

THEORETICAL INVESTIGATIONS IN  
NONLINEAR QUANTUM OPTICS,  
THEORY OF MEASUREMENT,  
AND  
PULSATIONS OF GENERAL RELATIVISTIC  
MODELS OF NEUTRON STARS

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Nature reveals herself willingly, I believe, but in proportion to our attentiveness. For a theoretical physicist, this attentiveness takes the form of a careful questioning of what he sees, or of what he thinks he sees, in nature. Often his discoveries, great or small, are a result of a persistent, demanding examination of issues previously neglected or assumed understood. It is in this spirit that this collection of papers is presented. While none of them represents a breakthrough in our understanding of how nature works, it is hoped that they offer a useful contribution to the description of a wide variety of physical systems, particularly those that can be modeled as harmonic oscillators.

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## ABSTRACT

This thesis is a collection of six papers. The first four constitute the heart of the thesis; they are concerned with quantum mechanical properties of certain harmonic-oscillator states. The first paper is a discourse on single-mode and two-mode Gaussian pure states (GPS), states produced when harmonic oscillators in their ground states are exposed to potentials that are linear or quadratic in oscillator position and momentum variables (creation and annihilation operators). The second and third papers develop a formalism for analyzing two-photon devices (e.g., parametric amplifiers and phase-conjugate mirrors), in which photons in the output modes arise from two-photon transitions, i.e., are created or destroyed two at a time. The states produced by such devices are single-mode and two-mode "squeezed states", special kinds of GPS whose low-noise properties make them attractive for applications in such fields as optical communications and gravitational wave detection. The fourth paper is an analysis of the noise in homodyne detection, a phase-sensitive detection scheme in which the special properties of (single-mode) squeezed states are revealed as an improved signal-to-noise ratio relative to that obtained with coherent states (the states produced, e.g., by a laser).

The fifth and sixth papers deal with problems of a different nature from that of the previous papers. The fifth paper considers the validity of the "standard quantum limit" (SQL) for measurements which monitor the position of a free mass. It shows specifically that when the pre-measurement wave functions of the free mass and the measuring apparatus(es) are Gaussian (in the general sense, which includes so-called "contractive states"), measurements described by linear couplings to the position or to both the position and momentum are limited by the SQL. The sixth paper develops

the mathematical theory of torsional (toroidal) oscillations in fully general relativistic, nonrotating, spherical stellar models, and of the gravitational waves they emit.

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## INTRODUCTION

The advent of the laser in the 1960's brought with it a host of new observable phenomena. The intense light from a laser operating far above threshold results from the coherent excitation of a single mode of the electric field; the fluctuations (noise) associated with the light reflect only quantum-mechanical zero-point fluctuations, and nothing more. Nonlinear effects, unobservable with ordinary light because they arise from couplings between matter and radiation that are second order and higher in the field strength, are observable with laser light. These effects give rise to a wide variety of phenomena, such as harmonic generation,<sup>1</sup> optical phase conjugation,<sup>2</sup> and squeezed-state light.<sup>3</sup> Squeezed-state light is the motivation for the first four papers of this thesis.

Quantum mechanics describes the output of a laser as a collection of modes (harmonic oscillators) of the electromagnetic field, each of which is in a "coherent state."<sup>4,5</sup> A (single-mode) coherent state is an eigenstate of the annihilation operator  $\mathbf{a}$  for the mode ( $[\mathbf{a}, \mathbf{a}^\dagger] = 1$ ). It therefore has the sharpest complex amplitude  $\langle \mathbf{a} \rangle$  allowed by quantum mechanics. The "total noise"

$$\langle |\Delta \mathbf{a}|^2 \rangle \equiv \langle \Delta \mathbf{a} \Delta \mathbf{a}^\dagger \rangle_{\text{sym}} \equiv \frac{1}{2} (\langle \Delta \mathbf{a} \Delta \mathbf{a}^\dagger \rangle + \langle \Delta \mathbf{a}^\dagger \Delta \mathbf{a} \rangle),$$

$$\Delta \mathbf{a} \equiv \mathbf{a} - \langle \mathbf{a} \rangle, \tag{1}$$

of a coherent state, equal to the sum of the variances of the real and imaginary parts of  $\mathbf{a}$ , is the minimum allowed by quantum mechanics (the half-quantum of "zero-point noise"). The vanishing of  $\langle (\Delta \mathbf{a})^2 \rangle$  implies that the variances (squared uncertainties) of the real and imaginary parts of  $e^{i\delta} \mathbf{a}$  for any  $\delta$  ("rotated" dimensionless position and momentum variables), are identical. The coordinate- and momentum-space wave functions of a

coherent state are therefore Gaussians, whose widths (in dimensionless units) are identical.

In the Fourier decomposition of an electric field (or other quantized field composed of bosonic modes), the annihilation operators appear as the Fourier components of the positive-frequency part of the field. Just as it is useful when discussing a single mode to distinguish between the real and imaginary parts of the annihilation operator, it is useful when discussing a collection of modes, i.e., an (electric) field, to distinguish between two (time-dependent) linear combinations of the positive- and negative-frequency parts of the field; these combinations are called quadrature-phase operators (or simply quadrature phases). They are the operators which appear when one decomposes the field into parts that vary as  $\cos\Omega(t - \mathbf{x})$  and  $\sin\Omega(t - \mathbf{x})$ , where the frequencies associated with the field are regarded as symmetric around a carrier frequency  $\Omega$ . For a field composed of modes in coherent states, the (time-averaged) variance of the field, or the sum of the variances of the quadrature phases, is the minimum allowed by quantum mechanics; this is because it is proportional to the sum of the total noises of each mode, each of which is equal to its minimum allowed value. Further, the vanishing of the complex number  $\langle(\Delta E^{(+)} )^2\rangle$ , where  $E^{(+)}$  is the positive-frequency part of the coherent-state field, implies that the variances of the quadrature phases (and all "rotated" quadrature phases) are identical.

Clearly coherent states do not describe all (pure) states with Gaussian wave functions ("Gaussian pure states", or "GPS"). Likewise, fields composed of modes in coherent states do not describe all fields with Gaussian quantum fluctuations. Gaussian pure states are produced when harmonic oscillators in their ground states are exposed to potentials (interaction



Hamiltonians) that are linear or quadratic in creation and annihilation operators. Hence, although they may affect  $N$  oscillators ( $N \geq 1$ ), thus producing an  $N$ -mode GPS, the interaction Hamiltonians that produce GPS are sums of Hamiltonians that either involve a single oscillator or couple two oscillators. This means that the properties of fields with Gaussian quantum fluctuations reflect the properties of single-mode and two-mode GPS. Paper 1 in this collection, entitled "Quantum mechanical pure states with Gaussian wave functions," is a treatise on single-mode and two-mode GPS. Many comments which would naturally appear in this Introduction have been omitted, in favor of referring the reader to the Introduction of that paper. That Introduction will also serve to familiarize the uninitiated reader with the material contained in Papers 2, 3, and 4 of this thesis.

Coherent states are produced from the vacuum state by interaction Hamiltonians that are linear in creation and annihilation operators ("one-photon" processes). Physically, this says that coherent states are produced when an oscillator in its ground state is subjected to a classical force (e.g., an electrical current). Gaussian pure states that are not coherent states are produced from coherent states by interaction Hamiltonians that are quadratic in creation and annihilation operators, i.e., by processes which involve two-photon transitions. The changes in the wave functions and noise properties of the field from those of coherent states and fields composed of modes in coherent states are a result of correlations between the photons in each pair. Single-mode GPS that are not coherent states are produced by degenerate two-photon interactions, in which two photons from the same mode are created or annihilated simultaneously. They are known in quantum optics as "single-mode squeezed states." The adjective "squeezed" alludes to the fact that the variance of the real or imaginary part of  $e^{i\delta} \alpha$ ,

for some  $\delta$ , is smaller than it would be in a coherent state. Two-mode GPS that are not coherent states are produced by all combinations of (i) degenerate two-photon interactions (one for each mode), (ii) frequency-converting interactions, in which a photon from one mode is annihilated while a photon from the other mode is created, and (iii) nondegenerate two-photon interactions, in which two photons from different modes are simultaneously created or annihilated. The frequency-converting interaction by itself cannot produce a state with noise properties that are different from those of a coherent state. The two-mode GPS that are produced from (two) coherent states by nondegenerate two-photon interactions are called "two-mode squeezed states." They (as distinguished from states that are products of two single-mode squeezed states) are the natural analog, both formally and physically, of single-mode squeezed states. For further discussion, please see the Introduction to Paper 1 (especially Section I.g). A thorough discussion of two-mode squeezed states and their significance for the noise properties of (multimode) electric fields is contained in Papers 2 and 3 of this thesis.

A brief summary of each of the papers in this collection follows.

Paper 1, entitled "Quantum mechanical pure states with Gaussian wave functions," is a comprehensive discussion of single-mode and two-mode Gaussian pure states (GPS). In it I investigate the physical and group theoretical significances of the Hamiltonians and unitary operators associated with GPS. These are used to develop a natural classification of all GPS. The properties of single-mode and two-mode GPS are discussed. Efficient vector notations are introduced, for both single-mode and two-mode GPS, which provide a powerful way to derive and describe properties of GPS and the unitary operators associated with them.

Papers 2 and 3 are the first two papers in a series that describes a new formalism for two-photon quantum optics; they appear in the May, 1985 issue of *Physical Review A*. The objective in this series of papers is to develop a formalism that can successfully describe real "two-photon" devices (e.g., parametric amplifiers and phase conjugate mirrors), which are of increasing theoretical and experimental interest because of their ability to produce squeezed states. The fundamental operators of the formalism are the (Fourier components of the) quadrature-phase operators described above, and the fundamental states are the two-mode squeezed states. Paper 2 (Paper I in the series) lays a foundation for the formalism; it defines the fundamental operators and states and describes the noise properties of fields produced by two-photon devices. These fields have a special kind of noise, called "time-stationary quadrature phase" (TSQP) noise, so named because for them all noise moments (moments with the mean excitations removed) of the quadrature-phase operators are time-independent. By comparison, the fields produced by one-photon devices such as the laser, i.e., coherent-state fields, have an even more special kind of noise, "time-stationary" (TS) noise, so named because all noise moments of the electric field, as well as of the quadrature phases, are time-independent. Paper 3 in this thesis (Paper II in the series) provides the mathematical foundation for the formalism. It introduces a two-component vector notation that naturally describes the properties of states (or fields) that have TSQP noise. This vector notation is a special case of the more general vector notation developed in Paper 1 of this thesis for describing all two-mode GPS. Properties of two-mode squeezed states and various unitary operators associated with them are investigated thoroughly. I am currently writing Paper III of this series (it does not appear in this thesis). That paper will define

quasiprobability distributions (QPDs) for fields with TSQP noise, based on the quadrature-phase operators. The use of QPDs is equivalent to a density operator description of the fields; it enables one to describe real devices, which exhibit losses and other nonideal effects.

Paper 4 of this collection is a short paper, published in *Optics Letters* in May, 1984, entitled "Noise in homodyne detection." It is a simple but rigorous analysis of the important sources of noise in homodyne detection. Homodyne detection is a phase-sensitive detection scheme, which works by combining at a beam splitter a monochromatic input field with a strong local-oscillator field, and monitoring one or both output ports. The relative strength of the local-oscillator field guarantees that (i) the dominant contribution to the output-field intensities is proportional to the mean field of that input-field quadrature phase which was in phase with the local oscillator (i.e., to the real or imaginary part of  $e^{i\delta}\langle a \rangle$ , since the input field is monochromatic), and (ii) the dominant noise in the output fields is proportional to the variance of that quadrature phase. A reduction in the noise in one quadrature phase relative to its coherent-state value, i.e., squeezing, is thus manifested as an improved signal-to-noise ratio over that obtained when the input field is a coherent state. This paper shows that "two-port" homodyning, in which one monitors both output ports of a 50-50 beam splitter with photodetectors and then subtracts the photodetector outputs, is insensitive to local-oscillator quadrature-phase noise; hence it provides (i) a means of detecting reduced quadrature-phase fluctuations (squeezing) that is more sensitive than conventional (one-port) homodyning, and (ii) an output signal-to-noise ratio that can be a modest to significant improvement over that of one-port homodyning and direct detection. The magnitude of the improvement is a function of how "squeezed" the input field is and how

efficient the photodetectors are.

Paper 5 is a short paper I wrote between January and March, 1984, but chose not to submit for publication. It examines the validity of the "standard quantum limit" (SQL) for certain kinds of measurements of the position of a free mass. The SQL says that one cannot predict the outcome of the second of two successive, identical measurements of the position of a free mass with an accuracy better than  $(\hbar\tau/m)^{1/2}$ , where  $\tau$  is the time interval between the measurements. My active interest in this subject was sparked by an article by Horace Yuen<sup>6</sup> that appeared in *Physical Review Letters* in August, 1983. In it he argued that the use of "contractive states" enables one to beat the SQL. Contractive states are free-mass states with the most general kind of Gaussian wave function (the wave function of a single-mode squeezed state); i.e., they are states in which the position and momentum variables are correlated with each other in a way that is independent of their mean values. The kinds of measurements that Yuen proposes are not measurements which we know how to realize (he has made this clear in published comments since his original Letter). There is no proof available, and perhaps no reason to believe, that the kinds of measurements of which he speaks could not beat the SQL. However, his article and others since his have revealed confusion regarding the precise statement of the SQL and the measurements to which it applies. In this paper I consider the validity of the SQL for measurements described by linear couplings to the position or to both the position and momentum of the mass.<sup>7</sup> It is shown that whenever the pre-measurement wave functions of the free mass and the measuring apparatus(es) are Gaussian (which includes Yuen's "contractive states"), such measurements are subject to the SQL.

Paper 6, written in collaboration with Kip S. Thorne, presents the mathematical theory of torsional oscillations in fully relativistic, nonrotating, spherical stellar models; and it examines the gravitational waves emitted as a consequence of these oscillations. The motivation for this analysis lies with neutron stars, which have crusts and perhaps also solid cores that could undergo such oscillations. These oscillations might be observed in pulsar timing data, as subpulses or in "post-glitch" data, or by gravitational wave detectors available ten or twenty years from now. The analysis uses first-order perturbation theory and ignores all damping except that due to gravitational radiation reaction. For each harmonic, the paper presents equations of motion, an action principle, and a proof that the oscillations are stable. An eigenvalue problem is posed for the eigenfunctions and eigenfrequencies of the normal modes with outgoing gravitational waves. Five methods of solving the eigenvalue problem are presented. An appendix develops a general theory of action principles for systems with radiative boundary conditions.

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## QUANTUM MECHANICAL PURE STATES WITH GAUSSIAN WAVE FUNCTIONS

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### *ABSTRACT*

This paper examines single-mode and two-mode Gaussian pure states (GPS), quantum mechanical pure states with Gaussian wave functions. These states are produced when harmonic oscillators in their ground states are exposed to potentials (interaction Hamiltonians) that are linear or quadratic in the position and momentum variables (annihilation and creation operators) of the oscillators. The physical and group theoretical properties of these Hamiltonians and the unitary operators they generate are investigated. They are used to develop a natural classification of all GPS. Properties of single-mode and two-mode GPS are discussed. An efficient vector notation is introduced and used to derive many of the important properties of GPS and the Hamiltonians and unitary operators associated with them.

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## I. INTRODUCTION AND OVERVIEW

This paper considers Gaussian pure states (GPS), quantum mechanical pure states that have Gaussian wave functions. These states are particularly relevant to the description of a harmonic oscillator with a nearly classical, coherent excitation whose intrinsic quantum mechanical fluctuations are important. Such a description arises, for example, in connection with the transmission or detection of coherent optical signals,<sup>1,2</sup> or high-precision measurements of a macroscopic oscillator's displacement, as in the detection of gravitational waves.<sup>3,4</sup> Gaussian pure states are familiar in quantum optics, where they describe the coherent output from a laser and the predicted "squeezed-state" light<sup>5,6</sup> from an optical parametric amplifier. For the theorist, these states have the satisfying feature that the Hamiltonians for the physical processes that produce them are known and have simple, easily interpreted forms.

Gaussian pure states are produced when harmonic oscillators in their ground states are subjected to particular kinds of time-dependent potentials, or interaction Hamiltonians. The oscillators might be mechanical or electrical, or they might be the normal (bosonic) modes of a quantized field such as the electromagnetic field; for purpose of illustration, I shall have the last of these examples in mind throughout this paper. The interaction Hamiltonians that produce GPS are polynomials that are linear and/or quadratic in the oscillators' position and momentum variables. Hence, although they may affect  $N$  oscillators ( $N \geq 1$ ), thus producing an  $N$ -mode GPS, the interaction Hamiltonians are sums of Hamiltonians that either involve a single oscillator or couple two oscillators. This has the important consequence that one need look only at single-mode and two-mode GPS in order to understand the fundamental features of all  $N$ -mode GPS. Single-mode GPS and

their subsets have been studied by many people during the last six decades.<sup>7-16</sup> The goal of this paper is to help make the less widely known and understood two-mode GPS as familiar as their single-mode counterparts.

Associated with any oscillator is a real, positive, constant frequency  $\Omega$ . The quantum mechanical free Hamiltonian for the oscillator is

$$H_0^{(1)} \equiv \Omega \mathbf{a}^\dagger \mathbf{a} , \quad \Omega = \Omega^* , \quad (1.1a)$$

where  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  are annihilation and creation operators for the mode ( $[\mathbf{a}, \mathbf{a}^\dagger] = 1$ ). (Here and throughout this paper I use units with  $\hbar \equiv c \equiv 1$ .) The expectation value of  $\mathbf{a}^\dagger \mathbf{a}$ , the photon-number operator for the mode, is the number of photons in the mode. The free Hamiltonian for  $N$  oscillators is the sum of  $N$  single-mode free Hamiltonians:

$$H_0^{(N)} \equiv \sum_{j=1}^N \Omega_j \mathbf{a}_j^\dagger \mathbf{a}_j , \quad \Omega_j = \Omega_j^* . \quad (1.1b)$$

The stationary states for each oscillator (eigenstates of  $H_0^{(1)}$ ) are the number states  $|n\rangle$ ,

$$|n\rangle \equiv (n!)^{-\frac{1}{2}} (\mathbf{a}^\dagger)^n |0\rangle , \quad (1.2a)$$

$$H_0^{(1)} |n\rangle = n \Omega |n\rangle , \quad (1.2b)$$

where the state vector  $|0\rangle$  represents the ground state. Throughout this paper the state vector  $|0\rangle$ , or the term "vacuum state", means the tensor product of the ground states of  $N$  oscillators, for any  $N \geq 1$ . The vacuum state, unlike the other number states, is also an eigenstate of the annihilation operators for all the modes. Its wave function is Gaussian, whereas the wave functions for the other number states  $|n\rangle$ ,  $n \geq 1$ , are not.<sup>17</sup>

The forms of the interaction Hamiltonians that produce (or preserve) Gaussian pure states are derived in this paper by considering the most general single-mode and two-mode Gaussian wave functions, in which all parameters are arbitrary, subject to normalization. The wave functions imply that Gaussian pure states are eigenstates of certain kinds of linear combinations of creation and annihilation operators. These linear combinations in turn determine the general form of the unitary operators that relate Gaussian pure states to the vacuum state, in the following way. Let the most general (normalized)  $N$ -mode GPS be expressed formally as the state vector  $U_G^{(N)} |0\rangle$ , where  $U_G^{(N)} \equiv e^{-iH_G^{(N)}}$  is a unitary operator with Hermitian generator  $H_G^{(N)}$ . Since the vacuum state is an eigenstate (with zero eigenvalue) of the annihilation operators  $a_j$ ,  $j = 1, 2, \dots, N$ , an  $N$ -mode GPS is an eigenstate (with zero eigenvalue) of the transformed annihilation operators  $g_j \equiv U_G^{(N)} a_j U_G^{(N)\dagger}$ . The (Gaussian) wave functions imply that the operators  $g_j$  are linear combinations of annihilation and creation operators plus a constant, which in turn implies that the Hermitian generator  $H_G^{(N)}$  consists only of linear and bilinear combinations of annihilation and creation operators. There are no further restrictions on the generator  $H_G^{(N)}$ , so  $H_G^{(N)}$  consists, in general, of all possible linear and bilinear combinations of annihilation and creation operators.

The unitary operators  $U_G^{(N)} \equiv e^{-iH_G^{(N)}}$  that relate  $N$ -mode GPS to the vacuum state factor naturally into unitary operators whose generators are (Hermitian) linear combinations of creation and annihilation operators, and unitary operators whose generators are (Hermitian) bilinear combinations of creation and annihilation operators. There are  $N$  unitary operators whose generators are linear in creation and annihilation operators, one for each mode, and they are identical to each other in form. They are called

"displacement operators".<sup>10</sup> In contrast, the unitary operators whose generators are bilinear combinations of creation and annihilation operators divide into four basic types, which differ fundamentally from each other in both their physical and group theoretical significance. In this paper they are referred to as rotation, mixing, single-mode squeeze, and two-mode squeeze operators. This division reflects the underlying structure of the  $N(2N + 1)$ -parameter Lie algebra consisting of all bilinear combinations of  $N$  creation and annihilation operators. These unitary operators and their generators are described below.

The proof (for  $N = 1$  and  $N = 2$ ) that the unitary operator  $U_G^{(N)}$  whose generator  $H_G^{(N)}$  is a sum of all linear and bilinear combinations of creation and annihilation operators factors into a product of displacement, rotation, mixing, and squeeze operators is subsumed by more general proofs given in Sections IIC, IIIC, and Appendix A. There each term in the generators  $H_G^{(1)}$  and  $H_G^{(2)}$  is allowed to have an arbitrary time dependence (subject to overall Hermiticity), and the unitary evolution operator  $U^{(N)}(t)$ , the solution to the Schrödinger equation  $i\partial_t U^{(N)}(t) = H_G^{(N)}(t) U^{(N)}(t)$ ,  $U^{(N)}(0) = 1$ , ( $N = 1, 2$ ), is shown to factor into a product of these unitary operators. The Hermitian forms associated with the displacement, rotation, mixing and squeeze operators thus take on a physical meaning, in addition to their group theoretical roles. When allowed to take on time dependences, they represent the interaction Hamiltonians that produce (or preserve) Gaussian pure states. Their properties are now described.

**a. Interaction Hamiltonians**

The interaction Hamiltonians that produce Gaussian pure states divide naturally into two broad categories: those that conserve the (total) number of photons in the mode(s), and those that do not. Those that conserve total photon number leave the vacuum state unchanged, and their effect on other GPS is merely to redistribute the photons among the different modes. Of greater interest here are those interactions that do not conserve the total photon number, but that do preserve the Gaussian nature of a state. As stated above, all interaction Hamiltonians that produce (or preserve) GPS are polynomials that are linear and/or quadratic in creation and annihilation operators (i.e., in position and momentum variables). Conversely, all such interaction Hamiltonians describe physical processes that produce (or preserve) Gaussian states. Those that conserve the total photon number must consist of products of equal numbers of creation and annihilation operators. The requirement that they also preserve Gaussians implies that they have the (normally-ordered) forms

$$H_R^{(N)}(t) \equiv \sum_{i,j=1}^N \Pi_{ij}(t) a_i^\dagger a_j, \quad \Pi_{ij} = \Pi_{ji}^* \quad (1.3)$$

where the  $\Pi_{ij}(t)$  are arbitrary complex-valued functions of time  $t$ . In contrast, the Hamiltonians that produce (or preserve) GPS but do not conserve the total number of photons have the forms

$$H_1^{(N)}(t) \equiv \sum_{j=1}^N i\lambda_j^*(t) a_j - i\lambda_j(t) a_j^\dagger, \quad (1.4a)$$

$$H_2^{(N)}(t) \equiv \sum_{i,j=1}^N \frac{1}{2} i\zeta_{ji}^*(t) a_i a_j - \frac{1}{2} i\zeta_{ij}(t) a_j^\dagger a_i^\dagger, \quad (1.4b)$$

where  $\lambda_j(t)$  and  $\zeta_{ij}(t)$  are arbitrary complex-valued functions of time.

The photon number-conserving interactions defined in Eq. (1.3) divide naturally into two types. The first is made up of the terms for which  $i = j$ ; for each mode  $i$  these Hamiltonians have the form

$$H_R^{(1)}(t) = \omega_i(t) a_i^\dagger a_i, \quad \omega_i(t) = \omega_i^*(t), \quad (1.5)$$

which looks like the free Hamiltonian for the mode but with a time-dependent real function (not restricted to positive values) in place of the frequency. These Hamiltonians conserve the number of photons in each mode; hence they conserve the total energy, as well as the total number of photons. They are referred to in this paper as rotation Hamiltonians. Like the free Hamiltonian, they cause a time-dependent exchange of kinetic and potential energy within each mode, but unlike the free Hamiltonian, the time dependence need not be harmonic.

The second type of photon number-conserving interaction is made up of the terms in Eq. (1.3) for which  $i \neq j$ . For each pair of modes  $i$  and  $j$ , these Hamiltonians have the form

$$H_{Rij}(t) \equiv \Pi_{ij}(t) a_i^\dagger a_j + \Pi_{ij}^*(t) a_j^\dagger a_i, \quad i \neq j. \quad (1.6)$$

These interactions conserve the total number of photons in each pair of modes, but not the number in each mode separately; i.e., the Hamiltonian  $H_{Rij}(t)$  commutes with the sum, but not the difference, of the photon-number operators for the two modes. Physically, they describe "ideal" frequency-converting interactions, in which a photon of frequency  $\Omega_j \leq \Omega_i$  and a "pump" photon of (or photons of total) frequency  $(\Omega_i - \Omega_j)$  are destroyed simultaneously to produce a photon of frequency  $\Omega_i$  (and vice-versa). The interaction is "ideal" if the pump(s) can be assumed to have an unlimited supply of photons, and so be described by a classical function  $\Pi_{ij}(t)$ .

The interactions defined in Eq. (1.4), which do not conserve photon number, are of three types. The first consists of the linear Hamiltonians  $H_1^{(1)}(t)$ , each of which describes the interaction of an oscillator with a classical force characterized by a function  $\lambda(t)$  (e.g., a classical current).<sup>10,11</sup> These will be seen to conserve quantities related to the noise (uncertainties) of a GPS. The second type of interaction consists of those quadratic Hamiltonians  $H_2^{(2)}(t)$  that couple two different modes ( $i \neq j$ ). For a pair of modes  $i, j$  these Hamiltonians have the form

$$H_{2ij}(t) \equiv i\zeta^*(t) a_i a_j - i\zeta(t) a_j^\dagger a_i^\dagger, \quad i \neq j. \quad (1.7)$$

These interactions conserve the difference in the number of photons in the two modes, but not the total number; i.e., the Hamiltonian  $H_{2ij}(t)$  commutes with the difference, but not the sum, of the photon-number operators for the two modes.<sup>18</sup> Physically, these describe "ideal" nondegenerate two-photon interactions, in which two photons of frequencies  $\Omega_i$  and  $\Omega_j$  are destroyed simultaneously to produce a pump photon of (or photons of total) frequency  $(\Omega_i + \Omega_j)$  (and vice-versa). The simplest example of a device that operates on such an interaction is a nondegenerate parametric amplifier,<sup>18-20</sup> which uses a single pump at frequency  $(\Omega_i + \Omega_j)$ ; the two modes are called the signal and the idler. Another example is a four-wave mixer,<sup>21,22</sup> which uses two pumps, the sum of whose frequencies is  $(\Omega_i + \Omega_j)$ ; here the two modes are the transmitted and reflected waves. The interaction is "ideal" if the pump(s) can be described by a classical function  $\zeta(t)$ .

The third type of interaction that does not conserve photon number consists of the quadratic Hamiltonians  $H_2^{(1)}(t)$  that involve single modes ( $i = j$ ). For each mode these Hamiltonians have the form

$$H_2^{(1)}(t) = \frac{1}{2}i\zeta^*(t) a^2 - \frac{1}{2}i\zeta(t) a^{\dagger 2}. \quad (1.8)$$

Physically, these interactions describe ideal degenerate two-photon interactions in which two photons of frequency  $\Omega$  from the same mode are destroyed simultaneously to produce a pump photon of (or photons of total) frequency  $2\Omega$  (and vice-versa). Such an interaction is used, for example, in a degenerate parametric amplifier. For specificity throughout the remainder of this paper, whenever I need an example of a device that operates on a two-photon interaction (degenerate or nondegenerate), I shall have in mind the simplest example -- an ideal parametric amplifier.

### b. Unitary operators

The unitary operators that relate one GPS to other GPS with the same total number of photons are generated by the photon number-conserving Hermitian forms  $H_R^{(N)}$ . They are of two types: rotation operators, which act on one mode at a time, and "mixing" operators, which couple two modes. For each mode a rotation operator  $R(\theta)$  is defined by

$$R(\theta) \equiv e^{-i\theta \mathbf{a}^\dagger \mathbf{a}}, \quad \theta = \theta^* \quad (1.9)$$

[Eq. (1.5)]. Formally,  $R(\theta)$  rotates the real and imaginary parts of  $\mathbf{a}$  (i.e., position and momentum) into each other. For each pair of modes  $i, j$  a mixing operator  $T(q, \chi)$  is defined by

$$T(q, \chi) \equiv \exp[q (e^{-2i\chi} \mathbf{a}_j^\dagger \mathbf{a}_i - e^{2i\chi} \mathbf{a}_i^\dagger \mathbf{a}_j)], \quad i \neq j \quad (1.10a)$$

[Eq. (1.6)], where  $q$  and  $\chi$  are real numbers defined on the intervals

$$0 \leq q \leq \frac{1}{2}\pi, \quad -\frac{1}{2}\pi < \chi \leq \frac{1}{2}\pi. \quad (1.10b)$$

Formally,  $T(q, \chi)$  unitarily transforms  $\mathbf{a}_i$  and  $\mathbf{a}_j$  into linear combinations of each other.



The unitary operators that relate one GPS to other GPS with different total photon number are generated by the (non-photon-number-conserving) Hermitian forms  $H_1^{(N)}$  and  $H_2^{(N)}$ . Again, they are of two types: those that act on one mode at a time, and those that couple two modes. For each mode a displacement operator<sup>10,11</sup> and a single-mode squeeze operator<sup>12,13,23</sup> are defined by

$$D(\mathbf{a}, \mu) \equiv \exp[\mu \mathbf{a}^\dagger - \mu^* \mathbf{a}] , \quad (1.11)$$

$$S_1(\mathbf{r}, \varphi) \equiv \exp[\frac{1}{2}r (e^{-2i\varphi} \mathbf{a}^2 - e^{2i\varphi} \mathbf{a}^{\dagger 2})] \quad (1.12a)$$

[Eqs. (1.4a), (1.8)]. Here  $\mu$  is a complex number, and  $r$  and  $\varphi$ , known as the squeeze factor and squeeze angle, are real numbers defined on the intervals

$$0 \leq r < \infty , \quad -\frac{1}{2}\pi < \varphi \leq \frac{1}{2}\pi . \quad (1.12b)$$

Formally, the displacement operator adds a constant ( $\mu$ ) to  $\mathbf{a}$ , thus changing the mean values of the position and momentum variables. The single-mode squeeze operator mixes  $\mathbf{a}$  with  $\mathbf{a}^\dagger$ . Consequently, it induces a correlation between the position and momentum variables that is independent of their mean values. This correlation can result in a narrowing of the coordinate-space wave function, with a corresponding broadening of the momentum-space wave function.

For each pair of modes a two-mode squeeze operator<sup>24-27</sup>  $S(\mathbf{r}, \varphi)$  is defined by

$$S(\mathbf{r}, \varphi) \equiv \exp[r(e^{-2i\varphi} \mathbf{a}_i \mathbf{a}_j - e^{2i\varphi} \mathbf{a}_j^\dagger \mathbf{a}_i^\dagger)] , \quad i \neq j \quad (1.13)$$

[Eq. (1.7)], where  $r$  and  $\varphi$  are defined as above [Eq. (1.12b)]. The two-mode squeeze operator mixes  $\mathbf{a}_i$  with  $\mathbf{a}_j^\dagger$  (and  $\mathbf{a}_j$  with  $\mathbf{a}_i^\dagger$ ). Consequently, it induces correlations between the positions and momenta of the two modes

(but not of each mode, as the single-mode squeeze operator would do); i.e., it causes  $\alpha_i$  and  $\alpha_j$  to become correlated.

### c. Gaussian pure states

Turn now from discussion of the interaction Hamiltonians and unitary operators associated with GPS to the states themselves. Although it is useful to classify the interaction Hamiltonians and unitary operators according to whether they conserve the total number of photons, it is not so useful to classify the states according to their total number of photons. More useful for classifying GPS is a quantity that ignores the mean excitation ( $\langle \alpha_j \rangle$ ,  $j = 1, 2, \dots, N$ ) and focuses exclusively on the total (second-moment) noise associated with the state. The total noise of a single-mode GPS is defined as the sum (hence the adjective "total") of the squared uncertainties (variances) in the real and imaginary parts of  $\alpha$ . The minimum total noise allowed by quantum mechanics (i.e., by the commutator  $[\alpha, \alpha^\dagger] = 1$ ) for each mode is therefore one half quantum ("zero-point noise"). This minimum is realized if and only if the state is an eigenstate of the annihilation operator for that mode. Note that although the total noise of a mode includes in its definition the half quantum of zero-point noise, this is not the significance of the adjective "total." The total noise of an  $N$ -mode GPS is defined as the sum of the contributions from ("total noises" of) each mode. The total noise of a GPS can be thought of as the noise content of the state in units of photon number; it is the number of photons, including the half quantum from each mode due to zero-point noise, that would be left in the state if the mean excitation were removed. The total noise of a state is a more fundamental quantity than the total number of photons. It is conserved if the total number of photons is conserved, but the converse is not

true. [For example, a classical force interacting with an oscillator(s) changes the total number of photons, but not the total noise; see Section IIA.3.]

When considering two or more modes one should note the distinction between the total noise and another quantity, the total noise energy. The total noise energy of a GPS is the noise content of the state in units of energy; it is the energy (including zero-point energy) that would be left in the state if the mean excitation were removed. For a single mode the distinction is not important, since the total noise energy is equal to the product of the total noise and the mode's frequency. But for two or more modes with different frequencies, the total noise and the total noise energy are not proportional to each other. They are proportional to each other only when the total noises of all the modes are identical. Just as photon number is a more convenient quantity than energy for classifying the potentials that produce GPS, so total noise is a more convenient quantity than total noise energy for classifying GPS.

It is shown below that all linear interaction Hamiltonians  $H_1^{(N)}(t)$ , as well as all photon number-conserving interaction Hamiltonians  $H_R^{(N)}(t)$ , conserve the total noise of an  $N$ -mode state. Further, these are the only interaction Hamiltonians that conserve both the total noise and the Gaussian nature of a state. This means that states unitarily related to each other by products of rotation, mixing, or displacement operators all have the same total noise. Conversely, all GPS with the same total noise are related to each other by (products of) rotation, mixing, and displacement operators. Only the quadratic, non-photon-number-conserving potentials  $H_2^{(N)}(t)$  can change the total noise of a state. There are, therefore, two broad classes of GPS. The first class consists of all states unitarily related to the vacuum

state by products of displacement, rotation, and mixing operators. These states have a total noise equal to that of the vacuum state, the minimum allowed by quantum mechanics ( $\frac{1}{2}N$ , for an  $N$ -mode state). Put another way, the first class consists of all (normalized) eigenstates of annihilation operators. The second class consists of all states unitarily related to states in the first class by products of single-mode and/or two-mode squeeze operators. The total noise of these states is necessarily greater than that of the vacuum state.

#### d. Coherent states

The single-mode GPS produced when an oscillator in its ground state is acted on by a classical force, i.e., subjected to the linear interaction Hamiltonian  $H_1^{(1)}(t)$ , is called a single-mode coherent state.<sup>10,11</sup> Formally, a single-mode coherent state, symbolized by the state vector  $|\mu\rangle_{\text{coh}}$ , is defined as that state unitarily related to the vacuum state by the single-mode displacement operator,

$$|\mu\rangle_{\text{coh}} \equiv D(\mathbf{a}, \mu) |0\rangle . \quad (1.14)$$

It is an eigenstate of the annihilation operator  $\mathbf{a}$  with eigenvalue  $\mu$ . An  $N$ -mode coherent state is simply a tensor product of  $N$  single-mode coherent states. For example, a two-mode coherent state, symbolized by the state vector  $|\underline{\mu}\rangle_{\text{coh}}$  (or  $|\mu_+, \mu_-\rangle_{\text{coh}}$ ), is defined by

$$\begin{aligned} |\underline{\mu}\rangle_{\text{coh}} &\equiv |\mu_+, \mu_-\rangle_{\text{coh}} \equiv |\mu_+\rangle_{\text{coh}} |\mu_-\rangle_{\text{coh}} \\ &\equiv D(\mathbf{a}_+, \mu_+) D(\mathbf{a}_-, \mu_-) |0\rangle \equiv D(\underline{\mathbf{a}}, \underline{\mu}) |0\rangle . \end{aligned} \quad (1.15)$$

It is an eigenstate of the annihilation operators  $\mathbf{a}_+$  and  $\mathbf{a}_-$  for each mode,

with eigenvalues  $\mu_+$  and  $\mu_-$ , respectively. All normalized  $N$ -mode states that are eigenstates of the annihilation operators for their modes can be described as  $N$ -mode coherent states. That is, all states unitarily related to a coherent state by products of rotation, displacement, and mixing operators can be described as another coherent state, with different eigenvalues. Glauber<sup>10,11</sup> and others<sup>18,28-32</sup> beginning in the early 1960's have used coherent states to build a powerful description of the electromagnetic field. Today these states are at the heart of quantum optics, providing the basis for a sophisticated theory of the laser, for example.

#### e. Single-mode squeezed states

For a single mode, there is only one interaction Hamiltonian,  $H_2^{(1)}(t)$ , that can produce a GPS whose total noise differs from (i.e., is greater than) that of a coherent state. The state produced when an oscillator in a coherent state is subjected to an interaction described by  $H_2^{(1)}(t)$  is called a "single-mode squeezed state"<sup>12,13,16,23</sup> (SMSS). Formally, a SMSS, symbolized by the state vector  $|\mu_\alpha\rangle_{(r,\varphi)}$ , is defined as that state unitarily related to the single-mode coherent state  $|\mu_\alpha\rangle_{\text{coh}}$  by the single-mode squeeze operator,

$$|\mu_\alpha\rangle_{(r,\varphi)} \equiv S_1(r,\varphi) |\mu_\alpha\rangle_{\text{coh}}. \quad (1.16)$$

The SMSS  $|\mu_\alpha\rangle_{(r,\varphi)}$  is an eigenstate of the "single-mode squeezed annihilation operator"<sup>24-26</sup>

$$\alpha(r,\varphi) \equiv S_1(r,\varphi) a S_1^\dagger(r,\varphi), \quad (1.17)$$

with complex eigenvalue  $\mu_\alpha$ . Any state unitarily related to the SMSS  $|\mu_\alpha\rangle_{(r,\varphi)}$  by products of single-mode rotation, displacement, and squeeze operators

can be expressed as another SMSS  $|\mu'_\alpha\rangle_{(r',\varphi)}$  (multiplied by an unobservable overall phase factor), with different squeeze factor, squeeze angle, and eigenvalue. As stated earlier, the unitary operator generated by any combination  $\mathbf{H}_G^{(1)}$  of the Hermitian forms  $H_R^{(1)}$ ,  $H_1^{(1)}$ , and  $H_2^{(1)}$  (i.e., the solution to the Schrödinger equation  $i\partial_t U(t) = \mathbf{H}_G^{(1)} U(t)$ ,  $U(0) \equiv 1$ ) can always be written as the product of a single-mode rotation, displacement and squeeze operator, and an overall phase factor (Section IIC and Appendix A). Since these Hermitian forms are the only ones associated with single-mode GPS (proved in Section IIB, by considering the most general single-mode Gaussian wave functions), the SMSS  $|\mu_\alpha\rangle_{(r,\varphi)}$  of Eq. (1.16), with  $r$  and  $\varphi$  defined over the ranges (1.12b), represents the most general (normalized) single-mode GPS.

Single-mode squeezed states were introduced independently by Stoler<sup>12</sup> ("minimum-uncertainty packets") and Lu<sup>13</sup> ("new coherent states"). They have been discussed in detail by Yuen<sup>16</sup> in the context of quantum optics under the name "two-photon coherent states" or "TCS". Their properties and possible application to back-action evading techniques<sup>4</sup> for gravitational-wave detection were first considered by Hollenhorst,<sup>23</sup> who coined the adjective "squeezed". For more recent discussions see, e.g., Refs. 5 and 6. "Generalizations" of coherent states, which include single-mode squeezed states, have been described from a group theoretical approach by Barut and Girardello,<sup>14</sup> Perelomov,<sup>15</sup> and others.<sup>33,34</sup> The properties of single-mode squeezed states are summarized briefly here and below in Sections IIA.5 and IIB.

Recall that the total noise of a single-mode GPS is the sum of the variances of the real and imaginary parts of the annihilation operator  $\alpha$ , or, equivalently, of  $e^{i\delta}\alpha$ , where  $\delta$  is any real number. The total noise of a

single-mode coherent state is equal to  $\frac{1}{2}$ , the smallest value allowed by quantum mechanics (the half quantum of zero-point noise). This implies that, for all choices of  $\delta$ , the two variances are equal to each other and equal to the minimum value allowed by quantum mechanics for the square root of their product. Contrast this with single-mode squeezed states. For certain ranges of the squeeze angle  $\varphi$  (or, equivalently, for those conjugate observables defined by certain ranges of  $\delta$ ), one of the variances is smaller than it would be in a coherent state. The other variance is greater than it would be in a coherent state, since the total noise of a SMSS is greater, but this does not alter the potential practical advantages offered by the reduced uncertainty in the one observable. These advantages are the impetus for the current experimental effort to produce squeezed states,<sup>35-37</sup> applications have been proposed in low-noise optical communications<sup>1,2</sup> and high-precision interferometric experiments,<sup>38-40</sup> for example. For a particular value of the squeeze angle  $\varphi$  ( $\varphi = -\delta$ ) the variance of one observable (the real part of  $e^{-i\varphi} \mathbf{a}$ ) is minimized and is a factor  $e^{-2r}$  smaller than its coherent-state value, while the variance of other (the imaginary part of  $e^{-i\varphi} \mathbf{a}$ ) is maximized and is a factor  $e^{2r}$  larger than its coherent-state value. Only for this value of  $\varphi$  is the product of the variances equal to its minimum allowed value, as in a coherent state.

The important parameter of a squeezed state is its squeeze factor  $r$ , not its squeeze angle  $\varphi$ . There are a number of ways to understand this. First, the conjugate observables defined as the real and imaginary parts of  $\mathbf{a}$  deserve no special status relative to the real and imaginary parts of  $e^{i\delta} \mathbf{a}$ . In actual experiments one would tune the apparatus to respond to whichever observable has the smallest uncertainty. Second, as the SMSS  $|\mu_{\mathbf{a}}\rangle_{(r, \varphi)}$  evolves freely, its squeeze angle changes, but its squeeze factor  $r$  does not.

The uncertainties oscillate between the conjugate observables (as does the energy between potential and kinetic), but the total noise, which depends only on  $\tau$ , is constant. Even if an oscillator in a SMSS is acted on by a classical force (i.e., multiplied by a product of rotation and displacement operators), its squeeze factor remains constant, and only its squeeze angle and eigenvalue (hence its complex amplitude  $\langle \alpha \rangle$ ) change. If, however, an oscillator in a SMSS is subjected to a new degenerate two-photon interaction [ $H_2^{(1)}(t)$ ] -- i.e., multiplied by another single-mode squeeze operator -- it will go into another SMSS, with different squeeze factor, squeeze angle, and eigenvalue.

#### f. Two-mode Gaussian pure states

For two modes, there are three interaction Hamiltonians in  $H_2^{(2)}(t)$  that can produce a GPS whose total noise is greater than that of a coherent state. Two of these are the degenerate two-photon interaction Hamiltonians  $H_2^{(1)}(t)$  of Eq. (1.8), one for each mode. The third is the nondegenerate (two-mode) two-photon interaction Hamiltonian  $H_{2+-}(t)$  defined in Eq. (1.7). The most general kind of (normalized) two-mode GPS is produced when two oscillators, each in a coherent state, are exposed to all three of these quadratic interaction Hamiltonians. Formally, this state, symbolized by the state vector  $|\underline{\mu}_g\rangle$  (or  $|\mu_{g+}, \mu_{g-}\rangle$ ), is related to a two-mode coherent state by a product of the three squeeze operators:

$$|\underline{\mu}_g\rangle \equiv S_{1+}(\tau_+, \varphi_+) S_{1-}(\tau_-, \varphi_-) S(\tau, \varphi) |\underline{\mu}_g\rangle_{\text{coh}} \equiv \mathbf{U}_g |\underline{\mu}_g\rangle_{\text{coh}}. \quad (1.18)$$

It is an eigenstate of the transformed annihilation operators  $g_{\pm} \equiv \mathbf{U}_g a_{\pm} \mathbf{U}_g^{\dagger}$ , with complex eigenvalues  $\mu_{g\pm}$ . The order of the three squeeze operators in Eq. (1.18) has been chosen for convenience only. All states unitarily related



to the GPS  $|\underline{\mu}_g\rangle$  defined by Eq. (1.18) by any product of rotation, displacement, mixing, and squeeze operators can be expressed as another two-mode GPS  $|\underline{\mu}'_g\rangle$  with the same form as Eq. (1.18), but with different parameters  $\tau_{\pm}$ ,  $\tau$ ,  $\varphi_{\pm}$ ,  $\varphi$ ,  $\mu_{g+}$ , and  $\mu_{g-}$ . Further, the unitary operator generated by any combination  $\mathbf{H}_G^{(2)}$  of the Hermitian forms  $H_R^{(2)}$ ,  $H_1^{(2)}$ , and  $H_2^{(2)}$  (i.e., the solution to the Schrödinger equation  $i\partial_t \mathbf{U}(t) = \mathbf{H}_G^{(2)} \mathbf{U}(t)$ ,  $\mathbf{U}(0) \equiv 1$ ) can always be written as the product of two single-mode rotation and displacement operators, a mixing operator, an operator like  $\mathbf{U}_g$ , and an overall phase factor (see Appendix A). It is for these reasons that the state  $|\underline{\mu}_g\rangle$  defined by Eq. (1.18) is said to represent the most general normalized two-mode GPS.

If two oscillators, each in a coherent state, are subjected only to degenerate two-photon interactions [ $H_2^{(1)}(t)$ ], the resulting (two-mode) state is simply a tensor product of two single-mode squeezed states. If, however, they are subjected only to a nondegenerate two-photon interaction [ $H_{2+-}(t)$ ], the resulting state is called a "two-mode squeezed state"<sup>24-26</sup> (TMSS). Formally, a TMSS, symbolized by the state vector  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$ , is defined as that state unitarily related to the two-mode coherent state  $|\underline{\mu}_\alpha\rangle_{\text{coh}}$  by the two-mode squeeze operator,

$$|\underline{\mu}_\alpha\rangle_{(r,\varphi)} \equiv |\mu_{\alpha+}, \mu_{\alpha-}\rangle_{(r,\varphi)} \equiv S(r, \varphi) |\underline{\mu}_\alpha\rangle_{\text{coh}}. \quad (1.19)$$

The TMSS  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  is an eigenstate of the "two-mode squeezed annihilation operators"<sup>24-26</sup>

$$\alpha_{\pm}(r, \varphi) \equiv S(r, \varphi) a_{\pm} S^\dagger(r, \varphi), \quad (1.20)$$

with complex eigenvalues  $\mu_{\alpha\pm}$ . The properties and importance of two-mode

squeezed states in the context of quantum optics are the subject of a recent series of papers by Caves and me.<sup>25,26</sup> They are motivated briefly here and discussed further in Section III.

**g. Two-mode squeezed states**

The two-mode squeezed states (1.19) are the natural two-mode analogs of single-mode squeezed states [Eq. (1.16)]. Formally, this is because they are unitarily related to two-mode coherent states in the same functional way that single-mode squeezed states are related to single-mode coherent states. More precisely, the operators  $a_i a_j$ ,  $a_j^\dagger a_i^\dagger$ , and  $(a_i^\dagger a_i + a_j^\dagger a_j + 1)$  generate the same (noncompact, pseudo-unitary) group  $SU(1,1)$  as the operators  $\frac{1}{2}a^2$ ,  $\frac{1}{2}a^{\dagger 2}$ , and  $(a^\dagger a + \frac{1}{2})$  (see Sections IIC, IIIA, and IIIC below).<sup>15,41</sup> Physically, a two-mode squeezed state can be produced in a parametric amplifier by using a single pump whose photons have energy  $\Omega_+ + \Omega_-$ , just as a single-mode squeezed state can be produced in the degenerate limit of a parametric amplifier by using a single pump with photons of energy  $2\Omega$ . In contrast, production of the general two-mode GPS (1.18) would require three separate parametric amplifiers -- i.e., three different pumps, with photon energies  $2\Omega_+$ ,  $2\Omega_-$ , and  $\Omega_+ + \Omega_-$ .

Like a single-mode squeezed state, a two-mode squeezed state is a state in which the variance of one of two conjugate observables is smaller than it would be in a coherent state. For a single-mode squeezed state the natural conjugate observables are the real and imaginary parts of  $a$  (or rotated versions thereof). But what are they for two-mode squeezed states? Analyses of optical heterodyning,<sup>1,42</sup> together with the properties of two-mode squeezed states, indicate that natural choices for these observables are the quadrature-phase operators  $E_1$  and  $E_2$  of the electric (or magnetic) field  $E$

(or similarly defined quantities if the oscillators are not modes of the electromagnetic field). The following qualitative remarks give a general idea of the nature and significance of the quadrature phases. For further discussion, the reader is referred to Refs. 1, 2, 24-27, 43, and 44.

In optical heterodyning an input field  $E \propto E_1 \cos\Omega(t - \mathbf{x}) + E_2 \sin\Omega(t - \mathbf{x})$ , composed of upper and lower sidebands of a carrier frequency  $\Omega$ , is combined at a beam splitter with a strong local-oscillator field at the carrier frequency. One or both of the beam-splitter output ports is then monitored with a photodetector(s).<sup>45,46</sup> The relative strength of the local-oscillator field guarantees that (i) the dominant contribution to the output-field intensities is proportional to the mean field of that quadrature phase which was in phase with the local oscillator, e.g.,  $\langle E_1 \rangle$ , and (ii) the dominant noise in the output fields is proportional to the noise in (variance of) that quadrature. A reduction in the noise in one quadrature phase relative to its coherent-state value is therefore manifested in heterodyning as an improved signal-to-noise ratio.

The upper and lower sidebands of the input field consist of modes with frequencies  $\Omega + \varepsilon$  and  $\Omega - \varepsilon$ , respectively, where the "modulation frequencies"  $\varepsilon$  take on all (positive) values in some bandwidth  $\Delta\varepsilon$  ( $0 \leq \Delta\varepsilon \ll \Omega$ ). The quadrature phases have no time dependence at the carrier frequency  $\Omega$ ; they carry only the time dependences at frequencies  $\varepsilon$ . The signal observed in heterodyning, e.g.,  $\langle E_1 \rangle$ , is an amplitude or phase modulation of the carrier wave at frequency  $\Omega$ , with modulation frequencies  $\varepsilon$ . One can filter the output of the photodetector(s) to pick out the contribution from a single modulation frequency  $\varepsilon$ , i.e., from one pair of modes, with frequencies  $\Omega \pm \varepsilon$ . The noise properties of this filtered output thus reflect the noise properties of a two-mode state.

When measured in units of energy, the minimum contribution that two modes with frequencies  $\Omega \pm \varepsilon$  can make to the total noise (variance) of the electric field is  $\frac{1}{2}(\Omega + \varepsilon) + \frac{1}{2}(\Omega - \varepsilon) = \Omega$ . This minimum is realized only if both modes are in coherent states; they then contribute a zero-point noise of  $\frac{1}{2}\Omega$  to each quadrature phase. The relations between the quadrature phases and the creation and annihilation operators [Eqs. (1.21) below, or Refs. 24-27] reveal that the minimum contribution that two modes of frequencies  $\Omega \pm \varepsilon$  can make to the noise in either quadrature phase is  $\frac{1}{2}\varepsilon$ , much smaller than that realized by a (two-mode) coherent state. Note that while a reduced noise in one quadrature phase relative to its coherent-state value implies that the noise in the other quadrature phase and the total noise of the electric field must both be larger than their respective coherent-state values, the latter do not degrade the output signal-to-noise ratio obtained in heterodyning. For the noise in one quadrature phase to be smaller than its coherent-state value of  $\frac{1}{2}\Omega$ , the two modes must be correlated with each other, in the way that would be produced by a nondegenerate two-photon interaction like (1.7); that is, the two modes must be in a state whose unitary relation to a (two-mode) coherent state includes a two-mode squeeze operator  $S(r, \varphi)$ . For a specific value of the squeeze factor  $r$ , this reduction may or may not be enhanced if, in addition, the two modes are also separately squeezed [by adding degenerate two-photon interactions like (1.8)]; in general, however, the reduction is greatest when the modes are correlated but not separately squeezed, i.e., when the two modes are in a two-mode squeezed state [Eq. (1.19)].

The obvious advantage heterodyning offers is that one can transmit a signal at frequencies  $\Omega \pm \varepsilon$  as amplitude or phase modulation of a carrier wave at frequency  $\Omega$  (modulation frequency  $\varepsilon \ll \Omega$ ), and have a noise

associated with that signal that is much smaller than the zero-point noises  $\frac{1}{2}(\Omega \pm \varepsilon)$  that would accompany the same signal if it were sent directly at the frequencies  $\Omega \pm \varepsilon$ . The "quadrature-phase zero-point noise"  $\frac{1}{2}\varepsilon$  is very small, and with real photodetectors essentially unobservable. That it is in principle nonzero, however, is consistent with what one might expect physically; it says that the zero-point noise of  $\frac{1}{2}\varepsilon$  associated with any signal transmitted directly at frequency  $\varepsilon$  cannot be made to vanish by "disguising" the signal as amplitude or phase modulation of a carrier wave at frequency  $\Omega \gg \varepsilon$ .

The properties of two-mode GPS can be described in terms of the annihilation and creation operators ( $\mathbf{a}_\pm$  and  $\mathbf{a}_\pm^\dagger$ ) of the two modes; this is the approach taken in Section III of this paper. However, more useful operators, at least for two-mode GPS that are not coherent states, are ones that reflect directly the statistics of the quadrature phases  $E_1$  and  $E_2$ , the natural conjugate observables associated with two-mode states. Such (non-Hermitian) operators have been defined,<sup>24-27</sup> and they are called "quadrature-phase amplitudes." Just as the annihilation operators  $\mathbf{a}_\pm$  are proportional to the positive-frequency Fourier components at frequencies  $\Omega \pm \varepsilon$  of the electric-field operator  $E$ , the quadrature-phase amplitudes  $\alpha_1$  and  $\alpha_2$  are proportional to the positive-frequency Fourier components at frequency  $\varepsilon$  of the quadrature-phase operators  $E_1$  and  $E_2$ . They are defined as the following linear combinations of  $\mathbf{a}_+$  and  $\mathbf{a}_-^\dagger$ :

$$\alpha_1 \equiv (2\Omega)^{-\frac{1}{2}}[(\Omega + \varepsilon)^{\frac{1}{2}}\mathbf{a}_+ + (\Omega - \varepsilon)^{\frac{1}{2}}\mathbf{a}_-^\dagger], \quad (1.21a)$$

$$\alpha_2 \equiv (2\Omega)^{-\frac{1}{2}}[-i(\Omega + \varepsilon)^{\frac{1}{2}}\mathbf{a}_+ + i(\Omega - \varepsilon)^{\frac{1}{2}}\mathbf{a}_-^\dagger]. \quad (1.21b)$$

The factors  $(\Omega \pm \varepsilon)^{\frac{1}{2}}$  arise from the requirement that the square of the electric field be proportional to the total energy of the field (see also the discussion of conjugate variables for two-mode GPS in Section IIIA.1a). The factor

$\Omega^{-1/2}$  is included so that  $\alpha_1^\dagger \alpha_1$  and  $\alpha_2^\dagger \alpha_2$  are dimensionless operators whose units are number of quanta at the carrier frequency  $\Omega$ . The factor  $2^{-1/2}$  is a convenient choice for the overall normalization.

The total noise of  $\alpha_1$  or  $\alpha_2$ , i.e., the sum of the variances of its real and imaginary parts, is a number proportional to the noise in (variance of)  $E_1$  or  $E_2$  (at frequency  $\varepsilon$ ), in units of number of quanta at the carrier frequency  $\Omega$ . The minimum contribution  $\frac{1}{2}\varepsilon$  which a pair of modes with frequencies  $\Omega \pm \varepsilon$  must make to the total noise of each quadrature phase is a consequence of the commutation relations of the quadrature-phase amplitudes:

$$\begin{aligned} [\alpha_1, \alpha_1^\dagger] &= [\alpha_2, \alpha_2^\dagger] = \varepsilon / \Omega ; \\ [\alpha_1, \alpha_2^\dagger] &= [\alpha_1^\dagger, \alpha_2] = i ; \\ [\alpha_1, \alpha_2] &= 0 . \end{aligned} \tag{1.22}$$

These imply that the minimum total noise of  $\alpha_1$  or  $\alpha_2$  is  $\frac{1}{2}\varepsilon/\Omega$ , or, equivalently, that the minimum total noise energy of  $\alpha_1$  or  $\alpha_2$  is  $\frac{1}{2}\varepsilon$ . They also imply that the minimum value for the product of the total noises in  $\alpha_1$  and  $\alpha_2$  is  $\frac{1}{4}$ ; this minimum is realized only by two-mode coherent states (see Refs. 25 and 26).

The correlations between the modes in a two-mode squeezed state are such that the only nonvanishing noise moments of  $\alpha_1$  and  $\alpha_2$  (moments with the mean excitations  $\langle \alpha_1 \rangle$  and  $\langle \alpha_2 \rangle$  removed) are those with equal numbers of quadrature-phase amplitudes and their Hermitian conjugates, e.g.,  $\langle \alpha_1 \alpha_2^\dagger \rangle$ ,  $\langle \alpha_1^\dagger \alpha_1 \rangle$ , etc. This implies that all time-dependent noise moments of the quadrature phases  $E_1$  and  $E_2$  vanish. Fields with this property are said to have "time-stationary quadrature-phase" (TSQP) noise.<sup>24-27</sup> The vanishing of certain noise moments of the quadrature-phase amplitudes or,

equivalently, of certain noise moments of the annihilation and creation operators  $a_{\pm}$  and  $a_{\pm}^{\dagger}$  for each pair of modes, allows the properties of two-mode squeezed states to be described with the same techniques (i.e., the same number of parameters, and the same group theoretical construction) used to describe single-mode squeezed states (see the discussions in Sections IIIA.5 and IIIC).

#### **h. Outline of this paper**

Section II of this paper is a review of single-mode Gaussian pure states. Section IIA looks at the unitary operators associated with single-mode GPS and reviews some of the properties of coherent states and single-mode squeezed states. Section IIB considers the most general single-mode Gaussian wave function and from it shows that the most general single-mode GPS is a single-mode squeezed state. Section IIC uses a two-component vector notation to provide a compact and powerful way to express the properties of single-mode GPS and their associated unitary operators. Section III is a detailed discussion of two-mode Gaussian pure states which parallels closely in structure but is necessarily more complicated than that of Section II.

Some useful details are relegated to appendices. Appendix A outlines the procedure and gives supporting details for writing the unitary evolution operator associated with the most general (time-dependent) linear combination of interaction Hamiltonians that can produce single-mode and two-mode GPS as a product of squeeze, rotation, mixing, and displacement operators. Appendix B derives the phase factors for the general single-mode and two-mode GPS coordinate-space wave functions. Appendix C elaborates on a point made in Section IIIB concerning the criterion for two arbitrary complex operators to have a complete (or overcomplete) set of

simultaneous, normalizable eigenstates.



## II. SINGLE-MODE GAUSSIAN PURE STATES

### A. Introduction and Review

#### 1. Notation and Definitions

The quantum mechanical operators naturally associated with a harmonic oscillator are the Schrödinger-picture (SP) annihilation operator  $\alpha$  and its adjoint  $\alpha^\dagger$ , the creation operator. Equivalent operators are the dimensionless position and momentum  $\hat{x}$  and  $\hat{p}$ ; these are Hermitian operators, constant in the SP and related to  $\alpha$  and  $\alpha^\dagger$  by

$$\hat{x} \equiv 2^{-\frac{1}{2}}(\alpha + \alpha^\dagger), \quad \hat{p} \equiv 2^{-\frac{1}{2}}(-i\alpha + i\alpha^\dagger); \quad (2A.1a)$$

$$\alpha \equiv 2^{-\frac{1}{2}}(\hat{x} + i\hat{p}). \quad (2A.1b)$$

The position and momentum are equal to  $2^{\frac{1}{2}}$  times the real and imaginary parts of  $\alpha$ , respectively. The creation and annihilation operators and the dimensionless position and momentum obey the standard commutation relations:

$$[\alpha, \alpha^\dagger] = 1, \quad [\hat{x}, \hat{p}] = i. \quad (2A.2)$$

The complex amplitude of a single-mode state, always denoted in this paper by the symbol  $\mu$ , is the expectation value of  $\alpha$ ; it is related to the mean position and momentum  $x_0$  and  $p_0$  by

$$\mu \equiv \langle \alpha \rangle = 2^{-\frac{1}{2}}(\langle \hat{x} \rangle + i \langle \hat{p} \rangle) \equiv 2^{-\frac{1}{2}}(x_0 + i p_0). \quad (2A.3)$$

The noise moments of the operators  $\alpha$  and  $\alpha^\dagger$  or  $\hat{x}$  and  $\hat{p}$  provide a useful way to characterize states associated with harmonic oscillators. Noise moments of any operator  $B$  are moments of  $\Delta B \equiv B - \langle B \rangle$ , the operator

minus its mean. Note that an operator  $\Delta B$  is defined only with reference to a particular state, which defines  $\langle B \rangle$ . All noise moments of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  (or  $\hat{x}$  and  $\hat{p}$ ) for Gaussian states are expressible in terms of the second-order noise moments. There are two second-order noise moments of the annihilation operator  $\mathbf{a}$ . These are the complex number

$$\langle (\Delta \mathbf{a})^2 \rangle \equiv \langle \mathbf{a}^2 \rangle - \langle \mathbf{a} \rangle^2 = \langle (\Delta \mathbf{a}^\dagger)^2 \rangle^* , \quad (2A.4a)$$

and the positive real number

$$\langle |\Delta \mathbf{a}|^2 \rangle \equiv \langle \Delta \mathbf{a} \Delta \mathbf{a}^\dagger \rangle_{\text{sym}} \equiv \frac{1}{2} \langle \Delta \mathbf{a} \Delta \mathbf{a}^\dagger + \Delta \mathbf{a}^\dagger \Delta \mathbf{a} \rangle \quad (2A.4b)$$

(the subscript "sym" denotes a symmetrized product). These second-order noise moments of  $\mathbf{a}$  are related to the three (real) second-order noise moments of  $\hat{x}$  and  $\hat{p}$  by

$$\langle (\Delta \mathbf{a})^2 \rangle = \frac{1}{2} (\langle (\Delta \hat{x})^2 \rangle - \langle (\Delta \hat{p})^2 \rangle) + i \langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}} , \quad (2A.5a)$$

$$\langle |\Delta \mathbf{a}|^2 \rangle = \frac{1}{2} (\langle (\Delta \hat{x})^2 \rangle + \langle (\Delta \hat{p})^2 \rangle) \quad (2A.5b)$$

[Eqs. (2A.1)]. The total (second-moment) noise of a single-mode GPS is  $\langle |\Delta \mathbf{a}|^2 \rangle$ , the sum of the variances (squared uncertainties) of the real and imaginary parts of  $\mathbf{a}$ .

The commutation relations (2A.2) enforce the following lower limits on the product and sum of the variances of  $\hat{x}$  and  $\hat{p}$ :<sup>47</sup>

$$\langle (\Delta \hat{x})^2 \rangle \langle (\Delta \hat{p})^2 \rangle \geq \frac{1}{4} + \langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}}^2 \geq \frac{1}{4} , \quad (2A.6a)$$

$$\frac{1}{4} (\langle (\Delta \hat{x})^2 \rangle + \langle (\Delta \hat{p})^2 \rangle)^2 = \langle |\Delta \mathbf{a}|^2 \rangle^2 \geq \frac{1}{4} + |\langle (\Delta \mathbf{a})^2 \rangle|^2 \geq \frac{1}{4} . \quad (2A.6b)$$

Equalities hold in the first of each of these inequalities if and only if the state is an eigenstate of certain linear combinations of  $\hat{x}$  and  $\hat{p}$  (or  $\mathbf{a}$  and  $\mathbf{a}^\dagger$ )

-- i.e., if and only if the state is a Gaussian pure state (see Section IIB).

## 2. Single-mode rotation operator

Consider now the single-mode rotation operator  $R(\theta)$ , defined by

$$R(\theta) \equiv e^{-i\theta \mathbf{a}^\dagger \mathbf{a}} = e^{-i\theta} e^{\frac{1}{2}i\theta(\hat{x}^2 + \hat{p}^2)} \quad (2A.7a)$$

[Eq. (1.9)]. It satisfies

$$R^{-1}(\theta) = R^\dagger(\theta) = R(-\theta) . \quad (2A.7b)$$

For an oscillator characterized by frequency  $\Omega$ ,  $R(\Omega t)$  is the evolution operator associated with the free Hamiltonian  $H_0^{(1)}$ ,

$$H_0^{(1)} \equiv \Omega \mathbf{a}^\dagger \mathbf{a} = \frac{1}{2}\Omega(\hat{x}^2 + \hat{p}^2 - 1) , \quad (2A.8a)$$

$$e^{-iH_0^{(1)}t} = R(\Omega t) . \quad (2A.8b)$$

The rotation operator acting on any number eigenstate  $|n\rangle$  simply multiplies it by the phase factor  $e^{-in\theta}$  [Eqs. (1.2)]; in particular, it leaves the vacuum state unchanged:

$$R(\theta) |0\rangle = |0\rangle . \quad (2A.9)$$

The rotation operator unitarily transforms the annihilation operator  $\mathbf{a}$  into  $e^{i\theta}\mathbf{a}$  -- i.e., it rotates  $\hat{x}$  and  $\hat{p}$  into each other:

$$R(\theta) \mathbf{a} R^\dagger(\theta) = e^{i\theta} \mathbf{a} \equiv \mathbf{a}(\theta) , \quad (2A.10a)$$

$$R(\theta) \hat{x} R^\dagger(\theta) = \hat{x} \cos\theta - \hat{p} \sin\theta \equiv \hat{x}(\theta) , \quad (2A.10b)$$

$$R(\theta) \hat{p} R^\dagger(\theta) = \hat{x} \sin\theta + \hat{p} \cos\theta \equiv \hat{p}(\theta) . \quad (2A.10c)$$

The unitarity of  $R(\theta)$  ensures that  $\hat{x}(\theta)$  and  $\hat{p}(\theta)$  are conjugate observables,  $[\hat{x}(\theta), \hat{p}(\theta)] = i$ . The transformation (2A.10a) shows that an eigenstate of  $\mathbf{a}$  remains an eigenstate of  $\mathbf{a}$  when operated on by a rotation operator, i.e., as it evolves freely. The rotation operator clearly preserves the total number of photons in the mode,

$$R^\dagger(\theta) \mathbf{a}^\dagger \mathbf{a} R(\theta) = \mathbf{a}^\dagger \mathbf{a} \quad (2A.11)$$

(hence also the total energy). The effect of the rotation operator is merely to transfer energy between kinetic ( $\hat{p}^2$ ) and potential ( $\hat{x}^2$ ). It therefore also preserves the total noise,

$$\langle R^\dagger(\theta) |\Delta \mathbf{a}|^2 R(\theta) \rangle = \langle |\Delta \mathbf{a}|^2 \rangle, \quad (2A.12a)$$

its effect on a state being merely to redistribute the noise between the position and momentum variables,

$$\langle R^\dagger(\theta) (\Delta \mathbf{a})^2 R(\theta) \rangle = \langle [\Delta \mathbf{a}(-\theta)]^2 \rangle = e^{-2i\theta} \langle (\Delta \mathbf{a})^2 \rangle. \quad (2A.12b)$$

Note in Eqs. (2.12) that the operator  $\Delta \mathbf{a}$  on the left-hand side of the equations is  $\Delta \mathbf{a} \equiv \mathbf{a} - \langle R^\dagger(\theta) \mathbf{a} R(\theta) \rangle$ , whereas on the right-hand side it is  $\Delta \mathbf{a} \equiv \mathbf{a} - \langle \mathbf{a} \rangle$ . A similar remark holds throughout this paper wherever the moments or noise moments of operators in a state  $|\Psi\rangle$  are compared with those in a state  $U|\Psi\rangle$ .

Finally, note that the simple form of  $R(\theta)$  implies that the product of an arbitrary number of rotation operators can be expressed trivially as a single rotation operator, using the rule

$$R(\theta) R(\theta') = R(\theta + \theta'). \quad (2A.13)$$

### 3. Single-mode displacement operator

The single-mode displacement operator<sup>10,11</sup> is defined by

$$D(\mathbf{a}, \mu) \equiv e^{\mu \mathbf{a}^\dagger - \mu^* \mathbf{a}} = e^{i(p_0 \hat{x} - z_0 \hat{p})} = e^{-\frac{i}{\hbar} p_0 z_0} e^{i p_0 \hat{x}} e^{-i z_0 \hat{p}} \quad (2A.14)$$

[Eq. (1.11)]. It satisfies the following equalities:

$$D^{-1}(\mathbf{a}, \mu) = D^\dagger(\mathbf{a}, \mu) = D(\mathbf{a}, -\mu) = D(-\mathbf{a}, \mu). \quad (2A.15)$$

Properties of  $D(\mathbf{a}, \mu)$  are discussed in Refs. 10, 11, 31, and 21. Most important is the way it unitarily transforms the annihilation operator:

$$D(\mathbf{a}, \mu) \mathbf{a} D^\dagger(\mathbf{a}, \mu) = \mathbf{a} - \mu. \quad (2A.16)$$

This shows that an eigenstate of  $\mathbf{a}$  remains an eigenstate of  $\mathbf{a}$  when operated on by a displacement operator. In particular, the single-mode coherent state  $|\mu\rangle_{\text{coh}}$ , defined as  $D(\mathbf{a}, \mu)$  acting on the vacuum state [Eq. (1.14)], is an eigenstate of  $\mathbf{a}$  with eigenvalue  $\mu$ . The additive nature of this transformation implies that when the displacement operator acts on a state it changes all moments of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  (e.g., the complex amplitude  $\langle \mathbf{a} \rangle$ , and the photon number  $\langle \mathbf{a}^\dagger \mathbf{a} \rangle$ ). However, since the transformation merely adds a complex number to  $\mathbf{a}$ , the noise moments of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  are left unchanged. Thus, when the displacement operator acts on a state, it displaces the wave function, but does not modify its shape.

Two other properties of the displacement operator are useful here. First, it is unitarily transformed by the rotation operator in the following way:

$$R(\theta) D(\mathbf{a}, \mu) R^\dagger(\theta) = D[\mathbf{a}(\theta), \mu] = D[\mathbf{a}, \mu(-\theta)], \quad (2A.17a)$$

$$\mu(\theta) \equiv e^{i\theta} \mu \quad (2A.17b)$$

[Eqs. (2A.10), (2A.14)]. That  $D(\mathbf{a}, \mu)$  doesn't commute with the rotation operator reveals why it doesn't preserve photon number. It also shows that the form of the displacement operator is invariant under a unitary transformation of  $\mathbf{a}$  generated by the rotation operator:

$$D(\mathbf{a}, \mu) = D[\mathbf{a}(\theta), \mu(\theta)] . \quad (2A.18)$$

Second, the transformation (2A.16) implies that the product of two displacement operators is another displacement operator, multiplied by a phase factor:

$$D(\mathbf{a}, \mu') D(\mathbf{a}, \mu) = e^{i\text{Im}(\mu\mu'^*)} D(\mathbf{a}, \mu + \mu') . \quad (2A.19)$$

These properties, like the transformations (2A.10a) and (2A.16), show that any eigenstate of  $\mathbf{a}$  remains an eigenstate of  $\mathbf{a}$  when displaced and/or allowed to evolve freely. For example, as a coherent state [Eq. (1.14)] evolves freely, it changes in the following way:

$$R(\Omega t) |\mu\rangle_{\text{coh}} = |\mu(-\Omega t)\rangle_{\text{coh}} \equiv |e^{-i\Omega t} \mu\rangle_{\text{coh}} . \quad (2A.20)$$

All single-mode states that are eigenstates of  $\mathbf{a}$  are unitarily related to the vacuum state by products of rotation and displacement operators. Conversely, all such states are eigenstates of  $\mathbf{a}$ . These states comprise the entire class of single-mode states whose total noise is equal to that of the vacuum state. The special properties of the rotation operator -- that it preserves the total number of photons, that it preserves the total noise, and that it preserves coherent states -- are a consequence of one essential property: the unitary transformation it induces on  $\mathbf{a}$  merely multiplies  $\mathbf{a}$  by a phase factor; i.e., it never mixes  $\mathbf{a}$  with  $\mathbf{a}^\dagger$ . To find unitary operators that do not conserve the total noise and that generate new states from coherent

states (states with a total noise greater than that of the vacuum state), one must consider operators -- single-mode squeeze operators -- that mix  $\alpha$  with  $\alpha^\dagger$ .

#### 4. Single-mode squeeze operator

The single-mode squeeze operator<sup>12,13,23</sup> is defined by

$$S_1(\tau, \varphi) \equiv \exp\left[\frac{1}{2}\tau (e^{-2i\varphi} \alpha^2 - e^{2i\varphi} \alpha^{\dagger 2})\right], \quad (2A.21a)$$

$$0 \leq \tau < \infty, \quad -\frac{1}{2}\pi < \varphi \leq \frac{1}{2}\pi \quad (2A.21b)$$

[Eqs. (1.12)]. It satisfies the following equalities:

$$S_1^{-1}(\tau, \varphi) = S_1^\dagger(\tau, \varphi) = S_1(-\tau, \varphi) = S_1(\tau, \varphi + \frac{1}{2}\pi). \quad (2A.22)$$

Properties of  $S_1(\tau, \varphi)$  are discussed in Refs. 23, 25, and 26. Most important is the way it unitarily transforms the annihilation operator:

$$S_1(\tau, \varphi) \alpha S_1^\dagger(\tau, \varphi) = \alpha \cosh \tau + \alpha^\dagger e^{2i\varphi} \sinh \tau \equiv \alpha(\tau, \varphi) \quad (2A.23a)$$

[Eq. (1.17)]. Inverting this relation gives  $\alpha$  in terms of  $\alpha$  and  $\alpha^\dagger$ :

$$\alpha = S_1^\dagger(\tau, \varphi) \alpha(\tau, \varphi) S_1(\tau, \varphi) = \alpha(\tau, \varphi) \cosh \tau - \alpha^\dagger(\tau, \varphi) e^{2i\varphi} \sinh \tau. \quad (2A.23b)$$

A state unitarily related to an eigenstate of  $\alpha$  by a single-mode squeeze operator is an eigenstate of the single-mode squeezed annihilation operator  $\alpha(\tau, \varphi)$  (sometimes denoted simply by  $\alpha$ ). The unitarity of  $S_1(\tau, \varphi)$  ensures that  $[\alpha(\tau, \varphi), \alpha^\dagger(\tau, \varphi)] = [\alpha, \alpha^\dagger] = 1$ .

The transformation (2A.23) implies that when the squeeze operator acts on a state it changes the noise moments of  $\alpha$  and  $\alpha^\dagger$ . That is, it modifies the shape of the wave function (and, if the mean position or momentum is nonzero, displaces it as well). In particular, it preserves neither the total

number of photons nor the total noise of a state,

$$\langle S_1^\dagger(\tau, \varphi) a^\dagger a S_1(\tau, \varphi) \rangle = \sinh^2 \tau + \cosh 2\tau \langle a^\dagger a \rangle - \sinh 2\tau \operatorname{Re}(e^{-2i\varphi} \langle a^2 \rangle) \quad (2A.24a)$$

$$\langle S_1^\dagger(\tau, \varphi) |\Delta a|^2 S_1(\tau, \varphi) \rangle = \cosh 2\tau \langle |\Delta a|^2 \rangle - \sinh 2\tau \operatorname{Re}(e^{-2i\varphi} \langle (\Delta a)^2 \rangle) \quad (2A.24b)$$

Equation (2A.24b) shows explicitly that any state whose unitary relation to the vacuum state (or to any eigenstate of  $a$ ) includes a single-mode squeeze operator has a total noise greater than that of the vacuum state.

A few other properties of the single-mode squeeze operator are useful here. First, it is unitarily transformed by the rotation operator in the following way:

$$R(\theta) S_1(\tau, \varphi) R^\dagger(\theta) = S_1(\tau, \varphi - \theta) \quad (2A.25)$$

[Eqs. (2A.10), (2A.21)]. That  $S_1(\tau, \varphi)$  doesn't commute with the rotation operator reveals why it doesn't preserve photon number [Eq. (2A.24a)].

Second, it unitarily transforms the displacement operator in the following way:

$$S_1^\dagger(\tau, \varphi) D(\alpha, \mu) S_1(\tau, \varphi) = D(\alpha, \mu_\alpha), \quad (2A.26)$$

$$\mu_\alpha \equiv \mu \cosh r + \mu^* e^{2i\varphi} \sinh r \quad (2A.27)$$

[Eqs. (2A.14), (2A.23a)]. This relation reflects the fact that the form of the displacement operator is invariant under unitary transformations of  $a$  that are linear in  $a$  and  $a^\dagger$  (and that do not add to  $a$  a constant). Such unitary transformations are generated only by (products of) rotation and single-mode squeeze operators. The invariance under rotations was noted in Eq. (2A.18). The invariance under transformations generated by the single-mode squeeze operator says that



$$D(\mathbf{a}, \mu) = D(\alpha, \mu_\alpha) . \quad (2A.28)$$

This equality implies that the SMSS  $|\mu_\alpha\rangle_{(r, \varphi)}$ , defined by Eq. (1.16) as the squeeze operator  $S_1(r, \varphi)$  acting on the coherent state  $|\mu_\alpha\rangle_{\text{coh}}$ , can as well be defined as the displacement operator  $D(\mathbf{a}, \mu)$  acting on the squeezed vacuum:

$$|\mu_\alpha\rangle_{(r, \varphi)} \equiv S_1(r, \varphi) |\mu_\alpha\rangle_{\text{coh}} = D(\mathbf{a}, \mu) S_1(r, \varphi) |0\rangle . \quad (2A.29)$$

The complex number  $\mu$  is equal to  $\langle \mathbf{a} \rangle$ , the state's complex amplitude. It is related to the eigenvalue  $\mu_\alpha$  by

$$\mu = \mu_\alpha \cosh r - \mu_\alpha^* e^{2i\varphi} \sinh r \quad (2A.30)$$

[ Eq. (2A.23b)]. With this definition one can easily verify the statement made in the Introduction: any state unitarily related to the SMSS  $|\mu_\alpha\rangle_{(r, \varphi)}$  by a product of rotation and displacement operators is equal to another SMSS (multiplied by an unobservable overall phase factor) with the same squeeze factor  $r$ , but with different squeeze angle and eigenvalue. For example,

$$\begin{aligned} R(\theta) D(\mathbf{a}, \mu') |\mu_\alpha\rangle_{(r, \varphi)} &= e^{i\text{Im}(\mu\mu'^*)} D(\mathbf{a}, \bar{\mu}) S_1(r, \varphi - \theta) |0\rangle \\ &= e^{i\text{Im}(\mu\mu'^*)} |\bar{\mu}_\alpha\rangle_{(r, \varphi - \theta)} , \end{aligned}$$

$$\bar{\mu} \equiv e^{-i\theta} (\mu + \mu') \quad (2A.31)$$

[Eqs. (2A.17), (2A.19), (2A.25)].

Finally, the product of two different single-mode squeeze operators is another single-mode squeeze operator, multiplied by a phase factor and a rotation operator [see Eqs. (2C.18) below]. For the case  $\varphi = \varphi'$  the relation simplifies to

$$S_1(\mathbf{r}', \varphi) S_1(\mathbf{r}, \varphi) = S_1(\mathbf{r} + \mathbf{r}', \varphi). \quad (2A.32)$$

It is proved in the next section, by considering the most general single-mode Gaussian wave function, that the generator  $H_G^{(1)}$  of the unitary operator  $U_G^{(1)} \equiv e^{-iH_G^{(1)}}$  that relates any single-mode GPS to the vacuum state is a sum of linear and bilinear combinations of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$ . In Section IIC and Appendix A it is shown that this unitary operator factors into a product of single-mode displacement, squeeze, and rotation operators. The properties described in this section ensure that any product of single-mode rotation, displacement, and squeeze operators can be expressed as the product of a displacement operator and a squeeze operator, multiplied on the right by a rotation operator (and an overall phase factor). Since the rotation operator has no effect on the vacuum state, one finds that the most general single-mode GPS is equal to a single-mode squeezed state, defined by Eq. (2A.29). It is produced when a harmonic oscillator in its ground state is exposed to the interaction Hamiltonians  $H_R^{(1)}(t)$ ,  $H_1^{(1)}(t)$ , and  $H_2^{(1)}(t)$  described in the Introduction [Eqs. (1.3) and (1.4)].

## 5. Single-mode GPS

Much of the interest in single-mode GPS has centered around the so-called "minimum-uncertainty states"<sup>12</sup> (MUS) -- states that minimize the product of the uncertainties in  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{p}}$ :

$$\langle (\Delta \hat{\mathbf{x}})^2 \rangle \langle (\Delta \hat{\mathbf{p}})^2 \rangle = \frac{1}{4} \quad (\text{MUS}) \quad (2A.33)$$

[Eq. (2A.6a)]. These are (single-mode) GPS that satisfy

$$\text{Im} \langle (\Delta \mathbf{a})^2 \rangle = \langle \Delta \hat{\mathbf{x}} \Delta \hat{\mathbf{p}} \rangle_{\text{sym}} = 0 \quad (2A.34)$$

[Eqs. (2A.5)]. It is shown below [see Eqs. (2B.5) or (2B.6)] that a single-mode

state is a MUS if and only if it is an eigenstate of the linear combination  $\hat{x} + i \gamma_1^{-1} \hat{p}$ ,  $\gamma_1 = \gamma_1^* > 0$ . Another set of single-mode states consists of those that have "random-phase noise", i.e., whose noise moments are invariant under rotations. Such states satisfy the condition

$$\langle [\Delta \hat{x}(\theta)]^2 \rangle = \langle [\Delta \hat{p}(\theta)]^2 \rangle \quad \text{for all } \theta \quad (2A.35a)$$

or, equivalently,

$$\langle (\Delta \alpha)^2 \rangle = 0. \quad (2A.35b)$$

The intersection between these two sets of states is the set of single-mode coherent states; i.e., coherent states are MUS that have random-phase noise. Coherent states therefore satisfy

$$\langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}} = \langle (\Delta \alpha)^2 \rangle = 0, \quad (2A.36a)$$

$$\langle (\Delta \hat{x})^2 \rangle = \langle (\Delta \hat{p})^2 \rangle = \langle |\Delta \alpha|^2 \rangle = \frac{1}{2}. \quad (2A.36b)$$

The last equality in Eq. (2A.36b) tells one that a coherent state is an eigenstate of the annihilation operator  $\alpha$ ; it has the minimum total noise allowed by quantum mechanics [Eqs. (2A.6)].

By extending the definition (2A.33) of a MUS to include all states related to MUS by the rotation operator  $R(\theta)$  one obtains all single-mode GPS. This is because the condition (2A.34) can always be met for some rotated annihilation operator  $e^{i\theta} \alpha \equiv R(\theta) \alpha R^\dagger(\theta)$  [Eq. (2A.10a)], with  $\theta$  chosen to make  $\text{Im} \langle e^{2i\theta} (\Delta \alpha)^2 \rangle = 0$ . The second-order noise moments for the most general normalized single-mode GPS, a single-mode squeezed state, follow directly from the transformation (2A.23) and those for a coherent state. Equivalently, they can be found by noting that the noise moments of the squeezed annihilation and creation operators  $\alpha(\tau, \varphi)$  and  $\alpha^\dagger(\tau, \varphi)$  for the

SMSS  $|\mu_\alpha\rangle_{(r,\varphi)}$  are identical to those of  $\alpha$  and  $\alpha^\dagger$  for a coherent state, since  $|\mu_\alpha\rangle_{(r,\varphi)}$  is an eigenstate of  $\alpha(r,\varphi)$ . They are

$$\langle (\Delta\alpha)^2 \rangle = -\frac{1}{2}e^{2i\varphi} \sinh 2r, \quad (2A.37a)$$

$$\langle |\Delta\alpha|^2 \rangle = \frac{1}{2} \cosh 2r. \quad (2A.37b)$$

Thus, the SMSS  $|\mu_\alpha\rangle_{(r,\varphi)}$  is a MUS for the conjugate variables  $\hat{x}(-\varphi)$  and  $\hat{p}(-\varphi)$ :

$$\langle [\Delta\hat{x}(-\varphi)]^2 \rangle = \frac{1}{2}e^{-2r}, \quad \langle [\Delta\hat{p}(-\varphi)]^2 \rangle = \frac{1}{2}e^{2r}, \quad (2A.38a)$$

$$\langle \Delta\hat{x}(-\varphi) \Delta\hat{p}(-\varphi) \rangle_{\text{sym}} = 0. \quad (2A.38b)$$

## B. Single-mode Gaussian wave functions

Consider now the coordinate-space wave function for an arbitrary single-mode Gaussian pure state, symbolized here for generality by the state vector  $|\mu_g\rangle$ . The GPS  $|\mu_g\rangle$  is an eigenstate of operators  $g$  whose general form is discussed below, with eigenvalue  $\mu_g$ . [Although  $g$  will be seen to have the form of single-mode squeezed annihilation operators  $\alpha(r, \varphi)$ , the symbol  $g$  is used here to refer to all operators of which single-mode GPS are eigenstates -- i.e., all multiples of single-mode squeeze annihilation operators, for all  $r$  and  $\varphi$ .] The wave function is written in terms of the dimensionless position variable  $x$ , the eigenvalue of the Hermitian operator  $\hat{x}$ . The most general (normalized) single-mode Gaussian coordinate-space wave function has the form

$$\langle x | \mu_g \rangle = N_g e^{\frac{1}{2}i\delta_x} e^{-\frac{1}{2}ip_0 x_0} e^{ip_0 x} e^{-\frac{1}{2}\gamma(x-x_0)^2}. \quad (2B.1)$$

Here

$$x_0 \equiv \langle \hat{x} \rangle = \int_{-\infty}^{\infty} dx x |\langle x | \mu_g \rangle|^2, \quad (2B.2a)$$

$$p_0 \equiv \langle \hat{p} \rangle = -i \int_{-\infty}^{\infty} dx \langle \mu_g | x \rangle \partial_x \langle x | \mu_g \rangle, \quad (2B.2b)$$

are the mean values of the position and momentum,  $\gamma$  is a complex number related to the second-order noise moments of  $\hat{x}$  and  $\hat{p}$ ,  $\delta_x$  is an unobservable phase angle (separated out for reasons discussed below), and  $N_g$  is a (real) normalization constant determined by the condition  $\langle \mu_g | \mu_g \rangle = 1$ . The subscript "x" on the phase angle  $\delta_x$  serves only to distinguish  $\delta_x$  from the phase angle  $\delta_p$  which appears in the momentum-space wave function [Eqs. (2B.40)-(2B.43) below];  $\delta_x$  has no dependence on  $x$ . Normalizability dictates that

$$\text{Re}(\gamma) \equiv \gamma_1 > 0, \quad (2B.3)$$

and the normalization constant  $N_g$  is equal to

$$N_g = (\pi / \gamma_1)^{-1/2}. \quad (2B.4)$$

The most important parameter in the wave function (2B.1) is the complex number  $\gamma$ . The form of the wave function tells one that the state  $|\mu_g\rangle$  is an eigenstate of the linear combination  $\hat{x} + i\gamma^{-1}\hat{p}$ , and hence that  $\gamma$  is related to the second-order noise moments of  $\hat{x}$  and  $\hat{p}$  by

$$\gamma \equiv \gamma_1 + i\gamma_2 = \frac{-i(\langle \Delta\hat{x} \Delta\hat{p} \rangle_{\text{sym}} + \frac{1}{2}i)}{\langle (\Delta\hat{x})^2 \rangle} = \frac{-i \langle (\Delta\hat{p})^2 \rangle}{\langle \Delta\hat{x} \Delta\hat{p} \rangle_{\text{sym}} - \frac{1}{2}i}. \quad (2B.5a)$$

The real and imaginary parts of  $\gamma$  are therefore equal to

$$\gamma_1 \equiv \text{Re}(\gamma) = \frac{1}{2\langle (\Delta\hat{x})^2 \rangle}, \quad \gamma_2 \equiv \text{Im}(\gamma) = \frac{-\langle \Delta\hat{x} \Delta\hat{p} \rangle_{\text{sym}}}{\langle (\Delta\hat{x})^2 \rangle}, \quad (2B.5b)$$

and the absolute square is

$$|\gamma|^2 = \frac{\langle (\Delta\hat{p})^2 \rangle}{\langle (\Delta\hat{x})^2 \rangle}. \quad (2B.5c)$$

Inverting these expressions gives the second-order noise moments of  $\hat{x}$  and  $\hat{p}$  in terms of  $\gamma$ :

$$\begin{aligned} \langle (\Delta\hat{x})^2 \rangle &= (2\gamma_1)^{-1}, & \langle (\Delta\hat{p})^2 \rangle &= [2\text{Re}(\gamma^{-1})]^{-1} = |\gamma|^2 (2\gamma_1)^{-1}, \\ \langle \Delta\hat{x} \Delta\hat{p} \rangle_{\text{sym}} &= -(2\gamma_1)^{-1} \gamma_2. \end{aligned} \quad (2B.6)$$

The normalization constant  $N_g$  can thus be rewritten as

$$N_g = (\pi / \gamma_1)^{-1/2} = (2\pi \langle (\Delta\hat{x})^2 \rangle)^{-1/2}. \quad (2B.7)$$

That the state  $|\mu_g\rangle$  is an eigenstate of  $\hat{x} + i\gamma^{-1}\hat{p}$  means that it is also an eigenstate of the operator  $\mathbf{a} + (\gamma + 1)^{-1}(\gamma - 1)\mathbf{a}^\dagger$ . The second-order noise moments of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  are therefore more naturally expressed in terms of the complex number

$$\Gamma \equiv \frac{\gamma - 1}{\gamma + 1} = \frac{-\langle(\Delta\mathbf{a})^2\rangle}{\langle|\Delta\mathbf{a}|^2\rangle + \frac{1}{2}} = \frac{\langle|\Delta\mathbf{a}|^2\rangle - \frac{1}{2}}{-\langle(\Delta\mathbf{a}^\dagger)^2\rangle}. \quad (2B.8)$$

Inverting these expressions gives the variance and symmetric variance of  $\mathbf{a}$  in terms of both  $\Gamma$  and  $\gamma$ :

$$\langle(\Delta\mathbf{a})^2\rangle = -(1 - |\Gamma|^2)^{-1}\Gamma = -(\gamma^* + 1)(4\gamma_1)^{-1}(\gamma - 1), \quad (2B.9a)$$

$$\langle|\Delta\mathbf{a}|^2\rangle = \frac{1}{2}(1 + |\Gamma|^2)(1 - |\Gamma|^2)^{-1} = (1 + |\gamma|^2)(4\gamma_1)^{-1}. \quad (2B.9b)$$

Note also that

$$1 - |\Gamma|^2 = 4\gamma_1 |\gamma + 1|^{-2}; \quad (2B.10)$$

hence normalizability dictates that  $|\Gamma| < 1$ .

From the above relations one can see how the three real pieces of information in the second-order noise moments for a single-mode GPS reduce to two independent pieces, since

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle = \frac{1}{4} + \langle\Delta\hat{x}\Delta\hat{p}\rangle_{\text{sym}}^2 = \langle\Delta\hat{x}\Delta\hat{p}\rangle\langle\Delta\hat{p}\Delta\hat{x}\rangle, \quad (2B.11a)$$

$$\langle|\Delta\mathbf{a}|^2\rangle^2 = \frac{1}{4} + |\langle(\Delta\mathbf{a})^2\rangle|^2 \quad (2B.11b)$$

[cf. Eqs. (2A.6)]. These relations are made more obvious below [Eqs. (2B.31), (2B.32)].

The remaining parameter in the wave function (2B.1) is the phase angle  $\delta_{\mathbf{x}}$ ; in general it can be any real number. The phase angle  $\delta_{\mathbf{x}}$  is unobservable, but for a state defined as a particular unitary operator acting on the

vacuum state it has a well-defined value, provided one assigns a phase angle to the vacuum-state wave function. The reason that the phase factor  $e^{\frac{\hbar i}{2} \delta_x}$  separates naturally from the rest of the overall phase factor in the wave function lies with the definition (2A.14) of the single-mode displacement operator. That definition, together with the correspondence  $\hat{p} \rightarrow -i\partial_x$  [Eq. (2A.2)], implies that if one "displaces" any single-mode pure state  $|\Psi\rangle$ , by operating on it with the single-mode displacement operator  $D(\alpha, \mu)$ , the resulting wave function is related to the original wave function  $\langle x | \Psi \rangle$  in the following way:

$$\langle x | D(\alpha, \mu) | \Psi \rangle = e^{-\frac{\hbar i}{2} p_0 x_0} e^{i p_0 x} \langle x - x_0 | \Psi \rangle . \quad (2B.12)$$

Formally, therefore, one way to obtain an arbitrary single-mode pure state  $|\Psi_\mu\rangle$  with complex amplitude  $\mu$  is to operate with the displacement operator  $D(\alpha, \mu)$  on a state  $|\Psi_0\rangle \equiv U_0 |0\rangle$  that has the desired noise properties but has zero complex amplitude ( $\langle 0 | U_0^\dagger \alpha U_0 | 0 \rangle \equiv 0$ ):

$$|\Psi_\mu\rangle \equiv D(\alpha, \mu) U_0 |0\rangle . \quad (2B.13)$$

The property (2A.16) of the displacement operator then ensures that  $|\Psi_\mu\rangle$  has complex amplitude  $\mu$ ,

$$\langle \Psi_\mu | \alpha | \Psi_\mu \rangle = \mu . \quad (2B.14)$$

Any normalized single-mode pure state with complex amplitude  $\mu$  can be defined by an expression like (2B.13). The advantage of this definition is that the state's mean values  $x_0$  and  $p_0$  (or the complex amplitude  $\mu$ ) are determined solely by the displacement operator  $D(\alpha, \mu)$ , and its noise moments of  $\alpha$  and  $\alpha^\dagger$  are determined solely by the unitary operator  $U_0$ . Any normalized single-mode GPS  $|\mu_g\rangle$  with complex amplitude  $\mu$  can therefore



be formally defined by

$$|\mu_g\rangle \equiv D(\alpha, \mu) U_g |0\rangle. \quad (2B.15)$$

Note the following three properties of  $U_g$ : First, it is uniquely defined only up to (right-hand) multiplication by a rotation operator  $R(\theta)$  and an overall phase factor. Second, since it defines the noise moments of  $\alpha$  and  $\alpha^\dagger$  (or  $\hat{x}$  and  $\hat{p}$ ) for the GPS  $|\mu_g\rangle$ , it has associated with it two independent real parameters (over and above that of a rotation operator and phase factor). Third, since the state  $|\mu_g\rangle$  has complex amplitude  $\mu$ , the expectation value  $\langle 0 | U_g^\dagger \alpha U_g | 0 \rangle$  must vanish.

The phase factor  $e^{i\delta_x}$  in the wave function  $\langle x | \mu_g \rangle$  is given, from Eqs. (2B.1) and (2B.12), by

$$e^{i\delta_x} = \frac{\langle x=0 | U_g | 0 \rangle}{|\langle x=0 | U_g | 0 \rangle|}. \quad (2B.16)$$

The phase angle  $\delta_x$  has no dependence on the complex amplitude  $\mu$ , provided  $U_g$  does not; any dependence of  $U_g$  on  $\mu$  is artificial, however, in the sense that it does not affect the state's complex amplitude  $\langle \alpha \rangle$ . Consider, for illustration, the coherent state  $|\mu\rangle_{\text{coh}} \equiv D(\alpha, \mu) |0\rangle$  [Eq. (1.14)], for which the operator  $U_g$  is the identity operator. Equation (2B.12) says that the wave function for the coherent state  $|\mu\rangle_{\text{coh}}$  is related to the vacuum-state wave function  $\langle x | 0 \rangle$  by

$$\langle x | \mu \rangle_{\text{coh}} = e^{-i p_0 x_0} e^{i x p_0} \langle x - x_0 | 0 \rangle, \quad (2B.17)$$

so the phase angle  $\delta_x$  for a coherent-state wave function is just equal to the phase angle  $\delta_0$  assigned to the vacuum-state coordinate-space wave function; conventionally,  $\delta_0$  is set equal to zero.

