_i \subseteq X$ be any countable basis of open sections. Let $x \in X$ and let $W \subseteq X$ be any open section containing x. Since $1_{p(x)} \cdot x = x$, by continuity of the action, there is a basic open section $x \in V_i \subseteq W$ and a basic open set $1_{p(x)} \in [\![\vec{b} \mapsto \vec{b}']\!] \cap \partial_1^{-1}([\![\psi(\vec{c})]\!]) \subseteq \mathrm{Iso}(\mathcal{L}, \mathcal{T})$ such that $([\![\vec{b} \mapsto \vec{b}']\!] \cap \partial_1^{-1}([\![\psi(\vec{c})]\!])) \cdot V_i \subseteq W$. That $1_{p(x)} \in [\![\vec{b} \mapsto \vec{b}']\!] \cap \partial_1^{-1}([\![\psi(\vec{c})]\!])$ means that $\vec{b} = \vec{b}'$ and that the model $p(x) \in \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ contains $\vec{b}$ and $\vec{c}$ and satisfies $\psi^{p(x)}(\vec{c})$. Since $p(x) \in p(V_i)$, there is a basic open set $p(x) \in [\![\theta(\vec{d})]\!] \subseteq p(V_i)$. Putting $\vec{a} := (\vec{b}, \vec{c}, \vec{d})$ and $\phi(\vec{x}, \vec{y}, \vec{z}) := \psi(\vec{y}) \wedge \theta(\vec{z})$, we have $p(x) \in [\![\phi(\vec{a})]\!] \subseteq [\![\theta(\vec{d})]\!] \subseteq p(V_i)$. Thus

$$U := V_i \cap p^{-1}([\![\phi(\vec{a})]\!])$$

is an open section over $[\![\phi(\vec{a})]\!]$ containing x, such that

$$[\![\vec{a} \mapsto \vec{a}]\!] \cdot U = ([\![\vec{a} \mapsto \vec{a}]\!] \cap \partial_1^{-1}([\![\phi(\vec{a})]\!])) \cdot U \subseteq ([\![\vec{b} \mapsto \vec{b}]\!] \cap \partial_1^{-1}([\![\psi(\vec{c})]\!])) \cdot V_i \subseteq W.$$

This implies $[\![\vec{a} \mapsto \vec{a}]\!] \cdot U \subseteq U$, since $U \subseteq W$, W is a section, and $[\![\vec{a} \mapsto \vec{a}]\!] \cdot p(U) = [\![\vec{a} \mapsto \vec{a}]\!] \cdot [\![\phi(\vec{a})]\!] \subseteq [\![\phi(\vec{a})]\!] = p(U)$. So we may take $\mathcal{U}$ to consist of all $U = V_i \cap p^{-1}([\![\phi(\vec{a})]\!])$ with ϕ a basic formula, $[\![\phi(\vec{a})]\!] \subseteq p(V_i)$, and $[\![\vec{a} \mapsto \vec{a}]\!] \cdot U \subseteq U$. *(Claim)*□

Now having found such a basis $\mathcal{U}$, fix some $U \in \mathcal{U}$ and associated $\phi, \vec{a}$ satisfying $(*)$, say with $|\vec{a}| = n$. Put

$$\alpha(\vec{x}) := (S_{\mathbb{X}} \cdot \vec{x} \ni \vec{a}) \wedge \phi(\vec{x}).$$

We claim that we have a $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-equivariant continuous map $f : [\![\alpha]\!] \to X$ given by

$$f(\mathcal{M},\vec{b}) = x \iff p(x) = \mathcal{M} \ \& \ \exists g \in S_M\,(g(\vec{b}) = \vec{a} \ \& \ g \cdot x \in U)$$

whose image contains U.

Proof of claim. First, we must check that f so defined is a function. For $(\mathcal{M},\vec{b}) \in [\![\alpha]\!]$, by definition of α, there is some isomorphism $g : \mathcal{M} \cong \mathcal{N}$ such that $g(\vec{b}) = \vec{a}$, whence $\mathcal{N} \in [\![\phi(\vec{a})]\!] = p(U)$; letting $y \in U \cap p^{-1}(\mathcal{N})$ be the unique element, $x := g^{-1} \cdot y$ is one value for $f(\mathcal{M},\vec{b})$. If $x, x' \in p^{-1}(\mathcal{M})$ and there are two isomorphisms $g : \mathcal{M} \cong \mathcal{N}$ and $g' : \mathcal{M} \cong \mathcal{N}'$ with $g(\vec{b}) = \vec{a} = g'(\vec{b})$ and $g\cdot x, g'\cdot x' \in U$, then $g' \circ g^{-1} \in [\![\vec{a} \mapsto \vec{a}]\!]$, whence (by $(*)$) $g'\cdot x = (g'\circ g^{-1})\cdot(g\cdot x) \in U$ with $p(g' \cdot x) = \mathcal{N}' = p(g' \cdot x')$, whence $g' \cdot x = g' \cdot x'$ since U is a section, whence $x = x'$; thus $f(\mathcal{M},\vec{b})$ is unique.

It is straightforward that f is $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-equivariant. Furthermore, for $x \in U$, clearly $f(p(x),\vec{a}) = x$ as witnessed by $1_{p(x)} \in S_{p(x)}$; hence the image of f contains U.

Finally, we must check that f is continuous. Since $[\![\alpha]\!] = \bigsqcup_{\vec{b}}[\![\alpha]\!]_{\vec{b}}$, it suffices to check that each $f|[\![\alpha]\!]_{\vec{b}}$ is continuous. For $\vec{b} \not\equiv \vec{a}$, we have $[\![\alpha]\!]_{\vec{b}} = \varnothing$. For $\vec{b} \equiv \vec{a}$, put $m := \max_i(a_i + 1, b_i + 1)$, and fix some $g \in S_\infty$ which is the identity on $\mathbb{N} \setminus m$ such that $g(\vec{b}) = \vec{a}$. The map

$$\begin{aligned} [\![|\mathbb{X}| \geq m]\!] &\longrightarrow \mathrm{Iso}(\mathcal{L},\mathcal{T}) \\ \mathcal{M} &\longmapsto (g|M : \mathcal{M} \cong g(\mathcal{M})) \end{aligned}$$

is easily seen to be continuous, whence for any continuous $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-space $q : Y \to \mathrm{Mod}(\mathcal{L},\mathcal{T})$, we have a continuous map (which we denote simply by g)

$$\begin{aligned} g : q^{-1}([\![|\mathbb{X}| \geq m]\!]) &\longrightarrow Y \\ y &\longmapsto (g|q(y)) \cdot y; \end{aligned}$$

similarly, we have a map $g^{-1} : q^{-1}([\![|\mathbb{X}| \geq m]\!]) \to Y$. Then $f|[\![\alpha]\!]_{\vec{b}}$ factors as the composite

$$[\![\alpha]\!]_{\vec{b}} \underset{\cong}{\xrightarrow{\pi}} [\![\alpha(\vec{b})]\!] \underset{\cong}{\xrightarrow{g}} [\![\alpha(\vec{a}) \wedge (|\mathbb{X}| \geq m)]\!] \underset{\cong}{\xrightarrow{p^{-1}}} U \cap p^{-1}([\![\mathbb{X} \geq m]\!]) \xrightarrow{g^{-1}} X,$$

which is continuous. $(\mathit{Claim})\square$

We can now finish the proof of essential surjectivity via a standard covering argument. For every basic open section $U \in \mathcal{U}$, we have found a formula α_U and an equivariant continuous map $f_U : [\![\alpha_U]\!] \to X$, defined as above, whose image contains U. Combining these yields an equivariant continuous surjection $f : [\![\bigsqcup_{U\in\mathcal{U}} \alpha_U]\!] \to X$. The kernel $[\![\bigsqcup_{U\in\mathcal{U}} \alpha_U]\!] \times_X [\![\bigsqcup_{U\in\mathcal{U}} \alpha_U]\!]$ of f is a sub-countable étalé $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-space of $[\![(\bigsqcup_{U\in\mathcal{U}} \alpha_U)^2]\!]$ (by Lemma 5.2.1(iv)), hence since (as shown above) $[\![-]\!]$ is full on subobjects, is given by $[\![\bigsqcup_{U,V\in\mathcal{U}} \varepsilon_{UV}]\!]$ for some family of formulas ε_{ij}, such that $\mathcal{T}$ proves that $\bigsqcup_{U,V} \varepsilon_{UV}$ is an equivalence relation on $\bigsqcup_U \alpha_U$ by conservativity of $[\![-]\!]$ (or completeness). Since f is an étalé surjection, the quotient of its kernel is X (by Lemma 5.2.1(i,v,vi)), i.e., putting $A := (\bigsqcup_U \alpha_U)/(\bigsqcup_{U,V} \varepsilon_{UV})$, we have $X \cong [\![A]\!]$, as desired. ◻

5.9 Naming fiberwise countable Borel actions

In this section, we prove Theorem 5.1.3 using Theorem 5.1.4 and (the proof of) Theorem 5.1.5.

Proof of Theorem 5.1.3. As in the preceding section, we prove that $[\![-]\!]$ is conservative, full on subobjects, and essentially surjective. The proofs of conservativity and fullness on subobjects are the same as before, except that for the latter we use Corollary 5.7.4 instead of Corollary 5.7.2. We now give the proof of essential surjectivity, i.e., of Theorem 5.1.2.

Let $\mathcal{F} \supseteq \mathcal{T}$ be any countable fragment, so that $\mathsf{Mod}(\mathcal{F},\mathcal{T})$ is an open Polish groupoid whose underlying standard Borel groupoid is $\mathsf{Mod}(\mathcal{L},\mathcal{T})$. Let $\mathcal{U}$ be the open basis for $\mathrm{Iso}(\mathcal{F},\mathcal{T})$ consisting of sets of the form

$$U = \partial_0^{-1}(C) \cap [\![\vec{a} \mapsto \vec{b}]\!]$$

where $C \subseteq \mathrm{Mod}(\mathcal{F},\mathcal{T})$ is basic open and $\vec{a} \equiv \vec{b} \in \mathbb{N}^n$ (recall the definition of $[\![\vec{a} \mapsto \vec{b}]\!]$ from Section 5.7). Note that for such U and for any Borel $B \subseteq \mathrm{Mod}(\mathcal{F},\mathcal{T})$, we have

$$B^{\triangle U} = [\![\phi(\vec{a})]\!] \tag{$*$}$$

for some $\mathcal{L}_{\omega_1\omega}$-formula ϕ; this follows from Lemma 5.7.3 (with $n = 0$) and the observation that $B^{\triangle U} = (B \cap \partial_0^{-1}(C))^{\triangle [\![\vec{a}\mapsto\vec{b}]\!]}$.

Now let $p : X \to \mathrm{Mod}(\mathcal{L},\mathcal{T})$ be a fiberwise countable Borel $\mathsf{Mod}(\mathcal{L},\mathcal{T})$-space, equivalently a fiberwise countable Borel $\mathsf{Mod}(\mathcal{F},\mathcal{T})$-space. We modify the proof of Theorem 5.1.5 in Section 5.3 for $\mathsf{Mod}(\mathcal{F}, \mathcal{T})$ and X, using the above basis $\mathcal{U}$ for $\mathrm{Iso}(\mathcal{F}, \mathcal{T})$, by imposing further conditions on the countable Boolean algebras $\mathcal{A}, \mathcal{B}$ used in that proof, while simultaneously keeping track of a countable fragment $\mathcal{F}' \supseteq \mathcal{F}$, as follows:

(vi) for each $B \in \mathcal{B}$ and $U \in \mathcal{U}$, there exist $\phi, \vec{a}$ so that $(*)$ holds and such that $\phi \in \mathcal{F}'$;

(vii) $\mathcal{B}$ contains each basic open set in $\mathrm{Mod}(\mathcal{F}', \mathcal{T})$.

Clearly these can be achieved by enlarging $\mathcal{A}, \mathcal{B}, \mathcal{F}'$ ω-many times. The new topology on $\mathrm{Mod}(\mathcal{F}, \mathcal{T})$ produced by Theorem 5.1.5 is then that of $\mathrm{Mod}(\mathcal{F}', \mathcal{T})$ by $(*)$ and Lemma 5.3.5, so that we have turned X into a countable étalé $\mathsf{Mod}(\mathcal{F}', \mathcal{T})$-space. By Theorem 5.1.4, X is isomorphic to $[\![A]\!]$ for some $\mathcal{F}'$-imaginary sort $A \in \overline{\langle \mathcal{F}' \mid \mathcal{T} \rangle}_{\omega_1} \subseteq \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^{B}_{\omega_1}$. □

5.10 Interpretations between theories

In this section, we consider the (2-)functorial aspects of the passage from theories to their groupoids of models; in particular, we define the notion of an interpretation *between theories*. This will enable us, in the next section, to compare our results to the various known strong conceptual completeness theorems mentioned in the Introduction, as well as to the main result of [HMM]. We would like to warn the reader that these two sections involve some rather technical 2-categorical notions.

For the basic theory of 2-categories, see [J02, B1.1] or [Bor, I Ch. 7]. In every 2-category we will consider, all 2-cells will be invertible.

The following definitions make precise the idea that the syntactic ω_1-pretopos $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ of an ω_1-coherent theory $(\mathcal{L}, \mathcal{T})$ is a category with algebraic structure "presented by $(\mathcal{L}, \mathcal{T})$"; more precisely, it is the "free category with the structures listed in Remark 5.4.2, generated by an object $\mathbb{X}$, together with subobjects $R \subseteq \mathbb{X}^n$ for each n-ary $R \in \mathcal{L}$, and satisfying the relations in $\mathcal{T}$".

Definition 5.10.1. An ω_1**-pretopos** is a category C with the following three kinds of structure (see [J02, A1.3–4], [MR, §3.4], [CLW], [Bor, II Ch. 2]):

- Finite limits (equivalently, finite products and equalizers) exist.
- Every countable family of objects $X_i \in \mathsf{C}$ has a coproduct $\bigsqcup_i X_i$ which is disjoint and pullback-stable. **Disjoint** means that the injections $X_i \to \bigsqcup_i X_i$

are monomorphisms and have pairwise empty intersections, i.e., the pullback $X_i \times_{\bigsqcup_k X_k} X_j$ is X_i for $i = j$ and the initial object for $i \neq j$. **Pullback-stable** means that for every morphism $f : Y \to \bigsqcup_i X_i$, the pullback of the injections $X_i \to \bigsqcup_j X_j$ along f exhibits Y as the coproduct $\bigsqcup_i (X_i \times_{\bigsqcup_j X_j} Y)$. Such a coproduct is also called a **disjoint union**.

- Every equivalence relation $E \subseteq X^2$ has a coequalizer (of the two projections) $E \rightrightarrows X \to X/E$ which is effective and pullback-stable. An **equivalence relation** $E \subseteq X^2$ is a subobject which is "reflexive", "symmetric", and "transitive", as expressed internally in C, i.e., E contains the diagonal subobject $X \subseteq X^2$ and is invariant under the "twist" automorphism $X^2 \to X^2$, and the pullback of E along the projection $X^3 \to X^2$ omitting the middle coordinate contains $E \times_X E \subseteq X^3$. **Effective** means that E is (via the two projections) the kernel of $X \to X/E$, i.e., the pullback of $X \to X/E$ with itself. **Pullback-stable** means that the coequalizer diagram $E \rightrightarrows X \to X/E$ is still a coequalizer diagram after pullback along any morphism $f : Y \to X/E$. The coequalizer X/E is then also called a **quotient**.

An ω_1**-coherent functor** $F : \mathsf{C} \to \mathsf{D}$ between two ω_1-pretoposes is a functor preserving these operations. By combining these operations, every ω_1-pretopos also has the following (and they are also preserved by every ω_1-coherent functor):

- The **image** $\mathrm{im}(f)$ of a morphism $f : X \to Y$ in C is the quotient of the kernel of f, and yields a factorization of f into a regular epimorphism $X \twoheadrightarrow \mathrm{im}(f)$ followed by a monomorphism $\mathrm{im}(f) \hookrightarrow Y$. This factorization is pullback-stable (along morphisms $Z \to Y$).
- Given countably many subobjects $A_i \subseteq X$, their **union** $\bigcup_i A_i \subseteq X$ is the image of the induced map from the disjoint union $\bigsqcup_i A_i \to X$. Unions are pullback-stable.