The form of its wave function shows that a single-mode GPS  $|\mu_g\rangle$  is an eigenstate of operators  $g$  which are proportional to the linear combinations  $\hat{x} + i\gamma^{-1}\hat{p}$  or  $\mathbf{a} + \Gamma\mathbf{a}^\dagger$ . The label  $\mu_g$  for the GPS  $|\mu_g\rangle$  is chosen to be the eigenvalue of  $g$ . Thus one can write the following relations:

$$g |\mu_g\rangle = \mu_g |\mu_g\rangle, \quad (2B.18a)$$

$$g \propto \hat{x} + i\gamma^{-1}\hat{p} \propto \mathbf{a} + \Gamma\mathbf{a}^\dagger, \quad (2B.18b)$$

$$\mu_g \propto x_0 + i\gamma^{-1}p_0 \propto \mu + \Gamma\mu^*. \quad (2B.18c)$$

It is instructive to consider the general form for the operators  $g$  of which the single-mode GPS  $|\mu_g\rangle$  is an eigenstate:

$$\begin{aligned} g &= \rho_c \mathbf{a} + \rho_s \mathbf{a}^\dagger \equiv \rho_c (\mathbf{a} + \Gamma\mathbf{a}^\dagger) \\ &= \rho_p \hat{x} + i\rho_z \hat{p} \equiv \rho_p (\hat{x} + i\gamma^{-1}\hat{p}). \end{aligned} \quad (2B.19a)$$

Here  $\rho_p$ ,  $\rho_z$ ,  $\rho_c$ , and  $\rho_s$  are complex numbers, related to each other by

$$\rho_c = 2^{-\frac{1}{2}}(\rho_p \pm \rho_z), \quad \rho_z = 2^{-\frac{1}{2}}(\rho_c \pm \rho_s). \quad (2B.20)$$

The eigenvalue  $\mu_g$  is related to the complex amplitude  $\mu$  and the mean position and momentum by similar relations,

$$\mu_g = \rho_c \mu + \rho_s \mu^* = \rho_p x_0 + i\rho_z p_0. \quad (2B.19b)$$

Inverting Eqs. (2B.19a,b) leads to the following expressions for  $\mathbf{a}$  and the complex amplitude  $\mu$  in terms of  $g$  and the eigenvalue  $\mu_g$ :

$$\mathbf{a} = [g, g^\dagger]^{-1}(\rho_c^* g - \rho_s g^\dagger), \quad (2B.21a)$$

$$\mu \equiv \langle \mathbf{a} \rangle = [g, g^\dagger]^{-1}(\rho_c^* \mu_g - \rho_s \mu_g^*). \quad (2B.21b)$$

These relations imply the important equality

$$D(\alpha, \mu) = D(g, [g, g^\dagger]^{-1} \mu_g). \quad (2B.22)$$

The equality (2B.22) enables one to see explicitly how the form of the unitary operator  $U_g$ , which defines a single-mode GPS  $|\mu_g\rangle$  through Eq. (2B.15), is determined by the form of the operators  $g$ . To see this, begin with an alternative definition for the GPS  $|\mu_g\rangle$  [the equality (2B.22) will be seen to ensure that this definition is equivalent to the definition (2B.15)]. First, assume that  $|\mu_g\rangle$  is related to the vacuum state by some unitary operator  $\bar{U}$ :

$$|\mu_g\rangle = \bar{U} |0\rangle. \quad (2B.23a)$$

It is convenient to define another unitary operator  $U_g$  by

$$\bar{U} \equiv U_g D(\alpha, \mu_g), \quad (2B.23b)$$

so that the state  $|\mu_g\rangle$  is equal to the operator  $U_g$  acting on the coherent state  $|\mu_g\rangle_{\text{coh}}$ ,

$$|\mu_g\rangle = U_g D(\alpha, \mu_g) |0\rangle = U_g |\mu_g\rangle_{\text{coh}}. \quad (2B.24)$$

It is then consistent with the eigenvalue equation (2B.18a) that the operators  $g$  be unitarily related to the annihilation operator  $\alpha$  through the operator  $U_g$ :

$$g = U_g \alpha U_g^\dagger. \quad (2B.25)$$

The form of  $U_g$  is thus determined by the form of the operators  $g$ . The equivalence of the definitions (2B.15) and (2B.24) for  $|\mu_g\rangle$  is a result of the unitarity of  $U_g$ , which ensures that  $[g, g^\dagger] = [\alpha, \alpha^\dagger] = 1$ , and the forms

(2B.19) of  $g$ ; these imply the equality

$$D(\mathbf{a}, \mu) = D(g, \mu_g) \quad (2B.26)$$

[Eq. (2B.22)]. Thus, any single-mode GPS  $|\mu_g\rangle$  has the following two equivalent definitions:

$$|\mu_g\rangle = D(\mathbf{a}, \mu) U_g |0\rangle = U_g D(\mathbf{a}, \mu_g) |0\rangle \equiv U_g |\mu_g\rangle_{\text{coh}}. \quad (2B.27)$$

Return now to the general forms (2B.19) for the operators  $g$  of which single-mode GPS are eigenstates. Two of the four degrees of freedom in the expressions (2B.19a) for  $g$  are removed by the wave function  $\langle \mathbf{x} | \mu_g \rangle$ , which specifies the ratios

$$\rho_p / \rho_x \equiv \gamma, \quad \rho_s / \rho_c \equiv \Gamma. \quad (2B.28)$$

The third degree of freedom in  $g$  has already been partially removed by the requirement that  $g$  have a complete (or overcomplete) set of normalizable eigenstates, i.e., that the commutator  $[g, g^\dagger]$  be a positive real number (see Appendix C). It is removed completely if one specifies that  $g$  be unitarily related to the annihilation operator  $\mathbf{a}$  [Eq. (2B.25)], which implies that

$$[g, g^\dagger] = [\mathbf{a}, \mathbf{a}^\dagger] = 1. \quad (2B.29)$$

The commutator  $[g, g^\dagger]$  can be written in the following different ways, using Eqs. (2B.5) and (2B.8):

$$\begin{aligned} [g, g^\dagger] &= 2\text{Re}(\rho_x^* \rho_p) = 2|\rho_x|^2 \gamma_1 = |\rho_x|^2 / \langle (\Delta \hat{x})^2 \rangle \\ &= 2|\rho_p|^2 \text{Re}(\gamma^{-1}) = |\rho_p|^2 / \langle (\Delta \hat{p})^2 \rangle \\ &= |\rho_c|^2 - |\rho_s|^2 = |\rho_c|^2 (1 - |\Gamma|^2) = |\rho_c|^2 (\langle |\Delta \mathbf{a}|^2 \rangle + \frac{1}{2})^{-1} \end{aligned}$$

$$\begin{aligned}
 \langle \Delta \hat{x} \rangle &= |\rho_s|^2 (|\Gamma|^{-2} - 1) = |\rho_s|^2 (\langle |\Delta \mathbf{a}|^2 \rangle - \frac{1}{2})^{-1} . \\
 \langle \Delta \hat{p} \rangle &= \rho_c^* \rho_s \Gamma^{-1} (1 - |\Gamma|^2) = -\rho_c^* \rho_s \langle (\Delta \mathbf{a})^2 \rangle^{-1} .
 \end{aligned} \tag{2B.30}$$

These expressions show that the operators  $g$  have normalizable eigenstates if and only if the wave function  $\langle \mathbf{x} | \mu_g \rangle$  is normalizable -- i.e., if and only if  $\gamma_1 > 0$ , or  $|\Gamma| < 1$  [Eqs. (2B.3) and (2B.10)]. Note also that normalizability requires that the numbers  $\rho_p$ ,  $\rho_x$ , and  $\rho_c$  be nonzero. The requirement that  $g$  be unitarily related to  $\mathbf{a}$  -- i.e., that  $[g, g^\dagger] = 1$  -- implies that

$$\text{Re}(\rho_x^* \rho_p) = \frac{1}{2} , \quad |\rho_c|^2 - |\rho_s|^2 = 1 . \tag{2B.31}$$

The only remaining degree of freedom in  $g$  is its overall phase. Multiplying  $g$  by a phase factor  $e^{i\theta}$  is equivalent to multiplying  $U_g$  (on the right) by a rotation operator  $R(\theta)$ . The definition (2B.27) of  $|\mu_g\rangle$  shows that this freedom reflects the fact that a coherent state remains a coherent state when multiplied by a rotation operator [Eq. (2A.20)].

The expressions (2B.30) for the commutator  $[g, g^\dagger]$  reveal the following simple relations between the second-order noise moments of  $\mathbf{a}$ ,  $\mathbf{a}^\dagger$ ,  $\hat{\mathbf{x}}$ , and  $\hat{\mathbf{p}}$  and the numbers  $\rho_p$ ,  $\rho_x$ ,  $\rho_c$ , and  $\rho_s$  for operators  $g$  that are unitarily related to  $\mathbf{a}$ :

$$\langle (\Delta \hat{\mathbf{x}})^2 \rangle = |\rho_x|^2 , \quad \langle (\Delta \hat{\mathbf{p}})^2 \rangle = |\rho_p|^2 ,$$

$$\langle \Delta \hat{\mathbf{x}} \Delta \hat{\mathbf{p}} \rangle_{\text{sym}} = -\text{Im}(\rho_x^* \rho_p) ; \tag{2B.32a}$$

$$\langle (\Delta \mathbf{a})^2 \rangle = -\rho_c^* \rho_s , \tag{2B.32b}$$

$$\langle |\Delta \mathbf{a}|^2 \rangle = \frac{1}{2} (|\rho_c|^2 + |\rho_s|^2) = |\rho_c|^2 - \frac{1}{2} = |\rho_s|^2 + \frac{1}{2} \tag{2B.32c}$$

[cf. Eqs. (2B.6) and (2B.9)]. These expressions make obvious the relations (2B.11) between the different second-order noise moments.

The form of the unitary operator  $U_g$  in the definition (2B.27) of the single-mode GPS  $|\mu_g\rangle$  is determined by the unitary transformation (2B.25) and the form of  $g$  [Eqs. (2B.19a), (2B.31)]. The linearity and absence of any additive constants in the transformation imply that  $U_g = e^{-iH_g^{(1)}}$ , where  $H_g^{(1)}$  is a (Hermitian) linear combination of the three operators  $a^\dagger a$ ,  $a^2$ , and  $a^{\dagger 2}$ . That is, the generator  $H_g^{(1)}$  of  $U_g$  has the general form  $[H_0^{(1)} + H_2^{(1)}]$  defined in the Introduction [Eqs. (1.1)-(1.4)]. It is shown in Section IIC and Appendix A that this operator  $U_g$  can always be written as a product of a single-mode squeeze operator and a single-mode rotation operator (and an unobservable overall phase factor). That the rotation operator can be neglected in the general form for  $U_g$  can be seen in a couple of ways. First, the rotation operator can be placed either to the right or left of the squeeze operator, without changing the general form of  $U_g$  [Eq. (2A.25)]; when placed to the right of the squeeze operator the rotation operator acts like the identity operator on the vacuum state and hence is inconsequential. Second, note that the parameter  $\theta$  in the rotation operator is related to the overall phase of the operator  $g$ , which can be chosen arbitrarily; it is zero if  $\rho_c \equiv \rho_c^*$ . Hence the operator  $U_g$  is equal to a single-mode squeeze operator, and the state defined by Eq. (2B.27) with  $U_g \equiv S_1(\tau, \varphi)$  is the SMSS

$$|\mu_\alpha\rangle_{(\tau, \varphi)} \equiv D(\alpha, \mu) S_1(\tau, \varphi) |0\rangle = S_1(\tau, \varphi) |\mu_\alpha\rangle_{\text{coh}} \quad (2B.33)$$

[Eq. (1.16)]. Thus, any (and all) single-mode GPS can be described as a SMSS, for some values of  $\tau$  and  $\varphi$  and some complex amplitude  $\mu$ .

The SMSS  $|\mu_\alpha\rangle_{(\tau, \varphi)}$  is an eigenstate of the squeezed annihilation operator

$$\alpha(\tau, \varphi) \equiv S_1(\tau, \varphi) a S_1^\dagger(\tau, \varphi) = a \cosh r + a^\dagger e^{2i\varphi} \sinh r \quad (2B.34a)$$

$$= 2^{-\frac{1}{2}}(\cosh r + e^{2i\varphi} \sinh r) \hat{x} + i 2^{-\frac{1}{2}}(\cosh r - e^{2i\varphi} \sinh r) \hat{p} \quad (2B.34b)$$

[Eq. (2A.24); cf. Eq. (2B.19a)]. The complex numbers  $\rho_c$ ,  $\rho_s$ ,  $\Gamma$  and  $\rho_p$ ,  $\rho_x$ ,  $\gamma$ , which define the noise moments of  $\alpha$ ,  $\alpha^\dagger$ ,  $\hat{x}$ , and  $\hat{p}$ , are therefore related to  $r$  and  $\varphi$  by

$$\rho_c \equiv \cosh r, \quad \rho_s \equiv e^{2i\varphi} \sinh r; \quad \rho_p \equiv 2^{-\frac{1}{2}}(\cosh r \pm e^{2i\varphi} \sinh r); \quad (2B.35a)$$

$$\Gamma \equiv \rho_s / \rho_c = e^{2i\varphi} \tanh r, \quad \gamma \equiv \rho_p / \rho_x = \frac{(\cosh r + e^{2i\varphi} \sinh r)}{(\cosh r - e^{2i\varphi} \sinh r)}. \quad (2B.35b)$$

The complex amplitude  $\mu$  and the eigenvalue  $\mu_\alpha$  are related to each other by

$$\mu \equiv \langle \alpha \rangle = \mu_\alpha \cosh r - \mu_\alpha^* e^{2i\varphi} \sinh r, \quad (2B.36a)$$

$$\mu_\alpha = \mu \cosh r + \mu^* e^{2i\varphi} \sinh r \quad (2B.36b)$$

[Eqs. (2B.19b) and (2B.21b)]. The second-order noise moments of a SMSS in terms of  $r$  and  $\varphi$  can be obtained by inserting the expressions (2B.35) into the relations (2B.32):

$$\langle (\Delta \hat{x})^2 \rangle = |\rho_x|^2 = \frac{1}{2}(\cosh 2r - \sinh 2r \cos 2\varphi), \quad (2B.37a)$$

$$\langle (\Delta \hat{p})^2 \rangle = |\rho_p|^2 = \frac{1}{2}(\cosh 2r + \sinh 2r \cos 2\varphi), \quad (2B.37b)$$

$$\langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}} = -\text{Im}(\rho_x^* \rho_p) = -\frac{1}{2} \sinh 2r \sin 2\varphi; \quad (2B.37c)$$

$$\langle (\Delta \alpha)^2 \rangle = -\rho_c^* \rho_s = -\frac{1}{2} e^{2i\varphi} \sinh 2r, \quad (2B.38a)$$

$$\langle |\Delta \alpha|^2 \rangle = \frac{1}{2}(|\rho_c|^2 + |\rho_s|^2) = \frac{1}{2} \cosh 2r. \quad (2B.38b)$$

The phase angle  $\delta_x$  in the coordinate-space wave function for the SMSS  $|\mu_\alpha\rangle_{(r,\varphi)}$  is obtained from Eq. (2B.16). The calculation is described in Appendix B. The result is

$$e^{\frac{1}{2}i\delta_x} = \frac{(\cosh r - e^{-2i\varphi} \sinh r)^{\frac{1}{2}}}{|\cosh r - e^{-2i\varphi} \sinh r|^{\frac{1}{2}}} \equiv \frac{(\rho_x^*)^{\frac{1}{2}}}{|\rho_x|^{\frac{1}{2}}} \quad (2B.39)$$

To conclude this discussion of single-mode Gaussian wave functions, consider briefly the momentum-space wave function for a single-mode Gaussian pure state,  $\langle p | \mu_g \rangle$ , obtained by Fourier transforming  $\langle x | \mu_g \rangle$  [Eq. (2B.1)]; here the dimensionless momentum variable  $p$  is the eigenvalue of the Hermitian operator  $\hat{p}$ . The momentum-space wave function has the following form:

$$\langle p | \mu_g \rangle \equiv \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{\frac{1}{2}}} e^{-ipx} \langle x | \mu_g \rangle \equiv M_g e^{-\frac{1}{2}i\delta_p} e^{\frac{1}{2}ip_0 x_0} e^{-ix_0 p} e^{-(p-p_0)^2/2\gamma} \quad (2B.40a)$$

where the (real) normalization constant  $M_g$  is

$$M_g \equiv (2\pi \langle (\Delta \hat{p})^2 \rangle)^{-\frac{1}{4}}, \quad (2B.40b)$$

and the phase angle  $\delta_p$  is related to the coordinate-space phase angle  $\delta_x$  by

$$e^{i\delta_p} = e^{-i\delta_x} \frac{\gamma}{|\gamma|} = e^{-i\delta_x} \frac{-i(\langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}} + \frac{1}{2}i)}{|\langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}} + \frac{1}{2}i|}. \quad (2B.41)$$

For the single-mode squeezed state  $|\mu_\alpha\rangle_{(r,\varphi)}$  the phase factor  $e^{-\frac{1}{2}i\delta_p}$  is

$$e^{-\frac{1}{2}i\delta_p} = \frac{\cosh r + e^{-2i\varphi} \sinh r)^{\frac{1}{2}}}{|\cosh r + e^{-2i\varphi} \sinh r|^{\frac{1}{2}}} = \frac{(\rho_p^*)^{\frac{1}{2}}}{|\rho_p|^{\frac{1}{2}}}. \quad (2B.42)$$

The position and momentum probabilities have the usual Gaussian forms:

$$|\langle x | \mu_g \rangle|^2 = (2\pi \langle (\Delta \hat{x})^2 \rangle)^{-\frac{1}{2}} e^{-(x-x_0)^2/2\langle (\Delta \hat{x})^2 \rangle}, \quad (2B.43a)$$

$$|\langle p | \mu_g \rangle|^2 = (2\pi \langle (\Delta \hat{p})^2 \rangle)^{-\frac{1}{2}} e^{-(p-p_0)^2/2\langle (\Delta \hat{p})^2 \rangle}. \quad (2B.43b)$$



### C. Two-component vector notation for single-mode GPS

The previous discussion has shown that the unitary operators that relate single-mode GPS to the vacuum state and to other single-mode GPS are rotation operators, displacement operators, and single-mode squeeze operators. Since these operators induce linear transformations on  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  (or  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{p}}$ ), it is useful to define the two-component operator column vectors<sup>29,18,33,26</sup>

$$\tilde{\mathbf{a}} \equiv \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}, \quad \tilde{\hat{\mathbf{x}}} \equiv \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}} \end{pmatrix} = A \tilde{\mathbf{a}}, \quad (2C.1a)$$

$$A \equiv 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = (A^\dagger)^{-1}. \quad (2C.1b)$$

The expectation values of these operator column vectors are column vectors whose components are complex numbers (for  $\tilde{\mathbf{a}}$ ), or real numbers (for  $\tilde{\hat{\mathbf{x}}}$ ):

$$\tilde{\boldsymbol{\mu}} \equiv \langle \tilde{\mathbf{a}} \rangle = \begin{pmatrix} \mu \\ \mu^* \end{pmatrix}, \quad \tilde{\boldsymbol{\xi}} \equiv \langle \tilde{\hat{\mathbf{x}}} \rangle = \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} = A \tilde{\boldsymbol{\mu}}. \quad (2C.2)$$

The adjoints of the operator column vectors are the row vectors

$$\tilde{\mathbf{a}}^\dagger \equiv (\mathbf{a}^\dagger \ \mathbf{a}), \quad \tilde{\hat{\mathbf{x}}}^\dagger \equiv (\hat{\mathbf{x}} \ \hat{\mathbf{p}}) = \tilde{\hat{\mathbf{x}}}^T, \quad (2C.3)$$

where a superscript "T" means transpose. The transpose of the adjoint of an operator column vector is denoted by a superscript "\*":

$$(\tilde{\mathbf{a}}^\dagger)^T \equiv \tilde{\mathbf{a}}^* = \begin{pmatrix} \mathbf{a}^\dagger \\ \mathbf{a} \end{pmatrix}, \quad (\tilde{\hat{\mathbf{x}}}^\dagger)^T \equiv \tilde{\hat{\mathbf{x}}}^* = \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}} \end{pmatrix} = \tilde{\hat{\mathbf{x}}}. \quad (2C.4)$$

Similar definitions hold for column vectors of complex numbers. Note that the product of a column vector and a row vector, e.g.,  $\tilde{\mathbf{a}} \tilde{\mathbf{a}}^\dagger$ , is a tensor product (i.e., a matrix), whereas the product of a row vector and a column

vector, e.g.,  $\underline{a}^\dagger \underline{a}$ , is a scalar product (i.e., an operator or number).

The two-dimensional matrices that arise naturally with this vector notation are the Pauli matrices  $\sigma_1, \sigma_2, \sigma_3$  and the identity matrix  $1$ , defined by

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2C.5a)$$

These satisfy

$$\sigma_i \sigma_j = \delta_{ij} 1 + i \varepsilon_{ijk} \sigma_k, \quad i, j, k = 1, 2, 3. \quad (2C.5b)$$

It is also useful to define rotated versions of  $\sigma_1$  and  $\sigma_2$ ,

$$\sigma_\varphi \equiv \sigma_1 \cos 2\varphi - \sigma_2 \sin 2\varphi = \begin{pmatrix} 0 & e^{2i\varphi} \\ e^{-2i\varphi} & 0 \end{pmatrix}, \quad (2C.6a)$$

$$\sigma_{\varphi - \frac{1}{2}\pi} \equiv \sigma_1 \sin 2\varphi + \sigma_2 \cos 2\varphi = \begin{pmatrix} 0 & -ie^{2i\varphi} \\ ie^{-2i\varphi} & 0 \end{pmatrix}. \quad (2C.6b)$$

Note that  $[\sigma_\varphi, \sigma_{\varphi - \frac{1}{2}\pi}] = [\sigma_1, \sigma_2] = 2i \sigma_3$ .

The commutation relations for  $\underline{a}, \underline{a}^\dagger$  and  $\hat{x}, \hat{p}$  are conveniently expressed by the Hermitian commutator matrices

$$[\underline{a}, \underline{a}^\dagger] \equiv \underline{a} \underline{a}^\dagger - (\underline{a}^* \underline{a}^T)^T = \sigma_3, \quad (2C.7a)$$

$$[\hat{x}, \hat{x}^T] \equiv \hat{x} \hat{x}^T - (\hat{x} \hat{x}^T)^T = A \sigma_3 A^\dagger = -\sigma_2. \quad (2C.7b)$$

The (single-mode) rotation, displacement, and squeeze operators are expressed in vector notation by

$$R(\theta) \equiv \exp[-i\theta \underline{a}^\dagger \underline{a}] = e^{\frac{1}{2}i\theta} \exp[-\frac{1}{2}i\theta \underline{a}^\dagger \underline{a}], \quad (2C.8a)$$

$$D(\underline{a}, \underline{\mu}) \equiv \exp[\underline{\mu} \underline{a}^\dagger - \underline{\mu}^* \underline{a}] = \exp[\underline{a}^\dagger \sigma_3 \underline{\mu}], \quad (2C.8b)$$

$$S_1(r, \varphi) \equiv \exp[\frac{1}{2}r (e^{-2i\varphi} \underline{a}^2 - e^{2i\varphi} \underline{a}^{\dagger 2})] = \exp[-\frac{1}{2}i r \underline{a}^{\dagger} \sigma_{\varphi} \underline{a}]. \quad (2C.8c)$$

A unitary transformation of the column vectors  $\underline{a}$  or  $\underline{\hat{x}}$  generated by the displacement operator results in the addition of a constant column vector:

$$D(\underline{a}, \mu) \underline{a} D^{\dagger}(\underline{a}, \mu) = \underline{a} - \underline{\mu}, \quad D(\underline{a}, \mu) \underline{\hat{x}} D^{\dagger}(\underline{a}, \mu) = \underline{\hat{x}} - \underline{\xi} \quad (2C.9)$$

[Eq. (2A.16)]. Unitary transformations generated by rotation operators and squeeze operators result in matrix transformations of  $\underline{a}$  and  $\underline{\hat{x}}$ . An easy way to obtain these transformation matrices is to note the following general relation, for arbitrary two-dimensional matrix  $K$ , which follows from the fact that the commutator matrix  $[\underline{a}, \underline{a}^{\dagger}] = \sigma_3$ :

$$[\underline{a}^{\dagger} K \underline{a}, \underline{a}] = K_0 \underline{a},$$

$$K_0 \equiv -\sigma_3 \text{Tr} K + [K, \sigma_3] = -\sigma_3 (K + \sigma_1 K^T \sigma_1). \quad (2C.10a)$$

This implies that

$$e^{\underline{a}^{\dagger} K \underline{a}} \underline{a} e^{-\underline{a}^{\dagger} K \underline{a}} = e^{K_0} \underline{a}. \quad (2C.10b)$$

The matrix transformations on  $\underline{a}$  and  $\underline{\hat{x}}$  generated by the rotation operator  $R(\theta)$  are therefore

$$R(\theta) \underline{a} R^{\dagger}(\theta) = e^{i\theta\sigma_3} \underline{a}, \quad (2C.11a)$$

$$R(\theta) \underline{\hat{x}} R^{\dagger}(\theta) = A e^{i\theta\sigma_3} A^{\dagger} \underline{\hat{x}} = e^{-i\theta\sigma_2} \underline{\hat{x}}, \quad (2C.11b)$$

$$e^{i\theta\sigma_3} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad e^{-i\theta\sigma_2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (2C.11c)$$

[cf. Eqs. (2A.10)].

The matrix transformations on  $\underline{a}$  and  $\hat{x}$  generated by the single-mode squeeze operator  $S_1(r, \varphi)$  are

$$S_1(r, \varphi) \underline{a} S_1^\dagger(r, \varphi) = C_{r, \varphi} \underline{a} \equiv \underline{\alpha}(r, \varphi), \quad (2C.12a)$$

$$S_1(r, \varphi) \hat{x} S_1^\dagger(r, \varphi) = AC_{r, \varphi} A^\dagger \hat{x}, \quad (2C.12b)$$

where

$$C_{r, \varphi} \equiv \begin{pmatrix} \cosh r & e^{2i\varphi} \sinh r \\ e^{-2i\varphi} \sinh r & \cosh r \end{pmatrix} = \cosh r \mathbf{1} + \sinh r \sigma_\varphi = e^{r\sigma_\varphi}, \quad (2C.13a)$$

$$\begin{aligned} AC_{r, \varphi} A^\dagger &\equiv \begin{pmatrix} \cosh r + \sinh r \cos 2\varphi & \sinh r \sin 2\varphi \\ \sinh r \sin 2\varphi & \cosh r - \sinh r \cos 2\varphi \end{pmatrix} \\ &= e^{r(\sigma_3 \cos 2\varphi + \sigma_1 \sin 2\varphi)} \end{aligned} \quad (2C.13b)$$

[Eq. (2C.1b)]. The Hermitian matrix  $C_{r, \varphi}$  has the following important properties:

$$C_{r, \varphi}^{-1} = C_{-r, \varphi} = C_{r, \varphi + \frac{1}{2}\pi} = \sigma_3 C_{r, \varphi} \sigma_3; \quad (2C.14a)$$

$$C_{r, 0} = e^{r\sigma_1} = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}; \quad C_{r, \frac{1}{2}\pi} = e^{-r\sigma_2} = \begin{pmatrix} \cosh r & i \sinh r \\ -i \sinh r & \cosh r \end{pmatrix}; \quad (2C.14b)$$

$$C_{r, \varphi} = e^{i\varphi' \sigma_3} C_{r, \varphi - \varphi'} e^{-i\varphi' \sigma_3}; \quad (2C.14c)$$

$$C_{r, \varphi} C_{r', \varphi} = C_{r+r', \varphi} \quad (2C.14d)$$

(see also Appendix A of Ref. 26).

The transformation matrices (2C.11) and (2C.13) arise naturally, without specific reference to the single-mode rotation and squeeze operators, from the requirement that a unitary transformation on  $\underline{a}$  (or  $\hat{x}$  and  $\hat{p}$ )

preserve the commutators (2C.7). This is most easily seen by considering the real, two-dimensional matrices  $\bar{M}$  that describe unitary transformations on the (real) column vector  $\hat{x}$ :  $\bar{M} \hat{x} \equiv U \alpha U^\dagger$ . The unitarity of  $U$  implies that these matrices preserve the (antisymmetric) commutator matrix  $[\hat{x}, \hat{x}^T] = -\sigma_2$  [Eq. (2C.6b)], i.e.,

$$\bar{M} \sigma_2 \bar{M}^T = \bar{M}^T \sigma_2 \bar{M} = \sigma_2 . \quad (2C.15a)$$

The real matrices  $\bar{M}$  that satisfy this condition have unity determinant. They comprise the three-parameter symplectic group  $\text{Sp}(2, \mathbb{R})$ .<sup>41</sup> The complex two-dimensional matrices  $M$  that describe unitary transformations on the column vector  $\alpha = A^\dagger \hat{x}$ ,  $M \alpha \equiv U \alpha U^\dagger$ , are unitarily related to the real matrices  $\bar{M}$  by the matrix  $A$  [Eq. (2C.1b)]:

$$M = A^\dagger \bar{M} A . \quad (2C.15b)$$

These matrices  $M$  comprise the three-parameter, noncompact group  $\text{SU}(1, 1)$ ,<sup>41</sup> isomorphic to  $\text{Sp}(2, \mathbb{R})$ ; it consists of all complex, two-dimensional matrices that have unitary determinant and that preserve the metric  $\sigma_3$  (i.e., the commutator matrix  $[\alpha, \alpha^\dagger] = \sigma_3$ ),

$$M \sigma_3 M^\dagger = \sigma_3 = M^\dagger \sigma_3 M . \quad (2C.15c)$$

The three free (real) parameters associated with the transformation matrices  $M$  and  $\bar{M}$  can be identified with the parameters of the unitary operators that induce the matrix transformations. The generators of these unitary operators are bilinear combinations of the annihilation and creation operator; i.e., the unitary operators are the single-mode rotation and squeeze operators. The underlying Lie algebra for these groups is that of (combinations of) the three operators  $\alpha^2$ ,  $\alpha^{\dagger 2}$ , and  $\alpha^\dagger \alpha$ . The preceding

discussion of the single-mode rotation and squeeze operators shows that the transformation matrix  $M$  has the general form

$$M = e^{i\theta\sigma_3} C_{r,\varphi} = C_{r,\varphi+\theta} e^{-i\theta\sigma_3} = \begin{pmatrix} e^{i\theta} \cosh r & e^{i(2\varphi+\theta)} \sinh r \\ e^{-i(2\varphi+\theta)} \sinh r & e^{-i\theta} \cosh r \end{pmatrix}, \quad (2C.15d)$$

where  $\theta$ ,  $r$ , and  $\varphi$  are real, continuous parameters [Eq. (2A.21b)].

The general forms for the transformation matrices  $M$  (hence also  $\bar{M}$ ) can also be obtained in other ways. For example, one might first note that any two-dimensional matrix  $M$  that describes a unitary transformation on  $\underline{a}$  must satisfy

$$M^* = \sigma_1 M \sigma_1, \quad (2C.16a)$$

since  $\underline{a} = \sigma_1 \underline{a}^*$ . This means that the matrix  $M$  has the general form

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad (2C.16b)$$

where  $\alpha$  and  $\beta$  are arbitrary complex numbers. It also implies that

$$M \sigma_3 M^\dagger \sigma_3 = M \sigma_2 M^T \sigma_2 = (\det M) 1. \quad (2C.16c)$$

The last equality in (2C.16c) is satisfied by all two-dimensional matrices  $M$ ; the first is satisfied by only (and all) those matrices  $M$  that satisfy (2C.16a). The unitarity of the transformation ensures that the Hermitian commutator matrix  $[\underline{a}, \underline{a}^\dagger] = \sigma_3$  is preserved [Eq. (2C.15c)] and that the antisymmetric commutator matrix  $[\underline{a}, \underline{a}^T] = i \sigma_2$  is preserved, i.e.,

$$M \sigma_2 M^T = \sigma_2 = M^T \sigma_2 M. \quad (2C.16d)$$

Either of these conditions, together with (2C.16a) or (2C.16c), implies that

$\det M = 1$ . Hence the matrices  $M$  have the general form (2C.15d), and the real matrices  $\bar{M}$  are given by  $\bar{M} = A M A^\dagger$ .

Many properties of the single-mode squeeze operator  $S_1(r, \varphi)$  that would otherwise be difficult to see can be found from properties of the transformation matrix  $C_{r, \varphi}$ . For example, one can factor  $S_1(r, \varphi)$  into a product of exponentials of  $a^\dagger a$ ,  $a^2$ , and  $a^{\dagger 2}$  simply by factoring the matrix  $C_{r, \varphi}$  into exponentials of other matrices (linear combinations of the Pauli matrices) which have a commutator algebra that is identical to that of  $(a^\dagger a)_{\text{sym}}$ ,  $\frac{1}{2}a^2$ , and  $\frac{1}{2}a^{\dagger 2}$  (see, e.g., Refs. 15, 48, or 26). These factored forms are useful, for example, for expressing a SMSS as a sum over number states, a technique useful for calculating the phase factor  $e^{\frac{1}{2}i\delta_z}$  in the wave function for a SMSS (see Appendix B). These factored forms are listed in Appendix B [Eq. (B.12)] of Ref. 26; one of the most useful is

$$S_1(r, \varphi) = (\cosh r)^{-\frac{1}{2}} e^{-\frac{1}{2}\Gamma a^{\dagger 2}} e^{-\ln(\cosh r) a^\dagger a} e^{\frac{1}{2}\Gamma a^2}, \quad \Gamma \equiv e^{2i\varphi} \tanh r. \quad (2C.17)$$

Also, the product of two different squeeze operators can be found from the product of two different matrices  $C_{r, \varphi}$ . In this way one finds that any product of single-mode squeeze operators can always be expressed as the product of one single-mode squeeze operator and one rotation operator, by virtue of the following rule:

$$S_1^\dagger(r', \varphi') S_1(r, \varphi) = e^{-i\Theta} R(\Theta) S_1(R, \Phi) = e^{-i\Theta} S_1(R, \Phi - \Theta) R(\Theta). \quad (2C.18a)$$

The real numbers  $\Theta$ ,  $R$ , and  $\Phi$  are related to  $r'$ ,  $\varphi'$ ,  $r$ , and  $\varphi$  by the matrix equality

$$C_{R, \Phi} e^{i\Theta \sigma_3} = \begin{pmatrix} e^{i\Theta} \cosh R & e^{i(2\Phi - \Theta)} \sinh R \\ e^{-i(2\Phi - \Theta)} \sinh R & e^{-i\Theta} \cosh R \end{pmatrix} = C_{r, \varphi} C_{-r', \varphi'} \quad (2C.18b)$$

[Eq. (B.14) of Ref. 26]. For the special case  $\varphi = \varphi'$  this gives the simple

relation

$$S_1(\mathbf{r}, \varphi) S_1(\mathbf{r}', \varphi) = S_1(\mathbf{r} + \mathbf{r}', \varphi) \quad (2C.18c)$$

[Eq. (2C.14d)].

The vector notation simplifies the task of ordering noncommuting unitary operators. For example, using the matrix transformations given above one finds that

$$R(\theta) D(\mathbf{a}, \mu) = \exp[\tilde{\mathbf{a}}^\dagger e^{-i\theta\sigma_3} \sigma_3 \tilde{\mu}] R(\theta) = D(\mathbf{a}, e^{-i\theta} \mu) R(\theta), \quad (2C.19a)$$

$$\begin{aligned} R(\theta) S_1(\mathbf{r}, \varphi) &= \exp[\frac{1}{2} \mathbf{r} \tilde{\mathbf{a}}^\dagger e^{-i\theta\sigma_3} \sigma_\varphi \frac{1}{2} \pi e^{i\theta\sigma_3} \tilde{\mathbf{a}}] R(\theta) \\ &= \exp[\frac{1}{2} \mathbf{r} \tilde{\mathbf{a}}^\dagger \sigma_{\varphi-\theta} \frac{1}{2} \pi \tilde{\mathbf{a}}] R(\theta) = S_1(\mathbf{r}, \varphi - \theta) R(\theta), \end{aligned} \quad (2C.19b)$$

$$\begin{aligned} D(\mathbf{a}, \mu) S_1(\mathbf{r}, \varphi) &= S_1(\mathbf{r}, \varphi) \exp[\tilde{\mathbf{a}}^\dagger C_{\mathbf{r}, \varphi} \sigma_3 \tilde{\mu}] \\ &= S_1(\mathbf{r}, \varphi) \exp[\tilde{\mathbf{a}}^\dagger \sigma_3 C_{\mathbf{r}, \varphi} \tilde{\mu}] \equiv S_1(\mathbf{r}, \varphi) D(\mathbf{a}, \mu_\alpha), \end{aligned}$$

$$\tilde{\mu}_\alpha \equiv \begin{pmatrix} \mu_\alpha \\ \mu_\alpha^* \end{pmatrix} \equiv C_{\mathbf{r}, \varphi} \tilde{\mu} \quad (2C.19c)$$

[cf. Eqs. (2A.17), (2A.25), (2A.26)-(2A.28)].

The vector notation is particularly useful for calculating second-order noise moments of  $\mathbf{a}$ ,  $\mathbf{a}^\dagger$ ,  $\hat{\mathbf{x}}$ , and  $\hat{\mathbf{p}}$ . The matrix of second-order noise moments of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  is the Hermitian matrix

$$\begin{aligned} \mathcal{Q} &\equiv \langle \Delta \tilde{\mathbf{a}} \Delta \tilde{\mathbf{a}}^\dagger \rangle_{\text{sym}} \equiv \frac{1}{2} (\langle \Delta \tilde{\mathbf{a}} \Delta \tilde{\mathbf{a}}^\dagger \rangle + \langle \Delta \tilde{\mathbf{a}}^* \Delta \tilde{\mathbf{a}}^T \rangle^T) \\ &= \begin{pmatrix} \langle |\Delta \mathbf{a}|^2 \rangle & \langle \langle \Delta \mathbf{a} \rangle^2 \rangle \\ \langle \langle \Delta \mathbf{a}^\dagger \rangle^2 \rangle & \langle |\Delta \mathbf{a}|^2 \rangle \end{pmatrix} = \mathcal{Q}^\dagger. \end{aligned} \quad (2C.20)$$



The matrix of second-order noise moments of  $\hat{x}$  and  $\hat{p}$  is the (real, symmetric) covariance matrix

$$\begin{aligned} \mathcal{S} &\equiv \langle \Delta \hat{x} \Delta \hat{x}^T \rangle_{\text{sym}} \equiv \frac{1}{2} (\langle \Delta \hat{x} \Delta \hat{x}^T \rangle + \langle \Delta \hat{x} \Delta \hat{x}^T \rangle^T) \\ &= \begin{pmatrix} \langle (\Delta \hat{x})^2 \rangle & \langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}} \\ \langle \Delta \hat{x} \Delta \hat{p} \rangle_{\text{sym}} & \langle (\Delta \hat{p})^2 \rangle \end{pmatrix} = A Q A^\dagger = \mathcal{S}^* = \mathcal{S}^T. \end{aligned} \quad (2C.21)$$

The relations (2B.11) imply that for single-mode GPS these matrices satisfy

$$Q \sigma_3 Q \sigma_3 = \frac{1}{2} 1, \quad (2C.22a)$$

$$\mathcal{S} \sigma_2 \mathcal{S} \sigma_2 = \frac{1}{2} 1. \quad (2C.22b)$$

Hence their determinants are equal to  $\frac{1}{4}$ . For a coherent state, both are proportional to the identity matrix:

$$Q_{\text{coh}} = \mathcal{S}_{\text{coh}} = \frac{1}{2} 1 \quad (2C.23)$$

[Eqs. (2A.36)].

The noise matrices  $Q$  and  $\mathcal{S}$  for a state  $|\Psi\rangle$  are related to those of a rotated state  $R(\theta)|\Psi\rangle$  by

$$\begin{aligned} \langle R^\dagger(\theta) (\Delta \underline{\alpha} \Delta \underline{\alpha}^\dagger)_{\text{sym}} R(\theta) \rangle &= e^{-i\theta \sigma_3} Q e^{i\theta \sigma_3} \\ &= \begin{pmatrix} \langle |\Delta \underline{\alpha}|^2 \rangle & e^{-2i\theta} \langle (\Delta \underline{\alpha})^2 \rangle \\ e^{2i\theta} \langle (\Delta \underline{\alpha}^\dagger)^2 \rangle & \langle |\Delta \underline{\alpha}|^2 \rangle \end{pmatrix}, \end{aligned} \quad (2C.24a)$$

$$\begin{aligned} \langle R^\dagger(\theta) (\Delta \hat{x} \Delta \hat{x}^T)_{\text{sym}} R(\theta) \rangle &= A e^{-i\theta \sigma_3} A^\dagger \mathcal{S} A e^{i\theta \sigma_3} A^\dagger \\ &= e^{i\theta \sigma_2} \mathcal{S} e^{-i\theta \sigma_2}. \end{aligned} \quad (2C.24b)$$

They are related to those of a state  $S_1(\mathbf{r}, \varphi)|\Psi\rangle$  by

$$\langle S_1^\dagger(r, \varphi) (\Delta \tilde{a} \Delta \tilde{a}^\dagger)_{\text{sym}} S_1(r, \varphi) \rangle = C_{-r, \varphi} Q C_{-r, \varphi}, \quad (2C.25a)$$

$$\langle S_1^\dagger(r, \varphi) (\Delta \tilde{\hat{x}} \Delta \tilde{\hat{x}}^T)_{\text{sym}} S_1(r, \varphi) \rangle = A C_{-r, \varphi} A^\dagger \mathcal{S} A C_{-r, \varphi} A^\dagger. \quad (2C.25b)$$

This immediately tells one, for example, that the noise matrices for a single-mode squeezed state  $|\mu_\alpha\rangle_{r, \varphi}$  are

$$Q_{\mu_\alpha} = \frac{1}{2} C_{-r, \varphi}^2 = \frac{1}{2} C_{-2r, \varphi} = \frac{1}{2} \sigma_3 C_{2r, \varphi} \sigma_3, \quad (2C.26a)$$

$$\mathcal{S}_{\mu_\alpha} = \frac{1}{2} A C_{-2r, \varphi} A^\dagger = \frac{1}{2} \sigma_2 A C_{2r, \varphi} A^\dagger \sigma_2 \quad (2C.26b)$$

[Eqs. (2C.13c), (2C.23)]. The squeezing effect is clearly visible in the transformation of the noise matrix  $\mathcal{S}$  [Eq. (2C.25b)]. When  $\varphi = 0$ , this transformation says that

$$\begin{aligned} \langle S_1^\dagger(r, \varphi) (\Delta \tilde{\hat{x}} \Delta \tilde{\hat{x}}^T)_{\text{sym}} S_1(r, \varphi) \rangle &= e^{-r \sigma_3} \mathcal{S} e^{-r \sigma_3} \\ &= \begin{pmatrix} e^{-2r} \langle (\Delta \tilde{\hat{x}})^2 \rangle & \langle \Delta \tilde{\hat{x}} \Delta \tilde{\hat{p}} \rangle_{\text{sym}} \\ \langle \Delta \tilde{\hat{x}} \Delta \tilde{\hat{p}} \rangle_{\text{sym}} & e^{2r} \langle (\Delta \tilde{\hat{p}})^2 \rangle \end{pmatrix}. \end{aligned} \quad (2C.27)$$

Finally, the vector notation enables one to show with relative ease how the unitary operator whose generator is a linear combination of the Hermitian forms  $H_R^{(1)}$ ,  $H_1^{(1)}$ , and  $H_2^{(1)}$  factors into the product of a single-mode squeeze, rotation and displacement operator (and an overall phase factor). More generally, by giving these generators arbitrary time dependences one can calculate the evolution operator associated with the most general combination of Hamiltonians that can produce single-mode GPS. This result is given here briefly; details that are important for the calculation are presented in Appendix A. Equivalent results have been obtained by Yuen.<sup>16</sup>

The single-mode rotation Hamiltonian  $H_R^{(1)}(t)$  is expressed in vector notation by

$$H_R^{(1)}(t) \equiv \omega(t) \mathbf{a}^\dagger \mathbf{a} = \frac{1}{2} \omega(t) (\mathbf{a}^\dagger \mathbf{a} - 1), \quad (2C.28a)$$

where  $\omega(t)$  is an arbitrary real-valued function of time  $t$  [Eqs. (1.3)]. The linear and quadratic Hamiltonians  $H_1^{(1)}(t)$  and  $H_2^{(1)}(t)$  are expressed by

$$H_1^{(1)}(t) = i \mathbf{a}^\dagger \sigma_3 \mathbf{a}, \quad \lambda \equiv \begin{bmatrix} \lambda(t) \\ \lambda^*(t) \end{bmatrix}; \quad (2C.28b)$$

$$H_2^{(1)}(t) = \frac{1}{2} \kappa(t) \mathbf{a}^\dagger \sigma_{\varphi_\kappa - \frac{1}{2}\pi} \mathbf{a}, \quad (2C.28c)$$

where  $\lambda$  is a complex-valued function of time, and  $\kappa$  and  $\varphi_\kappa$  are real-valued functions of time [Eqs. (1.4), (2C.6b)].

The evolution operator  $U(t)$  is the solution to the equation

$$i \partial_t U(t) = [H_R^{(1)}(t) + H_1^{(1)}(t) + H_2^{(1)}(t)] U(t),$$

$$U(0) = 1. \quad (2C.29)$$

It can be written as the product

$$U(t) = e^{i\delta} S_1(\mathbf{r}, \varphi) R(\theta) D(\mathbf{a}, \mu_g) \quad (2C.30a)$$

$$= e^{i\delta} D(\mathbf{a}, \mu) S_1(\mathbf{r}, \varphi) R(\theta), \quad (2C.30b)$$

where  $\delta$ ,  $\theta$ ,  $\mathbf{r}$ , and  $\varphi$  are real-valued function of time, and  $\mu$  is a complex-valued function of time [cf. Eq. (2B.27)]. For notational convenience here I often drop explicit reference to the time dependence of these functions, e.g.,  $\mathbf{r} \equiv \mathbf{r}(t)$ , etc. The state  $|\mu_g\rangle \equiv U(t) |0\rangle$  is an eigenstate of an operator  $g \equiv U(t) \mathbf{a} U^\dagger(t)$  (with eigenvalue  $\mu_g$ ), whose relation to  $\mathbf{a}$  is described by the vector relation

$$\underline{g} \equiv \begin{bmatrix} g \\ g^\dagger \end{bmatrix} = S_1(\mathbf{r}, \varphi) R(\theta) \underline{a} R^\dagger(\theta) S_1^\dagger(\mathbf{r}, \varphi) = e^{i\theta \sigma_3} C_{\mathbf{r}, \varphi} \underline{a} \quad (2C.31a)$$

[Eqs. (2C.10)]. The eigenvalue  $\mu_g$  is therefore related to the complex amplitude  $\mu \equiv \langle \alpha \rangle$  by

$$\begin{pmatrix} \mu_g \\ \mu_g^* \end{pmatrix} \equiv e^{i\theta\sigma_3} C_{r,\varphi} \mu. \quad (2C.31b)$$

The calculations in Appendix A show that the functions  $r$ ,  $\varphi$ ,  $\theta$ ,  $\mu_g$  (or  $\mu$ ), and  $\delta$  are related to the Hamiltonian functions  $\kappa$ ,  $\varphi_\kappa$ ,  $\omega$ , and  $\lambda$  by the following matrix, vector, and scalar equalities:

$$\dot{r} \sigma_{\varphi-\frac{1}{2}\pi} - \dot{\varphi} 1 + (\dot{\varphi} + \dot{\theta}) C_{2r,\varphi} = \omega 1 + \kappa \sigma_{\varphi_\kappa-\frac{1}{2}\pi}, \quad (2C.32a)$$

$$\dot{\mu}_g = e^{i\theta\sigma_3} C_{r,\varphi} \dot{\lambda}, \quad (2C.32b)$$

$$\delta + \frac{1}{2}(\dot{\theta} - \dot{\omega}) = -\frac{1}{2}i \mu_g^\dagger \sigma_3 \dot{\mu}_g = \text{Im}(\mu^* \dot{\lambda}). \quad (2C.32c)$$

(Dots denote derivatives with respect to time.) The initial conditions, dictated by  $U(0) = 1$ , are

$$\delta(0) = r(0) = \theta(0) = \mu_g(0) = \mu(0) = 0. \quad (2C.33)$$

For illustration, consider the case  $\varphi_\kappa \equiv \varphi_{\kappa_0} - \int_0^t \omega(t) dt$ ,  $\varphi_{\kappa_0} \equiv \text{constant}$ . Then

Eq. (2C.32a) gives

$$r(t) = \int_0^t \kappa(t) dt, \quad (2C.34a)$$

$$\varphi(t) = \varphi_\kappa = \varphi_{\kappa_0} - \int_0^t \omega(t) dt, \quad \theta(t) = \int_0^t \omega(t) dt. \quad (2C.34b)$$

If no driving is present in this case [ $\lambda(t) = 0$ ], then  $\mu_g = \delta = 0$ , and  $U(t) = S_1(r, \varphi_\kappa) R(\theta) = R(\theta) S_1(r, \varphi_{\kappa_0})$ . Now consider the case where both  $\kappa$

and  $\varphi_{\kappa_0}$  are constant, and  $\lambda(t) \equiv \lambda_0 e^{-i \int_0^t \omega(t) dt}$ ,  $\lambda_0 \equiv \text{constant}$  [when

$\omega(t) \equiv \omega = \text{constant}$ , this corresponds to driving the oscillator on resonance]. Then  $r(t) = \kappa t$ , the angles  $\varphi(t)$  and  $\theta(t)$  are still given by Eq. (2C.34b), and the function  $\mu_g$  is given by the vector relation

$$\begin{aligned} \underline{\mu}_g &= \int_0^t dt e^{i\theta\sigma_3} C_{\kappa t, \varphi_{\kappa_0} - \theta} e^{-i\theta\sigma_3} \underline{\lambda}_0 = \int_0^t dt C_{\kappa t, \varphi_{\kappa_0}} \underline{\lambda}_0 \\ &= \kappa^{-1} (C_{\tau, \varphi_{\kappa_0}} - 1) e^{2i\varphi_{\kappa_0}\sigma_3} \underline{\lambda}^* \equiv C_{\tau, \varphi_{\kappa_0}} e^{i\int_0^t \omega(t) dt \sigma_3} \underline{\mu}. \end{aligned} \quad (2C.35)$$

The phase angle  $\delta(t)$  is

$$\begin{aligned} \delta(t) &= (2i\kappa)^{-1} \underline{\lambda}_0^\dagger \sigma_{\varphi_{\kappa_0}} \sigma_3 (1 - C_{\kappa t, \varphi_{\kappa_0}}) \underline{\lambda}_0 \\ &= \kappa^{-1} (\cosh \kappa t - 1) \text{Im}(e^{2i\varphi_{\kappa_0}} \underline{\lambda}_0^{*2}). \end{aligned} \quad (2C.36)$$

Note that the solution to the problem of factoring the unitary operator whose generator is any linear combination of the Hermitian forms  $H_R^{(1)}$ ,  $H_1^{(1)}$ , and  $H_2^{(1)}$  can be obtained from the above equations, by setting all the Hamiltonian functions  $(\kappa, \varphi_\kappa, \omega, \lambda)$  equal to constants, solving the coupled differential equations (2C.32), and then setting the dummy parameter  $t$  to unity. Thus,

$$e^{-i[H_R^{(1)} + H_1^{(1)} + H_2^{(1)}]} \equiv e^{i\omega} \exp\left[-\frac{1}{2}i \underline{a}^\dagger (\omega 1 + \kappa \sigma_{\varphi_\kappa - \frac{1}{2}\pi}) \underline{a} + \underline{a}^\dagger \sigma_3 \underline{\lambda}\right] \quad (2C.37)$$

can be put in the factored forms (2C.30), with  $\delta$ ,  $r$ ,  $\varphi$ ,  $\theta$ , and  $\mu_g$  (or  $\mu$ ) the solutions to Eqs. (2C.32) when  $t \equiv 1$ . One immediately finds that

$$r = \kappa, \quad \varphi = \varphi_\kappa, \quad \theta = \omega. \quad (2C.38)$$

The parameters  $\mu_g$  and  $\delta$  are the solutions, at  $t = 1$ , to the vector and scalar equations

$$\tilde{\mu}_g = e^{i\omega t \sigma_3} C_{\kappa t, \varphi_{\kappa_0} - \omega t} \tilde{\lambda}, \quad (2C.39a)$$

$$\delta = \text{Im}(\mu_g^* \tilde{\mu}_g) = \text{Im}(\mu^* \lambda). \quad (2C.39b)$$

### III. TWO-MODE GAUSSIAN PURE STATES

#### A. Introduction

##### 1. Notation and Definitions

###### a. Dimensionless Position and Momentum Variables

Consider now two oscillators, with characteristic frequencies  $\omega_+$  and  $\omega_-$  ( $\omega_+ \geq \omega_-$ ). Each oscillator can be described by its own set of annihilation and creation operators --  $a_+$ ,  $a_+^\dagger$  and  $a_-$ ,  $a_-^\dagger$  -- or, equivalently, by the dimensionless coordinate and momentum operators  $\hat{x}_+$ ,  $\hat{p}_+$  and  $\hat{x}_-$ ,  $\hat{p}_-$ . These operators are related to each other by

$$\hat{x}_\pm \equiv 2^{-\frac{1}{2}}(a_\pm + a_\pm^\dagger), \quad \hat{p}_\pm \equiv 2^{-\frac{1}{2}}(-ia_\pm + ia_\pm^\dagger), \quad (3A.1a)$$

$$a_\pm \equiv 2^{-\frac{1}{2}}(\hat{x}_\pm + i\hat{p}_\pm) \quad (3A.1b)$$

[Eqs. (2A.1)]. They obey the commutation relations

$$[a_\pm, a_\pm^\dagger] = 1, \quad [\hat{x}_\pm, \hat{p}_\pm] = i; \quad (3A.2a)$$

$$[a_+, a_-] = [a_+, a_-^\dagger] = [\hat{x}_+, \hat{p}_+] = [\hat{x}_+, \hat{x}_-] = [\hat{p}_+, \hat{p}_-] = 0. \quad (3A.2b)$$

While the operators  $\hat{x}_+$  and  $\hat{x}_-$  ( $\hat{p}_+$  and  $\hat{p}_-$ ) are both dimensionless, they do not have the same "units", since the natural units of length (momentum) for the two oscillators differ. Dimensionless position and momentum operators that have compatible units for the two oscillators can be obtained by dividing the usual dimensional position and momentum operators by new units of length  $L_0$  and momentum  $P_0$ . In general,  $L_0$  and  $P_0$  can be chosen quite arbitrarily, subject to the dimensional restriction  $L_0 P_0 = 1$  ( $\equiv \hbar$ ), but

there is a natural choice for them. To see this, let the two oscillators be modeled as masses  $m_{\pm}$  on springs (the result will also hold, however, for the normal modes of a quantized field). Their natural units of length and momentum are  $(m_{\pm} \omega_{\pm})^{-\frac{1}{2}}$  and  $(m_{\pm} \omega_{\pm})^{\frac{1}{2}}$ , respectively. Now choose quantities  $m$  and  $\Omega$ , with dimensions of mass and (time) $^{-1}$ , respectively, and define  $L_0 \equiv (m \Omega)^{-\frac{1}{2}}$ ,  $P_0 \equiv (m \Omega)^{\frac{1}{2}}$ . The dimensionless position and momentum operators  $Q_{\pm}$  and  $P_{\pm}$  for this choice are

$$Q_{\pm} \equiv \lambda_{\pm}^{-1} \hat{x}_{\pm}, \quad P_{\pm} \equiv \lambda_{\pm} \hat{p}_{\pm}, \quad (3A.3a)$$

$$\lambda_{\pm} \equiv P_0^{-1} (m_{\pm} \omega_{\pm})^{\frac{1}{2}} = L_0 (m_{\pm} \omega_{\pm})^{\frac{1}{2}} = (m_{\pm}/m)^{\frac{1}{2}} (\omega_{\pm}/\Omega)^{\frac{1}{2}}. \quad (3A.3b)$$

Thus, for equal masses with  $m_{+} = m_{-} \equiv m$ , or for normal modes of a quantized field, the natural choice for  $\lambda_{\pm}$  is  $(\omega_{\pm}/\Omega)^{\frac{1}{2}}$ . For modes of the electromagnetic field, these conjugate variables  $Q_{\pm}$  and  $P_{\pm}$  are the Fourier components of the vector potential and its conjugate momentum, the electric field. The free Hamiltonian for the two modes, written in terms of  $Q_{\pm}$  and  $P_{\pm}$ , has the canonical form

$$H_0^{(2)} = \frac{1}{2} \Omega [P_+^2 + (\omega_+/\Omega)^2 Q_+^2 + P_-^2 + (\omega_-/\Omega)^2 Q_-^2]. \quad (3A.4)$$

In most of this discussion of two-mode Gaussian pure states I use the variables  $\hat{x}_{\pm}$  and  $\hat{p}_{\pm}$ , rather than  $Q_{\pm}$  and  $P_{\pm}$ , because they provide an easy comparison with the previous discussion of single-mode states. The field variables  $Q_{\pm}$  and  $P_{\pm}$  or, equivalently, the quadrature-phase amplitudes  $\alpha_1$  and  $\alpha_2$  [Eqs. (1.21) and (1.22)], are useful for describing the special noise properties of two-mode squeezed states. The field variables  $Q_{\pm}$  and  $P_{\pm}$  are related to the quadrature-phase amplitudes defined in Eqs. (1.21) by

$$\text{Re}(\alpha_1) = (2\Omega)^{-1} [(\Omega + \varepsilon) Q_+ + (\Omega - \varepsilon) Q_-],$$



$$\operatorname{Re}(\alpha_2) = (2\Omega)^{-1}(P_+ + P_-),$$

$$\operatorname{Im}(\alpha_1) = (2\Omega)^{-1}(P_+ - P_-),$$

$$\operatorname{Im}(\alpha_2) = -(2\Omega)^{-1}[(\Omega + \varepsilon)Q_+ - (\Omega - \varepsilon)Q_-], \quad (3A.5a)$$

$$\operatorname{Re}(\alpha_j) \equiv \frac{1}{2}(\alpha_j + \alpha_j^\dagger), \quad \operatorname{Im}(\alpha_j) \equiv -i\frac{1}{2}(\alpha_j - \alpha_j^\dagger), \quad j = 1, 2. \quad (3A.5b)$$

### b. Two-component Vector Notation

An obvious way to generalize one's mathematics from a single mode to two modes is to replace the single-mode annihilation and creation operators  $\alpha$  and  $\alpha^\dagger$  by the two-component operator column vectors<sup>29,1</sup>

$$\underset{\sim}{\alpha} \equiv \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}, \quad \underset{\sim}{\alpha}^* \equiv \begin{pmatrix} \alpha_+^\dagger \\ \alpha_-^\dagger \end{pmatrix}. \quad (3A.6)$$

These column vectors should not be confused with the two-component column vectors defined in Section IIC for a single mode. The same symbol is used here because it is natural and because the risk of confusion is low; the single-mode column vector never appears in conjunction with two-mode column vectors (i.e., it never appears in this section). The column vectors for the dimensionless position and momentum variables  $\hat{x}_\pm$  and  $\hat{p}_\pm$  are related to  $\underset{\sim}{\alpha}$  and  $\underset{\sim}{\alpha}^\dagger$  by

$$\underset{\sim}{\hat{x}} \equiv \begin{pmatrix} \hat{x}_+ \\ \hat{x}_- \end{pmatrix} \equiv 2^{-\frac{1}{2}}(\underset{\sim}{\alpha} + \underset{\sim}{\alpha}^*), \quad \underset{\sim}{\hat{p}} \equiv \begin{pmatrix} \hat{p}_+ \\ \hat{p}_- \end{pmatrix} \equiv 2^{-\frac{1}{2}}(-i\underset{\sim}{\alpha} + i\underset{\sim}{\alpha}^*); \quad (3A.7a)$$

$$\underset{\sim}{\alpha} \equiv 2^{-\frac{1}{2}}(\underset{\sim}{\hat{x}} + i\underset{\sim}{\hat{p}}) \quad (3A.7b)$$

[cf. Eqs. (2A.1)]. The adjoints and transposes of these column vectors are defined in the usual way (see Section II.C). Similar definitions hold for

column vectors of complex numbers; e.g., the column vector for the complex amplitudes  $\mu_+$  and  $\mu_-$  is

$$\underline{\mu} \equiv \langle \underline{a} \rangle \equiv \begin{pmatrix} \mu_+ \\ \mu_- \end{pmatrix} \equiv 2^{-\frac{1}{2}}(\underline{x}_0 + i \underline{p}_0), \quad (3A.8a)$$

where  $\underline{x}_0$  and  $\underline{p}_0$  are the column vectors for the mean positions  $x_{0\pm}$  and momentums  $p_{0\pm}$ , respectively,

$$\underline{x}_0 \equiv \langle \underline{\hat{x}} \rangle \equiv \begin{pmatrix} x_{0+} \\ x_{0-} \end{pmatrix} = 2^{-\frac{1}{2}}(\underline{\mu} + \underline{\mu}^*), \quad (3A.8b)$$

$$\underline{p}_0 \equiv \langle \underline{\hat{p}} \rangle \equiv \begin{pmatrix} p_{0+} \\ p_{0-} \end{pmatrix} = 2^{-\frac{1}{2}}(-i\underline{\mu} + i\underline{\mu}^*) \quad (3A.8c)$$

[cf. Eq. 2A.3]. This two-component vector notation is used throughout this section in order to present the two-mode results in a simple form that resembles as closely as possible the single-mode results. For example, the commutation relations (3A.2) take on the matrix form

$$[\underline{a}, \underline{a}^\dagger] \equiv \underline{a} \underline{a}^\dagger - (\underline{a}^* \underline{a}^T)^T = 1,$$

$$[\underline{a}, \underline{a}^T] \equiv \underline{a} \underline{a}^T - (\underline{a} \underline{a}^T)^T = [\underline{a}_+, \underline{a}_-] i \sigma_2 = 0; \quad (3A.9a)$$

$$[\underline{\hat{x}}, \underline{\hat{p}}^T] \equiv \underline{\hat{x}} \underline{\hat{p}}^T - (\underline{\hat{p}} \underline{\hat{x}}^T)^T = i 1,$$

$$[\underline{\hat{x}}, \underline{\hat{x}}^T] = [\underline{\hat{p}}, \underline{\hat{p}}^T] = 0. \quad (3A.9b)$$

**c. Second-order Noise Moments**

For a single mode there are only two relevant Hermitian operators,  $\hat{x}$  and  $\hat{p}$  (or one complex operator,  $\alpha$ ), and hence only three real second-order noise moments to consider --  $\langle(\Delta\hat{x})^2\rangle$ ,  $\langle(\Delta\hat{p})^2\rangle$ , and  $\langle\Delta\hat{x}\Delta\hat{p}\rangle_{\text{sym}}$ , or, equivalently,  $\langle(\Delta\alpha)^2\rangle$  and  $\langle|\Delta\alpha|^2\rangle$ . For two modes, there are four relevant Hermitian operators,  $\hat{x}_{\pm}$  and  $\hat{p}_{\pm}$ , and hence ten real second-order noise moments to consider; six of these are associated with each of the modes separately, and four describe correlations between the modes. The four correlated noise moments are  $\langle\Delta\hat{x}_+\Delta\hat{x}_-\rangle$ ,  $\langle\Delta\hat{p}_+\Delta\hat{p}_-\rangle$ , and  $\langle\Delta\hat{x}_+\Delta\hat{p}_-\rangle$ , or, equivalently, the complex noise moments  $\langle\Delta\alpha_+\Delta\alpha_-\rangle$  and  $\langle\Delta\alpha_+\Delta\alpha_-^\dagger\rangle$ . The two-mode analog of  $\langle(\Delta\alpha)^2\rangle$  is the complex, symmetric matrix  $T$ :

$$T \equiv \langle\Delta\tilde{\alpha}\Delta\tilde{\alpha}^T\rangle = \begin{bmatrix} \langle(\Delta\alpha_+)^2\rangle & \langle\Delta\alpha_+\Delta\alpha_-\rangle \\ \langle\Delta\alpha_+\Delta\alpha_-\rangle & \langle(\Delta\alpha_-)^2\rangle \end{bmatrix} = T^T. \quad (3A.10a)$$

The two-mode analog of  $\langle|\Delta\alpha|^2\rangle$  is the Hermitian matrix  $Q$ :

$$\begin{aligned} Q \equiv \langle\Delta\tilde{\alpha}\Delta\tilde{\alpha}^\dagger\rangle_{\text{sym}} &\equiv \frac{1}{2}\langle\Delta\tilde{\alpha}\Delta\tilde{\alpha}^\dagger\rangle + \frac{1}{2}\langle\Delta\tilde{\alpha}^*\Delta\tilde{\alpha}^T\rangle^T \\ &= \begin{bmatrix} \langle|\Delta\alpha_+|^2\rangle & \langle\Delta\alpha_+\Delta\alpha_-^\dagger\rangle \\ \langle\Delta\alpha_-\Delta\alpha_+^\dagger\rangle & \langle|\Delta\alpha_-|^2\rangle \end{bmatrix} = Q^\dagger \end{aligned} \quad (3A.10b)$$

[cf. Eqs. (2A.4)].

The two-mode analogs of the three real second-order noise moments of  $\hat{x}$  and  $\hat{p}$  are the three real, two-dimensional covariance matrices  $S_x$ ,  $S_p$ , and  $S_{xp}$ :

$$S_x \equiv \langle\Delta\tilde{x}\Delta\tilde{x}^T\rangle = \begin{bmatrix} \langle(\Delta\hat{x}_+)^2\rangle & \langle\Delta\hat{x}_+\Delta\hat{x}_-\rangle \\ \langle\Delta\hat{x}_+\Delta\hat{x}_-\rangle & \langle(\Delta\hat{x}_-)^2\rangle \end{bmatrix} = S_x^T, \quad (3A.11a)$$

$$S_p \equiv \langle\Delta\tilde{p}\Delta\tilde{p}^T\rangle = \begin{bmatrix} \langle(\Delta\hat{p}_+)^2\rangle & \langle\Delta\hat{p}_+\Delta\hat{p}_-\rangle \\ \langle\Delta\hat{p}_+\Delta\hat{p}_-\rangle & \langle(\Delta\hat{p}_-)^2\rangle \end{bmatrix} = S_p^T, \quad (3A.11b)$$

$$\begin{aligned}
 S_{xp} &\equiv \langle \Delta \hat{x} \Delta \hat{p}^T \rangle_{\text{sym}} \equiv \frac{1}{2} \langle \Delta \hat{x} \Delta \hat{p}^T \rangle + \frac{1}{2} \langle \Delta \hat{p} \Delta \hat{x}^T \rangle^T \\
 &= \begin{bmatrix} \langle \Delta \hat{x}_+ \Delta \hat{p}_+ \rangle_{\text{sym}} & \langle \Delta \hat{x}_+ \Delta \hat{p}_- \rangle \\ \langle \Delta \hat{x}_- \Delta \hat{p}_+ \rangle & \langle \Delta \hat{x}_- \Delta \hat{p}_- \rangle_{\text{sym}} \end{bmatrix}. \quad (3A.11c)
 \end{aligned}$$

The matrices  $S_x$  and  $S_p$  are positive semi-definite -- i.e., their traces and determinants are nonnegative;<sup>47</sup> they are positive-definite, if one excludes eigenstates of  $\hat{x}_+$  and  $\hat{x}_-$  or  $\hat{p}_+$  and  $\hat{p}_-$ . Such states are excluded here, for although they can be viewed in a formal sense as limiting cases of GPS, they are not normalizable, since their wave functions are delta functions. Throughout this paper, therefore,  $S_x$  and  $S_p$  are positive-definite (i.e., nonzero).

The noise matrices  $T$  and  $Q$  are related to the covariance matrices  $S_x$ ,  $S_p$ , and  $S_{xp}$  by

$$T = \frac{1}{2}(S_x - S_p) + \frac{1}{2}i(S_{xp}^T + S_{xp}), \quad (3A.12a)$$

$$Q = \frac{1}{2}(S_x + S_p) + \frac{1}{2}i(S_{xp}^T - S_{xp}) \quad (3A.12b)$$

[Eqs. (3A.1); cf. Eqs. (2A.5)]. These matrix equalities are a compact way of writing the following relations between the second-order noise moments:

$$\langle (\Delta a_{\pm})^2 \rangle = \frac{1}{2}[\langle (\Delta \hat{x}_{\pm})^2 \rangle - \langle (\Delta \hat{p}_{\pm})^2 \rangle + 2i \langle \Delta \hat{x}_{\pm} \Delta \hat{p}_{\pm} \rangle_{\text{sym}}], \quad (3A.13a)$$

$$\langle |\Delta a_{\pm}|^2 \rangle = \frac{1}{2}[\langle (\Delta \hat{x}_{\pm})^2 \rangle + \langle (\Delta \hat{p}_{\pm})^2 \rangle], \quad (3A.13b)$$

$$\langle \Delta a_+ \Delta a_- \rangle = \frac{1}{2}[\langle \Delta \hat{x}_+ \Delta \hat{x}_- \rangle - \langle \Delta \hat{p}_+ \Delta \hat{p}_- \rangle + i(\langle \Delta \hat{x}_- \Delta \hat{p}_+ \rangle + \langle \Delta \hat{x}_+ \Delta \hat{p}_- \rangle)], \quad (3A.13c)$$

$$\langle \Delta a_+ \Delta a_-^\dagger \rangle = \frac{1}{2}[\langle \Delta \hat{x}_+ \Delta \hat{x}_- \rangle + \langle \Delta \hat{p}_+ \Delta \hat{p}_- \rangle + i(\langle \Delta \hat{x}_- \Delta \hat{p}_+ \rangle - \langle \Delta \hat{x}_+ \Delta \hat{p}_- \rangle)]. \quad (3A.13d)$$

The total noise of a two-mode GPS is

$$\langle |\Delta a_+|^2 \rangle + \langle |\Delta a_-|^2 \rangle = \text{Tr} Q = \frac{1}{2} \text{Tr}(S_x + S_p). \quad (3A.14)$$

The analog for two-mode states of the uncertainty principles (2A.6) is a matrix relation between the noise matrices  $S_x$ ,  $S_p$ , and  $S_{xp}$ , or, equivalently, between  $Q$  and  $T$ .<sup>49</sup> This relation says that the real matrix

$$S_x S_p - \frac{1}{4}1 - S_{xp}^2 \quad (3A.15a)$$

and the Hermitian matrix

$$Q^2 - \frac{1}{4}1 - T T^* \quad (3A.15b)$$

are both positive semi-definite (psd), and they vanish identically if and only if the state is an eigenstate of two independent linear combinations of  $\hat{x}_+$ ,  $\hat{x}_-$ ,  $\hat{p}_+$ , and  $\hat{p}_-$  (or  $a_+$ ,  $a_-$ ,  $a_+^\dagger$ , and  $a_-^\dagger$ ) -- i.e., if and only if the state is a two-mode Gaussian pure state (see Section IIIB). Note that the diagonal elements of the matrices (3A.15a) and (3A.15b) are identical. A psd two-dimensional Hermitian matrix must have both diagonal elements nonnegative, and it is equal to the null matrix if and only if both diagonal elements vanish. This implies the following two (equivalent) sets of uncertainty principles:

$$\langle |\Delta a_\pm|^2 \rangle^2 \geq \frac{1}{4} + |\langle (\Delta a_\pm)^2 \rangle|^2 + |\langle \Delta a_+ \Delta a_- \rangle|^2 - |\langle \Delta a_+ \Delta a_-^\dagger \rangle|^2, \quad (3A.16a)$$

$$\begin{aligned} \langle (\Delta \hat{x}_\pm)^2 \rangle \langle (\Delta \hat{p}_\pm)^2 \rangle &\geq \frac{1}{4} + \langle \Delta \hat{x}_\pm \Delta \hat{p}_\pm \rangle_{\text{sym}}^2 + \langle \Delta \hat{x}_+ \Delta \hat{p}_- \rangle \langle \Delta \hat{x}_- \Delta \hat{p}_+ \rangle \\ &\quad - \langle \Delta \hat{x}_+ \Delta \hat{x}_- \rangle \langle \Delta \hat{p}_+ \Delta \hat{p}_- \rangle. \end{aligned} \quad (3A.16b)$$

Equations (3A.16a) and (3A.16b) each represent a pair of uncertainty principles, one for the upper ("+") sign, and one for the lower ("-") sign; the two sets (not the members of each pair) are equivalent. Equalities hold in these expressions if and only if the matrices (3A.15) vanish, i.e., if and only if the state is a two-mode Gaussian pure state. If the two modes are uncorrelated

with each other, these expressions reduce to the single-mode uncertainty principles (2A.6) for each mode.

## 2. Rotation Operators

Associated with each of the two modes is a single-mode rotation operator,

$$R_{\pm}(\theta) \equiv e^{-i\theta a_{\pm}^{\dagger} a_{\pm}}, \quad (3A.17)$$

whose properties were discussed in Section IIA.2. It is useful to define another pair of commuting unitary operators,  $U_R(\theta)$  and  $U_M(\theta)$ , equivalent to  $R_+(\theta)$  and  $R_-(\theta)$ , as follows:

$$U_R(\theta) \equiv R_+(\theta) R_-(\theta) = e^{-i\theta \tilde{a}^{\dagger} \tilde{a}}; \quad (3A.18a)$$

$$U_M(\theta) \equiv R_+(\theta) R_-^{\dagger}(\theta) = e^{-i\theta \tilde{a}^{\dagger} \sigma_3 \tilde{a}}. \quad (3A.18b)$$

For notational convenience denote the general product of two single-mode rotation operators (angles  $\theta_+$ ,  $\theta_-$ ) by the symbol  $\mathbf{R}(\theta)$ , and define angles  $\theta_s$  and  $\theta_d$  by

$$\mathbf{R}(\theta) \equiv R_+(\theta_+) R_-(\theta_-) = U_R(\theta_s) U_M(\theta_d) = \exp[-i \tilde{a}^{\dagger} \underline{\theta} \tilde{a}], \quad (3A.19a)$$

$$\underline{\theta} \equiv \theta_s \mathbf{1} + \theta_d \sigma_3 = \begin{pmatrix} \theta_+ & 0 \\ 0 & \theta_- \end{pmatrix}, \quad (3A.19b)$$

$$\theta_s \equiv \frac{1}{2}(\theta_+ + \theta_-), \quad \theta_d \equiv \frac{1}{2}(\theta_+ - \theta_-). \quad (3A.19c)$$

For two oscillators characterized by frequencies  $\Omega \pm \varepsilon$ ,  $U_R(\Omega t) U_M(\varepsilon t)$  is the evolution operator associated with the free Hamiltonian  $H_0^{(2)}$ ,<sup>50</sup>

$$H_0^{(2)} = \Omega \tilde{a}^{\dagger} \tilde{a} + \varepsilon \tilde{a}^{\dagger} \sigma_3 \tilde{a}, \quad (3A.20a)$$

$$e^{-iH_0^{(2)}t} = U_R(\Omega t) U_M(\varepsilon t) . \quad (3A.20b)$$

The operators  $U_R(\theta)$  and  $U_M(\theta)$  acting on any (two-mode) number eigenstate  $|\mathbf{n}_+, \mathbf{n}_-\rangle$  multiply it by the phase factors  $e^{-i(\mathbf{n}_+ + \mathbf{n}_-)\theta}$  and  $e^{-i(\mathbf{n}_+ - \mathbf{n}_-)\theta}$ , respectively; in particular, they leave the vacuum state unchanged:

$$U_R(\theta) |0\rangle = |0\rangle , \quad U_M(\theta) |0\rangle = |0\rangle . \quad (3A.21)$$

A unitary transformation generated by  $U_R(\theta)$  produces a common phase change of the annihilation operators -- i.e., it transforms  $\underline{a}$  into  $e^{i\theta} \underline{a}$ , and rotates  $\underline{\hat{x}}$  and  $\underline{\hat{p}}$  into each other:

$$U_R(\theta) \underline{a} U_R^\dagger(\theta) = e^{i\theta} \underline{a} = \begin{bmatrix} \underline{a}_+(\theta) \\ \underline{a}_-(\theta) \end{bmatrix} , \quad (3A.22a)$$

$$U_R(\theta) \underline{\hat{x}} U_R^\dagger(\theta) = \underline{\hat{x}} \cos\theta - \underline{\hat{p}} \sin\theta = \begin{bmatrix} \underline{\hat{x}}_+(\theta) \\ \underline{\hat{x}}_-(\theta) \end{bmatrix} , \quad (3A.22b)$$

$$U_R(\theta) \underline{\hat{p}} U_R^\dagger(\theta) = \underline{\hat{x}} \sin\theta + \underline{\hat{p}} \cos\theta = \begin{bmatrix} \underline{\hat{p}}_+(\theta) \\ \underline{\hat{p}}_-(\theta) \end{bmatrix} \quad (3A.22c)$$

[Eqs. (2A.10)]. The notation here means that each component of the column vector undergoes the (same) unitary transformation. A unitary transformation generated by  $U_M(\theta)$  produces an opposite phase change of the annihilation operators:

$$U_M(\theta) \underline{a} U_M^\dagger(\theta) = e^{i\theta\sigma_3} \underline{a} = \begin{bmatrix} \underline{a}_+(\theta) \\ \underline{a}_(-\theta) \end{bmatrix} , \quad (3A.23a)$$

$$U_M(\theta) \underline{\hat{x}} U_M^\dagger(\theta) = \underline{\hat{x}} \cos\theta - \sigma_3 \underline{\hat{p}} \sin\theta = \begin{bmatrix} \underline{\hat{x}}_+(\theta) \\ \underline{\hat{x}}_(-\theta) \end{bmatrix} , \quad (3A.23b)$$

$$U_M(\theta) \underline{\hat{p}} U_M^\dagger(\theta) = \sigma_3 \underline{\hat{x}} \sin\theta + \underline{\hat{p}} \cos\theta = \begin{bmatrix} \underline{\hat{p}}_+(\theta) \\ \underline{\hat{p}}_(-\theta) \end{bmatrix} . \quad (3A.23c)$$

The product of two single-mode rotation operators therefore unitarily transforms  $\underline{a}$  in the following way:

$$\mathbf{R}(\underline{\theta}) \underline{a} \mathbf{R}(\underline{\theta})^\dagger = \begin{pmatrix} \underline{a}_+(\theta_+) \\ \underline{a}_-(\theta_-) \end{pmatrix} = e^{i\theta_+} e^{i\theta_- \sigma_3} \underline{a} = e^{i\underline{\theta}} \underline{a} \equiv \underline{a}(\underline{\theta}) ; \quad (3A.24a)$$

$$\mathbf{R}(\underline{\theta}) \hat{x} \mathbf{R}(\underline{\theta})^\dagger = \begin{pmatrix} \hat{x}_+(\theta_+) \\ \hat{x}_-(\theta_-) \end{pmatrix} \equiv \hat{x}(\underline{\theta}) ; \quad (3A.24b)$$

$$\mathbf{R}(\underline{\theta}) \hat{p} \mathbf{R}(\underline{\theta})^\dagger = \begin{pmatrix} \hat{p}_+(\theta_+) \\ \hat{p}_-(\theta_-) \end{pmatrix} \equiv \hat{p}(\underline{\theta}) \quad (3A.24c)$$

[cf. Eqs. (2A.10)].

Both  $U_R(\theta)$  and  $U_M(\theta)$  preserve the total number of photons in each mode separately, i.e., they preserve both the sum and the difference of the number of photons:

$$\langle U_R^\dagger(\theta) \underline{a}^\dagger \underline{a} U_R(\theta) \rangle = \langle \underline{a}^\dagger \underline{a} \rangle , \quad \langle U_M^\dagger(\theta) \underline{a}^\dagger \underline{a} U_M(\theta) \rangle = \langle \underline{a}^\dagger \underline{a} \rangle ; \quad (3A.25a)$$

$$\langle U_R^\dagger(\theta) \underline{a}^\dagger \sigma_3 \underline{a} U_R(\theta) \rangle = \langle \underline{a}^\dagger \sigma_3 \underline{a} \rangle , \quad \langle U_M^\dagger(\theta) \underline{a}^\dagger \sigma_3 \underline{a} U_M(\theta) \rangle = \langle \underline{a}^\dagger \sigma_3 \underline{a} \rangle .$$

(3A.25b)

They therefore also preserve the total noise of each mode separately. This is seen by replacing  $\underline{a}^\dagger \underline{a}$  in Eq. (3A.25a) by the operator for the sum of the total noises,

$$(\Delta \underline{a}^\dagger \Delta \underline{a})_{\text{sym}} \equiv \frac{1}{2} (\Delta \underline{a}^\dagger \Delta \underline{a} + \Delta \underline{a}^T \Delta \underline{a}^*) = |\Delta \underline{a}_+|^2 + |\Delta \underline{a}_-|^2 , \quad (3A.26)$$

and replacing  $\underline{a}^\dagger \sigma_3 \underline{a}$  in Eq. (3A.25b) by the operator for the difference in the total noises of the two modes,

$$\Delta \underline{a}^\dagger \sigma_3 \Delta \underline{a} = \Delta \underline{a}^T \sigma_3 \Delta \underline{a}^* = |\Delta \underline{a}_+|^2 - |\Delta \underline{a}_-|^2 . \quad (3A.27)$$



[Recall that the operators  $\Delta \mathbf{a}_\pm$  are defined only with reference to a particular state, which defines  $\langle \mathbf{a}_\pm \rangle$ ; see comment after Eq. (2A.12b).] The noise matrices  $T$  and  $Q$  for a state  $|\Psi\rangle$  are related to those of the rotated state  $\mathbf{R}(\theta)|\Psi\rangle$  in the following ways:

$$\begin{aligned} \langle \mathbf{R}^\dagger(\theta) \Delta \mathbf{a} \Delta \mathbf{a}^T \mathbf{R}(\theta) \rangle &= \langle \Delta \mathbf{a}(-\theta) \Delta \mathbf{a}^T(-\theta) \rangle = e^{-i\boldsymbol{\theta}} T e^{-i\boldsymbol{\theta}} \\ &= \begin{pmatrix} e^{-2i\theta_+} \langle (\Delta \mathbf{a}_+)^2 \rangle & e^{-2i\theta_s} \langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_- \rangle \\ e^{-2i\theta_s} \langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_- \rangle & e^{-2i\theta_-} \langle (\Delta \mathbf{a}_-)^2 \rangle \end{pmatrix}; \end{aligned} \quad (3A.28a)$$

$$\begin{aligned} \langle \mathbf{R}^\dagger(\theta) (\Delta \mathbf{a} \Delta \mathbf{a}^\dagger)_{\text{sym}} \mathbf{R}(\theta) \rangle &= \langle \Delta \mathbf{a}(-\theta) \Delta \mathbf{a}^\dagger(-\theta) \rangle_{\text{sym}} = e^{-i\boldsymbol{\theta}} Q e^{i\boldsymbol{\theta}} \\ &= \begin{pmatrix} \langle |\Delta \mathbf{a}_+|^2 \rangle & e^{-2i\theta_d} \langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_-^\dagger \rangle \\ e^{2i\theta_d} \langle \Delta \mathbf{a}_- \Delta \mathbf{a}_+^\dagger \rangle & \langle |\Delta \mathbf{a}_-|^2 \rangle \end{pmatrix} \end{aligned} \quad (3A.28b)$$

[Eq. (3A.24a); cf. Eqs. (2A.12)].

### 3. Two-mode Displacement Operator

The two-mode displacement operator<sup>10,11</sup> is simply a product of two single-mode displacement operators,

$$\begin{aligned} D(\underline{\mathbf{a}}, \underline{\boldsymbol{\mu}}) &\equiv D(\mathbf{a}_+, \boldsymbol{\mu}_+) D(\mathbf{a}_-, \boldsymbol{\mu}_-) = \exp[\underline{\mathbf{a}}^\dagger \underline{\boldsymbol{\mu}} - \underline{\boldsymbol{\mu}}^\dagger \underline{\mathbf{a}}] \\ &= \exp[i(\underline{\mathbf{p}}_0^T \underline{\hat{\mathbf{x}}} - \underline{\mathbf{x}}_0^T \underline{\hat{\mathbf{p}}})] = e^{-\frac{1}{2}i \underline{\mathbf{p}}_0^T \underline{\mathbf{x}}_0} \exp[i \underline{\mathbf{p}}_0^T \underline{\hat{\mathbf{x}}} - i \underline{\mathbf{x}}_0^T \underline{\hat{\mathbf{p}}}] \end{aligned} \quad (3A.29)$$

[cf. Eq. (2A.14)]. It satisfies the following equalities:

$$D^{-1}(\underline{\mathbf{a}}, \underline{\boldsymbol{\mu}}) = D^\dagger(\underline{\mathbf{a}}, \underline{\boldsymbol{\mu}}) = D(\underline{\mathbf{a}}, -\underline{\boldsymbol{\mu}}) = D(-\underline{\mathbf{a}}, \underline{\boldsymbol{\mu}}). \quad (3A.30)$$

The properties of  $D(\underline{\mathbf{a}}, \underline{\boldsymbol{\mu}})$  follow directly from those of the single-mode dis-

placement operator  $D(\underline{a}, \underline{\mu})$  (Section IIA.3). Most important is the way it unitarily transforms the annihilation operators for the two modes:

$$D(\underline{a}, \underline{\mu}) \underline{a} D^\dagger(\underline{a}, \underline{\mu}) = \underline{a} - \underline{\mu}. \quad (3A.31)$$

This implies that when the displacement operator acts on a (two-mode) state, it preserves all noise moments of  $\underline{a}_\pm$  and  $\underline{a}_\pm^\dagger$ .

Two other properties of the two-mode displacement operator are useful here. They are the two-mode analogs of the properties (2A.17)-(2A.20). First, it is unitarily transformed by the product of two single-mode rotation operators in the following way:

$$\begin{aligned} \mathbf{R}(\underline{\theta}) D(\underline{a}, \underline{\mu}) \mathbf{R}(\underline{\theta})^\dagger &= D[\underline{a}(\underline{\theta}), \underline{\mu}] = D[\underline{a}, \underline{\mu}(-\underline{\theta})] = D(\underline{a}, e^{-i\underline{\theta}} \underline{\mu}) \\ &= D(\underline{a}_+, e^{-i\theta_+} \mu_+) D(\underline{a}_-, e^{-i\theta_-} \mu_-), \end{aligned} \quad (3A.32a)$$

$$\underline{\mu}(\underline{\theta}) \equiv e^{i\underline{\theta}} \underline{\mu} \equiv e^{i\theta_+} e^{i\theta_- \sigma_3} \underline{\mu} = \begin{pmatrix} \mu_+(\theta_+) \\ \mu_-(\theta_-) \end{pmatrix} = \begin{pmatrix} e^{i\theta_+} \mu_+ \\ e^{i\theta_-} \mu_- \end{pmatrix} \quad (3A.32b)$$

[Eqs. (3A.24a), (3A.29)]. This transformation shows the invariance of the form of the displacement operator under unitary transformations generated by the rotation operator:

$$D(\underline{a}, \underline{\mu}) = D[\underline{a}(\underline{\theta}), \underline{\mu}(\underline{\theta})]. \quad (3A.33)$$

Second, the product of two two-mode displacement operators is another displacement operator, multiplied by a phase factor:

$$D(\underline{a}, \underline{\mu}') D(\underline{a}, \underline{\mu}) = e^{i\text{Im}(\underline{\mu}'^\dagger \underline{\mu})} D(\underline{a}, \underline{\mu} + \underline{\mu}') \quad (3A.34)$$

[cf. Eq. (2A.19)]. These properties, like the transformations (3A.24a) and

(3A.31), show that any eigenstate of  $\mathbf{a}_+$  or  $\mathbf{a}_-$  remains an eigenstate of  $\mathbf{a}_+$  or  $\mathbf{a}_-$  when displaced and/or allowed to evolve freely. A two-mode coherent state [Eq. (1.15)], for example, changes in the following way as it evolves freely:

$$\mathbf{R}(\theta) |\underline{\mu}\rangle_{\text{coh}} = |\underline{\mu}(-\theta)\rangle_{\text{coh}} \equiv |e^{-i\theta_+} \mu_+, e^{-i\theta_-} \mu_-\rangle_{\text{coh}}, \quad (3A.35)$$

where  $\theta_{\pm} \equiv (\Omega \pm \varepsilon) t$  [cf. Eq. (2A.20)].

#### 4. Mixing Operator

The two-mode mixing operator  $T(q, \chi)$  is defined by

$$T(q, \chi) \equiv \exp[q(e^{-2i\chi} \mathbf{a}_-^\dagger \mathbf{a}_+ - e^{2i\chi} \mathbf{a}_+^\dagger \mathbf{a}_-)] = \exp[-i q \underline{\mathbf{a}}^\dagger \sigma_{\chi-\frac{1}{2}\pi} \underline{\mathbf{a}}], \quad (3A.36a)$$

$$0 \leq q \leq \frac{1}{2}\pi, \quad -\frac{1}{2}\pi < \chi \leq \frac{1}{2}\pi \quad (3A.36b)$$

[Eqs. (1.10), (2C.10b)]. It satisfies the following equalities:

$$T^{-1}(q, \chi) = T^\dagger(q, \chi) = T(-q, \chi) = T(q, \chi + \frac{1}{2}\pi). \quad (3A.37)$$

It is called a mixing operator because it unitarily transforms the annihilation operators for the two modes into each other:

$$T(q, \chi) \underline{\mathbf{a}} T^\dagger(q, \chi) = F_{q, \chi} \underline{\mathbf{a}}, \quad (3A.38a)$$

$$F_{q, \chi} \equiv \begin{pmatrix} \cos q & e^{2i\chi} \sin q \\ -e^{-2i\chi} \sin q & \cos q \end{pmatrix} = \cos q \mathbf{1} + i \sin q \sigma_{\chi-\frac{1}{2}\pi} = e^{iq} \sigma_{\chi-\frac{1}{2}\pi}. \quad (3A.38b)$$

[Note that the matrix  $\sigma_{\chi-\frac{1}{2}\pi}$  appears in both  $T(q, \chi)$  and the transformation matrix  $F_{q, \chi}$  because the matrix of commutators  $[\underline{\mathbf{a}}, \underline{\mathbf{a}}^\dagger] = \mathbf{1}$  is the identity matrix, i.e., because  $[\underline{\mathbf{a}}^\dagger K \underline{\mathbf{a}}, \underline{\mathbf{a}}] = -K \underline{\mathbf{a}}$ ; cf. Eqs. (2C.10).] The unitary matrix  $F_{q, \chi}$  has the following important properties:

$$F_{q,\chi}^{-1} = F_{q,\chi}^\dagger = F_{-q,\chi} = F_{q,\chi+\frac{1}{2}\pi} = \sigma_3 F_{q,\chi} \sigma_3, \quad (3A.39a)$$

$$F_{q,0} = e^{iq\sigma_2} = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix}, \quad F_{q,\frac{1}{2}\pi} = e^{iq\sigma_1} = \begin{bmatrix} \cos q & i \sin q \\ i \sin q & \cos q \end{bmatrix}, \quad (3A.39b)$$

$$F_{q,\chi} = e^{i\chi'\sigma_3} F_{q,\chi-\chi'} e^{-i\chi'\sigma_3}, \quad (3A.39c)$$

$$F_{q,\chi} F_{q',\chi} = F_{q+q',\chi}. \quad (3A.39d)$$

The transformation (3A.38a) ensures that states unitarily related to eigenstates of  $\mathbf{a}_+$  and  $\mathbf{a}_-$  by mixing operators are themselves eigenstates of  $\mathbf{a}_+$  and  $\mathbf{a}_-$ . This shows, for example, that the mixing operator, like the rotation operators, leave the vacuum state unchanged:

$$T(q,\chi) |0\rangle = |0\rangle. \quad (3A.40)$$

This can also be seen from the factored forms for  $T(q,\chi)$  given below [Eq. (3A.45)].

The unitarity of  $F_{q,\chi}$  ensures that the mixing operator preserves the total number of photons in the two modes:

$$\langle T^\dagger(q,\chi) \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} T(q,\chi) \rangle = \langle \tilde{\mathbf{a}}^\dagger F_{q,\chi} F_{q,\chi}^\dagger \tilde{\mathbf{a}} \rangle = \langle \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \rangle. \quad (3A.41a)$$

It therefore also preserves the total noise of the two modes:

$$\langle T^\dagger(q,\chi) (\Delta \tilde{\mathbf{a}}^\dagger \Delta \tilde{\mathbf{a}})_{\text{sym}} T(q,\chi) \rangle = \langle \Delta \tilde{\mathbf{a}}^\dagger \Delta \tilde{\mathbf{a}} \rangle_{\text{sym}} = \langle |\Delta \mathbf{a}_+|^2 \rangle + \langle |\Delta \mathbf{a}_-|^2 \rangle. \quad (3A.41b)$$

The noise matrices  $T$  and  $Q$  for a state  $|\Psi\rangle$  are related to those of the transformed state  $T(q,\chi) |\Psi\rangle$  in the following ways:

$$\langle T^\dagger(q,\chi) \Delta \tilde{\mathbf{a}} \Delta \tilde{\mathbf{a}}^T T(q,\chi) \rangle = F_{q,\chi}^\dagger T F_{q,\chi}^*, \quad (3A.42a)$$

$$\langle T^\dagger(q,\chi) (\Delta \tilde{\mathbf{a}} \Delta \tilde{\mathbf{a}}^\dagger)_{\text{sym}} T(q,\chi) \rangle = F_{q,\chi}^\dagger Q F_{q,\chi} \quad (3A.42b)$$

[Eqs. (3A.38)]. Equation (3A.42b) shows that for states whose noise matrix  $Q$  is proportional to the identity matrix  $1$ , the mixing operator also preserves the difference in the noises of the two modes, i.e., it preserves the total noise of each mode separately. Such states include all eigenstates of  $a_+$  and  $a_-$ , all two-mode squeezed states, and all products of two single-mode squeezed states with identical squeeze factors. For the general two-mode GPS (1.18), the total noise of each mode will be separately preserved under a mixing transformation only for a specific  $q$  and  $\chi$ , determined by the condition

$$(\langle |\Delta a_+|^2 \rangle - \langle |\Delta a_-|^2 \rangle) \sin q - 2 \operatorname{Re}(e^{-2i\chi} \langle \Delta a_+ \Delta a_-^\dagger \rangle) \cos q = 0. \quad (3A.43)$$

The mixing operator and the two rotation operators represent all the unitary operators that induce matrix transformations on the column vector  $\underline{a}$ . The transformation matrices associated with them comprise the group  $\tilde{U}(2)$  of two-dimensional, unitary matrices that preserve the identity matrix, i.e., that preserve the commutator matrix  $[\underline{a}, \underline{a}^\dagger] = 1$ . The most general element of this group has the form

$$\begin{aligned} M_U &= e^{i\theta_s} e^{i\theta_d \sigma_3} F_{q,\chi} = e^{i\theta_s} F_{q,\chi+\theta_d} e^{i\theta_d \sigma_3} \\ &= e^{i\theta_s} \begin{pmatrix} e^{i\theta_d} \cos q & e^{i(2\chi+\theta_d)} \sin q \\ -e^{-i(2\chi+\theta_d)} \sin q & e^{-i\theta_d} \cos q \end{pmatrix} \end{aligned} \quad (3A.44a)$$

for some real numbers  $\theta_s$ ,  $\theta_d$ ,  $q$ , and  $\chi$ . It is the transformation matrix that results from a unitary transformation of  $\underline{a}$  by a product of the two rotation operators and a mixing operator, i.e.,

$$M_U \underline{a} = T(q, \chi) \mathbf{R}(\vartheta) \underline{a} \mathbf{R}^\dagger(\vartheta) T^\dagger(q, \chi). \quad (3A.44b)$$

The unitary matrices associated with the unitary operators  $U_M(\theta_d)$  and  $T(q, \chi)$  form the three-parameter group  $SU(2)$ , i.e., they are the elements of  $U(2)$  with unity determinant. The underlying Lie algebra for these operators is that of the operators  $a_-^\dagger a_+$ ,  $a_+^\dagger a_-$ , and  $(a_-^\dagger a_- - a_+^\dagger a_+)$ .

Properties of the mixing operator can be obtained directly from properties of the matrix  $F_{q, \chi}$ , just as properties of the single-mode squeeze operator are obtained from properties of the transformation matrix  $C_{r, \phi}$ . For example, the mixing operator can be factored into a product of exponentials of the operators  $a_-^\dagger a_+$ ,  $a_+^\dagger a_-$ , and  $(a_-^\dagger a_- - a_+^\dagger a_+)$ , simply by factoring the matrix  $F_{q, \chi}$  into exponentials of matrices (linear combinations of the Pauli matrices) that have the same commutator algebra as those operators. These matrices are  $\sigma_-$ ,  $\sigma_+$ , and  $-\sigma_3$ , respectively, where  $\sigma_\pm \equiv \frac{1}{2}(\sigma_1 \pm i \sigma_2)$ . The mixing operator  $T(q, \chi)$  thus has the following equivalent factored forms:

$$\begin{aligned} T(q, \chi) &= e^{-\Lambda A^\dagger} e^{\Lambda^* e^{2f} A} e^{fB} = e^{\Lambda^* A} e^{-\Lambda e^{2f} A^\dagger} e^{-fB} \\ &= e^{-\Lambda A^\dagger} e^{fB} e^{\Lambda^* A} = e^{fB} e^{-\Lambda e^{2f} A^\dagger} e^{\Lambda^* A} \\ &= e^{-fB} e^{\Lambda^* e^{2f} A} e^{-\Lambda A^\dagger} = e^{\Lambda^* A} e^{-fB} e^{-\Lambda A^\dagger} , \end{aligned}$$

$$\Lambda \equiv e^{2i\chi} \tan q , \quad f \equiv \ln(\cos q) ,$$

$$A \equiv a_-^\dagger a_+ , \quad B \equiv a_-^\dagger a_- - a_+^\dagger a_+ . \quad (3A.45)$$

These factored forms show explicitly that the mixing operator leaves the vacuum state unchanged [Eq. (3A.40)]. The matrix equality

$$F_{q, \chi} F_{q', \chi'}^\dagger = F_{\zeta, \eta} e^{i\theta \sigma_3} = e^{i\theta \sigma_3} F_{\zeta, \eta - \theta} \quad (3A.46a)$$

shows that the product of two different mixing operators is another mixing

operator, multiplied by a rotation operator:

$$T^\dagger(q', \chi') T(q, \chi) = U_M(\Theta) T(\zeta, \eta) = T(\zeta, \eta - \Theta) U_M(\Theta); \quad (3A.46b)$$

here the real numbers  $\zeta$ ,  $\eta$ , and  $\Theta$  are related to  $q$ ,  $q'$ ,  $\chi$ , and  $\chi'$  by

$$e^{i\Theta} \cos \zeta = \cos q \cos q' + e^{2i(\chi - \chi')} \sin q \sin q', \quad (3A.46c)$$

$$e^{-i(\Theta - 2\eta)} \sin \zeta = e^{2i\chi} \sin q \cos q' - e^{2i\chi'} \sin q' \cos q. \quad (3A.46d)$$

For the special case  $\chi = \chi'$  this gives the simple relation

$$T(q, \chi) T(q', \chi) = T(q + q', \chi) \quad (3A.46e)$$

[Eq. (3A.39d)].

The property (3A.39c) of the transformation matrix  $F_{q, \chi}$  shows that the mixing operator  $T(q, \chi)$  is unitarily transformed by the rotation operators in the following way:

$$\mathbf{R}(\theta) T(q, \chi) \mathbf{R}^\dagger(\theta) = T(q, \chi - \theta_d); \quad (3A.47)$$

That  $T(q, \chi)$  commutes with  $\mathbf{U}_R(\theta)$  but not with  $\mathbf{U}_M(\theta)$  reveals why it preserves the total number of photons (hence the total noise), but not the difference in the number of photons in the two modes. The mixing operator unitarily transforms the two-mode displacement operator in the following way:

$$T^\dagger(q, \chi) D(\underline{\alpha}, \underline{\mu}) T(q, \chi) = D(F_{q, \chi}^\dagger \underline{\alpha}, \underline{\mu}) = D(\underline{\alpha}, F_{q, \chi} \underline{\mu}). \quad (3A.48a)$$

This shows that the form of the (two-mode) displacement operator is invariant under unitary transformations of  $\underline{\alpha}$  generated by a mixing operator:

$$D(\underline{\alpha}, \underline{\mu}) = D(F_{q, \chi} \underline{\alpha}, F_{q, \chi} \underline{\mu}). \quad (3A.48b)$$

These transformations, together with the properties described above, enable one to verify the statement made in the Introduction: a state unitarily related to a two-mode coherent state by any product of rotation, displacement, and mixing operators is equal to another coherent state (multiplied by an unobservable phase factor). For example,

$$\mathbf{R}(\theta) D(\alpha, \mu') T(q, \chi) |\mu\rangle_{\text{coh}} = e^{i\text{Im}(\mu^\dagger F_{q,\chi} \mu)} |\bar{\mu}\rangle_{\text{coh}},$$

$$\bar{\mu} \equiv \mu'(-\theta) + F_{q,\chi-\theta} \mu(-\theta). \quad (3A.49)$$

All states that are eigenstates of both  $\alpha_+$  and  $\alpha_-$  are unitarily related to the vacuum state by products of rotation, displacement, and mixing operators. Conversely, all such states are eigenstates of  $\alpha_+$  and  $\alpha_-$ . These states comprise the entire class of (two-mode) states whose total noise is equal to that of the vacuum state. The special properties of the rotation and mixing operators -- that they preserve the total number of photons, that they preserve the total noise, and that they preserve coherent states -- are a consequence of one essential property: these operators generate the most general unitary matrix transformation of the annihilation operators and nothing more; i.e., they never mix creation operators with annihilation operators. To find unitary operators that do not conserve the total noise and that generate new states from coherent states (states with a total noise greater than that of the vacuum state), one must consider operators -- the single-mode and two-mode squeeze operators -- that mix creation and annihilation operators.



## 5. Squeeze operators for two modes

The single-mode squeeze operators

$$S_{1\pm}(\mathbf{r}, \varphi) \equiv \exp[\frac{1}{2}\mathbf{r} (e^{-2i\varphi} \mathbf{a}_{\pm}^2 - e^{2i\varphi} \mathbf{a}_{\pm}^{\dagger 2})] \quad (3A.50)$$

were discussed in Section IIA.4. Each single-mode squeeze operator unitarily transforms the annihilation operator for its mode into a linear combination of the annihilation and creation operator for that mode:

$$S_{1\pm}(\mathbf{r}, \varphi) \mathbf{a}_{\pm} S_{1\pm}^{\dagger}(\mathbf{r}, \varphi) = \mathbf{a}_{\pm} \cosh r + \mathbf{a}_{\pm}^{\dagger} e^{2i\varphi} \sinh r \quad (3A.51)$$

[Eqs. (2A.21)-(2A.23)]. In vector notation these transformations are

$$S_{1+}(\mathbf{r}_+, \varphi_+) S_{1-}(\mathbf{r}_-, \varphi_-) \underline{\mathbf{a}} S_{1-}^{\dagger}(\mathbf{r}_-, \varphi_-) S_{1+}^{\dagger}(\mathbf{r}_+, \varphi_+) = P_{1c} \underline{\mathbf{a}} + P_{1s} \underline{\mathbf{a}}^*, \quad (3A.52a)$$

$$P_{1c} \equiv \begin{pmatrix} \cosh r_+ & 0 \\ 0 & \cosh r_- \end{pmatrix}, \quad P_{1s} \equiv \begin{pmatrix} e^{2i\varphi_+} \sinh r_+ & 0 \\ 0 & e^{2i\varphi_-} \sinh r_- \end{pmatrix}. \quad (3A.52b)$$

Recall that a single-mode squeeze operator preserves neither the total number of photons nor the total noise of a mode [Eqs. (2A.24)].

The two-mode squeeze operator  $S(\mathbf{r}, \varphi)$  is defined by

$$\begin{aligned} S(\mathbf{r}, \varphi) &\equiv \exp[\mathbf{r} (e^{-2i\varphi} \mathbf{a}_+ \mathbf{a}_- - e^{2i\varphi} \mathbf{a}_+^{\dagger} \mathbf{a}_-^{\dagger})] \\ &= \exp[\frac{1}{2}\mathbf{r} (e^{-2i\varphi} \underline{\mathbf{a}}^T \sigma_1 \underline{\mathbf{a}} - e^{2i\varphi} \underline{\mathbf{a}}^{\dagger} \sigma_1 \underline{\mathbf{a}}^*)], \end{aligned} \quad (3A.53a)$$

$$0 \leq r < \infty, \quad -\frac{1}{2}\pi < \varphi \leq \frac{1}{2}\pi \quad (3A.53b)$$

[Eq. (1.13)]. It satisfies the following equalities:

$$S^{-1}(\mathbf{r}, \varphi) = S^{\dagger}(\mathbf{r}, \varphi) = S(-\mathbf{r}, \varphi) = S(\mathbf{r}, \varphi + \frac{1}{2}\pi). \quad (3A.54)$$

Properties of  $S(\mathbf{r}, \varphi)$  are discussed in Refs. 25 and 26. Most important is

that it unitarily transforms  $\mathbf{a}_\pm$  into a linear combination of  $\mathbf{a}_\pm$  and  $\mathbf{a}_\mp^\dagger$ :

$$S(\mathbf{r}, \varphi) \mathbf{a}_\pm S^\dagger(\mathbf{r}, \varphi) = \mathbf{a}_\pm \cosh r + \mathbf{a}_\mp^\dagger e^{2i\varphi} \sinh r \equiv \underline{\alpha}_\pm(\mathbf{r}, \varphi) \quad (3A.55)$$

[Eq. (1.20)]. In vector notation, these transformations take the form

$$S(\mathbf{r}, \varphi) \underline{\alpha} S^\dagger(\mathbf{r}, \varphi) = \underline{\alpha} \cosh r + \sigma_1 \underline{\alpha}^* e^{2i\varphi} \sinh r \equiv \underline{\alpha}(\mathbf{r}, \varphi) . \quad (3A.56a)$$

Inverting this relation gives the vector expression for  $\underline{\alpha}$  in terms of  $\underline{\alpha}(\mathbf{r}, \varphi)$  and  $\underline{\alpha}^*(\mathbf{r}, \varphi)$ :

$$\underline{\alpha} = S^\dagger(\mathbf{r}, \varphi) \underline{\alpha}(\mathbf{r}, \varphi) S(\mathbf{r}, \varphi) = \underline{\alpha}(\mathbf{r}, \varphi) \cosh r + \sigma_1 \underline{\alpha}^*(\mathbf{r}, \varphi) e^{2i\varphi} \sinh r . \quad (3A.56b)$$

A state unitarily related to an eigenstate of  $\mathbf{a}_+$  and  $\mathbf{a}_-$  by a two-mode squeeze operator is an eigenstate of two "two-mode squeezed annihilation operators"<sup>24-26</sup>  $\underline{\alpha}_\pm(\mathbf{r}, \varphi)$  (sometimes denoted simply by  $\alpha_\pm$ , or by the column vector  $\underline{\alpha}$ ). The unitarity of  $S(\mathbf{r}, \varphi)$  ensures that the commutator algebra of  $\alpha_\pm$  and  $\alpha_\pm^\dagger$  is identical to that of  $\mathbf{a}_\pm$  and  $\mathbf{a}_\pm^\dagger$ :

$$[\underline{\alpha}(\mathbf{r}, \varphi), \underline{\alpha}^\dagger(\mathbf{r}, \varphi)] = [\underline{\alpha}, \underline{\alpha}^\dagger] = 1 , \quad (3A.57a)$$

$$[\underline{\alpha}(\mathbf{r}, \varphi), \underline{\alpha}^T(\mathbf{r}, \varphi)] = [\alpha_+, \alpha_-] i \sigma_2 = [\underline{\alpha}, \underline{\alpha}^T] = 0 \quad (3A.57b)$$

[cf. Eqs. (3A.8)].

The transformations (3A.55) or (3A.56) imply that the two-mode squeeze operator preserves neither the total number of photons nor the total noise of a (two-mode) state:

$$\langle S^\dagger(\mathbf{r}, \varphi) \underline{\alpha}^\dagger \underline{\alpha} S(\mathbf{r}, \varphi) \rangle = \sinh^2 r + \cosh 2r \langle \underline{\alpha}^\dagger \underline{\alpha} \rangle - 2 \sinh 2r \operatorname{Re}(e^{-2i\varphi} \langle \mathbf{a}_+ \mathbf{a}_- \rangle) , \quad (3A.58a)$$

$$\langle S^\dagger(\mathbf{r}, \varphi) (\Delta \tilde{\mathbf{a}}^\dagger \Delta \tilde{\mathbf{a}})_{\text{sym}} S(\mathbf{r}, \varphi) \rangle = \cosh 2r \langle \Delta \tilde{\mathbf{a}}^\dagger \Delta \tilde{\mathbf{a}} \rangle_{\text{sym}} - 2 \sinh 2r \operatorname{Re}(e^{-2i\varphi} \langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_- \rangle)$$

(3A.58b)

[cf. Eqs. (2A.24)]. Equation (3A.58b) shows explicitly that any (two-mode) state whose unitary relation to the vacuum state (or to any eigenstate of  $\mathbf{a}_+$  and  $\mathbf{a}_-$ ) involves a two-mode squeeze operator has a total noise greater than that of the vacuum state. The two-mode squeeze operator does, however, preserve the difference in the number of photons, and therefore also the difference in the total noises, of the two modes:

$$\langle S^\dagger(\mathbf{r}, \varphi) \tilde{\mathbf{a}}^\dagger \sigma_3 \tilde{\mathbf{a}} S(\mathbf{r}, \varphi) \rangle = \langle \tilde{\mathbf{a}}^\dagger \sigma_3 \tilde{\mathbf{a}} \rangle, \quad (3A.59a)$$

$$\langle S^\dagger(\mathbf{r}, \varphi) \Delta \tilde{\mathbf{a}}^\dagger \sigma_3 \Delta \tilde{\mathbf{a}} S(\mathbf{r}, \varphi) \rangle = \langle \Delta \tilde{\mathbf{a}}^\dagger \sigma_3 \Delta \tilde{\mathbf{a}} \rangle. \quad (3A.59b)$$

The relations between the noise matrices  $T$  and  $Q$  for a state  $|\Psi\rangle$  and for the state  $S(\mathbf{r}, \varphi) |\Psi\rangle$  follow from the transformation (3A.56). The matrices  $T$  and  $Q$  for a TMSS are given explicitly below (see part 7 of this section).

The two-mode squeeze operator and the two rotation operators represent all the unitary operators that induce matrix transformations on another two-component column vector,

$$\tilde{\mathbf{a}}' \equiv \begin{pmatrix} \mathbf{a}_+ \\ \mathbf{a}_-^\dagger \end{pmatrix}. \quad (3A.60)$$

The transformation matrices associated with them comprise the group  $U(1,1)$  of two-dimensional, complex matrices that preserve the metric  $\sigma_3$ , i.e., that preserve the commutator matrix  $[\tilde{\mathbf{a}}', \tilde{\mathbf{a}}'^\dagger] = \sigma_3$ . The elements of this group have the general form

$$M = e^{i\theta_d} e^{i\theta_s \sigma_3} C_{\mathbf{r}, \varphi} = e^{i\theta_d} C_{\mathbf{r}, \varphi + \theta_s} e^{-i\theta_s \sigma_3}$$

$$= e^{i\theta_d} \begin{pmatrix} e^{i\theta_s} \cosh r & e^{i(2\varphi+\theta_s)} \sinh r \\ e^{-i(2\varphi+\theta_s)} \sinh r & e^{-i\theta_s} \cosh r \end{pmatrix}, \quad (3A.61a)$$

for some real numbers  $\theta_d$ ,  $\theta_s$ ,  $r$ , and  $\varphi$  [cf. Eq. (2C.15g)]. It is the transformation matrix that results from a unitary transformation of the column vector  $\underline{\alpha}'$  by a product of two rotation operators and a two-mode squeeze operator, i.e.,

$$M \underline{\alpha}' = S(r, \varphi) R(\theta) \underline{\alpha}' R^\dagger(\theta) S^\dagger(r, \varphi) \quad (3A.61b)$$

[Eqs. (3A.24) and (3A.55)]. The transformation matrices associated with unitary transformations on  $\underline{\alpha}'$  by the rotation operator  $U_R(\theta_s)$  and the two-mode squeeze operator  $S(r, \varphi)$  form the three-parameter group  $SU(1,1)$ , the elements of  $U(1,1)$  with unity determinant. They are the same transformation matrices that arise in connection with the single-mode two-component column vector defined in Section IIC; there they describe unitary transformations induced by the single-mode rotation and squeeze operators. This similarity is the formal reason that two-mode squeezed states are the natural analog of single-mode squeezed states. The underlying group structure of the unitary operators associated with the most general two-mode GPS is considerably more complicated than the three-parameter group  $SU(1,1)$  (see Section IIIC). The group structure of the unitary operators associated with two-mode squeezed states, however, is  $SU(1,1)$ . This is a consequence of the fact that two-mode squeezed states have special noise properties (TSQP noise; see the discussion in Section I.g).

The underlying Lie algebra associated with the unitary operators  $U_R(\theta_s)$  and  $S(r, \varphi)$  is that of the operators  $\mathbf{a}_+ \mathbf{a}_-$ ,  $\mathbf{a}_-^\dagger \mathbf{a}_+^\dagger$ , and  $\mathbf{a}_+^\dagger \mathbf{a}_+ + \mathbf{a}_-^\dagger \mathbf{a}_- + 1$ . It is the same Lie algebra as that associated with the single-mode rotation operator  $R(\theta)$  and squeeze operator  $S_1(r, \varphi)$ , i.e., it is that of the operators

$\frac{1}{2}a^2$ ,  $\frac{1}{2}a^{\dagger 2}$ , and  $a^\dagger a + \frac{1}{2}$ . Hence all the results for single-mode GPS can be converted to analogous results for two-mode squeezed states simply by replacing these single-mode operators with the corresponding two-mode operators. This immediately enables one, for example, to write factored expressions for the two-mode squeeze operator [Eq. (2C.17)], and to write the product of two different two-mode squeeze operators as the product of a two-mode squeeze operator and a rotation operator  $U_R(\theta)$  [Eqs. (2C.18)]. States with TSQP noise can be described completely by a two-component vector notation, in which the fundamental operator column vector is  $\underline{\alpha}'$  [Eq. (3A.60)] or, equivalently, the column vector formed from the quadrature-phase amplitudes  $\alpha_1$  and  $\alpha_2$ :

$$\underline{\alpha} \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A \lambda \underline{\alpha}', \quad (3A.62a)$$

$$A \equiv 2^{-\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad \lambda \equiv \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \quad \lambda_{\pm} \equiv \left( \frac{\Omega \pm \varepsilon}{\Omega} \right)^{\frac{1}{2}} \quad (3A.62b)$$

[cf. Eqs. (1.21)]. Such a description is motivated and developed in Refs. 24-26. In contrast, more general two-mode GPS require a four-component vector notation, described in Section IIIC of this paper.

The property (2C.14c) of the transformation matrix  $C_{\tau, \varphi}$ , or the relation (3A.24a) together with the definition (3A.53), shows that the two-mode squeeze operator is unitarily transformed by the rotation operators in the following way:

$$\mathbf{R}(\theta) S(\tau, \varphi) \mathbf{R}^\dagger(\theta) = S(\tau, \varphi - \theta_s) \quad (3A.63)$$

[cf. Eqs. (2A.25) or (2C.19b)]. That  $S(\tau, \varphi)$  commutes with  $\mathbf{U}_M(\theta)$  but not with  $\mathbf{U}_R(\theta)$  reveals why it preserves the difference in the number of photons

(hence the difference in the total noises), but not the total number of photons in the two modes [contrast this with the mixing operator, Eq. (3A.47)]. It unitarily transforms the two-mode displacement operator in the following way:

$$S^\dagger(\tau, \varphi) D(\underline{\alpha}, \underline{\mu}) S(\tau, \varphi) = D(\underline{\alpha}, \underline{\mu}_\alpha) , \quad (3A.64)$$

$$\underline{\mu}_\alpha \equiv \underline{\mu} \cosh r + \sigma_1 \underline{\mu}^* e^{2i\varphi} \sinh r \quad (3A.65)$$

[Eqs. (3A.29), (3A.56a)]. This shows that the form of the displacement operator is invariant under unitary transformations of  $\underline{\alpha}$  generated by the two-mode squeeze operator:

$$D(\underline{\alpha}, \underline{\mu}) = D(\underline{\alpha}, \underline{\mu}_\alpha) . \quad (3A.66)$$

This equality implies that the TMSS  $|\underline{\mu}_\alpha\rangle_{(\tau, \varphi)}$ , defined by Eq. (1.19) as  $S(\tau, \varphi)$  acting on the two-mode coherent state  $|\underline{\mu}_\alpha\rangle_{\text{coh}}$ , can as well be defined as the displacement operator  $D(\underline{\alpha}, \underline{\mu})$  acting on the (two-mode) squeezed vacuum:

$$|\underline{\mu}_\alpha\rangle_{(\tau, \varphi)} \equiv S(\tau, \varphi) |\underline{\mu}_\alpha\rangle_{\text{coh}} = D(\underline{\alpha}, \underline{\mu}) S(\tau, \varphi) |0\rangle . \quad (3A.67)$$

The complex numbers  $\mu_+$ ,  $\mu_-$  are the complex amplitudes  $\langle \underline{a}_+ \rangle$ ,  $\langle \underline{a}_- \rangle$ , i.e.,  $\underline{\mu} \equiv \langle \underline{a} \rangle$ . They are related to the eigenvalues  $\mu_{\alpha+}$ ,  $\mu_{\alpha-}$  by

$$\underline{\mu} = \underline{\mu}_\alpha \cosh r - \sigma_1 \underline{\mu}_\alpha^* e^{2i\varphi} \sinh r \quad (3A.68)$$

[Eq. (3A.56b); cf. Eqs. (2A.36)-(2A.30)]. These transformations, together with the properties described above, enable one to verify the statement made in the Introduction: a state unitarily related to the TMSS  $|\underline{\mu}_\alpha\rangle_{(\tau, \varphi)}$  by a product of rotation and displacement operators is equal to another TMSS (multiplied

by an unobservable phase factor) with the same squeeze factor  $\tau$ , but with different squeeze angle and eigenvalues. For example,

$$\begin{aligned} \mathbf{R}(\theta) D(\underline{\alpha}, \underline{\mu}') |\underline{\mu}_\alpha\rangle_{(\tau, \varphi)} &= e^{i\text{Im}(\underline{\mu}'^\dagger \underline{\mu})} D(\underline{\alpha}, \underline{\mu}) S(\tau, \varphi - \theta_s) |0\rangle \\ &= e^{i\text{Im}(\underline{\mu}'^\dagger \underline{\mu})} |\underline{\mu}_\alpha\rangle_{(\tau, \varphi - \theta_s)}, \end{aligned}$$

$$\underline{\mu} \equiv \underline{\mu}(-\theta) + \underline{\mu}'(-\theta) \quad (3A.69)$$

[Eqs. (3A.32), (3A.34), (3A.65)].

Finally, one should note the formal relation between a product of two single-mode squeeze operators and a two-mode squeeze operator. Physically, these represent very different processes. The product of two single-mode squeeze operators is, roughly speaking, the evolution operator for a process in which two harmonic oscillators, each in a coherent state, are separately squeezed [i.e., each is subjected to a degenerate two-photon interaction Hamiltonian (1.8)]. In contrast, a two-mode squeeze operator is the evolution operator for a process in which two harmonic oscillators, each in a coherent state, are made to become correlated, through the nondegenerate two-photon interaction Hamiltonian (1.7). Although these operators differ profoundly from each other in a physical sense, in a purely formal sense they are (unitarily) equivalent. The unitary operator that transforms them into each other is a mixing operator. That is, by defining certain linear combinations of  $\alpha_+$  and  $\alpha_-$ , call them  $b_+$  and  $b_-$ , one can write the two-mode squeeze operator  $S(\tau, \varphi)$  as a product of two single-mode squeeze operators, one for the " $b_+$ -mode" and one for the " $b_-$ -mode".<sup>13,33,25</sup> This formal equivalence is described by the following general relation:

$$T(\pm\frac{1}{2}\pi, \varphi_d) S_{1+}(\tau_+, \varphi_+) S_{1-}(\tau_-, \varphi_-) T^\dagger(\pm\frac{1}{2}\pi, \varphi_d) = S_{1+}(\tau_s, \varphi_+) S_{1-}(\tau_s, \varphi_-) S(\pm\tau_d, \varphi_s)$$

$$r_s \equiv \frac{1}{2}(r_+ \pm r_-), \quad \varphi_s \equiv \frac{1}{2}(\varphi_+ \pm \varphi_-). \quad (3A.70a)$$

For the special case  $r_s \equiv 0$  ( $r_+ = -r_-$ ), this can be rewritten as

$$T(\frac{1}{2}\pi, \varphi_d + \frac{1}{2}\pi) S_{1+}(r, \varphi_+) S_{1-}(r, \varphi_-) T^\dagger(\frac{1}{2}\pi, \varphi_d + \frac{1}{2}\pi) = S(r, \varphi_s - \frac{1}{2}\pi). \quad (3A.70b)$$

These relations provide another way to see why the natural nondegenerate (two-mode) analog of degenerate (single-mode) squeezing is the squeezing produced by the nondegenerate two-photon interaction Hamiltonian (1.7) -- i.e., by the two-mode squeeze operator  $S(r, \varphi)$ ; it is not the separate squeezing of two single modes by degenerate two-photon interaction Hamiltonians like (1.8). A physical explanation was given in the Introduction: both single-mode and two-mode squeezing can be accomplished in a parametric amplifier by using a single pump; one simply moves away from degeneracy, and the resulting state is a two-mode squeezed state. These relations show in another way why a two-mode squeezed state is not equivalent physically to two single-mode squeezed states. They tell one that in order to produce a two-mode squeezed state by separately squeezing two single modes (or vice-versa), one would have to first turn on a mixing interaction (frequency converter), then separately squeeze the two modes, and then turn on another mixing interaction. Separately squeezing two single modes does not produce a state with the reduced noise properties of a two-mode squeezed state.

## 6. Product of three squeeze operators

Consider now the unitary operator  $U_g$  that relates the most general normalized two-mode GPS to a two-mode coherent state:

$$U_g \equiv S_{1+}(r_+, \varphi_+) S_{1-}(r_-, \varphi_-) S(r, \varphi) \quad (3A.71)$$



[Eq. (1.18)]. Note that the inverse of  $\mathbf{U}_g$  is obtained by changing the signs of  $r_+$ ,  $r_-$ , and  $r$  and reversing the order of the squeeze operators:

$$\mathbf{U}_g^{-1} = \mathbf{U}_g^\dagger = S(-r, \varphi) S_{1-}(-r_-, \varphi_-) S_{1+}(-r_+, \varphi). \quad (3A.72)$$

The operator  $\mathbf{U}_g$  transforms the operator column vector  $\underline{\alpha}$  into a linear combination of  $\underline{\alpha}$  and  $\underline{\alpha}^*$ :

$$\mathbf{U}_g \underline{\alpha} \mathbf{U}_g^\dagger = P_c \underline{\alpha} + P_s \underline{\alpha}^* \equiv \underline{g} \equiv \begin{pmatrix} g_+ \\ g_- \end{pmatrix}; \quad (3A.73a)$$

the complex matrices  $P_c$  and  $P_s$  are equal to

$$P_c \equiv \cosh r P_{1c} + e^{2i\varphi} \sinh r \sigma_1 P_{1s}^*, \quad (3A.73b)$$

$$P_s \equiv \cosh r P_{1s} + e^{2i\varphi} \sinh r \sigma_1 P_{1c}^* \quad (3A.73c)$$

[Eqs. (3A.52) and (3A.56); see also Eq. (3B.35) below]. States unitarily related to eigenstates of  $\alpha_+$  and  $\alpha_-$  by  $\mathbf{U}_g$  are eigenstates of the transformed annihilation operators  $g_+$  and  $g_-$ . The unitarity of  $\mathbf{U}_g$  ensures that the commutator algebra of  $g_\pm$  and  $g_\pm^\dagger$  is identical to that of  $\alpha_\pm$  and  $\alpha_\pm^\dagger$ :

$$[g, g^\dagger] = [\underline{\alpha}, \underline{\alpha}^\dagger] = 1. \quad (3A.74a)$$

$$[g, g^T] = [g_+, g_-] i \sigma_2 = [\underline{\alpha}, \underline{\alpha}^T] = 0 \quad (3A.74b)$$

[cf. Eqs. (3A.9)]. Because of the presence of both single-mode squeeze operators,  $\mathbf{U}_g$  does not, in general, preserve any of the noise moments of  $\alpha_\pm$  or  $\alpha_\pm^\dagger$ . The components of the noise matrices  $T$  and  $Q$  for a coherent state and for a general two-mode GPS  $|\underline{\mu}_g\rangle \equiv \mathbf{U}_g |\underline{\mu}_g\rangle_{\text{coh}}$  are given below [Eqs. (3A.84)].

Many properties of  $U_g$  follow trivially from the corresponding properties of the three squeeze operators (e.g., the way it is unitarily transformed by rotation operators); these are left to the reader. Two properties, however, deserve special attention. First,  $U_g$  unitarily transforms the two-mode displacement operator in the following way:

$$U_g^\dagger D(\underline{a}, \underline{\mu}) U_g = D(\underline{a}, \underline{\mu}_g) , \quad (3A.75)$$

$$\underline{\mu}_g \equiv P_c \underline{\mu} + P_s \underline{\mu}^* \quad (3A.76)$$

[Eqs. (3A.29), (3A.73)]. This relation reflects the fact the form of the two-mode displacement operator is invariant under unitary transformations of  $\underline{a}$  that are linear in  $\underline{a}$  and  $\underline{a}^*$  (and that do not add to  $\underline{a}$  a constant column vector). Such unitary transformations are generated only by (products of) rotation, mixing, and squeeze operators. The invariance under transformations generated by rotation, mixing, and two-mode squeeze operators has already been noted [Eqs. (3A.33), (3A.48), (3A.66)]. The invariance under transformations generated by  $U_g$  says that

$$D(\underline{a}, \underline{\mu}) = D(\underline{g}, \underline{\mu}_g) . \quad (3A.77)$$

This equality implies that the general two-mode GPS  $|\underline{\mu}_g\rangle$ , defined by Eq. (1.18) as the operator  $U_g$  acting on the two-mode coherent state  $|\underline{\mu}_g\rangle_{\text{coh}}$ , can as well be defined as the product of the displacement operator  $D(\underline{a}, \underline{\mu})$  and  $U_g$  acting on the vacuum state:

$$|\underline{\mu}_g\rangle \equiv U_g |\underline{\mu}_g\rangle_{\text{coh}} = D(\underline{a}, \underline{\mu}) U_g |0\rangle . \quad (3A.78)$$

The complex numbers  $\mu_+$ ,  $\mu_-$  are the complex amplitudes  $\langle \underline{a}_+ \rangle$ ,  $\langle \underline{a}_- \rangle$ , i.e.,

$\underline{\mu} \equiv \langle \underline{a} \rangle$ . They are related to the eigenvalues  $\mu_{g+}$ ,  $\mu_{g-}$  by

$$\underline{\mu} = P_c^\dagger \underline{\mu}_g - P_s^T \underline{\mu}_g^* . \quad (3A.79)$$

The second important property to note is that the product of two different operators  $\mathbf{U}_g$  and  $\mathbf{U}_g'$  can always be expressed as another operator  $\bar{\mathbf{U}}_g$  (i.e., another product of the three squeeze operators), multiplied (on the right) by rotation operators and a mixing operator. This property is not necessarily obvious, since it requires knowing that when single-mode and two-mode squeeze operators are commuted through each other, the result can always be expressed as a product of three squeeze operators (i.e., an operator like  $\mathbf{U}_g$ ), multiplied (on the right) by rotation and mixing operators. This fact follows as a special case of the general proof, given in Section IIIC and Appendix A, that the evolution operator associated with any two-mode GPS can always be expressed in this form.

It is proved in the next section, by considering the most general two-mode Gaussian wave function, that the generator  $H_G^{(2)}$  of the unitary operator  $U_G^{(2)} \equiv e^{-iH_G^{(2)}}$  that relates a two-mode GPS to the vacuum state is a sum of linear and bilinear combinations of  $\mathbf{a}_+$ ,  $\mathbf{a}_-$ ,  $\mathbf{a}_+^\dagger$ , and  $\mathbf{a}_-^\dagger$ . In Section IIIC and Appendix A it is shown that this unitary operator factors into a product of displacement, squeeze, mixing, and rotation operators. The properties described in this and the previous sections ensure that any product of rotation, mixing, squeeze and displacement operators can be expressed as the product of a displacement operator and an operator like  $\mathbf{U}_g$ , multiplied on the right by rotation operators and a mixing operator (and an overall phase factor). Since the rotation and mixing operators have no effect on the vacuum state, one finds that the most general two-mode GPS is the state  $|\underline{\mu}_g\rangle$  defined by Eqs. (3A.71) and (3A.78). It is produced when two harmonic

oscillators in their ground states are exposed to the interactions  $H_R^{(2)}(t)$ ,  $H_1^{(2)}(t)$ , and  $H_2^{(2)}(t)$  described in the Introduction [Eqs. (1.3) and (1.4)].

## 7. Two-mode GPS

The noise matrices  $T$  and  $Q$  (or  $S_x$ ,  $S_p$ , and  $S_{xp}$ ) for two-mode coherent states and products of two single-mode squeezed states can be obtained directly from the noise moments of single-mode coherent and squeezed states, given in Section II [Eqs. (2A.37) or (2B.39)]. For a product of two single-mode squeezed states,  $S_{1+}(r_+, \varphi_+) S_{1-}(r_-, \varphi_-) |\underline{\mu}_\alpha\rangle_{\text{coh}}$ , the noise matrices are

$$T = -\frac{1}{2} \begin{pmatrix} e^{2i\varphi_+} \sinh 2r_+ & 0 \\ 0 & e^{2i\varphi_-} \sinh 2r_- \end{pmatrix},$$

$$Q = \frac{1}{2} \begin{pmatrix} \cosh 2r_+ & 0 \\ 0 & \cosh 2r_- \end{pmatrix}. \quad (3A.80)$$

For a two-mode squeezed state the noise matrices  $T$  and  $Q$  are not diagonal, but they have particularly simple forms. This is a consequence of TSQP noise, which says that the noise moments  $\langle(\Delta\alpha_+)^2\rangle$ ,  $\langle(\Delta\alpha_-)^2\rangle$ , and  $\langle\Delta\alpha_+\Delta\alpha_-^\dagger\rangle$  vanish, or, equivalently, that the noise moments  $\langle(\Delta\alpha_1)^2\rangle$ ,  $\langle(\Delta\alpha_2)^2\rangle$ , and  $\langle\Delta\alpha_1\Delta\alpha_2\rangle$  of the quadrature-phase amplitudes vanish (see Section I.g). The noise matrices for a two-mode squeezed state can be obtained from the transformation (3A.56) and the noise matrices for a coherent state. For the TMSS  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  [Eq. (3A.67)] they are

$$T = -\frac{1}{2} e^{2i\varphi} \sinh 2r \sigma_1, \quad Q = \frac{1}{2} \cosh 2r \mathbf{1}. \quad (3A.81)$$

The second-order noise moments of the quadrature-phase amplitudes for a two-mode squeezed state are

$$\langle |\Delta\alpha_1|^2 \rangle = \frac{1}{2} [\cosh 2r - (1 - \varepsilon^2 / \Omega^2)^{1/2} \sinh 2r \cos 2\varphi] ,$$

$$\langle |\Delta\alpha_2|^2 \rangle = \frac{1}{2} [\cosh 2r + (1 - \varepsilon^2 / \Omega^2)^{1/2} \sinh 2r \cos 2\varphi] ,$$

$$\langle \Delta\alpha_1 \Delta\alpha_2^\dagger \rangle_{\text{sym}} = \frac{1}{2} (1 - \varepsilon^2 / \Omega^2)^{1/2} \sinh 2r \sin 2\varphi + \frac{1}{2} i \varepsilon / \Omega \cosh 2r , \quad (3A.82)$$

[Eqs. (1.21) or (3A.62); cf. Eqs. (2B.37)]. The noise matrices  $S_x$ ,  $S_p$ , and  $S_{xp}$  for a two-mode squeezed state also have simple forms:

$$S_x = \frac{1}{2} \begin{pmatrix} \cosh 2r & -\sinh 2r \cos 2\varphi \\ -\sinh 2r \cos 2\varphi & \cosh 2r \end{pmatrix} ,$$

$$S_p = \sigma_3 S_x \sigma_3 = \frac{1}{2} \begin{pmatrix} \cosh 2r & -\sinh 2r \cos 2\varphi \\ -\sinh 2r \cos 2\varphi & \cosh 2r \end{pmatrix} ,$$

$$S_{xp} = -\frac{1}{2} \sinh 2r \sin 2\varphi \sigma_1 . \quad (3A.83)$$

The noise matrices  $T$  and  $Q$  for the general two-mode GPS  $|\underline{\mu}_g\rangle$  [Eqs. (3A.71), (3A.78)] can be obtained from the transformation (3A.73) and the noise matrices for a coherent state, or from Eqs. (3B.32) and (3B.35) below. Their components, the noise moments of  $\alpha_\pm$  and  $\alpha_\pm^\dagger$ , are

$$\langle (\Delta\alpha_\pm)^2 \rangle = -\frac{1}{2} \cosh 2r e^{2i\varphi_\pm} \sinh 2r_\pm ,$$

$$\langle \Delta\alpha_+ \Delta\alpha_- \rangle = -\frac{1}{2} \sinh 2r (e^{2i\varphi} \cosh r_+ \cosh r_- + e^{2i(\varphi_+ + \varphi_- - \varphi)} \sinh r_+ \sinh r_-) ,$$

$$\langle |\Delta\alpha_\pm|^2 \rangle = \frac{1}{2} \cosh 2r \cosh 2r_\pm ,$$

$$\langle \Delta\alpha_+ \Delta\alpha_-^\dagger \rangle = \frac{1}{2} \sinh 2r (e^{2i(\varphi - \varphi_-)} \cosh r_+ \sinh r_- + e^{-2i(\varphi - \varphi_+)} \cosh r_- \sinh r_+ .$$

(3A.84)

The choice for the order of the three squeeze operators in  $\mathbf{U}_g$  [Eq. (3A.71)] was made so that the noise moments  $\langle (\Delta\alpha_\pm)^2 \rangle$  and  $\langle |\Delta\alpha_\pm|^2 \rangle$  would have this simple form.

By analogy with single-mode MUS, the natural definition for "minimum uncertainty states" (MUS) is those states for which

$$S_x S_p = \frac{1}{2} 1 \quad (3A.85a)$$

[Eq. (3A.15a); cf. Eq. (2A.33)]. These are (two-mode) GPS that satisfy

$$\text{Im} T = \text{Im} Q = S_{xp} = 0 \quad (3A.85b)$$

[Eqs. (3A.12)]. Thus, the states that satisfy Eqs. (3A.85) have Gaussian wave functions, and their second-order noise moments of  $\alpha_{\pm}$  and  $\alpha_{\pm}^{\dagger}$  satisfy the following four conditions:

$$\text{Im} \langle \Delta \alpha_{+} \Delta \alpha_{-}^{\dagger} \rangle = 0, \quad (3A.86a)$$

$$\text{Im} \langle \Delta \alpha_{+} \Delta \alpha_{-} \rangle = 0, \quad (3A.86b)$$

$$\text{Im} \langle (\Delta \alpha_{+})^2 \rangle = \text{Im} \langle (\Delta \alpha_{-})^2 \rangle = 0. \quad (3A.86c)$$

It is shown below [Eqs. (3B.5) or (3B.6)] that a two-mode state satisfies Eqs. (3A.85) if and only if it is an eigenstate of the (components of the) linear combination  $\hat{x} + i M_1^{-1} \hat{p}$ , where  $M_1$  is a real, symmetric, positive-definite matrix. The set of MUS consists of all two-mode coherent states, two-mode squeezed states with  $\varphi = 0$ , and products of single-mode squeezed states with  $\varphi_{+} = \varphi_{-} = 0$ . Milburn<sup>33</sup> has proposed a more restrictive definition of two-mode MUS; his definition does not include the two-mode squeezed states among the MUS.

Similarly, one could define the two-mode analogs of single-mode states with random-phase noise to be states whose noise moments are invariant under rotations by  $\mathbf{R}(\theta) = U_R(\theta_s) U_M(\theta_d)$ . Such states must have  $T = 0$  and  $Q$  diagonal [Eqs. (3A.28)]. Note that the condition

$$S_{\mathbf{z}}(\underline{\theta}) = S_{\mathbf{p}}(\underline{\theta}) \quad \text{for all } \theta_+, \theta_- , \quad (3A.87a)$$

where

$$S_{\mathbf{z}}(\underline{\theta}) \equiv \langle \Delta \hat{\mathbf{x}}(\underline{\theta}) \Delta \hat{\mathbf{x}}^T(\underline{\theta}) \rangle , \quad S_{\mathbf{p}}(\underline{\theta}) \equiv \langle \Delta \hat{\mathbf{p}}(\underline{\theta}) \Delta \hat{\mathbf{p}}^T(\underline{\theta}) \rangle , \quad (3A.87b)$$

is equivalent to the condition  $T = 0$ , but is not adequate to ensure invariance under rotations produced by  $U_{\mathbf{M}}(\theta_{\mathbf{d}})$ , and hence does not define random-phase noise for two-mode states [Eqs. (3A.12), (3A.24b,c); cf. Eqs. (2A.35)]. The intersection between these two sets of states, i.e., MUS that have random-phase noise, is the set of two-mode coherent states, which have

$$(S_{\mathbf{zp}})_{\text{coh}} = T_{\text{coh}} = 0 , \quad (3A.88a)$$

$$(S_{\mathbf{z}})_{\text{coh}} = (S_{\mathbf{p}})_{\text{coh}} = Q_{\text{coh}} = \frac{1}{2} \mathbf{1} \quad (3A.88b)$$

[cf. Eqs. (2A.36)].

By extending the definition (3A.85) of MUS to include all states related to MUS by the rotation operator  $U_{\mathbf{R}}(\theta_{\mathbf{s}})$ , one obtains all two-mode squeezed states. This is because TSQP noise, i.e., the vanishing of the noise moments  $\langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_-^\dagger \rangle$ ,  $\langle (\Delta \mathbf{a}_+)^2 \rangle$ , and  $\langle (\Delta \mathbf{a}_-)^2 \rangle$ , means that the noise moments of a two-mode squeezed state are invariant under rotations by  $U_{\mathbf{M}}(\theta_{\mathbf{d}})$ , but not by  $U_{\mathbf{R}}(\theta_{\mathbf{s}})$  [Eqs. (3A.28)]. For a two-mode squeezed state to be a MUS, the noise moment  $\langle \Delta \mathbf{a}_+ \Delta \mathbf{a}_- \rangle$  must be real. This means that two-mode squeezed states are MUS for an infinite set of rotated variables  $\hat{\mathbf{x}}_{\pm}(\theta_{\pm})$  and  $\hat{\mathbf{p}}_{\pm}(\theta_{\pm})$ , defined by the condition that  $\frac{1}{2}(\theta_+ + \theta_-) \equiv \theta_{\mathbf{s}} = -\varphi$ ; i.e., a two-mode squeezed state satisfies

$$S_{\mathbf{z}}(\theta_{\mathbf{s}} = -\varphi) S_{\mathbf{p}}(\theta_{\mathbf{s}} = -\varphi) = \frac{1}{4} \mathbf{1} ,$$

$$S_{\mathbf{zp}}(\theta_{\mathbf{s}} = -\varphi) = 0 . \quad (3A.89)$$

This fact is yet another indication that the noise in a two-mode squeezed state is the natural analog of the noise in a single-mode squeezed state, i.e., of the noise in all single-mode GPS. Invariance under rotations by  $U_M(\theta_d)$  has no meaning at degeneracy, where  $U_M(\theta_d)$  becomes the identity operator.

By extending the definition (3A.85) of MUS to include all states related to MUS by both rotation operators,  $U_R(\theta_s)$  and  $U_M(\theta_d)$ , one obtains, in addition to all two-mode squeezed states, all states that are products of two single-mode squeezed states. States that are products of two single-mode squeezed states are MUS for a particular set of rotated variables,  $\hat{x}_\pm(-\varphi_\pm)$  and  $\hat{p}_\pm(-\varphi_\pm)$ . Thus, products of single-mode squeezed states satisfy

$$S_x(-\varphi) S_p(-\varphi) = \frac{1}{2} \mathbf{1} ,$$

$$S_{xp}(-\varphi) = 0 . \tag{3A.90}$$

Finally, by extending the definition (3A.85) of MUS to include all states related to two-mode MUS by products of rotation and mixing operators, one obtains all two-mode GPS. This is so because the four conditions (3A.86) can always be met for some operators defined as linear combinations of  $\alpha_+$  and  $\alpha_-$  by a transformation like (3A.44b), with appropriate choices for the four parameters  $q$ ,  $\chi$ ,  $\theta_+$ , and  $\theta_-$ .



## B. Two-mode Gaussian wave functions

Consider now the coordinate-space wave function for an arbitrary two-mode Gaussian pure state symbolized by the state vector  $|\mu_g, \mu_g\rangle$  or, for compactness,  $|\mu_g\rangle$ . The wave function is written in terms of the dimensionless position variables  $x_{\pm}$ , the eigenvalues of the Hermitian operators  $\hat{x}_{\pm}$ . The most general (normalized) two-mode Gaussian coordinate-space wave function has the form

$$\begin{aligned} \langle x_+ x_- | \mu_g \rangle &= N_g e^{\mathcal{H}i\delta_x} e^{-\mathcal{H}i(p_{0+}x_{0+} + p_{0-}x_{0-})} e^{i(p_{0+}x_+ + p_{0-}x_-)} \\ &\quad \exp\left[-\frac{1}{2}M_{11}(\Delta x_+)^2 - \frac{1}{2}M_{22}(\Delta x_-)^2 - \frac{1}{2}(M_{12} + M_{21})\Delta x_+\Delta x_-\right] \\ &= N_g e^{\mathcal{H}i\delta_x} e^{-\mathcal{H}i p_0^T \tilde{x}_0} e^{i p_0^T \tilde{x}} e^{-\frac{1}{2}\Delta \tilde{x}^T M \Delta \tilde{x}}, \\ \Delta \tilde{x} &\equiv \begin{pmatrix} \Delta x_+ \\ \Delta x_- \end{pmatrix}, \quad \Delta x_{\pm} \equiv x_{\pm} - \langle \hat{x}_{\pm} \rangle \end{aligned} \quad (3B.1)$$

[cf. Eq. (2B.1)]. Here  $\tilde{x}_0$  and  $p_0$  are the column vectors for the mean position and momentum [Eqs. (3A.7)], with components defined by

$$x_{0\pm} \equiv \langle \hat{x}_{\pm} \rangle = \int_{-\infty}^{\infty} dx_+ \int_{-\infty}^{\infty} dx_- x_{\pm} |\langle x_+ x_- | \mu_g \rangle|^2, \quad (3B.2a)$$

$$p_{0\pm} \equiv \langle \hat{p}_{\pm} \rangle = -i \int_{-\infty}^{\infty} dx_+ \int_{-\infty}^{\infty} dx_- \langle \mu_g | x_+ x_- \rangle \partial_{x_{\pm}} \langle x_+ x_- | \mu_g \rangle, \quad (3B.2b)$$

$M$  is a two-dimensional, complex matrix whose components  $M_{ij}$  are related to the second-order noise moments of  $\hat{x}_{\pm}$  and  $\hat{p}_{\pm}$ ,  $\delta_x$  is an unobservable phase angle (separated out for reasons discussed below), and  $N_g$  is a (real) normalization constant determined by the condition  $\langle \mu_g | \mu_g \rangle = 1$ . The sub-

script "x" on the phase angle  $\delta_x$  serves only to distinguish  $\delta_x$  from the phase angle  $\delta_p$  which appears in the momentum-space wave function [Eqs. (3B.38)-(3B.41) below];  $\delta_x$  has no dependence on  $x_{\pm}$ . For convenience it is assumed henceforth that the matrix  $M$  is symmetric,  $M \equiv M^T$  ( $M_{12} \equiv M_{21}$ ); this assumption can be lifted if all statements about  $M$  that follow are interpreted as statements about  $\frac{1}{2}(M + M^T)$ . Normalizability dictates that the real part of  $M$  be positive-definite -- i.e., that

$$\begin{aligned} \text{Tr}M_1 > 0, \quad \det M_1 > 0, \\ M_1 \equiv \text{Re}M = \frac{1}{2}(M + M^*), \end{aligned} \quad (3B.3)$$

and the normalization constant  $N_g$  is equal to

$$N_g = (\pi^2 / \det M_1)^{-1/2} \quad (3B.4)$$

[cf. Eqs. (2B.2)-(2B.4)].

The most important parameters in the wave function (3B.1) are the three complex numbers that make up the (symmetric) matrix  $M$ . The form of the wave function tells one that the state  $|\mu_g\rangle$  is a simultaneous eigenstate of the components of the linear combination  $\hat{x} + iM^{-1}\hat{p}$ , and hence that the matrix  $M$  is related to the noise matrices  $S_x$ ,  $S_p$ , and  $S_{xp}$  by

$$M \equiv M_1 + iM_2 = -iS_x^{-1}(S_{xp} + \frac{1}{2}i1) = -iS_p(S_{xp} - \frac{1}{2}i1)^{-1}. \quad (3B.5a)$$

The real and imaginary parts of  $M$  are therefore equal to

$$M_1 \equiv \text{Re}M = \frac{1}{2}S_x^{-1}, \quad M_2 \equiv \text{Im}M = -S_x^{-1}S_{xp}, \quad (3B.5b)$$

and the absolute square of its determinant is

$$|\det M|^2 = \frac{\det S_p}{\det S_x} \quad (3B.5c)$$

[cf. Eqs. (2B.5)]. Equation (3B.5c) shows that for normalizable two-mode GPS the matrix  $M$  must be nonsingular [if  $\det M = 0$ , the wave function in the momentum representation is a delta function; see discussion below Eqs. (3A.11)]. Inverting these expressions gives the covariance matrices  $S_x$ ,  $S_p$ , and  $S_{xp}$  in terms of the matrix  $M$ :

$$\begin{aligned} S_x &= (2M_1)^{-1}, & S_p &= [2\text{Re}(M^{-1})]^{-1}, \\ S_{xp} &= -(2M_1)^{-1} M_2 \end{aligned} \quad (3B.6)$$

[cf. Eq. (2B.6)]. The normalization constant  $N_g$  can thus be rewritten as

$$N_g = (\pi^2 / \det M_1)^{-1/4} = (4\pi^2 \det S_x)^{-1/4} \quad (3B.7)$$

[cf. Eq. (2B.7)].

That the state  $|\underline{\mu}_g\rangle$  is an eigenstate of the components of  $\hat{x} + iM^{-1}\hat{p}$  means that it is also an eigenstate of the components of  $\underline{a} + (M+1)^{-1}(M-1)\underline{a}^*$ . The noise matrices  $T$  and  $Q$  are therefore more naturally expressed in terms of the symmetric complex matrix

$$\Gamma = \Gamma^T \equiv (M+1)^{-1}(M-1) = -(Q + \frac{1}{2}1)^{-1}T = -(Q - \frac{1}{2}1)T^{*-1} \quad (3B.8)$$

[cf. Eq. (2B.8)]. Inverting these expressions gives the noise matrices  $T$  and  $Q$  in terms of the matrices  $\Gamma$  and  $M$ :

$$\begin{aligned} T &= -\Gamma(1 - \Gamma^*\Gamma)^{-1} = -(1 - \Gamma\Gamma^*)^{-1}\Gamma \\ &= -(M^* + 1)(4M_1)^{-1}(M - 1), \end{aligned} \quad (3B.9a)$$

$$Q = \frac{1}{2}(1 + \Gamma\Gamma^*)(1 - \Gamma\Gamma^*)^{-1} = (M + 1)^{-1}(1 + MM^*)(4M_1)^{-1}(M + 1) \quad (3B.9b)$$

[cf. Eqs. (2B.9)]. Note also that

$$1 - \Gamma\Gamma^* = (M + 1)^{-1}4M_1(M^* + 1)^{-1}; \quad (3B.10)$$

hence normalizability requires that the (Hermitian) matrix  $(1 - \Gamma\Gamma^*)$  be positive-definite. The above expressions reveal, though not in a transparent way, how the ten real pieces of information in the second-order noise moments for a two-mode GPS reduce to six independent pieces, since

$$S_x S_p = \frac{1}{2}1 + S_{xp}^2, \quad (3B.11a)$$

$$Q^2 = \frac{1}{2}1 + TT^* \quad (3B.11b)$$

[cf. Eqs. (3A.13) and (2B.11)]. They also reveal that the following matrix products are symmetric:

$$S_{xp} S_x = (S_{xp} S_x)^T, \quad S_p S_{xp} = (S_p S_{xp})^T, \quad QT = (QT)^T = TQ^*. \quad (3B.11c)$$

These relations are made more obvious below [Eqs. (3B.30)-(3B.32)].

The remaining parameter in the wave function (3B.1) is the phase angle  $\delta_x$ ; in general it can be any real number. The phase angle  $\delta_x$  is unobservable, but for a state defined as a particular unitary operator acting on the (two-mode) vacuum state it has a well-defined value, provided one assigns a phase angle to the vacuum-state wave function. The properties of the displacement operator reveal why the phase factor  $e^{\frac{1}{2}i\delta_x}$  separates naturally from the overall phase factor in the wave function  $\langle \underline{x} | \underline{\mu}_g \rangle$ . The definition (3A.29) of the two-mode displacement operator implies that the wave function for a "displaced" two-mode state  $D(\underline{a}, \underline{\mu}) |\Psi\rangle$  is related to the wave function of the original state  $|\Psi\rangle$  in the following way:

$$\langle \underline{x} | D(\underline{a}, \underline{\mu}) | \Psi \rangle = e^{-\frac{i}{2} \underline{p}_0^T \underline{x}_0} e^{i \underline{p}_0^T \underline{x}} \langle \underline{x} - \underline{x}_0 | \Psi \rangle \quad (3B.12)$$

[cf. Eq. (2B.12)]. A natural way, therefore, to obtain an arbitrary two-mode pure state  $|\Psi_{\underline{\mu}}\rangle$  with complex amplitudes  $\mu_+$  and  $\mu_-$  is to operate with the two-mode displacement operator  $D(\underline{a}, \underline{\mu})$  on a state  $|\Psi_0\rangle \equiv \mathbf{U}_0|0\rangle$  that has the desired noise properties but has zero complex amplitudes ( $\langle 0 | \mathbf{U}_0^\dagger \underline{a} \mathbf{U}_0 | 0 \rangle = 0$ ):

$$|\Psi_{\underline{\mu}}\rangle \equiv D(\underline{a}, \underline{\mu}) \mathbf{U}_0 | 0 \rangle . \quad (3B.13)$$

The property (3A.31) of the displacement operator then ensures that  $|\Psi_{\underline{\mu}}\rangle$  has complex amplitudes  $\mu_+$  and  $\mu_-$ ,

$$\langle \Psi_{\underline{\mu}} | \underline{a} | \Psi_{\underline{\mu}} \rangle = \underline{\mu} . \quad (3B.14)$$

Any normalized two-mode pure state with complex amplitudes  $\mu_+$  and  $\mu_-$  can be defined by an expression like (3B.13). The advantage of this definition is that the state's mean values  $x_{0\pm}$  and  $p_{0\pm}$  (or complex amplitudes  $\mu_{\pm}$ ) are determined solely by the displacement operator  $D(\underline{a}, \underline{\mu})$ , and its noise moments of  $\underline{a}_{\pm}$  and  $\underline{a}_{\pm}^\dagger$  are determined solely by the unitary operator  $\mathbf{U}_0$ . The (normalized) two-mode GPS  $|\underline{\mu}_g\rangle$  with complex amplitudes  $\mu_+$ ,  $\mu_-$  can therefore be formally defined by

$$|\underline{\mu}_g\rangle \equiv D(\underline{a}, \underline{\mu}) \mathbf{U}_g | 0 \rangle . \quad (3B.15)$$

Note the following three properties of  $\mathbf{U}_g$ : First, it is uniquely defined only up to (right-hand) multiplication by rotation operators  $R_{\pm}(\theta_{\pm})$ , a mixing operator  $T(q, \chi)$ , and an overall phase factor. Second, since it defines the

noise moments of  $\mathbf{a}_\pm$  and  $\mathbf{a}_\pm^\dagger$  (or  $\hat{\mathbf{x}}_\pm$  and  $\hat{\mathbf{p}}_\pm$ ) for the GPS  $|\underline{\mu}_g\rangle$ , it has associated with it no more than six independent real parameters (over and above those of the rotation and mixing operators and phase factor). Third, since the state  $|\underline{\mu}_g\rangle$  has complex amplitudes  $\mu_+$ ,  $\mu_-$ , the expectation value  $\langle 0 | \mathbf{U}_g^\dagger \underline{\mathbf{a}} \mathbf{U}_g | 0 \rangle$  must vanish.

The phase factor  $e^{\frac{1}{2}i\delta_x}$  in the wave function  $\langle \underline{\mathbf{x}} | \Psi_{\underline{\mu}} \rangle$  is given, from Eqs. (3B.1) and (3B.12), by

$$e^{\frac{1}{2}i\delta_x} = \frac{\langle \mathbf{x}_+ = \mathbf{x}_- = 0 | \mathbf{U}_g | 0 \rangle}{|\langle \mathbf{x}_+ = \mathbf{x}_- = 0 | \mathbf{U}_g | 0 \rangle|} \quad (3B.16)$$

[cf. Eq. (2B.15)]. The phase angle  $\delta_x$  has no dependence on the complex amplitudes  $\mu_\pm$ , provided  $\mathbf{U}_g$  does not; any dependence of  $\mathbf{U}_g$  on  $\mu_\pm$  is artificial, however, since it does not affect the state's complex amplitudes  $\langle \mathbf{a}_\pm \rangle$ . Consider, for illustration, the two-mode coherent state  $|\underline{\mu}\rangle_{\text{coh}} \equiv D(\underline{\mathbf{a}}, \underline{\mu}) | 0 \rangle$  [Eq. (1.15)], for which the operator  $\mathbf{U}_g$  is the identity operator. Equation (3B.12) says that the wave function for  $|\underline{\mu}\rangle_{\text{coh}}$  is related to the vacuum-state wave function  $\langle \underline{\mathbf{x}} | 0 \rangle$  by

$$\langle \underline{\mathbf{x}} | \underline{\mu} \rangle_{\text{coh}} = e^{-\frac{1}{2}i \underline{p}_0^T \underline{x}_0} e^{i \underline{p}_0^T \underline{x}} \langle \underline{\mathbf{x}} - \underline{\mathbf{x}}_0 | 0 \rangle, \quad (3B.17)$$

so the phase angle  $\delta_x$  for a two-mode coherent-state wave function is equal to the phase angle assigned to the two-mode vacuum-state wave function.

The form of its wave function shows that a two-mode GPS  $|\underline{\mu}_g\rangle$  is a simultaneous eigenstate of linearly independent operators  $g_+$  and  $g_-$ , which are linear combinations of the two operators defined by the components of  $\hat{\mathbf{x}} + i M^{-1} \hat{\mathbf{p}}$  or  $\underline{\mathbf{a}} + \Gamma \underline{\mathbf{a}}^*$ . The label  $\underline{\mu}_g$  ( $\mu_{g+}$ ,  $\mu_{g-}$ ) for the GPS  $|\underline{\mu}_g\rangle$  is chosen to

be the eigenvalues of  $\underline{g}$  ( $g_+$ ,  $g_-$ ). Thus, one can write the following relations:

$$\underline{g} |\underline{\mu}_g\rangle = \underline{\mu}_g |\underline{\mu}_g\rangle, \quad (3B.18a)$$

$$\underline{g} = K(\underline{\hat{x}} + i M^{-1} \underline{\hat{p}}) = \bar{K}(\underline{a} + \Gamma \underline{a}^*), \quad (3B.18b)$$

$$\underline{\mu}_g = K(\underline{x}_0 + i M^{-1} \underline{p}_0) = \bar{K}(\underline{\mu} + \Gamma \underline{\mu}^*), \quad (3B.18c)$$

where  $K$  and  $\bar{K}$  are two-dimensional nonsingular matrices. It is useful to consider the general form for all independent operators  $g_+$ ,  $g_-$  of which the two-mode GPS  $|\underline{\mu}_g\rangle$  is an eigenstate:

$$\begin{aligned} \underline{g} &\equiv \begin{pmatrix} g_+ \\ g_- \end{pmatrix} = P_c \underline{a} + P_s \underline{a}^* \equiv P_c (\underline{a} + \Gamma \underline{a}^*) \\ &= P_p \underline{\hat{x}} + i P_x \underline{\hat{p}} \equiv P_p (\underline{\hat{x}} + i M^{-1} \underline{\hat{p}}). \end{aligned} \quad (3B.19a)$$

Here  $P_p$ ,  $P_x$ ,  $P_c$ , and  $P_s$  are two-dimensional complex matrices, related to each other by

$$P_c = 2^{-\frac{1}{2}}(P_p \pm P_x), \quad P_x = 2^{-\frac{1}{2}}(P_c \pm P_s). \quad (3B.20)$$

The eigenvalues  $\mu_{g+}$ ,  $\mu_{g-}$  are related to the complex amplitudes  $\mu_+$ ,  $\mu_-$  and the mean positions and momentums  $x_{0\pm}$ ,  $p_{0\pm}$  by similar relations,

$$\underline{\mu}_g = P_p \underline{x}_0 + i P_x \underline{p}_0 = P_c \underline{\mu} + P_s \underline{\mu}^*. \quad (3B.19b)$$

Inverting Eqs. (3B.19) leads to the following expressions for  $\underline{a}$  and  $\underline{\mu}$  in terms of  $\underline{g}$  and  $\underline{\mu}_g$ :

$$\underline{a} = P_c^\dagger \Gamma_g^{-1} \underline{g} - P_s^T (\Gamma_g^*)^{-1} \underline{g}^*, \quad (3B.21a)$$

$$\underline{\underline{\mu}} \equiv \langle \underline{\underline{a}} \rangle = P_c^\dagger \Upsilon_g^{-1} \underline{\underline{\mu}}_g - P_s^T (\Upsilon_g^*)^{-1} \underline{\underline{\mu}}_g^* , \quad (3B.21b)$$

where  $\Upsilon_g \equiv [g, g^\dagger]$  is the Hermitian commutator matrix

$$\Upsilon_g \equiv [g, g^\dagger] \equiv \underline{\underline{g}} \underline{\underline{g}}^\dagger - (\underline{\underline{g}}^* \underline{\underline{g}}^T)^T = \begin{bmatrix} [g_+, g_+^\dagger] & [g_+, g_-^\dagger] \\ [g_-, g_+^\dagger] & [g_-, g_-^\dagger] \end{bmatrix} \quad (3B.21c)$$

[cf. Eqs. (2B.21)]. These relations imply the important equality

$$D(\underline{\underline{a}}, \underline{\underline{\mu}}) = D(g, \Upsilon_g^{-1} \underline{\underline{\mu}}_g) \quad (3B.22)$$

[cf. Eq. (2B.22)].

The equality (3B.22) enables one to see explicitly how the form of the unitary operator  $\mathbf{U}_g$ , which defines a two-mode GPS through Eq. (3B.15), is determined by the forms of the operators  $g_+$  and  $g_-$ . To see this, begin with an alternative definition for the GPS  $|\underline{\underline{\mu}}_g\rangle$  [the equality (3B.22) will be seen to ensure that this definition is equivalent to the definition (3b.15)]. First, assume that  $|\underline{\underline{\mu}}_g\rangle$  is related to the vacuum state by some unitary operator  $\bar{\mathbf{U}}$ :

$$|\underline{\underline{\mu}}_g\rangle = \bar{\mathbf{U}} |0\rangle . \quad (3B.23a)$$

It is convenient to define another unitary operator  $\mathbf{U}_g$  by

$$\bar{\mathbf{U}} \equiv \mathbf{U}_g D(\underline{\underline{a}}, \underline{\underline{\mu}}_g) , \quad (3B.23b)$$

so that the state  $|\underline{\underline{\mu}}_g\rangle$  is equal to the operator  $\mathbf{U}_g$  acting on the two-mode coherent state  $|\underline{\underline{\mu}}_g\rangle_{\text{coh}}$ ,

$$|\underline{\underline{\mu}}_g\rangle = \mathbf{U}_g D(\underline{\underline{a}}, \underline{\underline{\mu}}_g) |0\rangle = \mathbf{U}_g |\underline{\underline{\mu}}_g\rangle_{\text{coh}} . \quad (3B.24)$$



It is then consistent with the eigenvalue equations (3B.18a) that the operators  $g_+$  and  $g_-$  be unitarily related to the annihilation operators  $a_+$  and  $a_-$ , respectively, through the operator  $U_g$ :

$$\underline{g} = U_g \underline{a} U_g^\dagger. \quad (3B.25)$$

The form of  $U_g$  is thus determined by the forms of the operators  $g_+$  and  $g_-$ . Note that this relation says that both  $a_+$  and  $a_-$  are transformed by the same unitary operator,  $U_g$ . The equivalence of the definitions (3B.15) and (3B.24) for  $|\underline{\mu}_g\rangle$  is a result of the unitarity of  $U_g$  and the fact that both  $a_+$  and  $a_-$  are transformed by the same unitary operator, which ensure that  $[\underline{g}, \underline{g}^\dagger] = [\underline{a}, \underline{a}^\dagger] = 1$ , and the forms (3B.19) of  $g_+$  and  $g_-$ ; these imply the equality

$$D(\underline{a}, \underline{\mu}) = D(\underline{g}, \underline{\mu}_g) \quad (3B.26)$$

[Eq. (3B.22)]. Thus, any two-mode GPS  $|\underline{\mu}_g\rangle$  has the following two equivalent definitions:

$$|\underline{\mu}_g\rangle = U D(\underline{a}, \underline{\mu}_g) |0\rangle = U |\underline{\mu}_g\rangle_{\text{coh}} = D(\underline{a}, \underline{\mu}) U |0\rangle \quad (3B.27)$$

[cf. Eq. (2B.27)].

Return now to the general forms (3B.19) for the operators  $g_+$  and  $g_-$  of which two-mode GPS are eigenstates. Six of the sixteen degrees of freedom in the expression (3B.19a) for  $\underline{g}$  are removed by the wave function  $\langle \underline{x} | \underline{\mu}_g \rangle$ , which specifies the (symmetric) matrix products

$$P_z^{-1} P_p \equiv M, \quad P_c^{-1} P_s \equiv \Gamma. \quad (3B.28)$$

Six more degrees of freedom have already been partially removed by the

requirement that  $g_+$  and  $g_-$  have a complete (overcomplete) set of simultaneous, normalizable eigenstates, i.e., that the commutator  $[g_+, g_-] = 0$  and the Hermitian commutator matrix  $[g, g^\dagger]$  be positive-definite (for further discussion of this requirement see Appendix C). These degrees of freedom are removed completely if one specifies that the operators  $g_+$  and  $g_-$  be unitarily related to  $a_+$  and  $a_-$ , respectively, by the same unitary operator [Eq. (3B.25)], which implies that

$$[g, g^\dagger] = [\alpha, \alpha^\dagger] = 1. \quad (3B.29)$$

The commutator  $[g_+, g_-]$  is related to the matrices in the expression (3B.24a) for  $g$  in the following way:

$$\begin{aligned} [g, g^\dagger] &\equiv g g^\dagger - (g g^\dagger)^T = [g_+, g_-] i \sigma_2 \\ &= P_x P_p^T - P_p P_x^T = P_p [M^{-1} - (M^{-1})^T] P_p^T \\ &= P_c P_s^T - P_s P_c^T = P_c (\Gamma^T - \Gamma) P_c^T. \end{aligned} \quad (3B.30a)$$

Thus, the requirement that  $[g_+, g_-] = 0$  is satisfied if and only if the matrix  $M$  in the wave function  $\langle x | \mu_g \rangle$  is symmetric. (Recall that I originally stipulated that  $M$  was symmetric. This condition merely reminds one that the two-mode state  $|\mu_g\rangle$  with wave function (3B.1) is an eigenstate of  $\hat{x} + i[\frac{1}{2}(M + M^T)]^{-1}\hat{p}$ , and not of  $\hat{x} + iM^{-1}\hat{p}$ , unless  $M = M^T$ .) The Hermitian commutator matrix  $[g, g^\dagger]$  can be written in the following different ways, using Eqs. (3B.5) and (3B.8):

$$[g, g^\dagger] = P_x P_p^\dagger + P_p P_x^\dagger = 2P_x M_1 P_x^\dagger = P_x S_x^{-1} P_x^\dagger$$

$$\begin{aligned}
&= 2P_p \operatorname{Re}(M^{-1}) P_p^\dagger = P_p S_p^{-1} P_p^\dagger \\
&= P_c P_c^\dagger - P_s P_s^\dagger = P_c (1 - \Gamma\Gamma^*) P_c^\dagger = P_c (Q + \frac{1}{2}1)^{-1} P_c^\dagger \\
&= P_s [(\Gamma^*\Gamma)^{-1} - 1] P_s^\dagger = P_s (Q^* - \frac{1}{2}1)^{-1} P_s^\dagger \\
&= P_s \Gamma^{-1} (1 - \Gamma\Gamma^*) P_c^\dagger = -P_s T^{-1} P_c^\dagger \tag{3B.30b}
\end{aligned}$$

[cf. Eq. (2B.30)]. These expressions show that the operators  $g_+$  and  $g_-$  have simultaneous normalizable eigenstates if and only if the wave function  $\langle \underline{x} | \underline{\mu}_g \rangle$  is normalizable -- i.e., the (symmetric) real matrix  $M_1$  [or  $(1 - \Gamma\Gamma^*)$ ] is positive-definite [Eqs. (3B.3) and (3B.10)]. Note also that  $g_+$  and  $g_-$  can have simultaneous normalizable eigenstates only if the matrices  $P_p$ ,  $P_z$ , and  $P_c$  are nonsingular. The requirement that  $g_+$  and  $g_-$  be unitarily related to  $\alpha_+$  and  $\alpha_-$  through the same unitary operator -- i.e., that  $[g, g^\dagger] = 1$  -- implies that

$$P_z P_p^\dagger + P_p P_z^\dagger = P_c P_c^\dagger - P_s P_s^\dagger = 1. \tag{3B.31a}$$

Two other useful properties of these matrices are revealed by using the expression (3B.21a) for  $\underline{a}$  in terms of  $\underline{g}$  and  $\underline{g}^*$  and setting  $[\underline{a}, \underline{a}^\dagger] = 1$ . For the matrices  $P_p$  and  $P_z$  these properties are

$$\operatorname{Im}(P_p^\dagger \Upsilon_g^{-1} P_p) = \operatorname{Im}(P_z^\dagger \Upsilon_g^{-1} P_z) = 0, \tag{3B.31b}$$

$$\operatorname{Re}(P_z^\dagger \Upsilon_g^{-1} P_p) = \frac{1}{2}1. \tag{3B.31c}$$

For the matrices  $P_c$  and  $P_s$  they are

$$P_c^\dagger \Upsilon_g^{-1} P_s = (P_c^\dagger \Upsilon_g^{-1} P_s)^T, \tag{3B.31d}$$

$$P_c^\dagger \Upsilon_g^{-1} P_c - P_s^T (\Upsilon_g^{-1})^* P_s^* = 1. \tag{3B.31e}$$

From now on I restrict attention to operators  $g_+$ ,  $g_-$  that satisfy  $[g, g^\dagger] = 1$ , i.e., to operators  $g_+$  and  $g_-$  that are unitarily related to  $a_+$  and  $a_-$  by the same unitary transformation. This entails no loss of generality, since by taking appropriate linear combinations of other operators  $g_+'$ ,  $g_-'$  for which  $[g', g'^\dagger]$  is positive-definite but not proportional to the identity matrix, one can always define operators  $g_+$ ,  $g_-$  that satisfy  $[g, g^\dagger] = 1$ . Henceforth, therefore, the operators  $g_+$  and  $g_-$  are assumed to be related to  $a_+$  and  $a_-$  by an expression like Eq. (3B.25).

The four remaining degrees of freedom in the expression (3B.19a) for  $g$  represent the freedom to multiply  $g$  by an arbitrary unitary matrix and an overall phase factor, acts which do not change the commutator matrix  $[g, g^\dagger] = 1$ . Multiplying  $g$  in this way is equivalent to (right-hand) multiplying the operator  $U_g$  of Eq. (3B.25) by a mixing operator  $T(q, \chi)$  and two rotation operators  $R_\pm(\theta_\pm)$  [Eqs. (3A.44)]. The definition (3B.27) of  $|\mu_g\rangle$  shows that this freedom reflects the fact noted in Section 3A.4 that a (two-mode) coherent state remains a coherent state when multiplied by rotation and mixing operators [Eq. (3A.49)].

The expressions (3B.30b) for the commutator matrix  $[g, g^\dagger]$ , together with the matrix properties described above, reveal the following simple relations between the noise matrices for a two-mode GPS and the complex matrices  $P_p$ ,  $P_z$ ,  $P_c$ , and  $P_s$  for operators  $g_+$ ,  $g_-$  unitarily related to  $a_+$ ,  $a_-$  by the same unitary operator:

$$S_z = P_z^\dagger P_z, \quad S_p = P_p^\dagger P_p, \quad S_{zp} = -\text{Im}(P_z^\dagger P_p); \quad (3B.32a)$$

$$T = -P_c^\dagger P_s, \quad (3B.32b)$$

$$Q = \frac{1}{2}(P_c^\dagger P_c + P_s^T P_s^*) = P_c^\dagger P_c - \frac{1}{2}1 = P_s^T P_s^* + \frac{1}{2}1 \quad (3B.32c)$$

[cf. Eqs. (3B.6), (3B.9), and (2B.32)]. These expressions, together with with Eqs. (3B.30) and (3B.31), make apparent the relations (3B.11) between the noise matrices.

The form of the unitary operator  $\mathbf{U}_g$  in the definition (3B.27) of the two-mode GPS  $|\mu_g\rangle$  is dictated by the transformation (3B.25) and the form of  $g_+$  and  $g_-$  [Eqs. (3B.19a), (3B.31)]. The linearity and absence of any additive constants in the transformation imply that  $\mathbf{U}_g = e^{-iH_g^{(2)}}$ , where  $H_g^{(2)}$  is a (Hermitian) linear combination of the ten bilinear products of  $\mathbf{a}_\pm$  and  $\mathbf{a}_\pm^\dagger$  ( $\mathbf{a}_\pm^\dagger \mathbf{a}_\pm$ ,  $\mathbf{a}_+ \mathbf{a}_-$ ,  $\mathbf{a}_\pm^2$ ,  $\mathbf{a}_+ \mathbf{a}_-^\dagger$ , and their adjoints). That is, the generator  $H_g^{(2)}$  of  $\mathbf{U}_g$  has the general form  $[H_0^{(2)} + H_R^{(2)} + H_2^{(2)}]$  defined in the Introduction [Eqs. (1.1)-(1.4)]. It is shown in Section IIIC and Appendix A that this operator  $\mathbf{U}_g$  can always be written as a product of two single-mode squeeze operators, a two-mode squeeze operator, two rotation operators, and a mixing operator (and an unobservable overall phase factor). That the rotation and mixing operators can be neglected in the general form for  $\mathbf{U}_g$  can be seen in a couple of ways. First, the rotation and mixing operators can be placed in any position relative to the squeeze operators, without changing the form of  $\mathbf{U}_g$  [Eqs. (2A.25), (3A.47), (3A.63), (3A.70)]; when placed to the right of the squeeze operators, they act like the identity operator on the vacuum state and hence are inconsequential. Second, the parameters  $\theta_\pm$ ,  $q$ , and  $\chi$  in the rotation and mixing operators are related to multiplication of  $\underline{g}$  by a unitary matrix, which can be chosen arbitrarily [Eqs. (3A.44)]; they are made zero if the diagonal elements of the transformation matrix  $P_c$  are chosen to be real. Hence the operator  $\mathbf{U}_g$  is equal to the product of two single-mode squeeze operators and a two-mode squeeze operator -- i.e., to the operator

$\mathbf{U}_g$  defined in the preceding section [Eq. (3A.71)].

The two-mode GPS defined by Eq. (3B.27) with  $\mathbf{U}_g$  equal to a product of three squeeze operators [Eq. (3A.71)] is an eigenstate of the operators  $\underline{g}$  defined by Eqs. (3A.73). The matrices  $P_c$  and  $P_s$  are therefore given by Eqs. (3A.73):

$$P_c = \begin{pmatrix} \cosh r \cosh r_+ & e^{2i(\varphi_+ - \varphi_-)} \sinh r \sinh r_- \\ e^{-2i(\varphi_+ - \varphi_-)} \sinh r \sinh r_+ & \cosh r \cosh r_- \end{pmatrix},$$

$$P_s = \begin{pmatrix} e^{2i\varphi_+} \cosh r \sinh r_+ & e^{2i\varphi} \sinh r \cosh r_- \\ e^{2i\varphi} \sinh r \cosh r_+ & e^{2i\varphi_-} \cosh r \sinh r_- \end{pmatrix}. \quad (3B.35)$$

The matrices  $P_p$  and  $P_x$  follow from these according to the relations (3B.25). The complex amplitudes  $\mu_+$ ,  $\mu_-$  and eigenvalues  $\mu_{g+}$ ,  $\mu_{g-}$  are related to each other by

$$\underline{\mu} \equiv \langle \underline{a} \rangle = P_c^\dagger \underline{\mu}_g - P_s^T \underline{\mu}_g^*,$$

$$\underline{\mu}_g = P_c \underline{\mu} + P_s \underline{\mu}^* \quad (3B.36)$$

[Eqs. (3B.19b) and (3B.21b)]. The noise matrices  $S_x$ ,  $S_p$ ,  $S_{xp}$ ,  $T$  and  $Q$  for the two-mode GPS  $|\underline{\mu}_g\rangle$  are obtained by inserting these expressions (3B.35) for  $P_c$ ,  $P_s$ ,  $P_p$ , and  $P_x$  into Eqs. (3B.32). The components of the noise matrices  $T$  and  $Q$  were given explicitly in the preceding section [Eq. (3A.84)].

The phase angle  $\delta_x$  in the coordinate-space wave function  $\langle x | \underline{\mu}_g \rangle$  for the GPS  $|\underline{\mu}_g\rangle \equiv \mathbf{U}_g |\underline{\mu}_g\rangle_{\text{coh}}$  is obtained from Eq. (3B.16). The calculation is described in Appendix B. The result is

$$e^{ik\delta_x} = \frac{(\det P_x^*)^{\frac{1}{2}}}{|\det P_x|^{\frac{1}{2}}} \quad (3B.37)$$

[cf. Eq. (2B.39)].

To conclude this discussion of two-mode Gaussian wave functions, consider briefly the momentum-space wave function for a two-mode Gaussian pure state,  $\langle \underline{p} | \underline{\mu}_g \rangle$ , obtained by Fourier transforming  $\langle \underline{x} | \underline{\mu}_g \rangle$  [Eq. (3B.1)]; here the dimensionless momentum variables  $p_{\pm}$  are the eigenvalues of the Hermitian operators  $\hat{p}_{\pm}$ . The momentum-space wave function has the following form:

$$\begin{aligned} \langle \underline{p} | \underline{\mu}_g \rangle &= \int_{-\infty}^{\infty} \frac{d\underline{x}_+ d\underline{x}_-}{2\pi} e^{-i\hat{p}^T \underline{\hat{x}}} \langle \underline{x} | \underline{\mu}_g \rangle \\ &= \mathcal{N}_g e^{-\frac{1}{2}i\delta_p} e^{\frac{1}{2}i p_j^T x_0} e^{-i x_j^T p} e^{-\frac{1}{2}\Delta p^T M^{-1} \Delta p}, \end{aligned} \quad (3B.38a)$$

where the (real) normalization constant  $\mathcal{N}_g$  is

$$\mathcal{N}_g = (4\pi^2 \det S_p)^{-1/2} \quad (3B.38b)$$

[cf. Eqs. (3B.1), (3B.7), and (2B.40)]. The phase angle  $\delta_p$  is related to the coordinate-space phase angle  $\delta_x$  by

$$e^{i\delta_p} = e^{-i\delta_x} \frac{\det M}{|\det M|} = e^{-i\delta_x} \frac{\det(S_{xp} + \frac{1}{2}i1)}{|\det(S_{xp} + \frac{1}{2}i1)|}. \quad (3B.39)$$

For the state  $|\underline{\mu}_g\rangle = \mathbf{U}_g |\underline{\mu}_g\rangle_{\text{coh}}$  the phase factor  $e^{-\frac{1}{2}i\delta_p}$  is

$$e^{-\frac{1}{2}i\delta_p} = \frac{(\det P_p^*)^{1/2}}{|\det P_p|^{1/2}} \quad (3B.40)$$

[cf. Eq. (2B.42)].

The position and momentum probabilities have the usual Gaussian forms:

$$|\langle \tilde{x} | \tilde{\mu}_g \rangle|^2 = (4\pi^2 \det S_x) \hbar e^{-\frac{1}{2} \Delta \tilde{x}^T S_x^{-1} \Delta \tilde{x}}, \quad (3B.41a)$$

$$|\langle \tilde{p} | \tilde{\mu}_g \rangle|^2 = (4\pi^2 \det S_p) \hbar e^{-\frac{1}{2} \Delta \tilde{p}^T S_p^{-1} \Delta \tilde{p}}, \quad (3B.41b)$$

[cf. Eqs. (2B.43)].



### C. Four-component vector notation for two-mode GPS

The previous discussion has shown that the unitary operators that relate two-mode GPS to the vacuum state and to other two-mode GPS are rotation, mixing, displacement, and squeeze operators. Since these operators induce linear transformations on  $\alpha_{\pm}$  and  $\alpha_{\pm}^{\dagger}$  (or  $\hat{x}_{\pm}$  and  $\hat{p}_{\pm}$ ), it is useful to define the four-component operator column vectors

$$\underline{\alpha}_{\pm} \equiv \begin{pmatrix} \alpha_{\pm} \\ \alpha_{\pm}^* \end{pmatrix}, \quad \underline{\hat{x}}_{\pm} = \begin{pmatrix} \hat{x}_{\pm} \\ \hat{p}_{\pm} \end{pmatrix} = \mathbf{A}_{\pm} \underline{\alpha}_{\pm}, \quad (3C.1a)$$

$$\mathbf{A}_{\pm} \equiv \mathcal{Q}^{-\hbar} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = (\mathbf{A}_{\pm}^{\dagger})^{-1} \quad (3C.1b)$$

[cf. Eqs. (2C.1)]. Here and throughout this section the components of four-component vectors are grouped into two two-component vectors, and the components of four-dimensional matrices are grouped into four two-dimensional block matrices. The symbol  $\mathbf{1}$  is used to denote both the two- and four-dimensional identity matrices. The expectation values of these operator column vectors are column vectors of complex numbers (for  $\underline{\alpha}_{\pm}$ ) or real numbers (for  $\underline{\hat{x}}_{\pm}$ ):

$$\underline{\mu}_{\pm} \equiv \langle \underline{\alpha}_{\pm} \rangle \equiv \begin{pmatrix} \mu_{\pm} \\ \mu_{\pm}^* \end{pmatrix}, \quad \underline{\xi}_{\pm} \equiv \langle \underline{\hat{x}}_{\pm} \rangle = \begin{pmatrix} \hat{x}_{\pm 0} \\ \hat{p}_{\pm 0} \end{pmatrix} = \mathbf{A}_{\pm} \underline{\mu}_{\pm}. \quad (3C.2)$$

The adjoints of the operator column vectors are the row vectors

$$\underline{\alpha}_{\pm}^{\dagger} \equiv (\alpha_{\pm}^{\dagger} \quad \alpha_{\pm}^T), \quad \underline{\hat{x}}_{\pm}^{\dagger} \equiv (\hat{x}_{\pm}^T \quad \hat{p}_{\pm}^T) = \underline{\hat{x}}_{\pm}^T, \quad (3C.3)$$

where a superscript "T" means transpose. The transpose of the adjoint of an operator column vector is denoted by a superscript "\*":

$$(\underline{a}^\dagger)^T \equiv \underline{a}^* = \begin{pmatrix} \underline{a}^* \\ \underline{a} \end{pmatrix}, \quad (\underline{\hat{x}}^\dagger)^T \equiv \underline{\hat{x}}^* = \begin{pmatrix} \underline{\hat{x}} \\ \underline{\hat{p}} \end{pmatrix} = \underline{\hat{x}}. \quad (3C.4)$$

Similar definitions hold for column vectors of complex numbers. Note that the product of a column vector and a row vector, e.g.,  $\underline{a} \underline{a}^\dagger$ , is a tensor product (i.e., a four-dimensional matrix), whereas the product of a row vector and a column vector, e.g.,  $\underline{a}^\dagger \underline{a}$ , is a scalar product (i.e., an operator or number).

There are six (Hermitian) four-dimensional matrices, in addition to the identity matrix, that arise naturally with this vector notation. They are

$$\Sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (3C.5a)$$

$$\Gamma_1 \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \Gamma_2 \equiv \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Gamma_3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}. \quad (3C.5b)$$

Different, but equivalent, matrices have been used by Milburn<sup>33</sup> to discuss a subset of two-mode GPS. Each of these two sets of matrices satisfies properties analogous to those of the two-dimensional Pauli matrices. For example,

$$\Sigma_i \Sigma_j = \delta_{ij} 1 + i \varepsilon_{ijk} \Sigma_k, \quad \Gamma_i \Gamma_j = \delta_{ij} 1 + i \varepsilon_{ijk} \Gamma_k, \quad i, j, k = 1, 2, 3. \quad (3C.5c)$$

It is useful to define rotated versions of  $\Sigma_1, \Sigma_2$  and  $\Gamma_1, \Gamma_2$ :

$$\Sigma_\varphi \equiv \Sigma_1 \cos 2\varphi - \Sigma_2 \sin 2\varphi = \begin{pmatrix} 0 & e^{2i\varphi} \\ e^{-2i\varphi} & 0 \end{pmatrix}, \quad (3C.6a)$$

$$\Sigma_{\varphi - \frac{\pi}{2}} \equiv \Sigma_1 \sin 2\varphi + \Sigma_2 \cos 2\varphi = \begin{pmatrix} 0 & -ie^{2i\varphi} \\ ie^{-2i\varphi} & 0 \end{pmatrix}; \quad (3C.6b)$$

$$\Gamma_\varphi \equiv \Gamma_1 \cos 2\varphi - \Gamma_2 \sin 2\varphi = \begin{pmatrix} \sigma_\varphi & 0 \\ 0 & -\sigma_\varphi^* \end{pmatrix}, \quad (3C.6c)$$

$$\Gamma_{\varphi-\frac{1}{2}\pi} \equiv \Gamma_1 \sin 2\varphi + \Gamma_2 \cos 2\varphi = \begin{pmatrix} \sigma_{\varphi-\frac{1}{2}\pi} & 0 \\ 0 & -\sigma_{\varphi-\frac{1}{2}\pi}^* \end{pmatrix} \quad (3C.6d)$$

[cf. Eqs. (2C.6)]. Note that  $[\Sigma_{\varphi}, \Sigma_{\varphi-\frac{1}{2}\pi}] = \Sigma_3$ , and  $[\Gamma_{\varphi}, \Gamma_{\varphi-\frac{1}{2}\pi}] = \Gamma_3$ . The following projection matrices are also useful:

$$\frac{1}{2}(1 + \Sigma_3 \Gamma_3) \equiv P_+, \quad \frac{1}{2}(1 - \Sigma_3 \Gamma_3) \equiv P_-. \quad (3C.6e)$$

Some of the most useful properties of these matrices follow:

$$[P_{\pm}, \Sigma_j] = [\Gamma_j, \Sigma_3] = 0; \quad \Sigma_{\varphi} \Gamma_3 \Sigma_{\varphi} = -\Gamma_3; \quad \Sigma_{\varphi} \Gamma_{\chi} \Sigma_{\varphi} = -\Gamma_{\chi}^*; \quad (3C.7a)$$

$$\Sigma_{\varphi} \Sigma_{\varphi'} = e^{2i(\varphi-\varphi')\Sigma_3}; \quad \Sigma_{\varphi_+} \Sigma_{\varphi} \Sigma_{\varphi_-} = \Sigma_{\varphi_+ + \varphi_- - \varphi}; \quad (3C.7b)$$

$$e^{2i\chi\Gamma_3} \Gamma_{\chi} = \Gamma_{\chi+\chi'}, \quad e^{2i\varphi'\Sigma_3} \Sigma_{\varphi} = \Sigma_{\varphi+\varphi'}; \quad (3C.7c)$$

$$P_{\pm} e^{2i\chi\Gamma_3} \Sigma_{\varphi} P_{\pm} = P_{\pm} \Sigma_{\varphi \pm \chi}, \quad P_{\pm} \Sigma_3 \Gamma_{\chi} P_{\mp} = P_{\pm} \Sigma_3 \Gamma_{\chi}; \quad (3C.7d)$$

$$P_{\pm} \Sigma_{\varphi} \Gamma_1 P_{\mp} = P_{\pm} \Sigma_{\varphi} \Gamma_1, \quad P_{\pm} \Sigma_3 \Gamma_3 = \pm P_{\pm}; \quad (3C.7e)$$

$$P_+ \Gamma_1 e^{2i\varphi\Sigma_3} - P_- \Gamma_1 e^{-2i\varphi\Sigma_3} = i \Sigma_3 \Gamma_{\varphi-\frac{1}{2}\pi}. \quad (3C.7f)$$

The commutation relations for  $\mathbf{a}_{\pm}$ ,  $\mathbf{a}_{\pm}^{\dagger}$  and  $\hat{\mathbf{x}}_{\pm}$ ,  $\hat{\mathbf{p}}_{\pm}$  are expressed by the Hermitian commutator matrices

$$[\mathbf{a}_{\pm}, \mathbf{a}_{\pm}^{\dagger}] \equiv \mathbf{a}_{\pm} \mathbf{a}_{\pm}^{\dagger} - (\mathbf{a}_{\pm}^* \mathbf{a}_{\pm}^T)^T = \Sigma_3, \quad (3C.8a)$$

$$[\hat{\mathbf{x}}_{\pm}, \hat{\mathbf{x}}_{\pm}^T] \equiv \hat{\mathbf{x}}_{\pm} \hat{\mathbf{x}}_{\pm}^T - (\hat{\mathbf{x}}_{\pm} \hat{\mathbf{x}}_{\pm}^T)^T = \mathbf{A} \Sigma_3 \mathbf{A}^{\dagger} = -\Sigma_2 \quad (3C.8b)$$

[Eqs. (3A.9); cf. Eqs. (2C.7)].

The rotation, mixing, displacement, and squeeze operators for two modes are expressed in this vector notation in the following ways:

$$\mathbf{R}(\theta) = e^{i\theta_3} \exp[-\frac{1}{2}i \mathbf{a}_{\pm}^{\dagger} N_{\theta} \mathbf{a}_{\pm}].$$

$$N_{\theta} \equiv \theta_+ P_+ + \theta_- P_- = \theta_s \mathbf{1} + \theta_d \Sigma_3 \Gamma_3 = \underline{d} \mathbf{1} \quad (3C.9a)$$

[cf. Eqs. (3A.19)];

$$T(q, \chi) = \exp[-\frac{1}{2}i q \underline{a}^\dagger \Gamma_\chi \underline{a} \Sigma_3] \quad (3C.9b)$$

[Eqs. (3A.36)];

$$D(\underline{a}, \underline{\mu}) = \exp[\underline{a}^\dagger \Sigma_3 \underline{\mu}] \quad (3C.9c)$$

[Eq. (3A.29)];

$$S_{1+}(r_+, \varphi_+) S_{1-}(r_-, \varphi_-) = \exp[-\frac{1}{2}i \underline{a}^\dagger N_1 \underline{a}] ,$$

$$N_1 \equiv r_+ P_+ \Sigma_{\varphi_+} \Gamma_1 + r_- P_- \Sigma_{\varphi_-} \Gamma_1 \quad (3C.9d)$$

[Eq. (3A.50)];

$$S(r, \varphi) = \exp[-\frac{1}{2}i \underline{a}^\dagger N_2 \underline{a}] ,$$

$$N_2 \equiv r \Sigma_{\varphi} \Gamma_1 \Sigma_3 = i r \Sigma_{\varphi} \Gamma_1 = \begin{pmatrix} 0 & -ir e^{2i\varphi} \sigma_1 \\ ir e^{-2i\varphi} \sigma_1 & 0 \end{pmatrix} \quad (3C.9e)$$

[Eqs. (3A.53)].

A unitary transformation of  $\underline{a}$  or  $\hat{\underline{x}}$  generated by the (two-mode) displacement operator results in the addition of a constant column vector:

$$D(\underline{a}, \underline{\mu}) \underline{a} D^\dagger(\underline{a}, \underline{\mu}) = \underline{a} - \underline{\mu} , \quad D(\underline{a}, \underline{\mu}) \hat{\underline{x}} D^\dagger(\underline{a}, \underline{\mu}) = \hat{\underline{x}} - \underline{\xi} \quad (3C.10)$$

[Eq. (3A.31)]. Unitary transformations generated by rotation, mixing, and squeeze operators result in matrix transformations of  $\underline{a}$  (and  $\hat{\underline{x}}$ ). One way to obtain these transformation matrices is to note the following general relation, valid for an arbitrary four-dimensional symmetric matrix  $K$ , which

follows from the fact that the commutator matrix  $[\underline{a}, \underline{a}^\dagger] = \Sigma_3$ :

$$[\underline{a}^\dagger K \underline{a}, \underline{a}] = K_0 \underline{a}, \quad (3C.11a)$$

$$K_0 = -\Sigma_3 (K + \Sigma_1 K^T \Sigma_1) = \begin{pmatrix} -(K_a + K_d^T) & -(K_b + K_b^T) \\ (K_c + K_c^T) & (K_a^T + K_d) \end{pmatrix}, \quad (3C.11b)$$

where the matrix  $K$  is defined to be

$$K \equiv \begin{pmatrix} K_a & K_b \\ K_c & K_d \end{pmatrix}, \quad (3C.11c)$$

with  $K_a$ ,  $K_b$ ,  $K_c$ , and  $K_d$  arbitrary two-dimensional matrices. Note that if  $K_a$ ,  $K_b$ ,  $K_c$ , and  $K_d$  are all symmetric,

$$K_0 = -(K_a + K_d) \Sigma_3 + [K, \Sigma_3]. \quad (3C.11d)$$

Equation (3C.11a) implies that

$$e^{\underline{a}^\dagger K \underline{a}} \underline{a} e^{-\underline{a}^\dagger K \underline{a}} = e^{K_0} \underline{a}. \quad (3C.11e)$$

This relation, together with the expressions (3C.9), implies that the matrix transformation induced on  $\underline{a}$  by the rotation operators is

$$\mathbf{R}(\theta) \underline{a} \mathbf{R}^\dagger(\theta) = e^{i N_\theta \Sigma_3} \underline{a} \quad (3C.12)$$

[Eqs. (3A.23)]. The matrix transformation on  $\underline{a}$  induced by the mixing operator is

$$T(q, \chi) \underline{a} T^\dagger(q, \chi) = e^{i q \Gamma_\chi - \chi \pi} \underline{a} \quad (3C.13)$$

[Eqs. (3A.39)]. A product of two single-mode squeeze operators induces the matrix transformation

$$S_{1+}(\tau_+, \varphi_+) S_{1-}(\tau_-, \varphi_-) \underline{\alpha} S_{1-}^\dagger(\tau_-, \varphi_-) S_{1+}^\dagger(\tau_+, \varphi_+) = P_1 \underline{\alpha},$$

$$P_1 \equiv P_+ e^{\tau_+ \Sigma_{\varphi_+}} + P_- e^{\tau_- \Sigma_{\varphi_-}} = \begin{pmatrix} P_{1c} & P_{1s} \\ P_{1s}^* & P_{1c} \end{pmatrix} \quad (3C.14)$$

[Eqs. (3A.52)]. Finally, the matrix transformation induced on  $\underline{\alpha}$  by a two-mode squeeze operator is

$$S(\tau, \varphi) \underline{\alpha} S^\dagger(\tau, \varphi) = P_2 \underline{\alpha},$$

$$P_2 \equiv e^{\tau \Sigma_{\varphi} \Gamma_1 \Sigma_3} = \begin{pmatrix} \cosh r & e^{2i\varphi} \sinh r \sigma_1 \\ e^{-2i\varphi} \sinh r \sigma_1 & \cosh r \end{pmatrix} \quad (3C.15)$$

[Eqs. (3A.56)]. The simple form of the transformation matrix  $P_2$  associated with the two-mode squeeze operator  $S(\tau, \varphi)$  or, equivalently, the simple form of the matrix  $N_2$  that defines  $S(\tau, \varphi)$  [Eq. (3C.9e)], shows why the properties of two-mode squeezed states can be described using only a two-component vector notation. This two-component vector notation is one which naturally groups  $\alpha_+$  with  $\alpha_-^\dagger$  and  $\alpha_-$  with  $\alpha_+^\dagger$  [see Eq. (3A.60) and surrounding discussion].