We denote the 2-category of (small) ω_1-pretoposes, ω_1-coherent functors, and natural *isomorphisms* by $\omega_1\mathfrak{PTop}$. Thus, given two ω_1-pretoposes C, D, the groupoid of ω_1-coherent functors $\mathsf{C} \to \mathsf{D}$ and natural isomorphisms between them is denoted

$$\omega_1\mathfrak{PTop}(\mathsf{C}, \mathsf{D}).$$

(We restrict to isomorphisms because we are only considering isomorphisms between models.)

A typical example of an ω_1-pretopos is the syntactic ω_1-pretopos $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ of an ω_1-coherent theory $(\mathcal{L}, \mathcal{T})$. A simpler example is the full subcategory

$$\mathsf{Count} := \{0, 1, 2, \dots, \mathbb{N}\} \subseteq \mathsf{Set},$$

which is a skeleton of the category of countable sets.

Definition 5.10.2. Let $\mathcal{L}$ be a countable relational language and C be an ω_1-pretopos. An **$\mathcal{L}$-structure in** C, $\mathcal{M} = (M, R^{\mathcal{M}})_{R \in \mathcal{L}}$, consists of an **underlying object** $M \in \mathsf{C}$ together with subobjects $R^{\mathcal{M}} \subseteq M^n$ for each n-ary $R \in \mathcal{L}$. An **isomorphism** between $\mathcal{L}$-structures $f : \mathcal{M} \to \mathcal{N}$ is an isomorphism $f : M \to N$ in C such that for each n-ary $R \in \mathcal{L}$, $f^n : M^n \to N^n$ restricts to an isomorphism $R^{\mathcal{M}} \to R^{\mathcal{N}}$. The groupoid of $\mathcal{L}$-structures in C and isomorphisms is denoted

$$\mathsf{Mod}_{\mathsf{C}}(\mathcal{L}).$$

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. For each ω_1-coherent $\mathcal{L}$-formula ϕ with n variables, we define its **interpretation in** $\mathcal{M}$, $\phi^{\mathcal{M}} \subseteq M^n$, by induction on ϕ in the expected manner (see [J02, D1.2]):

- For $\phi(\vec{x}) = R(\vec{x})$, $\phi^{\mathcal{M}} := R^{\mathcal{M}}$.
- For $\phi(\vec{x}) = (x_i = x_j)$, $\phi^{\mathcal{M}} := M^n$ if $i = j$, otherwise $\phi^{\mathcal{M}} := M^{n-1} \subseteq M^n$ is the diagonal which duplicates the ith coordinate into the jth.
- For $\phi = \psi \wedge \theta$, $\phi^{\mathcal{M}}$ is the intersection (i.e., pullback) of $\psi^{\mathcal{M}}, \theta^{\mathcal{M}} \subseteq M^n$. For $\phi = \top$, $\phi^{\mathcal{M}} := M^n$.
- For $\phi = \bigvee_i \psi_i$, $\phi^{\mathcal{M}}$ is the union $\bigcup_i \psi_i^{\mathcal{M}}$.
- For $\phi(\vec{x}) = \exists y\, \psi(\vec{x}, y)$, $\phi^{\mathcal{M}}$ is the image of the composite $\psi^{\mathcal{M}} \subseteq M^{n+1} \to M^n$ (where the second map is the projection onto the first n coordinates).

By the usual inductions, interpretations are sound with respect to provability (see [J02, D1.3.2]), and isomorphisms preserve interpretations of formulas (see [J02, D1.2.9]). An ω_1-coherent axiom $\forall \vec{x}\, (\phi(\vec{x}) \Rightarrow \psi(\vec{x}))$ (where $n = |\vec{x}|$) is **satisfied by** $\mathcal{M}$ if $\phi^{\mathcal{M}} \subseteq \psi^{\mathcal{M}}$ (as subobjects of M^n). For an ω_1-coherent $\mathcal{L}$-theory $\mathcal{T}$, we say that $\mathcal{M}$ is a **model of** $\mathcal{T}$ if $\mathcal{M}$ satisfies every axiom in $\mathcal{T}$. The groupoid of models of $\mathcal{T}$ in C and isomorphisms is the full subgroupoid

$$\mathsf{Mod}_{\mathsf{C}}(\mathcal{L}, \mathcal{T}) \subseteq \mathsf{Mod}_{\mathsf{C}}(\mathcal{L}).$$

Given a model $\mathcal{M}$ of $\mathcal{T}$ in C, we may also interpret imaginary sorts and definable functions in $\mathcal{M}$, exactly as expected:

- For an imaginary sort $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$, $\bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}} \subseteq (\bigsqcup_i \alpha_i^{\mathcal{M}})^2$ is an equivalence relation (by soundness and (5.4.Eqv)); the quotient object is $A^{\mathcal{M}}$.
- For a definable function $f = [(\phi_{ik})_{i,k}] : A \to B$, $f^{\mathcal{M}} : A^{\mathcal{M}} \to B^{\mathcal{M}}$ is the unique morphism whose graph, when pulled back along $(\bigsqcup_i \alpha_i^{\mathcal{M}}) \times (\bigsqcup_k \beta_k^{\mathcal{M}}) \to A^{\mathcal{M}} \times B^{\mathcal{M}}$, is $\bigsqcup_{i,k} \phi_{ik}$.

When $\mathsf{C} = \mathsf{Set}$, $\mathsf{Mod}_{\mathsf{Set}}(\mathcal{L}, \mathcal{T})$ is the usual groupoid of set-theoretic models of $\mathcal{T}$. When $\mathsf{C} = \mathsf{Count}$, we recover $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ as defined before (Section 5.4):

$$\mathsf{Mod}_{\mathsf{Count}}(\mathcal{L}, \mathcal{T}) \cong \mathsf{Mod}(\mathcal{L}, \mathcal{T}).$$

The general notion of model of $\mathcal{T}$ in C formalizes that of "an object in C equipped with subobjects for each $R \in \mathcal{L}$ which satisfy the relations in $\mathcal{T}$". By comparing the definition of $\phi^{\mathcal{M}}$ with Remark 5.4.2, we see that we have a model $\mathcal{X}$ of $(\mathcal{L}, \mathcal{T})$ in $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$, called the **universal model**, with underlying object $\mathbb{X}$ and

$$\phi^{\mathcal{X}} = \phi \subseteq \mathbb{X}^n$$

for all ω_1-coherent $\mathcal{L}$-formulas ϕ with n variables.

Let $F : \mathsf{C} \to \mathsf{D}$ be an ω_1-coherent functor between ω_1-pretoposes. Since the definitions of "model of $\mathcal{T}$ in C" and "isomorphism between models" use only the categorical structure found in an arbitrary ω_1-pretopos, which is preserved by an ω_1-coherent functor, we get a functor

$$\begin{aligned} F_* : \mathsf{Mod}_{\mathsf{C}}(\mathcal{L}, \mathcal{T}) &\longrightarrow \mathsf{Mod}_{\mathsf{D}}(\mathcal{L}, \mathcal{T}) \\ (M, R^{\mathcal{M}})_{R \in \mathcal{L}} &\longmapsto (F(M), F(R^{\mathcal{M}}))_{R \in \mathcal{L}}. \end{aligned}$$

We are finally ready to state the universal property of the syntactic ω_1-pretopos:

Proposition 5.10.3. *For any ω_1-coherent theory $(\mathcal{L}, \mathcal{T})$, the syntactic ω_1-pretopos $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}$ is the free ω_1-pretopos containing a model $\mathcal{X}$ of $\mathcal{T}$: for any other ω_1-pretopos C, we have an equivalence of groupoids*

$$\begin{aligned} \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1}, \mathsf{C}) &\longrightarrow \mathsf{Mod}_{\mathsf{C}}(\mathcal{L}, \mathcal{T}) \\ F &\longmapsto F_*(\mathcal{X}) \end{aligned}$$

between ω_1-coherent functors $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1} \to \mathsf{C}$ and models of $\mathcal{T}$ in C.

Proof. An inverse equivalence takes a model $\mathcal{M}$ to the functor $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1} \to \mathsf{C}$ which takes imaginary sorts and definable functions to their interpretations in $\mathcal{M}$. For details, see [J02, D1.4.7, D1.4.12] (which deals with finitary logic, but generalizes straightforwardly). $\square$

We define an **(ω_1-coherent) interpretation** $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ between two ω_1-coherent theories to mean an ω_1-coherent functor between their syntactic ω_1-pretoposes $F : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_{\omega_1} \to \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_{\omega_1}$. By Proposition 5.10.3, an interpretation is equivalently a model of $\mathcal{T}$ in $\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_{\omega_1}$, which can be rephrased in more familiar terms:

- an imaginary sort $F(\mathbb{X}) \in \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_{\omega_1}$;
- for each n-ary $R \in \mathcal{L}$, a subsort $F(R) \subseteq F(\mathbb{X})^n$;
- such that for each axiom $\forall \vec{x}\, (\phi(\vec{x}) \Rightarrow \psi(\vec{x}))$ in $\mathcal{T}$, the corresponding inclusion of subsorts $F(\phi) \subseteq F(\psi)$ is $\mathcal{T}'$-provable (where $F(\phi), F(\psi)$ are defined by induction in the obvious way).

We have analogous notions for $\mathcal{L}_{\omega_1\omega}$-theories. An ω_1-pretopos C is **Boolean** if for every object $X \in \mathsf{C}$, the lattice of subobjects of X is a Boolean algebra. Clearly, an ω_1-coherent functor automatically preserves complements of subobjects when they exist. We denote the 2-category of Boolean ω_1-pretoposes (a full sub-2-category of $\omega_1\mathfrak{PTop}$) by

$$\mathfrak{B}\omega_1\mathfrak{PTop}.$$

Given an $\mathcal{L}$-structure $\mathcal{M}$ in a Boolean ω_1-pretopos C, we may define the interpretation $\phi^{\mathcal{M}}$ not just for ω_1-coherent $\mathcal{L}$-formulas ϕ, but for all $\mathcal{L}_{\omega_1\omega}$-formulas ϕ; thus, we may speak of $\mathcal{M}$ being a **model of** $\mathcal{T}$ for an arbitrary $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, meaning that $\phi^{\mathcal{M}} = 1$ (the terminal object) for each $\phi \in \mathcal{T}$. The **universal model** $\mathcal{X} \in \mathsf{Mod}_{\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}}(\mathcal{L}, \mathcal{T})$ is defined in the same way as before. An easy generalization of Lemma 5.4.1 shows

Lemma 5.10.4. *Let $\mathcal{T}$ be an $\mathcal{L}_{\omega_1\omega}$-theory and $(\mathcal{L}', \mathcal{T}')$ be its Morleyization in the (uncountable) fragment of all $\mathcal{L}_{\omega_1\omega}$-formulas. Then for any Boolean ω_1-pretopos C, we have an isomorphism of groupoids $\mathsf{Mod}_{\mathsf{C}}(\mathcal{L}', \mathcal{T}') \cong \mathsf{Mod}_{\mathsf{C}}(\mathcal{L}, \mathcal{T})$, which takes a model of $\mathcal{T}'$ to its $\mathcal{L}$-reduct.* $\square$

The universal property of the syntactic Boolean ω_1-pretopos is analogous to Proposition 5.10.3:

Proposition 5.10.5. *For an $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, the syntactic Boolean ω_1-pretopos $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ is the free Boolean ω_1-pretopos containing a model $\mathcal{X}$ of $\mathcal{T}$: for any other Boolean ω_1-pretopos* C*, we have an equivalence of groupoids*

$$\begin{aligned} \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{C}) &\longrightarrow \mathsf{Mod}_{\mathsf{C}}(\mathcal{L}, \mathcal{T}) \\ F &\longmapsto F_*(\mathcal{X}). \end{aligned}$$

Proof. By Proposition 5.10.3, Lemma 5.10.4, and the definition $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1} = \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_{\omega_1}$ (for $(\mathcal{L}', \mathcal{T}')$ the Morleyization of $\mathcal{T}$ in the fragment of all $\mathcal{L}_{\omega_1\omega}$-formulas). Alternatively, we may directly define the inverse as in the proof of Proposition 5.10.3 (i.e., [J02, D1.4.7]). □

For an $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ and an $\mathcal{L}'_{\omega_1\omega}$-theory $\mathcal{T}'$, an **($\mathcal{L}'_{\omega_1\omega}$-)interpretation** $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ is an ω_1-coherent functor $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1} \to \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1}$, or equivalently a model of $\mathcal{T}$ in $\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1}$; this may be spelled out explicitly as with ω_1-coherent interpretations above. Let

$$\omega_1\omega\mathfrak{Thy}_{\omega_1}$$

denote the 2-category of countable $\mathcal{L}_{\omega_1\omega}$-theories, interpretations, and natural isomorphisms. Thus $\omega_1\omega\mathfrak{Thy}_{\omega_1}$ is equivalent, via $\overline{\langle - \rangle}^B_{\omega_1}$, to a full sub-2-category of $\mathfrak{B}\omega_1\mathfrak{PTop}$.

Given an interpretation $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$, precomposition with F yields a functor

$$F^* : \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1}, \mathsf{C}) \longrightarrow \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{C})$$

for any Boolean ω_1-pretopos C. By Proposition 5.10.5, this is equivalently a functor

$$F^* : \mathsf{Mod}_{\mathsf{C}}(\mathcal{L}', \mathcal{T}') \longrightarrow \mathsf{Mod}_{\mathsf{C}}(\mathcal{L}, \mathcal{T});$$

in other words, F gives a uniform way of defining a model of $\mathcal{T}$ from a model of $\mathcal{T}'$. However, this latter functor is canonically defined only up to isomorphism: given a model $\mathcal{M}$ of $\mathcal{T}'$ in C, turning it into an ω_1-coherent functor $\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1} \to \mathsf{C}$ (i.e., computing the inverse image of $\mathcal{M}$ under Proposition 5.10.5) requires a choice of representatives for the disjoint unions and quotients involved in the interpretation of $\mathcal{T}'$-imaginary sorts in $\mathcal{M}$.

We now specialize to the case $\mathsf{C} = \mathsf{Count}$, so that $\mathsf{Mod}_{\mathsf{C}}(\mathcal{L}, \mathcal{T}) = \mathsf{Mod}(\mathcal{L}, \mathcal{T})$. We will show that in this case, the functor $F^* : \mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ above can always be taken to be Borel; moreover, the assignment $F \mapsto F^*$ can be made (pseudo)functorial "in a Borel way". This is conceptually straightforward, although the details (which involve coding functions) are quite messy.

We regard Count as a standard Borel category by equipping its space of morphisms $\bigsqcup_{M,N \in \{0,1,\dots,\mathbb{N}\}} N^M$ with the obvious standard Borel structure. Recall that Count is also a Boolean ω_1-pretopos. The following lemma says that the Boolean ω_1-pretopos operations may be taken to be Borel:

Lemma 5.10.6. *There are Borel coding maps implementing the Boolean ω_1-pretopos operations on* Count*, i.e., which*

(a) given two objects $M, N \in \mathsf{Count}$, yields an object $P \in \mathsf{Count}$ together with a bijection $(p, q) : P \cong M \times N$;

(b) given two objects $M, N \in \mathsf{Count}$ and two morphisms $f, g : M \to N$, yields an object $E \in \mathsf{Count}$ together with an injection $h : E \to M$ whose image is the equalizer $\{m \in M \mid f(m) = g(m)\}$;

(c) given three objects L, M, N and morphisms $f : M \to L$ and $g : N \to L$, yields an object P together with a bijection $(p, q) : P \to M \times_L N$;

(d) given objects $(N_i)_{i \in \mathbb{N}}$, yields an object S together with injections $j_i : N_i \to S$ forming a bijection $\bigsqcup_i N_i \cong S$;

(e) given objects M, N and maps $p, q : M \to N$ such that $(p, q) : M \to N^2$ is injective with image an equivalence relation on N, yields an object Q together with a surjection $r : N \to Q$ exhibiting Q as the quotient;

(f) given objects M, N and a map $f : M \to N$, yields an object I, a surjection $g : M \to I$, and an injection $h : I \to N$, such that $f = h \circ g$;

(g) given an object M, objects $(N_i)_{i \in \mathbb{N}}$, and injections $f_i : N_i \to M$, yields an object U and an injection $g : U \to M$ with image the union of the images of the f_i;

(h) given objects M, N and an injection $f : M \to N$, yields an object C and an injection $g : C \to N$ whose image is the complement of that of f.