The product of the transformation matrices (3C.14) and (3C.15), i.e., the transformation matrix that results from unitarily transforming  $\underline{\alpha}$  with the product  $S_{1+} S_{1-} S \equiv \mathbf{U}_g$  of three squeeze operators, is denoted by the symbol  $\mathbf{P}$ ,

$$S_{1+} S_{1-} S \underline{\alpha} (S_{1+} S_{1-} S)^\dagger \equiv \mathbf{U}_g \underline{\alpha} \mathbf{U}_g^\dagger = \mathbf{P} \underline{\alpha},$$

$$\mathbf{P} \equiv P_2 P_1 = \begin{pmatrix} P_c & P_s \\ P_s^* & P_c^* \end{pmatrix} \quad (3C.16)$$

[Eqs. (3A.73)]. The transformation matrices for  $\hat{\underline{x}}$  are unitarily related to

those for  $\underline{\alpha}$  by the unitary operator  $\mathbf{A}$  [Eq. (3C.1b)].

The transformation matrices (3C.12)-(3C.16) arise naturally, without specific reference to the rotation, mixing, or squeeze operators, from the requirement that a unitary transformation on  $\alpha_+$  and  $\alpha_-$  (or  $\hat{x}_+$ ,  $\hat{x}_-$ ,  $\hat{p}_+$ , and  $\hat{p}_-$ ) preserve the commutators (3C.9). Consider first the real, four-dimensional matrices  $\bar{M}$  that describe unitary transformations on the (real) column vector  $\hat{x}$ :  $\bar{M} \hat{x} \equiv U \hat{x} U^\dagger$ . The unitarity of  $U$  implies that these matrices preserve the (antisymmetric) commutator matrix  $[\hat{x}, \hat{x}^T] = -\Sigma_2$  [Eq. (3C.8b)], i.e.,

$$\bar{M} \Sigma_2 \bar{M}^T = \bar{M}^T \Sigma_2 \bar{M} = \Sigma_2 . \quad (3C.17a)$$

The real matrices  $\bar{M}$  that satisfy this condition have unity determinant. They comprise the ten-parameter symplectic group  $\text{Sp}(4, \text{R})$ .<sup>41</sup> Milburn<sup>33</sup> has used the properties of  $\text{Sp}(4, \text{R})$  to describe a subset of two-mode GPS which he calls "minimum-uncertainty states" (a more restrictive subset than that described in Section IIIA.7 of this paper). The complex, four-dimensional matrices  $M$  that describe unitary transformations on the column vector  $\underline{\alpha} = \mathbf{A}^\dagger \hat{x}$ ,  $M \underline{\alpha} \equiv U \underline{\alpha} U^\dagger$ , are unitarily related to the real matrices  $\bar{M}$  by the matrix  $\mathbf{A}$  [Eq. (3C.1b)]:

$$M = \mathbf{A}^\dagger \bar{M} \mathbf{A} . \quad (3C.17b)$$

These matrices  $M$  comprise a ten-parameter subgroup of the fifteen-parameter, noncompact group  $\text{SU}(2, 2)$ . The latter consists of all complex, four-dimensional matrices that have unitary determinant and that preserve the metric  $\Sigma_3$  (i.e., the commutator matrix  $[\underline{\alpha}, \underline{\alpha}^\dagger] = \Sigma_3$ ),

$$M \Sigma_3 M^\dagger = \Sigma_3 = M^\dagger \Sigma_3 M . \quad (3C.17c)$$

The ten free (real) parameters associated with the transformation matrices  $M$  and  $\bar{M}$  can be identified with the parameters of the unitary operators  $U$  that induce the matrix transformations. The generators of these unitary operators are bilinear combinations of the annihilation and creation operators for the two modes; i.e., these unitary operators are the rotation, mixing, and squeeze operators for the two modes. The underlying Lie algebra for these groups is that of (combinations of) the ten bilinear products of  $\alpha_+$ ,  $\alpha_-$ ,  $\alpha_+^\dagger$ , and  $\alpha_-^\dagger$ . The preceding discussion of the rotation, mixing, and squeeze operators shows that the matrices  $M$  have the general form

$$M = e^{iN\theta\Sigma_3} e^{iq\Gamma_{\chi-\tau}} \mathbf{P}, \quad (3C.17d)$$

where  $\theta_\pm$ ,  $\tau_\pm$ ,  $\tau$ ,  $\varphi_\pm$ ,  $\varphi$ ,  $q$ , and  $\chi$  are real, continuous parameters [Eqs. (3C.12)-(3C.16)].

The general forms for the transformation matrices  $M$  and  $\bar{M}$  can also be obtained in other ways. For example, note that any four-dimensional matrix  $M$  that describes a unitary transformation on  $\underline{\alpha}$  must satisfy

$$M^* = \Sigma_1 M \Sigma_1. \quad (3C.18a)$$

since  $\underline{\alpha} = \Sigma_1 \underline{\alpha}^*$ . This means that the matrix  $M$  has the general form

$$M = \begin{pmatrix} M_a & M_b \\ M_b^* & M_a^* \end{pmatrix}, \quad (3C.18b)$$

where  $M_a$  and  $M_b$  are arbitrary two-dimensional (complex) matrices. It also implies that

$$M \Sigma_3 M^\dagger \Sigma_3 = M \Sigma_2 M^T \Sigma_2 \quad (3C.18c)$$

[cf. Eq. (2C.16c)]. The unitarity of the transformation ensures that the



Hermitian commutator matrix  $[\underline{a}, \underline{a}^\dagger] = \Sigma_3$  is preserved [Eq. (3C.17c)] and that the antisymmetric commutator matrix  $[\underline{a}, \underline{a}^T] = i \Sigma_2$  is preserved,

$$M \Sigma_2 M^T = \Sigma_2 = M^T \Sigma_2 M ; \quad (3C.18d)$$

i.e., both products in (3C.18c) are equal to the identity matrix. These conditions remove six of the sixteen degrees of freedom associated with the matrix  $M$  of Eq. (3C.18b), by imposing the following equivalent sets of conditions on the two-dimensional matrices  $M_a$  and  $M_b$ :

$$M_a M_a^\dagger - M_b M_b^\dagger = 1 ,$$

$$M_a M_b^T - (M_a M_b^T)^T = 0 \quad (3C.18e)$$

[imposed by the first equality in Eq. (3C.17c) or (3C.18d)], or

$$M_a^\dagger M_a - M_b^T M_b^* = 1 ,$$

$$M_a^\dagger M_b - (M_a^\dagger M_b)^T = 0 \quad (3C.18f)$$

[imposed by the second equality in Eq. (3C.17c) or (3C.18d)]. These conditions also ensure that  $\det M = 1$ . Hence the matrices  $M$  have the general form (3C.17d), with ten free real parameters.

The four-component vector notation is a powerful aid in calculating the (second-order) noise matrices  $Q$ ,  $T$ ,  $S_x$ ,  $S_p$ , and  $S_{xp}$  for two-mode Gaussian (pure or mixed) states. The four-dimensional matrix that contains all second-order noise moments of  $\underline{a}_\pm$  and  $\underline{a}_\pm^\dagger$  is the Hermitian matrix

$$Q \equiv \langle \Delta \underline{a} \Delta \underline{a}^\dagger \rangle_{\text{sym}} \equiv \frac{1}{2} (\langle \Delta \underline{a} \Delta \underline{a}^\dagger \rangle + \langle \Delta \underline{a}^* \Delta \underline{a}^T \rangle^T) = \begin{pmatrix} Q & T \\ T^* & Q^* \end{pmatrix} = \mathcal{Q} . \quad (3C.19a)$$

The four-dimensional matrix that contains all second-order noise moments

of  $\hat{x}_\pm$  and  $\hat{p}_\pm$  is the (real, symmetric) covariance matrix

$$\begin{aligned} \mathcal{S} &\equiv \langle \Delta \hat{x}_\pm \Delta \hat{x}_\pm^T \rangle_{\text{sym}} \equiv \frac{1}{2} (\langle \Delta \hat{x}_\pm \Delta \hat{x}_\pm^T \rangle + \langle \Delta \hat{x}_\pm \Delta \hat{x}_\pm^T \rangle^T) \\ &= \begin{pmatrix} S_x & S_{xp} \\ S_{xp}^T & S_p \end{pmatrix} = \mathbf{A} \mathcal{Q} \mathbf{A}^\dagger = \mathcal{S}^* = \mathcal{S}^T . \end{aligned} \quad (3C.19b)$$

The relations (3B.11) imply that for two-mode GPS these matrices satisfy

$$\mathcal{Q} \Sigma_3 \mathcal{Q} \Sigma_3 = \frac{1}{2} \mathbf{1} , \quad (3C.20a)$$

$$\mathcal{S} \Sigma_2 \mathcal{S} \Sigma_2 = \frac{1}{2} \mathbf{1} \quad (3C.20b)$$

[cf. Eqs. (2C.22)]. Hence their determinants are both equal to  $\frac{1}{2}$ . For a two-mode coherent state both are proportional to the identity matrix:

$$\mathcal{Q}_{\text{coh}} = \mathcal{S}_{\text{coh}} = \frac{1}{2} \mathbf{1} \quad (3C.21)$$

[Eqs. (3A.89)]

The noise matrix  $\mathcal{Q}$  for a (two-mode) state  $|\Psi\rangle$  is related to that of a rotated state  $\mathbf{R}(\theta) |\Psi\rangle$  by

$$\langle \mathbf{R}^\dagger (\Delta \hat{\alpha}_\pm \Delta \hat{\alpha}_\pm^\dagger)_{\text{sym}} \mathbf{R} \rangle = e^{-iN_\theta \Sigma_3} \mathcal{Q} e^{iN_\theta \Sigma_3} , \quad (3C.22)$$

in agreement with Eqs. (3A.28). It is related to that of a state  $T(q, \chi) |\Psi\rangle$  by

$$\langle T^\dagger(q, \chi) (\Delta \hat{\alpha}_\pm \Delta \hat{\alpha}_\pm^\dagger)_{\text{sym}} T(q, \chi) \rangle = e^{-iq \Gamma_x - \frac{1}{2}\pi} \mathcal{Q} e^{iq \Gamma_x - \frac{1}{2}\pi} . \quad (3C.23)$$

Finally, the noise matrix  $\mathcal{Q}$  for a state  $|\Psi\rangle$  is related to that of a state  $S_{1+} S_{1-} S |\Psi\rangle = \mathbf{U}_g |\Psi\rangle$  by

$$\langle \mathbf{U}_g^\dagger (\Delta \hat{\alpha}_\pm \Delta \hat{\alpha}_\pm^\dagger)_{\text{sym}} \mathbf{U}_g \rangle = \mathbf{P}^{-1} \mathcal{Q} (\mathbf{P}^{-1})^\dagger . \quad (3C.24)$$

This immediately tells one, for example, that the noise matrix  $\mathcal{Q}$  for the

(most general) two-mode GPS  $|\underline{\mu}_g\rangle$  defined by Eq. (3B.23) is

$$Q_{\underline{\mu}_g} = \frac{1}{2}(\mathbf{P}^\dagger \mathbf{P})^{-1} = \frac{1}{2}\Sigma_3 \mathbf{P}^\dagger \mathbf{P} \Sigma_3 . \quad (3C.25)$$

These are precisely the noise moments derived in the preceding sections [Eqs. (3B.32) or (3A.84)].

Finally, the four-component vector notation is a useful aid for seeing how the unitary operator whose generator is an arbitrary combination of the Hermitian forms  $H_R^{(2)}$ ,  $H_1^{(2)}$ , and  $H_2^{(2)}$  factors into a product of three squeeze operators, a mixing operator, a (two-mode) rotation operator, and a (two-mode) displacement operator (and an overall phase factor). By giving these generators arbitrary time dependences, one can calculate the evolution operator associated with the most general combination of Hamiltonians that can produce two-mode GPS. This result is given here, with some supporting details presented in Appendix A.

The rotation Hamiltonians associated with two-mode GPS are expressed in vector notation by

$$H_{R+}^{(1)}(t) + H_{R-}^{(1)}(t) = -\frac{1}{2}\omega_s + \frac{1}{2}\mathbf{a}^\dagger N_\omega \mathbf{a} ,$$

$$N_\omega \equiv \omega_+ P_+ + \omega_- P_- = \omega_s \mathbf{1} + \omega_d \Sigma_3 \Gamma_3 ,$$

$$\omega_\pm \equiv \omega_s \pm \omega_d , \quad \omega_s \equiv \frac{1}{2}(\omega \pm \omega_-) ; \quad (3C.26a)$$

$$H_{R+-}(t) = \frac{1}{2}\rho \mathbf{a}^\dagger \Gamma_{\chi\rho - \frac{1}{4}\pi} \Sigma_3 \mathbf{a} , \quad (3C.26b)$$

where  $\omega_\pm$  (or  $\omega_s$ ,  $\omega_d$ ),  $\rho$ , and  $\chi\rho$  are real-valued function of time  $t$  [Eqs. (1.3), (3A.20), (3A.36); cf. Eqs. (3C.9a,b)]. The linear and quadratic Hamiltonians associated with two-mode GPS have the forms

$$H_1^{(2)}(t) = i \underline{\alpha}^\dagger \Sigma_3 \underline{\lambda},$$

$$\underline{\lambda} \equiv \begin{bmatrix} \underline{\lambda} \\ \underline{\lambda}^* \end{bmatrix} \quad \underline{\lambda} \equiv \begin{bmatrix} \lambda_+ \\ \lambda_- \end{bmatrix}, \quad (3C.26c)$$

where  $\lambda_\pm$  are complex-valued function of time  $t$ ;

$$H_{2+}^{(1)}(t) + H_{2-}^{(1)}(t) = \frac{1}{2} \underline{\alpha}^\dagger N_1 \underline{\alpha},$$

$$N_1 \equiv \kappa_+ P_+ \Sigma_{\varphi_{\kappa_+} - \chi\pi} + \kappa_- P_- \Sigma_{\varphi_{\kappa_-}} \quad (3C.26d)$$

[Eqs. (1.4); cf. Eqs. (3C.9c,d)], where  $\kappa_\pm$  and  $\varphi_{\kappa_\pm}$  are real-valued functions of  $t$ ; and

$$H_{2+-}(t) = \frac{1}{2} \underline{\alpha}^\dagger N_2 \underline{\alpha}, \quad N_2 \equiv \kappa \Sigma_{\varphi_{\kappa} - \chi\pi} \Gamma_1 \Sigma_3 \quad (3C.26e)$$

[cf. Eq. (3C.9e)], where  $\kappa$  and  $\varphi$  are real-valued functions of time  $t$ .

The evolution operator  $U(t)$  is the solution to the equation

$$i \partial_t U(t) = H_G^{(2)}(t) U(t), \quad U(0) = 1,$$

$$H_G^{(2)} \equiv [H_{R+}^{(1)} + H_{R-}^{(1)} + H_{R+-} + H_1^{(2)} + H_{2+}^{(1)} + H_{2-}^{(1)} + H_{2+-}]. \quad (3C.27)$$

It can be written as the product

$$U(t) = e^{i\delta} S_{1+}(\tau_+, \varphi_+) S_{1-}(\tau_-, \varphi_-) S(\tau, \varphi) T(q, \chi) \mathbf{R}(\theta_+, \theta_-) D(\underline{\alpha}, \underline{\mu}_g) \quad (3C.28a)$$

$$= e^{i\delta} D(\underline{\alpha}, \underline{\mu}) S_{1+}(\tau_+, \varphi_+) S_{1-}(\tau_-, \varphi_-) S(\tau, \varphi) T(q, \chi) \mathbf{R}(\theta_+, \theta_-), \quad (3C.28b)$$

where  $\tau_\pm$ ,  $\tau$ ,  $\varphi_\pm$ ,  $\varphi$ ,  $q$ ,  $\chi$ , and  $\theta_\pm$  are real-valued functions of time, and  $\mu_{g\pm}$  (or  $\mu_\pm$ ) are complex-valued functions of time [cf. Eqs. (3B.27)]. [For notational convenience here, I often drop explicit reference both to the time

dependence of these functions and to the dependence on these functions of the unitary operators, e.g.,  $\mathbf{r}_\pm \equiv \mathbf{r}_\pm(t)$ , and  $S_{1\pm} \equiv S_{1\pm}(\mathbf{r}_\pm, \varphi_\pm)$ .] The state  $|\underline{\mu}_g\rangle \equiv U(t) |0\rangle$  is an eigenstate of operators  $g_\pm = U(t) \mathbf{a}_\pm U^\dagger(t)$  (with eigenvalues  $\mu_{g\pm}$ ), whose relations to  $\mathbf{a}_\pm$  are described by the vector relation

$$\begin{aligned} \underline{g} &\equiv \begin{pmatrix} \underline{g} \\ \underline{g}^* \end{pmatrix} = S_{1+} S_{1-} S T \mathbf{R} \underline{\mathbf{a}} \mathbf{R}^\dagger T^\dagger S^\dagger S_{1-}^\dagger S_{1+}^\dagger \\ &= e^{iN_\theta \Sigma_3} e^{iq \Gamma_{X-\mathcal{K}} \pi} \mathbf{P} \underline{\mathbf{a}} \end{aligned} \quad (3C.29a)$$

[Eqs. (3C.12)-(3C.16)]. The eigenvalues  $\mu_{g\pm}$  are therefore related to the complex amplitudes  $\mu_\pm \equiv \langle \mathbf{a}_\pm \rangle$  by

$$\underline{\mu}_g \equiv \begin{pmatrix} \underline{\mu}_g \\ \underline{\mu}_g^* \end{pmatrix} = e^{iN_\theta \Sigma_3} e^{iq \Gamma_{X-\mathcal{K}} \pi} \mathbf{P} \underline{\mu}. \quad (3C.29b)$$

The relations of the functions  $\mathbf{r}$ ,  $\mathbf{r}_\pm$ ,  $\varphi$ ,  $\varphi_\pm$ ,  $\theta_\pm$  (or  $\theta_s$ ,  $\theta_d$ ),  $\mu_{g\pm}$  (or  $\mu_\pm$ ), and  $\delta$  to the Hamiltonian functions  $\kappa$ ,  $\kappa_\pm$ ,  $\varphi_\kappa$ ,  $\varphi_{\kappa_\pm}$ ,  $\omega_\pm$  (or  $\omega_s$ ,  $\omega_d$ ), and  $\lambda_\pm$  take the form of matrix, vector, and scalar equalities. The vector and scalar equalities are

$$\dot{\underline{\mu}}_g = e^{iN_\theta \Sigma_3} e^{iq \Gamma_{X-\mathcal{K}} \pi} \mathbf{P} \underline{\lambda}, \quad (3C.30a)$$

$$\dot{\delta} + \dot{\theta}_s - \dot{\omega}_s = -\frac{1}{2} i \underline{\mu}_g^\dagger \Sigma_3 \dot{\underline{\mu}}_g = \text{Im}(\mu_+^* \dot{\lambda}_+ + \mu_-^* \dot{\lambda}_-). \quad (3C.30b)$$

(Dots denote derivatives with respect to time.) The matrix equality is given in its full generality in Appendix A [Eqs. (A.13)-(A.15)]. The initial conditions, dictated by  $U(0) = 1$ , are

$$\delta(0) = \mathbf{r}(0) = \mathbf{r}_\pm(0) = \mathbf{q}(0) = \theta_\pm(0) = \mu_{g\pm}(0) = \mu_\pm(0) = 0. \quad (3C.31)$$

For illustration, consider the case

$$\varphi_{\kappa_{\pm}} \equiv \varphi_{\kappa_{\pm 0}} - \int_0^t \omega_{\pm} dt \equiv \varphi_{\kappa} \pm \chi_{\rho},$$

$$\varphi_{\kappa} \equiv \varphi_{\kappa_0} - \int_0^t \omega_s dt, \quad \chi_{\rho} \equiv \chi_{\rho_0} - \int_0^t \omega_d dt, \quad (3C.32)$$

where  $\varphi_{\kappa_{\pm 0}}$ ,  $\varphi_{\kappa_0}$ , and  $\chi_{\rho_0}$  are constants. The matrix equality then implies that

$$\varphi_{\pm} = \varphi_{\kappa_{\pm}}, \quad \varphi = \varphi_{\kappa}, \quad \chi = \chi_{\rho}, \quad \theta_{\pm} = \int_0^t \omega_{\pm} dt; \quad (3C.33a)$$

$$\rho = \dot{q} \cosh 2r \cosh 2r_d - \dot{r} \sinh 2r_d,$$

$$\kappa_{\pm} = \dot{r}_{\pm} + \dot{q} \sinh 2r,$$

$$\kappa = \dot{r} \cosh 2r_d - \dot{q} \cosh 2r \sinh 2r_d, \quad (3C.33b)$$

$$r_{\pm} \equiv r_s \pm r_d. \quad (3C.33c)$$

If the mixing interaction is absent ( $\rho = 0$ ), and if  $\kappa_+ = \kappa_- \equiv \kappa'$ , the four coupled equations (3C.33b) have the following simple solutions:

$$r_+ = r_- = \int_0^t \kappa' dt, \quad r = \int_0^t \kappa dt, \quad q = 0. \quad (3C.34)$$

The phase angle  $\delta$  and complex numbers  $\mu_{g_{\pm}}$  (or  $\mu_{\pm}$ ) are obtained by using these solutions to solve the vector and scalar equalities (3C.30a,b).

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## Appendix A: Evolution operators for GPS

In this appendix I derive certain properties of the unitary operators associated with single-mode and two-mode GPS which are useful for calculating the general evolution operators described in Sections IIC and IIIC [Eqs. (2C.28)-(2C.39), Eqs. (3C.26)-(3C.33)].

### 1. Single-mode GPS

One way to find the single-mode evolution operator  $U(t)$  defined by Eqs. (2C.28)-(2C.30) is to take the derivative with respect to time of either of the factored expressions (2C.30) for  $U(t)$ , and match terms with the Hamiltonian. This is the approach described here. An alternative approach, which permits calculation of everything except the phase factor  $e^{i\delta}$  in the expressions (2C.30) for  $U(t)$ , is to solve in the Heisenberg picture the matrix equation  $\underline{\alpha}(t) = M(t)\underline{\alpha}(0)$ , and identify the unitary operator  $U_g$  that generates  $M$ ; then  $\underline{\alpha}(t) = U_g^\dagger \underline{\alpha}(0) U_g$ .<sup>29</sup> The product of  $U_g$  and a displacement operator is the evolution operator  $U(t)$ , up to an overall phase factor.

The first (and hardest) task involved with computing the time derivative of the expressions (2C.30) for  $U(t)$  is to compute the (first-order) derivatives of each of the unitary operators  $S_1(\tau, \varphi)$ ,  $R(\theta)$ , and  $D(\alpha, \mu)$ ; the second task is to commute these operators through each other. The time derivative of the expression (2C.30a) for  $U(t)$  is

$$\dot{U} U^{-1} = i\dot{\delta} + \dot{S}_1 S_1^\dagger + S_1 (\dot{R} R^\dagger) S_1^\dagger + S_1 R (\dot{D}_g D_g^\dagger) R^\dagger S_1^\dagger, \quad (\text{A.1})$$

where  $D_g \equiv D(\alpha, \mu_g)$ , and a superposed dot denotes a single derivative with respect to time. These time derivatives can be found using the general rule

$$[\partial_t e^{f(t)}] e^{-f(t)} = \sum_{n=0}^{\infty} \frac{\{f^n \dot{f}\}}{(n+1)!} \equiv \dot{f} + \frac{1}{2!}[f, \dot{f}] + \frac{1}{3!}[f, [f, \dot{f}]] + \dots \quad (\text{A.2})$$



(derived in Part 3 of this appendix). The time derivative of  $S_1(\tau, \varphi)$  can be calculated with the help of the following facts:

$$\dot{\sigma}_\varphi = -2\dot{\varphi} \sigma_{\varphi-\chi\pi}, \quad \dot{\sigma}_{\varphi-\chi\pi} = 2\dot{\varphi} \sigma_\varphi; \quad (\text{A.3a})$$

$$[\tilde{a}^\dagger \sigma_{\varphi-\chi\pi} \tilde{a}, \tilde{a}^\dagger \sigma_\varphi \tilde{a}] = 4i \tilde{a}^\dagger \tilde{a}, \quad (\text{A.3b})$$

$$[\tilde{a}^\dagger \sigma_{\varphi-\chi\pi} \tilde{a}, \tilde{a}^\dagger \tilde{a}] = 4i \tilde{a}^\dagger \sigma_\varphi \tilde{a}. \quad (\text{A.3c})$$

The result is

$$\dot{S}_1 S_1^\dagger = -\frac{1}{2}i \tilde{a}^\dagger [-\dot{\varphi} 1 + \dot{\varphi} C_{2\tau, \varphi} + \dot{\tau} \sigma_{\varphi-\chi\pi}] \tilde{a}. \quad (\text{A.4})$$

The time derivatives of  $R(\theta)$  and  $D(\alpha, \mu_g)$  are

$$\dot{R} R^\dagger = -\frac{1}{2}i \dot{\theta} (\tilde{a}^\dagger \tilde{a} - 1), \quad (\text{A.5})$$

$$\dot{D}_g D_g^\dagger = (\tilde{a}^\dagger - \frac{1}{2}\mu_g^\dagger) \sigma_3 \dot{\mu}_g. \quad (\text{A.6})$$

Note that  $\dot{\mu}_g \equiv \partial_t(\mu_g) \neq (\dot{\mu})_g$ .

Commuting the operators through each other to find the last two terms in Eq. (A.1) is accomplished using the transformations in Eqs. (2C.10) and (2C.11). Equating  $i \dot{U} U^\dagger$  to the sum of the Hamiltonians on the right-hand side of Eq. (2C.29) then results in the relations (2C.32), which define the functions  $\tau, \varphi, \theta, \mu_g$  (or  $\mu$ ), and  $\delta$  uniquely in terms of the Hamiltonian functions  $\kappa, \varphi_\kappa, \Omega$ , and  $\lambda$ .

## 2. Two-mode GPS

As in the single-mode case, the first task is to calculate the (first-order) derivatives of the various unitary operators, and the second is to commute the operators through each other. The time derivative of the expression (3C.28a) for  $U(t)$  is

$$\begin{aligned} \dot{U} U^\dagger &= i\dot{\delta} + \dot{U}_g U_g^\dagger + U_g (\dot{T} T^\dagger) U_g^\dagger + U_g T (\dot{\mathbf{R}}\mathbf{R}^\dagger) T^\dagger U_g^\dagger \\ &+ U_g T \mathbf{R} (\dot{D}_g D_g^\dagger) \mathbf{R}^\dagger T^\dagger U_g^\dagger, \end{aligned} \quad (\text{A.7})$$

where  $D_g \equiv D(\underline{a}, \underline{\mu}_g)$ ,  $U_g \equiv S_{1+} S_{1-} S$ , and a superposed dot denotes a single derivative with respect to time. Using the relation (A.2) one finds the following expressions for the derivatives of these operators:

$$\begin{aligned} \dot{\mathbf{R}}\mathbf{R}^\dagger &= i\dot{\theta}_s - \frac{1}{2}i \underline{a}^\dagger N_\phi \underline{a}, \\ N_\phi &\equiv \dot{\theta}_+ P_+ + \dot{\theta}_- P_- = \dot{\theta}_s \mathbf{1} + \dot{\theta}_d \Sigma_3 \Gamma_3; \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} \dot{T} T^\dagger &= -\frac{1}{2}i \underline{a}^\dagger M_T \underline{a}, \\ M_T &\equiv -\dot{\chi} \Sigma_3 \Gamma_3 (1 - e^{2iq\Gamma\chi - \chi\pi}) + \dot{q} \Gamma_{\chi - \chi\pi} \Sigma_3; \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} (\dot{\theta}_t S_{1+} S_{1-}) (S_{1+} S_{1-})^\dagger &= -\frac{1}{2}i \underline{a}^\dagger M_1 \underline{a}, \\ M_1 &\equiv \dot{\varphi}_+ P_+ (e^{2r_+ \Sigma_{\varphi_+}} - 1) + \dot{\varphi}_- P_- (e^{2r_- \Sigma_{\varphi_-}} - 1) \\ &+ \dot{r}_+ P_+ \Sigma_{\varphi_+ - \chi\pi} + \dot{r}_- P_- \Sigma_{\varphi_- - \chi\pi}; \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \dot{S} S^\dagger &= -\frac{1}{2}i \underline{a}^\dagger M_2 \underline{a}, \\ M_2 &\equiv \dot{\varphi} (e^{2r \Sigma_\varphi \Gamma_1 \Sigma_3} - 1) + \dot{r} \Sigma_{\varphi - \chi\pi} \Gamma_1 \Sigma_3; \end{aligned} \quad (\text{A.11})$$

$$\dot{D}_g D_g^\dagger = (\underline{\alpha}^\dagger - \frac{1}{2} \underline{\mu}_g^\dagger) \Sigma_3 \underline{\mu}_g . \quad (\text{A.12})$$

Commuting the operators through each other to find the last three terms in Eq. (A.7) can be accomplished with the help of the transformations described in Eqs. (3C.12)-(3C.16). Equating  $i\dot{U} U^\dagger$  to the sum of the Hamiltonians on the right-hand side of Eq. (3C.27) then results in matrix, vector, and scalar equalities, which define the functions  $\tau$ ,  $\tau_\pm$ ,  $\varphi$ ,  $\varphi_\pm$ ,  $q$ ,  $\chi$ ,  $\theta_\pm$ ,  $\mu_{g\pm}$  (or  $\mu_\pm$ ), and  $\delta$  uniquely in terms of the Hamiltonian functions  $\kappa$ ,  $\kappa_\pm$ ,  $\varphi_\kappa$ ,  $\varphi_{\kappa_\pm}$ ,  $\rho$ ,  $\chi_\rho$ ,  $\Omega_\pm$ , and  $\lambda_\pm$ . The vector and scalar equalities were given in Eqs. (3C.30). The matrix equality is

$$\begin{aligned} N_\omega + \rho \Sigma_3 \Gamma_{\chi_\rho - \frac{1}{2}\pi} + \kappa_+ P_+ \Sigma_{\varphi_{\kappa_+} - \frac{1}{2}\pi} + \kappa_- P_- \Sigma_{\varphi_{\kappa_-} - \frac{1}{2}\pi} + \kappa + \Sigma_{\varphi_\kappa - \frac{1}{2}\pi} \Gamma_1 \Sigma_3 \\ = M_1 + P_1 M_2 P_1 + \mathbf{P}^\dagger M_T \mathbf{P} + \mathbf{P}^\dagger e^{-iq \Gamma_{\chi - \frac{1}{2}\pi}} N_\phi e^{iq \Gamma_{\chi - \frac{1}{2}\pi}} \mathbf{P} \end{aligned} \quad (\text{A.13})$$

[Eqs. (3C.12)-(3C.16) and (A.8)-(A.12)]. The matrix transformations required in order to put the right-hand side of Eq. (A.13) into a form that is easily compared with the left-hand side (the Hamiltonian) can be accomplished fairly easily by making use of the properties of the matrices  $\Sigma_i$  and  $\Gamma_i$  noted in Eqs. (3C.5)-(3C.7). The terms that comprise the right-hand side of Eq. (A.13) are listed below, with the four-dimensional matrix that multiplies it listed at the left of each term. The following shorthand notations are used:

$$\varphi_\pm - \varphi \mp \chi \equiv \delta_\pm ,$$

$$\chi + \theta_d \equiv \gamma_d ,$$

$$\varphi + \theta_s \equiv \gamma_s ,$$

$$B_\varphi \equiv \Sigma_\varphi \Gamma_1 \Sigma_3 = i \Gamma_1 \Sigma_{\varphi - \frac{1}{2}\pi} . \quad (\text{A.14})$$

The terms are as follows:

$$\begin{aligned}
 P_+ : \quad & -\dot{\varphi}_+ + \cosh 2r_+ [\dot{\delta}_+ + \dot{\gamma}_s \cosh 2r + \dot{\gamma}_d \cos 2q] \\
 & + \sinh 2r \sinh 2r_+ [\dot{\gamma}_d \sin 2q \cos 2\delta_+ - \dot{q} \sin 2\delta_+], \quad (A.15a)
 \end{aligned}$$

$$\begin{aligned}
 P_- : \quad & -\dot{\varphi}_- + \cosh 2r_- [\dot{\delta}_- + \dot{\gamma}_s \cosh 2r - \dot{\gamma}_d \cos 2q] \\
 & + \sinh 2r \sinh 2r_- [\dot{\gamma}_d \sin 2q \cos 2\delta_- + \dot{q} \sin 2\delta_-], \quad (A.15b)
 \end{aligned}$$

$$\begin{aligned}
 \Sigma_3 \Gamma_{\chi - \frac{\chi}{2}\pi} : \quad & \dot{q} \cosh 2r \cosh r_+ \cosh r_-, \\
 \Sigma_3 \Gamma_{\delta_+ - \delta_- + \chi - \frac{\chi}{2}\pi} : \quad & -\dot{q} \cosh 2r \sinh r_+ \sinh r_-, \\
 \Sigma_3 \Gamma_{\delta_+ + \chi - \frac{\chi}{2}\pi} : \quad & -\dot{r} \sinh r_+ \cosh r_-, \\
 \Sigma_3 \Gamma_{\chi - \delta_- - \frac{\chi}{2}\pi} : \quad & \dot{r} \sinh r_- \cosh r_+; \quad (A.15c)
 \end{aligned}$$

$$\begin{aligned}
 P_+ \Sigma_{\varphi_+ - \frac{\chi}{2}\pi} : \quad & \dot{r}_+, \\
 P_+ \Sigma_{\varphi_+ + \delta_+ - \frac{\chi}{2}\pi} : \quad & -\dot{q} \sinh 2r \sinh^2 r_+, \\
 P_+ \Sigma_{\varphi_+ - \delta_+ - \frac{\chi}{2}\pi} : \quad & \dot{q} \sinh 2r \cosh^2 r_+; \quad (A.15d)
 \end{aligned}$$

$$\begin{aligned}
 P_- \Sigma_{\varphi_- - \frac{\chi}{2}\pi} : \quad & \dot{r}_-, \\
 P_- \Sigma_{\varphi_- + \delta_- - \frac{\chi}{2}\pi} : \quad & \dot{q} \sinh 2r \sinh^2 r_-, \\
 P_- \Sigma_{\varphi_- - \delta_- - \frac{\chi}{2}\pi} : \quad & -\dot{q} \sinh 2r \cosh^2 r_-; \quad (A.15e)
 \end{aligned}$$

$$\begin{aligned}
 B_{\varphi - \frac{\chi}{2}\pi} : \quad & \dot{r} \cosh r_+ \cosh r_-, \\
 B_{\varphi + \delta_+ + \delta_- - \frac{\chi}{2}\pi} : \quad & -\dot{r} \sinh r_+ \sinh r_-, \\
 B_{\varphi + \delta_+ - \frac{\chi}{2}\pi} : \quad & -\dot{q} \cosh 2r \sinh r_+ \cosh r_-.
 \end{aligned}$$

$$B_{\varphi+\delta_-\chi\pi} : \dot{q} \cosh 2r \sinh r_- \cosh r_+ ; \quad (\text{A.15f})$$

$$B_{\varphi} : \dot{\gamma}_s \sinh 2r \cosh r_+ \cosh r_- ,$$

$$B_{\varphi+\delta_+\delta_-} : \dot{\gamma}_s \sinh 2r \sinh r_+ \sinh r_- ,$$

$$B_{\varphi+\delta_+} : \dot{\gamma}_d \sin 2q \cosh 2r \sinh r_+ \cosh r_- ,$$

$$B_{\varphi+\delta_-} : \dot{\gamma}_d \sin 2q \cosh 2r \sinh r_- \cosh r_+ ; \quad (\text{A.15g})$$

$$\Sigma_3 \Gamma_{\chi} : \dot{\gamma}_d \sin 2q \cosh 2r \cosh r_+ \cosh r_- ,$$

$$\Sigma_3 \Gamma_{\chi+\delta_+\delta_-} : \dot{\gamma}_d \sin 2q \cosh 2r \sinh r_+ \sinh r_- ,$$

$$\Sigma_3 \Gamma_{\chi+\delta_+} : \dot{\gamma}_s \sinh 2r \sinh r_+ \cosh r_- ,$$

$$\Sigma_3 \Gamma_{\chi-\delta_-} : \dot{\gamma}_s \sinh 2r \sinh r_- \cosh r_+ ; \quad (\text{A.15h})$$

$$P_+ \Sigma_{\varphi_+} : \sinh 2r_+ [\dot{\delta}_+ + \dot{\gamma}_s \cosh 2r + \dot{\gamma}_d \cos 2q] ,$$

$$P_+ \Sigma_{\varphi_+\delta_+} : \dot{\gamma}_d \sin 2q \sinh 2r \sinh^2 r_+ ,$$

$$P_+ \Sigma_{\varphi_+\delta_-} : \dot{\gamma}_d \sin 2q \sinh 2r \cosh^2 r_+ ; \quad (\text{A.15i})$$

$$P_- \Sigma_{\varphi_-} : \sinh 2r_- [\dot{\delta}_- + \dot{\gamma}_s \cosh 2r - \dot{\gamma}_d \cos 2q] ,$$

$$P_- \Sigma_{\varphi_-\delta_-} : \dot{\gamma}_d \sin 2q \sinh 2r \sinh^2 r_- ,$$

$$P_- \Sigma_{\varphi_-\delta_+} : \dot{\gamma}_d \sin 2q \sinh 2r \cosh^2 r_- . \quad (\text{A.15j})$$

The obvious simplifying case is that considered in Section IIIC [Eq. (3C.32)],

i.e., the case with

$$\delta_+ = \delta_- = 0 . \quad (\text{A.16})$$

The results for this case and further simplifying cases are given in Section IIIC.

### 3. Derivation of Equation (A.2)

The formula (A.2) for  $\partial_t e^{f(t)}$  can be found as follows: First, use the standard definition for derivatives, and keep only terms of lowest order in  $\Delta t$ :

$$\partial_t e^{f(t)} = \lim_{\Delta t \rightarrow 0} \left[ \frac{e^{f(t+\Delta t)} - e^{f(t)}}{\Delta t} \right] = \lim_{\Delta t \rightarrow 0} \left[ \frac{e^{f(t)+\Delta t \dot{f}(t)} - e^{f(t)}}{\Delta t} \right]. \quad (\text{A.17})$$

Next, use the definition of  $e^x$  to write

$$e^{f+\Delta t \dot{f}} = \lim_{n \rightarrow \infty} \left[ e^f + \sum_{j=0}^{n-1} \left( 1 + \frac{f}{n} \right)^j \left( \frac{\Delta t \dot{f}}{n} \right) \left( 1 + \frac{f}{n} \right)^{n-j-1} + O(\Delta t)^2 \right]. \quad (\text{A.18})$$

As  $n \rightarrow \infty$ , let  $1/n \rightarrow dx$ , where the variable  $x (=j/n)$  runs from 0 to 1.

Then

$$e^{f+\Delta t \dot{f}} - e^f = \Delta t \int_0^1 dx e^{xf} \dot{f} e^{-xf} e^f + O(\Delta t)^2. \quad (\text{A.19})$$

The relation (A.2) is then proved by noting that

$$e^{xf} \dot{f} e^{-xf} = \sum_0^{\infty} \frac{x^n}{n!} \{f^n \dot{f}\} \equiv \dot{f} + x [f, \dot{f}] + \frac{x^2}{2!} [f, [f, \dot{f}]] + \dots \quad (\text{A.20})$$

### Appendix B: Phase factors for wave functions

The phase factors  $e^{\frac{1}{2}i\delta_x}$  and  $e^{\frac{1}{2}i\delta_p}$  in the single-mode and two-mode coordinate- and momentum-space wave functions can be found using Eqs. (2B.16) and (3B.16), respectively. The method described here for single-mode GPS (i.e., single-mode squeezed states) makes use of a factored form for the single-mode squeeze operator [Eq.(2C.17)]. Another method, not described here but straightforward, uses a differential equation approach.<sup>51</sup> For the most general two-mode GPS the calculation is more challenging. One could try to use the same method described here for single-mode GPS, which would require computing a factored forms(s) for the product of the three squeeze operators. These factored forms are not as convenient as their single-mode counterparts, however, because they involve many operators that do not leave the vacuum state unchanged. The differential equation approach, while possibly more promising, still involves a painful process of commuting operators through each other. Although no simple derivation is given here, the phase factor for a general two-mode GPS can be guessed with reasonable certainty. Of course, for two-mode squeezed states the calculation is no more difficult than for a single-mode squeezed state, since the two-mode squeeze operator factors just as easily. Similarly, for states that are a product of two single-mode squeezed states the phase factor is just the product of the single-mode phase factors.

The coordinate-space phase factor  $e^{\frac{1}{2}i\delta_x}$  for a single-mode squeezed state is found, from Eq. (2B.16), by calculating

$$\langle x = 0 | S_1(r, \varphi) | 0 \rangle . \quad (\text{B.1})$$

The factored form for  $S_s(r, \varphi)$  given in Eq. (2C.17) implies that

$$S_1(r, \varphi) | 0 \rangle = (\cosh r)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}e^{2i\varphi} \tanh r)^n}{n!} \sqrt{(2n)!} | 2n \rangle . \quad (\text{B.2})$$

The wave function for a number state  $|n\rangle \equiv (n!)^{-\frac{1}{2}}(\alpha^\dagger)^n |0\rangle$  is<sup>17</sup>

$$\langle x | n \rangle = (2^n n!)^{-\frac{1}{2}} H_n(x) \langle x | 0 \rangle, \quad (\text{B.3a})$$

where

$$\langle x | 0 \rangle = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}, \quad (\text{B.3b})$$

and  $H_n(x)$  is a Hermite polynomial with the property that

$$H_{2n}(0) = (-1)^n 2^n (2n-1)!! . \quad (\text{B.3c})$$

These relations then imply that

$$\begin{aligned} \langle x=0 | S_1(r, \varphi) | 0 \rangle &= \pi^{-\frac{1}{4}} (\cosh r)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!} (\frac{1}{2} e^{2i\varphi} \tanh r)^n \\ &= \pi^{-\frac{1}{4}} (\cosh r - e^{2i\varphi} \sinh r)^{-\frac{1}{2}}. \end{aligned} \quad (\text{B.4})$$

The phase factor in the coordinate-space wave function of a single-mode squeezed state  $|\mu_\alpha\rangle_{(r, \varphi)}$  is therefore

$$e^{\frac{1}{2}i\delta_x} = \frac{(\cosh r - e^{-2i\varphi} \sinh r)^{\frac{1}{2}}}{|\cosh r - e^{-2i\varphi} \sinh r|^{\frac{1}{2}}} = \frac{(\rho_x^*)^{\frac{1}{2}}}{|\rho_x|^{\frac{1}{2}}}, \quad (\text{B.5})$$

as given in Eq. (2B.39).

The phase factors in the wave functions of two-mode squeezed states can be calculated in this same way, since factored forms for the two-mode squeezed operator are known and result in no more complication than encountered above. Using the factored form analogous to that in Eq. (2C.17), i.e., with the relevant two-mode operators replacing the single-mode operators (see the discussion in Section IIIA.5), one obtains the following phase factor for the wave function of a two-mode squeezed state  $|\mu_\alpha\rangle_{(r, \varphi)}$ :



$$e^{\frac{1}{2}i\delta_z} = \frac{(\cosh^2 r - e^{-4i\varphi} \sinh^2 r)^{\frac{1}{2}}}{|\cosh^2 r - e^{-4i\varphi} \sinh^2 r|^{\frac{1}{2}}}. \quad (\text{B.6})$$

For a product of two single-mode squeezed states the phase factor is simply the product of the phase factors for two single-mode squeeze states, as given by Eq. (B.5). These results, and the form of the single-mode phase factor (B.5), suggest strongly that the phase factor for the coordinate-space wave function for the general two-mode GPS  $|\underline{\mu}_g\rangle \equiv S_{1+} S_{1-} S |\underline{\mu}\rangle_{\text{coh}}$  is given by the following expression:

$$e^{\frac{1}{2}i\delta_z} = \frac{(\det P_z^*)^{\frac{1}{2}}}{|\det P_z|^{\frac{1}{2}}}. \quad (\text{B.7})$$

### Appendix C: Simultaneous eigenstates of complex operators

The complex (non-Hermitian) operators relevant for Gaussian states are linear combinations of creation and annihilation operators. The commutators of such operators with their Hermitian conjugates are (real) numbers. The discussion in Sections IIB and IIIB showed that a single operator  $g$  of this type has a complete set of normalizable eigenstates if and only if the commutator  $[g, g^\dagger]$  is a positive real number, since the latter is equivalent to the condition that the wave function be normalizable (if  $[g, g^\dagger] = 0$ , the wave function is a delta function).

Two-mode Gaussian pure states are eigenstates of two linearly independent complex operators  $g_+$  and  $g_-$ , each of which is a linear combination of the creation and annihilation operators for the two modes. The discussion in Section IIIB showed that two such operators have a common, complete (overcomplete) set of normalizable eigenstates if and only if (i) the commutator  $[g_+, g_-] = 0$ , and (ii) the commutator matrix  $\Upsilon_g$  is positive-definite; these conditions followed from requiring that the (two-mode) wave function be normalizable. The commutator matrix  $\Upsilon_g$  is defined as

$$\Upsilon_g \equiv [g, g^\dagger] = \begin{pmatrix} [g_+, g_+^\dagger] & [g_+, g_-^\dagger] \\ [g_-, g_+^\dagger] & [g_-, g_-^\dagger] \end{pmatrix} = \Upsilon_{g^\dagger},$$

$$\underline{g} \equiv \begin{pmatrix} g_+ \\ g_- \end{pmatrix}. \tag{C.1}$$

The requirement that  $g_+$  and  $g_-$  commute with each other if they are to have a complete set of simultaneous eigenstates is obvious, without recourse to a wave function. The further requirement that  $\Upsilon_g$  be positive-definite, but not necessarily diagonal, is not so obvious; one might expect at first that the two operators must commute completely, i.e., that the commutator

$[g_+, g_-^\dagger]$  must also vanish. Following is a simple argument that shows why  $\Upsilon_g$  must be positive-definite, and why it need not be diagonal.

Let  $g_+$  and  $g_-$  be two complex operators (with  $c$ -number commutators) that commute with each other completely,

$$[g_+, g_-] = [g_+, g_-^\dagger] = 0. \quad (\text{C.2})$$

Suppose also that each has a complete (or overcomplete) set of normalizable eigenstates, i.e.,  $[g_\pm, g_\pm^\dagger] > 0$ . Clearly there exist normalizable states that are eigenstates of both  $g_+$  and  $g_-$ , and hence also of all linear combinations of  $g_+$  and  $g_-$ . Consider two such (independent) linear combinations,  $g_{+'}$  and  $g_{-}'$ , defined by

$$\underline{g}' \equiv \begin{pmatrix} g_{+'} \\ g_{-}' \end{pmatrix} = K \underline{g}, \quad (\text{C.3})$$

where  $K$  is any two-dimensional nonsingular matrix ( $\det K \neq 0$ ). The operators  $g_{+'}$  and  $g_{-}'$  will certainly commute with each other,  $[g_{+'}, g_{-}'] = 0$ , but the commutator  $[g_{+'}, g_{-}'^\dagger]$  will not, in general, be zero. It will vanish if, for example, the operators  $g_{+'}$  and  $g_{-}'$  are obtained by unitarily transforming  $g_+$  and  $g_-$  by the same unitary operator  $U$ ,  $g_\pm' \equiv U g_\pm U^\dagger$ , since then all commutators are preserved. If, however, the operators  $g_{+'}$  and  $g_{-}'$  are obtained by unitarily transforming  $g_+$  and  $g_-$  by different unitary operators,

$$g_\pm' \equiv U_\pm g_\pm U_\pm^\dagger, \quad (\text{C.4})$$

then only the commutators  $[g_\pm, g_\pm^\dagger]$  must be preserved, and the commutator  $[g_{+'}, g_{-}'^\dagger]$  need not vanish. The commutator matrix for the operators  $g_{+'}$  and  $g_{-}'$  is

$$\Upsilon_{g'} \equiv [g', g'^\dagger] = K \Upsilon_g K^\dagger. \quad (\text{C.5})$$

In general, all elements of  $\Upsilon_{g'}$  can differ from those of  $\Upsilon_g$ . The property of the commutator matrix that is preserved in the transformation (C.5) is positive-definiteness. A Hermitian matrix is positive-definite if and only if its eigenvalues are positive; i.e., if  $\underline{\mu}$  is a vector of complex numbers with components  $\mu_i$ ,  $i = 1, 2, \dots, N$  and  $M$  is an  $N$ -dimensional matrix with components  $M_{ij}$ , then  $M$  is positive-definite if and only if

$$\underline{\mu}^\dagger M \underline{\mu} = M_{ij} \mu_i^* \mu_j > 0 \quad (\text{C.6})$$

for all vectors  $\underline{\mu}$ . This shows clearly that  $\Upsilon_{g'}$  is positive-definite if and only if  $\Upsilon_g$  is positive-definite, since

$$\underline{\mu}^\dagger \Upsilon_g \underline{\mu} = (K \underline{\mu})^\dagger \Upsilon_{g'} (K \underline{\mu}) \geq 0. \quad (\text{C.7})$$

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The following paper, entitled

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New formalism for two-photon quantum optics. I. Quadrature phases and squeezed states

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This paper introduces a new formalism for analyzing two-photon devices (e.g., parametric amplifiers and phase-conjugate mirrors), in which photons in the output modes are created or destroyed two at a time. The key property of a two-photon device is that it excites pairs of output modes independently. Thus our new formalism deals with two modes at a time; a continuum multimode description can be built by integrating over independently excited pairs of modes. For a pair of modes at frequencies  $\Omega \pm \epsilon$ , we define (i) *quadrature-phase amplitudes*, which are complex-amplitude operators for modulation at frequency  $\epsilon$  of waves " $\cos[\Omega(t-x/c)]$ " and " $\sin[\Omega(t-x/c)]$ " and (ii) *two-mode squeezed states*, which are the output states of an ideal two-photon device. The quadrature-phase amplitudes and the two-mode squeezed states serve as the building blocks for our formalism; their properties and their physical interpretation are extensively investigated.

I. INTRODUCTION AND OVERVIEW

In this and the accompanying paper we introduce a new formalism for analyzing a particular class of nonlinear optical devices—devices that we call two-photon devices. The light produced by any optical system is an excitation of various modes of the electromagnetic field; the defining feature of a two-photon device is that its output light is generated by the simultaneous emission of two photons into two of the output modes. Examples of two-photon devices include parametric amplifiers, where the simultaneously excited output modes are called the signal and the idler, and phase-conjugate mirrors (four-wave mixers), where the output modes are the transmitted and reflected waves.

Two-photon devices can produce, in principle, special states of the electromagnetic field called squeezed states<sup>1</sup> or two-photon coherent states.<sup>2</sup> Squeezed states<sup>3,4</sup> have manifestly nonclassical properties; they might find application in low-noise optical communications<sup>5-7</sup> and in high-precision interferometric measurements.<sup>1,8,9</sup> Experiments to generate squeezed states and to investigate their properties are now underway in several laboratories.<sup>10-12</sup>

Two-photon devices are to be contrasted with one-photon devices, such as the laser, in which photons are emitted into the output modes one at a time. The analytical tools of quantum optics were developed to describe and analyze one-photon processes; thus they are designed to analyze situations in which the modes of the electromagnetic field are excited independently. These tools are, in general, not adequate for analyzing two-photon devices, because a two-photon device excites modes in pairs, instead of singly. This series of papers develops a new set of analytical tools, which are suited to the description and analysis of two-photon devices. A brief, preliminary account of our work can be found in Ref. 13.

To motivate our approach, we start by reviewing briefly the formalism of one-photon optics. This review is heuristic, with emphasis on the features that tailor the formalism to the description of one-photon processes; in particular, we treat the electromagnetic field classically, ignoring its quantum-mechanical commutation relations. Consider a beam of light produced by a one-photon device, and idealize the beam as a plane wave with a particular linear polarization. The electric field can be written as the sum of positive- and negative-frequency parts:

$$E(x, t) = E^{(+)}(x, t) + E^{(-)}(x, t), \tag{1.1}$$

where

$$E^{(+)}(x, t) \equiv \int_{\mathcal{S}} \frac{d\omega}{2\pi} E(\omega) e^{-i(\omega t - x/c)}, \tag{1.2}$$

$$E^{(-)} = (E^{(+)})^* .$$

Here  $E(\omega)$  is the complex amplitude of the plane-wave mode at (positive) frequency  $\omega$ , and the integration runs over the bandwidth  $\mathcal{S}$  of interest. That the photons in the beam are created one at a time means that the fluctuations in the electric field are due to random emission of single photons which have various frequencies and phases. As a result, the fluctuations at different frequencies are independent, and the fluctuations at each frequency are distributed randomly in phase. The mathematical embodiment of these two statements is

$$\langle \Delta E(\omega) \Delta E(\omega') \rangle = 0, \tag{1.3a}$$

$$\langle \Delta E(\omega) \Delta E^*(\omega') \rangle = \frac{b}{2c} \mathcal{S}(\omega) 2\pi \delta(\omega - \omega'), \tag{1.3b}$$

where  $\Delta E(\omega) \equiv E(\omega) - \langle E(\omega) \rangle$ ,  $\mathcal{S}(\omega)$  is the flux spectral density of the electric field fluctuations (dimensions of energy per area), and  $b$  is a units-dependent constant (e.g.,

$b \equiv 4\pi$  in cgs Gaussian units). In Eqs. (1.3) and throughout this first section, brackets denote a classical statistical average. The noise produced by a one-photon device is conveniently characterized by a single function of frequency  $\mathcal{S}(\omega)$ , derived from the second moments of the complex amplitudes. Equivalent to Eqs. (1.3) is the more compact statement that the electric field has *time-stationary* (TS) noise; i.e., the variance of the electric field is constant:

$$\frac{c}{b} \langle [\Delta E(x,t)]^2 \rangle = \int_{\mathcal{V}} \frac{d\omega}{2\pi} \mathcal{S}(\omega) \quad (1.4)$$

$$[\Delta E(x,t) \equiv E(x,t) - \langle E(x,t) \rangle].$$

Implicit in this discussion of TS noise is the assumption, made throughout this paper, that the noise is Gaussian, so that second moments are sufficient to characterize it. An important consequence of Gaussian TS noise, which does not hold for TS noise in general, is that the modes at different frequencies are statistically independent [Eqs. (1.3)]. The restriction to Gaussian noise will be lifted in a future paper (paper III of this series), where the relations among Gaussian noise, TS noise, and statistically independent modes will be considered.

*The key property of a one-photon device is that its output consists of independently excited modes with TS noise.* In terms of constructing a formalism, this property has two crucial consequences, which can be thought of as the cornerstones of one-photon optics: (i) one can deal with one plane-wave mode at a time, building a continuum multimode description by integrating over independently excited single modes; (ii) the natural variable to characterize the excitation of each mode is its complex amplitude  $E(\omega)$ .

One is now in a position to identify the fundamental "building blocks" of one-photon optics. Specialize to a single mode at frequency  $\omega$ . The natural quantum-mechanical operator for the mode is its annihilation operator

$$a(\omega) \equiv (2cA_q/b\hbar\omega)^{1/2} E(\omega), \quad (1.5)$$

which is just the mode's complex amplitude rewritten in "units" of square root of the number of quanta per root Hz. ( $A_q$  is an appropriate "quantization area" transverse to the propagation direction.) The natural quantum states for the mode are the coherent states<sup>14</sup>—the states generated from the vacuum by an ideal one-photon process (e.g., a classical current distribution radiating into the vacuum). The coherent states are eigenstates of the annihilation operator; thus they have the sharpest complex amplitude permitted by quantum mechanics. The formalism of one-photon optics is founded firmly on the annihilation operator as the fundamental operator and on the coherent states as the fundamental quantum states.

Real one-photon devices do not exhibit ideal behavior. Describing their nonideal behavior requires consideration of the complicated interaction of the light with atomic systems and of the effects of losses and their associated fluctuations. One approach to analyzing the light produced by a real one-photon device is to derive an equation for the evolution of the reduced density operator (quantum state) of the electromagnetic field. This equation,

which is called the master equation, is generally a complicated operator equation not directly amenable to analysis. A powerful technique for rendering the master equation more tractable is to convert it into an equivalent  $c$ -number partial differential equation—a Fokker-Planck equation—for the evolution of a quasiprobability distribution (QPD). A QPD is a rigorous and complete representation of a density operator (i.e., it contains all the quantum statistics associated with the density operator), but it retains the appearance and some of the interpretation of a classical probability distribution.

The definition and interpretation of the QPD's used in one-photon optics ("one-photon QPD's") are intimately related to the use of the annihilation operator and the coherent states as the fundamental building blocks.<sup>15-17</sup> More than one QPD is associated with a given quantum state, each QPD corresponding to a different way of ordering the creation and annihilation operators. For a single mode of the electromagnetic field, each one-photon QPD is a function of a complex number  $\mu$ , which is a  $c$ -number analog of the mode's annihilation operator. The expectation value of a suitably ordered product of creation and annihilation operators is calculated using the appropriate QPD as though it were a classical probability distribution. The one-photon QPD's are powerful tools for analyzing real one-photon devices, but based as they are on the annihilation operator and the coherent states, they are tools designed specifically for one-photon processes and are not necessarily suited to the analysis of two-photon devices. For example, one of the most useful and most used one-photon QPD's is the Glauber-Sudarshan  $P$  function,<sup>18,19,14</sup> which reproduces the normally ordered statistics of  $a$  and  $a^\dagger$ ; this QPD does not exist as a well-behaved distribution for the squeezed states that can be produced by two-photon devices.<sup>20</sup>

Our philosophy has been that a new task requires new tools. The first step is to identify new operators and new quantum states, which are suited to the description of two-photon processes; this task is carried out exhaustively in papers I and II of this series. The second step is to use these operators and states to define "two-photon QPD's" that can be used to analyze real two-photon devices; this task will be tackled in paper III.

To simplify the introduction of our formalism, consider as an example a parametric amplifier, the prototype for all two-photon devices. In a paramp an intense laser beam at frequency  $2\Omega$ —the pump beam—illuminates a suitable nonlinear medium. The nonlinearity couples the pump beam to other modes of the electromagnetic field in such a way that a pump photon at frequency  $2\Omega$  can be annihilated to create "signal" and "idler" photons at frequencies  $\Omega \pm \epsilon$  and, conversely, signal and idler photons can be annihilated to create a pump photon. Thus the light produced by a paramp consists of pairs of simultaneously emitted photons which excite pairs of modes at frequencies  $\Omega \pm \epsilon$ . In general, the modes in each pair have correlated complex amplitudes [i.e.,  $\langle \Delta E(\Omega + \epsilon) \Delta E(\Omega - \epsilon) \rangle \neq 0$ ; cf. Eq. (1.3a)]. This fact tells one immediately that the formalism of one-photon optics must be abandoned; the correlations produced by two-photon processes cannot be described in terms of indepen-

dently excited single modes.

The electric field at the output of a paramp has the same form as Eq. (1.1); the difference lies in the correlation between the modes in each pair. It is useful to rewrite the field by factoring out the time dependence at frequency  $\Omega$ . Define (real) quadrature phases  $E_1(x, t)$  and  $E_2(x, t)$  by

$$E^{(\pm)} \equiv \frac{1}{2}(E_1 \pm iE_2)e^{\mp i\Omega(t-x/c)}; \quad (1.6)$$

$E_1 + iE_2$  is the complex amplitude of the electric field, defined with respect to the carrier frequency  $\Omega$ . In terms of the quadrature phases, the electric field is given by

$$E(x, t) = E_1(x, t)\cos[\Omega(t-x/c)] + E_2(x, t)\sin[\Omega(t-x/c)]; \quad (1.7)$$

thus,  $E_1$  and  $E_2$  describe modulation of waves "cos $[\Omega(t-x/c)]$ " and "sin $[\Omega(t-x/c)]$ ." The quadrature phases can be written in terms of their Fourier components:

$$E_m(x, t) = \int_{\mathcal{A}} \frac{d\epsilon}{2\pi} [E_m(\epsilon)e^{-i\epsilon(t-x/c)} + E_m^*(\epsilon)e^{i\epsilon(t-x/c)}], \quad m=1, 2. \quad (1.8)$$

Here the integral runs over a suitable set  $\mathcal{A}$  of (positive) modulation frequencies  $\epsilon$ , and

$$E_1(\epsilon) = E(\Omega + \epsilon) + E^*(\Omega - \epsilon), \quad (1.9a)$$

$$E_2(\epsilon) = -iE(\Omega + \epsilon) + iE^*(\Omega - \epsilon). \quad (1.9b)$$

The Fourier component  $E_1(\epsilon)$  [ $E_2(\epsilon)$ ] is a complex amplitude for modulation at frequency  $\epsilon$ , of a wave  $\cos[\Omega(t-x/c)]$  [ $\sin[\Omega(t-x/c)]$ ]. Now consider the emission of a pair of photons at frequencies  $\Omega \pm \epsilon$ . The conventional view is that these photons excite a pair of modes that are sidebands of the carrier frequency  $\Omega$ ; an equally good alternative view is that they excite directly a modulation at frequency  $\epsilon$  of a wave at frequency  $\Omega$ . Roughly speaking, if the phases of the two photons are such that  $E(\Omega + \epsilon) = E^*(\Omega - \epsilon)$ , then they excite  $E_1(\epsilon)$ ; if the phases are such that  $E(\Omega + \epsilon) = -E^*(\Omega - \epsilon)$ , then they excite  $E_2(\epsilon)$ . Our message is that two-photon optics should be formulated in a different language from one-photon optics. In one-photon optics attention focuses on the electric field  $E(x, t)$  and its Fourier components  $E(\omega)$ ; emission of a photon excites a mode at a particular frequency. In two-photon optics attention shifts to the quadrature phases  $E_1(x, t)$  and  $E_2(x, t)$  and their Fourier components  $E_1(\epsilon)$  and  $E_2(\epsilon)$ ; emission of a pair of photons excites one of the quadrature phases at a particular modulation frequency.

With this new language in hand, the discussion of natural variables for two-photon optics is just a translation of the preceding review of one-photon optics. The fluctuations in the quadrature phases are due to random emission of pairs of photons, which excite the quadrature phases at various modulation frequencies with various phases [phase in this context is the phase of the (complex)

Fourier component  $E_1(\epsilon)$  or  $E_2(\epsilon)$ ]. As a result, the fluctuations at different modulation frequencies are independent, and the fluctuations at each modulation frequency are distributed randomly in phase. This means that the quadrature phases have time-stationary noise—a kind of noise that we call *time-stationary quadrature-phase (TSQP) noise*.<sup>13,21</sup> For Gaussian noise the conditions for TSQP noise are

$$\langle \Delta E_m(\epsilon) \Delta E_n(\epsilon') \rangle = 0, \quad (1.10a)$$

$$\langle \Delta E_m(\epsilon) \Delta E_n^*(\epsilon') \rangle = \frac{b}{c} \mathcal{S}_{mn}(\epsilon) 2\pi \delta(\epsilon - \epsilon'), \quad (1.10b)$$

where  $m, n = 1, 2$ ,  $\Delta E_m(\epsilon) \equiv E_m(\epsilon) - \langle E_m(\epsilon) \rangle$ , and  $\mathcal{S}_{mn}(\epsilon) = \mathcal{S}_{nm}^*(\epsilon)$  is the flux spectral-density matrix for the quadrature-phase fluctuations [dimensions of energy per area; cf. Eqs. (1.3)]. Equivalent to Eqs. (1.10) is the time independence of the covariance matrix of the quadrature phases:

$$\frac{c}{2b} \langle \Delta E_m(x, t) \Delta E_n(x, t) \rangle = \int_{\mathcal{A}} \frac{d\epsilon}{2\pi} \text{Re}[\mathcal{S}_{mn}(\epsilon)] \quad (1.11)$$

[ $m, n = 1, 2$ ;  $\Delta E_m(x, t) \equiv E_m(x, t) - \langle E_m(x, t) \rangle$ ; "Re" denotes "the real part of"]. Unlike TS noise, TSQP noise allows the quadratures to carry different amounts of noise ( $\mathcal{S}_{11} \neq \mathcal{S}_{22}$ ), and it allows them to have a nonvanishing time-stationary correlation [ $\text{Re}(\mathcal{S}_{12}) \neq 0$ ]. This means that the variance of the electric field is not, in general, constant:

$$\begin{aligned} \frac{c}{b} \langle [\Delta E(x, t)]^2 \rangle &= \int_{\mathcal{A}} \frac{d\epsilon}{2\pi} \{ \mathcal{S}_{11} + \mathcal{S}_{22} + (\mathcal{S}_{11} - \mathcal{S}_{22}) \cos[2\Omega(t-x/c)] \\ &\quad + 2\text{Re}(\mathcal{S}_{12}) \sin[2\Omega(t-x/c)] \} \quad (1.12) \end{aligned}$$

[cf. Eq. (1.4)]. Equations (1.11) and (1.12) can be interpreted as saying that the fluctuations in the electric field are not distributed randomly in phase, where phase is here defined relative to frequency  $\Omega$ .

*The key property of a two-photon device is that its output consists of independently excited pairs of modes with TSQP noise.* This property is the reason that two-photon optics is formulated more conveniently in terms of the quadrature phases and their Fourier components than in terms of the electric field and its Fourier components. The consequences of this property, and the cornerstones of two-photon optics, are the following: (i) one can deal with one pair of modes, i.e., one modulation frequency, at a time, building a continuum multimode description by integrating over independently excited pairs of modes; (ii) the natural variables for each pair of modes are the Fourier components  $E_1(\epsilon)$  and  $E_2(\epsilon)$ .

We can now identify the fundamental building blocks for two-photon optics. Specialize to a pair of modes at frequencies  $\Omega \pm \epsilon$ . The natural quantum-mechanical operators for the modes are the quadrature-phase amplitudes  $\alpha_1(\epsilon)$  and  $\alpha_2(\epsilon)$ ,<sup>13,21</sup> defined by

$$\begin{aligned}\alpha_1(\epsilon) &\equiv \left[ \frac{cA_q}{b\hbar\Omega} \right]^{1/2} E_1(\epsilon) \\ &= \left[ \frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a(\Omega+\epsilon) + \left[ \frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a^\dagger(\Omega-\epsilon),\end{aligned}\tag{1.13a}$$

$$\begin{aligned}\alpha_2(\epsilon) &\equiv \left[ \frac{cA_q}{b\hbar\Omega} \right]^{1/2} E_2(\epsilon) \\ &= -i \left[ \frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a(\Omega+\epsilon) + i \left[ \frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a^\dagger(\Omega-\epsilon)\end{aligned}\tag{1.13b}$$

[Eqs. (1.5) and (1.9)]. The quadrature-phase amplitudes are simply rescaled versions of  $E_1(\epsilon)$  and  $E_2(\epsilon)$ —rescaled to be in units of square root of the number of quanta, referred to the carrier frequency  $\Omega$ , per root Hz. The natural quantum states are the two-mode squeezed states<sup>3,13,21</sup>—the states generated from (two-mode) coherent states by an ideal two-photon device (e.g., an ideal paramp, with undepleted classical pump and no losses). The two-mode squeezed states have TSQP noise, and they have, in general, unequal amounts of noise in the two quadratures ( $\mathcal{S}_{11} \neq \mathcal{S}_{22}$ ). The present paper (paper I) focuses on the properties and the significance of the quadrature-phase amplitudes and the two-mode squeezed states; the goal is to achieve a good physical understanding of these fundamental entities. The accompanying paper (paper II) develops a mathematical formalism suited to manipulating the quadrature-phase amplitudes and the two-mode squeezed states, and it uses the formalism to write their properties in a compact form. With its emphasis on physical interpretation, this first paper omits many mathematical details, which are filled in by paper II.

These building blocks of two-photon optics have been used to construct new two-photon quasiprobability distributions.<sup>13</sup> More than one two-photon QPD is associated with a given (two-mode) quantum state, each QPD corresponding to a different way of ordering the quadrature-phase amplitudes and their Hermitian conjugates. Since the two-photon QPD's are written in a language tailored to the description of two-photon processes, we think they will be valuable tools for analyzing nonideal behavior of two-photon devices. A future paper (paper III) will describe in detail the new operator orderings and the two-photon QPD's.

In this paper Sec. II deals with a couple of minor notational issues; Sec. III reviews briefly the building blocks of one-photon optics; Sec. IV introduces the quadrature-phase amplitudes and the two-mode squeezed states, with emphasis on the physical significance of the quadrature-phase amplitudes; Sec. V considers in detail TSQP noise for the case of Gaussian noise; Sec. VI discusses uncertainty principles for the quadrature-phase amplitudes; Sec. VII lists important properties of the two-mode squeezed states; finally, Sec. VIII specializes our work to the previously explored degenerate limit ( $\epsilon=0$ ). An appendix

treats uncertainty principles for non-Hermitian operators. Throughout the remainder of this paper we use units with  $\hbar=c=1$ .

## II. NOTATIONAL ISSUES

For convenience we have adopted a notation that sometimes sacrifices precision for ease in use. To minimize confusion that might arise from our preference for convenience, we consider here a couple of notational issues.

Throughout our discussion of two-photon optics, we find that each physical quantity is most conveniently represented by its operator in a particular picture. For example, the creation and annihilation operators are most conveniently written in the Schrödinger picture (SP); field quantities, such as the electric field and the quadrature phases, are most conveniently written in the usual interaction picture (IP), in which all the free time dependence is incorporated in the operators; and the quadrature-phase amplitudes are most conveniently written in an interaction picture that we call the modulation picture (MP), which we define and discuss in Sec. IV. As a result, we have acquired the habit of mixing in the same equation various operators written in different pictures. This habit has the potential to cause confusion, which we seek to avoid by adhering strictly to the following procedure. For each physical quantity, the corresponding operators in different pictures are denoted differently. As each physical quantity is introduced in Secs. III and IV, we define its operator in a particular picture by a picture-consistent equation, i.e., an equation in which all operators are written in the same picture. The operators corresponding to the same physical quantity in other pictures are then defined as they are needed. The appropriate picture for a picture-consistent equation is indicated by writing SP, MP, or IP in parentheses next to the equation; of course, a picture-consistent equation retains the same form when all operators are transformed to another picture.

As an illustration of this procedure, consider a plane electromagnetic wave with a particular linear polarization, which propagates in the  $x$  direction. In the SP the creation and annihilation operators for the plane-wave mode at frequency  $\omega$  are denoted by  $a^\dagger(\omega)$  and  $a(\omega)$ ; they satisfy the continuum commutation relations

$$[a(\omega), a^\dagger(\omega')] = 2\pi\delta(\omega - \omega').\tag{2.1}$$

The electric field operator in the SP is given by

$$E(x) \equiv E^{(+)}(x) + E^{(-)}(x) \text{ (SP)},\tag{2.2a}$$

$$\begin{aligned}E^{(+)}(x) &\equiv \int_0^\infty \frac{d\omega}{2\pi} (b\omega/2A_q)^{1/2} a(\omega) e^{i\omega x} \\ &= [E^{(-)}(x)]^\dagger \text{ (SP)},\end{aligned}\tag{2.2b}$$

where  $E^{(+)}(x)$  and  $E^{(-)}(x)$  are the SP positive- and negative-frequency parts of the field,  $A_q$  is a suitable quantization area, and  $b$  is the units-dependent constant introduced in Eqs. (1.3). In the IP the electric field operator is given by

$$E(x,t) \equiv e^{iH_C t} E(x) e^{-iH_C t} \equiv E^{(+)}(x,t) + E^{(-)}(x,t), \quad (2.3a)$$

$$\begin{aligned} E^{(+)}(x,t) &\equiv e^{iH_C t} E^{(+)}(x) e^{-iH_C t} \\ &= \int_0^\infty \frac{d\omega}{2\pi} (b\omega/2A_q)^{1/2} a(\omega) e^{-i\omega t - ix} \\ &= [E^{(-)}(x,t)]^\dagger \end{aligned} \quad (2.3b)$$

[cf. Eqs. (1.1), (1.2), and (1.5)], where

$$H_C \equiv \int_0^\infty \frac{d\omega}{2\pi} \omega a^\dagger(\omega) a(\omega) \quad (2.4)$$

is the free Hamiltonian for the continuum of modes, and where we use the fact that the IP form of the annihilation operator for a particular mode has the harmonic time dependence of the mode, i.e.,  $e^{iH_C t} a(\omega) e^{-iH_C t} = a(\omega) e^{-i\omega t}$ .

A second notational issue concerns the way we use the symbol  $\Delta$ . In general, we use  $\Delta$  to designate the difference between a quantity and its mean value. Thus, for a quantum-mechanical operator  $R$ ,  $\Delta R$  is defined to be the operator

$$\Delta R \equiv R - \langle R \rangle. \quad (2.5)$$

For a Hermitian operator  $B$  this notation allows the variance (squared uncertainty) of  $B$  to be written as  $\langle (\Delta B)^2 \rangle$  ( $\equiv \langle B^2 \rangle - \langle B \rangle^2$ ); we always write the variance in this form. For a general, possibly non-Hermitian operator  $R$ , a fundamental quantity in our analysis is the mean-square uncertainty in  $R$ , by which we mean the sum of the variances of the Hermitian real and imaginary parts of  $R$  [ $\text{Re}(R) \equiv \frac{1}{2}(R + R^\dagger)$ ;  $\text{Im}(R) \equiv -\frac{1}{2}i(R - R^\dagger)$ ]. To define and write the mean-square uncertainty compactly, we use three shorthand notations: (i) for two operators  $R$  and  $S$ , the subscript "sym" denotes a symmetrically ordered product, i.e.,

$$(RS)_{\text{sym}} \equiv \frac{1}{2}(RS + SR); \quad (2.6)$$

(ii) the expectation value of a symmetrically ordered product is written

$$\langle (RS)_{\text{sym}} \rangle \equiv \langle RS \rangle_{\text{sym}}; \quad (2.7)$$

(iii)  $|\Delta R|^2$  denotes the operator

$$|\Delta R|^2 \equiv (\Delta R \Delta R^\dagger)_{\text{sym}} = \frac{1}{2}(\Delta R \Delta R^\dagger + \Delta R^\dagger \Delta R). \quad (2.8)$$

These shorthands allow us to write the mean-square uncertainty as

$$\langle |\Delta R|^2 \rangle = \langle \Delta R \Delta R^\dagger \rangle_{\text{sym}} = \langle RR^\dagger \rangle_{\text{sym}} - |\langle R \rangle|^2. \quad (2.9)$$

For a Hermitian operator the mean-square uncertainty is the variance; our notation is consistent because  $|\Delta B|^2 = (\Delta B)^2$  if  $B = B^\dagger$ .

### III. REVIEW OF ONE-PHOTON OPTICS

We turn now to a brief review of one-photon optics, briefer even than the review in Sec. I, but rigorous quantum-mechanically. Consider the light produced by a one-photon device such as a laser. As is discussed in Sec. I, one can specialize to a single (discrete) plane-wave mode

with frequency  $\omega$ ; a continuum multimode description is built by integrating over independently excited single modes. The mode's creation and annihilation operators in the SP are denoted by  $a^\dagger$  and  $a$ , which satisfy the usual (discrete) commutation relation

$$[a, a^\dagger] = 1. \quad (3.1)$$

We introduce an "electric field operator" for the mode, which is denoted in the SP by

$$E(x) \equiv E^{(+)}(x) + E^{(-)}(x) \quad (\text{SP}), \quad (3.2a)$$

$$E^{(+)}(x) \equiv (\omega/2)^{1/2} a e^{i\omega x} = [E^{(-)}(x)]^\dagger \quad (\text{SP}) \quad (3.2b)$$

[cf. Eqs. (2.2)]. In the IP the single-mode electric field operator becomes

$$E(x,t) \equiv e^{iH_S t} E(x) e^{-iH_S t} = E^{(+)}(x,t) + E^{(-)}(x,t), \quad (3.3a)$$

$$E^{(+)}(x,t) = (\omega/2)^{1/2} a e^{-i\omega t - ix} = [E^{(-)}(x,t)]^\dagger \quad (3.3b)$$

[cf. Eqs. (2.3)], where

$$H_S \equiv \omega a^\dagger a \quad (\text{SP}) \quad (3.4)$$

is the free Hamiltonian for a single mode.

Our motivation for introducing the single-mode electric field operators of Eqs. (3.2) and (3.3) is that we want to be able to calculate the statistics of fieldlike quantities associated with a single (discrete) plane-wave mode. The normalization of the electric field for a single plane-wave mode is somewhat arbitrary, so we have simply made a convenient choice that leaves our results uncluttered by irrelevant constants. The  $\omega^{1/2}$  in Eqs. (3.2b) and (3.3b) is the obligatory factor of root frequency that accompanies the annihilation operator [cf. Eqs. (2.2b) and (2.3b)]; it gives the single-mode electric field units of square root of energy. The  $2^{-1/2}$  in Eqs. (3.2b) and (3.3b) is chosen for convenience.

The natural states for describing the output of a one-photon device can be identified by considering the Hamiltonian for an ideal one-photon process:

$$H = H_S - ig^*(t) a e^{i\omega t} + ig(t) a^\dagger e^{-i\omega t} \quad (\text{SP}). \quad (3.5)$$

Here  $g(t) \equiv g$  is an arbitrary complex function of time. The interaction part of this Hamiltonian creates and destroys photons one at a time; the process is ideal because it is characterized by a  $c$ -number function  $g(t) e^{-i\omega t}$ , which can be regarded as a classical generalized force acting on the mode. The Hamiltonian (3.5) describes a classical current distribution radiating into the mode of interest.<sup>14,16</sup> The SP unitary evolution operator  $U(t,0)$  corresponding to the Hamiltonian (3.5) is<sup>18,22</sup>

$$\begin{aligned} U(t,0) &= e^{-ih(t)} e^{-iH_S t} D(a, \gamma) \\ &= e^{-ih(t)} D(a, \gamma e^{-i\omega t}) e^{-iH_S t}, \end{aligned} \quad (3.6a)$$

$$\gamma \equiv \gamma(t) \equiv \int_0^t g(t') dt', \quad (3.6b)$$

$$h(t) \equiv \frac{1}{2} i \int_0^t (\gamma^* g - \gamma g^*) dt'. \quad (3.6c)$$

In Eq. (3.6a)

$$D(a, \mu) \equiv \exp(\mu a^\dagger - \mu^* a) \quad (3.7)$$

is the (unitary) *single-mode displacement operator*,<sup>14</sup> so named because of the important property<sup>14</sup>

$$D^\dagger(a, \mu) a D(a, \mu) = a + \mu. \quad (3.8)$$

The natural states for one-photon optics are those generated from the vacuum state  $|0\rangle$  by an ideal one-photon process. These states, which are called (single-mode) *coherent states*,<sup>14</sup> are defined by

$$|\mu\rangle_{\text{coh}} \equiv D(a, \mu) |0\rangle. \quad (3.9)$$

A coherent state is an eigenstate of the annihilation operator with complex eigenvalue  $\mu$ :

$$a |\mu\rangle_{\text{coh}} = \mu |\mu\rangle_{\text{coh}} \quad (3.10)$$

[Eq. (3.8)]. The coherent states lie at the very core of one-photon optics; their properties have been extensively investigated.<sup>14,16</sup>

The natural variable for one-photon optics is the annihilation operator  $a$ , which is simply a complex-amplitude operator for the mode, written in units of square root of the number of quanta. The reason the annihilation operator is natural is that the states of interest in one-photon optics have *time-stationary* (TS) noise. To see what TS noise means, let the initial state of the mode be the density operator  $\rho$ . The noise associated with an arbitrary state  $\rho$  is completely characterized by the "noise moments" of  $a$  and  $a^\dagger$ , where by noise moments we mean moments of  $\Delta a \equiv a - \langle a \rangle$  and  $\Delta a^\dagger$  [Eq. (2.5)]. In this paper we consider only the lowest-order noise, which is described by the second-order noise moments

$$\langle (\Delta a)^2 \rangle \equiv \text{tr}[\rho (\Delta a)^2] = \langle a^2 \rangle - \langle a \rangle^2, \quad (3.11a)$$

$$\begin{aligned} \langle |\Delta a|^2 \rangle &\equiv \text{tr}[\rho (\Delta a \Delta a^\dagger)_{\text{sym}}] \\ &= \langle a a^\dagger \rangle_{\text{sym}} - |\langle a \rangle|^2. \end{aligned} \quad (3.11b)$$

The state  $\rho$  has (second-moment) TS noise if

$$\langle (\Delta a)^2 \rangle = 0 \quad (3.12)$$

[cf. Eqs. (1.3a) and (1.5)]; hence, for TS noise the lowest-order noise is described completely by the mean-square uncertainty  $\langle |\Delta a|^2 \rangle$  [Eq. (2.9); cf. Eqs. (1.3b) and (1.5)]. The physical content of Eq. (3.12) is that the noise in the single mode is distributed randomly in phase; thus TS noise can be characterized as random-phase noise or phase-insensitive noise. An immediate consequence of Eq. (3.12) is that the electric field has TS noise; i.e., if the mode undergoes free evolution (Hamiltonian  $H_S$ ), the variance of the electric field (3.3a) is constant:

$$\langle [\Delta E(x, t)]^2 \rangle = \omega \langle |\Delta a|^2 \rangle \quad (3.13)$$

[cf. Eq. (1.4)].

It is useful to emphasize here why the annihilation operator is the natural variable for describing TS noise. Under free evolution (evolution operator  $e^{-iH_S t}$ ), the noise moment  $\langle (\Delta a)^2 \rangle$  acquires a harmonic time dependence  $e^{-2i\omega t}$ , whereas the mean-square uncertainty  $\langle |\Delta a|^2 \rangle$  remains constant. The essence of TS noise is that the time-dependent noise moment  $\langle (\Delta a)^2 \rangle$  vanishes, so that the lowest-order noise is described by the time-

independent moment  $\langle |\Delta a|^2 \rangle$ . These considerations are the key to generalizing the notion of TS noise to noise moments of arbitrarily high order. The definition (3.12) considers only the lowest-order noise moments, the justification being an implicit assumption of Gaussian noise. The general definition of TS noise, which will be given explicitly and discussed in paper III, requires that all the time-dependent noise moments of  $a$  and  $a^\dagger$  vanish, so that the noise is completely characterized by the time-independent noise moments. *This, then, is the reason the annihilation operator is the natural variable for one-photon optics: the TS noise produced by one-photon devices is completely characterized by the time-independent noise moments of  $a$  and  $a^\dagger$ .*

The commutator  $[a, a^\dagger] = 1$  enforces an uncertainty principle,

$$\langle |\Delta a|^2 \rangle \geq \frac{1}{2} |\langle [a, a^\dagger] \rangle| = \frac{1}{2}. \quad (3.14)$$

[This and other uncertainty principles for non-Hermitian operators are derived and discussed in the Appendix; see Eq. (A9).] The lower limit in Eq. (3.14) is the half-quantum of zero-point noise. A coherent state  $|\mu\rangle_{\text{coh}}$  has mean complex amplitude  $\langle a \rangle = \mu$  and has TS noise with  $\langle |\Delta a|^2 \rangle = \frac{1}{2}$ ; it can be thought of as a classical excitation of the mode contaminated by zero-point noise.

The fundamental building blocks for one-photon optics are the annihilation operator and the coherent states. Although the coherent states arise from a consideration of ideal one-photon devices, they and the annihilation operator have been used to define quasiprobability distributions,<sup>15-17,20</sup> which are powerful tools for analyzing the nonideal behavior of real one-photon devices. Quasiprobability distributions will be considered in detail in a future paper (paper III).

#### IV. BUILDING BLOCKS OF TWO-PHOTON OPTICS

Attention shifts now to a discussion of the natural variables and natural quantum states for two-photon optics. As is made clear in Sec. I, one can analyze the light produced by a two-photon device by specializing to a pair of (discrete) plane-wave modes with frequencies  $\Omega \pm \epsilon$ , where  $\Omega$  is a carrier frequency and  $\epsilon < \Omega$  is a modulation frequency; a continuum multimode description is built by integrating over independently excited pairs of modes (i.e., integrating over  $\epsilon$ ). In optical applications it is always true that  $\epsilon \ll \Omega$ . The annihilation operators for the two modes in the SP are denoted by  $a_+$  and  $a_-$ ; they satisfy the usual (discrete) commutation relations

$$[a_+, a_-] = [a_+, a_-^\dagger] = 0, \quad (4.1a)$$

$$[a_+, a_+^\dagger] = [a_-, a_-^\dagger] = 1. \quad (4.1b)$$

The free Hamiltonian for the two modes is given by

$$\begin{aligned} H_0 &= (\Omega + \epsilon) a_+^\dagger a_+ + (\Omega - \epsilon) a_-^\dagger a_- \\ &= H_R + H_M \quad (\text{SP}), \end{aligned} \quad (4.2a)$$

$$H_R \equiv \Omega (a_+^\dagger a_+ + a_-^\dagger a_-) \quad (\text{SP}), \quad (4.2b)$$

$$H_M \equiv \epsilon (a_+^\dagger a_+ - a_-^\dagger a_-) \quad (\text{SP}). \quad (4.2c)$$

We find it useful to split  $H_0$  into two commuting pieces,  $H_R$  and  $H_M$  ( $[H_R, H_M] = 0$ ), which are the key to defining the pictures we use in our new formalism. In the usual *interaction picture* (IP), all the free time dependence is transferred from the states to the operators; the relation between operators (including density operators) in the IP and the SP is

$$R_{IP}(t) \equiv e^{iH_0 t} R_{SP}(t) e^{-iH_0 t}. \quad (4.3)$$

The *modulation picture*<sup>13</sup> (MP) is an interaction picture in which the free time dependence at the carrier frequency  $\Omega$  is transferred from the states to the operators, the states retaining the remaining free time dependence at modulation frequency  $\epsilon$ ; operators in all three pictures are related by

$$R_{MP}(t) \equiv e^{iH_R t} R_{SP}(t) e^{-iH_R t} = e^{-iH_M t} R_{IP}(t) e^{iH_M t}. \quad (4.4)$$

There is no reason why the two modes we consider need be plane-wave modes with the same polarization propagating in the same direction. Nonetheless, we assume they are so that we can introduce a "two-mode electric field operator," which in the IP is given by

$$E(x, t) \equiv E^{(+)}(x, t) + E^{(-)}(x, t) \quad (\text{IP}), \quad (4.5a)$$

$$E^{(+)}(x, t) = 2^{-1/2} [(\Omega + \epsilon)^{1/2} a_+ e^{-i(\Omega + \epsilon)(t-x)} + (\Omega - \epsilon)^{1/2} a_- e^{-i(\Omega - \epsilon)(t-x)}], \quad (4.5b)$$

$$E^{(-)}(x, t) = [E^{(+)}(x, t)]^\dagger \quad (4.5c)$$

[cf. Eqs. (3.3) and subsequent discussion].

#### A. Two-mode squeezed states

Consider now the Hamiltonian for an ideal two-photon process:

$$H = H_0 + i\kappa(t) [a_+ a_- e^{-2i(\varphi - \Omega t)} - a_+^\dagger a_-^\dagger e^{2i(\varphi - \Omega t)}] \quad (\text{SP}). \quad (4.6)$$

Here  $\kappa(t)$  is an arbitrary real function of time. The interaction part of this Hamiltonian creates or destroys a pair of photons in the two modes simultaneously; the process is ideal because it is characterized by a  $c$ -number function  $\kappa(t) e^{2i(\varphi - \Omega t)}$ . For convenience we choose this function to have a harmonic time dependence at frequency  $2\Omega$ , with fixed phase but time-varying amplitude. The Hamiltonian (4.6) describes, for example, an ideal parametric amplifier<sup>23-26</sup> with an undepleted classical pump, which has stable frequency  $2\Omega$  but whose amplitude varies in time. The unitary evolution operator for the Hamiltonian (4.6) is given by<sup>26,22</sup>

$$U(t, 0) = e^{-iH_0 t} S(\zeta, \varphi) = S(\zeta, \varphi - \Omega t) e^{-iH_0 t}, \quad (4.7)$$

$$\zeta \equiv \zeta(t) \equiv \int_0^t \kappa(t') dt', \quad (4.8)$$

where

$$S(r, \varphi) \equiv \exp[r(a_+ a_- e^{-2i\varphi} - a_+^\dagger a_-^\dagger e^{2i\varphi})] \quad (4.9)$$

is the (unitary) *two-mode squeeze operator*.<sup>13,21</sup> The real number  $r$  is called the *squeeze factor*. The most important

property of the two-mode squeeze operator is that

$$S(r, \varphi) a_\pm S^\dagger(r, \varphi) = a_\pm \cosh r + a_\mp^\dagger e^{2i\varphi} \sinh r, \quad (4.10)$$

a result which follows from Eq. (8.105) of Ref. 27.

To construct the natural states for two-photon optics, one begins with the *two-mode coherent states*<sup>14</sup>

$$|\mu_+, \mu_-\rangle_{\text{coh}} \equiv D(a_+, \mu_+) D(a_-, \mu_-) |0\rangle \quad (4.11)$$

[cf. Eq. (3.9)], which are eigenstates of  $a_+$  and  $a_-$  with eigenvalues  $\mu_+$  and  $\mu_-$ , respectively. Formally, a two-mode coherent state is obtained by applying the *two-mode displacement operator*<sup>14</sup>

$$D(a_-, \mu_-) D(a_+, \mu_+) = \exp(\mu_+ a_+^\dagger - \mu_+^* a_+ + \mu_- a_-^\dagger - \mu_-^* a_-) \quad (4.12)$$

to the vacuum state [cf. Eq. (3.7)]; physically, it could be created from the vacuum by an ideal one-photon process for each of the two modes. The natural states for two-photon optics are those generated from two-mode coherent states by the ideal two-photon process (4.6). Before defining these states, it is useful to define operators that we call *squeezed annihilation operators*. In the SP these operators have explicit time dependence and are defined by

$$\alpha_\pm(r, \varphi; t) \equiv e^{-iH_R t} S(r, \varphi) a_\pm S^\dagger(r, \varphi) e^{iH_R t} \\ = a_\pm e^{i\Omega t} \cosh r + a_\mp^\dagger e^{-i\Omega t} e^{2i\varphi} \sinh r \quad (\text{SP}) \quad (4.13)$$

[Eq. (4.10)]; in the MP [Eq. (4.4)] the squeezed annihilation operators are constant and are given by

$$\alpha_\pm(r, \varphi) \equiv \alpha_\pm(r, \varphi; 0) = S(r, \varphi) a_\pm S^\dagger(r, \varphi) \\ = a_\pm \cosh r + a_\mp^\dagger e^{2i\varphi} \sinh r. \quad (4.14)$$

The natural states for two-photon optics are the *two-mode squeezed states*,<sup>8,13,21</sup> which are defined by

$$|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r, \varphi)} \equiv S(r, \varphi) |\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{\text{coh}} \\ = S(r, \varphi) D(a_+, \mu_{\alpha_+}) D(a_-, \mu_{\alpha_-}) |0\rangle. \quad (4.15)$$

We label these states by the complex eigenvalues of  $\alpha_\pm(r, \varphi)$ :

$$\alpha_\pm(r, \varphi) |\mu_{\alpha_\pm}, \mu_{\alpha_\mp}\rangle_{(r, \varphi)} = \mu_{\alpha_\pm} |\mu_{\alpha_\pm}, \mu_{\alpha_\mp}\rangle_{(r, \varphi)} \quad (4.16)$$

[Eqs. (4.14) and (3.10)]. Using Eq. (4.10), one can write the two-mode squeezed states in the form

$$|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r, \varphi)} = D(a_+, \mu_+) D(a_-, \mu_-) S(r, \varphi) |0\rangle, \quad (4.17)$$

where

$$\mu_{\alpha_\pm} = \mu_\pm \cosh r + \mu_\mp^* e^{2i\varphi} \sinh r. \quad (4.18)$$

Two-mode squeezed states were introduced independently by Caves<sup>21</sup> in an analysis of quantum limits on the performance of linear amplifiers (see also Ref. 13) and by Unruh<sup>8</sup> in a quantum-mechanical analysis of an interferome-

er; they have also been considered formally by Barut and Girardello,<sup>28</sup> Perelomov,<sup>29</sup> and Milburn.<sup>30</sup> Properties of two-mode squeezed states are considered in Sec. VII.

Almost all previous work on squeezed states has dealt with the degenerate limit, in which the two modes we consider coalesce into one ( $\epsilon=0$ ,  $a_+ = a_-$ ). Our attitude is that the degenerate limit is not very important in describing real two-photon devices, because it is merely the  $\epsilon=0$  boundary for a more realistic and more general multimode description. The degenerate limit can, however, play a useful heuristic role, so we consider it in some detail in Sec. VIII.

### B. Quadrature-phase amplitudes

It is useful to decompose the electric field into its (Hermitian) *quadrature phases* defined with respect to the carrier frequency  $\Omega$ .<sup>13,21</sup> In the IP the quadrature phases are defined by

$$E_1(x,t) \equiv E^{(+)}(x,t)e^{i\Omega(t-x)} + E^{(-)}(x,t)e^{-i\Omega(t-x)} \quad (\text{IP}), \quad (4.19a)$$

$$E_2(x,t) \equiv -iE^{(+)}(x,t)e^{i\Omega(t-x)} + iE^{(-)}(x,t)e^{-i\Omega(t-x)} \quad (\text{IP}), \quad (4.19b)$$

$$E^{(\pm)}(x,t) \equiv \frac{1}{2}[E_1(x,t) \pm iE_2(x,t)]e^{\mp i\Omega(t-x)} \quad (\text{IP}) \quad (4.20)$$

[cf. Eq. (1.6)]. In terms of the quadrature phases the IP electric field operator (4.5a) becomes

$$E(x,t) = E_1(x,t)\cos[\Omega(t-x)] + E_2(x,t)\sin[\Omega(t-x)] \quad (\text{IP}) \quad (4.21)$$

[cf. Eq. (1.7)]; thus  $E_1(x,t)$  and  $E_2(x,t)$  describe modulation of waves "cos $[\Omega(t-x)]$ " and "sin $[\Omega(t-x)]$ ." The quadrature phases (4.19) or their multimode analogs [Eq. (1.8)] have been used in multimode analyses of optical homodyning,<sup>7</sup> resonance fluorescence,<sup>31-33</sup> parametric amplification,<sup>34-36</sup> and four-wave mixing.<sup>34</sup>

For two modes the concept of (second-moment) TS noise means that each mode has (second-moment) TS noise [Eq. (3.12)] and that the two modes have zero second-order correlation [cf. Eqs. (1.3)]; these conditions imply that the electric field has constant variance. One says that, for TS noise, the noise in the electric field is distributed randomly in phase, where phase is defined relative to frequency  $\Omega$ ; equivalently, one can say that TS noise means that the noise in the electric field is divided equally between the quadrature phases.

A two-mode squeezed state does not, in general, have TS noise. The two modes have correlated noise, and the quadrature phases carry different amounts of noise. Thus, in two-photon optics it is convenient to describe the noise in terms of the quadrature phases. In particular, the natural variables are the (two-mode) *quadrature-phase amplitudes*,<sup>13,21</sup> which are simply the Fourier components of the quadrature phases, normalized to be in units of square root of the number of quanta referred to the carrier frequency  $\Omega$ . In the SP the quadrature-phase amplitudes are explicitly time-dependent operators defined by

$$\alpha_1(t) \equiv \left[ \frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a_+ e^{i\Omega t} + \left[ \frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a_-^\dagger e^{-i\Omega t} \quad (\text{SP}), \quad (4.22a)$$

$$\alpha_2(t) \equiv -i \left[ \frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a_+ e^{i\Omega t} + i \left[ \frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a_-^\dagger e^{-i\Omega t} \quad (\text{SP}). \quad (4.22b)$$

Notice that the quadrature-phase amplitudes are not Hermitian. In the MP the quadrature-phase amplitudes are constant and are denoted by

$$\alpha_1 \equiv e^{iH_R t} \alpha_1(t) e^{-iH_R t} = \alpha_1(0) = \left[ \frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a_+ + \left[ \frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a_-^\dagger, \quad (4.23a)$$

$$\alpha_2 \equiv e^{iH_R t} \alpha_2(t) e^{-iH_R t} = \alpha_2(0) = -i \left[ \frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a_+ + i \left[ \frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a_-^\dagger \quad (4.23b)$$

[cf. Eqs. (1.13)]. We find it convenient to introduce the symbols

$$\lambda_\pm \equiv [(\Omega \pm \epsilon)/\Omega]^{1/2}, \quad (4.24)$$

so that Eqs. (4.23) and their inverse can be written in the compact forms

$$\alpha_1 = 2^{-1/2}(\lambda_+ a_+ + \lambda_- a_-^\dagger), \quad (4.25a)$$

$$\alpha_2 = 2^{-1/2}(-i\lambda_+ a_+ + i\lambda_- a_-^\dagger), \quad (4.25b)$$

$$\lambda_+ a_+ = 2^{-1/2}(\alpha_1 + i\alpha_2), \quad (4.26a)$$

$$\lambda_- a_- = 2^{-1/2}(\alpha_1^\dagger + i\alpha_2^\dagger). \quad (4.26b)$$

In the IP the quadrature-phase amplitudes acquire a harmonic time dependence at the modulation frequency:

$$e^{iH_M t} \alpha_m e^{-iH_M t} = e^{iH_0 t} \alpha_m(t) e^{-iH_0 t} = \alpha_m e^{-i\epsilon t}, \quad m=1,2. \quad (4.27)$$

Using Eqs. (4.5), (4.19), and (4.26), one can write the quadrature phases in the form

$$E_m(x,t) = \Omega^{1/2}[\alpha_m e^{-i\epsilon(t-x)} + \alpha_m^\dagger e^{i\epsilon(t-x)}], \quad m=1,2 \quad (4.28)$$

which shows explicitly that  $\alpha_m$  is a complex-amplitude operator at modulation frequency  $\epsilon$  for  $E_m(x,t)$  [i.e., it is the Fourier component of  $E_m(x,t)$  at positive frequency  $\epsilon$ ]. In our notation the MP is the most convenient picture for writing a picture-consistent equation relating the quadrature phases to their amplitudes; the MP quadrature phases are denoted by

$$E_m(x) \equiv e^{-iH_M t} E_m(x,t) e^{iH_M t}, \quad m=1,2 \quad (4.29)$$

so that



$$E_m(x) = \Omega^{1/2}(\alpha_m e^{i\epsilon x} + \alpha_m^\dagger e^{-i\epsilon x}), \quad m=1,2 \text{ (MP)}. \quad (4.30)$$

The two-mode quadrature-phase amplitudes have the following (discrete) commutator algebra:

$$[\alpha_1, \alpha_1^\dagger] = [\alpha_2, \alpha_2^\dagger] = \epsilon/\Omega, \quad (4.31a)$$

$$[\alpha_1, \alpha_2] = 0, \quad (4.31b)$$

$$[\alpha_1, \alpha_2^\dagger] = [\alpha_1^\dagger, \alpha_2] = i. \quad (4.31c)$$

These commutators enforce a set of uncertainty principles which we discuss in detail in Sec. VI.

All of our two-mode results thus far can easily be extended to a continuum description by using "continuum" quadrature-phase amplitudes and integrating over the positive modulation frequencies of interest [cf. Eqs. (1.8) and (1.13)]. The MP continuum quadrature-phase amplitudes<sup>21</sup>  $\alpha_1(\epsilon)$  and  $\alpha_2(\epsilon)$  are related to the continuum creation and annihilation operators [Eq. (2.1)] by Eqs. (1.13) [cf. Eqs. (4.23)]; they obey the commutation relations

$$[\alpha_1(\epsilon), \alpha_1(\epsilon')] = [\alpha_1(\epsilon), \alpha_2(\epsilon')] = [\alpha_2(\epsilon), \alpha_2(\epsilon')] = 0, \quad (4.32a)$$

$$[\alpha_1(\epsilon), \alpha_1^\dagger(\epsilon')] = [\alpha_2(\epsilon), \alpha_2^\dagger(\epsilon')] = \frac{\epsilon}{\Omega} 2\pi\delta(\epsilon - \epsilon'), \quad (4.32b)$$

$$[\alpha_1(\epsilon), \alpha_2^\dagger(\epsilon')] = [\alpha_1^\dagger(\epsilon), \alpha_2(\epsilon')] = i2\pi\delta(\epsilon - \epsilon'). \quad (4.32c)$$

The fundamental building blocks of two-photon optics are the quadrature-phase amplitudes and the two-mode squeezed states. In paper III these building blocks will be used to define new two-photon quasiprobability distributions.

### C. Pictorial convention

As is made clear by Eqs. (4.19)–(4.21),  $E_1(x,t) + iE_2(x,t)$  is a complex-amplitude operator for the two-mode electric field, defined with respect to frequency  $\Omega$ . The choice of phase for this complex amplitude is arbitrary, so one can ask what happens under a change of phase. The unitary operator

$$R(\theta) \equiv \exp[-i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)] \quad (4.33)$$

generates just such a phase change, i.e.,

$$\begin{aligned} R^\dagger(\theta)[E_1(x,t) + iE_2(x,t)]R(\theta) \\ \equiv E_1'(x,t) + iE_2'(x,t) \\ = [E_1(x,t) + iE_2(x,t)]e^{-i\theta}. \end{aligned} \quad (4.34)$$

We call  $R(\theta)$  the *rotation operator* because the transformation (4.34) is a rotation of the complex amplitude. This rotation corresponds to a common phase change for the annihilation operators,

$$R^\dagger(\theta)a_\pm R(\theta) \equiv a'_\pm = a_\pm e^{-i\theta}, \quad (4.35)$$

and to a rotation of the quadrature-phase amplitudes

$$R^\dagger(\theta)\alpha_1 R(\theta) \equiv \alpha'_1 = \alpha_1 \cos\theta + \alpha_2 \sin\theta, \quad (4.36a)$$

$$R^\dagger(\theta)\alpha_2 R(\theta) \equiv \alpha'_2 = -\alpha_1 \sin\theta + \alpha_2 \cos\theta. \quad (4.36b)$$

Notice that  $e^{-iH_R t} = R(\Omega t)$  [Eqs. (4.2b) and (4.33)]; thus the time dependence at the carrier frequency is simply a rotation of the complex amplitude.

One is now in a position to appreciate the importance of the MP. In two-photon optics one deals with the quadrature phases and their amplitudes as the fundamental quantities. The time dependence at frequency  $\Omega$  is trivial and uninteresting; the important free time dependence is at the modulation frequency. One would like to formulate the theory in such a way that the trivial time dependence at  $\Omega$  is suppressed. This goal is achieved in two steps: (i) one works in the MP, thereby transferring the time dependence at  $\Omega$  from the states to the operators; (ii) one defines the fundamental operators—the quadrature phases and their amplitudes—so that they are constant in the MP. The second step requires defining the quadrature phases and the quadrature-phase amplitudes with explicit time dependences in the SP [Eqs. (4.19) and (4.22)], which then disappear in the MP [Eqs. (4.23) and (4.30)]. The effect of the above two steps is to transform frequency  $\Omega$  to zero frequency, thereby removing it from the problem. In two-photon optics the MP in essence replaces the SP: in the MP the states carry the important time dependence, and the fundamental operators are constant.

With these remarks in mind, we introduce a set of conventions that we adhere to throughout the remainder of this paper and subsequent papers in this series. The creation and annihilation operators are always written in the SP (operators  $a_\pm^\dagger, a_\pm$ ); expectation values of the creation and annihilation operators are evaluated using the SP density operator  $\rho_{SP}(t)$ . The electric field and the quadrature phases are always written in the IP [operators  $E(x,t)$ ,  $E^{(\pm)}(x,t)$ ,  $E_1(x,t)$ , and  $E_2(x,t)$ ; Eqs. (4.5) and (4.19)]; expectation values of these field quantities are evaluated using the IP density operator  $\rho_{IP}(t)$ . Finally, the quadrature-phase amplitudes and the squeezed annihilation operators are always written in the MP [operators  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_\pm(r,\varphi)$ ; Eqs. (4.23) and (4.14)]; expectation values of these quantities are evaluated using the MP density operator  $\rho_{MP}(t)$ . The *MP free evolution operator* we dignify by a special notation,

$$U_M(t) \equiv e^{-iH_M t} = \exp[-i\epsilon t(a_+^\dagger a_+ - a_-^\dagger a_-)] \quad (4.37)$$

[Eq. (4.2c)], because of the importance of the MP in our formalism. In the SP the free evolution operator is  $e^{-iH_0 t} = R(\Omega t)U_M(t)$ , and in the IP the free evolution operator is the identity operator.

### D. Physical significance of quadrature-phase amplitudes

Throughout this subsection we are interested in expectation values of field quantities (the electric field and the quadrature phases) which are undergoing free evolution. Thus, in accordance with the conventions just described in Sec. IV C, all expectation values are evaluated with respect to the initial ( $t=0$ ) state.

We turn now to a detailed discussion of the meaning of the quadrature-phase amplitudes. To understand their close connection to experiment, it is useful first to look closely at how the expectation values of  $\alpha_1$  and  $\alpha_2$  determine the classical behavior of the electromagnetic field.

By classical behavior we mean simply the free time evolution of the expectation value of the electric field at a particular point in space, say  $x=0$ . Equivalent information, but with the rapid time dependence at frequency  $\Omega$  removed, is contained in the expectation value of the field's complex amplitude:

$$\langle E(0,t) \rangle = \text{Re}[\langle E_1(0,t) + iE_2(0,t) \rangle e^{-i\Omega t}]. \quad (4.38)$$

For present purposes it is more convenient to deal with a dimensionless complex amplitude, which is defined in the IP by

$$\begin{aligned} \mathcal{E}_1(x,t) + i\mathcal{E}_2(x,t) &\equiv (2\Omega)^{-1/2}[E_1(x,t) + iE_2(x,t)] \\ &= (2/\Omega)^{1/2} E^{(+)}(x,t) e^{i\Omega(t-x)}. \end{aligned} \quad (4.39)$$

This dimensionless complex amplitude is related to the annihilation operators by

$$\begin{aligned} \mathcal{E}_1(x,t) + i\mathcal{E}_2(x,t) &= \lambda_+ a_+ e^{-i\epsilon(t-x)} \\ &\quad + \lambda_- a_- e^{i\epsilon(t-x)} \end{aligned} \quad (4.40)$$

[Eqs. (4.5b) and (4.24)], and its components, dimensionless (Hermitian) quadrature phases, can be written as

$$\begin{aligned} \mathcal{E}_m(x,t) &\equiv (2\Omega)^{-1/2} E_m(x,t) \\ &= 2^{-1/2} (\alpha_m e^{-i\epsilon(t-x)} + \alpha_m^\dagger e^{i\epsilon(t-x)}), \\ &\quad m=1,2 \end{aligned} \quad (4.41)$$

[Eq. (4.28)].

For the simple case of a two-frequency field, the classical behavior is specified by

$$\begin{aligned} \langle \mathcal{E}_1(0,t) + i\mathcal{E}_2(0,t) \rangle &= \lambda_+ \langle a_+ \rangle e^{-i\epsilon t} + \lambda_- \langle a_- \rangle e^{i\epsilon t} \\ &= \text{Re}(2^{1/2} \langle \alpha_1 \rangle e^{-i\epsilon t}) \\ &\quad + i \text{Re}(2^{1/2} \langle \alpha_2 \rangle e^{-i\epsilon t}). \end{aligned} \quad (4.42)$$

Equation (4.42) says that the mean complex amplitude rotates about the origin, its tip tracing out an ellipse, the "signal ellipse," during each modulation period  $2\pi/\epsilon$ . The classical behavior of the field can be pictured on a complex-amplitude diagram (Fig. 1). On a complex-amplitude plane one draws the signal ellipse, indicates the initial ( $t=0$ ) complex amplitude by a vector whose tip lies on the signal ellipse, and shows the direction of rotation of the complex amplitude by arrows on the signal ellipse. Four pieces of information are required to specify the classical behavior: the two radii and the orientation of the signal ellipse, and the direction of the initial complex amplitude. Notice that the phase change (4.34) corresponds to rotating the axes of the complex-amplitude plane counterclockwise by an angle  $\theta$ . Notice also that in the degenerate limit ( $\epsilon=0$ ,  $a_+ = a_-$ ) the mean complex amplitude never changes; the signal ellipse collapses to a single point, which is just the unchanging complex amplitude of a single mode.

Simple though the representation in Fig. 1 may be, it is instructive to decompose the elliptical motion of the complex amplitude into even simpler parts. The obvious decomposition is in terms of the two Fourier components of the field, i.e., in terms of the mean complex amplitudes

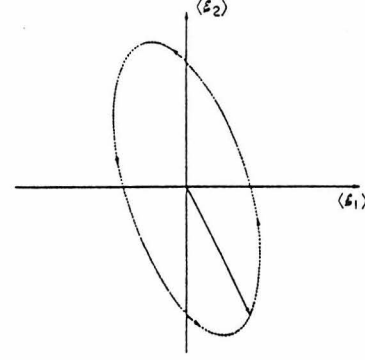


FIG. 1. Complex-amplitude diagram for the classical behavior of the electric field. The dotted ellipse is the signal ellipse traced out by the mean complex amplitude  $\langle \mathcal{E}_1(0,t) + i\mathcal{E}_2(0,t) \rangle$  during each modulation period  $2\pi/\epsilon$ . Arrows on the signal ellipse show the direction of rotation of the complex amplitude. A vector indicates the initial ( $t=0$ ) complex amplitude.

$\langle a_\pm \rangle$  of the two modes. In this decomposition [Eq. (4.42)], the mean complex amplitude is a sum of two vectors,  $\lambda_+ \langle a_+ \rangle e^{-i\epsilon t}$ , which rotates clockwise, and  $\lambda_- \langle a_- \rangle e^{i\epsilon t}$ , which rotates counterclockwise (see Fig. 2). The four classical pieces of information are given by the complex numbers  $\langle a_\pm \rangle$  and  $\langle a_\mp \rangle$ , each of which specifies the (real) amplitude and phase of one of the modes.

The other useful decomposition is in terms of the quadrature-phase amplitudes:

$$\langle \mathcal{E}_m(0,t) \rangle = \text{Re}(2^{1/2} \langle \alpha_m \rangle e^{-i\epsilon t}), \quad m=1,2. \quad (4.43)$$

In this decomposition the four required pieces of information are given by the complex numbers  $\langle \alpha_1 \rangle$  and  $\langle \alpha_2 \rangle$ , each of which is a complex amplitude for one of the quadrature phases. To represent this decomposition graphically, one draws separate complex planes for the vectors  $2^{1/2} \langle \alpha_1 \rangle e^{-i\epsilon t}$  and  $2^{1/2} \langle \alpha_2 \rangle e^{-i\epsilon t}$ . In each of these planes the vector  $2^{1/2} \langle \alpha_m \rangle e^{-i\epsilon t}$  rotates clockwise, and its projection on the real axis gives  $\langle \mathcal{E}_m(0,t) \rangle$  [Eq. (4.43); see Fig. 2]. These separate planes are phase planes for the quadrature phases; they show vividly how  $\langle \alpha_m \rangle$  specifies the (real) amplitude and phase of  $\langle \mathcal{E}_m(0,t) \rangle$ .

Figure 2 shows, at four separate times, the complex-amplitude plane for  $\langle \mathcal{E}_1(0,t) + i\mathcal{E}_2(0,t) \rangle$ , together with the two decompositions discussed above. Such a diagram at any particular time (usually chosen to be  $t=0$ ) contains all the information about the classical behavior of the field. In the next section we show how to include information about TSQP noise on such a diagram.

The physical significance of the quadrature-phase amplitudes can be demonstrated compellingly in two ways. The first is to consider their relation to amplitude and phase modulation. Superpose on the two-mode electric field (4.5) a strong, classical carrier wave at frequency  $\Omega$ ; let the carrier wave be given by  $(2\Omega)^{1/2} B \cos(\Omega(t-x))$ , where  $B$  is real. The two modes at frequencies  $\Omega \pm \epsilon$

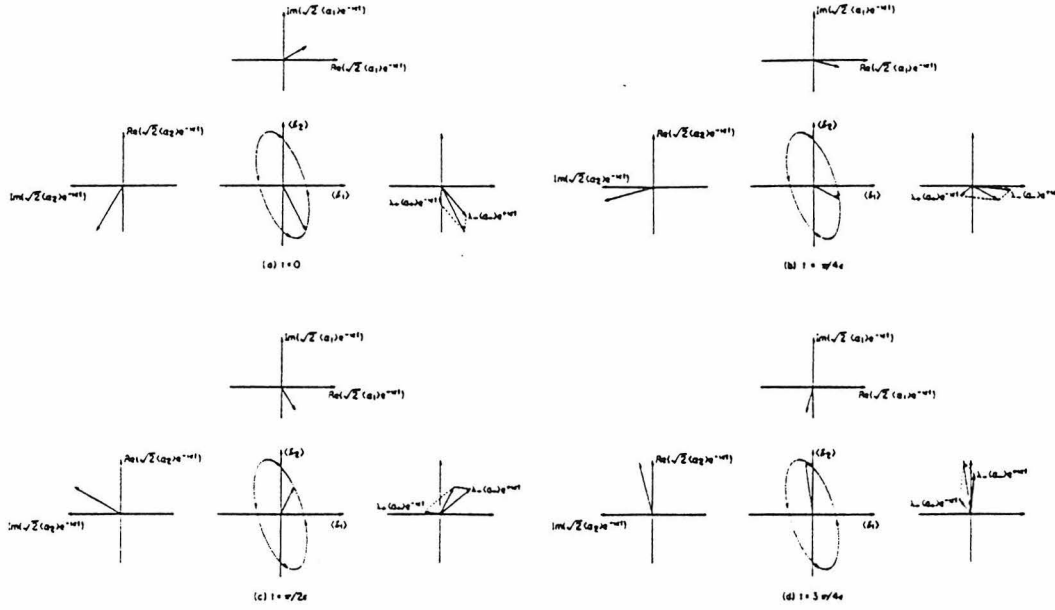


FIG. 2. Complex-amplitude diagrams at four times: (a)  $t=0$ , (b)  $t=\pi/4\epsilon$ , (c)  $t=\pi/2\epsilon$ , (d)  $t=3\pi/4\epsilon$ . At each time the central complex-amplitude diagram is the same as in Fig. 1, except that the vector indicates the mean complex amplitude at the appropriate time. To the right of the central diagram is a complex-amplitude plane which shows the decomposition of the mean complex amplitude into contributions from the two modes [Eq. (4.42)]. Above and to the left of the central diagram are phase planes for the quadrature phases. In the phase plane above the central diagram, a vector indicates the value of  $2^{1/2}\langle\alpha_1\rangle e^{-i\epsilon t}$ ; its real part is  $\langle\mathcal{F}_1(0,t)\rangle$  [Eq. (4.43)]. In the phase plane to the left a vector indicates the value of  $2^{1/2}\langle\alpha_2\rangle e^{-i\epsilon t}$ ; its real part is  $\langle\mathcal{F}_2(0,t)\rangle$ .

represent sidebands of the carrier. The positive-frequency part of the total field is given by

$$\begin{aligned} \bar{E}^{(+)}(x,t) &= (\Omega/2)^{1/2} B e^{-i\Omega(t-x)} + E^{(+)}(x,t) \\ &= \frac{1}{2} [(2\Omega)^{1/2} B + E_1(x,t) + iE_2(x,t)] e^{-i\Omega(t-x)} \end{aligned} \quad (4.44)$$

[Eq. (4.20)], corresponding to an electric field

$$\begin{aligned} \bar{E}(x,t) &= \bar{E}^{(+)}(x,t) + [\bar{E}^{(+)}(x,t)]^\dagger \\ &= [(2\Omega)^{1/2} B + E_1(x,t)] \cos[\Omega(t-x)] \\ &\quad + E_2(x,t) \sin[\Omega(t-x)]. \end{aligned} \quad (4.45)$$

In Eqs. (4.44) and (4.45) an overbar designates the total field, including both the carrier and the sidebands. Equation (4.45) shows that  $E_1(x,t)$  modulates a wave that is in phase with the carrier—amplitude modulation of the carrier—and  $E_2(x,t)$  modulates a wave that is  $90^\circ$  out of phase with the carrier—phase modulation of the carrier. Thus the quadrature-phase amplitudes are complex-amplitude operators for the amplitude and phase modulation. The expectation value of the total field's dimensionless complex amplitude is the sum of the constant amplitude  $B$  of the carrier and the modulated complex amplitude (4.42):

$$\begin{aligned} \langle \bar{\mathcal{F}}_1(0,t) + i\bar{\mathcal{F}}_2(0,t) \rangle &\equiv (2/\Omega)^{1/2} \langle \bar{E}^{(+)}(0,t) \rangle e^{i\Omega t} \\ &= B + \langle \mathcal{F}_1(0,t) + i\mathcal{F}_2(0,t) \rangle \end{aligned} \quad (4.46)$$

[Eqs. (4.39) and (4.44)]. Thus the effect of the carrier on the complex-amplitude diagrams of Figs. 1 and 2 is to displace the signal ellipse a distance  $B$  along the real axis. The resulting complex-amplitude diagram makes clear that the oscillation of  $\langle \mathcal{F}_1(0,t) \rangle$  is the amplitude-modulation signal and the oscillation of  $\langle \mathcal{F}_2(0,t) \rangle$  is the phase-modulation signal. The separate planes for  $2^{1/2}\langle\alpha_1\rangle e^{-i\epsilon t}$  and  $2^{1/2}\langle\alpha_2\rangle e^{-i\epsilon t}$  are phase planes for the amplitude and phase modulation.

The second way to demonstrate the significance of the quadrature-phase amplitudes is to note their relation to ideal heterodyning. In heterodyne detection the two-mode field (4.21) is mixed with (multiplied by) a strong local-oscillator field at frequency  $\Omega$ , and the result is filtered to pick out the Fourier component at frequency  $\epsilon$ . If the local-oscillator field is proportional to  $\cos[\Omega(t-x)]$  ( $\sin[\Omega(t-x)]$ ) and if the mixing and filtering are ideal, then the output of the heterodyne detector is proportional to  $E_1(x,t)$  [ $E_2(x,t)$ ], and its complex amplitude is proportional to  $\alpha_1$  ( $\alpha_2$ ).<sup>34</sup> In terms of the complex-amplitude diagrams of Fig. 2, heterodyning picks out the oscillation of  $\langle \mathcal{F}_1(0,t) \rangle$  [ $\langle \mathcal{F}_2(0,t) \rangle$ ]; the separate plane for  $2^{1/2}\langle\alpha_1\rangle e^{-i\epsilon t}$  ( $2^{1/2}\langle\alpha_2\rangle e^{-i\epsilon t}$ ) is a phase plane for the

heterodyned output.

At optical frequencies heterodyning is performed by combining the two-mode field (4.21) with a local-oscillator field at a beam splitter and then directing the combined field onto a photodetector; the mixing is a result of the photodetector's square-law response. Yuen and Shapiro<sup>7</sup> have analyzed optical heterodyning in detail. They assume  $\epsilon \ll \Omega$  so they can neglect  $\epsilon$  relative to  $\Omega$  in factors like  $\lambda_{\pm} = [(\Omega \pm \epsilon)/\Omega]^{1/2}$  [cf. Eqs. (4.25)]. In this approximation they find that ideal optical heterodyning does indeed produce an output whose complex amplitude is proportional to  $\alpha_1$ .

*The physical significance of the quadrature-phase amplitudes lies in their close connection to experimental techniques; they are the complex-amplitude operators for fields—the quadrature phases—that are directly accessible to measurement and experimental manipulation.* The quadrature phases are accessible because they describe the physical process of putting amplitude and phase modulation on a carrier signal and because they are the quantities detected by phase-sensitive detection techniques such as heterodyning.

In place of the quadrature-phase amplitudes, one might be tempted to use operators<sup>27</sup> defined in the MP by

$$\beta_1 \equiv 2^{-1/2}(a_+ + a_-^\dagger), \quad (4.47a)$$

$$\beta_2 \equiv 2^{-1/2}(-ia_+ + ia_-^\dagger) \quad (4.47b)$$

[cf. Eqs. (4.25)]. These operators have a simpler commutator algebra than  $\alpha_1$  and  $\alpha_2$ :

$$[\beta_1, \beta_1^\dagger] = [\beta_2, \beta_2^\dagger] = [\beta_1, \beta_2] = 0, \quad (4.48a)$$

$$[\beta_1, \beta_2^\dagger] = [\beta_1^\dagger, \beta_2] = i \quad (4.48b)$$

[cf. Eqs. (4.31)]; and under a unitary transformation generated by  $S(r, 0)$ , they transform very simply:

$$S^\dagger(r, 0)\beta_1 S(r, 0) = \beta_1 e^{-r}, \quad (4.49a)$$

$$S^\dagger(r, 0)\beta_2 S(r, 0) = \beta_2 e^r \quad (4.49b)$$

[Eqs. (4.10) and (4.47)]. Despite these simple properties,  $\beta_1$  and  $\beta_2$  are not the natural variables for two-photon optics because they have no close connection to experimental techniques; they are not complex-amplitude operators for fields that can be measured. Shapiro and Wagner<sup>27</sup> have argued that  $\beta_1$  or  $\beta_2$  is the quantity detected by optical heterodyning. Their contention is based on Cook's claim<sup>28</sup> that photodetectors respond to "photon flux." The detailed analysis of Kimble and Mandel<sup>29</sup> does not support Cook's claim. Recent work by Yurke<sup>29</sup> indicates that  $\alpha_1$  or  $\alpha_2$ , more nearly than  $\beta_1$  or  $\beta_2$ , is the quantity detected by ideal optical heterodyning.

One can understand why  $\beta_1$  and  $\beta_2$  are not the natural variables—and at the same time understand the factors  $\lambda_{\pm}$  which appear in the definition of  $\alpha_1$  and  $\alpha_2$  [Eqs. (4.25)]—by a simple units argument. The operators  $a_+$  and  $a_-^\dagger$  should not be added directly, as in Eqs. (4.47), because they have incompatible units; each has units of square root of the number of quanta, referred to its own frequency. Multiplication of  $a_+$  by  $(\Omega + \epsilon)^{1/2}$  and  $a_-^\dagger$  by

$(\Omega - \epsilon)^{1/2}$  converts the two quantities to common units of square root of energy; after this multiplication the two quantities may be added, as is done in the definitions of  $\alpha_1$  and  $\alpha_2$  [Eqs. (4.23)]. Division by  $(2\Omega)^{1/2}$  then leaves  $\alpha_1$  and  $\alpha_2$  with dimensionless units of square root of the number of quanta, referred to the carrier frequency  $\Omega$ . That  $\alpha_1$  and  $\alpha_2$  have these units is confirmed by writing the free Hamiltonian (4.2a) as

$$H_0 = \Omega[(\alpha_1 \alpha_1^\dagger)_{\text{sym}} + (\alpha_2 \alpha_2^\dagger)_{\text{sym}} - 1]. \quad (4.50)$$

Thus  $(\alpha_1 \alpha_1^\dagger)_{\text{sym}} + (\alpha_2 \alpha_2^\dagger)_{\text{sym}} = (H_0 + \Omega)/\Omega$  is the total energy, including the one quantum of zero-point energy, measured in units of the quantum at frequency  $\Omega$ .

## V. TIME-STATIONARY QUADRATURE-PHASE NOISE

### A. Definition and discussion

The states encountered in two-photon optics—in particular, two-mode squeezed states—can have electric field noise that is not distributed randomly in phase, where phase is defined relative to  $\Omega$ . This phase-sensitive noise is of a special sort, however, which we call *time-stationary quadrature-phase (TSQP) noise*.<sup>13,21</sup> The reason for the name is that the quadrature phases have time-stationary noise; this means that the natural variables to describe TSQP noise are the Fourier components of the quadrature phases, the quadrature-phase amplitudes.

To see what TSQP noise means, let  $\rho$  be the initial density operator for the pair of modes considered in Sec. IV. The noise associated with  $\rho$  can be characterized by the noise moments of  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_1^\dagger$ , and  $\alpha_2^\dagger$ . Just as we did for TS noise, we consider only the second-order noise moments—a specialization justified by the assumption that the noise is Gaussian; a complete description of TSQP and TS noise, based on all noise moments, will be presented in paper III. The state  $\rho$  is said to have (second-moment) TSQP noise if the quadrature-phase amplitudes have random-phase noise, i.e., if

$$\begin{aligned} \langle \Delta \alpha_m \Delta \alpha_n \rangle &\equiv \text{tr}(\rho \Delta \alpha_m \Delta \alpha_n) \\ &= \langle \alpha_m \alpha_n \rangle - \langle \alpha_m \rangle \langle \alpha_n \rangle = 0. \end{aligned} \quad (5.1)$$

where  $m, n = 1, 2$  and  $\Delta \alpha_m \equiv \alpha_m - \langle \alpha_m \rangle$  [cf. Eqs. (1.10a) and (1.13)]. In general, ten real numbers are required to specify all the second-order noise information, but the TSQP condition (5.1) eliminates six of those numbers. The remaining four numbers are contained in the "reduced" spectral-density matrix

$$\begin{aligned} \Sigma_{mn} &\equiv \langle \Delta \alpha_m \Delta \alpha_n^\dagger \rangle_{\text{sym}} \\ &\equiv \text{tr}[\rho(\Delta \alpha_m \Delta \alpha_n^\dagger)_{\text{sym}}] = \langle \alpha_m \alpha_n^\dagger \rangle_{\text{sym}} - \langle \alpha_m \rangle \langle \alpha_n^\dagger \rangle \end{aligned} \quad (5.2)$$

[cf. Eqs. (1.10b) and (1.13)], which is dimensionless (units of number of quanta at frequency  $\Omega$ ) and Hermitian:

$$\Sigma_{mn}^* = \Sigma_{nm}. \quad (5.3)$$

The spectral-density matrix, which has units of energy, is

defined by

$$S_{mn} \equiv \Omega \Sigma_{mn} . \quad (5.4)$$

The diagonal elements of  $\Sigma_{mn}$  are simply the mean-square uncertainties in  $\alpha_1$  and  $\alpha_2$ :

$$\Sigma_{mm} = \langle |\Delta\alpha_m|^2 \rangle, \quad m=1,2; \quad (5.5)$$

the off-diagonal element  $\Sigma_{12} = \Sigma_{21}^*$  is a complex correlation coefficient between the quadratures.

Under free evolution [MP evolution operator  $U_M(t)$ ; Eq. (4.37)] the noise moments  $\langle \Delta\alpha_m \Delta\alpha_n \rangle$  acquire a harmonic time dependence  $e^{-2i\epsilon t}$ , whereas the noise moments  $\langle \Delta\alpha_m \Delta\alpha_n^\dagger \rangle_{\text{sym}}$  are constant [Eq. (4.27)]. Just as for TS noise, the vanishing of the time-dependent noise moments is the key to generalizing the notion of TSQP noise to moments of arbitrary order and also to understanding why the quadrature-phase amplitudes are suited to two-photon optics. The general definition of TSQP noise, which will be given explicitly in paper III, requires that all time-dependent noise moments of  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_1^\dagger$ , and  $\alpha_2^\dagger$  vanish. *The quadrature-phase amplitudes are the natural variables for two-photon optics because the TSQP noise produced by two-photon devices is completely characterized by the time-independent noise moments of  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_1^\dagger$ , and  $\alpha_2^\dagger$ .*

It is often useful to have the TSQP condition (5.1) and the reduced spectral-density matrix (5.2) written in terms of creation and annihilation operators. The (second-moment) TSQP condition is equivalent to the following conditions on the second-order noise moments of the creation and annihilation operators:

$$\langle (\Delta a_\pm)^2 \rangle = 0, \quad (5.6a)$$

$$\langle \Delta a_+ \Delta a_-^\dagger \rangle = 0 \quad (5.6b)$$

[Eqs. (4.26); Eq. (5.6a) means that for TSQP' noise each mode by itself has random-phase noise]. The remaining second-order noise moments of the creation and annihilation operators are related to  $\Sigma_{mn}$ :

$$\begin{aligned} \lambda_+ \lambda_- \langle \Delta a_+ \Delta a_- \rangle &= \frac{1}{2}(\Sigma_{11} - \Sigma_{22}) + \frac{1}{2}i(\Sigma_{12} + \Sigma_{21}) \\ &= \frac{1}{2}(\Sigma_{11} - \Sigma_{22}) + i \text{Re}(\Sigma_{12}), \end{aligned} \quad (5.7a)$$

$$\begin{aligned} \lambda_\pm^2 \langle |\Delta a_\pm|^2 \rangle &= \frac{1}{2}(\Sigma_{11} + \Sigma_{22}) \mp \frac{1}{2}i(\Sigma_{12} - \Sigma_{21}) \\ &= \frac{1}{2}(\Sigma_{11} + \Sigma_{22}) \pm \text{Im}(\Sigma_{12}) \end{aligned} \quad (5.7b)$$

[Eqs. (4.26)]. Equations (5.7) can be recast in the form

$$\frac{1}{2}(\Sigma_{11} + \Sigma_{22}) = \frac{1}{2}(\lambda_+^2 \langle |\Delta a_+|^2 \rangle + \lambda_-^2 \langle |\Delta a_-|^2 \rangle), \quad (5.8a)$$

$$\frac{1}{2}(\Sigma_{11} - \Sigma_{22}) = \lambda_+ \lambda_- \text{Re}(\langle \Delta a_+ \Delta a_- \rangle), \quad (5.8b)$$

$$\frac{1}{2}(\Sigma_{12} + \Sigma_{21}) = \text{Re}(\Sigma_{12}) = \lambda_+ \lambda_- \text{Im}(\langle \Delta a_+ \Delta a_- \rangle), \quad (5.8c)$$

$$\begin{aligned} -\frac{1}{2}i(\Sigma_{12} - \Sigma_{21}) &= \text{Im}(\Sigma_{12}) = \frac{1}{2}(\lambda_+^2 \langle |\Delta a_+|^2 \rangle \\ &\quad - \lambda_-^2 \langle |\Delta a_-|^2 \rangle). \end{aligned} \quad (5.8d)$$

Notice that for TSQP noise the time-dependent noise moment  $\langle \Delta a_+ \Delta a_- \rangle$  (free time dependence  $e^{-2i\Omega t}$ ) need not vanish. Since it must vanish for TS noise, Eqs. (5.7) im-

ply that (second-moment) TSQP noise is (second-moment) TS noise if and only if

$$\Sigma_{11} = \Sigma_{22} = \frac{1}{2}(\lambda_+^2 \langle |\Delta a_+|^2 \rangle + \lambda_-^2 \langle |\Delta a_-|^2 \rangle), \quad (5.9a)$$

$$\Sigma_{12} = -\Sigma_{21} = \frac{1}{2}i(\lambda_+^2 \langle |\Delta a_+|^2 \rangle - \lambda_-^2 \langle |\Delta a_-|^2 \rangle). \quad (5.9b)$$

The reduced spectral-density matrix  $\Sigma_{mn}$  describes how the noise is distributed in phase, where phase is defined with respect to frequency  $\Omega$ . There are two good ways of seeing this—ways that make clear the meaning of the four pieces of information in  $\Sigma_{mn}$ . The first way looks at the two-point correlation matrix of the dimensionless quadrature phases  $\mathcal{E}_1(x,t)$  and  $\mathcal{E}_2(x,t)$  [Eq. (4.41)],

$$\mathcal{K}_{mn}(\tau) \equiv \langle \Delta \mathcal{E}_m(x, t+\tau) \Delta \mathcal{E}_n(x, t) \rangle_{\text{sym}}, \quad m, n = 1, 2 \quad (5.10)$$

which is a dimensionless, real matrix. If the two modes evolve freely, then

$$\mathcal{K}_{mn}(\tau) = \frac{1}{2}(\Sigma_{mn} e^{-i\epsilon\tau} + \Sigma_{nm} e^{i\epsilon\tau}) = \text{Re}(\Sigma_{mn} e^{-i\epsilon\tau}) \quad (5.11)$$

[Eqs. (4.41), (5.1), and (5.2)]. The two-point correlation matrix also satisfies

$$\mathcal{K}_{mn}(-\tau) = \mathcal{K}_{nm}(\tau) \quad (5.12)$$

[Eq. (5.3)]. That the two-point correlation matrix depends on the time delay  $\tau$ , but not on the retarded time  $t-x$ , is the essence of TSQP noise, and it is a direct consequence of the TSQP condition (5.1). Note, however, that TSQP noise does *not* mean that the two-point correlation matrix for the two-mode electric field depends only on the time delay  $\tau$ ; that condition is met only for TS noise [Eqs. (5.9)].

Consider now the zero time-delay ( $\tau=0$ ) correlation matrix

$$\mathcal{K}_{mn} \equiv \mathcal{K}_{mn}(0) = \langle \Delta \mathcal{E}_m(x, t) \Delta \mathcal{E}_n(x, t) \rangle_{\text{sym}} = \text{Re}(\Sigma_{mn}), \quad (5.13)$$

which is just the symmetric covariance matrix of the dimensionless quadrature phases [cf. Eq. (1.11)]. If the noise is distributed randomly in phase, then  $\mathcal{K}_{mn}$  is a multiple of the unit matrix. The covariance matrix  $\mathcal{K}_{mn}$  contains three of the pieces of information in  $\Sigma_{mn}$ . Two pieces of information are contained in the diagonal elements

$$\mathcal{K}_{mm} = \langle [\Delta \mathcal{E}_m(x, t)]^2 \rangle = \Sigma_{mm} = \langle |\Delta\alpha_m|^2 \rangle, \quad m=1,2 \quad (5.14)$$

which give the (constant) variances of  $\mathcal{E}_1(x,t)$  and  $\mathcal{E}_2(x,t)$ , and the third piece is contained in the off-diagonal element

$$\mathcal{K}_{12} = \mathcal{K}_{21} = \langle \Delta \mathcal{E}_1(x, t) \Delta \mathcal{E}_2(x, t) \rangle_{\text{sym}} = \text{Re}(\Sigma_{12}), \quad (5.15)$$

which is a correlation coefficient for  $\mathcal{E}_1(x,t)$  and  $\mathcal{E}_2(x,t)$ . These three pieces of information characterize the noise in the following way: the overall scale of the noise is set by  $\frac{1}{2}(\Sigma_{11} + \Sigma_{22})$ , which is the average noise in the quadra-

tures [Eq. (5.14)] or the average noise in the two modes [Eq. (5.8a)]; the extent to which the noise is not distributed randomly in phase is specified by  $\frac{1}{2}(\Sigma_{11}-\Sigma_{22})$  and  $\text{Re}(\Sigma_{12})$ . The roles of these quantities are immediately apparent in the variance of the electric field

$$\langle [\Delta E(x,t)]^2 \rangle = \Omega \{ \Sigma_{11} + \Sigma_{22} + (\Sigma_{11} - \Sigma_{22}) \cos[2\Omega(t-x)] + 2 \text{Re}(\Sigma_{12}) \sin[2\Omega(t-x)] \} \quad (5.16)$$

[Eqs. (4.21), (4.39), and (5.13); cf. Eqs. (1.12) and (3.13)]. A nonrandom distribution of noise in phase corresponds to a time-dependent electric field variance. The quantities that describe a nonrandom distribution,  $\frac{1}{2}(\Sigma_{11}-\Sigma_{22})$  and  $\text{Re}(\Sigma_{12})$ , are related to  $\langle \Delta a_+ \Delta a_- \rangle$  [Eqs. (5.8b) and (5.8c)]; if the electric field noise is not distributed randomly in phase, the two modes must be correlated.

The fourth piece of information in  $\Sigma_{mn}$  shows up in the time-delayed ( $\tau \neq 0$ ) correlation between the dimensionless quadrature phases. Specifically, for  $\tau = \pi/2\epsilon$ , the two-point correlation matrix becomes an antisymmetric matrix

$$\begin{aligned} \overline{\mathcal{X}}_{mn} &\equiv \mathcal{X}_{mn}(\pi/2\epsilon) \\ &= \langle \Delta \mathcal{E}_m(x, t + \pi/2\epsilon) \Delta \mathcal{E}_n(x, t) \rangle_{\text{sym}} \\ &= \text{Im}(\Sigma_{mn}). \end{aligned} \quad (5.17)$$

The diagonal elements of  $\overline{\mathcal{X}}_{mn}$  vanish. This result one expects for TSQP noise; it says that for each quadrature phase the noise at a particular time is uncorrelated with the noise a quarter cycle later. In contrast, the off-diagonal element of  $\overline{\mathcal{X}}_{mn}$  need not vanish. It gives the fourth piece of information in  $\Sigma_{mn}$ :

$$\begin{aligned} \overline{\mathcal{X}}_{12} &= -\overline{\mathcal{X}}_{21} = \langle \Delta \mathcal{E}_1(x, t + \pi/2\epsilon) \Delta \mathcal{E}_2(x, t) \rangle_{\text{sym}} \\ &= \text{Im}(\Sigma_{12}). \end{aligned} \quad (5.18)$$

This result is a bit mysterious; it says that the noise in one quadrature at a particular time is correlated with the noise in the other quadrature a quarter cycle later. The explanation lies in the definitions of  $\alpha_1$  and  $\alpha_2$  [Eqs. (4.25)]. A fluctuation in the upper mode  $a_+$  corresponds to identical fluctuations in  $\alpha_1$  and  $\alpha_2$ , but the fluctuation in  $\alpha_1$  lags that in  $\alpha_2$  by a quarter cycle; this produces a positive contribution to  $\overline{\mathcal{X}}_{12}$ . Similarly, a fluctuation in the lower mode  $a_-$  corresponds to a fluctuation in  $\alpha_1$  that leads the fluctuation in  $\alpha_2$  by a quarter cycle; this produces a negative contribution to  $\overline{\mathcal{X}}_{12}$ . Thus  $\overline{\mathcal{X}}_{12}$  should be related to the difference in noise in the two modes, an inference confirmed by Eqs. (5.8d) and (5.18), which show that

$$\overline{\mathcal{X}}_{12} = \text{Im}(\Sigma_{12}) = \frac{1}{2}(\lambda_+^2 \langle |\Delta a_+|^2 \rangle - \lambda_-^2 \langle |\Delta a_-|^2 \rangle).$$

The second way of investigating the meaning of  $\Sigma_{mn}$  is to look at how it transforms under a rotation (phase change) of the complex amplitude of the electric field [rotation produced by  $R(\theta)$ ; Eqs. (4.33)–(4.36)]. Recalling that a rotation produces a common phase change of the annihilation operators [Eq. (4.35)], one sees from Eq. (5.7b) that  $\frac{1}{2}(\Sigma_{11} + \Sigma_{22})$  and  $-\frac{1}{2}i(\Sigma_{12} - \Sigma_{21}) = \text{Im}(\Sigma_{12})$  are invariant under rotations; these quantities have nothing to do with the differential distribution of noise in phase. Similarly, one sees from Eq. (5.7a) that  $\frac{1}{2}(\Sigma_{11} - \Sigma_{22})$  and

$\frac{1}{2}(\Sigma_{12} + \Sigma_{21}) = \text{Re}(\Sigma_{12})$  transform as

$$\begin{aligned} &\frac{1}{2}(\Sigma_{11} - \Sigma_{22}) + \frac{1}{2}i(\Sigma_{12} + \Sigma_{21}) \\ &= e^{-2i\theta} \left[ \frac{1}{2}(\Sigma_{11} - \Sigma_{22}) + \frac{1}{2}i(\Sigma_{12} + \Sigma_{21}) \right]; \end{aligned} \quad (5.19)$$

these quantities characterize precisely the extent to which the noise is not distributed randomly in phase.

### B. Complex-amplitude diagrams

One can add information about TSQP noise to the complex-amplitude diagrams in Figs. 1 and 2. Start with the complex-amplitude diagram ( $t=0$ ) in Fig. 1, which describes the classical behavior of the electric field. To add information about TSQP noise, draw an "error ellipse" centered at the tip of the initial complex-amplitude vector (Fig. 3). The error ellipse displays the information

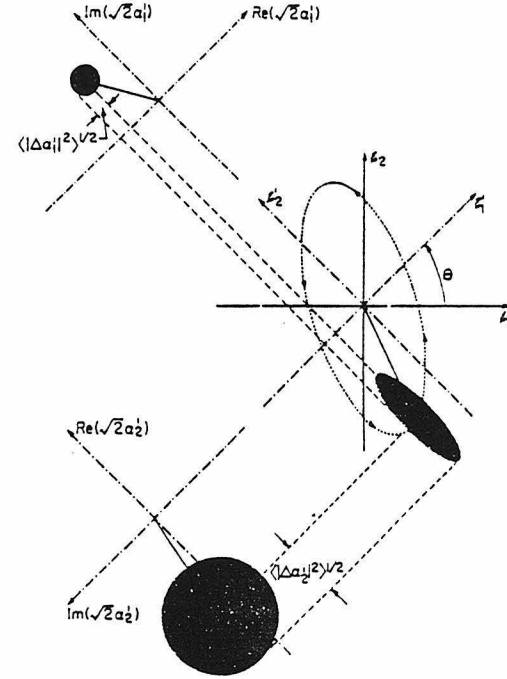


FIG. 3. Standard complex-amplitude diagram for TSQP noise. The behavior of the mean complex amplitude  $\langle \mathcal{E}_1(0, t) + i \mathcal{E}_2(0, t) \rangle$  is shown, as in Fig. 1, by a dotted signal ellipse and an initial ( $t=0$ ) complex-amplitude vector. The quadrature-phase noise is depicted by a shaded error ellipse. The principal axes of the error ellipse are the eigendirections of the covariance matrix  $\mathcal{X}_{mn}$  [Eq. (5.13)], and the principal radii are the square roots of the eigenvalues of  $\mathcal{X}_{mn}$ . The complex-amplitude diagram shows rotated (primed) axes that lie along the principal axes of the error ellipse. With respect to the rotated axes the covariance matrix  $\mathcal{X}'_{mn}$  is diagonal, its diagonal elements  $\mathcal{X}'_{mm} = \langle [\Delta \mathcal{E}'_m(x, t)]^2 \rangle = \Sigma'_{mm} = \langle |\Delta \alpha'_m|^2 \rangle$  giving the squares of the principal radii. Separate phase planes are drawn for the rotated quadrature phases (cf. Fig. 2). In each a vector indicates the initial ( $t=0$ ) value of  $2^{1/2} \langle \alpha'_m \rangle$ , and a shaded error circle, with radius  $\langle |\Delta \alpha'_m|^2 \rangle^{1/2}$ , depicts the noise in the quadrature phase.

contained in the covariance matrix  $\mathcal{X}_{mn} = \text{Re}(\Sigma_{mn})$  [Eq. (5.13)]; its principle axes are the eigendirections of  $\mathcal{X}_{mn}$ , and its principle radii are the square roots of the eigenvalues of  $\mathcal{X}_{mn}$ . It is convenient to rotate the axes of the complex-amplitude plane counterclockwise by an angle  $\theta$  [rotation defined by Eq. (4.34)] so that the new (primed) axes are parallel to the principal axes of the error ellipse (see Fig. 3), i.e., so that the covariance matrix is diagonal with respect to the new axes. The angle  $\theta$  is obtained from

$$\begin{aligned} \frac{1}{2}(\Sigma_{11} - \Sigma_{22}) + \frac{1}{2}i(\Sigma_{12} + \Sigma_{21}) \\ = -\frac{1}{2}[(\Sigma_{11} - \Sigma_{22})^2 + (\Sigma_{12} + \Sigma_{21})^2]^{1/2} e^{2i\theta}, \end{aligned}$$

where  $0 \leq \theta < \pi$  [Eq. (5.19)]. The diagonal elements of the rotated covariance matrix are given by  $\mathcal{X}'_{mm} = \langle [\Delta \mathcal{E}'_m(x, t)]^2 \rangle = \Sigma'_{mm} = \langle |\Delta \alpha'_m|^2 \rangle$ ; their square roots—the uncertainties in the rotated quadrature phases—are the principal radii of the error ellipse. The error ellipse is a convenient way to show graphically the nonrandom distribution of noise in phase.

Figure 3 also shows separate phase planes for the rotated quadrature phases (cf. Fig. 2). In each phase plane a vector indicates the initial expectation value of  $2^{1/2}\alpha'_m$ . The noise in each quadrature phase is depicted by an "error circle," which is centered at the tip of the vector  $2^{1/2}\langle \alpha'_m \rangle$  and whose radius is the root-mean-square uncertainty in  $\alpha'_m$ . That one uses a circle expresses the fact that the quadrature phases have time-stationary (random-phase) noise; i.e., the uncertainties in the Hermitian real and imaginary parts of  $2^{1/2}\alpha'_m$  are the same, and they are equal to the root-mean-square uncertainty  $\langle |\Delta \alpha'_m|^2 \rangle^{1/2}$ . Just as the projection of  $2^{1/2}\langle \alpha'_m \rangle$  onto its real axis gives the associated component  $\langle \mathcal{E}'_m(0, t) \rangle$  of the mean complex amplitude, so the projection of the error circle on the real axis gives the associated principal diameter of the error ellipse (Fig. 3).

We refer to Fig. 3 as the standard complex-amplitude diagram. The vectors in it are drawn at  $t=0$ , but a similar diagram could be constructed at any time. As time passes, the vector in each separate phase plane rotates clockwise with angular velocity  $\epsilon$ , dragging its error circle with it; the projection of the vector and its error circle on the real axis describes the oscillation of the associated quadrature phase with constant variance. These projections can also be used to construct the mean complex amplitude  $\langle \mathcal{E}_1(0, t) + i\mathcal{E}_2(0, t) \rangle$  and its error ellipse. The mean complex-amplitude vector rotates in the direction shown by the arrows. The error ellipse is dragged along as the mean complex-amplitude vector rotates, but it retains the same size, shape, and orientation—a consequence of TSQP noise.

The axes in Fig. 3 are somewhat loosely labeled by operators because the diagrams are supposed to indicate both the mean behavior and the fluctuations about the mean. The axes of the separate phase planes are labeled by the Hermitian real and imaginary parts of  $2^{1/2}\alpha'_m$ . Notice that the free time dependence  $e^{-i\epsilon t}$  is not indicated explicitly as in Fig. 2. The reason is that this time dependence is implicit; the expectation values of  $\alpha'_1$  and  $\alpha'_2$  are evaluated in the MP (Sec. IV C), where they have the free

time dependence  $e^{-i\epsilon t}$ . The standard complex-amplitude diagram can be put on a more rigorous footing after the two-photon quasiprobability distributions are introduced in a future paper (paper III). Then the axes can be labeled by variables of an appropriate quasiprobability distribution, and the error ellipse and the error circles become particular contours of the quasiprobability distribution.

TS noise is distributed randomly in phase ( $\mathcal{X}_{mn}$  is a multiple of the unit matrix). In the complex-amplitude diagram in Fig. 3 this means that the error ellipse is a circle [ $\Sigma_{11} = \Sigma_{22}$ ,  $\text{Re}(\Sigma_{12}) = 0$ ; Eqs. (5.9)] and the error circles in the separate phase planes have the same size. To go from TS noise to TSQP noise, one imagines "squeezing" the error circle of TS noise into the error ellipse that characterizes TSQP noise; noise is squeezed from one quadrature phase into the other so that the error circles in the separate phase planes have different sizes. The use of the term squeezed to describe a nonrandom distribution of noise in phase arose from this simple picture of a circle being squeezed into an ellipse. The term<sup>40</sup> was originally applied to the degenerate limit ( $\epsilon=0$ ,  $a_+ = a_-$ ), where one draws complex-amplitude diagrams very much like the central diagram in Fig. 3. In the degenerate limit the noise is depicted by an error ellipse just as in Fig. 3, but the signal ellipse collapses to a point, which is the unchanging complex amplitude of a single mode (see, for example, Fig. 1 of Ref. 1). It should be emphasized that squeezing is a consequence of correlation between the two modes [Eqs. (5.8b) and (5.8c)]; each mode by itself has random-phase noise [Eq. (5.6a)].

The standard complex-amplitude diagram (Fig. 3) does not display all the information about the second-order noise. It shows graphically the three pieces of information in the covariance matrix  $\mathcal{X}_{mn} = \text{Re}(\Sigma_{mn})$  [Eq. (5.13)], but it does not include any information about  $\text{Im}(\Sigma_{12})$  [Eq. (5.18)]. This omission is really not very serious. The purpose of the standard complex-amplitude diagram is to depict the nonrandom distribution of noise in phase, which does not depend on  $\text{Im}(\Sigma_{12})$ .

The relation of the standard complex-amplitude diagram to the behavior of the electric field and the quadrature phases is made clearer by the graphs in Fig. 4. Each part of Fig. 4 shows two complex-amplitude diagrams for a particular state of the field which has TSQP noise; one diagram is drawn at  $t=0$  and the other at  $t=\pi/2\epsilon$ . The states depicted in Fig. 4 are special in two ways: (i) All the signal is carried by  $\mathcal{E}_1(x, t)$ , i.e.,  $\langle \mathcal{E}_2(x, t) \rangle = 0$ . Thus the signal ellipse collapses to a line along the  $\mathcal{E}_1$  axis, and the mean electric field at  $x=0$  is given by

$$\langle E(0, t) \rangle = \langle E_1(0, t) \rangle \cos(\Omega t) \quad (5.20)$$

[Eq. (4.21)]. (ii) The quadrature phases have zero second-order correlation, i.e.,  $\mathcal{X}_{12} = \text{Re}(\Sigma_{12}) = 0$ . Thus the principal axes of the error ellipse are parallel to the  $\mathcal{E}_1$  and  $\mathcal{E}_2$  axes, and the uncertainty in the electric field at  $x=0$  is given by

$$\begin{aligned} \langle [\Delta E(0, t)]^2 \rangle^{1/2} = (2\Omega)^{1/2} [\Sigma_{11} \cos^2(\Omega t) \\ + \Sigma_{22} \sin^2(\Omega t)]^{1/2} \quad (5.21) \end{aligned}$$

[Eq. (5.16)]. Figure 4(a) depicts a state with TS noise

( $\Sigma_{11} = \Sigma_{22}$ ), Fig. 4(b) depicts a state with less noise in  $\mathcal{E}_1(x,t)$  than in  $\mathcal{E}_2(x,t)$  ( $\Sigma_{11} < \Sigma_{22}$ ), and Fig. 4(c) depicts a state with less noise in  $\mathcal{E}_2(x,t)$  than in  $\mathcal{E}_1(x,t)$  ( $\Sigma_{22} < \Sigma_{11}$ ). Below the complex-amplitude diagrams in each part of Fig. 4 are graphs for the electric field  $E(0,t)$  and the quadrature phases  $E_1(0,t)$  and  $E_2(0,t)$ . The dark central line in each graph is the expectation value of the appropriate field, and the width of the shaded region is twice the uncertainty in the same quantity. The graph for  $E_1(0,t)$  shows a sinusoidal oscillation at frequency  $\epsilon$  with constant uncertainty  $\langle [\Delta E_1(0,t)]^2 \rangle^{1/2} = (2\Omega\Sigma_{11})^{1/2}$ ; this behavior is described by the projection on the real axis of the rotating vector  $2^{1/2}\langle\alpha_1\rangle$  and its associated error circle. The graph for  $E_2(0,t)$  shows a zero expectation value with constant uncertainty  $\langle [\Delta E_2(0,t)]^2 \rangle^{1/2} = (2\Omega\Sigma_{22})^{1/2}$ ; this behavior is described by the unchanging projection on the real axis of the error circle in the phase plane for  $2^{1/2}\alpha_2$ . In the graph for  $E(0,t)$ , the mean electric field is modulated at frequency  $\epsilon$  [Eq. (5.20)], and the uncertainty oscillates as given by Eq. (5.21). Similar graphs for the behavior of the electric field have been drawn in the degenerate limit (see, for example, Fig. 2 of Ref. 1); the uncertainty oscillates just as in Eq. (5.21), but the mean elec-

tric field is unmodulated.

The graphs for  $E_1(0,t)$  and  $E_2(0,t)$  in Fig. 4 are closely related to the output of an ideal heterodyne detector and to amplitude and phase modulation of a carrier wave (see discussion in Sec. IV D). If the electric field in Fig. 4 is mixed with a local-oscillator wave proportional to  $\cos[\Omega(t-x)]$ , then the graph for  $E_1(0,t)$  characterizes the heterodyned output at frequency  $\epsilon$ , which has constant noise. If a strong classical carrier wave proportional to  $\cos[\Omega(t-x)]$  is added to the electric field in Fig. 4, then the graph for  $E_1(0,t)$  describes an amplitude-modulation signal with constant amplitude-modulation noise, and the graph for  $E_2(0,t)$  describes a zero phase-modulation signal with constant phase-modulation noise. The differences among the three parts of Fig. 4 lie in the different ratios of amplitude-modulation noise to phase-modulation noise.

### VI. UNCERTAINTY PRINCIPLES FOR QUADRATURE-PHASE AMPLITUDES

In this section we consider uncertainty principles that apply to the mean-square uncertainties in the quadrature-phase amplitudes. The analogous uncertainty principles

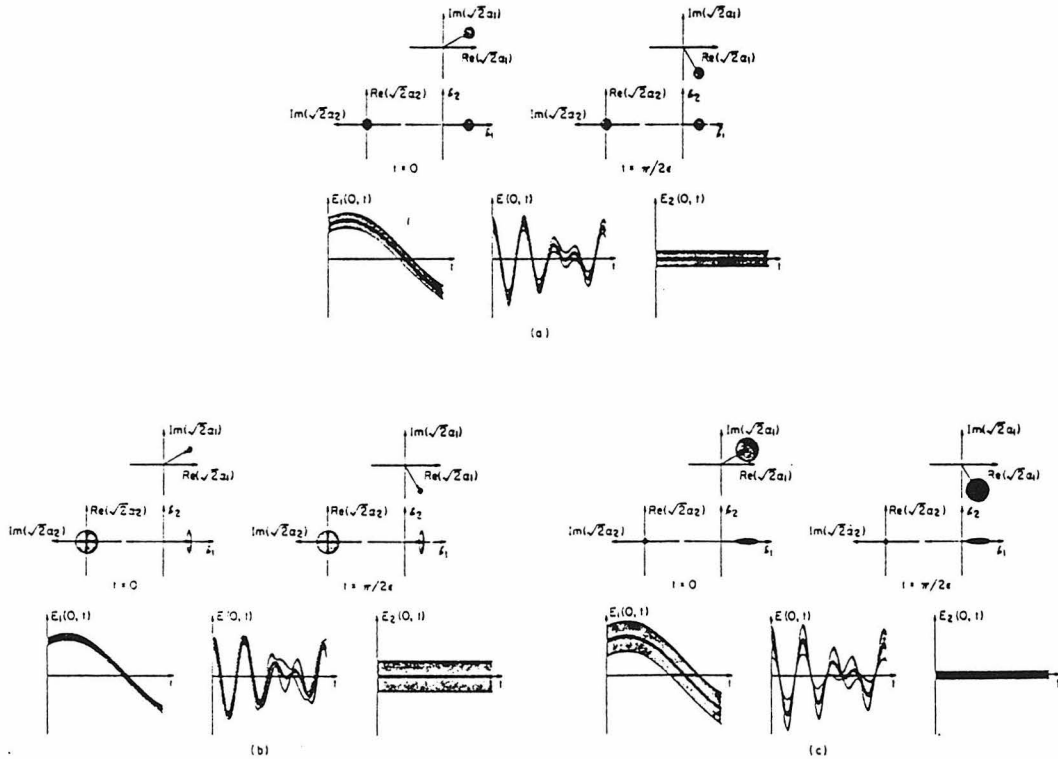


FIG. 4. Graphs of the electric field  $E(0,t)$  and the quadrature phases  $E_1(0,t)$  and  $E_2(0,t)$  for three states with TSQP noise: (a) a state with TS noise; (b) a state with less noise in  $\mathcal{E}_1(x,t)$  than in  $\mathcal{E}_2(x,t)$ ; (c) a state with less noise in  $\mathcal{E}_2(x,t)$  than in  $\mathcal{E}_1(x,t)$ . Above the graphs in each part are two complex-amplitude diagrams for the same state, one at  $t=0$  and one at  $t=\pi/2\epsilon$ . In each graph the dark central line is the expectation value of the appropriate field quantity, and the width of the shaded region at any time is twice the uncertainty in the same quantity. See the text for further discussion.



for more general non-Hermitian operators are derived and discussed in the Appendix; here we simply apply the more general results to the particular case of  $\alpha_1$  and  $\alpha_2$ .

The most important uncertainty principle<sup>13,21</sup> places a lower limit on the product of the root-mean-square uncertainties in  $\alpha_1$  and  $\alpha_2$ :

$$\langle |\Delta\alpha_1|^2 \rangle^{1/2} \langle |\Delta\alpha_2|^2 \rangle^{1/2} \geq \frac{1}{2} | \langle [\alpha_1, \alpha_2^\dagger] \rangle | = \frac{1}{2} \quad (6.1)$$

[Eqs. (A16) and (4.31c)]. In terms of the spectral-density matrix  $S_{mn}$  [Eq. (5.4)], the uncertainty principle (6.1) becomes  $S_{11}S_{22} \geq \frac{1}{4}\Omega^2$ . It should be noted that Eq. (6.1) does not require an assumption of TSQP noise, but it does rely on the fact that  $\alpha_1$  and  $\alpha_2$  commute [Eq. (4.31b)]. Yurke and Denker<sup>34,41</sup> have considered an uncertainty principle similar to Eq. (6.1), but in terms of the multimode quadrature phases [Eq. (1.6)].

What is the meaning of the uncertainty principle (6.1)? The zero-point noise in each mode corresponds to half a quantum at the mode's frequency. In units of energy the combined zero-point noise in the two modes is  $\frac{1}{2}(\Omega + \epsilon) + \frac{1}{2}(\Omega - \epsilon) = \Omega$ , which amounts to one quantum at the carrier frequency. If

$$\langle |\Delta\alpha_1|^2 \rangle = \langle |\Delta\alpha_2|^2 \rangle = \frac{1}{4} \quad (6.2)$$

( $S_{11} = S_{22} = \frac{1}{2}\Omega$ ), then each quadrature carries half of the one quantum of zero-point noise. The uncertainty principle (6.1) allows the uncertainty in one quadrature to be reduced below the level set by zero-point noise, but only at the expense of increasing the noise in the other quadrature above the zero-point level. Thus the uncertainty principle describes the squeezing referred to in Sec. VB: noise can be reduced below the zero-point level only by squeezing noise from one quadrature phase into the other.

Equation (6.1) is the two-mode analog of an uncertainty principle<sup>2</sup> that applies in the degenerate limit— $\epsilon = 0$ ,  $a_+ = a_- = a$ . This uncertainty principle, which is equivalent to the position-momentum uncertainty principle, is usually written in terms of  $a_1$  and  $a_2$ , the Hermitian real and imaginary parts of  $a = a_1 + ia_2$ :

$$\langle (\Delta a_1)^2 \rangle^{1/2} \langle (\Delta a_2)^2 \rangle^{1/2} \geq \frac{1}{2} | \langle [a_1, a_2] \rangle | = \frac{1}{4}. \quad (6.3)$$

Further discussion of the degenerate limit can be found in Sec. VIII.

Equality in Eq. (6.1) imposes very restrictive conditions on the state vector  $|\Psi\rangle$ ; indeed, Eqs. (A27), specialized to the case  $R = \alpha_1$  and  $S = \alpha_2$ , show that equality holds in Eq. (6.1) if and only if

$$(\Delta\alpha_1 + i\Delta\alpha_2)|\Psi\rangle = 0, \quad (6.4a)$$

$$(\Delta\alpha_1^\dagger + i\Delta\alpha_2^\dagger)|\Psi\rangle = 0 \quad (6.4b)$$

[Eqs. (4.31), (A27c), and (A29)]. Plugging in the definitions (4.25) of  $\alpha_1$  and  $\alpha_2$ , one finds that Eqs. (6.4) reduce to

$$\Delta a_\pm |\Psi\rangle = 0. \quad (6.5)$$

Thus the only states that yield equality in Eq. (6.1) are the simultaneous eigenstates of  $a_+$  and  $a_-$ , i.e., the two-mode coherent states (4.11).

In addition to the uncertainty principle (6.1), there is a

separate uncertainty principle for each quadrature-phase amplitude:<sup>13,21</sup>

$$\langle |\Delta\alpha_m|^2 \rangle \geq \frac{1}{2} | \langle [\alpha_m, \alpha_m^\dagger] \rangle | = \epsilon/2\Omega, \quad m = 1, 2 \quad (6.6)$$

[Eqs. (A9) and (4.31a)]. Equation (6.6) does not rely on an assumption of TSQP noise. Equality holds in Eq. (6.6) if and only if the state vector  $|\Psi\rangle$  is an eigenstate of  $\alpha_m$ , i.e.,

$$\Delta\alpha_m |\Psi\rangle = 0 \quad (6.7)$$

[Eq. (A12a)]. Since  $\epsilon < \Omega$ , it is immediately apparent from Eq. (6.1) that it is impossible to find a state  $|\Psi\rangle$  for which both  $\langle |\Delta\alpha_1|^2 \rangle$  and  $\langle |\Delta\alpha_2|^2 \rangle$  have the minimum value  $\epsilon/2\Omega$ . This means that there are no simultaneous eigenstates of  $\alpha_1$  and  $\alpha_2$ .

What can one learn from the uncertainty principle (6.6)? For each quadrature it says that the minimum noise is a factor  $\epsilon/\Omega$  smaller than the level set by zero-point noise [Eq. (6.2)]. If one writes Eq. (6.6) in units of energy— $S_{mm} \geq \frac{1}{2}\epsilon$ —one sees that the *minimum noise corresponds to half a quantum at the modulation frequency  $\epsilon$* . This suggests interpreting the minimum noise  $\frac{1}{2}\epsilon$  as a sort of zero-point noise for the quadrature phases; we call it the *quadrature-phase zero-point noise*. This interpretation is strengthened by noting that the quadrature phase  $E_m(x, t)$  is a "field operator" at frequency  $\epsilon$  [Eq. (4.23)]. The variance of  $2^{-1/2}E_m(x, t)$  for a state with TSQP noise,

$$\frac{1}{2} \langle [\Delta E_m(x, t)]^2 \rangle = \Omega \langle |\Delta\alpha_m|^2 \rangle \geq \frac{1}{2}\epsilon, \quad (6.8)$$

should be compared with the single-mode electric-field variance (3.13) for a state with TS noise, where the single mode has frequency  $\omega = \epsilon$ . In terms of energy the lower limit in Eq. (6.8), which is enforced by the quadrature-phase zero-point noise, is the same as the lower limit in Eq. (3.13), which is enforced by the ordinary zero-point noise at frequency  $\omega = \epsilon$  [Eq. (3.14)]. Physically the quadrature-phase zero-point noise means the following: *if one chooses to work at modulation frequency  $\epsilon$  about a high carrier frequency  $\Omega$ , then the noise in one quadrature phase can be made as small as, but no smaller than, the minimum noise that one would encounter if working directly at the low frequency  $\epsilon$ .*

The relation between the quadrature-phase amplitudes and the quadrature-phase zero-point noise is analogous to the relation between the creation and annihilation operators and the ordinary zero-point noise. The analogy becomes apparent if one writes the free Hamiltonian  $H_0$  [Eq. (4.2a)] in terms of various operator orderings. Orderings of the creation and annihilation operators give expressions that involve the ordinary zero-point energy  $\Omega$ :

$$(\Omega + \epsilon)(a_+ a_+^\dagger)_{\text{sym}} + (\Omega - \epsilon)(a_- a_-^\dagger)_{\text{sym}} = H_0 + \Omega, \quad (6.9a)$$

$$(\Omega + \epsilon)a_+^\dagger a_+ + (\Omega - \epsilon)a_-^\dagger a_- = H_0, \quad (6.9b)$$

$$(\Omega + \epsilon)a_+ a_+^\dagger + (\Omega - \epsilon)a_- a_-^\dagger = H_0 + 2\Omega. \quad (6.9c)$$

Symmetric ordering [Eq. (6.9a)] yields the total energy, including the one quantum of zero-point energy, normal ordering [Eq. (6.9b)] yields the total energy minus the zero-

point energy, and antinormal ordering [Eq. (6.9c)] yields the total energy plus the zero-point energy. Analogous orderings of the quadrature-phase amplitudes and their Hermitian conjugates involve the quadrature-phase zero-point energy  $\epsilon$  ( $\frac{1}{2}\epsilon$  from each quadrature):

$$\Omega[(\alpha_1^\dagger\alpha_1)_{\text{sym}} + (\alpha_2^\dagger\alpha_2)_{\text{sym}}] = H_0 + \Omega, \quad (6.10a)$$

$$\Omega(\alpha_1^\dagger\alpha_1 + \alpha_2^\dagger\alpha_2) = H_0 + \Omega - \epsilon, \quad (6.10b)$$

$$\Omega(\alpha_1\alpha_1^\dagger + \alpha_2\alpha_2^\dagger) = H_0 + \Omega + \epsilon. \quad (6.10c)$$

Symmetric ordering [Eq. (6.10a)] again yields the total energy. If one places  $\alpha_1^\dagger$  and  $\alpha_2^\dagger$  to the left of  $\alpha_1$  and  $\alpha_2$  [Eq. (6.10b)], an ordering which is analogous to ordinary normal ordering and which we call *quadrature-phase normal ordering*, then one obtains the total energy minus the quadrature-phase zero-point energy. Similarly, if one places  $\alpha_1$  and  $\alpha_2$  to the left of  $\alpha_1^\dagger$  and  $\alpha_2^\dagger$  [Eq. (6.10c)], an ordering which we call *quadrature-phase antinormal ordering*, then one obtains the total energy plus the quadrature-phase zero-point energy. These and other more general orderings for the quadrature-phase amplitudes will be considered in paper III.

One can also write an uncertainty principle for the operators  $\beta_1$  and  $\beta_2$  [Eqs. (4.47) and (4.48)]. Analogous to Eq. (6.1) is an uncertainty principle

$$\langle |\Delta\beta_1|^2 \rangle^{1/2} \langle |\Delta\beta_2|^2 \rangle^{1/2} \geq \frac{1}{2} |\langle [\beta_1, \beta_2^\dagger] \rangle| = \frac{1}{2}, \quad (6.11)$$

but there is no analog of Eq. (6.6); i.e.,  $\langle |\Delta\beta_m|^2 \rangle$  can be made arbitrarily small.

## VII. TWO-MODE SQUEEZED STATES

Two-mode squeezed states are the natural states for two-photon optics because they are the output states of an ideal two-photon device (see Sec. IV A). Here we discuss briefly the most important properties of two-mode squeezed states; our purpose is to show how they fit into the general framework developed in Secs. IV–VI. A more thorough investigation of their properties is undertaken in paper II.

A useful preliminary to the properties of two-mode squeezed states is a review of the most basic properties of two-mode coherent states [Eq. (4.11)]:

$$|\mu_+, \mu_-\rangle_{\text{coh}} \equiv D(a_+, \mu_+) D(a_-, \mu_-) |0\rangle. \quad (7.1)$$

Using the fact that  $|\mu_+, \mu_-\rangle_{\text{coh}}$  is an eigenstate of  $a_+$  and  $a_-$ , one can show, first, that the expectation values of the annihilation operators and the quadrature-phase amplitudes are given by

$$\langle a_\pm \rangle = \mu_\pm, \quad (7.2a)$$

$$\langle \alpha_1 \rangle = \xi_1 \equiv 2^{-1/2}(\lambda_+ \mu_+ + \lambda_- \mu_-^*), \quad (7.2b)$$

$$\langle \alpha_2 \rangle = \xi_2 \equiv 2^{-1/2}(-i\lambda_+ \mu_+ + i\lambda_- \mu_-^*) \quad (7.2c)$$

[Eqs. (4.25)] and, second, that  $|\mu_+, \mu_-\rangle_{\text{coh}}$  has TS noise— $\langle (\Delta a_\pm)^2 \rangle = \langle \Delta a_+ \Delta a_- \rangle = \langle \Delta a_- \Delta a_+ \rangle = 0$  [Eqs. (5.6) and (5.9)]—with  $\langle |\Delta a_+|^2 \rangle = \langle |\Delta a_-|^2 \rangle = \frac{1}{2}$ —i.e.,

$$\Sigma_{mm} = \langle |\Delta a_m|^2 \rangle = \frac{1}{2}, \quad m=1,2 \quad (7.3a)$$

$$\Sigma_{12} = -\Sigma_{21} = \frac{1}{2}i(\epsilon/\Omega) \quad (7.3b)$$

[Eq. (5.2)]. A two-mode coherent state can be regarded as a classical excitation of the two modes, contaminated by zero-point noise. The covariance matrix of the dimensionless quadrature phases [Eq. (5.13)] is a multiple of the unit matrix,

$$\mathcal{X}_{mn} = \text{Re}(\Sigma_{mn}) = \frac{1}{2}\delta_{mn}, \quad (7.4)$$

which shows that the noise associated with a coherent state is distributed randomly in phase. In the standard complex-amplitude diagram (see Sec. V B and Fig. 3), these properties of a two-mode coherent state show up in the following ways: the error ellipse in the central complex-amplitude plane is a circle, the two error circles in the separate phase planes have the same size, and all three circles have radius  $2^{-1/2}$ . Notice that  $\text{Im}(\Sigma_{12}) = \epsilon/2\Omega$  does not vanish for a coherent state—a consequence of the fact that the energy associated with the zero-point noise is different for the two modes [see discussion surrounding Eq. (5.18)].

Turn now to the two-mode squeezed states defined by Eqs. (4.15), (4.17), and (4.18):

$$\begin{aligned} |\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r,\varphi)} &\equiv S(r,\varphi) |\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{\text{coh}} \\ &= S(r,\varphi) D(a_+, \mu_{\alpha_+}) D(a_-, \mu_{\alpha_-}) |0\rangle \\ &= D(a_+, \mu_+) D(a_-, \mu_-) S(r,\varphi) |0\rangle, \end{aligned} \quad (7.5)$$

$$\mu_{\alpha_\pm} = \mu_\pm \cosh r + \mu_\mp^* e^{2i\varphi} \sinh r. \quad (7.6)$$

When  $r=0$  a two-mode squeezed state reduces to a two-mode coherent state. The unitary equivalence between the squeezed annihilation operators and the annihilation operators [Eq. (4.14)] provides an easy way to calculate first and second moments for a two-mode squeezed state; the moments of  $\alpha_\pm(r,\varphi)$  with respect to  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r,\varphi)}$  are the same as the moments of  $a_\pm$  with respect to  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{\text{coh}}$ . Using this approach, one can calculate the following expectation values for the two-mode squeezed state (7.5):

$$\langle a_\pm \rangle = \mu_\pm, \quad (7.7a)$$

$$\langle \alpha_m \rangle = \xi_m \quad (7.7b)$$

[cf. Eqs. (7.2)]. In addition, one can show that  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r,\varphi)}$  has TSQP noise [Eq. (5.1) or Eqs. (5.6)] with

$$\langle |\Delta a_+|^2 \rangle = \langle |\Delta a_-|^2 \rangle = \frac{1}{2} \cosh(2r), \quad (7.8a)$$

$$\langle \Delta a_+ \Delta a_- \rangle = -\frac{1}{2} e^{2i\varphi} \sinh(2r); \quad (7.8b)$$

translated into the language of the reduced spectral-density matrix (5.2), Eqs. (7.8) become

$$\begin{aligned} \Sigma_{11} &= \langle |\Delta \alpha_1|^2 \rangle \\ &= \frac{1}{2} \cosh(2r) - \frac{1}{2} (1 - \epsilon^2/\Omega^2)^{1/2} \sinh(2r) \cos(2\varphi), \end{aligned} \quad (7.9a)$$

$$\begin{aligned}\Sigma_{22} &= \langle |\Delta\alpha_2|^2 \rangle \\ &= \frac{1}{2} \cosh(2r) + \frac{1}{2} (1 - \epsilon^2/\Omega^2)^{1/2} \sinh(2r) \cos(2\varphi),\end{aligned}\quad (7.9b)$$

$$\begin{aligned}\Sigma_{12} = \Sigma_{21}^* &= -\frac{1}{2} (1 - \epsilon^2/\Omega^2)^{1/2} \sinh(2r) \sin(2\varphi) \\ &\quad + \frac{1}{2} i (\epsilon/\Omega) \cosh(2r)\end{aligned}\quad (7.9c)$$

[Eqs. (5.8)]. For  $r \neq 0$  a two-mode squeezed state does indeed display the nonrandom distribution of noise in phase which entitles it to be called squeezed. The standard complex-amplitude diagram looks like Fig. 3 with  $\theta = \varphi$ ; the error ellipse has principal radii  $2^{-1/2} [\cosh(2r) \mp (1 - \epsilon^2/\Omega^2)^{1/2} \sinh(2r)]^{1/2}$ , which also are the radii of the error circles in the separate phase planes.

An important subset of the two-mode squeezed states consists of those with  $\varphi = 0$ . For this subset the reduced spectral-density matrix (7.9) becomes

$$\begin{aligned}\Sigma_{11} &= \langle |\Delta\alpha_1|^2 \rangle \\ &= \frac{1}{2} e^{-2r} + \frac{1}{2} [1 - (1 - \epsilon^2/\Omega^2)^{1/2}] \sinh(2r),\end{aligned}\quad (7.10a)$$

$$\begin{aligned}\Sigma_{22} &= \langle |\Delta\alpha_2|^2 \rangle \\ &= \frac{1}{2} e^{2r} - \frac{1}{2} [1 - (1 - \epsilon^2/\Omega^2)^{1/2}] \sinh(2r),\end{aligned}\quad (7.10b)$$

$$\Sigma_{12} = -\Sigma_{21} = \frac{1}{2} i (\epsilon/\Omega) \cosh(2r). \quad (7.10c)$$

Letting  $\varphi = 0$  yields a diagonal covariance matrix  $\mathcal{X}_{mn} = \text{Re}(\Sigma_{mn})$ , which means that the squeezing of the error ellipse in Fig. 3 occurs along the  $\mathcal{S}_1$  and  $\mathcal{S}_2$  axes or, equivalently, that the quadrature phases  $E_1(x, t)$  and  $E_2(x, t)$  have zero second-order correlation. The reduced spectral-density matrix for any squeezed state can be put in the form (7.10) by using rotated quadrature-phase amplitudes  $\alpha'_1 = \alpha_1 \cos\varphi + \alpha_2 \sin\varphi$  and  $\alpha'_2 = -\alpha_1 \sin\varphi + \alpha_2 \cos\varphi$  [Eqs. (4.36) and (5.19)]. Thus the subset defined by  $\varphi = 0$  is not so much a special case as it is a convenient choice of phase for defining the quadrature phases—a choice that puts the information about squeezing wholly into the diagonal elements of  $\mathcal{X}_{mn}$ . For  $\varphi = 0$  the product of the root-mean-square uncertainties in  $\alpha_1$  and  $\alpha_2$  is given by

$$\langle |\Delta\alpha_1|^2 \rangle^{1/2} \langle |\Delta\alpha_2|^2 \rangle^{1/2} = \frac{1}{2} [1 + (\epsilon^2/\Omega^2) \sinh^2 r]^{1/2}. \quad (7.11)$$

In accordance with the proof in Sec. VI and the Appendix [Eq. (6.5)], the uncertainty product (7.11) achieves the minimum value of  $\frac{1}{2}$  if and only if  $r = 0$  (provided  $\epsilon \neq 0$ ).

Consider now what happens as the squeeze factor  $r$  increases from  $r = 0$ ; choose  $\varphi = 0$  for easy interpretation. For small  $r$  [ $\cosh(2r) \ll \Omega/\epsilon$ ], the mean-square uncertainties in  $\alpha_1$  and  $\alpha_2$  are given approximately by

$$\langle |\Delta\alpha_1|^2 \rangle \simeq \frac{1}{2} e^{-2r}, \quad \langle |\Delta\alpha_2|^2 \rangle \simeq \frac{1}{2} e^{2r}. \quad (7.12)$$

These mean-square uncertainties are the two-mode analog of the variances that apply in the degenerate limit [see Eq. (8.25)]. They show that  $\langle |\Delta\alpha_1|^2 \rangle$  is squeezed below the zero-point level; in accordance with the uncertainty principle (6.1),  $\langle |\Delta\alpha_2|^2 \rangle$  increases above the zero-point level. As long as  $\cosh(2r) < \Omega/\epsilon$ ,  $\langle |\Delta\alpha_1|^2 \rangle$  continues to de-

crease as  $r$  increases, but it departs more and more from  $\frac{1}{2} e^{-2r}$ . When  $r = r_0 > 0$ , where

$$\cosh(2r_0) \equiv \Omega/\epsilon, \quad (7.13a)$$

$$\cosh r_0 = (\Omega/2\epsilon)^{1/2} \lambda_+, \quad \sinh r_0 = (\Omega/2\epsilon)^{1/2} \lambda_- \quad (7.13b)$$

[Eq. (4.24)],  $\langle |\Delta\alpha_1|^2 \rangle$  achieves the minimum possible value  $\epsilon/2\Omega$  [Eq. (6.6)]; thus the state  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, 0)}$  yields a classical excitation of the quadrature phase  $E_1(x, t)$ , contaminated only by quadrature-phase zero-point noise. Equation (6.7) guarantees that  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, 0)}$  is an eigenstate of  $\alpha_1 = (\epsilon/\Omega)^{1/2} \alpha_+(r_0, 0)$  [Eqs. (4.14), (4.25a), and (7.13b)]:

$$\alpha_1 |\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, 0)} = \xi_1 |\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, 0)}, \quad (7.14a)$$

$$\xi_1 = (\epsilon/\Omega)^{1/2} \mu_{\alpha_+} \quad (7.14b)$$

[cf. Eq. (4.16)]. For  $r > r_0$ ,  $\langle |\Delta\alpha_1|^2 \rangle$  increases as  $r$  increases.

The state  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, 0)}$  belongs to a special class of two-mode squeezed states which we call *squashed states*.<sup>13</sup> The set of squashed states consists of the states  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, \varphi)}$  for all values of  $\varphi$ . The squashed state  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, \varphi)}$  is an eigenstate of the rotated quadrature-phase amplitude

$$\alpha'_1 = \alpha_1 \cos\varphi + \alpha_2 \sin\varphi = (\epsilon/\Omega)^{1/2} e^{-i\varphi} \alpha_+(r_0, \varphi)$$

with eigenvalue  $(\epsilon/\Omega)^{1/2} \mu_{\alpha_+} e^{-i\varphi}$  [Eqs. (4.14), (4.25), and (4.16)]; hence  $\alpha'_1$  has the minimum mean-square uncertainty  $\langle |\Delta\alpha'_1|^2 \rangle = \epsilon/2\Omega$ . In particular,  $|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r_0, \pi/2)}$  is an eigenstate of  $\alpha_2 = -i(\epsilon/\Omega)^{1/2} \times \alpha_+(r_0, \frac{1}{2}\pi)$  with eigenvalue  $\xi_2 = -i(\epsilon/\Omega)^{1/2} \mu_{\alpha_+}$ . Initially we hoped that the squashed states, as eigenstates of the quadrature-phase amplitudes, might play a fundamental role in two-photon optics, analogous to the role played by the eigenstates of the annihilation operator—the coherent states—in one-photon optics. Our initial hopes were dashed, however, by our inability to find any special role for the squashed states. In the formalism presented in this series of papers, therefore, the squashed states are on the same footing as all the other two-mode squeezed states.

The mean-square uncertainties in  $\beta_1$  and  $\beta_2$  [Eqs. (4.47)] for a two-mode squeezed state can be obtained from Eqs. (7.9) and (7.10) by setting  $\epsilon = 0$ . In particular, for  $\varphi = 0$  one finds that

$$\langle |\Delta\beta_1|^2 \rangle = \frac{1}{2} e^{-2r}, \quad \langle |\Delta\beta_2|^2 \rangle = \frac{1}{2} e^{2r}. \quad (7.15)$$

## VIII. DEGENERATE LIMIT

### A. Definition and discussion

We shift attention now to the degenerate limit of our two-mode formalism. By the degenerate limit we mean that the two modes we have dealt with coalesce into a single mode at frequency  $\Omega$ . Taking this limit is not an entirely trivial task. An obvious first step is to set  $\epsilon = 0$ , so we assume  $\epsilon = 0$  throughout the remainder of this subsec-

tion. This step alone, however, is not sufficient, because it leaves two degenerate, but distinct modes at frequency  $\Omega$ , which have distinct annihilation operators  $a_+$  and  $a_-$ . [Simply setting  $\epsilon=0$  would describe, for example, the case where the two modes are plane waves of the same frequency traveling in different directions; see discussion preceding Eqs. (4.5).] To take the desired degenerate limit, one must somehow reduce the number of modes from the two original modes to one mode that corresponds to the coalescence of the two original modes; out of the four original degrees of freedom, one must pick two relevant degrees of freedom and discard the other two.

The key to picking the relevant degrees of freedom is to define new annihilation operators  $a$  and  $b$ , which are unitarily related to  $a_+$  and  $a_-$ :

$$a \equiv 2^{-1/2}(a_+ + a_-), \quad b \equiv 2^{-1/2}(-ia_+ + ia_-), \quad (8.1a)$$

$$a_{\pm} = 2^{-1/2}(a \pm ib). \quad (8.1b)$$

The importance of these new operators becomes apparent when one writes the positive-frequency part of the two-mode electric field [Eq. (4.5b) with  $\epsilon=0$ ] in terms of  $a$  and  $b$ :

$$E^{(+)}(x,t) = \Omega^{1/2} a e^{-i\Omega(t-x)}. \quad (8.2)$$

One sees that  $a$  is the annihilation operator for a plane-wave mode at frequency  $\Omega$ ; it contains the relevant degrees of freedom. In contrast,  $b$  does not appear in the electric field; it contains the irrelevant degrees of freedom. One can write the operators introduced in Sec. IV in terms of  $a$  and  $b$ . For example, the quadrature-phase amplitudes (4.25) become

$$\alpha_m = a_m + ib_m, \quad m = 1, 2 \quad (8.3)$$

where  $a_1, a_2, b_1$ , and  $b_2$  are the Hermitian real and imaginary parts of  $a$  and  $b$ , i.e.,

$$a = a_1 + ia_2, \quad b = b_1 + ib_2. \quad (8.4)$$

Thus, another way to characterize the relevant degrees of freedom at degeneracy is that they are the real parts of  $\alpha_1$  and  $\alpha_2$ , whereas the irrelevant degrees of freedom are the imaginary parts. In terms of  $a$  and  $b$  the fundamental unitary operators become

$$U_M(t) |_{\epsilon=0} = 1, \quad (8.5a)$$

$$R(\theta) |_{\epsilon=0} = \exp(-i\theta a^\dagger a) \exp(-i\theta b^\dagger b), \quad (8.5b)$$

$$D(a_+, \mu_+) D(a_-, \mu_-) |_{\epsilon=0} = D(a, \mu) D(b, \gamma), \quad (8.5c)$$

$$S(r, \varphi) |_{\epsilon=0} = \exp\left[\frac{1}{2}r(a^2 e^{-2i\varphi} - a^{\dagger 2} e^{2i\varphi})\right] \\ \times \exp\left[\frac{1}{2}r(b^2 e^{-2i\varphi} - b^{\dagger 2} e^{2i\varphi})\right] \quad (8.5d)$$

[Eqs. (4.37), (4.33), (4.12), and (4.9)], where

$$\mu \equiv 2^{-1/2}(\mu_+ + \mu_-), \quad (8.6a)$$

$$\gamma \equiv 2^{-1/2}(-i\mu_+ + i\mu_-). \quad (8.6b)$$

Notice that Eq. (8.5a) implies that when  $\epsilon=0$  the MP and IP are the same.

The two-mode Hilbert space factors into a tensor (direct) product of Hilbert spaces for the  $a$  mode and the

$b$  mode. The  $a$ -mode Hilbert space is the Hilbert space for the relevant mode at degeneracy. We let  $\text{tr}_b$  denote a trace over the irrelevant  $b$ -mode Hilbert space. We use a subscript  $a$  to denote a state vector that lies in the  $a$ -mode space or an operator that operates in the  $a$ -mode space; a subscript  $b$  performs the same role for the  $b$ -mode space.

One is now in a position to define the degenerate limit: one reduces the Hilbert space from the two-mode space to the  $a$ -mode space; for a state vector or an operator, one extracts a piece that lies in or operates in the  $a$ -mode space. To make these notions precise, consider a two-mode density operator  $\rho$ . We say that  $\rho$  has a degenerate limit if the  $a$  mode is independent of the irrelevant  $b$  mode, so that no matter what operation is performed on the  $b$  mode, the  $a$ -mode is unaffected. Hence, a density operator  $\rho$  has a (unique) degenerate limit  $\rho_a \equiv \text{tr}_b(\rho)$  if  $\rho = \rho_a \rho_b$ ; we denote this limit by

$$\rho \xrightarrow[p]{} \rho_a. \quad (8.7)$$

Similarly, a state vector  $|\psi\rangle$  has a degenerate limit  $|\psi_a\rangle$ , denoted by

$$|\psi\rangle \xrightarrow[p]{} |\psi_a\rangle, \quad (8.8)$$

if  $|\psi\rangle = |\psi_a\rangle \otimes |\psi_b\rangle$ ; requiring that  $|\psi_a\rangle$  be normalized makes this limit unique up to an arbitrary phase factor. The limits (8.7) and (8.8) have an obvious extension to unitary operators. A unitary operator  $U$  has a degenerate limit  $U_a$ , denoted by

$$U \xrightarrow[p]{} U_a, \quad (8.9)$$

if  $U = U_a U_b$ ; requiring that  $U_a$  be unitary makes this limit unique up to an arbitrary phase factor. In Eqs. (8.7)–(8.9), the  $p$  under the arrow signifies that these are product degenerate limits; i.e., each requires that the relevant quantity factor into a product of an  $a$ -mode quantity times a  $b$ -mode quantity. The limits (8.7) and (8.9) could easily be extended to a product degenerate limit for arbitrary operators, but we have no need for such a generalization here. For present purposes the important properties of the product degenerate limit are that

$$U \xrightarrow[p]{} U_a, \quad \rho \xrightarrow[p]{} \rho_a \Rightarrow U \rho U^\dagger \xrightarrow[p]{} U_a \rho_a U_a^\dagger, \quad (8.10a)$$

$$U \xrightarrow[p]{} U_a, \quad |\psi\rangle \xrightarrow[p]{} |\psi_a\rangle \Rightarrow U |\psi\rangle \xrightarrow[p]{} U_a |\psi_a\rangle. \quad (8.10b)$$

For observable quantities or for non-Hermitian operators like the quadrature-phase amplitudes, a different degenerate limit is appropriate. Consider an arbitrary operator  $R$ . We say that  $R$  has a sum degenerate limit  $R_a$ , denoted by

$$R \xrightarrow[p]{} R_a, \quad (8.11)$$

if  $R = R_a + R_b$ . The motivation for this definition is that for a state  $\rho$  with a degenerate limit,  $R_a$  and  $R_b$  are uncorrelated. The sum degenerate limit (8.11) is defined only up to an arbitrary additive constant.

Having specified how to take degenerate limits, we now consider the limits of the two-mode quantities introduced in Sec. IV. We adopt the sensible convention that the lim-

it of an SP operator is an IP operator, and the limit of an IP or a MP operator is an IP operator. For normalization purposes we define the sum degenerate limits of the IP two-mode electric field operator [Eqs. (4.5)] and the IP quadrature phases [Eqs. (4.19)] to be  $2^{1/2}$  times a quantity denoted by the same symbol [e.g.,  $E^{(+)}(x,t) \rightarrow 2^{1/2}(\Omega/2)^{1/2} a e^{-i\Omega(t-x)} \equiv 2^{1/2} E^{(+)}(x,t)$ ; cf. Eq. (8.2)]; with this choice the degenerate limit of the IP two-mode electric field operator yields the IP single-mode electric field operator defined by Eqs. (3.3) with  $\omega = \Omega$ , and the relations between the electric field and the quadrature phases [Eqs. (4.19)–(4.21)] retain the same form in the degenerate limit. The sum degenerate limit of the SP annihilation operators [Eq. (8.1b)],

$$a_{\pm} \xrightarrow{\epsilon} 2^{-1/2} a_{\pm}, \quad (8.12)$$

suggests defining the sum degenerate limit of the MP squeezed annihilation operators (4.14) in the following way:

$$\alpha_{\pm}(r, \varphi) \xrightarrow{\epsilon} 2^{-1/2} (a \cosh r + a^{\dagger} e^{2i\varphi} \sinh r) \equiv 2^{-1/2} \alpha(r, \varphi). \quad (8.13)$$

The MP quadrature-phase amplitudes (4.25) have a Hermitian sum degenerate limit

$$\alpha_m \xrightarrow{\epsilon} a_m \equiv 2^{-1/2} x_m, \quad m = 1, 2 \quad (8.14)$$

[Eq. (8.3)].

The loss of two degrees of freedom at degeneracy erases the distinction between the quadrature phases and the quadrature-phase amplitudes: the IP quadrature phases, which are initially Hermitian operators with harmonic time dependence at frequency  $\epsilon$ , become constant in the degenerate limit; the MP quadrature-phase amplitudes, which are initially (constant) complex-amplitude operators, become Hermitian in the degenerate limit. As a result, at degeneracy there are three Hermitian IP operators,

$$E_m(x, t) = (2\Omega)^{1/2} a_m = \Omega^{1/2} x_m, \quad (8.15)$$

all of which are constant and any of which could be called a quadrature phase or a quadrature-phase amplitude.<sup>2</sup> We prefer to give  $x_1$  and  $x_2$  the distinction of being the (degenerate) *quadrature-phase amplitudes*, because their relation to the annihilation operator has the same form as Eqs. (4.25) with  $\epsilon = 0$ , i.e.,

$$x_1 = 2^{-1/2} (a + a^{\dagger}), \quad (8.16a)$$

$$x_2 = 2^{-1/2} (-ia + ia^{\dagger}), \quad (8.16b)$$

and because their commutator

$$[x_1, x_2] = i \quad (8.17)$$

enforces the same uncertainty principle as Eq. (6.1), i.e.,

$$\langle (\Delta x_1)^2 \rangle^{1/2} \langle (\Delta x_2)^2 \rangle^{1/2} \geq \frac{1}{2} \quad (8.18)$$

[cf. Eq. (6.3)].

The fundamental unitary operators introduced in Sec.

IV [Eqs. (8.5)] have the following (unitary) product degenerate limits:

$$U_M(t) \xrightarrow{\epsilon} 1, \quad (8.19a)$$

$$R(\theta) \xrightarrow{\epsilon} \exp(-i\theta a^{\dagger} a), \quad (8.19b)$$

$$D(a_+, \mu_+) D(a_-, \mu_-) \xrightarrow{\epsilon} D(a, \mu), \quad \mu \equiv 2^{-1/2} (\mu_+ + \mu_-), \quad (8.19c)$$

$$S(r, \varphi) \xrightarrow{\epsilon} \exp[\frac{1}{2} r (a^2 e^{-2i\varphi} - a^{\dagger 2} e^{2i\varphi})] \equiv S_1(r, \varphi). \quad (8.19d)$$

The MP free evolution operator  $U_M(t)$  becomes the identity operator, the rotation operator  $R(\theta)$  becomes a single-mode rotation operator, the two-mode displacement operator  $D(a_+, \mu_+) D(a_-, \mu_-)$  becomes the single-mode displacement operator (3.7), and the two-mode squeeze operator  $S(r, \varphi)$  becomes the *degenerate squeeze operator*,<sup>42,43</sup>  $S_1(r, \varphi)$ . Under a unitary transformation generated by  $S_1(r, \varphi)$  the annihilation operator  $a$  becomes the *squeezed annihilation operator*  $\alpha(r, \varphi)$  [Eq. (8.13)]:

$$\alpha(r, \varphi) = S_1(r, \varphi) a S_1^{\dagger}(r, \varphi) = a \cosh r + a^{\dagger} e^{2i\varphi} \sinh r \quad (8.20)$$

[cf. Eq. (4.14)]. For  $\varphi = 0$  the degenerate squeeze operator transforms the quadrature-phase amplitudes according to

$$S_1^{\dagger}(r, 0) x_1 S_1(r, 0) = x_1 e^{-r}, \quad (8.21a)$$

$$S_1^{\dagger}(r, 0) x_2 S_1(r, 0) = x_2 e^r. \quad (8.21b)$$

The degenerate limits (8.19) can be applied to obtain the degenerate limits of the special states defined in Sec. IV. The product degenerate limit of a two-mode coherent state [Eq. (4.11)] is a single-mode coherent state [Eq. (3.9)]:

$$|\mu_+, \mu_-\rangle_{\text{coh}} \xrightarrow{\epsilon} |\mu\rangle_{\text{coh}}, \quad \mu \equiv 2^{-1/2} (\mu_+ + \mu_-). \quad (8.22)$$

The product degenerate limit of a two-mode squeezed state [Eq. (4.15)] is a *degenerate squeezed state*<sup>42,44,2</sup>  $|\mu_{\alpha}\rangle_{(r, \varphi)}$ :

$$|\mu_{\alpha_+}, \mu_{\alpha_-}\rangle_{(r, \varphi)} \xrightarrow{\epsilon} S_1(r, \varphi) D(a, \mu_{\alpha}) |0\rangle = D(a, \mu) S_1(r, \varphi) |0\rangle \equiv |\mu_{\alpha}\rangle_{(r, \varphi)}, \quad (8.23a)$$

$$\mu_{\alpha} \equiv 2^{-1/2} (\mu_{\alpha_+} + \mu_{\alpha_-}) = \mu \cosh r + \mu^* e^{2i\varphi} \sinh r \quad (8.23b)$$

[Eqs. (4.17) and (4.18)]. A degenerate squeezed state is labeled by the eigenvalue of  $\alpha(r, \varphi)$  [Eq. (8.20)]:

$$\alpha(r, \varphi) |\mu_{\alpha}\rangle_{(r, \varphi)} = \mu_{\alpha} |\mu_{\alpha}\rangle_{(r, \varphi)} \quad (8.24)$$

[cf. Eq. (4.16)]. The quadrature-phase amplitudes have the following variances in a degenerate squeezed state with  $\varphi = 0$ :

$$\langle (\Delta x_1)^2 \rangle = \frac{1}{2} e^{-2r}, \quad \langle (\Delta x_2)^2 \rangle = \frac{1}{2} e^{2r} \quad (8.25)$$

[cf. Eqs. (7.12)].