Proof. All straightforward. $\square$

We now prove that for $\mathsf{C} = \mathsf{Count}$, the functor in Proposition 5.10.5 is an equivalence "in a Borel way" (which doesn't literally make sense, as $\omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count})$ is not standard Borel):

Lemma 5.10.7. *Let $\mathcal{L}$ be a countable relational language, $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$-theory. Let*

$$L : \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count}) \longrightarrow \mathsf{Mod}(\mathcal{L}, \mathcal{T})$$

be the functor $F \mapsto F^*(\mathcal{X})$ from Proposition 5.10.5. There is a functor

$$K : \mathsf{Mod}(\mathcal{L}, \mathcal{T}) \longrightarrow \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count})$$

and a natural isomorphism

$$\zeta : K \circ L \longrightarrow 1_{\omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} | \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count})}$$

such that

(i) $L \circ K = 1$ (on the nose) and $\zeta_{K(\mathcal{M})} = 1_{K(\mathcal{M})}$, so that K, L form an adjoint equivalence;

(ii) for each imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$, the functor $K(-)(A) : \mathsf{Mod}(\mathcal{L}, \mathcal{T}) \to$ Count is Borel;

(iii) for each definable function $f : A \to B$, the natural transformation $K(-)(f) :$ $K(-)(A) \to K(-)(B)$ is Borel;

(iv) for each imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$, there is a countable subcategory $\mathsf{S}_A \subseteq \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ such that the morphism $\zeta_{F,A} : K(L(F))(A) \to F(A)$ (in Count), as $F \in \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count})$ varies, depends only on $F|\mathsf{S}_A$ and in a Borel way.

Proof. Recall from the proof of Proposition 5.10.3 (i.e., [J02, D1.4.7]) that an inverse equivalence of L is defined by sending a model $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ to the functor $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1} \to \mathsf{Count}$ which takes imaginary sorts and definable functions to their interpretations in $\mathcal{M}$. This is how we will define K, except that when interpreting, we use the Borel ω_1-pretopos operations from Lemma 5.10.6.

Put $K(\mathcal{M})(\mathbb{X}) := M$. For each n, using Lemma 5.10.6(a) (repeatedly), let $K(\mathcal{M})(\mathbb{X}^n)$ be an nth power (in Count) of M equipped with projections to M which depend in a Borel way on $\mathcal{M}$. For n-ary $R \in \mathcal{L}$, let $K(\mathcal{M})(R) := |R^{\mathcal{M}}|$, equipped with a monomorphism $K(\mathcal{M})(R) \hookrightarrow K(\mathcal{M})(\mathbb{X}^n)$ which is Borel in $\mathcal{M}$ and such that the image of $K(\mathcal{M})(R) \hookrightarrow K(\mathcal{M})(\mathbb{X}^n) \cong M^n$ is $R^{\mathcal{M}} \subseteq M^n$ (for example, the unique order-preserving such monomorphism).

Next, for each $\mathcal{L}_{\omega_1\omega}$-formula α with n variables, define $K(\mathcal{M})(\alpha)$ equipped with a monomorphism $K(\mathcal{M})(\alpha) \hookrightarrow K(\mathcal{M})(\mathbb{X}^n)$ to be the interpretation $\alpha^{\mathcal{M}}$ as in Definition 5.10.2, but using Lemma 5.10.6(a–c,f–h) for the products, pullbacks, unions, etc., in that definition; then both $K(\mathcal{M})(\alpha)$ and the monomorphism to $K(\mathcal{M})(\mathbb{X}^n)$ are Borel in $\mathcal{M}$. Similarly define $K(\mathcal{M})(A) := A^{\mathcal{M}}$ using Lemma 5.10.6(d,e) for $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$, equipped with maps $K(\mathcal{M})(\alpha_i) \to K(\mathcal{M})(A)$ which are Borel in $\mathcal{M}$; and define $K(\mathcal{M})(f) := f^{\mathcal{M}} : K(\mathcal{M})(A) \to K(\mathcal{M})(B)$, Borel in $\mathcal{M}$, for a definable function $f : A \to B$. For an isomorphism $g : \mathcal{M} \cong \mathcal{N}$ in $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$, $K(g)(A) : K(\mathcal{M})(A) \cong K(\mathcal{N})(A)$ is defined by induction on the structure of A in the obvious way. This completes the definition of K; (ii) and (iii) are immediate.

From the definition of $K(\mathcal{M})(R)$, we have $L(K(\mathcal{M})) = \mathcal{M}$; likewise, $L(K(g)) = K(g)(\mathbb{X}) = g$, which proves the first part of (i).

As in [J02, D1.4.7], we define $\zeta_{F,A} : K(L(F))(A) \to F(A)$ by induction on the structure of A in the obvious way. That is, $\zeta_{F,\mathbb{X}} : K(L(F))(\mathbb{X}) = F(\mathbb{X}) \to F(\mathbb{X})$ is the identity; $\zeta_{F,\mathbb{X}^n} : K(L(F))(\mathbb{X}^n) = F(\mathbb{X})^n \to F(\mathbb{X}^n)$ is the comparison isomorphism, using that F preserves finite products; for n-ary $R \in \mathcal{L}$, $\zeta_{F,R} : K(L(F))(R) \to F(R)$ is such that

$$\begin{tikzcd}
K(L(F))(R) \arrow[r, "\zeta_{F,R}"] \arrow[d, hook] & F(R) \arrow[d, hook] \\
K(L(F))(\mathbb{X}^n) = F(\mathbb{X})^n \arrow[r, "\zeta_{F,\mathbb{X}^n}", "\cong"'] & F(\mathbb{X}^n)
\end{tikzcd}$$

commutes; $\zeta_{F,\alpha} : K(L(F))(\alpha) \to F(\alpha)$ is defined by induction on α, using that F preserves the ω_1-pretopos operations used to interpret α; and for $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$, $\zeta_{F,A} : K(L(F))(A) = (\bigsqcup_i K(L(F))(\alpha_i))/(\bigsqcup_{i,j} K(L(F))(\varepsilon_{ij})) \to F(A)$ is the comparison isomorphism, using that F preserves countable $\bigsqcup$ and quotients. When $F = K(\mathcal{M})$, it is easily verified by induction that $\zeta_{K(\mathcal{M}),A}$ is the identity morphism for all A (e.g., when $A = \mathbb{X}^n$, the comparison $K(\mathcal{M})(\mathbb{X})^n \to K(\mathcal{M})(\mathbb{X}^n)$ is the identity, since $K(\mathcal{M})(\mathbb{X}^n)$ is by definition $K(\mathcal{M})(\mathbb{X})^n$); this proves (i).

Finally, for (iv), we take S_A to consist of the various limits and colimits used to define the comparison isomorphisms in the definition $\zeta_{F,A}$. That is, for $A = \mathbb{X}$, we take $\mathsf{S}_{\mathbb{X}} = \{\mathbb{X}\}$; for $A = \mathbb{X}^n$, we take $\mathsf{S}_{\mathbb{X}^n}$ to be $\mathbb{X}$, $\mathbb{X}^n$, and the projections $\mathbb{X}^n \to \mathbb{X}$;

for n-ary $R \in \mathcal{L}$, we take S_R to be $\mathsf{S}_{\mathbb{X}^n}$ together with the inclusion $R \hookrightarrow \mathbb{X}^n$; for $\alpha = \phi \wedge \psi$ with n variables, we take S_α to be $\mathsf{S}_\phi, \mathsf{S}_\psi$ (which contain the inclusions $\phi \hookrightarrow \mathbb{X}^n$ and $\psi \hookrightarrow \mathbb{X}^n$) together with the inclusions $\alpha \hookrightarrow \phi$ and $\alpha \hookrightarrow \psi$; etc. It is straightforward to check that this works. □

Recall that a **pseudofunctor** $F : \mathfrak{C} \to \mathfrak{D}$ between two 2-categories $\mathfrak{C}, \mathfrak{D}$ consists of an object $F(X) \in \mathfrak{D}$ for each object $X \in \mathfrak{C}$, a morphism $F(f) : F(X) \to F(Y)$ for each morphism $f : X \to Y$ in $\mathfrak{C}$, and a 2-cell $F(\alpha) : F(f) \to F(g)$ for each 2-cell $\alpha : f \to g$ in $\mathfrak{C}$, together with unnamed (but specified) isomorphisms $F(g) \circ F(f) \cong F(g \circ f)$ and $1_{F(X)} \cong F(1_X)$ for $f : X \to Y$ and $g : Y \to Z$ in $\mathfrak{C}$, which are required to obey certain coherence conditions. A **pseudonatural transformation** $\tau : F \to G$ between two pseudofunctors $F, G : \mathfrak{C} \to \mathfrak{D}$ consists of a morphism $\tau_X : F(X) \to G(X)$ for each object $X \in \mathfrak{C}$ and an invertible 2-cell $\tau_f : G(f) \circ \tau_X \to \tau_Y \circ F(f)$ for each morphism $f : X \to Y$ in $\mathfrak{C}$, subject to certain coherence conditions. A **modification** $\Theta : \sigma \to \tau$ between two pseudonatural transformations $\sigma, \tau : F \to G$ consists of a 2-cell $\Theta_X : \sigma_X \to \tau_X$ for each object $X \in \mathfrak{C}$, subject to certain conditions. See [J02, 1.1.2] or [Bor, II 7.5.1–3] for details.

Let $\mathfrak{Gpd}$ denote the 2-category of small groupoids, functors, and natural isomorphisms, and $\mathfrak{BorGpd}$ denote the 2-category of standard Borel groupoids, Borel functors, and Borel natural isomorphisms; we have a forgetful 2-functor $\mathfrak{BorGpd} \to \mathfrak{Gpd}$. From above, we have a 2-functor

$$\begin{aligned}
\omega_1\mathfrak{PTop}(\overline{\langle - \rangle}^B_{\omega_1}, \mathsf{Count}) : \omega_1\omega\mathfrak{Thy}^{\mathrm{op}}_{\omega_1} &\longrightarrow \mathfrak{Gpd} \\
(\mathcal{L}, \mathcal{T}) &\longmapsto \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count}) \\
(F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')) &\longmapsto F^* : \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1}, \mathsf{Count}) \\
&\qquad \to \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count}).
\end{aligned}$$

Proposition 5.10.8. *The assignment $(\mathcal{L}, \mathcal{T}) \mapsto \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ extends to a pseudofunctor*

$$\mathsf{Mod} : \omega_1\omega\mathfrak{Thy}^{\mathrm{op}}_{\omega_1} \longrightarrow \mathfrak{BorGpd}$$

which is a "lifting of $\omega_1\mathfrak{PTop}(\overline{\langle - \rangle}^B_{\omega_1}, \mathsf{Count})$", in that there are pseudonatural transformations K, L as in the following diagram (given componentwise by K, L from Lemma 5.10.7) which form an adjoint equivalence between $\omega_1\mathfrak{PTop}(\overline{\langle - \rangle}^B_{\omega_1}, \mathsf{Count})$

and the composite $\omega_1\omega\mathfrak{Thy}_{\omega_1}^{\mathrm{op}} \xrightarrow{\mathsf{Mod}} \mathfrak{BorGpd} \to \mathfrak{Gpd}$.

$$\begin{tikzcd} & & \mathfrak{BorGpd} \arrow[d] \\ \omega_1\omega\mathfrak{Thy}_{\omega_1}^{\mathrm{op}} \arrow[urr, "\mathsf{Mod}"] \arrow[rr, "\omega_1\mathfrak{PTop}(\overline{\langle - \rangle}^{B}_{\omega_1}{,}\mathsf{Count})"'] \arrow[urr, phantom, shift right=4, "\Downarrow K \ \Uparrow L"] & & \mathfrak{Gpd} \end{tikzcd}$$

Proof. For two theories $(\mathcal{L},\mathcal{T}),(\mathcal{L}',\mathcal{T}')$, we define Mod on the hom-category between them by

$$\begin{aligned} \omega_1\omega\mathfrak{Thy}_{\omega_1}((\mathcal{L},\mathcal{T}),(\mathcal{L}',\mathcal{T}')) &\longrightarrow \mathfrak{BorGpd}(\mathsf{Mod}(\mathcal{L}',\mathcal{T}'),\mathsf{Mod}(\mathcal{L},\mathcal{T})) \\ (F:(\mathcal{L},\mathcal{T})\to(\mathcal{L}',\mathcal{T}')) &\longmapsto L_{\mathcal{T}}\circ F^*\circ K_{\mathcal{T}'}, \end{aligned}$$

where K, L are as in Lemma 5.10.7. To check that this lands in $\mathfrak{BorGpd}$: for a model $\mathcal{M}\in\mathsf{Mod}(\mathcal{L}',\mathcal{T}')$, the model $\mathcal{N} := L_{\mathcal{T}}(F^*(K_{\mathcal{T}'}(\mathcal{M}))) = (F^*(K_{\mathcal{T}'}(\mathcal{M})))_*(\mathcal{X}) \in \mathsf{Mod}(\mathcal{L},\mathcal{T})$ has underlying set $N = F^*(K_{\mathcal{T}'}(\mathcal{M}))(\mathbb{X}) = K_{\mathcal{T}'}(\mathcal{M})(F(\mathbb{X}))$, and for each $R\in\mathcal{L}$, $R^{\mathcal{N}} = F^*(K_{\mathcal{T}'}(\mathcal{M}))(R) = K_{\mathcal{T}'}(\mathcal{M})(F(R))$; these are Borel in $\mathcal{M}$ by Lemma 5.10.7(ii). Similarly, for an isomorphism of models $g:\mathcal{M}\cong\mathcal{M}'$ in $\mathsf{Mod}(\mathcal{L}',\mathcal{T}')$, the isomorphism $L_{\mathcal{T}}(F^*(K_{\mathcal{T}'}(g))) : \mathcal{N}\cong\mathcal{N}'$ (where $\mathcal{N}' := L_{\mathcal{T}}(F^*(K_{\mathcal{T}'}(\mathcal{M}'))))$ is given by $K_{\mathcal{T}'}(g)(\mathbb{X}) : N\cong N'$, which is Borel in g by Lemma 5.10.7(ii). Thus $L_{\mathcal{T}}\circ F^*\circ K_{\mathcal{T}'}$ is a Borel functor. Furthermore, for a natural transformation between two interpretations $f:F\to G:(\mathcal{L},\mathcal{T})\to(\mathcal{L}',\mathcal{T}')$, the natural transformation $L_{\mathcal{T}}\circ f^*\circ K_{\mathcal{T}'} : L_{\mathcal{T}}\circ F^*\circ K_{\mathcal{T}'} \to L_{\mathcal{T}}\circ G^*\circ K_{\mathcal{T}'}$ is Borel: its component at $\mathcal{M}\in\mathsf{Mod}(\mathcal{L}',\mathcal{T}')$ is $L_{\mathcal{T}}(f^*_{K_{\mathcal{T}'}(\mathcal{M})}) = K_{\mathcal{T}'}(\mathcal{M})(f_{\mathbb{X}}) : K_{\mathcal{T}'}(\mathcal{M})(F(\mathbb{X})) \to K_{\mathcal{T}'}(\mathcal{M})(G(\mathbb{X}))$, which is Borel in $\mathcal{M}$ by Lemma 5.10.7(iii). Thus the above definition of Mod on each hom-category lands in $\mathfrak{BorGpd}$.