## B. Review of previous work

Degenerate squeezed states were introduced independently by Stoler<sup>42,45</sup> ("minimum-uncertainty packets")

and Lu<sup>43,44</sup> ("new coherent states"), both of whom used the degenerate squeeze operator to generate squeezed states from coherent states. The first comprehensive treatment of squeezed states is due to Yuen,<sup>2</sup> who called them "two-photon coherent states" because of their generation by ideal two-photon processes. Yuen explored in detail the properties of degenerate squeezed states, and he discussed several physical mechanisms for generating them. In this series of papers we adopt Yuen's notational convention, which labels a degenerate squeezed state by the eigenvalue of the squeezed annihilation operator. Not long after Yuen's paper, Yuen, Shapiro,<sup>5,7</sup> and Machado Mata<sup>6</sup> developed the theory of optical communications using squeezed states. At about the same time Hollenhorst<sup>40</sup> introduced squeezed states into the theory of "quantum nondemolition measurements."<sup>46</sup> Hollenhorst coined the term squeezed and applied it to the degenerate squeeze operator (in Ref. 1 the term was extended in an obvious way to apply to the states themselves). Hollenhorst's work led to the realization<sup>1</sup> that squeezed states could be used to improve the sensitivity of laser interferometers used to detect gravitational waves. In the last few years there has been an explosion of interest in squeezed states.<sup>3,4</sup> Optical communications and high-precision measurements remain their primary potential applications, but interest is also fueled by a desire to investigate their nonclassical behavior.<sup>3</sup>

In unpublished work Yuen<sup>47</sup> has considered general multimode squeezed states. Yuen and Shapiro<sup>7</sup> and Milburn<sup>30</sup> have defined two-mode or multimode squeezed states, but the states they define are simply tensor (direct) products of degenerate squeezed states for each mode. There is a formal sense, realized by Lu<sup>43</sup> and pointed out explicitly by Milburn,<sup>30</sup> in which the two-mode squeezed states defined here can be regarded as a tensor product of two degenerate squeezed states. For any value of  $\epsilon$  one can define the operators  $a$  and  $b$  of Eqs. (8.1), and one can write the two-mode displacement operator and the two-mode squeeze operator in terms of  $a$  and  $b$  as in Eqs. (8.5c) and (8.5d). Thus a two-mode squeezed state (4.17) factors into a tensor product of degenerate squeezed states for the "a mode" and the "b mode."

The difficulty with this description is that unless  $\epsilon=0$  the operators  $a$  and  $b$  are not modal annihilation operators because they do not have a harmonic time dependence in the IP. The operators  $a_+$  and  $a_-$ —not  $a$  and  $b$ —appear in a modal decomposition of the electromagnetic field. Formally, it is correct to describe a two-mode squeezed state as a product of degenerate squeezed states for the "a mode" and the "b mode," and this description does permit one to obtain properties of two-mode squeezed states directly from properties of degenerate squeezed states. Physically, however, this description is very misleading, because it can easily lead one to believe that the way to produce nondegenerate (wide-band) squeezing is to squeeze separately two different modes. In reality, wide-band squeezing does not result from separately squeezing different modes [see Eq. (5.6a)]; rather, it is a consequence of a special sort of correlation between two modes symmetrically placed about a carrier frequency [Eqs. (5.8b) and (5.8c)]. Such correlation is

produced by ideal two-photon devices, and it is the feature that characterizes two-photon optics.

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#### APPENDIX: UNCERTAINTY PRINCIPLES FOR NON-HERMITIAN OPERATORS

In this Appendix we derive and discuss uncertainty principles that apply to the mean-square uncertainties of non-Hermitian operators. Our immediate objective is to derive the uncertainty principles for  $\alpha_1$  and  $\alpha_2$  which are given in Sec. VI. The derivations are more general,<sup>48</sup> however, than the special case of  $\alpha_1$  and  $\alpha_2$ , because we do not restrict ourselves to operators with  $c$ -number commutators. Since the uncertainty principles for non-Hermitian operators are based on the uncertainty principles for their Hermitian real and imaginary parts, we begin by reviewing the standard uncertainty principle for two Hermitian operators. The notation we use here is introduced in Sec. II.

##### 1. Two Hermitian operators

Consider two Hermitian operators  $B$  and  $C$ . They satisfy the ordinary uncertainty principle for the product of their uncertainties:

$$\langle(\Delta B)^2\rangle^{1/2}\langle(\Delta C)^2\rangle^{1/2} \geq \frac{1}{2} |\langle[B, C]\rangle|. \quad (\text{A1})$$

The derivation of Eq. (A1) can be found in most quantum-mechanics textbooks (see, e.g., Chap. 8.6 of Ref. 27). Equality holds in Eq. (A1) if and only if the state vector  $|\Psi\rangle$  is an eigenstate of a particular linear combination of  $B$  and  $C$ :

$$(\Delta B + i\beta \Delta C)|\Psi\rangle = 0, \quad (\text{A2a})$$

$$\begin{aligned} \beta &\equiv -i \frac{\langle[B, C]\rangle}{|\langle[B, C]\rangle|} \frac{\langle(\Delta B)^2\rangle^{1/2}}{\langle(\Delta C)^2\rangle^{1/2}} \\ &= 2i \frac{\langle(\Delta B)^2\rangle}{\langle[B, C]\rangle} = -\frac{1}{2}i \frac{\langle[B, C]\rangle}{\langle(\Delta C)^2\rangle}. \end{aligned} \quad (\text{A2b})$$

Notice that  $\beta$  is real because  $\langle[B, C]\rangle$  is pure imaginary. Equality in Eq. (A1) implies that  $B$  and  $C$  have zero second-order correlation, i.e.,

$$\langle\Delta B \Delta C\rangle_{\text{sym}} = \langle BC\rangle_{\text{sym}} - \langle B\rangle\langle C\rangle = 0. \quad (\text{A3})$$

##### 2. One non-Hermitian operator

Let  $R$  be a general, possibly non-Hermitian operator. We want to derive a lower limit for its mean-square

uncertainty  $\langle |\Delta R|^2 \rangle$  [Eq. (2.9)]. An instructive approach is to consider its Hermitian real and imaginary parts  $R_1 \equiv \text{Re}(R) = \frac{1}{2}(R + R^\dagger)$  and  $R_2 \equiv \text{Im}(R) = -\frac{1}{2}i(R - R^\dagger)$ , i.e.,

$$R = R_1 + iR_2. \quad (\text{A4})$$

It is useful to note the following relations among operators:

$$\langle \Delta R \rangle^2 = \langle \Delta R_1 \rangle^2 - \langle \Delta R_2 \rangle^2 + 2i \langle \Delta R_1 \Delta R_2 \rangle_{\text{sym}}, \quad (\text{A5})$$

$$|\Delta R|^2 = \langle \Delta R_1 \rangle^2 + \langle \Delta R_2 \rangle^2, \quad (\text{A6})$$

$$[R, R^\dagger] = -2i[R_1, R_2]. \quad (\text{A7})$$

Notice that  $\langle [R, R^\dagger] \rangle$  is real.

By noting that

$$\begin{aligned} \langle |\Delta R|^2 \rangle &= \langle (\Delta R_1)^2 \rangle + \langle (\Delta R_2)^2 \rangle \\ &\geq 2 \langle (\Delta R_1)^2 \rangle^{1/2} \langle (\Delta R_2)^2 \rangle^{1/2}, \end{aligned} \quad (\text{A8})$$

one can use the ordinary uncertainty principle (A1), applied to  $R_1$  and  $R_2$ , to establish a lower limit for  $\langle |\Delta R|^2 \rangle$ :

$$\langle |\Delta R|^2 \rangle \geq \frac{1}{2} |\langle [R, R^\dagger] \rangle| = |\langle [R_1, R_2] \rangle|. \quad (\text{A9})$$

This derivation makes clear that equality in Eq. (A9) is equivalent to each of the following: (i)  $R_1$  and  $R_2$  have equal uncertainties, which have the minimum-uncertainty product, i.e.,

$$\langle (\Delta R_1)^2 \rangle = \langle (\Delta R_2)^2 \rangle = \frac{1}{2} |\langle [R_1, R_2] \rangle|; \quad (\text{A10})$$

(ii) the state vector  $|\Psi\rangle$  satisfies

$$\left[ \Delta R_1 + i \frac{\langle [R, R^\dagger] \rangle}{|\langle [R, R^\dagger] \rangle|} \Delta R_2 \right] |\Psi\rangle = 0 \quad (\text{A11})$$

[Eqs. (A2) and (A7)]; (iii) the state vector  $|\Psi\rangle$  satisfies

$$\Delta R |\Psi\rangle = 0 \text{ if } \langle [R, R^\dagger] \rangle \geq 0, \quad (\text{A12a})$$

$$\Delta R^\dagger |\Psi\rangle = 0 \text{ if } \langle [R, R^\dagger] \rangle \leq 0. \quad (\text{A12b})$$

Equality in Eq. (A9) implies

$$\langle (\Delta R)^2 \rangle = 0. \quad (\text{A13})$$

The uncertainty principle (A9) can also be obtained directly without introducing  $R_1$  and  $R_2$ . One writes the mean-square uncertainty in two ways which imply two lower limits:

$$\langle |\Delta R|^2 \rangle = \langle \Delta R^\dagger \Delta R \rangle + \frac{1}{2} \langle [R, R^\dagger] \rangle \geq \frac{1}{2} \langle [R, R^\dagger] \rangle, \quad (\text{A14a})$$

$$\langle |\Delta R|^2 \rangle = \langle \Delta R \Delta R^\dagger \rangle - \frac{1}{2} \langle [R, R^\dagger] \rangle \geq -\frac{1}{2} \langle [R, R^\dagger] \rangle. \quad (\text{A14b})$$

Equations (A14) imply the uncertainty principle (A9). If  $\langle [R, R^\dagger] \rangle \geq 0$ , then equality holds in Eq. (A14a) if and only if  $\langle \Delta R^\dagger \Delta R \rangle = 0$ , which is equivalent to Eq. (A12a); similarly, if  $\langle [R, R^\dagger] \rangle \leq 0$ , then equality holds in Eq. (A14b) if and only if  $\langle \Delta R \Delta R^\dagger \rangle = 0$ , which is equivalent to Eq. (A12b).

### 3. Two commuting non-Hermitian operators

Consider now two general,<sup>48</sup> possibly non-Hermitian operators  $R$  and  $S$  which commute:

$$[R, S] = 0; \quad (\text{A15})$$

thus the important commutator is  $[R, S^\dagger] = -[R^\dagger, S]^\dagger$ . In analogy with the ordinary uncertainty principle (A1), one might expect  $|\langle [R, S^\dagger] \rangle|$  to set a lower limit on the product of the root-mean-square uncertainties in  $R$  and  $S$ . Indeed, the main result of this subsection is that

$$\langle |\Delta R|^2 \rangle^{1/2} \langle |\Delta S|^2 \rangle^{1/2} \geq \frac{1}{2} |\langle [R, S^\dagger] \rangle|, \quad (\text{A16})$$

an uncertainty principle that bears a striking resemblance to the ordinary uncertainty principle (A1).

The uncertainty principle (A16) is a consequence of the ordinary uncertainty principles for the real and imaginary parts of  $R$  and  $S$ . We therefore begin a proof of Eq. (A16) by introducing the Hermitian real and imaginary parts of  $R$  as in Eq. (A4) and by introducing the Hermitian real and imaginary parts of  $e^{i\lambda}S$ ,

$$S_1 \equiv \frac{1}{2}(e^{i\lambda}S + e^{-i\lambda}S^\dagger), \quad S_2 \equiv -\frac{1}{2}i(e^{i\lambda}S - e^{-i\lambda}S^\dagger), \quad (\text{A17a})$$

$$S = e^{-i\lambda}(S_1 + iS_2), \quad (\text{A17b})$$

where  $e^{i\lambda}$  is an arbitrary phase factor. For different values of  $\lambda$  the operators  $S_1$  and  $S_2$  are different linear combinations of the real and imaginary parts of  $S$ , but the mean-square uncertainty in  $S$  is still given by

$$\langle |\Delta S|^2 \rangle = \langle (\Delta S_1)^2 \rangle + \langle (\Delta S_2)^2 \rangle \quad (\text{A18})$$

[cf. Eq. (A8)]. In what follows we derive lower limits on  $\langle |\Delta R|^2 \rangle^{1/2} \langle |\Delta S|^2 \rangle^{1/2}$  which depend on  $\lambda$ , and we then choose  $\lambda$  to enforce the most stringent limit. Using Eq. (A15), one can derive the following commutators:

$$\begin{aligned} [R_1, S_1] &= [R_2, S_2] = \frac{1}{4}(e^{-i\lambda}[R, S^\dagger] + e^{i\lambda}[R^\dagger, S]) \\ &= \frac{1}{4}i \text{Im}(e^{-i\lambda}[R, S^\dagger]), \end{aligned} \quad (\text{A19a})$$

$$\begin{aligned} [R_1, S_2] &= -[R_2, S_1] = \frac{1}{4}i(e^{-i\lambda}[R, S^\dagger] - e^{i\lambda}[R^\dagger, S]) \\ &= \frac{1}{2}i \text{Re}(e^{-i\lambda}[R, S^\dagger]). \end{aligned} \quad (\text{A19b})$$

The notation is made less cumbersome by introducing the symbols

$$r_j \equiv \langle (\Delta R_j)^2 \rangle^{1/2} \geq 0, \quad s_j \equiv \langle (\Delta S_j)^2 \rangle^{1/2} \geq 0, \quad j = 1, 2. \quad (\text{A20})$$

The commutators (A19) enforce four uncertainty principles [Eq. (A1)],

$$r_1 s_1 \geq \frac{1}{4} c |\sin(\delta - \lambda)|, \quad (\text{A21a})$$

$$r_2 s_2 \geq \frac{1}{4} c |\sin(\delta - \lambda)|, \quad (\text{A21b})$$

$$r_1 s_2 \geq \frac{1}{4} c |\cos(\delta - \lambda)|, \quad (\text{A21c})$$

$$r_2 s_1 \geq \frac{1}{4} c |\cos(\delta - \lambda)|, \quad (\text{A21d})$$

where we define

$$\langle [R, S^\dagger] \rangle \equiv ce^{i\delta}, \quad c \equiv |\langle [R, S^\dagger] \rangle|. \quad (\text{A22})$$

Hence, the problem is to minimize

$$\begin{aligned} \langle |\Delta R|^2 \rangle \langle |\Delta S|^2 \rangle &= (r_1^2 + r_2^2)(s_1^2 + s_2^2) \\ &= |(r_1 + ir_2)(s_1 + is_2)|^2, \end{aligned} \quad (\text{A23})$$

subject to the constraints (A21). An easy way to do this is to write Eq. (A23) in two ways, which lead to two different lower limits:

$$\begin{aligned} \langle |\Delta R|^2 \rangle \langle |\Delta S|^2 \rangle &= (r_1 s_1 - r_2 s_2)^2 + (r_1 s_2 + r_2 s_1)^2 \\ &\geq \frac{1}{4} c^2 \cos^2(\delta - \lambda), \end{aligned} \quad (\text{A24a})$$

$$\begin{aligned} \langle |\Delta R|^2 \rangle \langle |\Delta S|^2 \rangle &= (r_1 s_1 + r_2 s_2)^2 + (r_1 s_2 - r_2 s_1)^2 \\ &\geq \frac{1}{4} c^2 \sin^2(\delta - \lambda). \end{aligned} \quad (\text{A24b})$$

If one chooses  $\lambda = \delta$  ( $\lambda = \delta - \pi/2$ ), then Eq. (A24a) [Eq. (A24b)] implies the uncertainty principle (A16).

The operators  $S_1$  and  $S_2$  defined by Eqs. (A17) with  $\lambda = \delta - \pi/2$  (or, equivalently, the operators  $S_2$  and  $-S_1$  defined by  $\lambda = \delta$ ) bear a special relationship to  $R_1$  and  $R_2$ . For  $\lambda = \delta - \pi/2$ , Eq. (A24b) shows that equality in Eq. (A16) is equivalent to each of the following two statements: (i)  $R_1, R_2, S_1$ , and  $S_2$  satisfy

$$\langle (\Delta R_1)^2 \rangle = \langle (\Delta R_2)^2 \rangle, \quad \langle (\Delta S_1)^2 \rangle = \langle (\Delta S_2)^2 \rangle, \quad (\text{A25a})$$

$$\begin{aligned} \langle (\Delta R_1)^2 \rangle^{1/2} \langle (\Delta S_1)^2 \rangle^{1/2} &= \langle (\Delta R_2)^2 \rangle^{1/2} \langle (\Delta S_2)^2 \rangle^{1/2} \\ &= \frac{1}{4} |\langle [R, S^\dagger] \rangle|; \end{aligned} \quad (\text{A25b})$$

(ii) the state vector  $|\Psi\rangle$  satisfies

$$(\Delta R_1 + i\gamma \Delta S_1) |\Psi\rangle = 0, \quad (\text{A26a})$$

$$(\Delta R_2 + i\gamma \Delta S_2) |\Psi\rangle = 0, \quad (\text{A26b})$$

$$\gamma \equiv \langle |\Delta R|^2 \rangle^{1/2} / \langle |\Delta S|^2 \rangle^{1/2} \quad (\text{A26c})$$

[Eqs. (A2)]. By taking appropriate linear combinations of Eqs. (A26a) and (A26b), one can show that equality holds in Eq. (A16) if and only if

$$(\Delta R + \gamma e^{i\delta} \Delta S) |\Psi\rangle = 0, \quad (\text{A27a})$$

$$(\Delta R^\dagger - \gamma e^{-i\delta} \Delta S^\dagger) |\Psi\rangle = 0, \quad (\text{A27b})$$

$$\begin{aligned} \gamma e^{i\delta} &\equiv \frac{\langle [R, S^\dagger] \rangle}{|\langle [R, S^\dagger] \rangle|} \frac{\langle |\Delta R|^2 \rangle^{1/2}}{\langle |\Delta S|^2 \rangle^{1/2}} \\ &= 2 \frac{\langle |\Delta R|^2 \rangle}{\langle [R, S^\dagger] \rangle} = \frac{1}{2} \frac{\langle [R, S^\dagger] \rangle}{\langle |\Delta S|^2 \rangle} \end{aligned} \quad (\text{A27c})$$

[Eqs. (A22) and (A26c); cf. Eqs. (A2)]. Equations (A27) do not depend on any special choice for  $\lambda$ . They can be used to show that equality in Eq. (A16) implies the fol-

lowing:

$$\langle (\Delta R)^2 \rangle = \langle (\Delta S)^2 \rangle = \langle \Delta R \Delta S \rangle = 0, \quad (\text{A28a})$$

$$\begin{aligned} \langle \Delta R \Delta S^\dagger \rangle_{\text{sym}} &= \frac{1}{2} \gamma e^{i\delta} \langle [S, S^\dagger] \rangle \\ &= \frac{1}{2} \gamma^{-1} e^{i\delta} \langle [R, R^\dagger] \rangle. \end{aligned} \quad (\text{A28b})$$

A simple, but important consequence of Eqs. (A27c) and (A28b) is that equality in the uncertainty principle (A16) implies

$$\gamma^2 = \frac{\langle |\Delta R|^2 \rangle}{\langle |\Delta S|^2 \rangle} = \frac{\langle [R, R^\dagger] \rangle}{\langle [S, S^\dagger] \rangle}, \quad (\text{A29})$$

provided that  $\langle [R, R^\dagger] \rangle \neq 0 \neq \langle [S, S^\dagger] \rangle$ .

Equation (A28a) shows that equality in Eq. (A16) implies  $\langle (\Delta R_1)^2 \rangle = \langle (\Delta R_2)^2 \rangle$  and  $\langle (\Delta S_1)^2 \rangle = \langle (\Delta S_2)^2 \rangle$ , regardless of the choice of  $\lambda$ . This tells one immediately that Eqs. (A25) are a consequence of equality in Eq. (A16), regardless of the choice of  $\lambda$ . Equally true is that Eqs. (A25) imply equality in Eq. (A16), regardless of the choice of  $\lambda$ . On the other hand, only for the special choices  $\lambda = \delta - \pi/2$  and  $\lambda = \delta$  (or their equivalents) are Eqs. (A25) equivalent to eigenvalue equations like Eqs. (A26), because only for these special choices is Eq. (A25b) a minimum-uncertainty product [cf. Eqs. (A21)]. Thus it is the eigenvalue equations (A26) that pick out the operators  $S_1$  and  $S_2$  defined by  $\lambda = \delta - \pi/2$ .

An alternative method of proving the uncertainty principle (A16) goes as follows. Choose for illustration  $\lambda = \delta - \pi/2$ ; the problem is then to minimize  $f(r_1, r_2, s_1, s_2) \equiv (r_1^2 + r_2^2)(s_1^2 + s_2^2)$  [Eq. (A23)], subject to the constraints  $r_1 s_1 \geq \frac{1}{4}c$  and  $r_2 s_2 \geq \frac{1}{4}c$  [Eqs. (A21a) and (A21b)]. As a first step, minimize  $f$  on the hypersurface  $r_1 r_2 s_1 s_2 = K^2 \geq c^2/16$ , where  $K$  is a constant. The minimum value  $f = 4K^2$  can be found by using a Lagrange multiplier to enforce the hypersurface constraint; the minimum occurs when  $r_1 = r_2, s_1 = s_2, r_1 s_1 = r_2 s_2 = K$ . Now vary  $K$  to find the absolute minimum consistent with the constraints; the obvious answer is  $K = \frac{1}{4}c$ , which yields an absolute minimum value  $f = \frac{1}{4}c^2$ .

It should be remembered that the uncertainty principle (A16) is not the whole story, since it is based only on the commutator  $[R, S^\dagger]$ . It is quite possible that the constraint

$$\langle |\Delta R|^2 \rangle \langle |\Delta S|^2 \rangle \geq \frac{1}{4} |\langle [R, R^\dagger] \rangle| |\langle [S, S^\dagger] \rangle|, \quad (\text{A30})$$

which follows from the separate uncertainty principles for  $R$  and  $S$  [Eq. (A9)], provides a more stringent lower limit than Eq. (A16).

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The following paper, entitled  
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New formalism for two-photon quantum optics.  
 II. Mathematical foundation and compact notation

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This paper provides the mathematical foundation for the two-mode formalism introduced in the preceding paper. A vector notation is introduced; it allows two-mode properties to be written as compactly as the comparable properties for a single mode. The fundamental unitary operators of the formalism are described and their properties are examined; particular attention is paid to the two-mode squeeze operator. Special quantum states associated with the formalism are considered, with emphasis on the two-mode squeezed states.

I. INTRODUCTION

The present series of papers introduces a new formalism for two-photon quantum optics. The goal is to develop a formalism suited to analyzing two-photon devices, such as parametric amplifiers and phase-conjugate mirrors, in which photons are created or destroyed in the output modes two at a time. In the preceding paper<sup>1,2</sup> (henceforth referred to as I) we introduced the basic building blocks of our two-mode formalism: (i) new operators, the quadrature phases and the quadrature-phase amplitudes, and (ii) new quantum states, the two-mode squeezed states. The emphasis in I was on developing a sound physical interpretation of these fundamental entities. A conversational style invited the reader to become familiar with the elementary, but most important properties of the quadrature-phase amplitudes and the two-mode squeezed states. In the present paper the emphasis shifts—from physical interpretation to mathematical details. We introduce a compact vector notation which simplifies the mathematical description and at the same time highlights the important physics underlying our two-mode formalism. With the help of this notation we examine in detail the components of the formalism. The reward for the persistent reader is to proceed to a future paper (paper III), where the notation and results of this paper are used to construct the working tools of the new formalism—a set of “two-photon” quasiprobability distributions.

The present paper is largely independent of I, but a complete understanding does require familiarity with some of the material in I. (Equations in I are referred to here by affixing I to the equation number.) Since we make no attempt in this paper to motivate the definitions of the quadrature-phase amplitudes and the two-mode squeezed states, the reader might find it helpful to be familiar with the physical interpretation developed in I. The reader should also be comfortable with our potentially confusing habit of writing equations which contain operators defined in different pictures (see Sec. II of I); in particular, he should be familiar with the relations among the Schrödinger picture (SP), the modulation picture (MP), and the interaction picture (IP) [Eqs. (I.4.3) and (I.4.4)]

and with the convention introduced in Sec. IV C of I by which we specify for each physical quantity the picture in which the operator corresponding to that quantity is always written.

Given this minimal familiarity with the material in I, we can cast aside the interpretative superstructure used in I and extract only the essentials needed in this paper. We deal with two electromagnetic field modes whose frequencies are  $\Omega \pm \epsilon$ , where  $\Omega$  is a carrier frequency and  $\epsilon < \Omega$  is a modulation frequency. The SP creation and annihilation operators for the two modes are denoted by  $a_{\pm}^{\dagger}$  and  $a_{\pm}$ ; they satisfy the standard commutation relations

$$[a_{+}, a_{-}] = [a_{+}, a_{-}^{\dagger}] = 0, \quad (1.1a)$$

$$[a_{+}, a_{+}^{\dagger}] = [a_{-}, a_{-}^{\dagger}] = 1. \quad (1.1b)$$

The free Hamiltonian for the two modes is

$$H_0 = H_R + H_M, \quad (1.2)$$

where

$$H_R \equiv \Omega(a_{+}^{\dagger}a_{+} + a_{-}^{\dagger}a_{-}), \quad (1.3a)$$

$$H_M \equiv \epsilon(a_{+}^{\dagger}a_{+} - a_{-}^{\dagger}a_{-}), \quad (1.3b)$$

$$[H_R, H_M] = 0. \quad (1.4)$$

The MP quadrature-phase amplitudes are defined by

$$\alpha_1 \equiv 2^{-1/2}(\lambda_{+}a_{+} + \lambda_{-}a_{-}^{\dagger}), \quad (1.5a)$$

$$\alpha_2 \equiv 2^{-1/2}(-i\lambda_{+}a_{+} + i\lambda_{-}a_{-}^{\dagger}), \quad (1.5b)$$

$$\lambda_{\pm} \equiv [(\Omega \pm \epsilon)/\Omega]^{1/2} \quad (1.6)$$

[Eqs. (I.4.25)]; they obey the following commutation relations:

$$[\alpha_1, \alpha_1^{\dagger}] = [\alpha_2, \alpha_2^{\dagger}] = \epsilon/\Omega, \quad (1.7a)$$

$$[\alpha_1, \alpha_2] = 0, \quad (1.7b)$$

$$[\alpha_1, \alpha_2^{\dagger}] = [\alpha_1^{\dagger}, \alpha_2] = i. \quad (1.7c)$$

The important new unitary operator in our formalism is the two-mode squeeze operator

$$S(r, \varphi) \equiv \exp[r(a_+ a_- e^{-2i\varphi} - a_+^\dagger a_-^\dagger e^{2i\varphi})] \quad (1.8)$$

[Eq. (I.4.9)], where  $r$  is a real number called the *squeeze factor* and  $\varphi$  is a (real) phase angle. The two-mode squeeze operator satisfies

$$S^{-1}(r, \varphi) = S^\dagger(r, \varphi) = S(-r, \varphi) = S(r, \varphi + \pi/2), \quad (1.9)$$

and it generates the *squeezed annihilation operators*, which in the MP are defined by

$$\alpha_\pm(r, \varphi) \equiv S(r, \varphi) a_\pm S^\dagger(r, \varphi) = a_\pm \cosh r + a_\mp^\dagger e^{2i\varphi} \sinh r \quad (1.10)$$

[Eq. (I.4.14)]. The squeezed annihilation operators are unitarily equivalent to the annihilation operators, so they have the same commutator algebra [Eqs. (1.1)].

This paper is built on Eqs. (1.1)–(1.10). Section II introduces the compact vector notation which is used throughout this and subsequent papers. The components of our formalism are a set of fundamental unitary operators and a set of special quantum states. Section III examines in detail the fundamental unitary operators, and Sec. IV does the same for the special quantum states, with emphasis on the two-mode squeezed states. A concluding section meditates on the formalism developed here and hints at the results to come in subsequent papers. Some of the important results are developed in appendices: Appendix A lists properties of various transformation matrices associated with the vector notation; Appendix B derives useful factored forms for the degenerate and two-mode squeeze operators and an expression for the product of two different squeeze operators; Appendix C considers the inner product of two squeezed states. Throughout this paper we use units with  $\hbar = c = 1$ .

## II. VECTOR NOTATION

The most important feature of the two-mode squeeze operator  $S(r, \varphi)$  [Eq. (1.8)] is that under a unitary transformation generated by  $S(r, \varphi)$ ,  $a_\pm$  is transformed into a linear combination of  $a_\pm$  and  $a_\mp^\dagger$ . This association of  $a_+$  with  $a_-^\dagger$  (and  $a_-$  with  $a_+^\dagger$ ) is evident in the definitions of the squeezed annihilation operators [Eq. (1.10)] and the quadrature-phase amplitudes [Eqs. (1.5)]. We have found it natural and useful to introduce an operator column vector<sup>2</sup>

$$\underline{a} \equiv \begin{pmatrix} a_+ \\ a_-^\dagger \\ a_- \end{pmatrix} \quad (2.1)$$

which recognizes explicitly this association. This vector notation has been used by Collett and Gardiner<sup>3</sup> in an analysis of parametric amplification. Mollow<sup>4</sup> and Yuen and Shapiro<sup>5</sup> have also used a two-component vector notation, but they use a column vector formed from  $a_+$  and  $a_-$ . The adjoint of the vector (2.1) is the row vector

$$\underline{a}^\dagger \equiv (a_+^\dagger \ a_-) \quad (2.2)$$

Products of the vectors (2.1) and (2.2) are calculated using the usual rules for matrix multiplication, i.e.,

$$\underline{a}^\dagger \underline{a} = a_+^\dagger a_+ + a_- a_-^\dagger, \quad (2.3a)$$

$$\underline{a} \underline{a}^\dagger = \begin{pmatrix} a_+ a_+^\dagger & a_+ a_-^\dagger \\ a_-^\dagger a_+^\dagger & a_-^\dagger a_- \end{pmatrix}. \quad (2.3b)$$

Also useful is an operator column vector for the quadrature-phase amplitudes,

$$\underline{\alpha} \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \underline{A} \underline{\lambda} \underline{a}, \quad (2.4)$$

where

$$\underline{A} \equiv 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = (\underline{A}^\dagger)^{-1}, \quad (2.5)$$

$$\underline{\lambda} \equiv \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \underline{\lambda}^\dagger \quad (2.6)$$

[Eqs. (1.5) and (1.6)]. A list of useful properties of  $\underline{A}$  and  $\underline{\lambda}$  appears in Appendix A; many of the properties are most conveniently written in terms of the unit matrix  $\underline{1}$  and the Pauli matrices

$$\underline{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \underline{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.7)$$

The matrix  $\underline{\lambda}^2 = \underline{1} + (\epsilon/\Omega)\underline{\sigma}_3$  plays an important role because it appears in the vector expression for the free Hamiltonian (1.2),

$$\begin{aligned} H_0 + \Omega - \epsilon &= \Omega \underline{a}^\dagger \underline{\lambda}^2 \underline{a} = (\Omega + \epsilon) a_+^\dagger a_+ + (\Omega - \epsilon) a_- a_-^\dagger \\ &= \Omega \underline{\alpha}^\dagger \underline{\alpha} = \Omega (\alpha_1^\dagger \alpha_1 + \alpha_2^\dagger \alpha_2) \end{aligned} \quad (2.8)$$

[cf. Eq. (I.6.10b)]. Other matrices that turn up repeatedly in the following are

$$\underline{\Delta} \equiv \underline{A} \underline{\lambda}^2 \underline{A}^\dagger = \underline{1} - (\epsilon/\Omega)\underline{\sigma}_2 = \begin{pmatrix} 1 & i\epsilon/\Omega \\ -i\epsilon/\Omega & 1 \end{pmatrix} = \underline{\Delta}^\dagger \quad (2.9)$$

[Eqs. (A.17) and (A.6)] and

$$\begin{aligned} \underline{\Pi} &\equiv \underline{A} \underline{\lambda} \underline{\sigma}_3 \underline{\lambda} \underline{A}^\dagger = -\underline{\sigma}_2 \underline{\Delta} = (\epsilon/\Omega)\underline{1} - \underline{\sigma}_2 \\ &= \begin{pmatrix} \epsilon/\Omega & i \\ -i & \epsilon/\Omega \end{pmatrix} = \underline{\Pi}^\dagger \end{aligned} \quad (2.10)$$

[Eqs. (A.20) and (A.6)]. The matrix  $\underline{\Pi}$  is the matrix of commutators for the quadrature-phase amplitudes:

$$\Pi_{mn} = [\alpha_m, \alpha_n^\dagger] \quad (2.11)$$

[cf. Eqs. (1.7)].

The naturalness of this vector notation is revealed most clearly by examining the operator matrix

$$(\underline{\Delta} \underline{\alpha} \underline{\Delta} \underline{\alpha}^\dagger)_{\text{sym}} = \begin{pmatrix} (\Delta \alpha_1 \Delta \alpha_1^\dagger)_{\text{sym}} & (\Delta \alpha_1 \Delta \alpha_2^\dagger)_{\text{sym}} \\ (\Delta \alpha_2 \Delta \alpha_1^\dagger)_{\text{sym}} & (\Delta \alpha_2 \Delta \alpha_2^\dagger)_{\text{sym}} \end{pmatrix}, \quad (2.12)$$

where  $\Delta \underline{\alpha} \equiv \underline{\alpha} - \langle \underline{\alpha} \rangle$ ,  $\Delta \alpha_m \equiv \alpha_m - \langle \alpha_m \rangle$ , and "sym" denotes a symmetrized product (see Sec. II of I). The expectation value of the matrix (2.12) is the (Hermitian) reduced spectral-density matrix

$$\underline{\Sigma} \equiv \langle \underline{\Delta} \underline{\alpha} \underline{\Delta} \underline{\alpha}^\dagger \rangle_{\text{sym}} \quad (2.13)$$

[Eq. (I.5.2)]; the components of  $\underline{\Sigma}$ ,  $\Sigma_{mn} = \langle \Delta\alpha_m \Delta\alpha_n^\dagger \rangle_{\text{sym}} = \Sigma_{nm}^*$ , are the second-order noise moments that characterize time-stationary quadrature-phase (TSQP) noise (see Sec. V of I). Thus the vector notation is tailored to describing TSQP noise—the kind of noise produced by two-photon devices—because it generates naturally the second-order noise moments that characterize TSQP noise. In contrast, the noise moments  $\langle \Delta\alpha_m \Delta\alpha_n \rangle$ , which vanish for TSQP noise [Eq. (I.5.1)], are not generated naturally by the vector notation.

Corresponding to the matrix (2.12) is a matrix involving the creation and annihilation operators,

$$(\Delta\mathbf{a} \Delta\mathbf{a}^\dagger)_{\text{sym}} = \begin{pmatrix} |\Delta a_+|^2 & \Delta a_+ \Delta a_- \\ \Delta a_-^\dagger \Delta a_+^\dagger & |\Delta a_-|^2 \end{pmatrix}, \quad (2.14)$$

where  $|\Delta a_\pm|^2 \equiv (\Delta a_\pm \Delta a_\pm^\dagger)_{\text{sym}}$  [cf. Eq. (I.2.8)]. Its expectation value is the Hermitian matrix

$$\underline{\bar{\Sigma}} \equiv \langle \Delta\mathbf{a} \Delta\mathbf{a}^\dagger \rangle_{\text{sym}}, \quad (2.15)$$

which gives the second-order noise moments that characterize TSQP noise in terms of the creation and annihilation operators instead of in terms of the quadrature-phase amplitudes. The relation between the two kinds of noise moments can be written in the compact matrix form

$$\underline{\Sigma} = \mathcal{A} \lambda \underline{\bar{\Sigma}} \lambda \mathcal{A}^\dagger, \quad (2.16)$$

which is equivalent to Eqs. (I.5.8).

Natural decompositions of  $\underline{\Sigma}$  and  $\underline{\bar{\Sigma}}$  are afforded by the unit matrix  $\mathbf{1}$  and the Pauli matrices  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ :

$$\underline{\Sigma} = \Sigma_0 \mathbf{1} + \Sigma_j \sigma_j, \quad (2.17)$$

$$\underline{\bar{\Sigma}} = \bar{\Sigma}_0 \mathbf{1} + \bar{\Sigma}_j \sigma_j, \quad (2.18)$$

where repeated indices are summed over  $j=1,2,3$ . The coefficients  $\Sigma_0, \Sigma_j$  and  $\bar{\Sigma}_0, \bar{\Sigma}_j$ , which are guaranteed to be real by the Hermiticity of  $\underline{\Sigma}$  and  $\underline{\bar{\Sigma}}$ , are related to the noise moments as follows:

$$\begin{aligned} \Sigma_0 &= \frac{1}{2}(\Sigma_{11} + \Sigma_{22}), \quad \Sigma_1 = \text{Re}(\Sigma_{12}), \\ \Sigma_2 &= -\text{Im}(\Sigma_{12}), \quad \Sigma_3 = \frac{1}{2}(\Sigma_{11} - \Sigma_{22}), \end{aligned} \quad (2.19)$$

$$\begin{aligned} \bar{\Sigma}_0 &= \frac{1}{2}(\langle |\Delta a_+|^2 \rangle + \langle |\Delta a_-|^2 \rangle), \\ \bar{\Sigma}_1 &= \text{Re}(\langle \Delta a_+ \Delta a_- \rangle), \\ \bar{\Sigma}_2 &= -\text{Im}(\langle \Delta a_+ \Delta a_- \rangle), \end{aligned} \quad (2.20)$$

$$\bar{\Sigma}_3 = \frac{1}{2}(\langle |\Delta a_+|^2 \rangle - \langle |\Delta a_-|^2 \rangle);$$

they are related to each other by

$$\begin{aligned} \Sigma_0 &= \bar{\Sigma}_0 + (\epsilon/\Omega) \bar{\Sigma}_3, \quad \Sigma_1 = -(1 - \epsilon^2/\Omega^2)^{1/2} \bar{\Sigma}_2, \\ \Sigma_2 &= -\bar{\Sigma}_2 - (\epsilon/\Omega) \bar{\Sigma}_0, \quad \Sigma_3 = (1 - \epsilon^2/\Omega^2)^{1/2} \bar{\Sigma}_1 \end{aligned} \quad (2.21)$$

[Eq. (2.16); cf. Eqs. (I.5.8)]. The differential distribution of noise in phase is specified by  $\Sigma_1$  and  $\Sigma_3$  or, equivalently, by  $\bar{\Sigma}_1$  and  $\bar{\Sigma}_2$ . TSQP noise that is distributed randomly in phase is called time-stationary (TS) noise. For TS noise  $\text{Re}(\underline{\Sigma})$  is a multiple of the unit matrix ( $\Sigma_1 = \Sigma_3 = 0$ ), and  $\underline{\bar{\Sigma}}$  is diagonal [ $\bar{\Sigma}_1 = \bar{\Sigma}_2 = 0$ ; cf. Eqs. (I.5.9)].

The final important operator column vector is a vector

for the squeezed annihilation operators (1.10),

$$\underline{\alpha}_{r,\varphi} \equiv \begin{pmatrix} \alpha_+(r,\varphi) \\ \alpha_-^\dagger(r,\varphi) \end{pmatrix} = S(r,\varphi) \underline{\mathbf{a}} S^\dagger(r,\varphi) = \underline{\mathcal{C}}_{r,\varphi} \underline{\mathbf{a}}, \quad (2.22)$$

where

$$\underline{\mathcal{C}}_{r,\varphi} \equiv \begin{pmatrix} \cosh r & e^{2i\varphi} \sinh r \\ e^{-2i\varphi} \sinh r & \cosh r \end{pmatrix} = \underline{\mathcal{C}}_{r,\varphi}^\dagger. \quad (2.23)$$

Notice that  $\underline{\mathbf{a}} = \underline{\alpha}_{0,\varphi}$ . In the expression  $S(r,\varphi) \underline{\mathbf{a}} S^\dagger(r,\varphi)$  the operators  $S(r,\varphi)$  and  $S^\dagger(r,\varphi)$  act separately on each component of  $\underline{\mathbf{a}}$ . Hence the adjoint of  $\underline{\alpha}_{r,\varphi}$  is given by

$$\underline{\alpha}_{r,\varphi}^\dagger = S(r,\varphi) \underline{\mathbf{a}}^\dagger S^\dagger(r,\varphi) = \underline{\mathbf{a}}^\dagger \underline{\mathcal{C}}_{r,\varphi}^\dagger. \quad (2.24)$$

The inverse of Eq. (2.22) takes the form

$$\underline{\mathbf{a}} = S^\dagger(r,\varphi) \underline{\alpha}_{r,\varphi} S(r,\varphi) = \underline{\mathcal{C}}_{r,\varphi}^{-1} \underline{\alpha}_{r,\varphi}. \quad (2.25)$$

The matrix  $\underline{\mathcal{C}}_{r,\varphi}$  describes the matrix transformation of  $\underline{\mathbf{a}}$  that is induced by a unitary transformation of  $\underline{\mathbf{a}}$  generated by  $S(r,\varphi)$ . Useful properties of  $\underline{\mathcal{C}}_{r,\varphi}$  are listed in Appendix A. Any unitary transformation  $U$  which generates a matrix transformation of  $\underline{\mathbf{a}}$  (linear transformation of  $a_+$  and  $a_-^\dagger$ ) is a canonical transformation, described by a matrix  $\underline{\mathcal{M}}$ :

$$U \underline{\mathbf{a}} U^\dagger = \underline{\mathcal{M}} \underline{\mathbf{a}}. \quad (2.26)$$

Since a canonical transformation preserves commutators,  $\underline{\mathcal{M}}$  must satisfy

$$\underline{\mathcal{M}} \sigma_j \underline{\mathcal{M}}^\dagger = \sigma_j, \quad (2.27)$$

which is equivalent to

$$\underline{\mathcal{M}}^\dagger \sigma_j \underline{\mathcal{M}} = \sigma_j. \quad (2.28)$$

If  $\underline{\mathcal{M}}$  has unity determinant, then it is an element of the group  $SU(1,1)$ .<sup>6</sup> The most general element of  $SU(1,1)$  [generated by  $U = S(r,\varphi) R^\dagger(\theta)$ ; see Eq. (3.12)] is  $\underline{\mathcal{M}} = e^{-i\theta \sigma_3} \underline{\mathcal{C}}_{r,\varphi}$ . The matrices  $\underline{\mathcal{C}}_{r,\varphi}$ , generated by  $U = S(r,\varphi)$ , are the Hermitian elements of  $SU(1,1)$ . They must satisfy Eq. (2.28):

$$\underline{\mathcal{C}}_{r,\varphi}^\dagger \sigma_j \underline{\mathcal{C}}_{r,\varphi} = \sigma_j. \quad (2.29)$$

Equation (2.29) is the key property of  $\underline{\mathcal{C}}_{r,\varphi}$ . It says that  $\underline{\mathcal{C}}_{r,\varphi}$  preserves the scalar product with respect to the "metric"  $\sigma_j$ ; it is an expression of the fact that a unitary transformation generated by  $S(r,\varphi)$  preserves the difference in the number of quanta in the two modes.<sup>7</sup> In terms of the vector notation this fact is most easily written as preservation of the scalar product  $\underline{\mathbf{a}}^\dagger \sigma_j \underline{\mathbf{a}} = a_+^\dagger a_+ - a_- a_-^\dagger$ , i.e.,

$$\begin{aligned} S(r,\varphi) \underline{\mathbf{a}}^\dagger \sigma_j \underline{\mathbf{a}} S^\dagger(r,\varphi) &= \underline{\alpha}_{r,\varphi}^\dagger \sigma_j \underline{\alpha}_{r,\varphi} \\ &= \underline{\mathbf{a}}^\dagger \underline{\mathcal{C}}_{r,\varphi}^\dagger \sigma_j \underline{\mathcal{C}}_{r,\varphi} \underline{\mathbf{a}} = \underline{\mathbf{a}}^\dagger \sigma_j \underline{\mathbf{a}}. \end{aligned} \quad (2.30)$$

In addition to the above operator vectors [Eqs. (2.1), (2.4), and (2.22)], it is useful to have available the  $c$ -number vectors defined in Table I. The components of each  $c$ -number vector are complex numbers. With each operator vector we associate two  $c$ -number vectors, an "active-role" vector and a "passive-role" vector. The

TABLE I. Two-mode vector notation.

Operator vector	Associated <i>c</i> -number vectors	
	Active role	Passive role
$\underline{a} \equiv \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$	$\underline{\mu} \equiv \begin{pmatrix} \mu_+ \\ \mu_- \end{pmatrix}$	$\underline{v} \equiv \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$
$\underline{\alpha}_{r,\varphi} \equiv \underline{\alpha} \equiv \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \mathcal{C}_{r,\varphi} \underline{a}$	$\underline{\mu}_\alpha \equiv \begin{pmatrix} \mu_{\alpha_+} \\ \mu_{\alpha_-} \end{pmatrix} \equiv \mathcal{C}_{r,\varphi} \underline{\mu}$	$\underline{v}_\alpha \equiv \begin{pmatrix} v_{\alpha_+} \\ v_{\alpha_-} \end{pmatrix} \equiv \mathcal{C}_{r,\varphi} \underline{v}$
$\underline{\mathcal{A}} \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = d \lambda \underline{a}$	$\underline{\xi} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \equiv d \lambda \underline{\mu}$	$\underline{\eta} \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \equiv d \lambda^{-1} \underline{v}$

active-role vectors are used in contexts where the *c*-numbers act as surrogates for the corresponding operators, e.g., as eigenvalues or expectation values of the operators or as variables of a quasiprobability distribution. The passive-role vectors are used when the *c*-numbers appear as variables of a characteristic function. (Characteristic functions and quasiprobability distributions will be discussed in paper III.) Notice that there is no real difference between the active-role and passive-role vectors in the first two rows of Table I; nonetheless, we maintain the distinction because of the difference encountered in the third row. The second row of Table I introduces a further notational convenience: for the squeezed annihilation operators and their vectors we drop explicit reference to a particular *r* and  $\varphi$  unless this leads to confusion. When we need additional *c*-number vectors for either role, we denote them by attaching primes to all vectors in the appropriate column of Table I.

The crucial properties of the vectors in Table I are the following invariants:

$$a_+^\dagger v_+ - a_- v_-^\dagger = \underline{a}^\dagger \underline{\sigma} \underline{v} = \underline{\alpha}^\dagger \underline{\sigma} \underline{v}_\alpha = \underline{\mathcal{A}}^\dagger \underline{\eta} = \alpha_1^\dagger \eta_1 + \alpha_2^\dagger \eta_2, \quad (2.31a)$$

$$\mu_+^\dagger v_+ - \mu_- v_-^\dagger = \underline{\mu}^\dagger \underline{\sigma} \underline{v} = \underline{\mu}_\alpha^\dagger \underline{\sigma} \underline{v}_\alpha = \underline{\xi}^\dagger \underline{\eta} = \xi_1^\dagger \eta_1 + \xi_2^\dagger \eta_2. \quad (2.31b)$$

In Eqs. (2.31) the second equality follows from Eq. (2.29)—that  $\mathcal{C}_{r,\varphi}$  preserves the scalar product with respect to  $\underline{\sigma}$ ; it is the analog of Eq. (2.30). The desire to have the third equality in Eqs. (2.31) is responsible for the peculiar definition of  $\underline{\eta}$  in Table I. In addition to the invariants (2.31) it is useful to note the relations

$$\underline{\xi}^\dagger \underline{\xi}' = \underline{\mu}^\dagger \lambda^2 \underline{\mu}', \quad (2.32)$$

$$\underline{v}^\dagger \underline{v}' = \underline{\eta}^\dagger \Delta \underline{\eta}', \quad (2.33)$$

$$\underline{v}^\dagger \underline{\sigma} \underline{v}' = \underline{\eta}^\dagger \Pi \underline{\eta}', \quad (2.34)$$

which reveal the significance of the matrices  $\lambda^2$ ,  $\Delta$ , and  $\Pi$  [Eqs. (2.8)–(2.10)].

The vector notation introduced in this section allows us to manipulate easily the components of our formalism. Good examples are provided by the relations

$$S(r,\varphi) \underline{a}^\dagger \underline{\sigma} \underline{v} S^\dagger(r,\varphi) = \underline{\alpha}^\dagger \underline{\sigma} \underline{v}_\alpha = v_+ \alpha_+^\dagger - v_- \alpha_-^\dagger, \quad (2.35a)$$

$$S^\dagger(r,\varphi) \underline{a}^\dagger \underline{\sigma} \underline{v} S(r,\varphi) = \underline{a}^\dagger \underline{\sigma} \underline{v}_\alpha = v_{\alpha_+}^\dagger a_+^\dagger - v_{\alpha_-}^\dagger a_-^\dagger, \quad (2.35b)$$

the first of which follow directly from Eq. (2.24) and the second of which requires the invariant (2.31a) and Eq. (2.25). Another example is the commutator

$$[\underline{v}^\dagger \underline{\sigma} \underline{v}, \underline{a}^\dagger \underline{\sigma} \underline{v}] = \underline{v}^\dagger \underline{\sigma} \underline{v} \quad (2.36)$$

[Eqs. (1.1)], which the invariants (2.31) and Eq. (2.34) allow us to write immediately in the equivalent forms

$$[\underline{v}^\dagger \underline{\sigma} \underline{v}, \underline{\alpha}^\dagger \underline{\sigma} \underline{v}_\alpha] = \underline{v}^\dagger \underline{\sigma} \underline{v}_\alpha, \quad (2.37)$$

$$[\underline{\eta}^\dagger \underline{\mathcal{A}}, \underline{\mathcal{A}}^\dagger \underline{\eta}] = \underline{\eta}^\dagger \Pi \underline{\eta}. \quad (2.38)$$

The space on which quasiprobability distributions are defined is a complex phase space, and the space on which characteristic functions are defined is the corresponding complex Fourier space. An active-role vector and the corresponding passive-role vector form a pair of vectors under a complex Fourier transform. It is useful to note here the relations among integration measures for the *c*-number vectors in each column of Table I. Begin by defining, for a complex integration variable  $\xi$ , an integration measure  $d^2 \xi \equiv d(\text{Re} \xi) d(\text{Im} \xi)$ .<sup>3</sup> For a pair of complex numbers  $\xi_1$  and  $\xi_2$ , which form the *c*-number vector  $\underline{\xi}$ , define an integration measure

$$d^4 \xi \equiv d^2 \xi_1 d^2 \xi_2 = d(\text{Re} \xi_1) d(\text{Im} \xi_1) d(\text{Re} \xi_2) d(\text{Im} \xi_2). \quad (2.39)$$

The relations among integration measures are then given by

$$d^4 \mu = d^4 \mu_\alpha = (1 - \epsilon^2 / \Omega^2)^{-1} d^4 \xi, \quad (2.40a)$$

$$d^4 v = d^4 v_\alpha = (1 - \epsilon^2 / \Omega^2) d^4 \eta. \quad (2.40b)$$

There are analogous relations among the  $\delta$  functions of the *c*-number vectors in each column of Table I. For a complex number  $\xi$ , let  $\delta^2(\xi) \equiv \delta(\text{Re} \xi) \delta(\text{Im} \xi)$ , and for a *c*-number vector  $\underline{\xi}$ , let

$$\delta^4(\underline{\xi}) \equiv \delta^2(\xi_1) \delta^2(\xi_2) = \delta(\text{Re} \xi_1) \delta(\text{Im} \xi_1) \delta(\text{Re} \xi_2) \delta(\text{Im} \xi_2). \quad (2.41)$$

Then one finds that

$$\delta^4(\underline{\mu}) = \delta^4(\underline{\mu}_\alpha) = (1 - \epsilon^2 / \Omega^2) \delta^4(\underline{\xi}), \quad (2.42a)$$

$$\delta^4(\underline{y}) = \delta^4(\underline{y}_a) = (1 - \epsilon^2/\Omega^2)^{-1} \delta^4(\underline{\eta}) . \quad (2.42b)$$

Equations (2.39)–(2.42) find application in Sec. III C 2, where we define the complex Fourier transform, and they will be used extensively in paper III.

We take the remainder of this section to consider the degenerate limit of our two-mode formalism, because most of the current work on two-photon devices deals with the degenerate limit. By the degenerate limit we mean the limit in which the two modes at frequencies  $\Omega \pm \epsilon$  coalesce into a single mode at frequency  $\Omega$  ( $\epsilon=0, a_+ = a_-$ ). The formal method for taking this limit is described in Sec. VIII A of I. Here we list the quantities—analogs of the corresponding two-mode quantities—that are used in the degenerate limit. The SP creation and annihilation operators for the single mode are denoted by  $a^\dagger$  and  $a$ . The (Hermitian) IP *degenerate quadrature-phase amplitudes* are defined by

$$x_1 \equiv 2^{-1/2}(a + a^\dagger) , \quad (2.43a)$$

$$x_2 \equiv 2^{-1/2}(-ia + ia^\dagger) , \quad (2.43b)$$

$$a = 2^{-1/2}(x_1 + ix_2) \quad (2.44)$$

[Eqs. (I.8.16); cf. Eqs. (1.5)]; their commutator is

$$[x_1, x_2] = i \quad (2.45)$$

[cf. Eqs. (1.7)]. Analogous to the two-mode squeeze operator (1.8) is the *degenerate squeeze operator*<sup>3,10</sup>

$$S_1(r, \varphi) \equiv \exp\left\{\frac{1}{2}r[a^2 e^{-2i\varphi} - (a^\dagger)^2 e^{2i\varphi}]\right\} \quad (2.46)$$

[Eq. (I.8.19d)], which satisfies

$$S_1^{-1}(r, \varphi) = S_1^\dagger(r, \varphi) = S_1(-r, \varphi) = S_1(r, \varphi + \pi/2) \quad (2.47)$$

[cf. Eq. (1.9)]. The IP *squeezed annihilation operator* is defined by

$$\alpha(r, \varphi) \equiv S_1(r, \varphi) a S_1^\dagger(r, \varphi) = a \cosh r + a^\dagger e^{2i\varphi} \sinh r \quad (2.48)$$

[Eq. (I.8.20); cf. Eq. (1.10)].

The degenerate quadrature-phase amplitudes (2.43) look deceptively like a dimensionless coordinate and momentum. To avoid confusion, we remind the reader that Eqs. (2.43) and (2.44) are written in mixed pictures:  $a$  and  $a^\dagger$  are (constant) SP operators, whereas  $x_1$  and  $x_2$  are (constant) IP operators. In the SP the degenerate quadrature-phase amplitudes are explicitly time-dependent operators, which we denote by

$$\begin{aligned} x_1(t) &\equiv e^{-i\Omega t} a^\dagger x_1 e^{i\Omega t} a \\ &= 2^{-1/2}(a e^{i\Omega t} + a^\dagger e^{-i\Omega t}) , \end{aligned} \quad (2.49a)$$

$$\begin{aligned} x_2(t) &\equiv e^{-i\Omega t} a^\dagger x_2 e^{i\Omega t} a \\ &= 2^{-1/2}(-ia e^{i\Omega t} + ia^\dagger e^{-i\Omega t}) \end{aligned} \quad (2.49b)$$

[cf. Eqs. (I.4.22)]. In contrast, the dimensionless coordinate and momentum are constant operators in the SP, defined by

$$x \equiv 2^{-1/2}(a + a^\dagger) , \quad (2.50a)$$

$$p \equiv 2^{-1/2}(-ia + ia^\dagger) , \quad (2.50b)$$

$$a = 2^{-1/2}(x + ip) . \quad (2.51)$$

Thus, although  $x_1 = x$  and  $x_2 = p$ , a picture-consistent equation relating the degenerate quadrature-phase amplitudes to the coordinate and momentum takes the form

$$x_1(t) = x \cos(\Omega t) - p \sin(\Omega t) , \quad (2.52a)$$

$$x_2(t) = x \sin(\Omega t) + p \cos(\Omega t) . \quad (2.52b)$$

Table II summarizes the operators and associated  $c$ -number quantities which are used in the degenerate limit. In Table II we introduce a vector notation for a single mode analogous to the two-mode vector notation summarized in Table I. It is hoped that use of nearly the same symbols for the two cases will not lead to confusion, because we never deal with the two cases simultaneously. Similar single-mode vector notations have been used by Yuen,<sup>11</sup> Milburn,<sup>12</sup> and Collett and Gardiner.<sup>3</sup> The single mode that exists in the degenerate limit has two degrees of freedom—two fewer than in the original two modes. In the first two rows of Table II this reduction shows up in that the components of the vectors are not independent quantities; in the third row it shows up in that the components of the vectors are, reading across the table from left to right, Hermitian operators, real numbers, and pure imaginary numbers. In the degenerate limit it is sometimes convenient to use a different passive-role vector

$$\underline{\zeta} \equiv \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \equiv i\underline{\eta} = i\underline{d} \underline{\alpha} ; \underline{\alpha} = 2^{1/2} \begin{pmatrix} -\text{Im}(v) \\ \text{Re}(v) \end{pmatrix} = \underline{\zeta}^* , \quad (2.53)$$

whose components are real.

In the degenerate limit the reduced spectral-density matrix (2.13) becomes an ordinary (real, symmetric) covariance matrix

$$\begin{aligned} \underline{\Sigma} &\equiv \langle \Delta \underline{x} \Delta \underline{x}^\dagger \rangle_{\text{sym}} \\ &= \begin{bmatrix} \langle (\Delta x_1)^2 \rangle & \langle \Delta x_1 \Delta x_2 \rangle_{\text{sym}} \\ \langle \Delta x_2 \Delta x_1 \rangle_{\text{sym}} & \langle (\Delta x_2)^2 \rangle \end{bmatrix} . \end{aligned} \quad (2.54)$$

The corresponding (Hermitian) matrix that gives the second-order noise moments in terms of the creation and annihilation operators is

$$\underline{\bar{\Sigma}} \equiv \langle \Delta \underline{a} \Delta \underline{a}^\dagger \rangle_{\text{sym}} = \begin{bmatrix} \langle |\Delta a|^2 \rangle & \langle (\Delta a)^2 \rangle \\ \langle (\Delta a^\dagger)^2 \rangle & \langle |\Delta a^\dagger|^2 \rangle \end{bmatrix} \quad (2.55)$$

[cf. Eq. (2.15)]. These two matrices are related by

$$\underline{\Sigma} = \underline{d} \underline{\bar{\Sigma}} \underline{d}^\dagger \quad (2.56)$$

[cf. Eq. (2.16)]. Just as in the two-mode case, one can decompose the matrices  $\underline{\Sigma}$  and  $\underline{\bar{\Sigma}}$  in terms of the unit matrix and the Pauli matrices. Equations (2.17)–(2.20) retain their forms in the degenerate limit, but note that  $\Sigma_2 = 0$  [Eq. (2.19)] and  $\bar{\Sigma}_3 = 0$  [Eq. (2.20)]. Equations (2.21) reduce to the simple equations

$$\Sigma_0 = \bar{\Sigma}_0, \quad \Sigma_1 = -\bar{\Sigma}_2, \quad \Sigma_3 = \bar{\Sigma}_1 . \quad (2.57)$$

For TS noise the matrices  $\underline{\Sigma}$  and  $\underline{\bar{\Sigma}}$  are identical and equal to a multiple of the unit matrix ( $\underline{\Sigma} = \underline{\bar{\Sigma}} = \Sigma_0 \mathbb{1}$ ).

The invariants in the degenerate limit are very much

TABLE II. Single-mode vector notation.

Operator vector	Associated c-number vectors	
	Active role	Passive role
$\underline{a} \equiv \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$	$\underline{\mu} \equiv \begin{pmatrix} \mu \\ \mu^* \end{pmatrix}$	$\underline{v} \equiv \begin{pmatrix} v \\ v^* \end{pmatrix}$
$\underline{a}_{r,\varphi} \equiv \underline{a} \equiv \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} \equiv \underline{C}_{r,\varphi} \underline{a}$	$\underline{\mu}_\alpha \equiv \begin{pmatrix} \mu_\alpha \\ \mu_\alpha^* \end{pmatrix} \equiv \underline{C}_{r,\varphi} \underline{\mu}$	$\underline{v}_\alpha \equiv \begin{pmatrix} v_\alpha \\ v_\alpha^* \end{pmatrix} \equiv \underline{C}_{r,\varphi} \underline{v}$
$\underline{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = d \underline{a}$	$\underline{\xi} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = d \underline{\mu} = \underline{\xi}^*$	$\underline{\eta} \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = d \underline{v} = -\underline{\eta}^*$

like the invariants (2.31):

$$\begin{aligned} a^\dagger v - a v^* &= \underline{a}^\dagger \underline{v} - \underline{a} \underline{v}^* = \underline{\xi}^\dagger \underline{\eta} = -i \underline{\xi}^\dagger \underline{\xi} \\ &= -i(x_1 \xi_1 + x_2 \xi_2), \end{aligned} \quad (2.58a)$$

$$\begin{aligned} \mu^* v - \mu v^* &= \underline{\mu}^\dagger \underline{v} - \underline{\mu} \underline{v}^* = \underline{\xi}^\dagger \underline{\eta} = -i \underline{\xi}^\dagger \underline{\xi} \\ &= -i(\xi_1 \xi_1 + \xi_2 \xi_2). \end{aligned} \quad (2.58b)$$

In the degenerate limit the relations (2.32)–(2.34) become

$$\underline{\xi}^\dagger \underline{\xi}' = \underline{\mu}^\dagger \underline{\mu}', \quad (2.59)$$

$$\underline{v}^\dagger \underline{v}' = \underline{\eta}^\dagger \underline{\eta}' = \underline{\xi}^\dagger \underline{\xi}', \quad (2.60)$$

$$\underline{v}^\dagger \underline{v}' = -\underline{\eta}^\dagger \underline{\eta}' = -\underline{\xi}^\dagger \underline{\xi}', \quad (2.61)$$

because when  $\epsilon=0$  ( $\lambda=1$ ),  $\lambda^2=\Delta=1$  and  $\Pi=-\sigma_2$ . Equations (2.35)–(2.38) retain their forms in the degenerate limit, with the two-mode vectors replaced by the corresponding single-mode vectors.

The integration measures corresponding to the c-number vectors in Table II deserve special comment; they do not have the form of Eqs. (2.40) with  $\epsilon=0$  because of the reduction in the number of degrees of freedom at degeneracy. Defining  $d^2\mu \equiv d(\text{Re}\mu) d(\text{Im}\mu)$ , one finds that

$$d^2\mu = d^2\mu_\alpha = \frac{1}{2} d\xi_1 d\xi_2, \quad (2.62a)$$

$$d^2v = d^2v_\alpha = \frac{1}{2} d\xi_1 d\xi_2. \quad (2.62b)$$

Notice that  $d^2\mu/\pi = d\xi_1 d\xi_2/2\pi$  is the usual phase-space volume element. Corresponding to Eqs. (2.62) are the following relations among  $\delta$  functions:

$$\delta^2(\mu) = \delta^2(\mu_\alpha) = 2\delta(\xi_1)\delta(\xi_2), \quad (2.63a)$$

$$\delta^2(v) = \delta^2(v_\alpha) = 2\delta(\xi_1)\delta(\xi_2), \quad (2.63b)$$

$$[\delta^2(\mu) \equiv \delta(\text{Re}\mu) \delta(\text{Im}\mu)].$$

### III. FUNDAMENTAL UNITARY OPERATORS

#### A. Modulation-picture free evolution operator

The basic picture in our formalism is the modulation picture (see discussion in Sec. IV C of I), so the fundamental free evolution operator is the *modulation-picture free evolution operator*

$$\begin{aligned} U_M(t) &\equiv e^{-iH_M t} = \exp[-i\epsilon t(a^\dagger_+ a_+ - a^\dagger_- a_-)] \\ &= e^{-i\epsilon t} \exp(-i\epsilon t \underline{a}^\dagger \underline{a}) \end{aligned} \quad (3.1)$$

[Eq. (I.4.37)], which satisfies

$$U_M^{-1}(t) = U_M^\dagger(t) = U_M(-t). \quad (3.2)$$

The MP free evolution operator is used to evolve states in the MP when the two modes are evolving freely. It unitarily transforms  $a_\pm$  as

$$U_M^\dagger(t) a_\pm U_M(t) = a_\pm e^{\mp i\epsilon t}, \quad (3.3)$$

which in vector notation becomes

$$U_M^\dagger(t) \underline{a} U_M(t) = \underline{a} e^{-i\epsilon t}. \quad (3.4)$$

Multiplying Eq. (3.4) first by  $\underline{C}_{r,\varphi}$  and then by  $d \underline{\lambda}$ , one finds that

$$U_M^\dagger(t) \underline{a} U_M(t) = \underline{a} e^{-i\epsilon t}, \quad (3.5)$$

$$U_M^\dagger(t) \underline{a}' U_M(t) = \underline{a}' e^{-i\epsilon t} \quad (3.6)$$

[cf. Eq. (I.4.27)]. An important property of  $U_M(t)$  is that it commutes with  $S(r,\varphi)$ :

$$U_M(t) S(r,\varphi) U_M^\dagger(t) = S(r,\varphi). \quad (3.7)$$

In the degenerate limit  $U_M(t)$  becomes the identity operator, i.e.,

$$U_M(t) \xrightarrow{p} 1 \quad (3.8)$$

[Eq. (I.8.19a)], which means that the MP and the IP coincide.

#### B. Rotation operator

An important feature of our formalism is the phase freedom in the definition of the quadrature-phase amplitudes (see discussion in Sec. IV C of I). The operator that describes this phase freedom is the *rotation operator*

$$\begin{aligned} R(\theta) &\equiv \exp[-i\theta(a^\dagger_+ a_+ + a^\dagger_- a_-)] \\ &= e^{i\theta} \exp(-i\theta \underline{a}^\dagger \underline{a}) \end{aligned} \quad (3.9)$$

[Eq. (I.4.33)], which satisfies

$$R^{-1}(\theta) = R^\dagger(\theta) = R(-\theta). \quad (3.10)$$

A unitary transformation generated by  $R(\theta)$  produces a common phase change of the annihilation operators,



$$R^\dagger(\theta)a_\pm R(\theta) = a_\pm e^{-i\theta} \quad (3.11)$$

[Eq. (I.4.35)], which in vector notation becomes

$$R^\dagger(\theta)\underline{a}R(\theta) = e^{-i\theta\alpha_3}\underline{a}, \quad (3.12)$$

$$e^{-i\theta\alpha_3} = \underline{1} \cos\theta - i\alpha_3 \sin\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \quad (3.13)$$

and it produces a rotation of the quadrature-phase amplitudes,

$$R^\dagger(\theta)\underline{a}'R(\theta) = e^{i\theta\alpha_2}\underline{a}', \quad (3.14)$$

$$e^{i\theta\alpha_2} = \underline{1} \cos\theta + i\alpha_2 \sin\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad (3.15)$$

[Eq. (A.6); cf. Eqs. (I.4.36)].

Important properties of the rotation operator include the following: (i)  $R(\theta)$  "rotates" the squeeze operator, i.e.,

$$R^\dagger(\theta)S(r,\varphi)R(\theta) = S(r,\varphi+\theta) \quad (3.16)$$

[Eq. (3.11)], a property which is the operator analog of Eq. (A.27); (ii)  $R(\Omega t) = e^{-iH_R t}$  is the unitary transformation that connects the SP and the MP [Eq. (I.4.4)]; (iii)  $R(\theta)$  commutes with  $U_M(t)$  [cf. Eq. (1.4)]; and (iv) the SP free evolution operator is  $e^{-iH_0 t} = R(\Omega t)U_M(t)$ . Equation (3.16) implies immediately that

$$R^\dagger(\theta)\underline{a}_{r,\varphi}R(\theta) = e^{-i\theta\alpha_3}\underline{a}_{r,\varphi+\theta}. \quad (3.17)$$

In the degenerate limit  $R(\theta)$  becomes a single-mode rotation operator, i.e.,

$$R(\theta) \xrightarrow{\rho} \exp(-i\theta a^\dagger a) \quad (3.18)$$

[Eq. (I.8.19b)].

### C. Two-mode displacement operator

At the heart of one-photon optics lies the displacement operator,<sup>3</sup> which generates coherent states from the vacuum. It continues to occupy an important place in two-photon optics. Here we begin by reviewing some well-known properties of the displacement operator for a single mode. We then proceed to the displacement operator for two modes and write its properties in terms of the vector notation.

#### 1. Single-mode displacement operator

The single-mode displacement operator<sup>3</sup> is defined by

$$D(a,\mu) \equiv \exp(\mu a^\dagger - \mu^* a) \quad (3.19)$$

[Eq. (I.3.7)]. It satisfies the following string of equalities:

$$D^{-1}(a,\mu) = D^\dagger(a,\mu) = D(a,-\mu) = D(-a,\mu). \quad (3.20)$$

The key property of the displacement operator is that it displaces the annihilation operator,<sup>3</sup> i.e.,

$$D^\dagger(a,\mu)aD(a,\mu) = a + \mu. \quad (3.21)$$

One can write Eq. (3.21) in an equivalent form involving

the degenerate quadrature-phase amplitudes (2.43):

$$D^\dagger(a,\mu)\underline{x}D(a,\mu) = \underline{x} + \underline{\xi} \quad (3.22)$$

(see Table II).

We use a two-slot notation for the displacement operator:  $D(a,\mu)$  can be regarded as an operator-valued function of an operator  $a$  (first slot) and a complex number  $\mu$  (second slot). The most important reason for this two-slot notation is that one can replace  $a$  with another operator that has the same commutator with its adjoint and the resulting "displacement operator" has the same properties as the original. In practice, it is sometimes useful to replace  $a$  with the squeezed annihilation operator  $\alpha(r,\varphi)$  [Eq. (2.48)]. The resulting operator

$$D(\alpha,\mu_\alpha) = e^{\mu_\alpha \alpha^\dagger - \mu_\alpha^* \alpha}, \quad (3.23)$$

which we conventionally write with  $\mu_\alpha$  as the label for the complex variable, displaces  $\alpha$ :

$$D^\dagger(\alpha,\mu_\alpha)\alpha D(\alpha,\mu_\alpha) = \alpha + \mu_\alpha. \quad (3.24)$$

Notice that  $D(\alpha,\mu)$  is unitarily equivalent to  $D(a,\mu)$ :

$$S_1(r,\varphi)D(a,\mu)S_1^\dagger(r,\varphi) = D(\alpha,\mu). \quad (3.25)$$

Further, the invariant (2.58a) implies

$$D(a,\mu) = D(\alpha,\mu_\alpha). \quad (3.26)$$

Equations (3.25) and (3.26) can be used to obtain the result

$$S_1^\dagger(r,\varphi)D(a,\mu)S_1(r,\varphi) = D(\alpha,\mu_\alpha). \quad (3.27)$$

One can also define the operator

$$\mathcal{D}(\underline{x},\underline{\eta}) \equiv \exp(\underline{x}^\dagger \underline{\eta}) = \exp(-i\underline{x}^\dagger \underline{\zeta}) = e^{-i(\zeta_1 x_1 + \zeta_2 x_2)}, \quad (3.28)$$

which is the displacement operator written in terms of  $x_1$  and  $x_2$ , i.e.,

$$D(a,\nu) = D(\alpha,\nu_\alpha) = \mathcal{D}(\underline{x},\underline{\eta}) \quad (3.29)$$

[Eq. (2.58a)].

A second reason for the two-slot notation is that one can replace the operator  $a$  with a complex number  $\mu$  to obtain a complex-valued function of two complex variables,

$$D(\mu,\nu) \equiv e^{\mu^* \nu - \nu^* \mu}, \quad (3.30)$$

which satisfies

$$D^{-1}(\mu,\nu) = D^*(\mu,\nu) = D(\nu,\mu) = D(\mu,-\nu) = D(-\mu,\nu), \quad (3.31a)$$

$$D(\mu,\nu)D(\mu,\nu') = D(\mu,\nu+\nu'). \quad (3.31b)$$

The importance of  $D(\mu,\nu)$  lies in its role as the expansion factor for complex Fourier transforms.<sup>13</sup> A function  $f(\mu)$  is related to its complex Fourier transform  $F(\nu)$  by

$$f(\mu) = \int \frac{d^2\nu}{\pi} F(\nu)D(\nu,\mu) \quad (3.32)$$

[ $d^2\nu \equiv d(\text{Re}\nu)d(\text{Im}\nu)$ ]. Employing the property

$$\int \frac{d^2\mu}{\pi} D(\mu, \nu) = \int \frac{d^2\mu}{\pi} D(\nu, \mu) = \pi \delta^2(\nu) \quad (3.33)$$

[ $\delta^2(\nu) \equiv \delta(\text{Re}\nu)\delta(\text{Im}\nu)$ ], one can invert Eq. (3.32) to give

$$F(\nu) = \int \frac{d^2\mu}{\pi} f(\mu) D(\mu, \nu). \quad (3.34)$$

Equations (3.32) and (3.34) are a neat, symmetric way to write the relations between a function and its complex Fourier transform. The invariant (2.58b) implies

$$D(\mu, \nu) = D(\mu_\alpha, \nu_\alpha) = e^{\xi^\dagger \eta} \equiv \mathcal{D}(\xi, \eta) \quad (3.35)$$

[cf. Eq. (3.29)].

We end this review of the single-mode displacement operator by listing a few other important properties:

$$D^\dagger(a, \mu) D(a, \nu) D(a, \mu) = D(a + \nu, \nu) = D(\mu, \nu) D(a, \nu), \quad (3.36)$$

$$D^\dagger(a, \mu') D(a, \mu) = D(\frac{1}{2}\mu', \mu) D(a, \mu - \mu'), \quad (3.37)$$

$$D(a, \mu) = e^{-i|\mu|^2/2} e^{\mu a} e^{-\mu^* a} = e^{i|\mu|^2/2} e^{-\mu^* a} e^{\mu a^\dagger}. \quad (3.38)$$

## 2. Two-mode displacement operator

We turn now to the displacement operator for two modes, our objective being to generalize (trivially) the properties of the single-mode displacement operator and to write the resulting two-mode properties in terms of the vector notation. Use of the vector notation gives the two-mode properties an appearance as compact and elegant as the single-mode properties.

We begin by writing the two-mode displacement operator<sup>3</sup> [Eq. (I.4.12)] in the form

$$\begin{aligned} D(\underline{a}, \underline{\mu}) &\equiv \exp(\underline{a}^\dagger \underline{\sigma}_+ \underline{\mu} - \underline{\mu}^\dagger \underline{\sigma}_+ \underline{a}) \\ &= D(\alpha_+, \mu_+) D(\alpha_-, \mu_-) \end{aligned} \quad (3.39)$$

[cf. Eq. (3.19)], which satisfies

$$D^{-1}(\underline{a}, \underline{\mu}) = D^\dagger(\underline{a}, \underline{\mu}) = D(\underline{a}, -\underline{\mu}) = D(-\underline{a}, \underline{\mu}) \quad (3.40)$$

[cf. Eq. (3.20)]. The two-mode displacement operator displaces  $\underline{a}$ , i.e.,

$$D^\dagger(\underline{a}, \underline{\mu}) \underline{a} D(\underline{a}, \underline{\mu}) = \underline{a} + \underline{\mu} \quad (3.41)$$

[cf. Eq. (3.21)], and it generates two-mode coherent states from the vacuum (see Sec. IV A 2). Multiplying Eq. (3.41) by  $\underline{A} \underline{\lambda}$  yields

$$D^\dagger(\underline{a}, \underline{\mu}) \underline{A} D(\underline{a}, \underline{\mu}) = \underline{A} + \underline{\xi} \quad (3.42)$$

[see Table I; cf. Eq. (3.22)].

We use a two-slot notation for the two-mode displacement operator:  $D(\underline{a}, \underline{\mu})$  can be regarded as an operator-valued function of an operator vector  $\underline{a}$  and a  $c$ -number vector  $\underline{\mu}$ . Just as in the single-mode case, the main reason for this two-slot notation is that we can also consider the operator

$$\begin{aligned} D(\underline{\alpha}, \underline{\mu}_\alpha) &= e^{\alpha_+^\dagger \mu_{\alpha+} - \mu_{\alpha+}^\dagger \alpha_+} \\ &= D(\alpha_+, \mu_{\alpha+}) D(\alpha_-, \mu_{\alpha-}) \end{aligned} \quad (3.43)$$

[cf. Eq. (3.23)]. An important connection between  $D(\underline{a}, \underline{\mu})$  and  $D(\underline{\alpha}, \underline{\mu}_\alpha)$  is that properties of  $D(\underline{\alpha}, \underline{\mu}_\alpha)$  can be obtained directly from those of  $D(\underline{a}, \underline{\mu})$  because the squeezed annihilation operators have the same commutation algebra as the annihilation operators. For example, one can say immediately that  $D(\underline{\alpha}, \underline{\mu}_\alpha)$  displaces  $\underline{\alpha}$ :

$$D^\dagger(\underline{\alpha}, \underline{\mu}_\alpha) \underline{\alpha} D(\underline{\alpha}, \underline{\mu}_\alpha) = \underline{\alpha} + \underline{\mu}_\alpha. \quad (3.44)$$

An equivalent way of stating this connection is that  $D(\underline{\alpha}, \underline{\mu})$  is unitarily equivalent to  $D(\underline{a}, \underline{\mu})$ , i.e.,

$$S(r, \varphi) D(\underline{a}, \underline{\mu}) S^\dagger(r, \varphi) = D(\underline{\alpha}, \underline{\mu}) \quad (3.45)$$

[cf. Eqs. (2.35a) and (3.25)]. Thus Eq. (3.44) could be obtained by unitarily transforming Eq. (3.41) with  $S(r, \varphi)$  and replacing  $\underline{\mu}$  with  $\underline{\mu}_\alpha$ . A different and crucial connection between  $D(\underline{a}, \underline{\mu})$  and  $D(\underline{\alpha}, \underline{\mu}_\alpha)$  is that they are the same operator, a consequence of the invariant (2.31a):

$$D(\underline{a}, \underline{\mu}) = D(\underline{\alpha}, \underline{\mu}_\alpha) \quad (3.46)$$

[cf. Eq. (3.26)]. Equation (3.46) means that Eq. (3.44) results from multiplying Eq. (3.41) by  $S(r, \varphi)$ . A further important relation is a consequence of Eqs. (3.46) and (2.25):

$$S^\dagger(r, \varphi) D(\underline{a}, \underline{\mu}) S(r, \varphi) = D(\underline{a}, \underline{\mu}_\alpha) \quad (3.47)$$

[cf. Eqs. (2.35b) and (3.27)].

We find it useful to write the two-mode displacement operator in terms of the quadrature-phase amplitudes  $\alpha_1$  and  $\alpha_2$ . Therefore, we define the operator

$$\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) \equiv \exp(\underline{\mathcal{A}}^\dagger \underline{\eta} - \underline{\eta}^\dagger \underline{\mathcal{A}}) = e^{\eta_1 \alpha_1^\dagger + \eta_2 \alpha_2^\dagger - \eta_1^\dagger \alpha_1 - \eta_2^\dagger \alpha_2} \quad (3.48)$$

[cf. Eq. (3.28)], which satisfies

$$\mathcal{D}^{-1}(\underline{\mathcal{A}}, \underline{\eta}) = \mathcal{D}^\dagger(\underline{\mathcal{A}}, \underline{\eta}) = \mathcal{D}(\underline{\mathcal{A}}, -\underline{\eta}) = \mathcal{D}(-\underline{\mathcal{A}}, \underline{\eta}) \quad (3.49)$$

and which can be regarded as an operator-valued function of an operator vector  $\underline{\mathcal{A}}$  and a  $c$ -number vector  $\underline{\eta}$ . Since it is not the same function as  $D(\underline{a}, \underline{\nu})$ , we distinguish it by using a script letter. Nonetheless, the invariant (2.31a) guarantees that  $\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})$  and  $D(\underline{a}, \underline{\nu})$  are the same operator:

$$D(\underline{a}, \underline{\nu}) = D(\underline{\alpha}, \underline{\nu}_\alpha) = \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) \quad (3.50)$$

[cf. Eq. (3.29)].

The introduction of  $\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})$  provides a good opportunity to elucidate the distinction between the active-role and passive-role vectors introduced in Table I. As noted in Sec. II, an active-role vector is used as a surrogate for the corresponding operator vector, e.g., as an eigenvalue or an expectation value of the operator vector or as the vector variable of a quasiprobability distribution. Thus the active-role vectors  $\underline{\mu}$  and  $\underline{\mu}_\alpha$  are used in the second slot of the two-mode displacement operator when it is used in its active role, i.e., as a unitary operator that transforms states and operators. A passive-role vector is used as the vector variable of a characteristic function. Thus the passive-role vectors  $\underline{\nu}$  and  $\underline{\nu}_\alpha$  are used in the second slot of the two-mode displacement operator

when it is used in its passive role, i.e., when one takes its expectation value to obtain a characteristic function.

In the first two rows of Table I there is no real difference between the active-role and passive-role vectors, since  $\underline{v}_\alpha$  is related to  $\underline{v}$  in the same way that  $\underline{\mu}_\alpha$  is related to  $\underline{\mu}$ ; the difference is merely a matter of choosing different labels for vectors in the two roles. To find a real difference, one must proceed to the third row. The definition of the active-role vector  $\underline{\xi} = \underline{d} \underline{\lambda} \underline{\mu}$  is determined by the fact that  $\underline{\xi}$  and  $\underline{\mu}$  can stand for the expectation values of  $\underline{\mathcal{A}}$  and  $\underline{a}$ , respectively; thus  $\underline{\xi}$  must be related to  $\underline{\mu}$  in the same way that  $\underline{\mathcal{A}}$  is related to  $\underline{a}$ . This natural definition of  $\underline{\xi}$  is to be contrasted with the definition of the passive-role vector  $\underline{\eta} = \underline{d} \underline{\lambda}^{-1} \underline{\sigma} \underline{v}$ , which at first appears very peculiar indeed. The explanation for this peculiar definition lies in the form of the operator  $\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})$ , which is the displacement operator written in terms of quadrature-phase amplitudes. The simple form of  $\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})$  is a consequence of the invariant (2.31a) and, hence, of the definition of  $\underline{\eta}$ . More illuminating is to put things the other way around: the peculiar definition of  $\underline{\eta}$  is dictated by the desire to have a simple form for the two-mode displacement operator when it is written in terms of the quadrature-phase amplitudes; thus this desire is ultimately responsible for the distinction we make between the active role and the passive role. This discussion also makes clear why  $\underline{v}_\alpha$  is related to  $\underline{v}$  in the same way that  $\underline{\mu}_\alpha$  is related to  $\underline{\mu}$ . The definition of  $\underline{\mu}_\alpha = \underline{C}_{r,\varphi} \underline{\mu}$  is determined by the relation  $\underline{\alpha} = \underline{C}_{r,\varphi} \underline{a}$ ; the same transformation  $\underline{v}_\alpha = \underline{C}_{r,\varphi} \underline{v}$  is appropriate for the passive-role vector because of the property (2.29) of  $\underline{C}_{r,\varphi}$ .

The operator  $\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})$  is defined in terms of the passive-role vector  $\underline{\eta}$ , and it is used exclusively in the passive role. We could write the two-mode displacement operator in terms of  $\underline{\mathcal{A}}$  and the active-role vector  $\underline{\xi}$  simply by substituting  $\underline{a} = \underline{\lambda}^{-1} \underline{d}^\dagger \underline{\mathcal{A}}$  and  $\underline{\mu} = \underline{\lambda}^{-1} \underline{d}^\dagger \underline{\xi}$  into  $D(\underline{a}, \underline{\mu})$ . The result does not have a simple form, nor do we find it useful, so we do without it.

A particularly important form of Eq. (3.41) can be obtained by using the passive-role vector  $\underline{v}$  and then writing Eq. (3.41) as the commutator

$$[\underline{a}, D(\underline{a}, \underline{v})] = \underline{v} D(\underline{a}, \underline{v}). \quad (3.51)$$

Multiplying Eq. (3.51) by  $\underline{d} \underline{\lambda}$  and substituting  $\underline{v} = \underline{\sigma} \underline{\lambda} \underline{d}^\dagger \underline{\eta}$ , one finds

$$[\underline{\mathcal{A}}, \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})] = \underline{\eta} \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) \quad (3.52)$$

[Eqs. (2.10) and (3.50)]. Equations (3.51) and (3.52) will play a crucial role in the operator-ordering formalism of paper III. Equation (3.52) expresses the same relation that Eq. (3.42) does; the apparent difference is due to the use of the passive-role vector  $\underline{\eta}$  in Eq. (3.52), in contrast to the use of the active-role vector  $\underline{\xi}$  in Eq. (3.42).

A second reason for the two-slot notation is that one can replace the operator vector in the first slot of  $D(\underline{a}, \underline{v})$  or  $\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})$  with a c-number vector. Hence, one can define the following complex-valued functions of two c-number vectors:

$$D(\underline{\mu}, \underline{v}) \equiv e^{\underline{\mu}^\dagger \underline{\sigma} \underline{v} - \underline{v}^\dagger \underline{\sigma} \underline{\mu}} = D(\underline{\mu}_+, \underline{v}_+) D(\underline{\mu}_-, \underline{v}_-), \quad (3.53)$$

$$\mathcal{D}(\underline{\xi}, \underline{\eta}) \equiv e^{\underline{\xi}^\dagger \underline{\eta} - \underline{\eta}^\dagger \underline{\xi}} = D(\underline{\xi}_1, \underline{\eta}_1) D(\underline{\xi}_2, \underline{\eta}_2) \quad (3.54)$$

[cf. Eq. (3.30)]. These functions satisfy

$$D^{-1}(\underline{\mu}, \underline{v}) = D^*(\underline{\mu}, \underline{v}) = D(\underline{v}, \underline{\mu}) = D(\underline{\mu}, -\underline{v}) = D(-\underline{\mu}, \underline{v}), \quad (3.55a)$$

$$D(\underline{\mu}, \underline{v}) D(\underline{\mu}, \underline{v}') = D(\underline{\mu}, \underline{v} + \underline{v}'), \quad (3.55b)$$

$$\begin{aligned} \mathcal{D}^{-1}(\underline{\xi}, \underline{\eta}) &= \mathcal{D}^*(\underline{\xi}, \underline{\eta}) = \mathcal{D}(\underline{\eta}, \underline{\xi}) = \mathcal{D}(\underline{\xi}, -\underline{\eta}) \\ &= \mathcal{D}(-\underline{\xi}, \underline{\eta}), \end{aligned} \quad (3.56a)$$

$$\mathcal{D}(\underline{\xi}, \underline{\eta}) \mathcal{D}(\underline{\xi}, \underline{\eta}') = \mathcal{D}(\underline{\xi}, \underline{\eta} + \underline{\eta}') \quad (3.56b)$$

[cf. Eqs. (3.31)]. The invariant (2.31b) implies

$$D(\underline{\mu}, \underline{v}) = D(\underline{\mu}_\alpha, \underline{v}_\alpha) = \mathcal{D}(\underline{\xi}, \underline{\eta}) \quad (3.57)$$

[cf. Eqs. (3.35) and (3.50)]. Either  $D(\underline{\mu}, \underline{v})$  or  $\mathcal{D}(\underline{\xi}, \underline{\eta})$  can serve as the expansion factor for complex Fourier transforms. For example, a function  $f(\underline{\mu})$  is related to its complex Fourier transform  $F(\underline{v})$  by

$$f(\underline{\mu}) = \int \frac{d^4 \underline{v}}{\pi^2} F(\underline{v}) D(\underline{v}, \underline{\mu}), \quad (3.58a)$$

$$F(\underline{v}) = \int \frac{d^4 \underline{\mu}}{\pi^2} f(\underline{\mu}) D(\underline{\mu}, \underline{v}) \quad (3.58b)$$

[Eq. (2.39); cf. Eqs. (3.32) and (3.34)]. The orthonormality and completeness relations for  $D(\underline{\mu}, \underline{v})$  and  $\mathcal{D}(\underline{\xi}, \underline{\eta})$  are subsumed in the equations

$$\int \frac{d^4 \underline{\mu}}{\pi^2} D(\underline{\mu}, \underline{v}) = \int \frac{d^4 \underline{\mu}}{\pi^2} D(\underline{v}, \underline{\mu}) = \pi^2 \delta^4(\underline{v}), \quad (3.59a)$$

$$\int \frac{d^4 \underline{\xi}}{\pi^2} \mathcal{D}(\underline{\xi}, \underline{\eta}) = \int \frac{d^4 \underline{\xi}}{\pi^2} \mathcal{D}(\underline{\eta}, \underline{\xi}) = \pi^2 \delta^4(\underline{\eta}) \quad (3.59b)$$

[Eq. (2.41); cf. Eq. (3.33)].

Further properties of the two-mode displacement operator include the way it is transformed by the MP free evolution operator,

$$U_M^\dagger(t) D(\underline{a}, \underline{v}) U_M(t) = D(\underline{a} e^{-i\epsilon t}, \underline{v}) = D(\underline{a}, \underline{v} e^{i\epsilon t}), \quad (3.60a)$$

$$U_M^\dagger(t) D(\underline{\alpha}, \underline{v}_\alpha) U_M(t) = D(\underline{\alpha} e^{-i\epsilon t}, \underline{v}_\alpha) = D(\underline{\alpha}, \underline{v}_\alpha e^{i\epsilon t}), \quad (3.60b)$$

$$U_M^\dagger(t) \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) U_M(t) = \mathcal{D}(\underline{\mathcal{A}} e^{-i\epsilon t}, \underline{\eta}) = \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta} e^{i\epsilon t}) \quad (3.60c)$$

[Eqs. (3.4)–(3.6)], and the way it is transformed by the rotation operator,

$$R^\dagger(\theta) D(\underline{a}, \underline{v}) R(\theta) = D(e^{-i\theta \alpha_3} \underline{a}, \underline{v}) = D(\underline{a}, e^{i\theta \alpha_3} \underline{v}), \quad (3.61a)$$

$$\begin{aligned} R^\dagger(\theta) D(\underline{\alpha}_{r,\varphi}, \underline{v}_\alpha) R(\theta) &= D(e^{-i\theta \alpha_3} \underline{\alpha}_{r,\varphi+\theta}, \underline{v}_\alpha) \\ &= D(\underline{\alpha}_{r,\varphi+\theta}, e^{i\theta \alpha_3} \underline{v}_\alpha) \\ &= D(\underline{\alpha}_{r,\varphi}, e^{i\theta \alpha_3} \underline{C}_{r,\varphi-\theta} \underline{v}), \end{aligned} \quad (3.61b)$$

$$R^\dagger(\theta) \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) R(\theta) = \mathcal{D}(e^{i\theta \alpha_3} \underline{\mathcal{A}}, \underline{\eta}) = \mathcal{D}(\underline{\mathcal{A}}, e^{-i\theta \alpha_3} \underline{\eta}) \quad (3.61c)$$

[Eqs. (3.12), (3.14), (3.17), and (A27)]. The two-mode versions of Eq. (3.36) are

$$D^\dagger(\underline{a}, \underline{\mu})D(\underline{a}, \underline{\nu})D(\underline{a}, \underline{\mu}) = D(\underline{a} + \underline{\mu}, \underline{\nu}) = D(\underline{\mu}, \underline{\nu})D(\underline{a}, \underline{\nu}), \quad (3.62a)$$

$$D^\dagger(\underline{a}, \underline{\mu})\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta})D(\underline{a}, \underline{\mu}) = \mathcal{D}(\underline{\mathcal{A}} + \underline{\xi}, \underline{\eta}) = \mathcal{D}(\underline{\xi}, \underline{\eta})\mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) \quad (3.62b)$$

[Eqs. (3.41) and (3.42)]. The product of two displacement operators can be written in two useful forms:

$$D(\underline{a}, \underline{\mu}')D(\underline{a}, \underline{\mu}) = D^*(\frac{1}{2}\underline{\mu}', \underline{\mu})D(\underline{a}, \underline{\mu} + \underline{\mu}'), \quad (3.63a)$$

$$D^\dagger(\underline{a}, \underline{\mu}')D(\underline{a}, \underline{\mu}) = D(\frac{1}{2}\underline{\mu}', \underline{\mu})D(\underline{a}, \underline{\mu} - \underline{\mu}') \quad (3.63b)$$

[cf. Eq. (3.37)].

In the degenerate limit the two-mode displacement operator reduces to the single-mode displacement operator:

$$D(\underline{a}, \underline{\mu}) \xrightarrow{p} D(a, \mu), \quad \mu \equiv 2^{-1/2}(\mu_+ + \mu_-) \quad (3.64)$$

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$$S(r, \varphi) = (\cosh r)^{-1} e^{-a^\dagger_+ a^\dagger_- e^{2i\varphi} \tanh r} e^{-(a^\dagger_+ a_+ + a^\dagger_- a_-) \ln(\cosh r)} e^{a_+ a_- e^{-2i\varphi} \tanh r} \quad (3.66)$$

[Eq. (B9b)]. In Appendix B we derive Eq. (3.66) and other factored forms in which the exponentials appear in different orders. Also in Appendix B we use the factored forms to write the product of two squeeze operators as a rotation operator times a squeeze operator:

$$S^\dagger(r', \varphi')S(r, \varphi) = e^{-i\Theta} R(\Theta)S(R, \Phi) = e^{-i\Theta} S(R, \Phi - \Theta)R(\Theta) \quad (3.67)$$

[Eq. (B16)]; here  $R$ ,  $\Phi$ , and  $\Theta$  are defined by the matrix equation

$$\underline{C}_{R, \Phi} e^{i\Theta \underline{\alpha}_3} = \underline{C}_{r, \varphi} \underline{C}_{r', \varphi'}^{-1} = \underline{C}_{r, \varphi} \underline{C}_{-r', \varphi'} \quad (3.68)$$

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$$S_1(r, \varphi) = (\cosh r)^{-1/2} e^{-(a^\dagger)^2 e^{2i\varphi} (\tanh r)/2} e^{-a^\dagger a \ln(\cosh r)} e^{a^2 e^{-2i\varphi} (\tanh r)/2} \quad (3.70)$$

[cf. Eq. (3.66); for the derivation of this and other factored forms see Appendix B]. The product of two degenerate squeeze operators is given by

$$S^\dagger_1(r', \varphi')S_1(r, \varphi) = e^{-i\Theta/2} e^{-i\Theta a^\dagger a} S_1(R, \Phi) = e^{-i\Theta/2} S_1(R, \Phi - \Theta) e^{-i\Theta a^\dagger a} \quad (3.71)$$

[Eq. (B16); cf. Eq. (3.67)], where  $R$ ,  $\Phi$ , and  $\Theta$  are again defined by Eq. (3.68).

#### IV. SPECIAL QUANTUM STATES

Any discussion of special quantum states begins with the two-mode vacuum state  $|0\rangle$ , the state annihilated by  $a_+$  and  $a_-$  ( $a_\pm |0\rangle = 0$ ). A useful associated state is the

[Eq. (I.8.19c)].

#### D. Two-mode squeeze operator

The last important unitary operator in our formalism is the *two-mode squeeze operator*

$$S(r, \varphi) \equiv \exp[r(a_+ a_- e^{-2i\varphi} - a^\dagger_+ a^\dagger_- e^{2i\varphi})] = \exp[-ir \underline{a}^\dagger [\sigma_1 \sin(2\varphi) + \sigma_2 \cos(2\varphi)] \underline{a}] \quad (3.65)$$

[Eq. (1.8)]. It squeezes the annihilation operators to give the squeezed annihilation operators (1.10), and it generates two-mode squeezed states from coherent states (see Sec. IV B 1).

Almost all the important properties of  $S(r, \varphi)$  have been listed elsewhere in Secs. I–III. Little is left to note here, except two properties—factorization of the squeeze operator and the product of two different squeeze operators. The two-mode squeeze operator can be factored into a product of exponentials of  $a^\dagger_+ a_-$ ,  $a_+ a_-$ , and  $a^\dagger_+ a_+ + a^\dagger_- a_-$ . The most useful factored form is<sup>14,15</sup>

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[Eq. (B14)]. Notice that if  $\varphi = \varphi'$ , then  $\Theta = 0$ ,  $\Phi = \varphi$ , and  $R = r - r'$ , i.e.,  $S^\dagger(r', \varphi')S(r, \varphi) = S(r - r', \varphi)$  [cf. Eq. (A26)].

In the degenerate limit the two-mode squeeze operator becomes the *degenerate squeeze operator*  $S_1(r, \varphi)$  [Eq. (2.46)]:

$$S(r, \varphi) \xrightarrow{p} S_1(r, \varphi) = e^{(r/2)[a^2 e^{-2i\varphi} - (a^\dagger)^2 e^{2i\varphi}]} \quad (3.69)$$

[Eq. (I.8.19d)]. The degenerate squeeze operator can be factored in the same way as the two-mode squeeze operator. In particular, its most useful factored form is<sup>15–17</sup>

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(two-mode) *squeezed vacuum state*

$$|0\rangle_{(r, \varphi)} \equiv S(r, \varphi) |0\rangle, \quad (4.1)$$

which is the two-mode squeezed state (see Sec. IV B) with  $\langle a_\pm \rangle = 0$ . A convenient basis is provided by the (two-mode) *number eigenstates*

$$|n_+, n_-\rangle \equiv [(n_+!)(n_-!)]^{-1/2} (a^\dagger_+)^{n_+} (a^\dagger_-)^{n_-} |0\rangle, \quad (4.2)$$

$$a^\dagger_\pm a_\pm |n_+, n_-\rangle = n_\pm |n_+, n_-\rangle. \quad (4.3)$$

Another basis, unitarily equivalent to the number-eigenstate basis under the two-mode squeeze operator, consists of the (two-mode) *squeezed number eigenstates*

$$\begin{aligned} |n_+, n_-\rangle_{(r, \varphi)} &\equiv S(r, \varphi) |n_+, n_-\rangle \\ &= [(n_+!)(n_-!)]^{-1/2} (\alpha_+^\dagger)^{n_+} \\ &\quad \times (\alpha_-^\dagger)^{n_-} |0\rangle_{(r, \varphi)}, \end{aligned} \quad (4.4)$$

$$\alpha_\pm^\dagger |n_+, n_-\rangle_{(r, \varphi)} = n_\pm |n_+, n_-\rangle_{(r, \varphi)}. \quad (4.5)$$

### A. Coherent states

If the displacement operator is the heart of one-photon optics, then the soul is the set of states it generates—the coherent states. Here we review briefly some well-known properties<sup>3,13</sup> of coherent states for a single mode, generalize (trivially) those properties to two-mode coherent states, and write the two-mode properties in our vector notation.

#### 1. Single-mode coherent states

The *single-mode coherent states*<sup>3</sup> are generated from the single-mode vacuum state  $|0\rangle$  by the displacement operator:

$$|\mu\rangle_{\text{coh}} \equiv D(a, \mu) |0\rangle \quad (4.6)$$

[Eq. (I.3.9)]. Their most important property is that they are eigenstates of the annihilation operator:

$$a |\mu\rangle_{\text{coh}} = \mu |\mu\rangle_{\text{coh}} \quad (4.7)$$

[Eq. (3.21)]. Equation (3.38) can be used to obtain an expansion of  $|\mu\rangle_{\text{coh}}$  in terms of the single-mode number eigenstates  $|n\rangle \equiv (n!)^{-1/2} (a^\dagger)^n |0\rangle$ :

$$\begin{aligned} |\mu\rangle_{\text{coh}} &= e^{-|\mu|^2/2} e^{\mu a^\dagger} |0\rangle \\ &= e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{\mu^n}{(n!)^{1/2}} |n\rangle. \end{aligned} \quad (4.8)$$

A coherent state  $|\mu\rangle_{\text{coh}}$  has time-stationary noise [ $\langle(\Delta a)^2\rangle=0$ ; Eq. (I.3.12)]; its important nonzero first- and second-order moments are

$$\langle a \rangle = \mu, \quad \langle |\Delta a|^2 \rangle = \frac{1}{2}, \quad (4.9a)$$

$$\langle \alpha(r, \varphi) \rangle = \mu_\alpha, \quad (4.9b)$$

$$\langle \underline{x} \rangle = \underline{\xi}, \quad \langle (\Delta x_1)^2 \rangle = \langle (\Delta x_2)^2 \rangle = \frac{1}{2} \quad (4.9c)$$

(see Table II). The expectation value and variance of the number of quanta are

$$\langle a^\dagger a \rangle = |\mu|^2, \quad \langle [\Delta(a^\dagger a)]^2 \rangle = |\mu|^2. \quad (4.10)$$

The symmetrically ordered characteristic function for a coherent state  $|\mu\rangle_{\text{coh}}$  is the expectation value of the displacement operator,

$$\begin{aligned} \text{coh} \langle \mu | D(a, \nu) | \mu \rangle_{\text{coh}} &= e^{-|\nu|^2/2} D(\mu, \nu) \\ &= e^{-\nu^\dagger \nu/4} D(\mu, \nu) \end{aligned} \quad (4.11)$$

[Eq. (3.38)]. A Taylor expansion of Eq. (4.11) with respect to  $\nu$  and  $\nu^*$  yields the symmetrically ordered moments<sup>13</sup> of  $a$  and  $a^\dagger$ . Using Eqs. (3.29) and (3.35), one can write the characteristic function (4.11) in terms of the variables  $\eta_1$  and  $\eta_2$  (see Table II):

$$\begin{aligned} \text{coh} \langle \mu | \mathcal{D}(\underline{x}, \underline{\eta}) | \mu \rangle_{\text{coh}} &= e^{(\eta_1^\dagger + \eta_2^\dagger)/4} \mathcal{D}(\underline{\xi}, \underline{\eta}) \\ &= e^{-\eta^\dagger \eta/4} \mathcal{D}(\underline{\xi}, \underline{\eta}). \end{aligned} \quad (4.12)$$

A Taylor expansion of Eq. (4.12) with respect to  $\eta_1$  and  $\eta_2$  yields the symmetrically ordered moments of  $x_1$  and  $x_2$ . Characteristic functions will be considered in detail in paper III.

The coherent states are not orthonormal,

$$\begin{aligned} \text{coh} \langle \mu' | \mu \rangle_{\text{coh}} &= \langle 0 | D^\dagger(a, \mu') D(a, \mu) | 0 \rangle \\ &= D(\frac{1}{2} \mu', \mu) e^{-|\mu - \mu'|^2/2} \end{aligned} \quad (4.13)$$

[Eqs. (3.37) and (4.11)], but they are complete in the following sense, pointed out by Klauder:<sup>19</sup>

$$1 = \int \frac{d^2 \mu}{\pi} |\mu\rangle_{\text{coh}} \text{coh} \langle \mu|. \quad (4.14)$$

The completeness relation (4.14) is the starting point for developing expansions in terms of the coherent states. It can also be used to demonstrate that the trace of an operator  $f$  is given by

$$\text{tr} f = \int \frac{d^2 \mu}{\pi} \text{coh} \langle \mu | f | \mu \rangle_{\text{coh}}, \quad (4.15)$$

which in turn shows that

$$\text{tr}[D(a, \nu)] = e^{-|\nu|^2/2} \int \frac{d^2 \mu}{\pi} D(\mu, \nu) = \pi \delta^2(\nu) \quad (4.16)$$

[Eqs. (4.11) and (3.33)].

#### 2. Two-mode coherent states

A *two-mode coherent state*<sup>3</sup> [Eq. (I.4.11)] is generated from the vacuum state by the two-mode displacement operator:

$$|\underline{\mu}\rangle_{\text{coh}} \equiv |\mu_+, \mu_-\rangle_{\text{coh}} \equiv D(\underline{a}, \underline{\mu}) |0\rangle \quad (4.17)$$

[cf. Eq. (4.6)]. It is an eigenstate of both  $a_+$  and  $a_-$ , i.e.,

$$a_\pm |\underline{\mu}\rangle_{\text{coh}} = \mu_\pm |\underline{\mu}\rangle_{\text{coh}} \quad (4.18)$$

[cf. Eq. (4.7)], and its number-eigenstate expansion is given by

$$\begin{aligned} |\underline{\mu}\rangle_{\text{coh}} &= e^{-\mu_+^\dagger \mu_+/2} e^{\mu_+ a_+^\dagger} e^{-\mu_-^\dagger \mu_-/2} e^{\mu_- a_-^\dagger} |0\rangle \\ &= e^{-\mu_+^\dagger \mu_+/2} \sum_{n_+, n_-} \frac{(\mu_+)^{n_+} (\mu_-)^{n_-}}{[(n_+!)(n_-!)]^{1/2}} |n_+, n_-\rangle \end{aligned} \quad (4.19)$$

[cf. Eq. (4.8)]. A coherent state  $|\underline{\mu}\rangle_{\text{coh}}$  has time-stationary noise [Eqs. (I.5.6) and (I.5.9)]; its nonzero first- and second-order moments are

$$\langle \underline{a} \rangle = \underline{\mu}, \quad \overline{\Sigma}_{\text{coh}} = \langle \Delta \underline{a} \Delta \underline{a}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \underline{1}, \quad (4.20a)$$

$$\langle \underline{\alpha}_{r, \varphi} \rangle = \underline{\mu}_\alpha, \quad \langle \Delta \underline{\alpha}_{r, \varphi} \Delta \underline{\alpha}_{r, \varphi}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \underline{C}_{2r, \varphi}, \quad (4.20b)$$

$$\begin{aligned} \langle \underline{\mathcal{A}} \rangle &= \underline{\xi}, \quad \overline{\Sigma}_{\text{coh}} = \langle \Delta \underline{\mathcal{A}} \Delta \underline{\mathcal{A}}^\dagger \rangle_{\text{sym}} \\ &= \frac{1}{2} \underline{\Delta} = \frac{1}{2} [\underline{1} - (\epsilon/\Omega) \underline{g}_2] \end{aligned} \quad (4.20c)$$

[Eqs. (2.9), (2.13), (2.15), and (2.16); cf. Eqs. (4.9), (I.7.2), and (I.7.3)].

For a coherent state  $|\underline{\mu}\rangle_{\text{coh}}$  the expectation value and variance of the free Hamiltonian (1.2) are

$$\langle H_0 \rangle = \Omega \underline{\mu}^\dagger \underline{\lambda}^2 \underline{\mu} = \Omega \underline{\xi}^\dagger \underline{\xi}, \quad (4.21a)$$

$$\langle (\Delta H_0)^2 \rangle = \Omega^2 \underline{\mu}^\dagger \underline{\lambda}^4 \underline{\mu} = \Omega^2 \underline{\xi}^\dagger \Delta \underline{\xi} \quad (4.21b)$$

[Eqs. (2.32) and (2.9)]. Equations (4.21) follow easily from the single-mode expectation value and variance of the number of quanta [Eq. (4.10)], but they can also be obtained from the following rules. Let  $\underline{M}$  be any two-dimensional matrix

$$\underline{M} \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \quad (4.22)$$

One wants to evaluate the expectation value and variance of the quadratic form

$$\underline{a}^\dagger \underline{M} \underline{a} = M_{11} a_+^\dagger a_- + M_{22} a_-^\dagger a_+ + M_{12} a_+^\dagger a_- + M_{21} a_-^\dagger a_+ + M_{22}. \quad (4.23)$$

It is an easy task to show that

$$\text{coh} \langle \underline{\mu} | \underline{a}^\dagger \underline{M} \underline{a} | \underline{\mu} \rangle_{\text{coh}} = \underline{\mu}^\dagger \underline{M} \underline{\mu} + M_{22}, \quad (4.24a)$$

$$\text{coh} \langle \underline{\mu} | [\Delta(\underline{a}^\dagger \underline{M} \underline{a})]^2 | \underline{\mu} \rangle_{\text{coh}} = \underline{\mu}^\dagger \underline{M}^2 \underline{\mu} + M_{12} M_{21}. \quad (4.24b)$$

By using Eq. (2.8), one can obtain Eqs. (4.21) directly from Eqs. (4.24).

The symmetrically ordered characteristic function for a two-mode coherent state  $|\underline{\mu}\rangle_{\text{coh}}$  is obtained easily from the analogous single-mode quantity [Eq. (4.11)]:

$$\begin{aligned} \text{coh} \langle \underline{\mu} | D(\underline{a}, \underline{v}) | \underline{\mu} \rangle_{\text{coh}} &= e^{-\underline{v}^\dagger \underline{v}/2} D(\underline{\mu}, \underline{v}) \\ &= e^{-\underline{\eta}^\dagger \Delta \underline{\eta}/2} \mathcal{D}(\underline{\xi}, \underline{\eta}) \\ &= \text{coh} \langle \underline{\mu} | \mathcal{D}(\underline{a}, \underline{\eta}) | \underline{\mu} \rangle_{\text{coh}} \end{aligned} \quad (4.25)$$

[Eqs. (2.33), (3.50), and (3.57)]. In Eq. (4.25) we write the characteristic function first in terms of the vector variable  $\underline{v}$  and then in terms of the vector variable  $\underline{\eta}$ .

The two-mode coherent states are not orthonormal:

$$\text{coh} \langle \underline{\mu}' | \underline{\mu} \rangle_{\text{coh}} = D(\frac{1}{2} \underline{\mu}', \underline{\mu}) e^{-\frac{1}{2} \underline{\mu}'^\dagger \underline{\mu}' - \frac{1}{2} \underline{\mu}^\dagger \underline{\mu}} \quad (4.26)$$

[Eqs. (3.63b) and (4.25); cf. Eq. (4.13)]. They do, however, satisfy a completeness relation which follows trivially from the single-mode completeness relation (4.14):

$$1 = \int \frac{d^4 \underline{\mu}}{\pi^2} |\underline{\mu}\rangle_{\text{coh}} \text{coh} \langle \underline{\mu} |. \quad (4.27)$$

Hence the trace of an operator  $f$  is given by

$$\text{tr} f = \int \frac{d^4 \underline{\mu}}{\pi^2} \text{coh} \langle \underline{\mu} | f | \underline{\mu} \rangle_{\text{coh}}, \quad (4.28)$$

and the trace of  $D(\underline{a}, \underline{v})$  is

$$\text{tr}[D(\underline{a}, \underline{v})] = e^{-\underline{v}^\dagger \underline{v}/2} \int \frac{d^4 \underline{\mu}}{\pi^2} D(\underline{\mu}, \underline{v}) = \pi^2 \delta^4(\underline{v}) \quad (4.29)$$

[Eqs. (4.25) and (3.59a)].

In the MP a two-mode coherent state evolves freely in

the following way:

$$U_M(t) |\underline{\mu}\rangle_{\text{coh}} = |\underline{\mu} e^{-i\epsilon t}\rangle_{\text{coh}} = |\underline{\mu}_+ e^{-i\epsilon t}, \underline{\mu}_- e^{-i\epsilon t}\rangle_{\text{coh}} \quad (4.30)$$

[Eq. (3.60a)]. A rotation transforms a coherent state according to

$$R(\theta) |\underline{\mu}\rangle_{\text{coh}} = |e^{-i\theta \alpha} \underline{\mu}\rangle_{\text{coh}} = |\mu_+ e^{-i\theta}, \mu_- e^{-i\theta}\rangle_{\text{coh}} \quad (4.31)$$

[Eq. (3.61a)]. Combining Eqs. (4.30) and (4.31) yields the SP free evolution

$$\begin{aligned} e^{-iH_0 t} |\underline{\mu}\rangle_{\text{coh}} &= |e^{-i(\Omega \alpha + \epsilon)t} \underline{\mu}\rangle_{\text{coh}} \\ &= |\mu_+ e^{-i(\Omega + \epsilon)t}, \mu_- e^{-i(\Omega - \epsilon)t}\rangle_{\text{coh}}. \end{aligned} \quad (4.32)$$

In the degenerate limit a two-mode coherent state reduces to a single-mode coherent state:

$$|\underline{\mu}\rangle_{\text{coh}} \xrightarrow{\rho} |\mu\rangle_{\text{coh}}, \quad \mu \equiv 2^{-1/2}(\mu_+ + \mu_-) \quad (4.33)$$

[Eq. (I.8.22)].

## B. Squeezed states

### 1. Two-mode squeezed states

The most important states in two-photon optics—the states produced by an ideal two-photon device (see Sec. IV A of I)—are the *two-mode squeezed states*, which can be defined by

$$\begin{aligned} |\underline{\mu}_\alpha\rangle_{(r,\varphi)} &\equiv |\mu_{\alpha+}, \mu_{\alpha-}\rangle_{(r,\varphi)} \\ &\equiv D(\underline{a}, \underline{\mu}) S(r, \varphi) |0\rangle \end{aligned} \quad (4.34)$$

[Eq. (I.4.17)]. Using Eqs. (3.46) and (4.1), one can write  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  in the form

$$|\underline{\mu}_\alpha\rangle_{(r,\varphi)} = D(\underline{a}, \underline{\mu}) |0\rangle_{(r,\varphi)} = D(\underline{a}, \underline{\mu}_\alpha) |0\rangle_{(r,\varphi)}; \quad (4.35)$$

hence, a two-mode squeezed state can be obtained by applying the "squeezed" displacement operator  $D(\underline{a}, \underline{\mu}_\alpha) = S(r, \varphi) D(\underline{a}, \underline{\mu}_\alpha) S^\dagger(r, \varphi)$  to the squeezed vacuum. Using Eq. (3.47) in the definition of  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$ , one shows that a two-mode squeezed state can be generated by applying  $S(r, \varphi)$  to a two-mode coherent state:

$$\begin{aligned} |\underline{\mu}_\alpha\rangle_{(r,\varphi)} &= S(r, \varphi) D(\underline{a}, \underline{\mu}_\alpha) |0\rangle \\ &= S(r, \varphi) |\underline{\mu}_\alpha\rangle_{\text{coh}} \end{aligned} \quad (4.36)$$

[Eq. (I.4.15)]. Notice that  $|\underline{\mu}_\alpha\rangle_{(0,\varphi)} = |\underline{\mu}_\alpha\rangle_{\text{coh}}$ . The unitary equivalence between squeezed states and coherent states is a powerful tool for generating properties of the two-mode squeezed states. For example, using Eqs. (4.18) and (4.36), one can tell immediately that  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  is an eigenstate of the squeezed annihilation operators (1.10):

$$\alpha_\pm(r, \varphi) |\underline{\mu}_\alpha\rangle_{(r,\varphi)} = \mu_{\alpha\pm} |\underline{\mu}_\alpha\rangle_{(r,\varphi)}. \quad (4.37)$$

As another example, one can use Eqs. (4.19) and (4.36) to obtain an expansion of  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  in terms of the squeezed number eigenstates (4.4):

$$\begin{aligned} |\underline{\mu}_\alpha\rangle_{(r,\varphi)} &= e^{-\underline{\mu}_\alpha^\dagger \underline{\mu}_\alpha / 2} e^{\underline{\mu}_\alpha + \alpha_+^\dagger} e^{\underline{\mu}_\alpha - \alpha_-^\dagger} |0\rangle_{(r,\varphi)} \\ &= e^{-\underline{\mu}_\alpha^\dagger \underline{\mu}_\alpha / 2} \sum_{n_+, n_-} \frac{(\mu_{\alpha_+})^{n_+} (\mu_{\alpha_-})^{n_-}}{[(n_+!)(n_-!)]^{1/2}} |n_+, n_-\rangle_{(r,\varphi)}. \end{aligned} \quad (4.38)$$

The expansion of  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  in terms of the number eigenstates  $|n_+, n_-\rangle$  is, in general, quite complicated, yet neither interesting nor enlightening. It does, however, have a simple form for the squeezed vacuum  $|0\rangle_{(r,\varphi)}$ —a form obtained by using the factorization (3.66):

$$\begin{aligned} S(r,\varphi)|0\rangle &= (\cosh r)^{-1} e^{-\alpha_+^\dagger \alpha_+^\dagger e^{2i\varphi} \tanh r} |0\rangle \\ &= (\cosh r)^{-1} \sum_{n=0}^{\infty} (-e^{2i\varphi} \tanh r)^n |n, n\rangle. \end{aligned} \quad (4.39)$$

Thus the squeezed vacuum is a superposition of number eigenstates which have equal numbers of quanta in the  $\alpha_+$  mode and the  $\alpha_-$  mode.

The two-mode squeezed state  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  has TSQP noise [Eqs. (I.5.1)] and (I.5.6). Its nonzero first- and second-order moments are most easily obtained by noting that Eqs. (4.20a) and (4.36) imply

$$\langle \underline{\alpha}_{r,\varphi} \rangle = \underline{\mu}_\alpha, \quad \langle \Delta \underline{\alpha}_{r,\varphi} \Delta \underline{\alpha}_{r,\varphi}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \mathbb{1}. \quad (4.40)$$

One can obtain the moments of the creation and annihilation operators and the quadrature-phase amplitudes simply by making matrix transformations of Eqs. (4.40):

$$\begin{aligned} \langle \underline{a} \rangle &= \underline{\mu}, \quad \langle \Delta \underline{a} \Delta \underline{a}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \underline{C}_{-2r,\varphi}, \quad (4.41a) \\ \langle \underline{\mathcal{A}} \rangle &= \underline{\xi}, \quad \langle \Delta \underline{\mathcal{A}} \Delta \underline{\mathcal{A}}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \underline{d} \underline{\lambda} \underline{C}_{-2r,\varphi} \underline{\lambda} \underline{d}^\dagger \end{aligned} \quad (4.41b)$$

[cf. Eqs. (4.20)]. Using Eqs. (A.25), (A.17)–(A.19), and (A.4)–(A.6), one can expand  $\underline{\Sigma}_{r,\varphi}$  and  $\underline{\Xi}_{r,\varphi}$  as

$$\begin{aligned} \underline{\Xi}_{r,\varphi} &= \frac{1}{2} \mathbb{1} \cosh(2r) \\ &\quad - \frac{1}{2} [\underline{\sigma}_1 \cos(2\varphi) - \underline{\sigma}_2 \sin(2\varphi)] \sinh(2r), \quad (4.42a) \\ \underline{\Sigma}_{r,\varphi} &= \frac{1}{2} \mathbb{1} \cosh(2r) - \frac{1}{2} (1 - \epsilon^2 / \Omega^2)^{1/2} \underline{\sigma}_3 \sinh(2r) \cos(2\varphi) \\ &\quad - \frac{1}{2} (1 - \epsilon^2 / \Omega^2)^{1/2} \underline{\sigma}_1 \sinh(2r) \sin(2\varphi) \\ &\quad - \frac{1}{2} (\epsilon / \Omega) \underline{\sigma}_2 \cosh(2r) \end{aligned} \quad (4.42b)$$

[cf. Eqs. (I.7.8) and (I.7.9)].

The rules (4.24), which give the expectation value and variance of an arbitrary quadratic form  $\underline{a}^\dagger \underline{M} \underline{a}$  with respect to a coherent state, can easily be generalized to squeezed states, once again by using Eq. (4.36):

$$\langle \underline{\mu}_\alpha | \underline{a}^\dagger \underline{M} \underline{a} | \underline{\mu}_\alpha \rangle_{(r,\varphi)} = \underline{\mu}_\alpha^\dagger \underline{M} \underline{\mu}_\alpha + \overline{M}_{22} = \underline{\mu}_\alpha^\dagger \underline{M} \underline{\mu}_\alpha + \overline{M}_{22}, \quad (4.43a)$$

$$\begin{aligned} \langle \underline{\mu}_\alpha | [\Delta(\underline{a}^\dagger \underline{M} \underline{a})]^2 | \underline{\mu}_\alpha \rangle_{(r,\varphi)} \\ &= \underline{\mu}_\alpha^\dagger \underline{M}^2 \underline{\mu}_\alpha + \overline{M}_{12} \overline{M}_{21} \\ &= \underline{\mu}_\alpha^\dagger \underline{M} \underline{C}_{-2r,\varphi} \underline{M} \underline{\mu}_\alpha + \overline{M}_{12} \overline{M}_{21}, \end{aligned} \quad (4.43b)$$

$$\overline{M} \equiv \underline{C}_{-r,\varphi} \underline{M} \underline{C}_{-r,\varphi}. \quad (4.44)$$

Applying Eqs. (4.43) to Eq. (2.8) yields the expectation value and variance of the free Hamiltonian with respect to  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$ :

$$\langle H_0 \rangle = \Omega \underline{\mu}_\alpha^\dagger \underline{\lambda}^2 \underline{\mu}_\alpha + 2\Omega \sinh^2 r = \Omega \underline{\xi}^\dagger \underline{\xi} + 2\Omega \sinh^2 r, \quad (4.45a)$$

$$\begin{aligned} \langle (\Delta H_0)^2 \rangle &= \Omega^2 \underline{\mu}_\alpha^\dagger \underline{\lambda}^2 \underline{C}_{-2r,\varphi} \underline{\lambda}^2 \underline{\mu}_\alpha + \Omega^2 \sinh^2(2r) \\ &= 2\Omega^2 \underline{\xi}^\dagger \underline{\Sigma}_{r,\varphi} \underline{\xi} + \Omega^2 \sinh^2(2r) \end{aligned} \quad (4.45b)$$

[Eq. (4.41b); cf. Eqs. (4.21)]. The reader should note the appearance of  $\underline{\Sigma}_{r,\varphi}$  in Eq. (4.45b). Its presence there is no accident: for any state the highest-order term in the expression for  $\langle (\Delta H_0)^2 \rangle$  is quadratic in the mean quadrature-phase amplitudes; for any state with TSQP noise [Eq. (I.5.1)], it is easy to demonstrate that the highest-order term is given by  $2\Omega^2 \underline{\xi}^\dagger \underline{\Sigma} \underline{\xi}$ , where  $\underline{\xi} = \langle \underline{\mathcal{A}} \rangle$ .

The symmetrically ordered characteristic function for a two-mode squeezed state can be obtained immediately from Eqs. (3.47), (4.25), and (4.36):

$$\begin{aligned} \langle \underline{\mu}_\alpha | \underline{D}(\underline{a}, \underline{y}) | \underline{\mu}_\alpha \rangle_{(r,\varphi)} &= \text{coth} \langle \underline{\mu}_\alpha | \underline{D}(\underline{a}, \underline{y}_\alpha) | \underline{\mu}_\alpha \rangle_{\text{coh}} \\ &= e^{-\underline{y}^\dagger \underline{\Sigma}_{r,\varphi} \underline{y} / 2} \underline{D}(\underline{\mu}_\alpha, \underline{y}_\alpha). \end{aligned} \quad (4.46)$$

This result can be transformed so that the characteristic function is written in terms of  $\underline{y} = \underline{C}_{r,\varphi} \underline{y}_\alpha$  or  $\underline{\eta} = \underline{d} \underline{\lambda}^{-1} \underline{\sigma}_3 \underline{y}$ :

$$\begin{aligned} \langle \underline{\mu}_\alpha | \underline{D}(\underline{a}, \underline{y}) | \underline{\mu}_\alpha \rangle_{(r,\varphi)} &= e^{-\underline{y}^\dagger \underline{\Sigma}_{r,\varphi} \underline{y} / 2} \underline{D}(\underline{\mu}_\alpha, \underline{y}) \\ &= e^{-\underline{\eta}^\dagger \underline{\Sigma}_{r,\varphi} \underline{\eta}} \mathcal{D}(\underline{\xi}, \underline{\eta}) \\ &= \langle \underline{\mu}_\alpha | \underline{D}(\underline{\mathcal{A}}, \underline{\eta}) | \underline{\mu}_\alpha \rangle_{(r,\varphi)} \end{aligned} \quad (4.47)$$

[Eqs. (3.50), (3.57), and (A.24)]. Notice the presence of  $\underline{\Sigma}_{r,\varphi}$  [Eq. (4.41b)] in the expression for the characteristic function. Its presence signals the fact that a two-mode squeezed state has Gaussian TSQP noise: the noise moments of arbitrary order are determined by the second-order noise moments contained in  $\underline{\Sigma}_{r,\varphi}$ .

The two-mode squeezed states are not orthonormal. The inner product of two squeezed states with the same  $r$  and  $\varphi$  is given by

$$\begin{aligned} \langle \underline{\mu}'_\alpha | \underline{\mu}_\alpha \rangle_{(r,\varphi)} &= \text{coth} \langle \underline{\mu}'_\alpha | \underline{\mu}_\alpha \rangle_{\text{coh}} \\ &= \underline{D}(\frac{1}{2} \underline{\mu}'_\alpha, \underline{\mu}_\alpha) e^{-\underline{\mu}'_\alpha - \underline{\mu}_\alpha^\dagger / 2} e^{(\underline{\mu}'_\alpha - \underline{\mu}_\alpha^\dagger) / 2} \\ &= \underline{D}(\frac{1}{2} \underline{\mu}', \underline{\mu}) e^{-\underline{(\mu}' - \underline{\mu})^\dagger \underline{\Sigma}_{r,\varphi} (\underline{\mu}' - \underline{\mu}) / 2} \end{aligned} \quad (4.48)$$

[Eq. (4.26)]. For squeezed states with different  $r$  and/or  $\varphi$ , the inner product is considerably more complicated than Eq. (4.48); it is derived in Appendix C. The set of two-mode squeezed states with a particular  $r$  and  $\varphi$  does satisfy a completeness relation

$$1 = \int \frac{d^4 \underline{\mu}_\alpha}{\pi^2} |\underline{\mu}_\alpha\rangle_{(r,\varphi)} \langle \underline{\mu}_\alpha|, \quad (4.49)$$

which is just a unitary transformation of the completeness

relation (4.27) for coherent states. Equation (4.49) allows us to write the trace of an operator  $f$  as an integral over squeezed states with the same  $r$  and  $\varphi$ :

$$\text{tr}f = \int \frac{d^4\mu_\alpha}{\pi^2} \langle \underline{\mu}_\alpha | f | \underline{\mu}_\alpha \rangle_{(r,\varphi)}. \quad (4.50)$$

The MP free evolution of  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  is given by

$$\begin{aligned} U_M(t) |\underline{\mu}_\alpha\rangle_{(r,\varphi)} &= |\underline{\mu}_\alpha e^{-i\epsilon t}\rangle_{(r,\varphi)} \\ &= |\mu_{\alpha+} e^{-i\epsilon t}, \mu_{\alpha-} e^{i\epsilon t}\rangle_{(r,\varphi)}. \end{aligned} \quad (4.51)$$

[Eqs. (3.7) and (4.30)]; under a rotation  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  transforms according to

$$\begin{aligned} R(\theta) |\underline{\mu}_\alpha\rangle_{(r,\varphi)} &= |e^{-i\theta\alpha}; \underline{\mu}_\alpha\rangle_{(r,\varphi-\theta)} \\ &= |\mu_{\alpha+} e^{-i\theta}, \mu_{\alpha-} e^{-i\theta}\rangle_{(r,\varphi-\theta)}. \end{aligned} \quad (4.52)$$

[Eqs. (3.16) and (4.31)]. Equations (4.51) and (4.52) together give the SP free evolution

$$\begin{aligned} e^{-iH_0 t} |\underline{\mu}_\alpha\rangle_{(r,\varphi)} &= |e^{-i(\Omega\alpha_3 + \epsilon)t}; \underline{\mu}_\alpha\rangle_{(r,\varphi-\Omega t)} \\ &= |\mu_{\alpha+} e^{-i(\Omega + \epsilon)t}, \mu_{\alpha-} e^{-i(\Omega - \epsilon)t}\rangle_{(r,\varphi-\Omega t)}. \end{aligned} \quad (4.53)$$

[cf. Eq. (4.32)].

In a separate paper one of us (BLS) considers the wave functions for two-mode squeezed states in the usual coordinate and momentum representations.<sup>20</sup>

## 2. Degenerate squeezed states

In the degenerate limit a two-mode squeezed state  $|\underline{\mu}_\alpha\rangle_{(r,\varphi)}$  becomes a *degenerate squeezed state*<sup>9,21,11</sup>

$$|\underline{\mu}_\alpha\rangle_{(r,\varphi)} \rightarrow |\mu_\alpha\rangle_{(r,\varphi)} \equiv D(a, \mu_\alpha) S_1(r, \varphi) |0\rangle, \quad (4.54a)$$

$$\mu_\alpha \equiv 2^{-1/2}(\mu_{\alpha+} + \mu_{\alpha-}) = \mu \cosh r + \mu^* e^{2i\varphi} \sinh r \quad (4.54b)$$

[Eqs. (I.8.23)]. Equation (3.27) can be used to show that  $S_1(r, \varphi)$  transforms a single-mode coherent state into a degenerate squeezed state:

$$|\mu_\alpha\rangle_{(r,\varphi)} = S_1(r, \varphi) D(a, \mu_\alpha) |0\rangle = S_1(r, \varphi) |\mu_\alpha\rangle_{\text{coh}}. \quad (4.55)$$

A degenerate squeezed state is an eigenstate of the squeezed annihilation operator (2.48):

$$\alpha(r, \varphi) |\mu_\alpha\rangle_{(r,\varphi)} = \mu_\alpha |\mu_\alpha\rangle_{(r,\varphi)} \quad (4.56)$$

[cf. Eq. (4.37)].

The properties of degenerate squeezed states can be derived in the same way as the properties of two-mode squeezed states. Here we content ourselves with providing a list of properties of the state  $|\mu_\alpha\rangle_{(r,\varphi)}$ . Above each equation in the list we give the equation number of the analogous two-mode property. All the results in the list, except Eqs. (4.62)–(4.65), can be found in Yuen's comprehensive paper<sup>11</sup> on "two-photon coherent states;" some of the results are also given in Refs. 9, 21, and 16.

Many of the properties are most conveniently stated in terms of the single-mode vector notation introduced in Table II. The list of properties is as follows:

Eq. (4.39):

$$\begin{aligned} S_1(r, \varphi) |0\rangle &= (\cosh r)^{-1/2} e^{-\frac{1}{2}e^{2i\varphi} \tanh r} |0\rangle \\ &= (\cosh r)^{-1/2} \\ &\quad \times \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{n!} \left(-\frac{1}{2}e^{2i\varphi} \tanh r\right)^n |2n\rangle, \end{aligned} \quad (4.57)$$

Eq. (4.40):

$$\langle \alpha(r, \varphi) \rangle = \mu_\alpha, \quad \langle \Delta \underline{\alpha}_{r,\varphi} \Delta \underline{\alpha}_{r,\varphi}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \underline{1}, \quad (4.58)$$

Eq. (4.41a):

$$\langle a \rangle = \mu, \quad \underline{\Sigma}_{r,\varphi} \equiv \langle \Delta \underline{a} \Delta \underline{a}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \underline{C}_{-2r,\varphi}, \quad (4.59a)$$

Eq. (4.41b):

$$\langle \underline{x} \rangle = \underline{\xi}, \quad \underline{\Sigma}_{r,\varphi} \equiv \langle \Delta \underline{x} \Delta \underline{x}^\dagger \rangle_{\text{sym}} = \frac{1}{2} \underline{d} \underline{C}_{-2r,\varphi} \underline{d}^\dagger, \quad (4.59b)$$

Eqs. (I.7.3):

$$\langle |\Delta \underline{a}|^2 \rangle = \frac{1}{2} \cosh(2r), \quad (4.60)$$

$$\langle (\Delta \underline{a})^2 \rangle = -\frac{1}{2} e^{2i\varphi} \sinh(2r),$$

Eq. (4.42b):

$$\begin{aligned} \underline{\Sigma}_{r,\varphi} &= \frac{1}{2} \underline{1} \cosh(2r) - \frac{1}{2} \underline{\sigma}_3 \sinh(2r) \cos(2\varphi) \\ &\quad - \frac{1}{2} \underline{\sigma}_1 \sinh(2r) \sin(2\varphi), \end{aligned} \quad (4.61)$$

Eq. (4.22):

$$\underline{M} \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_{11} = M_{22}, \quad (4.62)$$

Eq. (4.44):

$$\underline{M} \equiv \underline{C}_{-r,\varphi} \underline{M} \underline{C}_{-r,\varphi}, \quad \overline{M}_{11} = \overline{M}_{22}, \quad (4.63)$$

Eq. (4.23):

$$\frac{1}{2} \underline{a}^\dagger \underline{M} \underline{a} = M_{11} a^\dagger a + \frac{1}{2} M_{12} (a^\dagger)^2 + \frac{1}{2} M_{21} a^2 + \frac{1}{2} M_{11}, \quad (4.64)$$

Eq. (4.43a):

$${}_{(r,\varphi)} \langle \mu_\alpha | \frac{1}{2} \underline{a}^\dagger \underline{M} \underline{a} | \mu_\alpha \rangle_{(r,\varphi)} = \frac{1}{2} \underline{\mu}^\dagger \underline{M} \underline{\mu} + \frac{1}{2} \overline{M}_{11}, \quad (4.65a)$$

Eq. (4.43b):

$$\begin{aligned} {}_{(r,\varphi)} \langle \mu_\alpha | [\Delta(\frac{1}{2} \underline{a}^\dagger \underline{M} \underline{a})]^2 | \mu_\alpha \rangle_{(r,\varphi)} &= \frac{1}{2} \underline{\mu}^\dagger \underline{M} \underline{C}_{-2r,\varphi} \underline{M} \underline{\mu} \\ &\quad + \frac{1}{2} \overline{M}_{12} \overline{M}_{21}, \end{aligned} \quad (4.65b)$$

Eq. (4.45a):

$$\langle a^\dagger a \rangle = |\mu|^2 + \sinh^2 r = \frac{1}{2} \underline{\xi}^\dagger \underline{\xi} + \sinh^2 r, \quad (4.66a)$$



Eq. (4.45b):

$$\begin{aligned} \langle [\Delta(a^\dagger a)]^2 \rangle &= \frac{1}{2} \mu^\dagger C_{-2r, \varphi} \mu + \frac{1}{2} \sinh^2(2r) \\ &= \xi^\dagger \Sigma_{r, \varphi} \xi + \frac{1}{2} \sinh^2(2r), \end{aligned} \quad (4.66b)$$

Eq. (4.46):

$$\begin{aligned} {}_{(r, \varphi)} \langle \mu_\alpha | D(a, \nu) | \mu_\alpha \rangle_{(r, \varphi)} &= \text{coh} \langle \mu_\alpha | D(a, \nu_\alpha) | \mu_\alpha \rangle_{\text{coh}} \\ &= e^{-|\nu_\alpha|^2/2} D(\mu_\alpha, \nu_\alpha) = e^{-\nu_\alpha^\dagger \nu_\alpha/4} D(\mu_\alpha, \nu_\alpha), \end{aligned} \quad (4.67)$$

Eq. (4.47):

$$\begin{aligned} {}_{(r, \varphi)} \langle \mu_\alpha | D(a, \nu) | \mu_\alpha \rangle_{(r, \varphi)} &= e^{-\nu^\dagger C_{2r, \varphi} \nu/4} D(\mu, \nu) \\ &= e^{-\eta^\dagger \Sigma_{r, \varphi} \eta/2} \mathcal{D}(\underline{\xi}, \underline{\eta}) \\ &= {}_{(r, \varphi)} \langle \mu_\alpha | \mathcal{D}(\underline{x}, \underline{\eta}) | \mu_\alpha \rangle_{(r, \varphi)}, \end{aligned} \quad (4.68)$$

Eq. (4.48):

$$\begin{aligned} {}_{(r, \varphi)} \langle \mu'_\alpha | \mu_\alpha \rangle_{(r, \varphi)} &= \text{coh} \langle \mu'_\alpha | \mu_\alpha \rangle_{\text{coh}} \\ &= D(\frac{1}{2} \mu'_\alpha, \mu_\alpha) e^{-i \mu_\alpha - \mu'_\alpha|^2/2} \\ &= D(\frac{1}{2} \mu', \mu) e^{-i \mu - \mu'}^\dagger C_{2r, \varphi} (\mu - \mu')/4}, \end{aligned} \quad (4.69)$$

Eq. (4.49):

$$1 = \int \frac{d^2 \mu_\alpha}{\pi} | \mu_\alpha \rangle_{(r, \varphi)} {}_{(r, \varphi)} \langle \mu_\alpha |. \quad (4.70)$$

### V. CONCLUSION

This concluding section is a good place to recapitulate the key ideas behind papers I and II and to hint at what lies ahead. The goal of this series of papers is to develop a formalism suited to the analysis of two-photon devices. The crucial feature of a two-photon device is that its output consists of pairs of simultaneously emitted photons. Hence the starting point for our formalism is a pair of electromagnetic-field modes which are excited by emission of a pair of photons. The natural variables for describing the excitation of these two modes are the quadrature-phase amplitudes, and the natural quantum states are the two-mode squeezed states—the states generated by an ideal two-photon device. These basic building blocks were the focus of paper I, where our objective was to develop a physical understanding of the quadrature-phase amplitudes and the two-mode squeezed states. In the present paper we have described the mathematical structure of the formalism and developed techniques for manipulating its fundamental components. We introduced a vector notation which simplifies the mathematical description and makes it easy to learn and use the language of the quadrature-phase amplitudes. The vector notation also provides quick translation into the conventional language of creation and annihilation operators.

An important feature of the vector notation—built into

it right at the start—is that it recognizes the quadrature-phase amplitudes as the fundamental variables and, hence, it naturally associates  $a_+$  with  $a_-^\dagger$ . This feature has profound consequences for the operator orderings that are preferred in two-photon optics. One natural ordering for the quadrature-phase amplitudes and their Hermitian conjugates places  $\alpha_1^\dagger$  and  $\alpha_2^\dagger$  to the left of  $\alpha_1$  and  $\alpha_2$  (recall that  $[\alpha_1, \alpha_2] = 0$ ); this kind of ordering, which we call *quadrature-phase normal ordering*, is equivalent to normal ordering of the  $a_+$  mode and antinormal ordering of the  $a_-$  mode. Another natural ordering, which we call *quadrature-phase antinormal ordering*, places  $\alpha_1$  and  $\alpha_2$  to the left of  $\alpha_1^\dagger$  and  $\alpha_2^\dagger$ ; it is equivalent to antinormal ordering of the  $a_+$  mode and normal ordering of the  $a_-$  mode. Using the commutators (2.36) and (2.38), one can write the two-mode displacement operator in terms of these two orderings:

$$\begin{aligned} D(\underline{a}, \underline{\nu}) &= e^{-\nu^\dagger \alpha_2 \nu/2} e^{\alpha_1^\dagger \nu_2} e^{-\nu^\dagger \alpha_1 \nu} \\ &= e^{-\eta^\dagger \Pi \eta/2} e^{\mathcal{A}^\dagger \eta} e^{-\eta^\dagger \mathcal{A}} = \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}), \end{aligned} \quad (5.1a)$$

$$\begin{aligned} D(\underline{a}, \underline{\nu}) &= e^{\nu^\dagger \alpha_1 \nu/2} e^{-\nu^\dagger \alpha_2 \nu} e^{\alpha_1^\dagger \nu} \\ &= e^{\eta^\dagger \Pi \eta/2} e^{-\eta^\dagger \mathcal{A}} e^{\mathcal{A}^\dagger \eta} = \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) \end{aligned} \quad (5.1b)$$

[cf. Eq. (3.38)]. The lesson is that the natural orderings for two-photon optics, which are based on the quadrature-phase amplitudes, require opposite ordering of the two modes.

These operator orderings will play a prominent role in paper III, where the focus will be on characteristic functions and their complex Fourier transforms, quasiprobability distributions. The expectation value  $\langle \mathcal{D}(\underline{\mathcal{A}}, \underline{\eta}) \rangle \equiv \Phi(\underline{\eta})$  is a characteristic function whose Taylor expansion yields the symmetrically ordered moments of  $\alpha_1, \alpha_2, \alpha_1^\dagger,$  and  $\alpha_2^\dagger$ ; its complex Fourier transform is a two-mode version of the well-known Wigner distribution function.<sup>22</sup> The expectation values  $\langle e^{\mathcal{A}^\dagger \eta} e^{-\eta^\dagger \mathcal{A}} \rangle = e^{\eta^\dagger \Pi \eta/2} \Phi(\underline{\eta})$  and  $\langle e^{-\eta^\dagger \mathcal{A}} e^{\mathcal{A}^\dagger \eta} \rangle = e^{-\eta^\dagger \Pi \eta/2} \Phi(\underline{\eta})$  are characteristic functions whose Taylor expansions yield moments of  $\alpha_1, \alpha_2, \alpha_1^\dagger,$  and  $\alpha_2^\dagger$  that are, respectively, quadrature-phase normally ordered and quadrature-phase antinormally ordered. The complex Fourier transforms of these characteristic functions are new two-photon quasiprobability distributions, whose definitions build into them the association of  $a_-$  with  $a_-^\dagger$  which is responsible for the squeezing of the output of two-photon devices. Paper III will generalize quadrature-phase normal and antinormal orderings to a continuum of intermediate orderings and will explore the characteristic functions and quasiprobability distributions that arise from these general operator orderings.

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APPENDIX A: PROPERTIES  
OF TRANSFORMATION MATRICES

$$1. \mathcal{A} \equiv 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$\mathcal{A}^{-1} = \mathcal{A}^\dagger = 2^{-1/2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad (\text{A1})$$

$$\det \mathcal{A} = i, \quad (\text{A2})$$

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} e^{i\pi/4} [1 - i(\sigma_1 - \sigma_2 + \sigma_3)] \\ &= 2^{-1/2} e^{i\pi/4} (1 - \sigma_1 + \sigma_2 - \sigma_3), \end{aligned} \quad (\text{A3})$$

$$\mathcal{A} \sigma_1 \mathcal{A}^\dagger = \sigma_3, \quad (\text{A4})$$

$$\mathcal{A} \sigma_2 \mathcal{A}^\dagger = -\sigma_1, \quad (\text{A5})$$

$$\mathcal{A} \sigma_3 \mathcal{A}^\dagger = -\sigma_2, \quad (\text{A6})$$

$$\mathcal{A}^\dagger \sigma_1 \mathcal{A} = -\sigma_2, \quad (\text{A7})$$

$$\mathcal{A}^\dagger \sigma_2 \mathcal{A} = -\sigma_3, \quad (\text{A8})$$

$$\mathcal{A}^\dagger \sigma_3 \mathcal{A} = \mathcal{A}^T \mathcal{A} = \sigma_1 \quad (\text{A9})$$

(superscript  $T$  denotes a transpose).

$$2. \hat{\lambda} \equiv \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

$$\hat{\lambda}^\dagger = \hat{\lambda}, \quad (\text{A10})$$

$$\det \hat{\lambda} = \lambda_+ \lambda_- = (1 - \epsilon^2 / \Omega^2)^{1/2}, \quad (\text{A11})$$

$$\hat{\lambda}^{-1} = (1 - \epsilon^2 / \Omega^2)^{-1/2} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}. \quad (\text{A12})$$

Equation (A12) allows one to obtain properties of  $\hat{\lambda}^{-1}$  from properties of  $\hat{\lambda}$  by multiplying by factors of  $(1 - \epsilon^2 / \Omega^2)^{-1/2}$  and reversing the sign of  $\epsilon$ .

$$\hat{\lambda} = \frac{1}{2} (\lambda_+ + \lambda_-) \mathbb{1} + \frac{1}{2} (\lambda_+ - \lambda_-) \sigma_3, \quad (\text{A13})$$

$$[\hat{\lambda}, \sigma_1] = i(\lambda_+ - \lambda_-) \sigma_2, \quad (\text{A14})$$

$$[\hat{\lambda}, \sigma_2] = -i(\lambda_+ - \lambda_-) \sigma_1, \quad (\text{A15})$$

$$[\hat{\lambda}, \sigma_3] = 0, \quad (\text{A16})$$

$$\hat{\lambda}^2 = \mathbb{1} + (\epsilon / \Omega) \sigma_3, \quad (\text{A17})$$

$$\hat{\lambda} \sigma_1 \hat{\lambda} = (1 - \epsilon^2 / \Omega^2)^{1/2} \sigma_1, \quad (\text{A18})$$

$$\hat{\lambda} \sigma_2 \hat{\lambda} = (1 - \epsilon^2 / \Omega^2)^{1/2} \sigma_2, \quad (\text{A19})$$

$$\hat{\lambda} \sigma_3 \hat{\lambda} = \sigma_3 \hat{\lambda}^2 = (\epsilon / \Omega) \mathbb{1} + \sigma_3. \quad (\text{A20})$$

$$3. \mathcal{C}_{r,\varphi} \equiv \begin{pmatrix} \cosh r & e^{2i\varphi} \sinh r \\ e^{-2i\varphi} \sinh r & \cosh r \end{pmatrix}$$

$$\mathcal{C}_{r,\varphi}^\dagger = \mathcal{C}_{r,\varphi}, \quad (\text{A21})$$

$$\det \mathcal{C}_{r,\varphi} = 1, \quad (\text{A22})$$

$$\mathcal{C}_{r,\varphi}^\dagger \mathcal{C}_{r,\varphi} = \mathcal{C}_{r,\varphi} \mathcal{C}_{r,\varphi} = \sigma_3, \quad (\text{A23})$$

$$\mathcal{C}_{r,\varphi}^{-1} = \sigma_3 \mathcal{C}_{r,\varphi} \sigma_3 = \mathcal{C}_{-r,\varphi} = \mathcal{C}_{r,\varphi + \pi/2}. \quad (\text{A24})$$

The last two equalities in Eq. (A24) are the analogs of the last two equalities in Eq. (1.9).

$$\begin{aligned} \mathcal{C}_{r,\varphi} &= \mathbb{1} \cosh r + [\sigma_1 \cos(2\varphi) - \sigma_2 \sin(2\varphi)] \sinh r \\ &= e^{r(\sigma_1 \cos(2\varphi) - \sigma_2 \sin(2\varphi))}, \end{aligned} \quad (\text{A25})$$

$$\mathcal{C}_{r,\varphi} \mathcal{C}_{r',\varphi} = \mathcal{C}_{r+r',\varphi}, \quad (\text{A26})$$

$$e^{i\theta \sigma_3} \mathcal{C}_{r,\varphi} e^{-i\theta \sigma_3} = \mathcal{C}_{r,\varphi + \theta}. \quad (\text{A27})$$

APPENDIX B: FACTORIZATION AND PRODUCTS  
OF SQUEEZE OPERATORS

In this appendix we first factor the degenerate and the two-mode squeeze operators,  $S_1(r,\varphi)$  and  $S(r,\varphi)$  [Eqs. (2.46) and (1.8)], into products of exponentials; we then use these factored expressions to show explicitly that the product of two or more different squeeze operators (differing both in their magnitudes and directions of squeezing) is equal to the product of a rotation operator and a squeeze operator.

Factoring the two-mode squeeze operator turns out to be the same task as factoring the degenerate squeeze operator. In either case the problem reduces to factoring the expression

$$\mathcal{M}(r,\varphi) \equiv \exp[r(Ae^{-2i\varphi} - A^\dagger e^{2i\varphi})], \quad (\text{B1})$$

where the operator  $A$  and its adjoint  $A^\dagger$  obey the commutation relations

$$[A, A^\dagger] \equiv B = B^\dagger, \quad [A, B] = 2A, \quad [A^\dagger, B] = -2A^\dagger. \quad (\text{B2})$$

For the two-mode squeeze operator  $\mathcal{M} = S(r,\varphi)$ , one sets  $A = a_+ a_-$  and  $B = 1 + a_- a_+ + a_+ a_-$ ; for the degenerate squeeze operator  $\mathcal{M} = S_1(r,\varphi)$ , one sets  $A = \frac{1}{2} a^2$  and  $B = \frac{1}{2} + a^\dagger a$ . The commutation relations (B2) immediately imply the following useful relations (and their Hermitian conjugates):

$$e^{tA} A^\dagger e^{-tA} = A^\dagger + tB + t^2 A, \quad (\text{B3a})$$

$$e^{tA} B e^{-tA} = B + 2tA, \quad (\text{B3b})$$

$$e^{tB} A e^{-tB} = e^{-2t} A, \quad (\text{B3c})$$

where  $t$  is any complex number. The relations (B3) follow from the general relation

$$e^{iR} S e^{-iR} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{R^n S\}, \quad (\text{B4})$$

$$\{R^n S\} \equiv [R, \{R^{n-1} S\}], \quad \{R^0 S\} \equiv S.$$

[for a derivation see, for example, Eq. (8.105) of Ref. 23].

There are many approaches one can take to factor the operator  $\mathcal{M}(r,\varphi)$  into products of exponentials involving the operators  $A$ ,  $A^\dagger$ , and  $B$ ; here we briefly describe two approaches. In the "differential equation" approach<sup>24</sup> one multiplies the exponent of  $\mathcal{M}(r,\varphi)$  by a parameter  $t$  and

sets the resulting expression equal to the desired product of exponentials involving the operators  $A$ ,  $A^\dagger$ , and  $B$ , with coefficients in the exponents which are functions (to be determined) of the parameter  $t$ . One then takes the derivative with respect to  $t$  of both expressions, equates the two expressions by using the relations (B3) to put them in the same form, and solves the resulting coupled first-order differential equations (subject to boundary conditions at  $t=0$ ) to find the coefficients as functions of  $t$ ; the task is completed by setting the parameter  $t$  equal to one. This procedure is straightforward and, for this particular problem, not difficult.

A more elegant and more versatile approach,<sup>25,15,17</sup> however, makes use of the fact that the factored forms for  $M(r,\varphi)$  are consequences only of the commutation relations (B2); this means that the operators  $A$ ,  $A^\dagger$ , and  $B$  can be replaced by matrices which obey the commutation relations (B2), and the problem of factoring  $M(r,\varphi)$  can be reduced to factoring an exponential of a sum of matrices. The problem becomes particularly simple if one uses for this purpose the Pauli spin matrices (2.7), which have the following properties:

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k, \quad \mathbb{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{B5a})$$

$$e^{\gamma_j \sigma_j} = \mathbb{1} \cosh \gamma + \sigma_j \gamma_j \gamma^{-1} \sinh \gamma, \\ \gamma \equiv (\gamma_1^2 + \gamma_2^2 + \gamma_3^2)^{1/2} \quad (\text{B5b})$$

where  $i, j, k = 1, 2$ , or  $3$ ,  $\gamma_j$  are arbitrary complex numbers, and a summation over repeated indices is implied. For the matrices  $\sigma_+$  and  $\sigma_-$  defined by

$$\sigma_+ \equiv \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (\text{B6a})$$

$$\sigma_- \equiv \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_+^\dagger,$$

the properties (B5) have the following important consequences:

$$[\sigma_+, \sigma_-] = \sigma_3, \quad [\sigma_\pm, \sigma_3] = \mp 2\sigma_\pm, \quad (\text{B6b})$$

$$\sigma_+^2 = \sigma_-^2 = 0, \quad \sigma_\pm \sigma_\mp = \frac{1}{2}(\mathbb{1} \pm \sigma_3),$$

$$\sigma_\pm \sigma_3 = -\sigma_3 \sigma_\pm = \mp \sigma_\pm, \quad (\text{B6c})$$

$$e^{t\sigma_\pm} = \mathbb{1} + t\sigma_\pm. \quad (\text{B6d})$$

The commutation relations (B2) and (B6b) admit the formal correspondence

$$A \rightarrow -\sigma_-, \quad A^\dagger \rightarrow \sigma_+, \quad B \rightarrow \sigma_3. \quad (\text{B7})$$

This correspondence implies that

$$M(r,\varphi) \rightarrow e^{-r(\sigma_+ e^{2i\varphi} + \sigma_- e^{-2i\varphi})} \\ = \mathbb{1} \cosh r - (\sigma_+ e^{2i\varphi} + \sigma_- e^{-2i\varphi}) \sinh r = \underline{C}_{r,\varphi}^{-1} \quad (\text{B8})$$

[Eqs. (B5b) and (B6); cf. Eq. (A25)]. The correspondence (B7) is not unique; we choose it because it leads to the correspondence (B8), which is the correspondence induced naturally by the definition (2.22) of  $\underline{C}_{r,\varphi}$ . Note that while the operator  $M(r,\varphi)$  is unitary, the matrix  $\underline{C}_{r,\varphi}^{-1}$  is not.

The matrix  $\underline{C}_{r,\varphi}^{-1}$  is easily factored into exponentials of  $\sigma_+$ ,  $\sigma_-$ , and  $\sigma_3$ . For example, one such factorization is

$$\underline{C}_{r,\varphi}^{-1} = e^{-\Gamma \sigma_+} e^{-g \sigma_3} e^{-\Gamma \sigma_-}, \quad (\text{B9a})$$

$$\Gamma \equiv e^{2i\varphi} \tanh r, \quad g \equiv \ln(\cosh r)$$

[Eqs. (B5b), (B6), and (B8)]. This implies, through the correspondence (B7), that<sup>14-17</sup>

$$M(r,\varphi) = e^{-\Gamma A^\dagger} e^{-g B} e^{-\Gamma A}. \quad (\text{B9b})$$

The five other factored forms for  $M(r,\varphi)$ , which correspond to all other orderings of the operators  $A$ ,  $A^\dagger$ , and  $B$ , are easily derived from Eq. (B9b) with the help of the following rules:

$$e^{s\sigma_\pm} e^{t\sigma_\pm} = e^{t\sigma_\pm} e^{se^{\mp 2i\varphi} \sigma_\pm}, \quad (\text{B10a})$$

$$e^{s\sigma_+} e^{t\sigma_-} = e^{te^{-\omega} \sigma_-} e^{se^{\omega} \sigma_+} e^{\omega \sigma_3} \\ = e^{\omega \sigma_3} e^{te^{\omega} \sigma_-} e^{se^{-\omega} \sigma_+}, \quad \omega \equiv \ln(1 + st) \quad (\text{B10b})$$

[Eqs. (B5b) and (B6)]. Equation (B10a) follows from the matrix version of Eq. (B3c), and Eq. (B10b) corresponds to the rule for interchanging the order of exponentials of  $A$  and  $A^\dagger$ :

$$e^{sA^\dagger} e^{-tA} = e^{-te^{-\omega} A} e^{se^{\omega} A^\dagger} e^{\omega B} \\ = e^{\omega B} e^{-te^{\omega} A} e^{se^{-\omega} A^\dagger}. \quad (\text{B11})$$

The final result is that the operator  $M(r,\varphi)$  defined by Eqs. (B1) and (B2), i.e., the squeeze operator, has the following six equivalent factored forms:

$$M(r,\varphi) = e^{-\Gamma A^\dagger} e^{-g B} e^{-\Gamma A} = e^{-\Gamma A^\dagger} e^{\Gamma e^{2i\varphi} A} e^{-g B} = e^{-g B} e^{-\Gamma e^{2i\varphi} A^\dagger} e^{-\Gamma A} \\ = e^{\Gamma e^{2i\varphi} A} e^{-\Gamma e^{2i\varphi} A^\dagger} e^{g B} = e^{\Gamma e^{2i\varphi} A} e^{-g B} e^{-\Gamma A^\dagger} = e^{g B} e^{\Gamma e^{2i\varphi} A} e^{-\Gamma A^\dagger}. \quad (\text{B12})$$

The above rules and factored forms for the degenerate and two-mode squeeze operators allow us to prove what one would expect intuitively: the product of two different squeeze operators is equal to the product of a rotation operator and a squeeze operator. Equation (B8) implies the correspondence

$$M^\dagger(r',\varphi') M(r,\varphi) \rightarrow \underline{C}_{r',\varphi'} \underline{C}_{r,\varphi}^{-1} = (\underline{C}_{r,\varphi} \underline{C}_{r',\varphi'}^{-1})^{-1}. \quad (\text{B13})$$

The product  $\underline{C}_{r,\varphi} \underline{C}_{r',\varphi'}^{-1}$  can be written as the product of another  $C$  matrix,  $\underline{C}_{R,\Phi}$ , and  $e^{i\Theta \sigma_3}$ , giving

$$\underline{C}_{r,\varphi} \underline{C}_{r',\varphi}^{-1} = \underline{C}_{R,\Phi} e^{i\Theta \alpha_3} = e^{i\Theta \alpha_3} \underline{C}_{R,\Phi-\Theta} = \begin{pmatrix} e^{i\Theta} \cosh R & e^{i(2\Phi-\Theta)} \sinh R \\ e^{-i(2\Phi-\Theta)} \sinh R & e^{-i\Theta} \cosh R \end{pmatrix}, \quad (\text{B14})$$

where  $R$ ,  $\Phi$ , and  $\Theta$  are defined explicitly by

$$e^{i\Theta} \cosh R = \cosh r \cosh r' - e^{2i(\varphi-\varphi')} \sinh r \sinh r', \quad (\text{B15a})$$

$$e^{i(2\Phi-\Theta)} \sinh R = e^{2i\varphi} \sinh r \cosh r' - e^{2i\varphi'} \sinh r' \cosh r. \quad (\text{B15b})$$

The correspondences (B7) and (B8) then imply that

$$M^\dagger(r',\varphi') M(r,\varphi) = e^{-i\Theta \beta} M(R,\Phi) = M(R,\Phi-\Theta) e^{-i\Theta \beta}. \quad (\text{B16})$$

By using the fact that  $\underline{C}_{r,\varphi}^{-1} = \underline{C}_{r,\varphi}^\dagger$  [Eq. (A24)] and noting that  $\underline{C}_{r,\varphi} \underline{C}_{r',\varphi}$  is the matrix of commutators defined by  $\underline{C}_{r,\varphi}$  and  $\underline{C}_{r',\varphi}$ , one can write the defining relation for  $R$ ,  $\Phi$ , and  $\Theta$  [Eq. (B14)] in terms of the nonvanishing commutators of the squeezed annihilation operators:

$$\underline{C}_{R,\Phi} e^{i\Theta \alpha_3} = \underline{C}_{r,\varphi} \underline{C}_{r',\varphi} = \begin{pmatrix} [\alpha_+(r,\varphi), \alpha_+^\dagger(r',\varphi')] & -[\alpha_+(r,\varphi), \alpha_-(r',\varphi')] \\ [\alpha_-(r,\varphi), \alpha_+^\dagger(r',\varphi')] & -[\alpha_-(r,\varphi), \alpha_-(r',\varphi')] \end{pmatrix}. \quad (\text{B17})$$

### APPENDIX C: INNER PRODUCTS OF SQUEEZED STATES

In this appendix we derive the inner product of arbitrary squeezed states. The derivation is sketched for two-mode squeezed states. The same derivation works for degenerate squeezed states, so for them we merely list the main result.

One way to derive the general inner product is to begin with the matrix element  $\langle 0 | \underline{\mu}_\alpha \rangle_{(r,\varphi)}$ . This matrix element is easily obtained by using the number-eigenstate ex-

pansions of a two-mode coherent state [Eq. (4.19)] and the squeezed vacuum state [Eq. (4.39)]:

$$\begin{aligned} \langle 0 | \underline{\mu}_\alpha \rangle_{(r,\varphi)} &= \langle 0 | D(\underline{a}, \underline{\mu}) S(r, \varphi) | 0 \rangle \\ &= (\cosh r)^{-1} e^{-\underline{\mu}^\dagger \underline{\mu} / 2} e^{-\underline{\mu}_+^\dagger \underline{\mu}_+ e^{2i\varphi} \tanh r}. \end{aligned} \quad (\text{C1})$$

It is instructive to write the exponents in Eq. (C1) in terms of the vector notation:

$$\begin{aligned} \langle 0 | \underline{\mu}_\alpha \rangle_{(r,\varphi)} &= (\cosh r)^{-1} \exp\left[-\frac{1}{2} (\cosh r)^{-1} \underline{\mu}^\dagger (\underline{C}_{r,\varphi} + i D_{r,\varphi}) \underline{\mu}\right] \\ &= (\cosh r)^{-1} \exp\left[-\frac{1}{2} (\cosh r)^{-1} \underline{\mu}_\alpha^\dagger (\underline{C}_{r,\varphi}^{-1} + i D_{r,\varphi}) \underline{\mu}_\alpha\right]. \end{aligned} \quad (\text{C2})$$

The matrix  $D_{r,\varphi}$  is defined by

$$\begin{aligned} D_{r,\varphi} &\equiv \begin{pmatrix} 0 & -ie^{2i\varphi} \sinh r \\ ie^{-2i\varphi} \sinh r & 0 \end{pmatrix} \\ &= D_{r,\varphi}^\dagger = -D_{-r,\varphi}; \end{aligned} \quad (\text{C3})$$

it has the following easily verified properties:

$$D_{r,\varphi} = -\frac{1}{2} \frac{\partial \underline{C}_{r,\varphi}}{\partial \varphi} = [\underline{C}_1 \sin(2\varphi) + \underline{C}_2 \cos(2\varphi)] \sinh r, \quad (\text{C4a})$$

$$D_{r,\varphi} e^{i\Theta \alpha_3} = e^{-i\Theta \alpha_3} D_{r,\varphi}, \quad (\text{C4b})$$

$$D_{r,\varphi} = \underline{C}_{r',\varphi} D_{r,\varphi} \underline{C}_{r,\varphi}^{-1}. \quad (\text{C4c})$$

Equation (C2) decomposes  $\langle 0 | \underline{\mu}_\alpha \rangle_{(r,\varphi)}$  neatly into a magnitude times a phase factor.

Consider now the general inner product

$$\begin{aligned} (r',\varphi') \langle \underline{\mu}'_\alpha | \underline{\mu}_\alpha \rangle_{(r,\varphi)} \\ = \langle 0 | S^\dagger(r',\varphi') D^\dagger(\underline{a}, \underline{\mu}') D(\underline{a}, \underline{\mu}) S(r, \varphi) | 0 \rangle, \end{aligned} \quad (\text{C5})$$

where

$$\underline{\mu}_\alpha = \underline{C}_{r,\varphi} \underline{\mu}, \quad \underline{\mu}'_\alpha = \underline{C}_{r',\varphi} \underline{\mu}'. \quad (\text{C6})$$

Equations (3.63b), (3.47), and (3.67) imply

$$\begin{aligned} (r',\varphi') \langle \underline{\mu}'_\alpha | \underline{\mu}_\alpha \rangle_{(r,\varphi)} \\ = e^{-i\Theta} D\left(\frac{1}{2} \underline{\mu}', \underline{\mu}\right) \langle 0 | \underline{C}_{r,\varphi} (\underline{\mu} - \underline{\mu}') \rangle_{(R,\Phi)}, \end{aligned} \quad (\text{C7})$$

where  $R$ ,  $\Phi$ , and  $\Theta$  are defined by Eq. (3.68) [see also Eqs. (B15)]:

$$\underline{C}_{R,\Phi} e^{i\Theta \alpha_3} = \underline{C}_{r,\varphi} \underline{C}_{r',\varphi}^{-1}. \quad (\text{C8a})$$

The Hermitian conjugate of Eq. (C8a) is the useful relation

$$e^{-i\Theta \alpha_3} \underline{C}_{R,\Phi} = \underline{C}_{r',\varphi}^{-1} \underline{C}_{r,\varphi}. \quad (\text{C8b})$$

Note that

$$D_{R,\Phi} e^{i\Theta \alpha_3} = e^{-i\Theta \alpha_3} D_{R,\Phi} = D_{r,\varphi} \cosh r' - D_{r',\varphi} \cosh r \quad (\text{C9})$$

[Eq. (B15b)]. The matrix element (C2) can now be used in Eq. (C7) to give the desired result,

$$\begin{aligned} \langle \underline{\mu}'_{\alpha} | \underline{\mu}_{\alpha} \rangle_{(r,\varphi)} &= ND(\frac{1}{2}\underline{\mu}', \underline{\mu}) \exp[-\frac{1}{2}(\underline{\mu}-\underline{\mu}')^{\dagger}(E+i\mathcal{G})(\underline{\mu}-\underline{\mu}')] , \\ & \end{aligned} \tag{C10}$$

where

$$\begin{aligned} N &\equiv (e^{i\Theta} \cosh R)^{-1} \\ &= (\cosh r \cosh r' - e^{2i(\varphi-\varphi')} \sinh r \sinh r')^{-1} \end{aligned} \tag{C11a}$$

[Eq. (B15a)], and

$$\begin{aligned} E \cosh R &\equiv \mathcal{C}_{r,\varphi} \mathcal{C}_{R,\Phi}^{-1} \mathcal{C}_{r,\varphi} = \mathcal{C}_{r',\varphi'} e^{-i\Theta \alpha_3} \mathcal{C}_{r,\varphi} \\ &= \mathcal{C}_{r,\varphi} e^{i\Theta \alpha_3} \mathcal{C}_{r',\varphi'} , \end{aligned} \tag{C11b}$$

$$\begin{aligned} \mathcal{G} \cosh R &\equiv \mathcal{C}_{r,\varphi} \mathcal{D}_{R,\Phi} \mathcal{C}_{r,\varphi} \\ &= \mathcal{C}_{r,\varphi} (\mathcal{D}_{r,\varphi} \cosh r' - \mathcal{D}_{r',\varphi} \cosh r) e^{-i\Theta \alpha_3} \mathcal{C}_{r,\varphi} \\ &= \mathcal{C}_{r',\varphi'} (\mathcal{D}_{r,\varphi} \cosh r' - \mathcal{D}_{r',\varphi} \cosh r) e^{i\Theta \alpha_3} \mathcal{C}_{r',\varphi'} . \end{aligned} \tag{C11c}$$

[ $E = E^{\dagger}$ ,  $\mathcal{G} = \mathcal{G}^{\dagger}$ ; Eqs. (C8) and (C9)]. Note the relation

$$\frac{e^{-i\Theta \alpha_3}}{\cosh R} = \begin{pmatrix} N & 0 \\ 0 & N^* \end{pmatrix} . \tag{C11d}$$

Equations (C11) allow one to write the inner product (C10) in terms of the primary variables  $r$ ,  $r'$ ,  $\varphi$ , and  $\varphi'$ . Three special cases deserve attention: (i) if  $\underline{\mu} = \underline{\mu}'$ , then Eq. (C10) reduces to

$$\langle \underline{\mu}'_{\alpha} | \underline{\mu}_{\alpha} \rangle_{(r,\varphi)} = N = (\cosh r \cosh r' - e^{2i(\varphi-\varphi')} \sinh r \sinh r')^{-1} (\underline{\mu} = \underline{\mu}') ; \tag{C12}$$

(ii) if  $\varphi = \varphi'$ , then  $\Theta = 0$ ,  $\Phi = \varphi$ , and  $R = r - r'$ , so

$$\langle \underline{\mu}'_{\alpha} | \underline{\mu}_{\alpha} \rangle_{(r,\varphi)} = [\cosh(r - r')]^{-1} D(\frac{1}{2}\underline{\mu}', \underline{\mu}) \exp[-\frac{1}{2}[\cosh(r - r')]^{-1}(\underline{\mu} - \underline{\mu}')^{\dagger}(\mathcal{C}_{r+r',\varphi} + i\mathcal{D}_{r-r',\varphi})(\underline{\mu} - \underline{\mu}')] \tag{C13}$$

[cf. Eq. (4.48)]; (iii) if  $r' = 0$ , then  $\Theta = 0$ ,  $\Phi = \varphi$ , and  $R = r$ , and Eq. (C10) gives the inner product of the two-mode coherent state  $|\underline{\mu}'\rangle_{\text{coh}}$  with the two-mode squeezed state  $|\underline{\mu}_{\alpha}\rangle_{(r,\varphi)}$

$$\text{coh} \langle \underline{\mu}' | \underline{\mu}_{\alpha} \rangle_{(r,\varphi)} = (\cosh r)^{-1} D(\frac{1}{2}\underline{\mu}', \underline{\mu}) \exp[-\frac{1}{2}(\cosh r)^{-1}(\underline{\mu} - \underline{\mu}')^{\dagger}(\mathcal{C}_{r,\varphi} + i\mathcal{D}_{r,\varphi})(\underline{\mu} - \underline{\mu}')] \tag{C14}$$

[cf. Eq. (C2)].

For degenerate squeezed states the general inner product is given by<sup>11</sup>

$$\langle \underline{\mu}'_{\alpha} | \underline{\mu}_{\alpha} \rangle_{(r,\varphi)} = N^{1/2} D(\frac{1}{2}\underline{\mu}', \underline{\mu}) \exp[-\frac{1}{2}(\underline{\mu} - \underline{\mu}')^{\dagger}(E+i\mathcal{G})(\underline{\mu} - \underline{\mu}')] , \tag{C15}$$

where we use the single-mode vector notation introduced in Table II, with  $\underline{\mu}_{\alpha} = \mathcal{C}_{r,\varphi} \underline{\mu}$  and  $\underline{\mu}'_{\alpha} = \mathcal{C}_{r',\varphi'} \underline{\mu}'$ .

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The following paper, entitled  
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# Noise in homodyne detection

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A simple but rigorous analysis of the important sources of noise in homodyne detection is presented. Output noise and signal-to-noise ratios are compared for direct detection, conventional (one-port) homodyning, and two-port homodyning, in which one monitors both output ports of a 50–50 beam splitter. It is shown that two-port homodyning is insensitive to local-oscillator quadrature-phase noise and hence provides (1) a means of detecting reduced quadrature-phase fluctuations (squeezing) that is perhaps more practical than one-port homodyning and (2) an output signal-to-noise ratio that can be a modest to significant improvement over that of one-port homodyning and direct detection.

It has been known for some time that phase-sensitive detection schemes, such as homodyning and heterodyning, provide a means of measuring one of a signal's two quadratures.<sup>1–3</sup> Recently a clever scheme, referred to in this Letter as two-port homodyning, was proposed by Yuen and Chan (see Fig. 1)<sup>4,5</sup>; this scheme permits direct observation of one of the input signal's quadratures without placing such rigorous demands on local-oscillator (LO) performance as does conventional one-port homodyning. Hence the two-port scheme might provide a more practical means of observing the phenomenon of squeezing,<sup>6–8</sup> in which the noise in one quadrature phase of a signal is reduced below the level required by quantum mechanics of a coherent state. This noise can be observed because the output noise in two-port homodyning can be made insensitive to all LO quadrature-phase noise, provided the LO power is much larger than the input-signal power. In this Letter I compare output noise and signal-to-noise ratios for three different detection schemes—one-port homodyning, two-port homodyning, and direct detection.

A single plane-wave mode of the electromagnetic field at frequency  $\Omega_s$  (the input-signal frequency) has an electric-field operator described by

$$E(t, x) \propto \frac{1}{2}[a \exp[-i\Omega_s(t-x)] + a^\dagger \exp[i\Omega_s(t-x)]] \\ = a_1 \cos \Omega_s(t-x) + a_2 \sin \Omega_s(t-x). \quad (1)$$

Here  $a \equiv a_1 + ia_2$  is the annihilation operator for the mode and  $a_1$  and  $a_2$  are its Hermitian quadrature-phase amplitudes. Homodyne schemes use a beam splitter to combine an input-signal field with a LO field of large power compared with the input-signal field (Fig. 1). The dominant signal-carrying term in the intensities from each of the two beam-splitter output ports is proportional to the mean field of that quadrature of the input-signal field ( $a_1$ , say) that is in phase with the LO mean field after the beam splitter. One-port homodyne schemes use a photodetector to monitor the intensity from one output port of the beam splitter. Two-port homodyne schemes monitor the intensities from both output ports of a 50–50 beam splitter and then subtract the two photodetector outputs.

Each quadrature of the input-signal field and the LO field has associated with it a certain amount of noise (fluctuations), characterized by the variance of its quadrature-phase amplitude, i.e.,  $\Delta a_1^2 \equiv \langle (a_1 - \langle a_1 \rangle)^2 \rangle$  and  $\Delta a_2^2$  for the input-signal field. Quantum mechanics requires that  $\Delta a_1^2 \Delta a_2^2 \geq 1/16$ . Coherent states and the vacuum state have  $\Delta a_1^2 = \Delta a_2^2 = 1/4$  (random-phase noise). Single-mode squeezed states<sup>6–8</sup> satisfy  $\Delta a_1^2 \Delta a_2^2 = 1/16$  but have  $\Delta a_1^2 < 1/4$ , where  $a'_1 + ia'_2 \equiv e^{-i\varphi}(a_1 + ia_2)$ ,  $0 \leq \varphi < \pi$ .

Aside from that added by nonideal photodetectors, the dominant output noise in homodyning comes from interference between the (large) LO mean field and the noise in the in-phase quadratures of the input-signal and LO fields ( $a_1$  and  $b_1$ , say). In direct detection the output noise reflects only intensity fluctuations in the input signal [ $\Delta N_a^2 \equiv \langle (N_a - \langle N_a \rangle)^2 \rangle$ ], but in homodyning it reflects the variances  $\Delta a_1^2$  and  $\Delta b_1^2$ . In one-port homodyning with a lossless beam splitter of power transmissivity  $T$  the dominant output noise is proportional to the sum  $T\Delta a_1^2 + (1-T)\Delta b_1^2$ ; in two-port homodyning it can be made proportional to  $\Delta a_1^2$  alone. The two-port scheme can yield a better output signal-to-noise ratio (SNR) than both direct detection and one-port homodyning, the improvement over the latter being most significant when  $\Delta a_1^2 \ll \Delta b_1^2$ .

The two-port scheme owes its success to the law of energy conservation, which dictates that the interference terms between the LO field and the input-signal field contribute with opposite signs to the two outputs of the (lossless) beam splitter and that the noninterference terms contribute with the same sign. The two-port scheme, by using a 50–50 beam splitter and subtracting the two outputs, retains only the interference terms. Its output signal (intensity) is therefore a product of the LO and input-signal mean fields, and its dominant output noise (intensity fluctuations) is due to interference between the input-signal noise  $\Delta a_1^2$  and the LO power and between the LO noise  $\Delta b_1^2$  and the input-signal power. Thus, for large enough LO power, the output noise in the two-port scheme can be made proportional to  $\Delta a_1^2$ .

The analysis begins with the beam splitter, assumed linear and lossless with power transmissivity  $T$  and re-

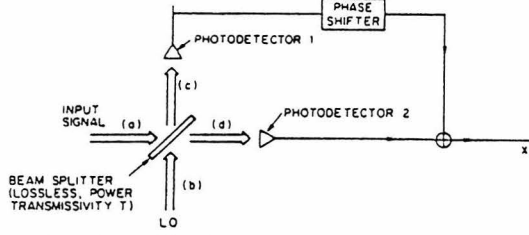


Fig. 1. Yuen-Chan<sup>1</sup> two-port homodyne detection scheme.

reflectivity  $1 - T$ . The law of energy conservation, together with the invariance of a (linear) lossless beam splitter under the combined operations of time reversal and reflection about the  $y = -x$  axis, leads to the following matrix transformation between the two in-mode annihilation operators, denoted by  $a$  for the signal mode and  $b$  for the LO mode, and the two out-mode annihilation operators, denoted by  $c$  and  $d$  (see Fig. 1):

$$\begin{bmatrix} c \\ d \end{bmatrix} = U \begin{bmatrix} a \\ b \end{bmatrix}, \quad (1)$$

$$U = e^{i\Delta} \begin{bmatrix} \sqrt{1-T} e^{i\mu} & \sqrt{T} \\ \sqrt{T} & -\sqrt{1-T} e^{-i\mu} \end{bmatrix} = (U^\dagger)^{-1}. \quad (2)$$

For present purposes, the overall phase  $\Delta$  and the relative phase  $\mu$ —both inherent properties of the beam splitter—need not be known; in fact, the relative phase can effectively be made to take on any desired value by putting phase delays in the path to one or both detectors or by adjusting the relative path lengths.

In order to understand physically the roles played by the various noises as distinct from the mean fields that they accompany, it is useful to separate each annihilation operator into a signal-carrying part and a noise-carrying part. To use the input-signal field as an example, the signal-carrying part is the mean field ( $a$ )  $\equiv A$ , a complex number; the noise-carrying part is the annihilation operator minus its mean:  $a - \langle a \rangle \equiv \Delta a$ . Thus

$$a = A + \Delta a \equiv A_1 + iA_2 + \Delta a_1 + i\Delta a_2. \quad (3)$$

For simplicity, choose the phase of the LO field so that the quantity  $e^{-i\mu} \langle b \rangle$  is equal to the real number  $B$ ; then define the operator

$$e^{-i\mu} b \equiv B + \Delta b = B + \Delta b_1 + i\Delta b_2. \quad (4)$$

With this LO phase, homodyning will see only that part of the input signal whose mean is equal to  $A_1$ , so we can consider  $A_2$  to be zero; thus  $A_1$  is the signal to be detected, and its inherent noise is characterized by the variance  $\Delta a_1^2$ .

Any single-mode state of a radiation field has a noise intensity, a total intensity, and intensity fluctuations (to second order) proportional to these expressions:

$$\langle N_{\bar{a}} \rangle \equiv \langle \Delta a^\dagger \Delta a \rangle = \Delta a_1^2 + \Delta a_2^2 - 1/2; \quad (5a)$$

$$\langle N_a \rangle \equiv \langle a^\dagger a \rangle = |A|^2 + \langle N_{\bar{a}} \rangle; \quad (5b)$$

$$\Delta N_a^2 = \Delta N_{\bar{a}}^2 + 4 \langle (A_1 \Delta a_1 + A_2 \Delta a_2)^2 \rangle + 4 \langle \Delta N_{\bar{a}} (A_1 \Delta a_1 + A_2 \Delta a_2) \rangle_{\text{sym}}, \quad (5c)$$

where the subscript sym means symmetrized. All states of interest have  $\langle N_{\bar{a}} \Delta a_1 \rangle_{\text{sym}} = \langle N_{\bar{a}} \Delta a_2 \rangle_{\text{sym}} = 0$ . Coherent light with Gaussian excess noise has

$$\Delta a_1^2 = \Delta a_2^2 = 1/4(2\bar{n} + 1), \quad \langle \Delta a_1 \Delta a_2 \rangle_{\text{sym}} = 0, \quad (6a)$$

$$\langle N_{\bar{a}} \rangle = \bar{n}, \quad \Delta N_{\bar{a}}^2 = \bar{n}(\bar{n} + 1), \quad (6b)$$

$$\Delta N_a^2 = |A|^2(2\bar{n} + 1) + \bar{n}(\bar{n} + 1) = \langle N_a \rangle + \bar{n}(\bar{n} + 2|A|^2). \quad (6c)$$

When  $\bar{n} = 0$ , the above expressions describe a coherent state or the vacuum state ( $|A| = 0$ ). A single-mode squeezed state<sup>6-8</sup> (with  $\varphi = 0$ ) has

$$\Delta a_{1,2}^2 = 1/4 e^{\pm 2r}, \quad \langle \Delta a_1 \Delta a_2 \rangle_{\text{sym}} = 0, \quad (7a)$$

$$\langle N_{\bar{a}} \rangle = \sinh^2 r, \quad \Delta N_{\bar{a}}^2 = 1/2 \sinh^2 2r = \langle N_{\bar{a}} \rangle (1 + \cosh 2r), \quad (7b)$$

$$\Delta N_a^2 = 1/2 \sinh^2 2r + A_1^2 e^{-2r} + A_2^2 e^{2r} = \langle N_a \rangle + \cosh 2r \sinh^2 r - 2 \sinh r (A_1^2 e^{-r} - A_2^2 e^r). \quad (7c)$$

The beam-splitter outputs are described by the photon-number operators  $N_c \equiv c^\dagger c$  and  $N_d \equiv d^\dagger d$ , where

$$N_c = (1 - T)N_a + TN_b + \sqrt{T(1 - T)}P, \quad (8a)$$

$$N_d = TN_a + (1 - T)N_b - \sqrt{T(1 - T)}P, \quad (8b)$$

$$P \equiv 2 \text{Re}(e^{-i\mu} b a^\dagger) = 2(A_1 + \Delta a_1)(B + \Delta b_1) + 2\Delta a_2 \Delta b_2 \quad (8c)$$

[Eqs. (2)–(4)]. For the usual situation of a strong LO these expressions reduce to

$$N_c \cong TB^2 + 2\sqrt{T(1 - T)}A_1B + 2\sqrt{TB}(\sqrt{T}\Delta b_1 + \sqrt{1 - T}\Delta a_1), \quad (9a)$$

$$N_d \cong (1 - T)B^2 - 2\sqrt{T(1 - T)}A_1B + 2\sqrt{1 - T}B(\sqrt{1 - T}\Delta b_1 - \sqrt{T}\Delta a_1), \quad (9b)$$

where here and below  $\cong$  means in the strong-LO limit,  $(1 - T)B^2 \gg TA_1^2$ . In all the following equations I assume that the power contributed by the fluctuations in the LO and input-signal fields is negligible compared with the power contributed by the mean fields.

One-port homodyne schemes look only at the output signal  $\langle N_d \rangle$ :

$$\langle N_d \rangle = (\sqrt{1 - TB} - \sqrt{T}A_1)^2 \cong (1 - T)B^2 - 2\sqrt{T(1 - T)}A_1B; \quad (10a)$$

the part of this output that contains input-signal information is

$$\langle N_d \rangle_{\text{sig}} \cong -2\sqrt{T(1 - T)}A_1B. \quad (10b)$$

The output noise in one-port homodyning is given by

$$\Delta N_d^2 = 4(\sqrt{1 - TB} - \sqrt{T}A_1)^2[(1 - T)\Delta b_1^2 + T\Delta a_1^2] + 4(\sqrt{1 - TB} - \sqrt{T}A_1)[(1 - T)^{3/2} \langle \Delta N_{\bar{b}} \Delta b_1 \rangle_{\text{sym}} - T^{3/2} \langle \Delta N_{\bar{a}} \Delta a_1 \rangle_{\text{sym}}] + (1 - T)^2 \Delta N_{\bar{b}}^2 + T^2 \Delta N_{\bar{a}}^2 + 4T(1 - T)\langle X \rangle, \quad (11a)$$

$$\langle X \rangle \equiv \Delta a_1^2 \Delta b_1^2 + \Delta a_2^2 \Delta b_2^2 - 1/8 + 2 \langle \Delta a_1 \Delta a_2 \rangle_{\text{sym}} \langle \Delta b_1 \Delta b_2 \rangle_{\text{sym}}. \quad (11b)$$

Neglecting pure noise terms and taking the usual strong-LO limit gives



$$\begin{aligned}\Delta N_d^2 &= 4(\sqrt{1-T}B - \sqrt{T}A_1)^2 T\Delta a_1^2 \\ &\quad + (1-T)\Delta b_1^2 \\ &\cong 4(1-T)B^2[T\Delta a_1^2 + (1-T)\Delta b_1^2] \quad (1c)\end{aligned}$$

The SNR for one-port homodyning is defined as the ratio of  $\langle N_d \rangle_{\text{sig}}$  to  $(\Delta N_d^2)^{1/2}$ :

$$\begin{aligned}(\text{SNR})_1 &\equiv \frac{|\langle N_d \rangle_{\text{sig}}|}{(\Delta N_d^2)^{1/2}} \\ &\cong \sqrt{T} \frac{A_1}{[T\Delta a_1^2 + (1-T)\Delta b_1^2]^{1/2}} \quad (12)\end{aligned}$$

Equations (11c) and (12) suggest that the output noise and SNR for one-port homodyning might be made insensitive to LO quadrature-phase noise if  $T \cong 1$  with  $(1-T)B^2$  finite; however, under these conditions other noise sources (those independent of LO power) may become important, so in practice it may not be possible to let  $T$  be close to 1.<sup>4</sup> Regardless of LO and input-signal power, one-port homodyning is insensitive to LO quadrature-phase noise only if  $(1-T)\Delta b_1^2 \ll T\Delta a_1^2$ .

The two-port scheme suggested by Yuen and Chan<sup>4</sup> uses a 50-50 beam splitter and looks at the quantity  $N_c - N_d = P$ , for which [Eqs. (8c) and (11b)]

$$\begin{aligned}\langle P \rangle &= 2BA_1, \\ \Delta P^2 &= 4B^2\Delta a_1^2 + 4A_1^2\Delta b_1^2 + 4\langle X \rangle \\ &\cong 4B^2\Delta a_1^2, \quad (13a)\end{aligned}$$

$$(\text{SNR})_2 \equiv \frac{\langle P \rangle}{(\Delta P^2)^{1/2}} \cong \frac{A_1}{(\Delta a_1^2)^{1/2}} \quad (13b)$$

By adjusting  $\theta$  (phase shifter in Fig. 1) in the combination  $P(\theta) = N_c \cos \theta + N_d \sin \theta$  of the two photodetector outputs, one can observe  $\Delta a_1^2$  and  $\Delta b_1^2$  individually ( $\theta = \mp \pi/4$ ) or in any linear combination.

By contrast, direct detection of the same input signal ( $A_2 = 0$ ) would give the following output signal, noise, and SNR [Eqs. (5)]:

$$\langle N_a \rangle \cong A_1^2, \quad \Delta N_a^2 \cong 4A_1^2\Delta a_1^2, \quad (14a)$$

$$(\text{SNR})_D \equiv \frac{\langle N_a \rangle}{(\Delta N_a^2)^{1/2}} \cong \frac{A_1}{2(\Delta a_1^2)^{1/2}} = \frac{1}{2}(\text{SNR})_2. \quad (14b)$$

For a weak, highly squeezed input signal, in which the power associated with the increased fluctuations of the unsqueezed (conjugate) quadrature rivals the mean-field power, the approximate expressions (14a) must be replaced by the exact expressions [Eqs. (5) and (7)].

The other important source of noise in homodyning is nonideal photodetectors. A photodetector with quantum efficiency  $\eta$  can be modeled as a lossless beam splitter with power transmissivity  $T \cong \eta$  followed by an ideal photodetector. The two inputs to the beam splitter are the signal and the vacuum state, described by annihilation operators  $d$  and  $d_0$ , respectively. The output of a nonideal photodetector is described by the annihilation operator  $d'$ , where [Eq. (2)]

$$d' = e^{i\Lambda}(\sqrt{\eta}d - e^{-i\nu}\sqrt{1-\eta}d_0) \quad (15)$$

and the phases  $\Lambda$  and  $\nu$  are inconsequential here. One-port homodyning with a nonideal photodetector gives the following output signal, noise, and SNR:

$$\langle N_d' \rangle_{\text{sig}} = \eta \langle N_d \rangle_{\text{sig}} \cong -2\eta\sqrt{T(1-T)}A_1B; \quad (16a)$$

$$\begin{aligned}\Delta N_d'^2 &= \eta^2\Delta N_d^2 + \eta(1-\eta)\langle N_d \rangle \\ &\cong (1-T)\eta^2B^2[4T\Delta a_1^2 \\ &\quad + 4(1-T)\Delta b_1^2 + (1-\eta)/\eta]; \quad (16b)\end{aligned}$$

$$(\text{SNR})_1' = \frac{2\sqrt{T}A_1}{[4T\Delta a_1^2 + 4(1-T)\Delta b_1^2 + (1-\eta)/\eta]^{1/2}} \quad (16c)$$

Two-port homodyning with identical nonideal photodetectors gives this output signal, noise, and SNR:

$$\langle P' \rangle = \eta \langle P \rangle = 2\eta BA_1, \quad (17a)$$

$$\begin{aligned}\Delta P'^2 &= \eta^2\Delta P^2 + \eta(1-\eta)[\langle N_c \rangle + \langle N_d \rangle] \\ &\cong \eta^2B^2[4\Delta a_1^2 + (1-\eta)/\eta]; \quad (17b)\end{aligned}$$

$$(\text{SNR})_2' \cong \frac{2A_1}{[4\Delta a_1^2 + (1-\eta)/\eta]^{1/2}} \quad (17c)$$

Direct detection of the same input signal would give the following output signal, noise, and SNR:

$$\begin{aligned}\langle N_a' \rangle &\equiv \eta \langle N_a \rangle \cong \eta A_1^2, \\ \Delta N_a'^2 &\cong \eta^2 A_1^2 [4\Delta a_1^2 + (1-\eta)/\eta]; \quad (18a)\end{aligned}$$

$$(\text{SNR})_D' \cong \frac{A_1}{[4\Delta a_1^2 + (1-\eta)/\eta]^{1/2}} = \frac{1}{2}(\text{SNR})_2. \quad (18b)$$

Equations (16) and (17) show that the absence of any contribution from LO quadrature-phase noise in the output noise of two-port homodyning can be a distinct advantage when one wants to detect squeezing in the input signal. For example, for two-port homodyning with efficient photodetectors ( $\eta \cong 0.9$ ) the ratio between the output noise produced by a coherent-state input signal ( $4\Delta a_1^2 = 1$ ) and that produced by an input signal that has been squeezed by a factor of 10 ( $4\Delta a_1^2 = e^{-2r} \cong 0.1$ ) is about 5:1. The same ratio for one-port homodyning with an ideal (coherent-state) LO ( $4\Delta b_1^2 = 1$ ) and a 90/10 beam splitter is about 3.5:1. For the same squeezed input signal, two-port homodyning offers an improvement in output SNR over one-port homodyning by a factor of roughly 1.3, and it has the additional advantage of not requiring a quiet local oscillator. More-efficient photodetectors and/or a more highly squeezed input signal would make this comparison more dramatic.

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## Validity of the Standard Quantum Limit for Monitoring Free-Mass Position

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### *ABSTRACT*

The validity of the standard quantum limit (SQL) for measurements which monitor the position of a free mass is examined. A necessary condition for such a measurement scheme to surpass the SQL is derived. This condition is then used to show that, at least for typical situations, measurements of position alone cannot beat the SQL, and simultaneous measurements of  $x$  and  $p$  might enable one to reach the SQL but not to surpass it.

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The standard quantum limit<sup>1,2</sup> (SQL) for monitoring the position of a free mass  $m$  says that in two successive, identical measurements of position  $x$  spaced a time  $\tau$  apart, the result of the second measurement cannot be predicted with an accuracy better than  $(\hbar\tau/m)^{1/2}$ . In this Letter I derive the SQL and discuss the conditions under which it limits any measurement scheme designed to monitor the position of a free mass. I then show that for measurements described by linear couplings to the position  $x$  or to both  $x$  and the momentum  $p$ , and at least for cases where the wave functions of the measuring apparatus(es) are Gaussian, the SQL cannot be surpassed.

The origin of the SQL will be shown to lie ultimately in the fact that under free evolution the Heisenberg operators  $x(0) \equiv x$  and  $x(\tau) \equiv x + (\hbar\tau/m)p$  do not commute:

$$[x(0), x(\tau)] = i\hbar\tau/m . \quad (1a)$$

From this fact it follows that

$$\sigma_x^2(0) + \sigma_x^2(\tau) \geq \hbar\tau/m , \quad (1b)$$

where  $\sigma_x^2(\tau) \equiv \langle(\Delta x)^2\rangle(\tau)$  and  $\Delta x \equiv x - \langle x \rangle$ ; the equality holds only when  $\sigma_x^2(0) = \sigma_x^2(\tau) = \hbar\tau/2m$ . [Henceforth I set  $\hbar \equiv 1$ .] In the following paragraphs I examine first measurements of  $x$  alone and then simultaneous measurements of  $x$  and  $p$ , using as illustrations for each the simple cases of linear coupling.

Consider first a measurement of  $x$  alone, in which a single measuring apparatus ("meter") is coupled, through a constant interaction Hamiltonian  $V$ , to the position  $x$  of the free mass. I assume that at time  $\tau = 0$   $V$  is turned on, and it is turned off a short time later at  $\tau = \tau^*$ ; by "short time" (or "strong coupling") I mean that during the measurement the free evolution of

the system and the meter may be ignored ("impulse approximation"). After the first measurement, the system is allowed to evolve freely until a time  $\tau \gg \tau^*$ , at which time the second measurement begins. The task is to find with what certainty one can predict the outcome of the second measurement, using the information one has about the state of the system after the result of the first measurement has been recorded.

Denote by  $X_1$  an observable of the meter whose measured value provides direct information about the position  $x$  of the free mass. The first measured value for  $X_1$  --  $X_1'$ , say -- is our best indicator for the value  $x_0$  of the position of the free mass just before the measurement. The uncertainty in  $X_1'$  -- i.e., the amount by which it might differ from  $x_0$  -- is equal to the sum of the intrinsic "width" of the free mass wave function before the measurement,  $\sigma_x^2(0)$  (in general unknown), and the finite resolution of our meter, denoted by  $\sigma_1^2$ :

$$\sigma_{X_1'}^2(\tau^*) = \sigma_x^2(0) + \sigma_1^2. \quad (2)$$

The important quantity to calculate, however, is the intrinsic uncertainty of the free mass position after we have completed and recorded our measurement; this involves a collapse of the joint wave function for meter-plus-free mass, whereby the  $X_1$ -dependence becomes an  $X_1'$ -dependence. Suppose the wave functions for the free mass and the meter before the first measurement are  $F(x;0)$  and  $M(X_1;0)$ , respectively. The joint wave function before the measurement is the product (meter and free mass initially uncoupled)

$$\Psi(x, X_1; 0) = F(x; 0) M(X_1; 0); \quad (3)$$

After the measurement the joint wave function is given by  $\Psi(x, X_1; \tau^*)$ . The (renormalized) wave function for the system, just after we have obtained the value  $X_1'$  from the first measurement, is given by

$$F'(x; \tau^*) = \frac{\Psi(x, X_1'; \tau^*)}{[P(X_1'; \tau^*)]^{1/2}}, \quad (4a)$$

where the probability distribution for the measured value  $X_1'$  is

$$P(X_1'; \tau^*) \equiv \int dx |\Psi(x, X_1'; \tau^*)|^2. \quad (4b)$$

The average position and its variance for the free mass after we have obtained the value  $X_1'$  are given by

$$\langle x \rangle'(\tau^*) = \int dx x |F'(x; \tau^*)|^2, \quad (5a)$$

$$\sigma_x^2(\tau^*) = \int dx |F'(x; \tau^*)|^2 (x - \langle x \rangle'(\tau^*))^2. \quad (5b)$$

Other relevant post-measurement quantities for the free mass which are similarly calculated are the mean momentum  $\langle p \rangle'(\tau^*)$ , the variance of the momentum  $\sigma_p^2(\tau^*)$ , and the correlation  $\sigma_{xp}'(\tau^*)$ , where  $\sigma_{xp}'(\tau^*) \equiv \langle \Delta x' \Delta p' \rangle_{\text{sym}}(\tau^*)$ ,  $\Delta x' \equiv x - \langle x \rangle'(\tau^*)$ , and "sym" means "symmetrized." In terms of these quantities the position and variance of the free mass at time  $\tau$ , just before the second measurement, are given by:

$$\langle x \rangle'(\tau) = \langle x \rangle'(\tau^*) + (\tau/m) \langle p \rangle'(\tau^*). \quad (6a)$$

$$\sigma_x^2(\tau) = \sigma_x^2(\tau^*) + (\tau/m)^2 \sigma_p^2(\tau^*) + 2(\tau/m) \sigma_{xp}'(\tau^*); \quad (6b)$$

The result of our second measurement,  $X_1'$ , will be our best indicator for  $\langle x \rangle'(\tau)$ ; the uncertainty in  $X_1'$  -- i.e., the amount by which it may differ from the predicted  $\langle x \rangle'(\tau)$  -- is equal to the sum of the intrinsic "width" of the free mass just before the second measurement and the resolution of our meter, denoted now (allowing for a new meter) by  $\sigma_2^2$  [cf. Eq.(2)]:

$$\sigma_{X_1'}^2(\tau + \tau^*) = \sigma_x^2(\tau) + \sigma_2^2; \quad (7)$$

The primes present in Eqs. (5)-(7) are reminders that these quantities depend on one's having obtained the result  $X_1'$  in the first measurement -- i.e. that they have been calculated using the renormalized wave function of Eq.(4a).