The isomorphism $1_{\mathsf{Mod}(\mathcal{L},\mathcal{T})} \cong \mathsf{Mod}(1_{(\mathcal{L},\mathcal{T})}) = L_{\mathcal{T}}\circ K_{\mathcal{T}}$ is the identity, using Lemma 5.10.7(i). For interpretations $F:(\mathcal{L},\mathcal{T})\to(\mathcal{L}',\mathcal{T}')$ and $G:(\mathcal{L}',\mathcal{T}')\to(\mathcal{L}'',\mathcal{T}'')$, the isomorphism

$$\begin{aligned} \mathsf{Mod}(F)\circ\mathsf{Mod}(G) &= L_{\mathcal{T}}\circ F^*\circ K_{\mathcal{T}'}\circ L_{\mathcal{T}'}\circ G^*\circ K_{\mathcal{T}''} \\ &\cong L_{\mathcal{T}}\circ F^*\circ G^*\circ K_{\mathcal{T}''} = \mathsf{Mod}(G\circ F) \end{aligned}$$

is given by $L_{\mathcal{T}}\circ F^*\circ\zeta_{\mathcal{T}'}\circ G^*\circ K_{\mathcal{T}''}$, where $\zeta_{\mathcal{T}'} : K_{\mathcal{T}'}\circ L_{\mathcal{T}'}\to 1$ is from Lemma 5.10.7. The coherence conditions are straightforward (the one corresponding to unitality uses the triangle identities, Lemma 5.10.7(i)). To complete the definition of Mod, we need only verify that the natural isomorphism $\mathsf{Mod}(F)\circ\mathsf{Mod}(G)\cong\mathsf{Mod}(G\circ F)$ is Borel. For a model $\mathcal{M}\in\mathsf{Mod}(\mathcal{L}'',\mathcal{T}'')$, the component of the isomorphism at $\mathcal{M}$ is

$L_{\mathcal{T}}(F^*(\zeta_{\mathcal{T}',G^*(K_{\mathcal{T}''}(\mathcal{M}))})) = \zeta_{\mathcal{T}',K_{\mathcal{T}''}(\mathcal{M})\circ G,F(\mathbb{X})}$; letting $\mathsf{S}_{F(\mathbb{X})} \subseteq \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1}$ be the countable subcategory given by Lemma 5.10.7(iv), $\zeta_{\mathcal{T}',K_{\mathcal{T}''}(\mathcal{M})\circ G,F(\mathbb{X})}$ is Borel in $K_{\mathcal{T}''}(\mathcal{M}) \circ G|\mathsf{S}_{F(\mathbb{X})}$, which is Borel in $\mathcal{M}$ by Lemma 5.10.7(ii,iii), as desired.

The components of K, L on objects are given by Lemma 5.10.7. On a morphism $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$, $K_F : F^* \circ K_{\mathcal{T}'} \to K_{\mathcal{T}} \circ \mathsf{Mod}(F) = K_{\mathcal{T}} \circ L_{\mathcal{T}} \circ F^* \circ K_{\mathcal{T}'}$ is given by $\zeta_{\mathcal{T}}^{-1} \circ F^* \circ K_{\mathcal{T}'}$, while $L_F : \mathsf{Mod}(F) \circ L_{\mathcal{T}'} = L_{\mathcal{T}} \circ F^* \circ K_{\mathcal{T}'} \circ L_{\mathcal{T}'} \to L_{\mathcal{T}} \circ F^*$ is given by $L_{\mathcal{T}} \circ F^* \circ \zeta_{\mathcal{T}'}$. The coherence conditions are again straightforward (again using Lemma 5.10.7(i) for units). We have $L \circ K = 1$ by Lemma 5.10.7(i) (both parts), as well as a modification $\zeta : K \circ L \to 1$ given componentwise by Lemma 5.10.7; by Lemma 5.10.7(i), these make K, L into an adjoint equivalence. □

Henceforth we will denote $\mathsf{Mod}(F) : \mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ also by F^*, whenever there is no risk of confusion with $F^* : \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1}, \mathsf{Count}) \to \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count})$.

5.11 Stone duality

In this section, we explain how Theorem 5.1.3 may be viewed as one half of a Stone-type duality, yielding a "strong conceptual completeness" theorem for $\mathcal{L}_{\omega_1\omega}$. We then use this viewpoint to deduce the Borel version of the main result of [HMM]. This section depends on the previous section, and like it, involves some tedious 2-categorical technicalities.

First, we briefly recall the abstract setup of the original Stone duality between the categories Bool of Boolean algebras and KZHaus of compact Hausdorff zero-dimensional spaces; our point of view here can be found in e.g., [J82, VI §4]. We have a **dualizing object**, the set $2 = \{0, 1\}$, which is both a Boolean algebra and a compact Hausdorff zero-dimensional space; and these two types of structure commute, meaning that the Boolean operations (e.g., $\wedge : 2 \times 2 \to 2$) are continuous. As a consequence, for every other $A \in \mathsf{Bool}$ and $X \in \mathsf{KZHaus}$, the set of Boolean homomorphisms $\mathsf{Bool}(A, 2)$ inherits the pointwise KZHaus-topology from 2, and the set of continuous maps $\mathsf{KZHaus}(X, 2)$ inherits the pointwise Boolean structure from 2; and we have natural bijections

$$\mathsf{Bool}(A, \mathsf{KZHaus}(X, 2)) \cong (\mathsf{Bool}, \mathsf{KZHaus})(A \times X, 2) \cong \mathsf{KZHaus}(X, \mathsf{Bool}(A, 2)),$$

where $(\mathsf{Bool}, \mathsf{KZHaus})(A \times X, 2)$ denotes the set of **bihomomorphisms** $A \times X \to 2$, i.e., maps which are continuous for each fixed $a \in A$ and Boolean homomorphisms for each fixed $x \in X$. This yields a contravariant adjunction between the functors

$$\mathsf{Bool}(-,2) : \mathsf{Bool}^{\mathrm{op}} \longrightarrow \mathsf{KZHaus}, \qquad \mathsf{KZHaus}(-,2) : \mathsf{KZHaus}^{\mathrm{op}} \longrightarrow \mathsf{Bool},$$

whose adjunction units are the "evaluation" maps

$$\begin{aligned} A &\longrightarrow \mathsf{KZHaus}(\mathsf{Bool}(A,2),2) & X &\longrightarrow \mathsf{Bool}(\mathsf{KZHaus}(X,2),2) \\ a &\longmapsto (x \mapsto x(a)) & x &\longmapsto (a \mapsto a(x)). \end{aligned}$$

The Stone duality theorem states that these maps are isomorphisms, i.e., the adjunction is an adjoint equivalence $\mathsf{Bool}^{\mathrm{op}} \cong \mathsf{KZHaus}$.

In the case where $A = \langle \mathcal{L} \mid \mathcal{T} \rangle$ is the Lindenbaum–Tarski algebra of a finitary propositional theory $(\mathcal{L}, \mathcal{T})$, $\mathsf{Bool}(A, 2)$ is the space $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ of models of $\mathcal{T}$; and the half of Stone duality asserting that the unit at A is an isomorphism is the completeness theorem for the theory $\mathcal{T}$, plus the "definability" theorem that every clopen set of models is named by some formula. The significance of the dualizing object 2 is that the syntax of propositional logic (i.e., propositional formulas) is to be interpreted as elements of 2.

In first-order logic, the syntax is assigned values of sets, functions, and relations; thus, the dualizing object for a first-order analog of Stone duality is naturally taken to be some variant of the category Set. We listed several such (half-)duality theorems in the Introduction, notably those of Makkai [M87][M88] who introduced the term **strong conceptual completeness** for this kind of logical interpretation of duality theorems. Here, our goal is to interpret Theorem 5.1.3 as such a (half-)duality. We take the dualizing object to be Count (from Section 5.10, equivalent to the category of countable sets), equipped with the structure of a Boolean ω_1-pretopos as well as that of a standard Borel groupoid (by forgetting the non-isomorphisms); Lemma 5.10.6 can be seen as showing that these two kinds of structure commute. However, there are some technical difficulties in directly copying the setup of Stone duality.

Let $(\mathcal{L}, \mathcal{T})$ be an $\mathcal{L}_{\omega_1\omega}$-theory and $\mathsf{G} = (G^0, G^1)$ be a standard Borel groupoid. We would like to say, on the basis of the two commuting structures on Count, that we have a 2-adjunction

$$\text{“}\,\omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Count})) \cong \mathfrak{BorGpd}(\mathsf{G}, \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Count}))\,\text{”}$$

as in Stone duality. It is easily seen that $\mathfrak{BorGpd}(\mathsf{G}, \mathsf{Count})$ is a Boolean ω_1-pretopos (with the pointwise operations from Count). The problem is that the groupoid

$\omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathsf{Count})$ is not standard Borel. Instead, we must replace it with the equivalent standard Borel groupoid $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$, using Proposition 5.10.8.

We proceed as follows. Forgetting for now the Borel structure on Count, we have an isomorphism

$$\begin{aligned}\omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathfrak{Gpd}(\mathsf{G}, \mathsf{Count})) &\cong (\omega_1\mathfrak{PTop}, \mathfrak{Gpd})(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1} \times \mathsf{G}, \mathsf{Count}) \\ &\cong \mathfrak{Gpd}(\mathsf{G}, \omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathsf{Count}))\end{aligned}$$

as in Stone duality, where $(\omega_1\mathfrak{PTop}, \mathfrak{Gpd})(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1} \times \mathsf{G}, \mathsf{Count})$ denotes the category of "bihomomorphisms" $\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1} \times \mathsf{G} \to \mathsf{Count}$, i.e., functors which are ω_1-coherent for each fixed object $x \in G^0$; this isomorphism is clearly (strictly) natural in $(\mathcal{L}, \mathcal{T})$ and G. Composing this isomorphism with the postcomposition functors

$$\begin{aligned}K_* &: \mathfrak{Gpd}(\mathsf{G}, \mathsf{Mod}(\mathcal{L}, \mathcal{T})) \longrightarrow \mathfrak{Gpd}(\mathsf{G}, \omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathsf{Count})), \\ L_* &: \mathfrak{Gpd}(\mathsf{G}, \omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathsf{Count})) \longrightarrow \mathfrak{Gpd}(\mathsf{G}, \mathsf{Mod}(\mathcal{L}, \mathcal{T}))\end{aligned}$$

induced by the functors K, L from Lemma 5.10.7 yields an adjoint equivalence consisting of

$$\begin{aligned}\Phi : \mathfrak{Gpd}(\mathsf{G}, \mathsf{Mod}(\mathcal{L}, \mathcal{T})) &\longrightarrow \omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathfrak{Gpd}(\mathsf{G}, \mathsf{Count})) \\ F &\longmapsto (A \mapsto (x \mapsto K(F(x))(A))), \\ \Psi : \omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathfrak{Gpd}(\mathsf{G}, \mathsf{Count})) &\longrightarrow \mathfrak{Gpd}(\mathsf{G}, \mathsf{Mod}(\mathcal{L}, \mathcal{T})) \\ G &\longmapsto (x \mapsto L(G(-)(x))),\end{aligned}$$

such that $\Psi \circ \Phi = 1$, and a natural isomorphism $\xi : \Phi \circ \Psi \to 1$ satisfying the triangle identities (induced by $\zeta : K \circ L \to 1$ from Lemma 5.10.7), given by

$$\xi_{G,A,x} = \zeta_{G(-)(x),A} : \Phi(\Psi(G))(A)(x) = K(L(G(-)(x)))(A) \longrightarrow G(A)(x).$$

Since K, L are pseudonatural and ζ is a modification by Proposition 5.10.8, this adjoint equivalence remains natural in G and pseudonatural $(\mathcal{L}, \mathcal{T})$.

Lemma 5.11.1. Φ, Ψ, ξ *restrict to a pseudonatural adjoint equivalence*

$$\omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Count})) \cong \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Mod}(\mathcal{L}, \mathcal{T})).$$

Proof. First, we check that Ψ, Ψ, ξ restrict for fixed $\mathsf{G}, \mathcal{T}$. To check that Ψ restricts, let $G, G' \in \omega_1\mathfrak{PTop}(\overline{\langle\mathcal{L} \mid \mathcal{T}\rangle}^B_{\omega_1}, \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Count}))$ and $\gamma : G \to G'$ with γ_A Borel for each A; we must check that $\Psi(G), \Psi(G'), \Psi(\gamma)$ are Borel. $\Psi(G)(x) = L(G(-)(x))$ is the model with underlying set $G(\mathbb{X})(x)$ and with $R^{\Psi(G)(x)}$ = the image of $G(R)(x) \hookrightarrow G(\mathbb{X}^n)(x)$ for n-ary $R \in \mathcal{L}$; thus $\Psi(G)(x)$ is Borel in x. For a morphism $g : x \to y$ in G, we have $\Psi(G)(x) = G(\mathbb{X})(g)$, which is Borel in g. Thus, $\Psi(G)$ is Borel; similarly, $\Psi(G')$ is Borel. And $\Psi(\gamma)(x) = \gamma_{\mathbb{X},x}$; so $\Psi(\gamma)$ is Borel.

To check that Φ restricts, let $F, F' \in \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Mod}(\mathcal{L}, \mathcal{T}))$ and $\phi : F \to F'$ be Borel; we must check that $\Phi(F), \Phi(F'), \Phi(\phi)$ are pointwise Borel. For $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$, we have $\Phi(F)(A)(x) = K(F(x))(A)$ which is Borel in x (and similarly when x is replaced by $g : x \to y$) by Lemma 5.10.7(ii) and Borelness of F; similarly, for a definable function $f : A \to B$, $\Phi(F)(f)(x) = K(F(x))(f)$ is Borel in x by Lemma 5.10.7(iii). Similarly, $\Phi(\phi)(A)(x) = K(\phi_x)(A)$ which is Borel in x by Lemma 5.10.7(ii) and Borelness of ϕ.

To check that ξ restricts, let $G \in \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Count}))$; then $\xi_{G,A,x} = \zeta_{G(-)(x),A}$ is Borel in $G(-)(x)|\mathsf{S}_A$ which is Borel in x, where $\mathsf{S}_A \subseteq \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ is given by Lemma 5.10.7(iv).

Finally, we must check that the pseudonaturality isomorphisms for Φ, Ψ as $(\mathcal{L}, \mathcal{T})$ varies are Borel (there is nothing to check as G varies, since Φ, Ψ are natural in G). Let $H : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ be an interpretation. From the proof of Proposition 5.10.8, it is easily seen that the pseudonaturality isomorphism $\Psi_H : \mathsf{Mod}(H)_* \circ \Psi_{\mathcal{T}'} \to \Psi_{\mathcal{T}} \circ H^*$ (induced by that for L) is given by

$$\Psi_{H,G,x} = \zeta_{\mathcal{T}',G(-)(x),H(\mathbb{X})} : K_{\mathcal{T}'}(L_{\mathcal{T}'}(G(-)(x)))(H(\mathbb{X})) \longrightarrow G(H(\mathbb{X}))(x)$$

which is Borel in x using Lemma 5.10.7(iv) as above, while the pseudonaturality isomorphism $\Phi_H : H^* \circ \Phi_{\mathcal{T}'} \to \Phi_{\mathcal{T}} \circ \mathsf{Mod}(H)_*$ (induced by that for K) is given by

$$\Phi_{H,F,A,x} = \zeta^{-1}_{\mathcal{T},K_{\mathcal{T}'}(F(x))\circ H,A} : K_{\mathcal{T}'}(F(x))(H(A)) \longrightarrow K_{\mathcal{T}}(L_{\mathcal{T}}(K_{\mathcal{T}'}(F(x)) \circ H))(A)$$

which is Borel in x using Lemma 5.10.7(ii–iv). □

Thus, in place of the adjunction in Stone duality, we have a "relative pseudoadjunction" between the pseudofunctors

$$\mathsf{Mod} : \omega_1\omega\mathfrak{Thy}^{\mathrm{op}}_{\omega_1} \longrightarrow \mathfrak{BorGpd}, \quad \mathfrak{BorGpd}(-, \mathsf{Count}) : \mathfrak{BorGpd}^{\mathrm{op}} \longrightarrow \mathfrak{B}\omega_1\mathfrak{PTop};$$

"relative" means that Mod is not defined on all of $\mathfrak{B}\omega_1\mathfrak{PTop}$, but only on $\omega_1\omega\mathfrak{Thy}_{\omega_1}$ (which, recall, is equivalent to a full sub-2-category of the former). See [LMV] for basic facts on pseudoadjunctions (also called biadjunctions), and [Ulm] for relative adjunctions. We still have one adjunction unit, namely the transpose across Lemma 5.11.1 of the identity $\mathsf{Mod}(\mathcal{L},\mathcal{T}) \to \mathsf{Mod}(\mathcal{L},\mathcal{T})$:

$$\begin{aligned} \eta_{\mathcal{T}} = \Phi(1_{\mathsf{Mod}(\mathcal{L},\mathcal{T})}) : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^{B}_{\omega_1} &\longrightarrow \mathfrak{BorGpd}(\mathsf{Mod}(\mathcal{L},\mathcal{T}), \mathsf{Count}) \\ A &\longmapsto K_{\mathcal{T}}(-)(A). \end{aligned}$$

We next verify that this unit functor is none other than the interpretation functor $[\![-]\!]$ from Theorem 5.1.3, under the following standard identification.

A Borel functor $F : \mathsf{G} \to \mathsf{Count}$ determines a fiberwise countable Borel G-space, namely

$$\Sigma(F) := \{(x,a) \mid x \in G^0 \ \& \ a \in F(x)\} \subseteq G^0 \times \mathbb{N},$$

with the projection $p : \Sigma(F) \to G^0$ and the obvious action of G: $g \cdot (x,a) := (y, F(g)(a))$ for $g : x \to y$. Given another Borel functor $G : \mathsf{G} \to \mathsf{Count}$ and a Borel natural transformation $f : F \to G$, we have the G-equivariant map $\Sigma(f) : \Sigma(F) \to \Sigma(G)$ given fiberwise by the components of f. Thus, we have a functor

$$\Sigma = \Sigma_{\mathsf{G}} : \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Count}) \longrightarrow \mathsf{Act}^{B}_{\omega_1}(\mathsf{G}).$$

Note that $\mathsf{Act}^{B}_{\omega_1}(-)$ is contravariantly pseudofunctorial: given a Borel functor $F : \mathsf{G} \to \mathsf{H}$, we may pull back H-spaces $p : X \to H^0$ along F to obtain G-spaces $F^*(X) = G^0 \times_{H^0} X$ (temporarily denote this by $G^0 \times^{F}_{H^0} X$); and given a Borel natural isomorphism $\theta : F \cong F' : \mathsf{G} \to \mathsf{H}$ and a fiberwise countable Borel H-space $p : X \to H^0$, we have a Borel G-equivariant isomorphism

$$\begin{aligned} \theta^*(X) : G^0 \times^{F}_{H^0} X &\cong G^0 \times^{F'}_{H^0} X \\ (y,x) &\mapsto (y, \theta_y \cdot x). \end{aligned}$$

Lemma 5.11.2. $\Sigma : \mathfrak{BorGpd}(-, \mathsf{Count}) \to \mathsf{Act}^{B}_{\omega_1}(-)$ *is a pseudonatural equivalence between pseudofunctors* $\mathfrak{BorGpd}^{\mathrm{op}} \to \mathfrak{B}\omega_1\mathfrak{PTop}$.

Proof. First, we check that for fixed G, Σ_{G} is an equivalence. Faithfulness and fullness are clear from the definition of $\Sigma_{\mathsf{G}}(f)$. For essential surjectivity, given an arbitrary fiberwise countable Borel G-space $p : X \to G^0$, we may use Lusin–Novikov to enumerate each fiber $p^{-1}(x)$ in a Borel way, yielding bijections $e_x : p^{-1}(x) \cong |p^{-1}(x)|$ such that e_x and $|p^{-1}(x)|$ are Borel in x; then defining $F : \mathsf{G} \to \mathsf{Count}$

by $F(x) := |p^{-1}(x)|$ and $F(g) := e_y \circ g \circ e_x^{-1}$ for $g : x \to y$ in G, we clearly have $X \cong \Sigma(F)$. (This was noted at the end of Section 5.3.)

For a Borel functor $F : \mathsf{G} \to \mathsf{H}$, the isomorphism $\Sigma_F : F^* \circ \Sigma_{\mathsf{H}} \cong \Sigma_{\mathsf{G}} \circ F^*$ is given by

$$\begin{aligned}\Sigma_{F,G} : F^*(\Sigma_{\mathsf{H}}(G)) = G^0 \times^F_{H^0} \Sigma_{\mathsf{H}}(G) &\longrightarrow \Sigma_{\mathsf{G}}(G \circ F) = \Sigma_{\mathsf{G}}(F^*(G)) \\ (x, (y, a)) &\longmapsto (x, a)\end{aligned}$$

for $G : \mathsf{H} \to \mathsf{Count}$. It is straightforward to check that this works. $\square$

Lemma 5.11.3. *For a countable $\mathcal{L}_{\omega_1\omega}$-theory $(\mathcal{L}, \mathcal{T})$, we have $\Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})} \circ \eta_{\mathcal{T}} \cong [\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1} \to \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$.*

Proof. By Lemma 5.11.2 and the definition of $\eta_{\mathcal{T}}$, $\Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})} \circ \eta_{\mathcal{T}}$ is an ω_1-coherent functor; from the definition of $[\![-]\!]$, it is easy to check that $[\![-]\!]$ is also an ω_1-coherent functor. Thus, by Proposition 5.10.5, it suffices to check that the models of $\mathcal{T}$ in $\mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$ corresponding to $\Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})} \circ \eta_{\mathcal{T}}$ and $[\![-]\!]$ are isomorphic. The former has underlying object

$$\begin{aligned}\Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})}(\eta_{\mathcal{T}}(\mathbb{X})) &= \Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})}(K_{\mathcal{T}}(-)(\mathbb{X})) \\ &= \{(\mathcal{M}, a) \mid \mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \;\&\; a \in M\} = [\![\mathbb{X}]\!];\end{aligned}$$

and the interpretation of n-ary $R \in \mathcal{L}$ is (the image of) $\Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})}(\eta_{\mathcal{T}}(R)) \hookrightarrow \Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})}(\eta_{\mathcal{T}}(\mathbb{X}^n)) \cong \Sigma_{\mathsf{Mod}(\mathcal{L},\mathcal{T})}(\eta_{\mathcal{T}}(\mathbb{X}))^n$, which from the definition of $K_{\mathcal{T}}$ (from Lemma 5.10.7) is easily seen to be $[\![R]\!] \subseteq [\![\mathbb{X}^n]\!]$. Thus, the two models are the same. $\square$

Combining Lemmas 5.11.1 to 5.11.3 and Theorem 5.1.3, we get

Proposition 5.11.4. *We have a contravariant relative pseudoadjunction*

$$\omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Act}^B_{\omega_1}(\mathsf{G})) \cong \mathfrak{BorGpd}(\mathsf{G}, \mathsf{Mod}(\mathcal{L}, \mathcal{T}))$$

between the pseudofunctors

$$\mathsf{Mod} : \omega_1\omega\mathfrak{Thy}^{\mathrm{op}}_{\omega_1} \longrightarrow \mathfrak{BorGpd}, \qquad \mathsf{Act}^B_{\omega_1} : \mathfrak{BorGpd}^{\mathrm{op}} \longrightarrow \mathfrak{B}\omega_1\mathfrak{PTop};$$

and the adjunction unit $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1} \to \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$ is (isomorphic to $[\![-]\!]$, and hence) an equivalence for every countable $\mathcal{L}_{\omega_1\omega}$-theory $(\mathcal{L}, \mathcal{T})$. $\square$

This is our promised interpretation of Theorem 5.1.3 as a half-duality. One consequence is the following reformulation of Theorem 5.1.3, which contains Theorem 5.1.1:

Corollary 5.11.5. *For any two theories* $(\mathcal{L}, \mathcal{T}), (\mathcal{L}', \mathcal{T}')$*, the functor*

$$\begin{aligned}\mathsf{Mod}_{\mathcal{T},\mathcal{T}'} : \omega_1\omega\mathfrak{Thy}_{\omega_1}(\mathcal{T}, \mathcal{T}') &\longrightarrow \mathfrak{BorGpd}(\mathsf{Mod}(\mathcal{L}', \mathcal{T}'), \mathsf{Mod}(\mathcal{L}, \mathcal{T})) \\ (F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')) &\longmapsto \mathsf{Mod}(F) = F^* : \mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})\end{aligned}$$

is an equivalence of groupoids.

Proof. This follows from a version (for relative pseudoadjunctions) of the standard fact that a left adjoint is full and faithful iff the unit is a natural isomorphism. See e.g., [LMV, 1.3].

(In more detail, $\mathsf{Mod}_{\mathcal{T},\mathcal{T}'}$ is easily seen to be isomorphic to the composite

$$\begin{aligned}\omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}^B_{\omega_1}) &\cong \omega_1\mathfrak{PTop}(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}, \mathsf{Act}^B_{\omega_1}(\mathsf{Mod}(\mathcal{L}', \mathcal{T}'))) \\ &\cong \mathfrak{BorGpd}(\mathsf{Mod}(\mathcal{L}', \mathcal{T}'), \mathsf{Mod}(\mathcal{L}, \mathcal{T})),\end{aligned}$$

where the second equivalence is given by Proposition 5.11.4 and the first equivalence is induced by the adjunction unit (i.e., $[\![-]\!]$) for $\mathcal{T}'$.) $\square$

Finally in this section, we explain how the Borel version of the main result of Harrison-Trainor–Miller–Montalbán [HMM, Theorem 9] can be viewed as a special case of Theorem 5.1.1.

Recall (see e.g., [Gao, §12.1]) that for a countable $\mathcal{L}$-structure $\mathcal{M}$, the **Scott sentence** of $\mathcal{M}$ is an $\mathcal{L}_{\omega_1\omega}$-sentence $\sigma_{\mathcal{M}}$ whose countable models are precisely the isomorphic copies of $\mathcal{M}$.

Let $\mathcal{M} \in \mathsf{Mod}(\mathcal{L})$ and $\mathcal{N} \in \mathsf{Mod}(\mathcal{L}')$ be countable structures (on initial segments of $\mathbb{N}$) in possibly different languages $\mathcal{L}, \mathcal{L}'$. According to [HMM], an **interpretation** $\mathcal{I}$ of $\mathcal{M}$ in $\mathcal{N}$ consists of:

(i) a subset $\mathcal{D}om^{\mathcal{N}}_{\mathcal{M}} \subseteq N^{<\omega}$, **definable (without parameters)** in $\mathcal{N}$, i.e., for each n we have $N^n \cap \mathcal{D}om^{\mathcal{N}}_{\mathcal{M}} = \phi^{\mathcal{N}}$ for some $\mathcal{L}'_{\omega_1\omega}$-formula with n variables;

(ii) a definable equivalence relation $\sim$ on $\mathcal{D}om^{\mathcal{N}}_{\mathcal{M}}$;

(iii) for each n-ary $R \in \mathcal{L}$, a $\sim$-invariant definable subset $R^{\mathcal{I}} \subseteq (\mathcal{D}om^{\mathcal{N}}_{\mathcal{M}})^n$;

(iv) an isomorphism of $\mathcal{L}$-structures $g_{\mathcal{I}} : (\mathcal{D}om^{\mathcal{N}}_{\mathcal{M}}/{\sim}, R^{\mathcal{I}}/{\sim})_{R\in\mathcal{L}} \cong \mathcal{M}$.

We may rephrase this in our terminology as follows. By completeness and the defining property of $\sigma_{\mathcal{N}}$, a definable subset $S \subseteq N^n$ is defined by a unique $\mathcal{L}'_{\omega_1\omega}$-formula modulo $\sigma_{\mathcal{N}}$-equivalence. Thus, definable subsets of N^n are in bijection with subobjects (i.e., subsorts) of $\mathbb{X}^n$ in $\overline{\langle \mathcal{L}' \mid \sigma_{\mathcal{N}} \rangle}^B_{\omega_1}$, and similarly for definable subsets of $N^{<\omega}$, etc. Furthermore, the conditions on the definable sets $\sim$ and $R^{\mathcal{I}}$ imposed by (ii–iv) above are equivalent to the corresponding syntactic conditions on the defining formulas being $\sigma_{\mathcal{N}}$-provable. Using this, it is easily seen that an interpretation $\mathcal{I}$ of $\mathcal{M}$ in $\mathcal{N}$ is equivalently given by

(i′) a subsort $D_{\mathcal{I}} \subseteq \bigsqcup_{n\in\mathbb{N}} \mathbb{X}^n$ in $\overline{\langle \mathcal{L}' \mid \sigma_{\mathcal{N}} \rangle}^B_{\omega_1}$;

(ii′) an equivalence relation $E_{\mathcal{I}}$ on $D_{\mathcal{I}}$;

(iii′) an interpretation $F_{\mathcal{I}} : (\mathcal{L}, \sigma_{\mathcal{M}}) \to (\mathcal{L}', \sigma_{\mathcal{N}})$ (in our sense), i.e., model of $\sigma_{\mathcal{M}}$ in $\overline{\langle \mathcal{L}' \mid \sigma_{\mathcal{N}} \rangle}^B_{\omega_1}$, with underlying object $F_{\mathcal{I}}(\mathbb{X}) = D_{\mathcal{I}}/E_{\mathcal{I}}$;

(iv′) an isomorphism $g_{\mathcal{I}} : F^*_{\mathcal{I}}(\mathcal{N}) \cong \mathcal{M}$ in $\mathsf{Mod}(\mathcal{L}, \sigma_{\mathcal{M}})$.

Remark 5.11.6. Note that $D_{\mathcal{I}}, E_{\mathcal{I}}, g_{\mathcal{I}}$ are in some sense irrelevant. Indeed, for any imaginary sort $A = (\bigsqcup_i \alpha_i)/(\bigsqcup_{i,j} \varepsilon_{ij}) \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}^B_{\omega_1}$ (in any theory $(\mathcal{L}, \mathcal{T})$), where α_i has n_i variables, we may express A as a quotient of a subsort of $\bigsqcup_{n\in\mathbb{N}} \mathbb{X}^n$, by picking $n'_0 < n'_1 < \cdots$ with $n'_i \geq n_i$ and then embedding $\alpha_i \subseteq \mathbb{X}^{n_i}$ into $\mathbb{X}^{n'_i}$ diagonally, so that $\bigsqcup_i \alpha_i \subseteq \bigsqcup_i \mathbb{X}^{n'_i} \subseteq \bigsqcup_n \mathbb{X}^n$. Thus, given $F_{\mathcal{I}} : (\mathcal{L}, \sigma_{\mathcal{M}}) \to (\mathcal{L}', \sigma_{\mathcal{N}})$, we can always find $D_{\mathcal{I}}$ and $E_{\mathcal{I}}$ as in (i′,ii′) such that $F_{\mathcal{I}}(\mathbb{X}) \cong D_{\mathcal{I}}/E_{\mathcal{I}}$. And since all countable models of $\sigma_{\mathcal{M}}$ are isomorphic to $\mathcal{M}$, we can always find $g_{\mathcal{I}}$ as in (iv′).

Given an interpretation $\mathcal{I}$ of $\mathcal{M}$ in $\mathcal{N}$, [HMM] defines the induced functor $\mathsf{Mod}(\mathcal{L}', \sigma_{\mathcal{N}}) \to \mathsf{Mod}(\mathcal{L}, \sigma_{\mathcal{M}})$ to take an isomorphic copy of $\mathcal{N}$ to the isomorphic copy of $\mathcal{M}$ given by $\mathcal{I}$, with domain replaced by (an initial segment of) $\mathbb{N}$ via some canonical coding of quotients of subsets of $\mathbb{N}^{<\omega}$. This is also how we defined $F^*_{\mathcal{I}} : \mathsf{Mod}(\mathcal{L}', \sigma_{\mathcal{N}}) \to \mathsf{Mod}(\mathcal{L}, \sigma_{\mathcal{M}})$ in Proposition 5.10.8, with the coding given by Lemma 5.10.6. Note that in accordance with the above remark, $D_{\mathcal{I}}, E_{\mathcal{I}}, g_{\mathcal{I}}$ are not used here.

The Borel version of [HMM, Theorem 9] states that every Borel functor $\mathsf{Mod}(\mathcal{L}', \sigma_{\mathcal{N}}) \to \mathsf{Mod}(\mathcal{L}, \sigma_{\mathcal{M}})$ is induced by some interpretation of $\mathcal{M}$ in $\mathcal{N}$. By the above, this is equivalent to Theorem 5.1.1 in the case where $\mathcal{T}, \mathcal{T}'$ are both (equivalent to) Scott sentences, i.e., when they both have a single countable model up to isomorphism.

5.12 κ-coherent frames and locales

In the rest of this paper, we sketch a proof of a generalization of Theorem 5.8.1 (itself a generalization of Theorem 5.1.4) using the Joyal–Tierney representation theorem for Grothendieck toposes. In this section, we review some concepts from locale theory; see [J82], [J02, C1], or [JT].

In this and the following sections, let κ be a regular *uncountable* cardinal or the symbol ∞ (bigger than all cardinals). By **κ-ary** we mean of size less than κ.

A **frame** is a poset with finite meets and arbitrary joins, the former distributing over the latter. A **locale** X is the same thing as a frame $\mathcal{O}(X)$, except that we think of X as a generalized topological space whose **frame of opens** is $\mathcal{O}(X)$. A **continuous map** or **locale morphism** $f : X \to Y$ between locales is a frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Thus, the category Loc of locales is the opposite of the category Frm of frames. A topological space X is regarded as the locale with $\mathcal{O}(X) = \{\text{open sets in } X\}$; thus we have a forgetful functor $\mathsf{Top} \to \mathsf{Loc}$. A locale X is **spatial** if it is isomorphic to a topological space. The **spatialization** $\mathrm{Sp}(X)$ of a locale X is the space of all locale morphisms $1 \to X$ or **points** (where $\mathcal{O}(1) = \{0 < 1\}$), with topology consisting of

$$[U] := \{x \in \mathrm{Sp}(X) \mid x^*(U) = 1\}$$

for $U \in \mathcal{O}(X)$; we have a canonical locale morphism $\varepsilon : \mathrm{Sp}(X) \to X$ given by $\varepsilon^* = [-] : \mathcal{O}(X) \to \mathcal{O}(\mathrm{Sp}(X))$, which is an isomorphism iff X is spatial.

A **sublocale** $Y \subseteq X$ is given by a quotient frame $\mathcal{O}(X) \twoheadrightarrow \mathcal{O}(Y)$. The **image sublocale** of a locale morphism $f : X \to Y$ is given by the image of the corresponding frame homomorphism f^*. Every open $U \in \mathcal{O}(X)$ gives rise to the **open sublocale** $U \subseteq X$ corresponding to the quotient frame $U \wedge (-) : \mathcal{O}(X) \twoheadrightarrow {\downarrow}U =: \mathcal{O}(U)$. A locale morphism $f : X \to Y$ is **open** if the image of every open $U \subseteq X$ is an open sublocale $f_+(U) \subseteq Y$, and **étalé** if X (i.e., the top element $\top \in \mathcal{O}(X)$) is the union of open sublocales $U \subseteq X$ to which the restriction $f|U : U \to Y$ is an open sublocale inclusion (with image $f_+(U)$; we call such U an **open section over** $f_+(U)$).

A **sheaf** on a frame L is a functor $L^{\mathrm{op}} \to \mathsf{Set}$ preserving limits (i.e., colimits in L) of the form $\bigvee A$ where $A \subseteq L$ is downward-closed; the category of sheaves on L is denoted $\mathsf{Sh}(L)$. A **sheaf** on a locale X is a sheaf on $\mathcal{O}(X)$; we also put $\mathsf{Sh}(X) := \mathsf{Sh}(\mathcal{O}(X))$. By standard sheaf theory (see [J02, C1.3]), a sheaf on X is equivalently given by an étalé locale over X.

A κ**-frame** is a poset with finite meets and κ-ary joins, the former distributing over the latter. For a κ-frame K, the κ**-ideal completion** $\mathrm{Idl}_\kappa(K)$ is a frame; a frame L (or locale X) is κ**-coherent** if L ($\mathcal{O}(X)$) is isomorphic to $\mathrm{Idl}_\kappa(K)$ for some κ-frame K. The κ**-compact** elements of L, denoted $L_\kappa \subseteq L$, are those $u \in L$ such that whenever $u \le \bigvee_i v_i$ then $u \le \bigvee_j v_{i_j}$ for a κ-ary subfamily $\{v_{i_j}\}_j$; for a κ-frame K we have $\mathrm{Idl}_\kappa(K)_\kappa \cong K$, hence for a κ-coherent frame L we have $L \cong \mathrm{Idl}_\kappa(L_\kappa)$. For a locale X, put $\mathcal{O}_\kappa(X) := \mathcal{O}(X)_\kappa$. A locale morphism $f : X \to Y$ between κ-coherent X, Y is κ**-coherent** if $f^*(\mathcal{O}_\kappa(Y)) \subseteq \mathcal{O}_\kappa(X)$, i.e., f^* is the join-preserving extension of a κ-frame homomorphism $\mathcal{O}_\kappa(Y) \to \mathcal{O}_\kappa(X)$. Thus, the category $\kappa\mathsf{Loc}$ of κ-coherent locales and κ-coherent locale morphisms is equivalent to the opposite of the category $\kappa\mathsf{Frm}$ of κ-frames. Because of this, we also refer to a κ-coherent locale (morphism) simply as a κ**-locale (morphism)**.

Let X, Y be κ-locales. A locale morphism $f : X \to Y$ is κ**-étalé** if X is a κ-ary union of κ-compact open sections. It is easy to see that a κ-étalé morphism is automatically κ-coherent. Note that when $\kappa = \omega_1$, the notion of ω_1-étalé locale morphism is not quite analogous to the notion of countable étalé map from Section 5.2: the present notion requires the open sections to be ω_1-compact. (By Proposition 5.12.3 below, the two notions agree when restricted to quasi-Polish spaces.) Nonetheless, we have analogs of the basic properties in Lemma 5.2.1 (with "κ-étalé" in place of "countable étalé"), proved in exactly the same way. For a κ-locale Y, we write $\mathsf{Sh}_\kappa(Y) \subseteq \mathsf{Sh}(Y)$ for the subcategory of κ-étalé locales over Y (identified with sheaves).

A (κ-)frame L is κ**-presented** if it has a κ-ary presentation, i.e., there are $< \kappa$-many $u_i \in L$ and $< \kappa$-many equations between (κ-)frame terms involving the u_i such that L is the free (κ-)frame generated by the u_i subject to these equations. Let $\mathsf{Frm}_\kappa \subseteq \mathsf{Frm}$ (resp., $\kappa\mathsf{Frm}_\kappa \subseteq \kappa\mathsf{Frm}$) denote the full subcategory of κ-presented (κ-)frames. The following is straightforward:

Lemma 5.12.1. *For a κ-presented κ-frame K, the principal ideal embedding $\downarrow : K \to \mathrm{Idl}_\kappa(K)$ is an isomorphism; and $\mathrm{Idl}_\kappa : \kappa\mathsf{Frm} \to \mathsf{Frm}$ restricts to an equivalence of categories $\kappa\mathsf{Frm}_\kappa \cong \mathsf{Frm}_\kappa$.* □

We call a locale X κ**-copresented** if $\mathcal{O}(X)$ is κ-presented as a frame, or equivalently as a κ-frame. By Lemma 5.12.1, a κ-copresented locale is κ-coherent, with $\mathcal{O}_\kappa(X) = \mathcal{O}(X)$.

In the next lemma (a generalization of Lemma 5.2.2, by Proposition 5.12.3), we adopt the point of view featured prominently in [JT], where a (κ-)frame is viewed as analogous to a commutative ring; thus, a κ-frame homomorphism $f : K \to L$ exhibits L as a "K-algebra".

Lemma 5.12.2. *Let $f : X \to Y$ be a κ-étalé locale morphism, and let $\mathcal{U} \subseteq \mathcal{O}(X)$ be $< \kappa$-many κ-compact open sections covering X and closed under binary meets. Then $f^* : \mathcal{O}_\kappa(Y) \to \mathcal{O}_\kappa(X)$ exhibits $\mathcal{O}_\kappa(X)$ as the $\mathcal{O}_\kappa(Y)$-algebra presented by the generators $U \in \mathcal{U}$ and the relations*

$$\begin{aligned} U &= f^*(f_+(U)) \wedge V && \text{for } U \le V \in \mathcal{U}, \\ U \wedge V &= W && \text{for } U, V \in \mathcal{U} \text{ and } W = U \wedge V, \\ \top &= \textstyle\bigvee_{U \in \mathcal{U}} U. \end{aligned}$$

Thus, if Y is κ-copresented, then so is X.

Proof. Given another $\mathcal{O}_\kappa(Y)$-algebra (i.e., κ-frame homomorphism) $g : \mathcal{O}_\kappa(Y) \to L$, and a map $h : \mathcal{U} \to L$ such that the above relations (with f^* replaced by g) hold after applying h to $\mathcal{U}$, the unique κ-frame homomorphism $h' : \mathcal{O}_\kappa(X) \to L$ extending h such that $h' \circ f^* = g$ is given by

$$h'(W) := \textstyle\bigvee_{U \in \mathcal{U}} (g(f_+(U \wedge W)) \wedge h(U)).$$

It is straightforward to check that this works. □

A basic intuition regarding (κ-)locales is that they are (quasi-)Polish spaces generalized by removing countability requirements. This is made precise by the following. It can be found in [Hec], who states that similar results have been proved before by various authors.

Proposition 5.12.3. *The forgetful functor $\mathsf{Top} \to \mathsf{Loc}$ restricts to an equivalence $\mathsf{QPol} \to \mathsf{Loc}_{\omega_1}$ between the category of quasi-Polish spaces and the category of ω_1-copresented locales.*

5.13 Locally κ-presentable categories

This section collects some basic facts we will need on locally κ-presentable categories; see [AR].

Let C be a category with (small) κ-filtered colimits. An object $X \in \mathsf{C}$ is **κ-presentable** if the representable functor $\mathsf{C}(X, -) : \mathsf{C} \to \mathsf{Set}$ preserves κ-filtered colimits. Let $\mathsf{C}_\kappa \subseteq \mathsf{C}$ denote the full subcategory of κ-presentable objects. C is **κ-accessible** if C_κ is essentially small and generates C under κ-filtered colimits, and **locally κ-presentable** if it is furthermore cocomplete (equivalently, C_κ has κ-ary colimits).

An arbitrary category C has a **κ-ind-completion** $\mathsf{Ind}_\kappa(\mathsf{C})$, which is the free cocompletion of C under (small) κ-filtered colimits; see [AR, 2.26]. When C is small and has κ-ary colimits, $\mathsf{Ind}_\kappa(\mathsf{C})$ can be constructed as the full subcategory of $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ on the functors preserving κ-ary limits. For a small category K, $\mathsf{Ind}_\kappa(\mathsf{K})$ is κ-accessible, with $\mathsf{Ind}_\kappa(\mathsf{K})_\kappa$ equivalent to the Cauchy completion of K; conversely, for a κ-accessible category C, we have $\mathsf{C} \cong \mathsf{Ind}_\kappa(\mathsf{C}_\kappa)$.

Lemma 5.13.1. *A κ-ary product $\prod_i \mathsf{C}_i$ of locally κ-presentable categories C_i is locally κ-presentable, with $(\prod_i \mathsf{C}_i)_\kappa = \prod_i (\mathsf{C}_i)_\kappa$.*

Proof. See [AR, 2.67] (which is missing the hypothesis that the product must be κ-ary). ◻

Lemma 5.13.2. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be cocontinuous functors between locally κ-presentable categories such that $F(\mathsf{C}_\kappa), G(\mathsf{C}_\kappa) \subseteq \mathsf{D}_\kappa$, and let $\alpha, \beta : F \to G$ be natural transformations. Then the **equifier** of α, β, i.e., the full subcategory $\mathsf{Eq}(\alpha, \beta) \subseteq \mathsf{C}$ of those $X \in \mathsf{C}$ for which $\alpha_X = \beta_X$, is locally κ-presentable, with $\mathsf{Eq}(\alpha, \beta)_\kappa = \mathsf{Eq}(\alpha, \beta) \cap \mathsf{C}_\kappa$.*

Proof. See [AR, 2.76]. Alternatively, here is a direct proof. Since F, G are cocontinuous, $\mathsf{Eq}(\alpha, \beta) \subseteq \mathsf{C}$ is closed under colimits. Since $\mathsf{Eq}(\alpha, \beta) \subseteq \mathsf{C}$ is full, $\mathsf{Eq}(\alpha, \beta) \cap \mathsf{C}_\kappa \subseteq \mathsf{Eq}(\alpha, \beta)_\kappa$. So it suffices to show that every $X \in \mathsf{Eq}(\alpha, \beta)$ is a κ-filtered colimit of objects in $\mathsf{Eq}(\alpha, \beta) \cap \mathsf{C}_\kappa$; for this, it suffices to show that every morphism $f : Y \to X$ with $Y \in \mathsf{C}_\kappa$ factors through some $g : Z \to X$ with $Z \in \mathsf{Eq}(\alpha, \beta) \cap \mathsf{C}_\kappa$. We have $G(f) \circ \alpha_Y = \alpha_X \circ F(f) = \beta_X \circ F(f) = G(f) \circ \beta_Y : F(Y) \to G(X) = \varinjlim_{\mathsf{C}_\kappa \ni Z \to X} G(Z)$, so since $F(Y) \in \mathsf{D}_\kappa$, f factors as $Y =: Z_0 \xrightarrow{g_0} Z_1 \xrightarrow{h_1} X$ with $Z_1 \in \mathsf{C}_\kappa$ such that $G(g_0) \circ \alpha_{Z_0} = G(g_0) \circ \beta_{Z_0}$. Similarly, h_1 factors as $Z_1 \xrightarrow{g_1} Z_2 \xrightarrow{h_2} X$ with $Z_2 \in \mathsf{C}_\kappa$ such that $G(g_1) \circ \alpha_{Z_1} = G(g_1) \circ \beta_{Z_1}$. Continue finding $Z_0 \xrightarrow{g_0} Z_1 \xrightarrow{g_1} Z_2 \xrightarrow{g_2} \cdots \xrightarrow{h_i} X$ in this way, then put $Z := \varinjlim_i Z_i$, to get $\alpha_Z = \beta_Z$. Since κ is uncountable, $Z \in \mathsf{C}_\kappa$. ◻

Lemma 5.13.3. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be cocontinuous functors between locally κ-presentable categories such that $F(\mathsf{C}_\kappa), G(\mathsf{C}_\kappa) \subseteq \mathsf{D}_\kappa$. Then the **inserter** category $\mathsf{Ins}(F, G)$, whose objects are pairs (X, α) where $X \in \mathsf{C}$ and $\alpha : F(X) \to G(X)$, is locally κ-presentable, with $\mathsf{Ins}(F, G)_\kappa$ consisting of those (X, α) with $X \in \mathsf{C}_\kappa$.*

Proof. This result is almost certainly well-known, although we could not find a precise reference for it. Here is a proof sketch.

First, one shows that under the hypotheses of Lemma 5.13.2, the **inverter** $\mathsf{Inv}(\alpha) \subseteq \mathsf{C}$, i.e., the full subcategory of X such that α_X is invertible, is locally κ-presentable, with $\mathsf{Inv}(\alpha)_\kappa = \mathsf{Inv}(\alpha) \cap \mathsf{C}_\kappa$. This is done by an ω-step construction as in the proof of Lemma 5.13.2 (see also [J02, B3.4.9]).

Now consider $(1_\mathsf{C}, F), (1_\mathsf{C}, G) : \mathsf{C} \to \mathsf{C} \times \mathsf{D}$. By [AR, 2.43], the comma category $(1_\mathsf{C}, F){\downarrow}(1_\mathsf{C}, G) = \{(X, Y, f : X \to Y, g : F(X) \to G(Y))\}$ is locally κ-presentable, with an object (X, Y, f, g) locally presentable iff $X, Y \in \mathsf{C}_\kappa$. Clearly, $\mathsf{Ins}(F, G)$ is equivalent to the inverter of the natural transformation ϕ between the two projections $(1_\mathsf{C}, F){\downarrow}(1_\mathsf{C}, G) \to \mathsf{C}$ given by $\phi_{(X,Y,f,g)} := f$. □

5.14 κ-coherent theories

In this section, we briefly define the κ-ary analogs of the concepts from Sections 5.4 to 5.6.

Let $\mathcal{L}$ be a first-order (relational) language. Recall that $\mathcal{L}_{\kappa\omega}$ is the extension of finitary first-order logic with κ-ary conjunctions $\bigwedge$ and disjunctions $\bigvee$. A proof system for $\mathcal{L}_{\kappa\omega}$ may be found in [J02, D1.3]. Note that this proof system is *not* complete with respect to set-theoretic models.

The notions of **κ-coherent formula**, **κ-coherent axiom**, **κ-coherent theory**, **κ-coherent imaginary sort**, and **κ-coherent definable function** are defined as in Section 5.4, with κ-ary disjunctions/disjoint unions replacing countable ones throughout. The κ-coherent imaginary sorts and definable functions of a κ-coherent theory $(\mathcal{L}, \mathcal{T})$ form the **syntactic κ-pretopos**, denoted

$$\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa,$$

which is the free **κ-pretopos** (defined as in Definition 5.10.1 but with κ-ary disjoint unions; functors preserving the κ-pretopos structure are called **κ-coherent**) containing a **model of $\mathcal{T}$** (defined as in Definition 5.10.2). An **interpretation** between two κ-coherent theories $(\mathcal{L}, \mathcal{T}), (\mathcal{L}', \mathcal{T}')$ is a κ-coherent functor $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa \to \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\kappa$; the theories are **($\kappa$-coherently) Morita equivalent** if their syntactic κ-pretoposes are equivalent.

In the case $\kappa = \infty$, "∞-coherent" is better known as **geometric**, while the syntactic ∞-pretopos is better known as the **classifying topos** (and usually denoted by $\mathsf{Set}[\mathcal{T}]$

instead of $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$; see e.g., [J02, D3]). Note that when we speak of an ∞-coherent (i.e., geometric) theory $(\mathcal{L}, \mathcal{T})$, we still mean that $\mathcal{L}, \mathcal{T}$ form sets (and not proper classes).

A κ-coherent theory $(\mathcal{L}, \mathcal{T})$ may also be regarded as a geometric theory. The link between the syntactic κ-pretopos and the classifying topos is provided by

Lemma 5.14.1. *For a κ-coherent theory $(\mathcal{L}, \mathcal{T})$, we have $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty \cong \mathsf{Ind}_\kappa(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa)$ (more precisely, the inclusion $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa \subseteq \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$ exhibits the latter as the κ-ind-completion of the former).*

Proof. Recall that $\mathsf{Ind}_\kappa(\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa)$ can be taken as the κ-ary limit-preserving functors $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa^{\mathrm{op}} \to \mathsf{Set}$. On the other hand, by the theory of syntactic sites (see [J02, D3.1]), $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$ can be taken as the functors $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa^{\mathrm{op}} \to \mathsf{Set}$ preserving limits (i.e., colimits in $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa$) of the form $(\bigsqcup_i A_i)/(\bigsqcup_{i,j} E_{ij})$ where $A_i \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa$ are $< \kappa$-many objects and $\bigsqcup_{i,j} E_{ij} \subseteq (\bigsqcup_i A_i)^2$ is an equivalence relation. Clearly this includes κ-ary coproducts in $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa$. Since κ is uncountable, we may compute general coequalizers in $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa$ by imitating the usual procedure in Set (to compute the coequalizer of $f, g : A \to B$, take the quotient of the equivalence relation generated $(f, g) : A \to B^2$; see [J02, A1.4.19] for details). This reduces coequalizers in $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa$ to colimits of the form $(\bigsqcup_i A_i)/(\bigsqcup_{i,j} E_{ij})$. □

There is a more general notion of a **multi-sorted κ-coherent theory** $(\mathcal{S}, \mathcal{L}, \mathcal{T})$, where $\mathcal{S}$ is a set of sorts, $\mathcal{L}$ consists of $\mathcal{S}$-sorted relation symbols, and formulas have $\mathcal{S}$-sorted variables and quantifiers; see [J02, D1.1]. So far we have been considering the case of a single-sorted theory, where $\mathcal{S} = \{\mathbb{X}\}$. The definitions of syntactic ω_1-pretopos, etc., have obvious multi-sorted generalizations. Other than single-sorted theories, we will consider 0-sorted or **propositional κ-coherent theories** $(\mathcal{L}, \mathcal{T})$, where $\mathcal{L}$ is a set of proposition symbols (i.e., 0-ary relation symbols) and $\mathcal{T}$ is a set of implications between **propositional κ-coherent $\mathcal{L}$-formulas** (i.e., $\mathcal{L}$-formulas built with finite $\wedge$ and κ-ary $\bigvee$). The **Lindenbaum–Tarski algebra** $\langle \mathcal{L} \mid \mathcal{T} \rangle_\kappa$ of a propositional theory $(\mathcal{L}, \mathcal{T})$ is the κ-frame presented by $(\mathcal{L}, \mathcal{T})$; it is also the **syntactic category** of $(\mathcal{L}, \mathcal{T})$, as defined in Section 5.4. Recall from there that the syntactic κ-pretopos is a certain completion of the syntactic category; for propositional theories, this takes the form

$$\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa \cong \mathsf{Sh}_\kappa(\langle \mathcal{L} \mid \mathcal{T} \rangle_\kappa).$$

Let $(\mathcal{L}, \mathcal{T})$ be a (single-sorted) geometric theory. We define the **locale of countable $\mathcal{L}$-structures** $\mathrm{Mod}(\mathcal{L})$ by formally imitating the definition in Section 5.5. That is, we take the frame $\mathcal{O}(\mathrm{Mod}(\mathcal{L}))$ to be freely generated by the symbols

$$[\![|\mathbb{X}| \geq n]\!] \quad \text{for } n \in \mathbb{N}, \qquad [\![R(\vec{a})]\!] \quad \text{for } n\text{-ary } R \in \mathcal{L} \text{ and } \vec{a} \in \mathbb{N}^n,$$

subject to the relations (compare with the proof in Section 5.5 that $\mathrm{Mod}(\mathcal{L})$ is quasi-Polish)

$$\top \leq [\![|\mathbb{X}| \geq 0]\!], \quad [\![|\mathbb{X}| \geq n+1]\!] \leq [\![|\mathbb{X}| \geq n]\!], \quad [\![R(\vec{a})]\!] \leq [\![|\mathbb{X}| \geq \max_i (a_i + 1)]\!].$$

Clearly, a point of $\mathrm{Mod}(\mathcal{L})$ (i.e., a frame homomorphism $\mathcal{O}(\mathrm{Mod}(\mathcal{L})) \to 2$) is the same thing as an $\mathcal{L}$-structure on an initial segment of $\mathbb{N}$; thus the spatialization $\mathrm{Sp}(\mathrm{Mod}(\mathcal{L}))$ is just the space of countable $\mathcal{L}$-structures, as defined in Section 5.5. For each geometric $\mathcal{L}$-formula ϕ with n variables and $\vec{a} \in \mathbb{N}^n$, we define $[\![\phi(\vec{a})]\!] \in \mathcal{O}(\mathrm{Mod}(\mathcal{L}))$ by induction on ϕ in the obvious manner:

$$\begin{aligned}
[\![\top]\!] &:= [\![|\mathbb{X}| \geq \max_i a_i]\!], \\
[\![a_i = a_j]\!] &:= [\![|\mathbb{X}| \geq \max_i a_i]\!] \text{ if } a_i = a_j, \text{ else } \bot, \\
[\![\phi(\vec{a}) \wedge \psi(\vec{a})]\!] &:= [\![\phi(\vec{a})]\!] \wedge [\![\psi(\vec{a})]\!], \\
[\![\textstyle\bigvee_i \phi_i(\vec{a})]\!] &:= \textstyle\bigvee_i [\![\phi_i(\vec{a})]\!], \\
[\![\exists x\, \phi(\vec{a}, x)]\!] &:= \textstyle\bigvee_{b \in \mathbb{N}} [\![\phi(\vec{a}, b)]\!].
\end{aligned}$$

We define the **locale of countable models of $\mathcal{T}$** to be the sublocale $\mathrm{Mod}(\mathcal{L}, \mathcal{T}) \subseteq \mathrm{Mod}(\mathcal{L})$ determined by the relations (i.e., $\mathcal{O}(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))$ is the quotient of $\mathcal{O}(\mathrm{Mod}(\mathcal{L}))$ by these relations)

$$[\![\phi(\vec{a})]\!] \leq [\![\psi(\vec{a})]\!] \quad \text{for an axiom } \forall \vec{x}\, (\phi(\vec{x}) \Rightarrow \psi(\vec{x})) \text{ in } \mathcal{T}, \text{ where } \vec{a} \in \mathbb{N}^{|\vec{x}|}.$$

We similarly define the locale $\mathrm{Iso}(\mathcal{L})$ "of pairs $(g, \mathcal{M})$ where $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ and $g \in S_M$" by imitating Section 5.5: $\mathcal{O}(\mathrm{Iso}(\mathcal{L}))$ is the frame generated by the symbols

$$\partial_1^*(U) \quad \text{for (a generator) } U \in \mathcal{O}(\mathrm{Mod}(\mathcal{L})), \qquad [\![a \mapsto b]\!] \quad \text{for } a, b \in \mathbb{N},$$

subject to the relations in $\mathcal{O}(\mathrm{Mod}(\mathcal{L}))$ between generators of the first kind, and the relations

$$\begin{gathered}
[\![a \mapsto b]\!] \leq \partial_1^*([\![|\mathbb{X}| \geq a+1]\!]) \wedge \partial_1^*([\![|\mathbb{X}| \geq b+1]\!]), \\
([\![a \mapsto b]\!] \wedge [\![a \mapsto c]\!]) \vee ([\![b \mapsto a]\!] \wedge [\![c \mapsto a]\!]) \leq \bot \quad \text{for } b \neq c, \\
\partial_1^*([\![|\mathbb{X}| \geq a+1]\!]) \leq (\textstyle\bigvee_b [\![a \mapsto b]\!]) \wedge (\textstyle\bigvee_b [\![b \mapsto a]\!])
\end{gathered}$$

which say that "$g \in S_M$". Clearly, $\mathrm{Sp}(\mathrm{Iso}(\mathcal{L}))$ is the space of isomorphisms as defined in Section 5.5. We clearly have a locale morphism $\partial_1 : \mathrm{Iso}(\mathcal{L}) \to \mathrm{Mod}(\mathcal{L})$; we also have $\partial_0 : \mathrm{Iso}(\mathcal{L}) \to \mathrm{Mod}(\mathcal{L})$, $\iota : \mathrm{Mod}(\mathcal{L}) \to \mathrm{Iso}(\mathcal{L})$, $\mu : \mathrm{Iso}(\mathcal{L}) \times_{\mathrm{Mod}(\mathcal{L})} \mathrm{Iso}(\mathcal{L}) \to \mathrm{Iso}(\mathcal{L})$, and $\nu : \mathrm{Iso}(\mathcal{L}) \to \mathrm{Iso}(\mathcal{L})$, given by

$$\begin{aligned}
\partial_0^*(\llbracket |\mathbb{X}| \geq n \rrbracket) &:= \llbracket |\mathbb{X}| \geq n \rrbracket, \\
\partial_0^*(\llbracket R(\vec{a}) \rrbracket) &:= \textstyle\bigvee_{\vec{b}}(\bigwedge_i \llbracket b_i \mapsto a_i \rrbracket \wedge \partial_1^*(\llbracket R(\vec{b}) \rrbracket), \\
\iota^*(\partial_1^*(U)) &:= U, \\
\iota^*(\llbracket a \mapsto b \rrbracket) &:= \llbracket |\mathbb{X}| \geq a+1 \rrbracket \text{ if } a = b, \text{ else } \bot, \\
\nu^*(\partial_1^*(U)) &:= \partial_0^*(U), \\
\nu^*(\llbracket a \mapsto b \rrbracket) &:= \llbracket b \mapsto a \rrbracket, \\
\mu^*(\partial_1^*(U)) &:= \pi_1^*(\partial_1^*(U)), \\
\mu^*(\llbracket a \mapsto b \rrbracket) &:= \textstyle\bigvee_c(\pi_1^*(\llbracket a \mapsto c \rrbracket) \wedge \pi_0^*(\llbracket c \mapsto b \rrbracket)),
\end{aligned}$$

where $\pi_0, \pi_1 : \mathrm{Iso}(\mathcal{L}) \times_{\mathrm{Mod}(\mathcal{L})} \mathrm{Iso}(\mathcal{L}) \to \mathrm{Iso}(\mathcal{L})$ are the two projections. It is straightforward to check that these are well-defined and form the structure maps of a localic groupoid $\mathsf{Mod}(\mathcal{L})$, the **localic groupoid of countable $\mathcal{L}$-structures**. For a geometric $\mathcal{L}$-theory $\mathcal{T}$, the **localic groupoid of countable models of $\mathcal{T}$**, $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$, is the full subgroupoid on $\mathrm{Mod}(\mathcal{L}, \mathcal{T}) \subseteq \mathrm{Mod}(\mathcal{L})$.

Note that when $(\mathcal{L}, \mathcal{T})$ is a κ-coherent theory, $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$, $\mathrm{Iso}(\mathcal{L}, \mathcal{T})$, and the structure maps are κ-coherent. If furthermore $(\mathcal{L}, \mathcal{T})$ is κ-ary (i.e., $|\mathcal{L}|, |\mathcal{T}| < \kappa$), then $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ and $\mathrm{Iso}(\mathcal{L}, \mathcal{T})$ are κ-presented. When $\kappa = \omega_1$, for a countable ω_1-coherent theory $(\mathcal{L}, \mathcal{T})$, it is easily verified that the definition of $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ given here corresponds under Proposition 5.12.3 to that given in Section 5.5.

For κ-coherent $(\mathcal{L}, \mathcal{T})$, we let $\mathsf{Act}_\kappa(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$ denote the **category of κ-étalé actions of** $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$, i.e., κ-étalé locales $X \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ over $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ equipped with an action $\mathrm{Iso}(\mathcal{L}, \mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L}, \mathcal{T})} X \to X$. Note that $\mathsf{Act}_\kappa(\mathsf{Mod}(\mathcal{L}, \mathcal{T})) \subseteq \mathsf{Act}_\infty(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$.

Finally, for a geometric theory $(\mathcal{L}, \mathcal{T})$, we define the **interpretation** $\llbracket A \rrbracket$ of an imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$ by imitating Section 5.6. That is, for a geometric formula α with n variables, we define $\llbracket \alpha \rrbracket$ to be the disjoint union of open sublocales $\llbracket \alpha \rrbracket_{\vec{a}} \subseteq \llbracket \alpha \rrbracket$ for $\vec{a} \in \mathbb{N}^n$, where each $\llbracket \alpha \rrbracket_{\vec{a}}$ is an isomorphic copy of $\llbracket \alpha(\vec{a}) \rrbracket \subseteq \mathrm{Mod}(\mathcal{L}, \mathcal{T})$, equipped with the étalé morphism $\pi : \llbracket \alpha \rrbracket = \bigsqcup_{\vec{a}} \llbracket \alpha \rrbracket_{\vec{a}} \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ induced by the inclusions $\llbracket \alpha \rrbracket_{\vec{a}} \cong \llbracket \alpha(\vec{a}) \rrbracket \subseteq \mathrm{Mod}(\mathcal{L}, \mathcal{T})$. Thus, $\mathcal{O}(\llbracket \alpha \rrbracket)$ is generated (as a frame) by

$$[\![\alpha]\!]_{\vec{a}} \quad \text{for } \vec{a} \in \mathbb{N}^n, \qquad \pi^*(U) \quad \text{for } U \in \mathcal{O}(\mathrm{Mod}(\mathcal{L},\mathcal{T})) \text{ (a generator).}$$

We let $\mathsf{Mod}(\mathcal{L},\mathcal{T})$ act on $[\![\alpha]\!]$ via $\rho_\alpha : \mathrm{Iso}(\mathcal{L},\mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} [\![\alpha]\!] \to [\![\alpha]\!]$, given by

$$\rho_\alpha^*([\![\alpha]\!]_{\vec{a}}) := \bigvee_{\vec{b}}(\pi_0^*(\textstyle\bigwedge_i [\![b_i \mapsto a_i]\!]) \wedge \pi_1^*([\![\alpha]\!]_{\vec{b}}))$$

where $\pi_0 : \mathrm{Iso}(\mathcal{L},\mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} [\![\alpha]\!] \to \mathrm{Iso}(\mathcal{L},\mathcal{T})$ and $\pi_1 : \mathrm{Iso}(\mathcal{L},\mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} [\![\alpha]\!] \to [\![\alpha]\!]$ are the projections. We then extend this definition to $[\![A]\!]$ for an arbitrary imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$, as well as $[\![f]\!] : [\![A]\!] \to [\![B]\!]$ for a definable function $f : A \to B$, exactly as in Section 5.6. This defines a geometric functor

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty \longrightarrow \mathsf{Act}_\infty(\mathsf{Mod}(\mathcal{L},\mathcal{T})).$$

When $(\mathcal{L},\mathcal{T})$ is κ-coherent, $[\![A]\!]$ is κ-étalé (over $\mathsf{Mod}(\mathcal{L},\mathcal{T})$) for κ-coherent $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa \subseteq \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$; thus $[\![-]\!]$ restricts to a κ-coherent functor

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa \longrightarrow \mathsf{Act}_\kappa(\mathsf{Mod}(\mathcal{L},\mathcal{T})).$$

For $\kappa = \omega_1$ and $(\mathcal{L},\mathcal{T})$ countable, we recover via Proposition 5.12.3 the definition in Section 5.6.

5.15 The Joyal–Tierney theorem for decidable theories

In this section, we sketch a proof of the following generalization of Theorem 5.8.1 using the Joyal–Tierney representation theorem.

As in Section 5.4, we call a κ-coherent theory $(\mathcal{L},\mathcal{T})$ **decidable** if there is a κ-coherent $\mathcal{L}$-formula with two variables (denoted $x \neq y$) which $\mathcal{T}$ proves is the negation of equality.

Theorem 5.15.1. *Let $\mathcal{L}$ be a relational language and $\mathcal{T}$ be a decidable κ-coherent $\mathcal{L}$-theory. Then*

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa \longrightarrow \mathsf{Act}_\kappa(\mathsf{Mod}(\mathcal{L},\mathcal{T}))$$

is an equivalence of categories.

Proof. We begin with the case $\kappa = \infty$, which is a straightforward variant of the usual proof of the Joyal–Tierney theorem; see [JT] or [J02, C5.2].

Let $(\mathcal{L}_d, \mathcal{T}_d)$ be the (single-sorted) **theory of decidable sets**, where $\mathcal{L}_d$ consists of a single binary relation symbol $\neq$ and $\mathcal{T}_d$ consists of the (ω-coherent) axioms

$$\forall x\,(x \neq x \Rightarrow \bot), \qquad \forall x, y\,(\top \Rightarrow (x = y) \vee (x \neq y)).$$

That $(\mathcal{L}, \mathcal{T})$ is decidable means that we have an interpretation

$$F_d : (\mathcal{L}_d, \mathcal{T}_d) \to (\mathcal{L}, \mathcal{T})$$

given by $F_d(\mathbb{X}) := \mathbb{X}$ and $F_d(\neq) := (\neq)$ (where $(\neq) \subseteq \mathbb{X}^2 \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$ is the geometric formula with 2 variables witnessing decidability). The classifying topos $\overline{\langle \mathcal{L}_d \mid \mathcal{T}_d \rangle}_\infty$ is computed in [J02, D3.2.7]: it is the presheaf topos $\mathsf{Set}^{\mathsf{Set}_{fm}}$ where Set_{fm} is the category of finite sets and injections, with the home sort $\mathbb{X} \in \mathsf{Set}^{\mathsf{Set}_{fm}}$ given by the inclusion.

Let $(\mathcal{L}_i, \mathcal{T}_i)$ be the **propositional theory of initial segments of** $\mathbb{N}$, where $\mathcal{L}_i$ consists of the proposition symbols $(|\mathbb{X}| \geq n)$ for each $n \in \mathbb{N}$ (here $\mathbb{X}$ is merely part of the notation, and does *not* denote a home sort, as the theory is propositional), and $\mathcal{T}_i$ consists of the axioms

$$\top \Rightarrow (|\mathbb{X}| \geq 0), \qquad (|\mathbb{X}| \geq n+1) \Rightarrow (|\mathbb{X}| \geq n).$$

The Lindenbaum–Tarski algebra $\langle \mathcal{L}_i \mid \mathcal{T}_i \rangle_\infty$ is the frame

$$\langle \mathcal{L}_i \mid \mathcal{T}_i \rangle_\infty = \{\bot < \cdots < [|\mathbb{X}| \geq 2] < [|\mathbb{X}| \geq 1] < [|\mathbb{X}| \geq 0] = \top\}.$$

Thus the classifying topos $\overline{\langle \mathcal{L}_i \mid \mathcal{T}_i \rangle}_\infty = \mathsf{Sh}(\langle \mathcal{L}_i \mid \mathcal{T}_i \rangle_\infty)$ is the presheaf topos $\mathsf{Set}^{\mathbb{N}}$, where $[|\mathbb{X}| \geq n] \in \overline{\langle \mathcal{L}_i \mid \mathcal{T}_i \rangle}_\infty$ is identified with the functor $\mathbb{N} \to \mathsf{Set}$ which is 1 on $m \geq n$ and 0 on $m < n$.

We have a geometric functor $\mathsf{Set}^{\mathsf{Set}_{fm}} \to \mathsf{Set}^{\mathbb{N}}$ induced by the inclusion $\mathbb{N} \to \mathsf{Set}_{fm}$ (mapping $m \leq n$ to the inclusion $m \subseteq n$). Using [J02, A4.2.7(b), C3.1.2], this geometric functor is easily seen to be the inverse image part of a surjective open geometric morphism; surjectivity follows from essential surjectivity of the inclusion $\mathbb{N} \to \mathsf{Set}_{fm}$, while for openness, given $U \in \mathbb{N}$, $V \in \mathsf{Set}_{fm}$, and $b : U \hookrightarrow V$ as in [J02, C3.1.2], let $U' := |V|$, $r : U' \cong V$ be any bijection such that $r|U = b$, and $i := r^{-1}$, so that $r \circ i = 1_V$ and $i \circ b$ is the inclusion $U \subseteq U'$, as required by [J02, C3.1.2]. This functor is the interpretation

$$\begin{aligned} F_i : (\mathcal{L}_d, \mathcal{T}_d) &\longrightarrow (\mathcal{L}_i, \mathcal{T}_i) \\ \mathbb{X} &\longmapsto \textstyle\bigsqcup_{n \in \mathbb{N}} (|\mathbb{X}| \geq n+1). \end{aligned}$$

In terms of models (in arbitrary Grothendieck toposes), this interpretation takes a model of $\mathcal{T}_i$, i.e., an initial segment of $\mathbb{N}$, to the model of $\mathcal{T}_d$, i.e., decidable set, given by that initial segment.

We now compute the "(2-)pushout of theories" $(\mathcal{L}', \mathcal{T}')$ of the two interpretations F_d, F_i (in the 2-category of ∞-pretoposes):

$$\begin{tikzcd} (\mathcal{L}', \mathcal{T}') & (\mathcal{L}_i, \mathcal{T}_i) \arrow[l] \\ (\mathcal{L}, \mathcal{T}) \arrow[u, "E"] & (\mathcal{L}_d, \mathcal{T}_d) \arrow[l, "F_d"] \arrow[u, "F_i"'] \end{tikzcd}$$

By definition, this is a geometric theory $(\mathcal{L}', \mathcal{T}')$ such that a model of $\mathcal{T}'$ (in an arbitrary Grothendieck topos) is the same thing as a model of $\mathcal{T}$, a model of $\mathcal{T}_i$, and an isomorphism between the two models of $\mathcal{T}_d$ given by the interpretations F_d, F_i; this is equivalently an initial segment of $\mathbb{N}$ together with a model of $\mathcal{T}$ on that initial segment. Thus, we may take $(\mathcal{L}', \mathcal{T}')$ to be the propositional theory presenting the frame of opens $\mathcal{O}(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))$ given in Section 5.14, i.e., $\mathcal{L}'$ consists of the proposition symbols $(|\mathbb{X}| \geq n)$ and $R(\vec{a})$ for n-ary $R \in \mathcal{L}$ and $\vec{a} \in \mathbb{N}$, and $\mathcal{T}'$ consists of the axioms in $\mathcal{T}_i$ together with the axioms

$$R(\vec{a}) \Rightarrow (|\mathbb{X}| \geq \max_i(a_i + 1)), \qquad \phi(\vec{a}) \Rightarrow \psi(\vec{a}),$$

where $\forall \vec{x}\, (\phi(\vec{x}) \Rightarrow \psi(\vec{x}))$ is an axiom in $\mathcal{T}$ and $\phi(\vec{a}), \psi(\vec{a})$ are the propositional $\mathcal{L}'$-formulas defined by induction on ϕ, ψ in the obvious way. The classifying topos is

$$\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\infty = \mathsf{Sh}(\langle \mathcal{L}' \mid \mathcal{T}' \rangle_\infty) = \mathsf{Sh}(\mathcal{O}(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))) = \mathsf{Sh}(\mathrm{Mod}(\mathcal{L}, \mathcal{T})),$$

and the interpretation $E : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty \to \mathsf{Sh}(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))$ in the diagram above is simply the functor $[\![-]\!]$ (identifying sheaves on $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ with étalé locales over $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$, and forgetting the $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-action for now).

Similarly, the pushout $(\mathcal{L}'', \mathcal{T}'')$ of E with itself

$$\begin{tikzcd} (\mathcal{L}'', \mathcal{T}'') & (\mathcal{L}', \mathcal{T}') \arrow[l, "D_1"'] \\ (\mathcal{L}', \mathcal{T}') \arrow[u, "D_0"] & (\mathcal{L}, \mathcal{T}) \arrow[l, "E"] \arrow[u, "E"'] \end{tikzcd}$$

is the theory of pairs of models of $\mathcal{T}'$ together with an isomorphism between them, or equivalently of pairs $(\mathcal{M}, g)$ of a model $\mathcal{M}$ of $\mathcal{T}'$ together with a permutation g of its underlying initial segment of $\mathbb{N}$; so we may take $(\mathcal{L}'', \mathcal{T}'')$ to be the propositional theory presenting $\mathrm{Iso}(\mathcal{L}, \mathcal{T})$ from Section 5.14, and D_0, D_1 to be induced by the frame homomorphisms $\partial_0^*, \partial_1^* : O(\mathrm{Mod}(\mathcal{L}, \mathcal{T})) \to O(\mathrm{Iso}(\mathcal{L}, \mathcal{T}))$. We have an interpretation

$$I : (\mathcal{L}'', \mathcal{T}'') \to (\mathcal{L}', \mathcal{T}')$$

induced (via the universal property of the pushout) by the identity $1_{\mathcal{T}'} : (\mathcal{L}', \mathcal{T}') \to (\mathcal{L}', \mathcal{T}')$ (so $I \circ D_0 \cong 1 \cong I \circ D_1$), which takes a model $\mathcal{M}$ of $\mathcal{T}'$ to the model $(\mathcal{M}, 1_M)$ of $\mathcal{T}''$; so I is induced by the frame homomorphism $\iota^* : O(\mathrm{Iso}(\mathcal{L}, \mathcal{T})) \to O(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))$.

We also have the triple pushout $(\mathcal{L}''', \mathcal{T}''')$ of three copies of E (equivalently of D_0, D_1), the theory of triples of models $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ with isomorphisms $\mathcal{M}_1 \cong \mathcal{M}_2 \cong \mathcal{M}_3$, which presents the frame $O(\mathrm{Iso}(\mathcal{L}, \mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L}, \mathcal{T})} \mathrm{Iso}(\mathcal{L}, \mathcal{T}))$. The three injections

$$P_0, M, P_1 : (\mathcal{L}'', \mathcal{T}'') \to (\mathcal{L}''', \mathcal{T}''')$$

take a model $(\mathcal{M}_1 \cong \mathcal{M}_2 \cong \mathcal{M}_3)$ of $\mathcal{T}'''$ to (respectively) the first isomorphism $\mathcal{M}_1 \cong \mathcal{M}_2$, the composite isomorphism $\mathcal{M}_1 \cong \mathcal{M}_3$, and the second isomophism $\mathcal{M}_2 \cong \mathcal{M}_3$, so are induced by $\pi_0^*, \mu^*, \pi_1^* : O(\mathrm{Iso}(\mathcal{L}, \mathcal{T})) \to O(\mathrm{Iso}(\mathcal{L}, \mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L}, \mathcal{T})} \mathrm{Iso}(\mathcal{L}, \mathcal{T}))$. All these data form a diagram

$$\begin{tikzcd}
(\mathcal{L}''', \mathcal{T}''') & (\mathcal{L}'', \mathcal{T}'') \arrow[l, shift right=2, "P_0"'] \arrow[l, "M"] \arrow[l, shift left=2, "P_1"] \arrow[r, "I"] & (\mathcal{L}', \mathcal{T}') \arrow[l, shift right=2, "D_0"'] \arrow[l, shift left=2, "D_1"] & (\mathcal{L}, \mathcal{T}) \arrow[l, "E"']
\end{tikzcd}$$

which is an augmented 2-**truncated simplicial topos**, in the sense of [J02, B3.4, C5.1].

Descent data (see [JT, VIII §1] or [J02, B3.4, C5.1]) on an object $A \in \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\infty$ consists of a morphism $\theta : D_1(A) \to D_0(A)$ obeying the **unit condition** that

$$A \cong I(D_1(A)) \xrightarrow{I(\theta)} I(D_0(A)) \cong A$$

is the identity 1_A, and the **cocycle condition** that the following two morphisms are equal:

$$P_1(D_1(A)) \cong M(D_1(A)) \xrightarrow{M(\theta)} M(D_0(A)) \cong P_0(D_0(A)),$$
$$P_1(D_1(A)) \xrightarrow{P_1(\theta)} P_1(D_0(A)) \cong P_0(D_1(A)) \xrightarrow{P_0(\theta)} P_0(D_0(A)).$$

Let $\mathsf{Desc}(\mathcal{L}^{(-)}, \mathcal{T}^{(-)})$ denote the category of objects in $\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\infty$ equipped with descent data (and morphisms which commute with the descent data in the obvious sense). For every $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$, we have an isomorphism $D_1(E(A)) \cong D_0(E(A))$ by definition of $(\mathcal{L}'', \mathcal{T}'')$, which is easily verified to be descent data on $E(A)$; this defines a lift of $E : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty \to \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\infty$ to a geometric functor

$$E' : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty \longrightarrow \mathsf{Desc}(\mathcal{L}^{(-)}, \mathcal{T}^{(-)}).$$

Since E is the pushout of F_i which is the inverse image part of a surjective open geometric morphism, so is E (see [JT, VII 1.3] or [J02, C3.1.26]). Thus by the **Joyal–Tierney descent theorem** (see [JT, VIII 2.1] or [J02, C5.1.6]), E' is an equivalence of categories.

Under the above identifications $\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\infty \cong \mathsf{Sh}(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))$ and $\overline{\langle \mathcal{L}'' \mid \mathcal{T}'' \rangle}_\infty \cong \mathsf{Sh}(\mathrm{Iso}(\mathcal{L}, \mathcal{T}))$ as well as the identification between sheaves and étalé locales, $\mathsf{Desc}(\mathcal{L}^{(-)}, \mathcal{T}^{(-)})$ is equivalently the category of étalé locales $p : X \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ equipped with a morphism $t : \mathrm{Iso}(\mathcal{L}, \mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} X \to X \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} \mathrm{Iso}(\mathcal{L}, \mathcal{T})$ commuting with the projections to $\mathrm{Iso}(\mathcal{L}, \mathcal{T})$ and satisfying the obvious analogs of the unit and cocycle conditions. By an easy calculation, via composition with the projection $X \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} \mathrm{Iso}(\mathcal{L}, \mathcal{T}) \to X$, such t are in bijection with $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$-actions $\mathrm{Iso}(\mathcal{L}, \mathcal{T}) \times_{\mathrm{Mod}(\mathcal{L},\mathcal{T})} X \to X$. Also morphisms preserving the descent data correspond to equivariant morphisms; so we have

$$\mathsf{Desc}(\mathcal{L}^{(-)}, \mathcal{T}^{(-)}) \cong \mathsf{Act}_\infty(\mathsf{Mod}(\mathcal{L}, \mathcal{T})).$$

Finally, it is straightforward to verify that under this equivalence, the functor E' above is just $[\![-]\!]$, which completes the proof of the theorem in the case $\kappa = \infty$.

Now consider general κ. Let $\mathcal{T}$ be a decidable κ-coherent $\mathcal{L}$-theory. We know that

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty \longrightarrow \mathsf{Act}_\infty(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$$

is an equivalence, and restricts to

$$[\![-]\!] : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa \longrightarrow \mathsf{Act}_\kappa(\mathsf{Mod}(\mathcal{L}, \mathcal{T})).$$

By Lemma 5.14.1, $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\kappa$ consists of the κ-presentable objects in $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}_\infty$; so it suffices to verify that every $X \in \mathsf{Act}_\kappa(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$ is κ-presentable in $\mathsf{Act}_\infty(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$. For $X \in \mathsf{Act}_\kappa(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$, we have $X \in \mathsf{Sh}_\kappa(\mathsf{Mod}(\mathcal{L}, \mathcal{T})) \cong \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\kappa$, whence by Lemma 5.14.1 again, X is κ-presentable in $\overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}_\infty \cong \mathsf{Sh}(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))$. This implies that X is κ-presentable in $\mathsf{Act}_\infty(\mathsf{Mod}(\mathcal{L}, \mathcal{T}))$ by

Lemmas 5.13.2 and 5.13.3, since the definition of $\mathsf{Desc}(\mathcal{L}^{(-)}, \mathcal{T}^{(-)})$ above can be rephrased as first taking the inserter $\mathsf{Ins}(D_1, D_0)$ and then taking two equifiers to enforce the unit and cocycle conditions, where the functors involved, namely D_i, P_i, I, M, clearly preserve colimits and κ-presentable objects. □

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