If it is the case that  $\sigma_2^2 \geq \sigma_x^2(\tau^*)$ , then the SQL is obeyed, by Eqs.(1b), (7) and the impulse approximation. Suppose now that the meter used in the second measurement has the same resolution as that used in the first:  $\sigma_2^2 = \sigma_1^2$  (this is the situation to which the SQL always refers). Through Eq. (5b) one has a definite relation between  $\sigma_1^2$  and  $\sigma_x^2(\tau^*)$ , and that relation dictates whether or not the SQL can be surpassed. In particular, if  $\sigma_x^2(\tau^*) \leq \sigma_1^2$ , the SQL cannot be surpassed.

In general, one does not know the free mass wave function  $F(x;0)$ , but the meter wave function  $M(x_1;0)$  is under one's control. Let us assume that the meter is a one-dimensional system with two degrees of freedom described by the conjugate observables  $X_1$  and  $P_1$ , and consider the case where the measurement process is described by the linear interaction Hamiltonian  $V = KxP_1$ . In units with  $K\tau^* = 1$ , measurement of  $X_1$  then corresponds to a measurement of  $x$ . The intrinsic resolution of the meter [ $\sigma_1^2$  in Eq.(2)] is equal to  $\sigma_{X_1}^2(0)$ . In this context the simplest case to analyze is when both  $F(x;0)$ , and  $M(X_1;0)$  are Gaussian:

$$F(x;0) \equiv N_x(0) e^{ip_0(x-x_0/2)} e^{-(x-x_0)^2/b(0)} ; \quad (8a)$$

$$N_x(0) \equiv [2\pi\sigma_x^2(0)]^{-1/4} e^{-i\delta_0/2} , \quad b(0) \equiv \frac{4\sigma_x^2(0)}{1+i\varepsilon(0)} ,$$

$$\varepsilon_0 = -\tan 2\delta_0 \equiv -2\sigma_{xp}(0) ; \quad (8b)$$

analogous relations hold for  $M(X_1;0)$  [quantities  $N_{X_1}(0)$ ,  $b_1(0)$ , and  $\varepsilon_1(0)$ ]. A Gaussian "contractive" state<sup>3</sup> is one for which  $\varepsilon_0 > 0$ . Under free evolution,

$\varepsilon(\tau) = \varepsilon_0 - 2(\tau/m)\sigma_p^2(0)$ ,  $\sigma_p^2(\tau) = \sigma_p^2(0)$ , and  $\sigma_x^2(\tau) = [1 + \varepsilon^2(\tau)]/4\sigma_p^2(0)$ ; hence  $\varepsilon(\tau)$  always decreases during free evolution (correlation of  $\Delta x$  and  $\Delta p$  always increases), and when it passes through zero, [ $\varepsilon(\tau_m) \equiv 0$ ],  $\sigma_x^2(\tau_m) = 1/4\sigma_p^2(0)$  is equal to its minimum possible value. These and other special states are not discussed further here, since they are relevant to the SQL only through Eqs. (5) and, at least for Gaussian states, the contractive quality does not help one beat the SQL.

For a measurement described by the linear coupling  $V = KxP_1$ , the joint wave function after the measurement is given by

$$\Psi(x, X_1; \tau^*) = F(x; 0) M(X_1 - x; 0) \quad ; \quad (9a)$$

For Gaussian wave functions  $F(x; 0)$  and  $M(X_1; 0)$  [Eqs.(8)], Eq.(9a), together with Eqs.(4), tells one that the variance of the free mass position at time  $\tau^*$  just after the first measurement is equal to

$$\sigma_x^2(\tau^*) = \frac{\sigma_x^2(0)\sigma_1^2}{\sigma_x^2(0) + \sigma_1^2} \leq \sigma_1^2. \quad (9b)$$

Hence, if  $\sigma_2^2 = \sigma_1^2$  (meters with identical resolutions), the SQL is obeyed [Eqs (9b), (7) and (1b)]. Eq. (9b) also says that the best one can do in this type of measurement is to *reach* the SQL, which is accomplished when one has essentially no information about the position of the free mass before the first measurement [ $\sigma_x^2(0) \gg \sigma_1^2$ ]. The result  $\sigma_x^2(\tau^*) \leq \sigma_1^2$  agrees with our intuition, and it is typical of measurements of  $x$  alone, due to the fact that the free-mass wave function  $F(x; 0)$  (typically of finite width) contributes to the post-measurement renormalized wave function  $F'(x; \tau^*)$  [Eqs.(4) and (9a)]. However, a more thorough examination of the restrictions on meter wavefunctions  $M(x; 0)$  necessary to ensure that  $\sigma_x^2(\tau^*) \leq \sigma_1^2$  may be in order.

Yuen<sup>3</sup> has implied that a simultaneous measurement of  $x$  and  $p$ <sup>4-6</sup> might enable one to beat the SQL. At this point the discussion is in some sense academic, for most experiments which attempt to monitor  $x$  do so for the purpose of detecting a classical force  $F(x,t)$ ;<sup>2</sup> this task would be much easier if we simply coupled to  $p$  alone, but we don't know how to do that. Nevertheless, the idea of simultaneous measurements of noncommuting observables is an important one, and such measurements do play an important role in quantum optics, for example.<sup>6,7</sup> In the following paragraphs I show that, at least for the case of a linear coupling to  $x$  and  $p$  and for Gaussian meter wave functions, the use of an optimally-arranged simultaneous measurement of  $x$  and  $p$  might enable one to reach the SQL, but not to surpass it.

I will assume that the simultaneous measurement is accomplished by using two one-dimensional "meters", described by the conjugate observables  $X_1, P_1$  and  $X_2, P_2$ , respectively. The prototype for such a measurement is again one in which the coupling is linear, with  $V = K(xP_1 + pP_2)$ . In units with  $K\tau^* \equiv 1$  and in the impulse approximation, measurement of  $X_1$  [ $X_2$ ] corresponds to a measurement of  $x$  [ $p$ ]. The values  $X_1'$  and  $X_2'$  obtained in the first measurement are our best indicators for the position  $x_0$  and momentum  $p_0$  of the free mass just before the measurement. The uncertainties in  $X_1'$  and  $X_2'$  (most easily calculated in the Heisenberg picture) are

$$\begin{aligned}\sigma_{X_1'}^2(\tau^*) &= \sigma_x^2(0) + \sigma_1^2, \\ \sigma_{X_2'}^2(\tau^*) &= \sigma_p^2(0) + \lambda_1^2,\end{aligned}\tag{10a}$$

where the quantities  $\sigma_1^2$  and  $\lambda_1^2$  describe the position and momentum resolution, respectively, of the two-meter measuring apparatus. In particular, for the linear coupling just mentioned,



$$\sigma_1^2 = \sigma_{X_1}^2(0) + \frac{1}{4}\sigma_{P_2}^2(0) ,$$

$$\lambda_1^2 = \sigma_{X_2}^2(0) + \frac{1}{4}\sigma_{P_1}^2(0) . \quad (10b)$$

Let the initial wave functions for the free mass and the two meters be  $F(\mathbf{x};0)$ ,  $M(X_1;0)$  and  $N(X_2;0)$ . Straightforward calculation of the evolution operator gives for the joint wave function after the first measurement

$$\Psi(\mathbf{x}, X_1, X_2; \tau^*) = \int \frac{dp_2}{2\pi} \bar{N}(p_2;0) F(\mathbf{x} - p_2;0) M(X_1 - \mathbf{x} + \frac{1}{2}p_2;0) , \quad (11)$$

where  $\bar{N}(p_2;0)$  is the Fourier transform of  $N(X_2;0)$ . Earlier discussion [Eqs.(9)] indicated that in order to make  $\sigma_{\mathbf{x}}^2(\tau^*) \geq \sigma_1^2$  (and hence have a chance of beating the SQL) one needs to find a way of making the wave function of the free mass just after the measurement be independent of its wave function just before the measurement. Arthurs and Kelly <sup>4</sup> have given a prescription for accomplishing this: one chooses the two meters to have suitably "balanced", Gaussian wave functions, and finds that just after the measurement the free mass wave function is also a Gaussian, with mean position and momentum equal to the measured values

$$\langle \mathbf{x} \rangle(\tau^*) = X_1' , \quad \langle \mathbf{p} \rangle(\tau^*) = X_2' , \quad (12a)$$

and position and momentum variances equal to the measurement resolutions

$$\sigma_{\mathbf{x}}^2(\tau^*) = \sigma_1^2 , \quad \sigma_{\mathbf{p}}^2(\tau^*) = \lambda_1^2 . \quad (12b)$$

Specifically, require that  $M(X_1;0)$  and  $N(X_2;0)$  have the general Gaussian form defined by Eqs. (8), and balance them by making  $b_1(0) b_2(0) \equiv 1$ , so that

$$\varepsilon_1(0) + \varepsilon_2(0) = 0 \quad \text{and} \quad 4\sigma_{x_2}^2(0) = \sigma_{p_1}^2(0) . \quad (13)$$

The joint wave function after the first measurement then has a simple  $x$ -dependence:

$$\Psi(x, X_1, X_2; \tau^*) \propto e^{-(x-X_1)^2/2b_1(0)} e^{iK_2 x} \quad (14a)$$

which tells one immediately that

$$\sigma_x^2(\tau^*) = 2\sigma_{X_1}^2(0) \equiv \sigma_1^2 \quad (14b)$$

[cf. Eqs. (9)]. It is also clear from Eq. (14a) that the system will be left in a contractive state if the  $X_1$ -meter was in one, since  $\varepsilon'(\tau^*) = \varepsilon_1(0)$ . However, this fact does not help us beat the SQL, for if our second measurement uses the same meter set-up as the first measurement ( $\sigma_2^2 = \sigma_1^2$ ), then Eqs. (14b), (7) and (1b) tell us that a simultaneous measurement of  $x$  and  $p$  can at best allow us to *reach* the SQL:

$$\sigma_{X_1}^2(\tau + \tau^*) = \sigma_2^2 + \sigma_x^2(\tau) = \sigma_x^2(\tau^*) + \sigma_x^2(\tau) \geq (\tau/m) \quad (15)$$

From Eq. (14a) it is clear that this result is true regardless of whether we look at the second meter ( $X_2'$ ). Again, however, a more thorough examination of the restrictions on the meter wave functions necessary to ensure that  $\sigma_x^2(\tau^*) \leq \sigma_1^2$  may be in order.

I have stated in this letter the criterion which must be satisfied if there is to be any chance of beating the SQL. In the context of measurements described by linear couplings to  $x$  or  $x$  and  $p$ , I have shown that for measuring apparatuses described by Gaussian wave functions (which includes the Gaussian contractive states proposed recently by Yuen<sup>3</sup> as a means of surpassing the SQL) the SQL *cannot* be surpassed, though it might be approached by performing a balanced simultaneous measurement of  $x$  and  $p$ . A more detailed examination of the general requirements on the meter wave function(s) necessary to enforce (or

not enforce) the SQL would be interesting and useful.

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## Torsional oscillations of neutron stars\*

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**Summary.** Motivated by the possibility that torsional oscillations of neutron stars may be observable in the timing of pulsar subpulses and/or in future gravitational-wave detectors, this paper develops the detailed mathematical theory of such torsional oscillations and of the gravitational waves they emit. The oscillations are analysed using the formulation of first-order perturbations of a fully general relativistic spherical stellar model. All sources of damping are ignored except gravitational radiation reaction. The perturbations are resolved into spherical harmonics, which decouple from each other. For each harmonic this paper presents equations of motion, an action principle, an energy conservation law and a Liapunov-type proof that the oscillations are always stable. Each harmonic is then resolved into normal modes with outgoing gravitational waves (time dependence  $e^{i\omega t}$  with  $\omega$  complex) and an eigenvalue problem is posed for the eigenfunctions and the eigenfrequencies  $\omega$ . Five methods of solving the eigenvalue problem are presented; three methods are valid in general (the method of resonances, the variational method and the method of energy conservation); one is valid in the slow-motion approximation (wavelength of waves large compared to star) and one is valid in the weak-gravity approximation. For stellar models with weak gravity and with radially constant density and shear modulus the eigenvalue problem is solved analytically.

An appendix develops a general theory of action principles for systems with radiative boundary conditions – a theory which is then used to derive the action principles in the body of the paper and which could be useful for a variety of other problems involving physical systems coupled to radiation.

### 1 Introduction

If torsional oscillations of neutron stars could be observed, then comparisons of their measured periods and  $Q$ s with theoretical models would give valuable information not only

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about neutron star structure, but also about the physics of matter at subnuclear and supranuclear densities. There are two hopes for such observations: pulsar timing data and gravitational radiation. Van Horn (1980) has pointed out that the ‘marching subpulses’ observed in some pulsars have the same range of periods, 10–50 ms, as low-order torsional oscillations of neutron star crusts; and on this basis he has argued that such oscillations may be the clock which regulates the marching subpulses. And Dyson (1972) has pointed out that, if neutron stars have solid cores, then quakes in those cores should generate torsional oscillations which might produce gravitational waves strong enough to detect on Earth.

With these two applications in mind, and with hope that they or others will materialize, we construct in this paper the detailed mathematical theory of torsional oscillations of non-rotating, general relativistic stellar models with isotropic shear moduli  $\mu$ .

The analogous general relativistic theory of non-spherical compressional oscillations of non-rotating perfect-fluid stars was laid out a number of years ago by Thorne & Campolattaro (1967), Price & Thorne (1969), Thorne (1969a, b), Campolattaro & Thorne (1970), Ipser & Thorne (1973), Detweiler & Ipser (1973), Thorne (1983, in preparation). Those eight papers developed many facets of the theory. This paper is rather long because it attempts to develop, all at once, all of those same facets for the theory of torsional oscillations, and several more facets besides.

To set the stage for our analysis, we shall review briefly the structures of neutron stars and the characteristic magnitudes of various quantities associated with them; for further detail see, e.g., Baym & Pethick (1975, 1979) and references therein.

Observation and theory agree that typical neutron stars have masses  $M \sim 1M_{\odot}$  and radii  $R \sim 10$  km. Theory predicts with great confidence that within minutes after the star is born, its crust will cool enough to solidify into a crystal governed by Coulomb forces between atomic nuclei. This crystalline crust should extend from the star’s atmosphere inward to a depth of order 1 km, where the density is within a factor 2 of nuclear,  $\rho \approx (1.5-3) \times 10^{14} \text{ g cm}^{-3}$ . Throughout the crust the shear modulus  $\mu$  is computed to be nearly proportional to density  $\rho$ , with

$$(\mu/\rho)^{1/2} = v_s \approx 1 \times 10^8 \text{ cm s}^{-1}. \quad (1)$$

Here  $v_s$  is the speed of non-relativistic shear waves (Ruderman 1968; Pandharipande, Pines & Smith 1976; Hansen & Cioffe 1980) (see equation 20 for a relativistic correction).

It is now widely believed that below the solid crust resides a superfluid mantle, which extends inward through a thickness of roughly 5 km and through a density range of  $(1.5-3) \times 10^{14}$  to  $(5-10) \times 10^{14} \text{ g cm}^{-3}$ , until it meets the star’s  $\sim 4$  km core. The physical state of the core is highly uncertain. Possibilities include a pion-condensed state, which might or might not be a solid governed by nuclear forces; an ‘abnormal state’ in which the nucleons become practically massless; a degenerate Fermi liquid of quarks, etc. The possibility of a solid core was viewed with much favour between 1971 and 1974, both on grounds of nuclear many-body calculations and on grounds of a reasonable fit between the theory of core quakes and observations of glitches in the timing of the Vela pulsar (Pines, Shaham & Ruderman 1974, see Hansen 1974, p. 189). However, by 1975 improved many-body calculations had cast doubt on the likelihood that supranuclear matter will solidify. The doubt remains today, but the calculations are far from convincing either way; see Baym & Pethick (1975, 1979) for details and references. If the core is a solid, then its shear modulus  $\mu$  could be as large as its pressure  $P$ , or it might be somewhat smaller:

$$(\mu/\rho)^{1/2} = v_s \leq (P/\rho)^{1/2} \approx 1 \times 10^{10} \text{ cm s}^{-1}. \quad (2)$$

Hansen & Cioffe (1980) have used Newtonian theory to compute the torsional oscillation

periods of neutron star crusts. As one might expect, they obtain for modes with no radial nodes (so transverse wavenumber dominates)

$$\text{Period} = 2\pi/\omega \approx 2\pi [l(l+1)]^{-1/2} R/v_s \sim 20 \text{ ms} \quad \text{for } l=2, \quad (3a)$$

where  $l = 1, 2, 3, \dots$  is the spherical-harmonic index. Relativistic effects (especially gravitational redshifts and the dragging of inertial frames) are likely to change these periods by  $\sim 10$ – $50$  per cent. These periods are a factor  $\sim 10$  longer than would be compressional-oscillation periods for the crust, because the electrostatic forces which govern the crystal and its torsional oscillations are  $\sim 100$  times weaker than the degeneracy forces and nucleon-nucleon forces which govern compressional oscillations. Because the crust's torsional oscillations are so slow,  $v_s/c \ll 1$ , they can be described very accurately by the 'slow-motion approximation' to general relativity (Thorne 1980; Section 4.5 of this paper) which predicts gravitational waves so weak that it is hopeless to ever detect them:

$$h \sim 6 \left( \frac{GM_{\text{cr}}}{rc^2} \right) \left( \frac{v_s}{c} \right)^3 \beta \sim 10^{-28} \left( \frac{10 \text{ kpc}}{r} \right) \left( \frac{\beta}{10^{-3}} \right) \quad \text{for } l=2. \quad (3b)$$

Here  $h$  is the dimensionless gravity-wave amplitude,  $r$  is the distance from the Earth to the star,  $\beta$  is the dimensionless amplitude of the star's shearing oscillations,  $M_{\text{cr}} \approx 0.1 M_\odot$  is the mass of the crust, and we have specialized to quadrupole modes which are the strongest emitters. Gravitational radiation reaction will damp the crustal oscillations with an  $e$ -folding time

$$\tau \sim 0.3 (GM_{\text{cr}}/Rc^2)^{-1} (v_s/c)^{-5} \omega^{-1} \sim 10^4 \text{ yr}; \quad (3c)$$

cf. equations (76).

If the core is solid and has  $\mu \sim P$  (as was widely believed in the early 1970s), then the periods of its torsional oscillations would be roughly the same as those of its compressional oscillations:

$$\text{Period} = 2\pi/\omega \approx 2\pi R_{\text{co}}/v_s \sim 0.3 \text{ ms} \quad \text{for } l=2. \quad (4a)$$

where  $R_{\text{co}} \approx 4 \text{ km}$  is the core radius. Because the torsional oscillations emit 'current quadrupole' gravitational waves (gravitational analogue of magnetic quadrupole), whereas the compressional oscillations emit 'mass quadrupole' waves (analogue of electric quadrupole), the waves from torsional oscillations will be weaker by  $(v_s/c) \sim 1/3$  and will be damped more slowly by  $(v_s/c)^{-2} \sim 10$  than those from compressional oscillations:

$$h \sim 0.3 \left( \frac{GM_{\text{co}}}{rc^2} \right) \left( \frac{v_s}{c} \right)^3 \beta \sim 3 \times 10^{-23} \left( \frac{10 \text{ kpc}}{r} \right) \left( \frac{\beta}{10^{-3}} \right), \quad (4b)$$

$$\tau \sim 30 \left( \frac{GM_{\text{co}}}{R_{\text{co}}c^2} \right)^{-1} \left( \frac{v_s}{c} \right)^{-5} \omega^{-1} \sim 1 \text{ s}. \quad (4c)$$

(The coefficients used here are extrapolated from strong-gravity, fast-motion calculations of compressional oscillations by Thorne (1969a); the coefficients used in equations (3) for crustal oscillations are based on the weak-gravity, slow-motion calculations of equations (76) of this paper.) Assuming that the Vela pulsar has a solid core, and that the glitches observed every 2 or 4 yr in the Vela pulse arrival times are due to core quakes, Pines *et al.* (1974) have estimated that the total strain energy released in each quake is  $\sim 10^{45}$  erg corresponding to  $\beta \sim 10^{-4}$ , which at a distance  $r \sim 500 \text{ pc}$  would produce  $h \sim 6 \times 10^{-23}$ . Other, younger neutron stars might be stronger emitters. For comparison, the best currently operating gravitational-



wave detector (Stanford's bar; Boughn *et al.* 1982) has a burst sensitivity  $h \sim 5 \times 10^{-18}$  (rms noise  $h \sim 1 \times 10^{-18}$ ) at a period  $P \sim 10^{-3}$  s; the design sensitivity of a multikilometre laser-interferometer gravity-wave detector being planned for the late 1980s (Drever *et al.* 1982) for 10 kHz waves that last 1 s, would be  $h \sim 3 \cdot 10^{-23}$ . Thus, it is not inconceivable that corequakes in neutron stars could be detected and studied routinely in the 1990s.

We turn now to the detailed analysis of torsional oscillations of spherical, non-rotating, relativistic stellar models. The spherical symmetry of the unperturbed star guarantees that the oscillations can be decoupled into modes of definite spherical-harmonic indices ( $l, m$ ) and definite parity. In the language of previous analyses (e.g. Regge & Wheeler 1957) pure torsional oscillations are the normal modes of odd-type or magnetic-type parity,  $\pi = (-1)^{l+1}$ . Such modes do not exist for  $l=0$  (monopole). They exist for  $l=1$  (dipole) but cannot generate gravitational waves. For  $l \geq 2$  they do generate waves. The differences between  $l=1$  and  $l \geq 2$  are so fundamental that they are best analysed in different gauges and with different mathematical techniques. Sections 2–4 of this paper are devoted to modes with  $l \geq 2$ ; Section 5 treats  $l=1$ . Section 2 lays the foundation for the analysis with  $l \geq 2$ , including the description of the unperturbed star (Section 2.1), the coordinates, metric and Ricci tensor for the perturbed star (Section 2.2) and the description of the material motion – i.e. the displacement function, four-velocity and stress-energy tensor (Section 2.3). Section 3 presents the details of the analysis, including the equations of motion for the matter and the gravitational field (Section 3.1), the boundary conditions on the matter and field variables (Section 3.2), the form of the gravitational waves emitted and their energy loss rate (Section 3.3), an action principle and local law of energy conservation for the pulsations and their waves (Section 3.4), and a Liapunov-type proof that so long as the shear modulus is positive the star is stable against arbitrary (but first-order) torsional perturbations (Section 3.5). Section 4 analyses the star's outgoing wave modes (pulsations with sinusoidal time dependences and complex frequencies), including a formulation of the eigenvalue problem for the normal modes (Section 4.1) and various methods of solving the eigenvalue problem: the method of resonances (Section 4.2), a variational principle method (Section 4.3), an energy conservation method (Section 4.4), a method valid in the slow-motion approximation (Section 4.5), and a method for stars with weak internal gravity (Section 4.6, which also includes an analytic solution of the eigenvalue problem for weak-gravity stars with constant density and shear modulus). The analysis of dipole oscillations in Section 5 follows a similar outline – but with all issues of gravitational radiation absent. Some mathematical details are relegated to appendices. Of special interest may be Appendix B which elaborates and extends an elegant formulation (by Friedman & Schutz 1975) of the general theory of action principles for systems that can radiate waves – any kind of waves – to infinity.

Throughout this paper we use the mathematical conventions of Misner, Thorne & Wheeler (1973, cited henceforth as MTW), including setting the speed of light and Newton's gravitation constant to unity and denoting covariant derivatives by semicolons and partial derivatives by commas.

## 2 Foundations for the analysis: $l \geq 2$

### 2.1 THE UNPERTURBED STAR

The unperturbed spherical star is described in the standard manner (see, e.g. MTW). The metric, in Schwarzschild coordinates, is

$$\begin{aligned} ds^2 &= (ds^2)_0 \equiv -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2) \\ &\equiv \gamma_{\mu\nu} dx^\mu dx^\nu, \end{aligned} \tag{5a}$$

where  $\Phi(r)$  and  $\Lambda(r)$  are functions of the radial coordinate,  $r$ . The 'mass inside radius  $r$ ',  $m(r)$ , is defined by

$$e^{2\Lambda} \equiv 1 - 2m/r. \quad (5b)$$

The density of total mass-energy and the (isotropic) pressure are denoted by  $\rho$  and  $P$ , respectively. The standard equations of structure for the equilibrium star are (i) the mass equation

$$m \equiv \int_0^r 4\pi r^2 \rho(r) dr. \quad (5c)$$

(ii) The Oppenheimer–Volkov equation of hydrostatic equilibrium

$$\frac{dP}{dr} = - \frac{(\rho + P)(m + 4\pi r^3 P)}{r^2(1 - 2m/r)}, \quad P(R) = 0 \quad (5d)$$

where  $R =$  (value of  $r$  at surface of star), and (iii) the source equation for  $\Phi(r)$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{r^2(1 - 2m/r)}, \quad \Phi(\infty) = 0. \quad (5e)$$

From equations (5b) to (5e) the following useful relations are easily derived:

$$\Lambda' = 1/2 r^{-1} (1 - e^{2\Lambda}) + 4\pi r \rho e^{2\Lambda}, \quad (5f)$$

$$\Phi' = -1/2 r^{-1} (1 - e^{2\Lambda}) + 4\pi r P e^{2\Lambda}, \quad (5g)$$

$$\Phi' + \Lambda' = 4\pi r (\rho + P) e^{2\Lambda} \quad \text{or} \quad (\rho + P) = - (4\pi r)^{-1} (e^{-\Phi - \Lambda})' e^{\Phi - \Lambda}, \quad (5h)$$

$$e^{2\Phi} = e^{-2\Lambda} = 1 - 2M/r \quad \text{outside the star,} \quad (5i)$$

where primes denote radial derivatives,  $\partial/\partial r$ , and where  $M \equiv m(R)$  is the star's total mass and  $R$  is its radius. The complete unperturbed model is specified by giving the radial distributions of  $\rho$ ,  $P$ ,  $\Phi$ ,  $\Lambda$  (or  $m$ ) and the shear modulus  $\mu(r)$ . We assume in this paper that  $\mu$  is isotropic ('scalar field').

## 2.2 COORDINATES, METRIC AND RICCI TENSOR FOR PERTURBED STAR

For the perturbed star we introduce coordinates  $(t, r, \vartheta, \phi)$  which reduce to those of the unperturbed star when the oscillations vanish. We linearize our entire analysis about the unperturbed configuration and resolve the oscillations into spherical harmonics of definite indices  $l, m$  and parity  $\pi$ . The spherical symmetry of the unperturbed configuration guarantees that modes of different  $l, m, \pi$  superpose linearly (i.e. no mixing). Therefore, we can restrict attention to modes with fixed  $l, m, \pi$  (pure modes). In this paper we do not consider 'even-parity' modes [ $\pi = (-1)^l$ ] because they represent compressional oscillations rather than pure torsional oscillations; see Thorne & Campolattaro (1967) for discussion. The odd-parity torsional modes with fixed  $l$  but different  $m$  can be obtained from each other by linear combinations of rotations about the star's centre. Thus, without loss of generality, we can specialize to an odd-parity mode with definite  $l$  and with  $m = 0$ ; and we henceforth use  $m$  exclusively to denote the mass inside radius  $r$  (equation 5b) and not a spherical harmonic index.

The metric  $g_{\mu\nu}$  for our oscillating star consists of the unperturbed metric  $\gamma_{\mu\nu}$  plus components  $h_{\mu\nu}$  which describe our odd-parity perturbation:

$$ds^2 = (ds^2)_0 + h_{\mu\nu} dx^\mu dx^\nu. \quad (6a)$$

Clearly,  $h_{tt}$ ,  $h_{rr}$  and  $h_{\theta\theta}$  are scalars under rotation and thus have even parity, which means they must vanish. Further (*cf.* appendix A of Thorne & Campolattaro 1967) we are free to specialize our coordinates (choose our gauge) so as to make all other components of  $h_{\mu\nu}$  vanish except the following:

$$\begin{aligned} h_{t\phi} = h_{\phi t} &\equiv -r^2 \dot{y}(t, r) b_\phi \equiv -r^2 \dot{y} \sin \vartheta \partial_\vartheta P_l(\cos \vartheta), \\ h_{r\phi} = h_{\phi r} &\equiv -r e^{\Lambda - \Phi} Q(t, r) b_\phi. \end{aligned} \quad (6b)$$

Here  $b_\phi$  is equal to  $[4\pi/(2l+1)]^{1/2}$  times the Regge–Wheeler (1957) odd-parity vector spherical harmonic  $\Phi_\phi^{l0}$ ; in future equations we shall raise and lower the index on  $b_\phi$  with the metric of the unit sphere:

$$b_\phi \equiv \sin^2 \vartheta b^\phi = \sin \vartheta \partial_\vartheta P_l(\cos \vartheta); \quad (6c)$$

the indices of the metric perturbation functions  $h_{\mu\nu}$  are raised and lowered with the unperturbed metric,  $\gamma_{\mu\nu}$  (equation 5a). In equations (6b, c),  $P_l(\cos \vartheta)$  is the Legendre polynomial of order  $l$ , and the dot over  $y$  denotes a time-derivative  $\partial/\partial t \equiv \partial_t$ . The perturbation function  $\dot{y} b^\phi$  is equal to the angular velocity of a zero-angular momentum observer (ZAMO; *cf.*, e.g., Bardeen, Press & Teukolsky 1972); thus  $y b^\phi$  is the angular displacement of a ZAMO and is dimensionless. Outside the star the perturbation function  $Q$  is equal to the Regge–Wheeler (1957) gravitational-wave variable, aside from a multiplicative constant.

The metric perturbation (6b) produces a perturbation of the Ricci tensor with the following non-vanishing components (Thorne & Campolattaro 1967, equation B3 as corrected in the erratum – but note the different notation and signature used there):

$$\delta R_{t\phi} = \delta R_{\phi t} = \left\{ \frac{e^{\Phi - \Lambda}}{2r^2} [r^4 e^{-\Phi - \Lambda} (\dot{y}' - e^{\Lambda - \Phi} \dot{Q}/r)]' + e^{-\Phi - \Lambda} (r e^{\Phi - \Lambda})' \dot{y} - \frac{l(l+1)}{2} \dot{y} \right\} b_\phi, \quad (7a)$$

$$\delta R_{r\phi} = \delta R_{\phi r} = \left\{ 1/2 r^2 e^{-2\Phi} (\dot{y}' - e^{\Lambda - \Phi} \dot{Q}/r) + r^{-1} e^{-2\Phi} (r e^{\Phi - \Lambda})' \dot{Q} - \frac{l(l+1)}{2r} e^{\Lambda - \Phi} \dot{Q} \right\} b_\phi, \quad (7b)$$

$$\delta R_{\vartheta\phi} = \delta R_{\phi\vartheta} = \{ 1/2 r^2 e^{-2\Phi} \ddot{y} - 1/2 e^{-\Lambda - \Phi} (rQ)' \} \sin^2 \vartheta b^\phi_{,\vartheta}. \quad (7c)$$

Here and below primes denote radial derivatives and dots denote time derivatives,  $Q' \equiv \partial Q/\partial r \equiv Q_{,r}$  and  $\dot{Q} \equiv \partial Q/\partial t \equiv Q_{,t}$ . Note that  $\sin^2 \vartheta b^\phi_{,\vartheta} \equiv \sin^2 \vartheta \partial_\vartheta b^\phi$  is equal to  $[16\pi/(2l+1)]^{1/2}$  times the Regge–Wheeler odd-parity tensor spherical harmonic  $\chi^l_{\vartheta\phi}$ .

We now go on to consider the motion of the star and the interaction of its matter with the surrounding spacetime geometry.

### 2.3 DISPLACEMENT FUNCTION, FOUR-VELOCITY AND STRESS-ENERGY TENSOR FOR PERTURBED STAR

In the perturbed star, the coordinate location of a specific particle of stellar matter oscillates. We describe its oscillating location by a displacement vector  $\xi$  whose components  $\xi^r$ ,  $\xi^\vartheta$  and  $\xi^\phi$  are functions of the particle's original location ( $r$ ,  $\vartheta$ ,  $\phi$ ) and of time  $t$ :

$$r_{\text{pert}} = r + \xi^r(t, r, \vartheta, \phi); \quad \vartheta_{\text{pert}} = \vartheta + \xi^\vartheta(t, r, \vartheta, \phi); \quad \phi_{\text{pert}} = \phi + \xi^\phi(t, r, \vartheta, \phi). \quad (8a)$$

Because  $\xi^r$  is a scalar under rotations about the centre of the star and thus has even parity, it must vanish. The angular displacements form a vector field on the unit sphere,  $\xi = \xi^\vartheta \partial_\vartheta +$

$\xi^\phi \partial_\phi$ , and must therefore have the angular dependence of a vector spherical harmonic of definite  $l$ , of  $m = 0$  and of parity  $\pi = (-1)^{l+1}$ :

$$\xi^r = 0; \quad \xi^\vartheta = 0; \quad \xi^\phi \equiv Y(t, r) (\sin \vartheta)^{-1} \partial_\vartheta P_l(\cos \vartheta) = Y(t, r) b^\phi. \quad (8b)$$

Note that just as  $yb^\phi$  is the angular displacement of a ZAMO, so  $Yb^\phi$  is the angular displacement of the stellar matter.

The four-velocity  $u^\mu$  of a particle with world line (8a, b) is obtained from the relations  $u^j/u^t = dx^j_{\text{pert}}/dt = \xi^j_{,t}$ ;  $u^\mu u^\nu g_{\mu\nu} = -1$ . The result, linearized in the perturbation functions  $y, Q, Y$ , is

$$u^t = e^{-\Phi}, \quad u^r = 0, \quad u^\vartheta = 0, \quad u^\phi = e^{-\Phi} \dot{Y} b^\phi. \quad (9)$$

The radial and angular variations of the azimuthal displacement  $\xi^\phi$  produce deformations (shears) of the star's crystal lattice. These deformations are described by a shear tensor  $S_{\alpha\beta}$ . When viewed in the orthonormal comoving frame of a particle of the stellar material  $S_{\alpha\beta}$  is purely spatial ( $S_{00} = S_{0j} = S_{j0} = 0$ ), and its spatial components  $S_{jk} = S_{kj}$  are precisely those of the non-relativistic theory of a stressed medium (see, e.g. Landau & Lifshitz 1970). Hence, in this proper reference frame of the particle, the shearing motion produces a restoring stress given by the standard non-relativistic formula  $T_{jk}^{\text{shear}} = -2\mu S_{jk}$ , where  $\mu$  is the shear modulus. By general covariance (*cf.* MTW, chapter 16) this equation can be rewritten in the coordinate-independent form

$$T_{\alpha\beta}^{\text{shear}} = -2\mu S_{\alpha\beta}. \quad (10)$$

To calculate the components  $S_{\alpha\beta}$  of the shear tensor in our Regge–Wheeler coordinate system we proceed as follows: First, we calculate the rate of shear  $\sigma_{\alpha\beta}$  from standard formulae (see, e.g., MTW, exercise 22.6):

$$\sigma_{\alpha\beta} = 1/2 (u_{\alpha;\mu} P^\mu_\beta + u_{\beta;\mu} P^\mu_\alpha) - 1/3 P_{\alpha\beta} u^\mu{}_{;\mu}, \quad (11)$$

where

$$P_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta.$$

The result is

$$\sigma_{r\phi} = \sigma_{\phi r} = 1/2 r^2 e^{-\Phi} (\dot{Y}' - e^{\Lambda - \Phi} \dot{Q}/r) b_\phi; \quad (12a)$$

$$\sigma_{\vartheta\phi} = \sigma_{\phi\vartheta} = 1/2 r^2 e^{-\Phi} \dot{Y} \sin^2 \vartheta b^\phi{}_{,\vartheta}; \quad (12b)$$

$$\text{all other components vanish.} \quad (12c)$$

(Notice that even for a fluid at rest in the  $(r, \vartheta, \phi)$  coordinate system ( $\dot{Y} = 0$ ) there is a changing radial shear  $\sigma_{r\phi}$  associated with the changing metric ( $\dot{Q} \neq 0$ ; 'deformation of coordinates'). It is only because we are in the Regge–Wheeler gauge where  $h_{\vartheta\phi} \equiv 0$  that the non-radial shear  $\sigma_{\vartheta\phi}$  vanishes when  $\dot{Y} = 0$ .) Next, we write in explicit form the relationship

$$\sigma = \mathcal{L}_u S, \quad (13)$$

that the rate of shear is the Lie derivative of the shear along the world lines – a relationship which is best derived in the proper reference frame of a fiducial material particle; *cf.* Carter & Quintana (1972). The result, to first order in the oscillations, is

$$\sigma_{\alpha\beta} = e^{-\Phi} S_{\alpha\beta,t}. \quad (14)$$

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Finally, we combine equations (12) and (14) and integrate with respect to time, using the initial condition that the shear  $S_{\alpha\beta}$  is equal to zero in the unperturbed star (i.e. when  $Y$  and  $h_{\mu\nu}$  vanish). The result is

$$S_{r\phi} = S_{\phi r} = 1/2 r^2 (Y' - e^{\Lambda - \Phi} Q/r) b_\phi = \xi_{(r;\phi)} + 1/2 h_{r\phi}; \quad (15a)$$

$$S_{\vartheta\phi} = S_{\phi\vartheta} = 1/2 r^2 Y \sin^2 \vartheta b^{\phi, \vartheta} = \xi_{(\vartheta;\phi)}. \quad (15b)$$

Note that the shear  $S_{\alpha\beta}$  is generated both by the deformation of the crystal relative to the coordinate system (non-zero  $Y$ ) and by the deformation of the coordinate system itself (non-zero  $h_{r\phi}$ ).

The shear stress of equations (10) and (15) is only one contributor to the stress-energy tensor of the stellar material. The other contributors are the total density of mass-energy  $\rho$  and the isotropic pressure  $P$ , both of which maintain their unperturbed values because they are scalar fields and therefore cannot undergo odd-parity perturbations. The stress-energy tensor associated with  $\rho$  and  $P$  (the bulk part of the stress-energy tensor) has the standard perfect fluid form

$$T_{\alpha\beta}^{\text{bulk}} = (\rho + P) u_\alpha u_\beta + P g_{\alpha\beta}. \quad (16)$$

Using equations (5), (6), (9), (10), (15) and (16), we obtain for the total stress-energy tensor  $T_{\alpha\beta} = T_{\alpha\beta}^{\text{bulk}} + T_{\alpha\beta}^{\text{shear}}$  of the oscillating star

$$T_{tt} = \rho e^{2\Phi}, \quad T_{rr} = P e^{2\Lambda}, \quad T_{\vartheta\vartheta} = P r^2, \quad T_{\phi\phi} = P r^2 \sin^2 \vartheta; \quad (17a)$$

$$T_{t\phi} = T_{\phi t} = -r^2 [(\rho + P) \dot{Y} - \rho \dot{y}] b_\phi; \quad (17b)$$

$$T_{r\phi} = T_{\phi r} = -r [\mu r Y' - (\mu - P) e^{\Lambda - \Phi} Q] b_\phi; \quad (17c)$$

$$T_{\vartheta\phi} = T_{\phi\vartheta} = -r^2 \mu Y \sin^2 \vartheta b^{\phi, \vartheta}; \quad (17d)$$

$$\text{all other components vanish.} \quad (17e)$$

For evaluation of the Einstein field equations,  $R_{\mu\nu} = 8\pi (T_{\mu\nu} - 1/2 T g_{\mu\nu})$ , we shall need the first-order perturbations of  $(T_{\mu\nu} - 1/2 T g_{\mu\nu})$ . These are easily found from equations (5), (6) and (17):

$$\delta(T_{t\phi} - 1/2 T g_{t\phi}) = r^2 [1/2(\rho + 3P)\dot{y} - (\rho + P)\dot{Y}] b_\phi; \quad (18a)$$

$$\delta(T_{r\phi} - 1/2 T g_{r\phi}) = r [-\mu r Y' + (\mu - 1/2\rho + 1/2P) e^{\Lambda - \Phi} Q] b_\phi; \quad (18b)$$

$$\delta(T_{\vartheta\phi} - 1/2 T g_{\vartheta\phi}) = -\mu r^2 Y \sin^2 \vartheta b^{\phi, \vartheta}, \quad (18c)$$

where we have used the fact that  $T \equiv T^\alpha_\alpha = 3P - \rho$ .

### 3 Details of the analysis: $l \geq 2$

#### 3.1 EQUATIONS OF MOTION

Because our stellar oscillations are described by three functions of  $t$  and  $r$  -  $Y, y, Q$  - our analysis will require three equations of motion. Our chosen versions of these equations are obtained from the perturbed Einstein field equations  $\delta R_{\mu\nu} = 8\pi\delta(T_{\mu\nu} - 1/2 T g_{\mu\nu})$  by

manipulations described in Appendix A.

Our first equation is an initial-value equation for the ZAMO angular displacement function  $y$ :

$$\begin{aligned} & -\frac{e^{\Phi+\Lambda}}{r^4} (r^4 e^{-\Phi-\Lambda} y')' + e^{2\Lambda} \left[ 16\pi(\rho + P) + \frac{(l+2)(l-1)}{r^2} \right] y \\ & - 16\pi(\rho + P) e^{2\Lambda} Y + \frac{e^{\Phi+\Lambda}}{r^4} (r^3 e^{-2\Phi} Q)' = 0. \end{aligned} \quad (19a)$$

This equation can be solved at any moment of time to give  $y$  in terms of  $Y$  and  $Q$ .

Our second equation is a wave equation for the angular displacement  $Y$  of the stellar material:

$$\begin{aligned} & (\rho + P) e^{-2\Phi} \ddot{Y} - \frac{e^{-\Phi-\Lambda}}{r^4} (\mu r^4 e^{\Phi-\Lambda} Y')' + \left[ 16\pi(\rho + P) + \frac{(l+2)(l-1)}{r^2} \right] \mu Y \\ & - (\rho + P) \frac{e^{-\Phi-\Lambda}}{r^2} (rQ)' + \frac{e^{-\Phi-\Lambda}}{r^4} (\mu r^3 Q)' = 0. \end{aligned} \quad (19b)$$

The characteristics of this equation (the world lines of high-frequency, radially propagating wave packets) have a propagation speed, as measured by an observer at rest in the star, given by

$$v_s = \frac{e^\Lambda dr}{e^\Phi dt} = \left( \frac{\mu}{\rho + P} \right)^{1/2}. \quad (20)$$

When one recalls that  $(\rho + P)$  is inertial mass per unit volume in relativity (see, e.g., exercise 5.4 of MTW), one recognizes this as the standard expression for the speed of propagation of shear waves in an isotropic solid; cf. Carter (1973a).

Our third equation is a wave equation for the Regge–Wheeler gravitational-wave-function,  $Q$ :

$$\begin{aligned} & e^{-2\Phi} \ddot{Q} - e^{-\Phi-\Lambda} (e^{\Phi-\Lambda} Q')' + \left[ 16\pi\mu + \frac{(l+2)(l-1)}{r^2} - r e^{-\Phi-\Lambda} \left( \frac{e^{\Phi-\Lambda}}{r^2} \right)' \right] Q \\ & + 16\pi r e^{-\Phi-\Lambda} (\mu e^{2\Phi})' Y = 0. \end{aligned} \quad (19c)$$

In the vacuum outside the star this reduces to the Regge–Wheeler (1957) equation for gravitational waves propagating in Schwarzschild spacetime. Both inside the star and out the characteristics of this equation are radial null lines (propagation speed equal to speed of light). Note that the ZAMO displacement function  $y$  has been completely decoupled from the wave equations (19b, c); they are coupled wave equations for  $Y$  and  $Q$  alone.

One can show (see Appendix A) that our equations of motion (19) are ‘complete’ in the sense that the set of all physically acceptable solutions of (19) is identical to the set of all physically acceptable solutions of the perturbed Einstein field equations – physical acceptability being defined as satisfaction of the boundary conditions as given in the next section of this paper.

One can also show from our equations of motion (19) plus boundary conditions (or, more easily, from equations 19c, 19b and  $\epsilon_{\theta\phi} = 0$  in A.3) that in any region of the star where the shear modulus vanishes,  $\mu = 0$ , the perturbed gravitational field is decoupled from

the stellar matter, and the matter cannot support torsional oscillations. More specifically equation (19c) then becomes a homogeneous wave equation for the decoupled gravitational-wave variable  $Q$ ; equation (19a) or (A.3) determines the ZAMO displacement  $y$  in terms of  $Q$ ; and (A.3) together with (19b) guarantees that the fluid is at rest relative to the ZAMOs,  $Y = y$ . This decoupling has been noted previously by Thorne & Campolattaro (1967).

### 3.2 BOUNDARY CONDITIONS

The equations of motion (19) must be solved subject to suitable boundary conditions at the star's centre and surface, and at infinity.

At the star's centre the fluid motions and the spacetime geometry must be suitably smooth. Roughly speaking, smoothness means that the more rapid are the angular variations of  $Y$ ,  $y$ ,  $Q$  – i.e. the larger the value of  $l$  – the more rapidly must  $Y$ ,  $y$  and  $Q$  approach zero at  $r = 0$ . To make this quantitative we introduce local Cartesian coordinates  $\{x^a\}$  near  $r = 0$ :

$$x^1 \equiv r \sin \vartheta \cos \phi, \quad x^2 \equiv r \sin \vartheta \sin \phi, \quad x^3 \equiv r \cos \vartheta. \quad (21)$$

Because  $\Lambda \sim \rho r^2$  near  $r = 0$  the components of the unperturbed spatial metric (5a) are Cartesian at  $r = 0$  in this coordinate system:  $\gamma_{ab} = \delta_{ab} + O(r^2)$ .

Consider the three-dimensional vector and tensor fields

$$\xi = \xi^\phi \partial_\phi, \quad \alpha = h_t^\phi \partial_\phi, \quad \beta = h^{r\phi} (\partial_r \otimes \partial_\phi + \partial_\phi \otimes \partial_r); \quad (22)$$

$\xi$  is the material displacement vector,  $\alpha$  is the time–space part of the metric perturbation,  $\beta$  is the spatial part of the metric perturbation, indices on  $h_{\alpha\beta}$  have been raised with the unperturbed metric  $\gamma^{\alpha\beta}$  and  $\xi$ ,  $\alpha$  and  $\beta$  can all be regarded as solutions of the perturbed Einstein field equations. The ‘smoothness’ of the Einstein equations at  $r = 0$  implies that the Cartesian components of  $\xi$ ,  $\alpha$  and  $\beta$  will have power series expansions near  $r = 0$  whose leading terms are infinitely differentiable – or, equivalently, whose leading terms are expressible as products of non-negative powers of  $x^1, x^2, x^3$ ; e.g.

$$\xi^1 = (x^1)^2 (x^2)^{13} (x^3)^4 (1 + \text{terms which vanish as } r \rightarrow 0).$$

Using equations (6) and (8) we can write  $\xi$ ,  $\alpha$  and  $\beta$  as

$$\xi = rYA, \quad \alpha = -rjA, \quad \beta = -e^{-\Phi - \Lambda} Q [\partial_r \otimes A + A \otimes \partial_r]; \quad (23a)$$

$$A \equiv r \times \nabla P_l(\cos \vartheta). \quad (23b)$$

By writing  $P_l(\cos \vartheta)$  in terms of Cartesian coordinates (*cf.* equation 33 below; Section II.C of Thorne 1980) we can bring the Cartesian components of equations (23) near  $r = 0$  into the following form:

$$\xi^b = Yr^{-l+1} P_l^{a_1 \dots a_l} \epsilon^{bcd} x^c (\partial/\partial x^d) (x^{a_1} \dots x^{a_l}), \quad (24a)$$

$$\alpha^b = -j r^{-l+1} P_l^{a_1 \dots a_l} \epsilon^{bcd} x^c (\partial/\partial x^d) (x^{a_1} \dots x^{a_l}), \quad (24b)$$

$$\beta^{bc} = -Q e^{-\Phi} r^{-l-1} P_l^{a_1 \dots a_l} 2x^{(b} \epsilon^{c)d} x^d (\partial/\partial x^f) (x^{a_1} \dots x^{a_l}). \quad (24c)$$

Here  $\epsilon^{abc}$  is the Levi–Civita tensor,  $P_l^{a_1 \dots a_l}$  is a constant, symmetric, trace-free tensor (*cf.* equation 33 below), and the parentheses in the superscript indicate symmetrization.

These Cartesian components will be non-negative products of  $x^1$ ,  $x^2$  and  $x^3$  near  $r=0$  if and only if

$$Y(t, r) = r^{l-1} [\text{constant} + \text{terms which vanish as } r \rightarrow 0], \quad (25a)$$

$$y(t, r) = r^{l-1} [\text{constant} + \text{terms which vanish as } r \rightarrow 0]. \quad (25b)$$

$$Q(t, r) = r^{l+1} [\text{constant} + \text{terms which vanish as } r \rightarrow 0]. \quad (25c)$$

Thus, our boundary conditions near  $r=0$  are

$$Y \sim r^{l-1}; \quad y \sim r^{l-1}; \quad Q \sim r^{l+1} \quad \text{as } r \rightarrow 0. \quad (26a)$$

It is straightforward to show that, so long as the unperturbed star is smooth at  $r=0$  ( $\rho$ ,  $p$ ,  $\mu$  and  $\mu'$  finite there), these asymptotic forms satisfy the equations of motion (19). However, there also exist solutions to (19) which violate these boundary conditions ( $Y \sim r^{-k}$  and/or  $y \sim r^{-k}$  and/or  $Q \sim r^{-l}$ ) and which thus are physically unacceptable.

At the star's surface,  $r=R$ , the normal (radial) components of the stress tensor must vanish (there is no matter outside  $r=R$  to support a stress):  $T_{\alpha}^r \rightarrow 0$  as  $r \rightarrow R_- \equiv$  inner edge of stellar surface. Inspection of the stress-energy tensor (equation 17) shows that this condition is satisfied if and only if (i) the unperturbed pressure  $P$  approaches 0 as  $r \rightarrow R_-$ , and (ii) the material motions and shear modulus  $\mu$  satisfy the 'zero-torque-at-surface' condition

$$T_{\phi}^r = -r^2 e^{-2\Lambda} \mu (Y' - e^{\Lambda-\Phi} Q/r) b_{\phi} \rightarrow 0 \quad \text{as } r \rightarrow R_-. \quad (26b)$$

For a star with a solid surface (e.g. iron),  $\mu$  is finite at  $r=R$ , so  $Y'$  must equal  $e^{\Lambda-\Phi} Q/R$  there.

At the star's surface the gravitational potentials  $y$  and  $Q$  must be sufficiently continuous that (i) the intrinsic geometry of the star's surface

$$ds^2 \equiv -e^{2\Phi} dt^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2) - 2r^2 \dot{y} b_{\phi} dt d\phi$$

is continuous, and (ii) the extrinsic curvature,

$$K_{AB} \equiv e^{\Lambda} \Gamma_{AB}^r \quad (A, B \text{ ranging over } t, \vartheta, \phi),$$

is continuous (see, e.g. section 21.13 of MTW). Straightforward calculation shows that

$$\begin{aligned} K = & (e^{2\Phi-\Lambda} \Phi') dt^2 + e^{-\Lambda} [(r^2 \dot{y})' - r e^{\Lambda-\Phi} \dot{Q}] b_{\phi} dt d\phi \\ & - r e^{-\Phi} Q \sin^2 \vartheta b^{\phi}_{,\vartheta} d\vartheta d\phi - r e^{-\Lambda} (d\vartheta^2 + \sin^2 \vartheta d\phi^2). \end{aligned} \quad (27)$$

Therefore, continuity of the intrinsic and extrinsic geometries is satisfied if and only if – in addition to the familiar equilibrium conditions of continuous  $\Phi$ ,  $\Phi'$  and  $\Lambda$  –

$$y, y' \text{ and } Q \text{ are continuous across } r=R. \quad (26c)$$

At the interfaces of solid regions (crust and/or core) with fluid regions (mantle) the shear modulus  $\mu$  may go to zero discontinuously. There one must be sure that the zero-torque condition and the continuous intrinsic and extrinsic curvature conditions are satisfied:

$$T_{\phi}^r = -r^2 e^{-2\Lambda} \mu (Y' - e^{\Lambda-\Phi} Q/r) b_{\phi} \rightarrow 0 \quad \text{at solid-fluid interfaces,} \quad (26d)$$

$$y, y' \text{ and } Q \text{ are continuous across solid-fluid interfaces.} \quad (26e)$$



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Far from the star,  $h_{t\phi}$  and  $h_{r\phi}$  must describe outgoing gravitational waves. In this region our equations of motion [(19c) for  $Q$ ; (A3), which follows from (19a, b, c). for  $y$ ] become

$$\ddot{Q} = \partial_{r^*} \partial_{r^*} Q - (1 - 2M/r) [l(l+1)/r^2 - 6M/r^3] Q, \quad (28a)$$

$$\ddot{y} = r^{-2} \partial_{r^*} (rQ), \quad (28b)$$

where

$$r_* \equiv r + 2M \ln(r/2M - 1) \quad (28c)$$

is Wheeler's 'tortoise coordinate', and where no approximations have yet been made. Note that equation (28a) is the Regge-Wheeler (1957) equation for odd-parity gravitational waves. The general outgoing-wave solution to these equations has the asymptotic form at large radii

$$Q = F^{(l+1)}(u) + \frac{l(l+1)}{2r} F^{(l)}(u) + O(r^{-2}) \quad \text{as } r \rightarrow \infty, \quad (26f)$$

$$y = -\frac{F^{(l)}(u)}{r} - \frac{(l+2)(l-1)}{2r^2} F^{(l-1)}(u) + O(r^{-3}) \quad \text{as } r \rightarrow \infty, \quad (26g)$$

where  $u \equiv t - r_*$ . Here,  $F(u)$  is an arbitrary function of  $u$  to be determined by integrating the equations of motion, and  $F^{(l)}(u) \equiv d^l F/du^l$  denotes the  $l$ th derivative of  $F(u)$ . We shall see later that  $F(u)$ , aside from a multiplicative constant, is the star's current  $l$ -pole moment. One can show that, in addition to the physically acceptable outgoing-wave solutions (26f, g), the equations of motion (19) possess unacceptable incoming-wave solutions of the form (26f, g) with  $u$  replaced by  $v \equiv t + r_*$  and with the signs of the second term of  $Q$  and first term of  $y$  reversed, and also unacceptable solutions with mixtures of outgoing and incoming waves.

### 3.3 RADIATION FIELD AND ENERGY LOSS RATE

The radiation field far from the star is described by the metric perturbations (obtained by combining equations 6b and 26f, g)

$$h_{t\phi} = h_{\phi t} = [rF^{(l+1)}(u) + 1/2(l+2)(l-1)F^{(l)}(u)]b_\phi + O(r^{-1}); \quad (29a)$$

$$h_{r\phi} = h_{\phi r} = [- (r + 2M)F^{(l+1)}(u) - 1/2l(l+1)F^{(l)}(u)]b_\phi + O(r^{-1}); \quad (29b)$$

$$\text{all other components vanish.} \quad (29c)$$

The physical components of these perturbations,

$$h_{i\phi} = e^{-\Phi} (r \sin \vartheta)^{-1} h_{t\phi}, \quad h_{r\phi} = e^{-\Lambda} (r \sin \vartheta)^{-1} h_{r\phi},$$

have amplitudes which are independent of  $r$ , rather than amplitudes which die out like  $1/r$ . This is because the Regge-Wheeler gauge is badly behaved in the radiation zone (*cf.* Price & Thorne 1969). A more reasonable behaviour is obtained by making a gauge change (infinitesimal coordinate transformation; Box 18.2 of MTW) with the generating vector

$$\eta_\phi = rF^{(l)}(u)b_\phi; \quad \text{all other } \eta_\mu \text{ vanish.} \quad (30)$$

The metric perturbation in the new gauge,

$$h_{\alpha\beta}^{\text{new}} = h_{\alpha\beta}^{\text{old}} - \eta_{\alpha|\beta} - \eta_{\beta|\alpha},$$

where the bar | denotes covariant derivative with respect to the flat metric, has components

$$h_{t\phi}^{\text{new}} = 1/2 (l+2)(l-1)F^{(l)}(u)b_\phi + O(r^{-1}); \quad (31a)$$

$$h_{r\phi}^{\text{new}} = -1/2 (l+2)(l-1)F^{(l)}(u)b_\phi + O(r^{-1}); \quad (31b)$$

$$h_{\vartheta\phi}^{\text{new}} = -rF^{(l)}(u)\sin^2\vartheta b^\phi_{,\vartheta}; \quad (31c)$$

$$\text{all other components vanish.} \quad (31d)$$

To leading order the new metric perturbation is in Lorentz gauge ( $h^{\text{new}}_{\alpha\nu|\nu} = 0$ ), and its physical components die off like  $1/r$  in the radiation zone.

Any gravitational wave can be characterized in a gauge-invariant way by the transverse-traceless (TT) part of its metric perturbation (see chapter 35 of MTW). Only  $h_{\vartheta\phi}^{\text{new}}$  contributes to the TT part of our wave (31). By combining equation (6c) for  $b_\phi$  with (31c) for  $h_{\vartheta\phi}^{\text{new}}$ , and by converting to covariant notation in the three-dimensional Euclidean space far from the star, we obtain

$$(h_{jk}^{\text{new}})^{\text{TT}} = [-2rF^{(l)}(t-r_*)n^p \epsilon_{pqj} P_l^{|q|k}]^S. \quad (32)$$

Here  $\mathbf{n} \equiv \mathbf{r}/r$  is the unit radial vector,  $\epsilon_{pqj}$  is the Levi-Civita tensor,  $P_l \equiv P_l(\cos\vartheta)$  is regarded as a scalar field in flat space, | denotes covariant derivative,  $S$  means symmetrize on indices  $j$  and  $k$ , and TT means take the transverse-traceless part using the techniques of Box 35.1 of MTW.

One of the authors has attempted to introduce a standardized formalism for multipole expansions of gravitational radiation fields (Thorne 1980). In that formalism the mass and current multipoles are represented by completely symmetric, trace-free tensors. To make the connection between equation (32) and that formalism we introduce into equation (32) the symmetric, trace-free representation of the Legendre polynomial

$$P_l(\cos\vartheta) = P_l^{a_1 \dots a_l} n_{a_1} \dots n_{a_l} \quad (33)$$

(cf. Section II.C of Thorne 1980, where  $P_l^{a_1 \dots a_l}$  is denoted by  $\mathcal{Y}_{a_1 \dots a_l}^{l0}/C^{l0}$ ) and we then perform the differentiations denoted by  $P_l^{|q|k}$ . The result is

$$(h_{jk}^{\text{new}})^{\text{TT}} = r^{-1} [2l(l-1)F^{(l)}(t-r_*)\epsilon_{pqj} P_l^{p a_1 \dots a_l} n_{a_1} \dots n_{a_l}]^S. \quad (34)$$

Direct comparison with equation (4.8) of Thorne (1980) shows that the radiation field is that of a current  $l$ -pole with  $l$ -pole moment

$$\mathcal{S}^{a_1 \dots a_l}(t-r_*) = \frac{(l-1)(l+1)!}{4} F(t-r_*) P_l^{a_1 \dots a_l}. \quad (35)$$

This radiation field carries off energy at a rate given by (cf. Thorne 1980, equation 4.16)

$$\begin{aligned} \frac{dE_{\text{star}}}{dt} &= - \frac{4l(l+2)}{(l-1)(l+1)!(2l+1)!!} \langle \mathcal{S}^{a_1 \dots a_l(l+1)} \mathcal{S}^{a_1 \dots a_l(l+1)} \rangle \\ &= - \frac{(l-1)l(l+1)(l+2)}{4(2l+1)} \langle [F^{(l+1)}(t-r_*)]^2 \rangle, \end{aligned} \quad (36)$$

where  $\langle \rangle$  means averaged over several characteristic periods of the radiation and where equation (2.26a) or (2.5) of Thorne (1980) has been used to obtain the second line.

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### 3.4 ACTION PRINCIPLE AND ENERGY CONSERVATION LAWS

Our star's torsional oscillations are governed by an action principle. The action's Lagrangian density can be derived either by second variation of the Einstein Lagrangian density  $(-g)^{1/2} R + \mathcal{L}_{\text{matter}}$  (method of Taub 1969) or by multiplying the star's equations of motion by carefully selected functions and removing a divergence (method of Chandrasekhar 1964a, b; see also Detweiler & Ipser 1973 and Appendix B of this paper). The Lagrangian density is

$$\begin{aligned} \mathcal{L} = & \frac{2\pi l(l+1)}{(2l+1)} [(\rho + P)r^4 e^{\Lambda-\Phi} (\dot{Y} - \dot{y})^2 + (1/16\pi)r^4 e^{-\Phi-\Lambda} (\dot{y}' - e^{\Lambda-\Phi} \dot{Q}/r)^2 \\ & + (1/16\pi)(l+2)(l-1)r^2 e^{\Lambda-\Phi} \dot{y}^2 - \mu r^4 e^{\Phi-\Lambda} (Y' - e^{\Lambda-\Phi} Q/r)^2 \\ & - \mu(l+2)(l-1)r^2 e^{\Phi+\Lambda} Y^2 - (1/16\pi)(l+2)(l-1)e^{\Lambda-\Phi} Q^2], \end{aligned} \quad (37)$$

and the action principle is

$$\delta \int_{\Omega} \mathcal{L} dt dr = 0, \quad (38)$$

where  $\Omega$  is any compact region of spacetime, and where the functions to be varied ( $Y$ ,  $y$  and  $Q$ ) must be held fixed on the boundary  $\partial\Omega$  (i.e.  $\delta Y = \delta y = \delta Q = 0$  there). If  $\Omega$  includes the star's centre or surface or a solid-fluid interface, then  $Y$ ,  $y$  and  $Q$  must satisfy the smoothness and continuity equations (26a, b, c, d, e) there. By varying  $Y$ ,  $y$  and  $Q$  in this action we obtain, respectively, the perturbed Einstein field equations  $\epsilon_T = 0$  (equation A.5),  $\dot{\epsilon}_{t\phi} = 0$  (equation A.2) and  $\epsilon_{r\phi} = 0$  (equation A.4). Our equations of motion (19) are linear combinations of these equations and their derivatives and time integrals; cf. equations (A.7)–(A.9).

Because our Lagrangian density (37) is time-independent,  $(\partial \mathcal{L} / \partial t)_{Y, y, Q \text{ fixed}} = 0$ , there is a conserved quantity associated with it:

$$S^\alpha_{, \alpha} = 0, \quad (39)$$

where

$$\begin{aligned} S^t \equiv & \frac{2\pi l(l+1)}{(2l+1)} \left\{ (\rho + P)r^4 e^{\Lambda-\Phi} (\dot{Y} - \dot{y})^2 + \frac{r^4 e^{-\Phi-\Lambda}}{16\pi} (\dot{y}' - e^{\Lambda-\Phi} \dot{Q}/r)^2 \right. \\ & + \frac{(l+2)(l-1)}{16\pi} r^2 e^{\Lambda-\Phi} \dot{y}^2 + \mu r^4 e^{\Phi-\Lambda} (Y' - e^{\Lambda-\Phi} Q/r)^2 \\ & \left. + \mu(l+2)(l-1)r^2 e^{\Phi+\Lambda} Y^2 + \frac{(l+2)(l-1)}{16\pi} e^{\Lambda-\Phi} Q^2 \right\}, \end{aligned} \quad (40a)$$

and

$$S^r \equiv -\frac{4\pi l(l+1)}{(2l+1)} \left\{ \frac{r^4 e^{-\Phi-\Lambda}}{16\pi} \dot{y} (\ddot{y}' - e^{\Lambda-\Phi} \ddot{Q}/r) + \mu r^4 e^{\Phi-\Lambda} \dot{Y} (Y' - e^{\Lambda-\Phi} Q/r) \right\}. \quad (40b)$$

(For a derivation, and for a discussion of how we have selected this specific  $S^\alpha$  from among an infinity of such divergence-free quantities, see Appendix B.) Note that the energy density,  $S^t$ , is just the Lagrangian density  $\mathcal{L}$  with the signs of the potential energy terms converted from minus to plus.

If we regard  $Q$  and  $y$  as gravitational fields which reside in the unperturbed spacetime and which couple to the matter displacement  $Y$ , then we can associate with the perturbations a

stress-energy tensor  $\tilde{T}^{\mu\nu}$  which resides in the unperturbed spacetime. The law of energy-momentum conservation  $\tilde{T}^{\mu\nu}{}_{;\nu} = 0$  (where the semicolon denotes covariant derivative with respect to the unperturbed metric  $\gamma_{\mu\nu}$ ), together with Killing's equation for the generator  $\partial/\partial t$  of time translations, guarantees that  $\tilde{T} \cdot \partial/\partial t$  has vanishing covariant derivative – i.e. in component notation and in the  $(t, r, \vartheta, \phi)$  coordinate system of equation (5a)

$$[(-\gamma)^{1/2} \tilde{T}_t^\alpha]_{,\alpha} = 0. \quad (41)$$

Here  $\gamma$  is the determinant of the unperturbed metric components  $\gamma_{\mu\nu}$ , and  $(-\gamma)^{1/2}$  is equal to  $r^2 e^{\Phi + \Lambda} \sin \vartheta$ . After integrating this equation over angles  $\vartheta$  and  $\phi$  we obtain the conservation law (39), with

$$S^\alpha = - \int \tilde{T}_t^\alpha (-\gamma)^{1/2} d\vartheta d\phi. \quad (42)$$

The perturbation stress-energy tensor  $\tilde{T}^{\mu\nu}$  can be computed in the canonical manner from the Lagrangian for the perturbations (albeit a Lagrangian in which, unlike (37), the angular dependences have not yet been integrated out). There is an infinity of resulting  $\tilde{T}^{\mu\nu}$ 's depending on the gauge in which the Lagrangian is written (i.e. depending on one's choice of infinitesimal ripples in the perturbed star's coordinate system). If one only wants to know the components  $\tilde{T}_t^\alpha$  one can evaluate them by undoing the angular integrations in equation (42), a process which contains some arbitrariness corresponding to part of the gauge-dependent arbitrariness in  $\tilde{T}^{\mu\nu}$ . With a choice for this arbitrariness which we regard as optimal, the equations (40a), (42) and

$$\int_0^\pi \sin \vartheta [\partial_\vartheta P_l(\cos \vartheta)]^2 d\vartheta = \frac{2l(l+1)}{(2l+1)},$$

$$\int_0^\pi \sin^3 \vartheta \left[ \partial_\vartheta \left( \frac{\partial_\vartheta P_l(\cos \vartheta)}{\sin \vartheta} \right) \right]^2 d\vartheta = (l+2)(l-1) \frac{2l(l+1)}{2l+1}, \quad (43)$$

give the result

$$\tilde{T}^{ii} = -\tilde{T}_t^t = 1/2 (\rho + P) \mathbf{v}^2 + \mu (S_{j\hat{k}})^2 + \frac{1}{16\pi} (\sigma^Z_{j\hat{k}})^2 + \frac{1}{16\pi} (B_{j\hat{k}})^2, \quad (44a)$$

$$\tilde{T}^{i\hat{r}} = -e^{\Lambda - \Phi} \tilde{T}_t^r = -2\mu S_{\hat{r}\hat{j}} u^{\hat{j}} + \frac{1}{16\pi} A_{\hat{r}\hat{j}\hat{k}} \sigma^Z_{j\hat{k}}, \quad (44b)$$

where there is an implied summation over  $\hat{j}$  and  $\hat{k}$ . These equations make use of the orthonormal basis of an observer at rest in the unperturbed star:

$$e_{\hat{t}} = e^{-\Phi} \partial_t, \quad e_{\hat{r}} = e^{-\Lambda} \partial_r, \quad e_{\hat{\vartheta}} = r^{-1} \partial_\vartheta, \quad e_{\hat{\phi}} = (r \sin \vartheta)^{-1} \partial_\phi. \quad (45a)$$

The quantity  $\tilde{T}^{ii}$  (equation 44a) is the energy density measured by this static observer. The term  $1/2 (\rho + P) \mathbf{v}^2$  in  $\tilde{T}^{ii}$  is the kinetic energy density of the matter;  $\mathbf{v}$  is the velocity of the matter relative to the ZAMO's (see discussion following equations 6c and 8b),

$$\mathbf{v} = r e^{-\Phi} (\dot{Y} - \dot{y}) b_{\hat{\phi}} e_{\hat{\phi}}; \quad b_{\hat{\phi}} \equiv (\sin \vartheta)^{-1} b_\phi = \partial_\vartheta P_l(\cos \vartheta). \quad (45b)$$

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The term  $\mu(S_{j\hat{k}})^2$  is the standard expression for the potential energy density of a deformed elastic solid;  $S_{j\hat{k}}$  is the shear

$$\begin{aligned} S_{\hat{r}\hat{\phi}} &= S_{\hat{\phi}\hat{r}} = 1/2 r e^{-\Lambda} (Y' - e^{\Lambda-\Phi} Q/r) b_{\hat{\phi}}, \\ S_{\hat{\theta}\hat{\phi}} &= S_{\hat{\phi}\hat{\theta}} = 1/2 Y \sin \vartheta b^{\hat{\phi}}_{,\vartheta} \end{aligned} \quad (45c)$$

(equations 15) and by virtue of the stress-strain relation (10)

$$\mu(S_{j\hat{k}})^2 = -1/2 T_{j\hat{k}}^{\text{shear}} S^{j\hat{k}}. \quad (45d)$$

The term  $(16\pi)^{-1}(\sigma^Z_{j\hat{k}})^2$  is the kinetic energy density of the gravitational field;  $\sigma^Z_{j\hat{k}}$  is the rate of shear of the congruence of ZAMO observers (equations 12 with the matter displacement  $Y$  replaced by the ZAMO displacement  $y$ )

$$\begin{aligned} \sigma^Z_{\hat{r}\hat{\phi}} &= \sigma^Z_{\hat{\phi}\hat{r}} = 1/2 r e^{-\Phi-\Lambda} (y' - e^{\Lambda-\Phi} \dot{Q}/r) b_{\hat{\phi}}, \\ \sigma^Z_{\hat{\theta}\hat{\phi}} &= \sigma^Z_{\hat{\phi}\hat{\theta}} = 1/2 e^{-\Phi} \dot{y} \sin \vartheta b^{\hat{\phi}}_{,\vartheta}. \end{aligned} \quad (45e)$$

The term  $(16\pi)^{-1}(B_{j\hat{k}})^2$  is the potential energy density of the gravitational field;  $B_{j\hat{k}}$  is defined to have as its only non-zero components

$$B_{\hat{\theta}\hat{\phi}} = B_{\hat{\phi}\hat{\theta}} \equiv -1/2 r^{-1} e^{-\Phi} Q \sin \vartheta b^{\hat{\phi}}_{,\vartheta}. \quad (45f)$$

We have not found a simple, physical description of the quantity  $B_{j\hat{k}}$  whose square is the gravitational potential energy, analogous to the description  $\sigma^Z_{j\hat{k}}$  ( $\equiv$  ZAMO rate of shear) of the quantity whose square is the gravitational kinetic energy.

The quantity  $\tilde{T}^{\hat{r}\hat{r}}$  (equation 44b) is the energy flux measured by a static observer. The term  $-2\mu S_{rj} u^j = 2l_{rj}^{\text{shear}} u^j$  is the standard expression for the radial energy flux carried by the matter's shear stress;  $u^j$  is the matter velocity relative to the static observer

$$\mathbf{u} \equiv u^{\hat{\phi}} \mathbf{e}_{\hat{\phi}} = r e^{-\Phi} \dot{Y} b^{\hat{\phi}}_{,\vartheta} \mathbf{e}_{\hat{\phi}}. \quad (45g)$$

The term  $A_{\hat{r}j\hat{k}} \sigma^Z_{j\hat{k}}$  is the radial energy flux carried by the gravitational waves;  $A_{j\hat{k}}$  is defined to have as its only non-zero components

$$A_{\hat{r}\hat{\theta}\hat{\phi}} = A_{\hat{\theta}\hat{\phi}\hat{r}} = -\frac{r^2 e^{-2\Phi-\Lambda}}{(l+2)(l-1)} (y' - e^{\Lambda-\Phi} \dot{Q}/r) \sin \vartheta b^{\hat{\phi}}_{,\vartheta}; \quad (45h)$$

and as with  $B_{j\hat{k}}$  we have not found a simple physical description of  $A_{j\hat{k}}$ .

In the radiation zone the energy density  $\tilde{T}^{\hat{r}\hat{r}}$  and energy flux  $\tilde{T}^{\hat{r}\hat{\theta}}$  are carried entirely by the gravitational waves (which we assume to be outgoing):

$$\begin{aligned} \tilde{T}^{\hat{r}\hat{r}} &= \tilde{T}^{\hat{r}\hat{r}} = (32\pi)^{-1} (\hat{h}^{\text{new}}_{j\hat{k}})^{\text{TT}} (\hat{h}^{\text{new}}_{j\hat{k}})^{\text{TT}} \\ &= (16\pi r^2)^{-1} [F^{(l+1)}(u)]^2 (\sin \vartheta b^{\hat{\phi}}_{,\vartheta})^2. \end{aligned} \quad (46)$$

Here  $(\hat{h}^{\text{new}}_{j\hat{k}})^{\text{TT}}$  is the transverse traceless gravitational-wave field of equations (31) and (32); equation (46) can be derived by combining equations (44), (45), (A.4) with  $\epsilon_{r\vartheta} = 0$ , (26) and (31c). When averaged over several wavelengths, expression (46) reduces to the standard Isaacson stress-energy tensor for the waves (see, e.g. Sections 35.7 and 35.15 of MTW).

The differential law of energy conservation  $S^\alpha_{,\alpha} = 0$  or  $[(-\gamma)^{1/2} \tilde{T}^{\hat{r}\hat{\alpha}}]_{,\alpha} = 0$ , when spatially integrated over the star's interior and on out to some radius  $R_\infty$  in the wave zone, becomes a

law of global energy conservation:

$$\begin{aligned} dE_{\text{star}}/dt &= -S^r(r=R_\infty) = - \left[ \int \tilde{T}^{ii} r^2 d\vartheta d\phi \right]_{r=R_\infty} \\ &= - \frac{(l-1)l(l+1)(l+2)}{4(2l+1)} [F^{(l+1)}(t-r^*)]^2. \end{aligned} \quad (47a)$$

$$E_{\text{star}} = \int_0^{R_\infty} S^r dr = \iiint_0^{R_\infty} \tilde{T}^{ii} e^\Phi d\text{vol}, \quad (47b)$$

where  $d\text{vol} \equiv e^{-\Lambda} r^2 \sin \vartheta d\vartheta d\phi dr$  is the spatial volume element and  $e^\Phi$  is the gravitational redshift factor.

### 3.5 STABILITY OF THE OSCILLATING STAR

The law of global energy conservation (47) is a foundation for proving that our oscillating star is stable: So long as the shear modulus  $\mu$  is non-negative, the energy density  $\tilde{T}^{ii}$  is everywhere positive (equation 44a), and therefore  $E_{\text{star}}$  (equation 47b) is a positive definite functional of  $Y$ ,  $y$  and  $Q$ . Since  $dE_{\text{star}}/dt < 0$  (equation 47a), no choice of initial conditions  $Y(t=0, r)$ ,  $y(t=0, r)$ ,  $Q(t=0, r)$  can produce  $Y$ ,  $y$ ,  $Q$  which grow arbitrarily large at later times. Therefore our star with outgoing-wave boundary conditions is stable against arbitrary initial perturbations (Liapunov stability; cf. LaSalle & Lefschetz 1961).

## 4 The outgoing-wave normal modes: $l \geq 2$

### 4.1 THE EIGENVALUE PROBLEM

For most applications of the theory developed in this paper one will want to resolve the torsional oscillations into normal modes with complex vibrational frequencies

$$\omega = \sigma + i/2\tau. \quad (48)$$

In a normal mode the perturbation functions have the forms

$$Y(t, r) \equiv Y_\omega(r) e^{i\omega t}; \quad y(t, r) \equiv y_\omega(r) e^{i\omega t}; \quad Q(t, r) \equiv Q_\omega(r) e^{i\omega t}. \quad (49)$$

The real part of the frequency,  $\sigma$ , describes sinusoidal oscillations; the imaginary part,  $1/2\tau$ , describes damping due to radiation reaction. (The factor 2 appears in  $\omega \equiv \sigma + i/2\tau$  so that  $\tau$  will be the  $e$ -folding time of the star's oscillation energy, not of its amplitude.)

For a normal mode the two dynamical equations (19b, c) form a fourth-order system of linear ordinary differential equations for the eigenfunctions  $Y_\omega(r)$  and  $Q_\omega(r)$  (hereafter we omit the subscript  $\omega$ ):

$$\begin{aligned} (\mu r^4 e^{\Phi-\Lambda} Y')' - r^4 e^{\Phi+\Lambda} [16\pi(\rho+P) + (l+2)(l-1)r^{-2}] \mu Y \\ - (\mu r^3 Q)' + (\rho+P)r^2 (rQ)' = -\omega^2(\rho+P)r^4 e^{\Lambda-\Phi} Y; \end{aligned} \quad (50a)$$

$$\begin{aligned} (e^{\Phi-\Lambda} Q')' - [16\pi e^{\Phi+\Lambda} \mu + (l+2)(l-1)r^{-2} e^{\Phi+\Lambda} - r(r^{-2} e^{\Phi-\Lambda})'] Q \\ - 16\pi r (\mu e^{2\Phi})' Y = -\omega^2 e^{\Lambda-\Phi} Q. \end{aligned} \quad (50b)$$

These equations must be solved subject to the boundary conditions (26a, b, f):

$$Y \sim r^{l-1}, \quad Q \sim r^{l+1} \quad \text{as} \quad r \rightarrow 0, \quad (51a)$$

$$\mu(Y' - e^{\Lambda-\Phi} Q/r) \rightarrow 0 \quad \text{as} \quad r \rightarrow R_-, \quad (51b)$$

and for the physically realistic case of outgoing waves at infinity (outgoing-wave normal mode)

$$Q = (i\omega)^{l+1} F_\omega e^{-i\omega r_*} \quad \text{as } r \rightarrow \infty, \quad (51c)$$

where  $F_\omega$  is the amplitude of the oscillatory  $l$ -pole moment at  $t = 0$ ,  $F(t) \equiv F_\omega e^{i\omega t}$ . Equations (50) and (51) together form an eigenvalue problem for the oscillation frequency  $\omega$  and eigenfunctions  $Y$ ,  $Q$ . Once the eigenvalue problem has been solved, the remaining metric perturbation function  $y$  can be computed most easily from the initial-value equation  $\epsilon_{\theta\phi} = 0$  (equation A.3), which gives

$$y = \frac{1}{\omega^2} \left[ \frac{-e^{\Phi-\Lambda}}{r^2} (rQ)' + 16\pi\mu e^{2\Phi} Y \right]; \quad (52)$$

alternatively (and equivalently)  $y$  can be computed from the initial value equation (19a).

In posing the eigenvalue problem (50)–(52) we have omitted some of the boundary conditions (26). It is straightforward to show (*cf.* discussion of equations 26) that, so long as the unperturbed star is well behaved at its centre and surface ( $\rho$ ,  $P$ ,  $\mu$  and  $\mu'$  finite at  $r = 0$ ;  $\rho$ ,  $\mu$  finite but perhaps non-zero and  $P \rightarrow 0$  as  $r \rightarrow R_-$ ), the omitted boundary conditions

$$y \sim r^{l-1} \quad \text{as } r \rightarrow 0, \quad (53a)$$

$$\mu(Y' - e^{\Lambda-\Phi} Q/r) \rightarrow 0 \text{ at solid-fluid interfaces,} \quad (53b)$$

$$y, y', Q \text{ continuous across } r = R \text{ and across interfaces,} \quad (53c)$$

$$y \sim -(i\omega)^l r^{-1} F_\omega e^{-i\omega r_*} \quad \text{as } r \rightarrow \infty \quad (53d)$$

are automatically satisfied by any solution of equations (50)–(52).

In order to understand the spectrum of eigenfrequencies of our torsionally oscillating star, we must first understand the asymptotic behaviours of the solutions of the eigenequations (50) just below the star's surface. Those behaviours depend on the asymptotic forms of the star's density  $\rho$  and shear modulus  $\mu$ . If the star's surface is solid,  $\rho$  will be finite; otherwise it may go to zero as a power law. In general  $\mu$  will go to zero at least as fast as  $\rho$ . Hence, it is reasonable to suppose that

$$\rho \sim (R-r)^N, \quad P \sim (R-r)^{N+1}, \quad \mu \sim (R-r)^{N+S}; \quad N \geq 0, \quad S \geq 0. \quad (54)$$

where the form of  $P$  follows from the equation of hydrostatic equilibrium. One can show that, so long as  $S < 2$  [i.e. so long as the speed of shear waves  $(\mu/\rho)^{1/2}$  goes to zero no faster than  $(R-r)$ ], one solution of the eigenequations (50) will have  $\mu(Y' - e^{\Lambda-\Phi} Q/r)$  finite and non-zero at  $R_-$  and will thus be physically unacceptable. All other solutions will be acceptable. For  $S > 2$  all solutions have  $\mu(Y' - e^{\Lambda-\Phi} Q/r)$  zero at  $R_-$ , but they also all have  $Y$  divergent, which would lead to a breaking of the crystal – a complication we are not prepared to face in this paper. Thus, we shall restrict ourselves henceforth to the case  $S < 2$ ; and we shall impose a similar restriction at interfaces of solid regions with the fluid mantle. In this case the spectrum of eigenfrequencies will be discrete, as the following argument shows.

Imagine a trial integration of the eigenequations (50). One selects a complex trial frequency  $\omega$  and complex starting values  $A$  and  $B$  for  $Y/r^{l-1}$  and  $Q/r^{l+1}$  near  $r = 0$ . (The eigenequations (50) have the general solution  $Y = Ar^{l-1} + Dr^{-l-2}$ ,  $Q = Br^{l+1} + Er^{-l}$  near  $r = 0$ ; one makes sure that  $D$  and  $E$  vanish.) One then integrates the eigenequations (50) outward from  $r = 0$  to the star's surface  $r = R$  and examines the value of the complex

number  $\mu(Y' - e^{\Lambda-\Phi} Q/r)$  there; it will turn out to be non-zero, unless the starting ratio  $A/B$  has been chosen to have some special value (or one of a discrete set of special values). That choice must be made. One then continues the integration on outward into the radiation zone, where one finds for  $Q$  (general solution of 50b)

$$Q = C^{(O)} e^{-i\omega r_*} + C^{(I)} e^{+i\omega r_*}. \quad (55)$$

To get an outgoing-wave normal mode one must ensure that the complex ingoing-wave amplitude  $C^{(I)}$  vanishes. One cannot do so by adjusting the starting product  $AB$ ; that product merely fixes the overall amplitude and phase of the oscillations. Instead, to make  $C^{(I)}$  vanish one must carefully adjust the complex eigenfrequency  $\omega$  to one of a discrete set of values. Thus, the spectrum is discrete.

The Liapunov proof of stability in Section 3.5 guarantees that the outgoing-wave normal modes are all damped, i.e. all have positive values of  $\text{Im}(\omega) = 1/2\tau$ .

We now describe five methods for solving the eigenvalue problem (50) and (51): the method of resonances (Section 4.2), the variational method (Section 4.3), the energy method (Section 4.4), the method of the slow-motion approximation (Section 4.5) and the method of the weak-field approximation (Section 4.6).

#### 4.2 METHOD OF RESONANCES

In the method of resonances (Thorne 1969a) one studies the unrealistic problem of an oscillating star inside a large spherical cavity whose walls reflect gravitational waves perfectly. This requires replacing the outgoing-wave boundary condition (51c) by a standing-wave boundary condition. The star and standing wave can oscillate with any desired *real* frequency  $\omega \equiv \sigma$ . For each value of the frequency  $\omega$  one can calculate (on a computer) the ratio

$$\mathcal{R} \equiv \frac{\text{(amplitude of star's oscillating motions)}}{\text{(amplitude of waves far from the star)}}. \quad (56)$$

As  $\omega$  varies,  $\mathcal{R}$  will go through a sequence of sharp resonances. These resonances, on the real frequency axis, are induced by nearby complex eigenfrequencies of the discrete, outgoing-wave normal modes; i.e. when  $\omega$  nears the oscillation frequency  $\omega_n$  of an outgoing-wave normal mode, the standing gravitational waves will excite the star's fluid into large-amplitude motions. From the locations, half-widths and phase-shifts of the resonances one can compute the complex frequencies  $\omega_n = \sigma_n + i/2\tau_n$  of the outgoing-wave normal modes. Thorne (1969a) has discussed these calculations in detail for compressional oscillations; calculations for our case of torsional oscillations would be the same in concept and method.

#### 4.3 VARIATIONAL METHOD

The normal-mode eigenfunctions and eigenfrequencies can be evaluated using a Detweiler-Ipser (1973) type action principle, which is closely related to the Lagrangian density  $\mathcal{L}$  of equation (37). The relationship to  $\mathcal{L}$  and a derivation of the action principle are sketched in Appendix B. The action principle utilizes integrals from the centre of the star  $r=0$  to a sphere  $r=R_\infty$  far out in the radiation zone, and it utilizes complex trial functions  $Y, Q$  which are constrained to satisfy the smoothness and continuity conditions (51a,b) and (53b,c) at  $r=0$ , on the star's surface  $r=R$ , and across solid-fluid interfaces. For any choice of such trial functions  $Y, Q$  a corresponding complex function  $y$  is to be computed by



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solving the initial-value equation (19a) subject to the smoothness boundary condition (53a) at  $r = 0$ , and subject to the demand that

$$\chi/y = \text{some fixed value, } (\chi/y)_\infty, \quad \text{at } r = R_\infty: \quad (57a)$$

here

$$\chi \equiv -r^2 e^{-\Phi - \Lambda} (y' - e^{\Lambda - \Phi} Q/r). \quad (57b)$$

(Recall that outside the star  $\Lambda = -\Phi$ .) The quantity

$$\Omega^2 \equiv B/A \quad (58a)$$

must then be computed, where

$$A \equiv \int_0^{R_\infty} \left[ (\rho + P)r^4 e^{\Lambda - \Phi} (Y - y)^2 + \frac{r^4 e^{-\Phi - \Lambda}}{16\pi} (y' - e^{\Lambda - \Phi} Q/r)^2 + \frac{(l+2)(l-1)}{16\pi} r^2 e^{\Lambda - \Phi} y^2 \right] dr + \frac{1}{16\pi} [r^2 y \chi]_{r=R_\infty}, \quad (58b)$$

$$B \equiv \int_0^{R_\infty} \left[ \mu r^4 e^{\Phi - \Lambda} (Y' - e^{\Lambda - \Phi} Q/r)^2 + \mu(l+2)(l-1)r^2 e^{\Phi + \Lambda} Y^2 + \frac{(l+2)(l-1)}{16\pi} e^{\Lambda - \Phi} Q^2 \right] dr. \quad (58c)$$

The quantity  $\Omega^2 \equiv B/A$  is an action for the normal modes. Those trial functions  $Q$  and  $Y$ , which make  $\Omega^2$  stationary ( $\delta\Omega^2 = 0$ ) with respect to all variations  $\delta Q$  and  $\delta Y$  that satisfy our smoothness and continuity conditions, are normal-mode eigenfunctions; and the stationary value of  $\Omega$  is their complex eigenfrequency  $\omega$ . (The Euler–Lagrange equations associated with this action principle are our eigenequations 50 with  $\omega^2 = \Omega^2$ .)

The specific normal modes obtained from this action principle depend on the chosen boundary value  $(\chi/y)_\infty$ . To obtain standing-wave normal modes, one chooses  $(\chi/y)_\infty$  real and all trial functions real. For a given real  $(\chi/y)_\infty$  there will be a discrete set of standing-wave modes (analogue of discrete normal modes of a violin string with ends clamped). To obtain the full continuous set of standing-wave modes (one mode for each real  $\omega$ ), the action principle must be used time and again, with various values of  $(\chi/y)_\infty$  and fixed  $R_\infty$ ; or with fixed  $(\chi/y)_\infty$  and various  $R_\infty$  (analogue of changing the clamping location of the violin string).

If one chooses  $(\chi/y)_\infty$  complex rather than real and uses complex trial functions, then the action principle (57) and (58) will produce a discrete set of normal modes, each with a different mixture of ingoing and outgoing waves – a mixture that cannot be predicted in advance. Only by an iterative application of the action principle (procedure devised by Detweiler (1975) for compressional oscillations of stars) can one be sure of obtaining a pure outgoing-wave mode. For an outgoing-wave mode, if one knew the complex frequency  $\omega$  in advance, one could solve the eigenequations (50) and initial-value equation (19a) far from the star to find the asymptotic forms of  $Q$ ,  $y$  and  $\chi$ :

$$Q = (i\omega)^{l+1} F_\omega \left[ 1 + \frac{l(l+1)}{2i\omega r} + O\left(\frac{1}{r^2}\right) \right] e^{-i\omega r_*}. \quad (59a)$$

$$y = -\frac{(i\omega)^l}{r} F_\omega \left[ 1 + \frac{(l+2)(l-1)}{2i\omega r} + O\left(\frac{1}{r^2}\right) \right] e^{-i\omega r_*}, \quad (59b)$$

$$\chi = -(l+2)(l-1) \frac{(i\omega)^{l-1}}{r} F_\omega \left[ 1 + \frac{l(l+1)}{2i\omega r} + O\left(\frac{1}{r^2}\right) \right] e^{-i\omega r_*}, \quad (59c)$$

where the complex number  $F_\omega$  is the (arbitrary) Fourier amplitude of the  $l$ -pole moment; cf. equations (26f, g). The corresponding boundary value of  $\chi/y$  is

$$(\chi/y)_\infty = \frac{(l+2)(l-1)}{i\omega} \left[ 1 + \frac{1}{i\omega R_\infty} + O\left(\frac{1}{R_\infty^2}\right) \right]. \quad (59d)$$

Detweiler's procedure is to guess a value of  $\omega$ ; choose the boundary value  $(\chi/y)_\infty$  equal to (59d); apply the action principle using trial functions with the asymptotic forms (59a, b, c), thereby obtaining a stationary  $\Omega$ ; if  $\Omega$  is equal to  $\omega$ , stop with joy; if not, reiterate using a new trial value of  $\omega$ . (One can show that if  $\Omega$  and  $\omega$  differ by a small amount, the normal modes of frequency  $\Omega$  with boundary condition 59d contain a mixture of ingoing and outgoing waves of relative amplitude

$$C^{(l)}/C^{(0)} = (\Omega - \omega)/(\Omega + \omega). \quad (60)$$

This is a measure of the error in an unconverged iteration by Detweiler's procedure.)

As cumbersome as this procedure may seem, it is the best method now known for computing outgoing-wave normal modes from an action principle; and it actually has been made to give reasonably accurate results for compressional oscillations of neutron stars (Detweiler 1975).

#### 4.4 ENERGY METHOD

If one has obtained reasonable approximations to the eigenfunctions  $Q$ ,  $Y$ ,  $y$  and to the real part  $\sigma$  of the eigenfrequency of a complex normal mode, one can then compute the imaginary part of the eigenfrequency,  $i/2\tau$ , using the law of energy conservation (39), (40). In integral form, and averaged over time, that law says (cf. equations 47 and B.19–B.23):

$$\tau = \frac{\bar{E}_{\text{star}}}{\bar{S}^r(r = R_\infty)}, \quad (61)$$

where

$$\begin{aligned} \bar{E}_{\text{star}} = & \frac{\pi l(l+1)}{(2l+1)} \int_0^{R_\infty} \left\{ (\sigma^2 + 1/4\tau^2) \left[ (\rho + p)r^4 e^{\Lambda - \Phi} |Y - y|^2 + \frac{r^4 e^{-\Phi - \Lambda}}{16\pi} |y' - e^{\Lambda - \Phi} Q/r|^2 \right. \right. \\ & \left. \left. + \frac{(l+2)(l-1)}{16\pi} r^2 e^{\Lambda - \Phi} |y|^2 \right] + \mu r^4 e^{\Phi - \Lambda} |Y' - e^{\Lambda - \Phi} Q/r|^2 \right. \\ & \left. + \mu(l+2)(l-1)r^2 e^{\Phi + \Lambda} |Y|^2 + \frac{(l+2)(l-1)}{16\pi} e^{\Lambda - \Phi} |Q|^2 \right\} dr, \end{aligned} \quad (62a)$$

$$\bar{S}^r(r = R_\infty) = -\frac{l(l+1)}{8(2l+1)} \text{Im} \left[ \left( \sigma + \frac{i}{2\tau} \right) \left( \sigma^2 + \frac{1}{4\tau^2} \right) r^2 y^* \chi \right]_{R_\infty} \quad (62b)$$

for  $R_\infty$  anywhere outside star,

$$= \frac{(l-1)l(l+1)(l+2)}{8(2l+1)} |Q|^2 \text{ for } R_\infty \text{ far out in wave zone.} \quad (62b)$$

Here  $y^*$  is the complex conjugate of  $y$ .

In applying this energy method one can place  $R_\infty$  anywhere one wishes outside the star, even in the near zone, if one uses the first line of equation (62b) for the energy flux.

#### 4.5 SLOW-MOTION METHOD

A generator of gravitational waves is said to be a slow-motion source if and only if the characteristic reduced wavelength of the waves,  $\lambda \equiv \lambda/2\pi = 1/\sigma$ , is much larger than both the source itself and the source's strong-field region:

$$\lambda \gg R, \quad \lambda \gg 2M \equiv (\text{gravitational radius}). \quad (63)$$

Thorne (1980) has given a detailed formalism for calculating the gravitational waves from slow-motion sources. Here we specialize that formalism to the case of torsional oscillations of a neutron star. (A forthcoming paper by Thorne will specialize it to  $g$ -mode compressional oscillations of a neutron star.)

The discussion in the Introduction of this paper gave reduced wavelengths of  $\lambda \approx 10^3$  km for crustal oscillations of neutron stars and  $\lambda \approx 10$  km for core oscillations. Thus the slow-motion approximation is accurate for crustal oscillations but probably not very accurate for core oscillations.

If the slow-motion condition (63) is satisfied, we can neglect retardation of the gravitational fields across the source, i.e. we can neglect  $\ddot{x} = -\omega^2 x$  compared to  $x''$ ,  $x'/r$  or  $x/r^2$  ( $x \equiv Q$  or  $y$ ) throughout the interior of the near-zone region

$$r \ll \lambda \equiv \sigma^{-1}. \quad (64)$$

(We cannot, of course, neglect retardation of the shear waves; i.e. we cannot neglect  $\ddot{x} = -\omega^2 x$  compared to  $[\mu/(\rho + P)] x''$ .) By neglecting gravitational retardation we convert our gravitational variables  $Q$  and  $y$  into action-at-a-distance potentials analogous to that of Newton; their wave equations become Poisson-like equations.

From  $(\mu/\rho) \sim (\text{speed of shear waves})^2 \lesssim (\sigma R)^2$  we learn that

$$\mu \lesssim (R/\lambda)^2 \rho \lesssim (R/\lambda)^2 r^{-2}; \quad (65a)$$

and from equation (19a) for  $y$  and (50b) for  $Q$  we learn the relative magnitudes of  $y$ ,  $Q$  and  $Y$  in the slow-motion approximation:

$$y \sim (M/R) Y, \quad Q \sim (R/\lambda)^2 y \ll y. \quad (65b)$$

Taking account of the extreme smallness of  $\mu$  compared to  $\rho$  and  $r^{-2}$  and of the extreme smallness of  $Q$  compared to  $y$  and  $Y$  and neglecting gravitational retardation, we can bring the equations governing normal-mode oscillations into the form

$$(\mu r^4 e^{\Phi-\Lambda} Y')' - (l+2)(l-1)r^2 e^{\Phi+\Lambda} \mu Y = -\omega^2(\rho+P)r^4 e^{\Lambda-\Phi}(Y-y), \quad (66a)$$

$$(r^4 e^{-\Phi-\Lambda} y')' - (l+2)(l-1)r^2 e^{\Lambda-\Phi} y = -16\pi(\rho+P)r^4 e^{\Lambda-\Phi}(Y-y). \quad (66b)$$

$$(e^{\Phi-\Lambda} Q')' - [(l+2)(l-1)r^{-2} e^{\Phi+\Lambda} - r(r^{-2} e^{\Phi-\Lambda})'] Q = 16\pi r(\mu e^{2\Phi})' Y. \quad (66c)$$

Equation (66a) is (50a) with (52) used to replace a term involving  $Q$  by one involving  $y$ . Equation (66b) is (19a); and equation (66c) is (50b).

Outside the star, and at radii  $M \ll r \ll \lambda$  where  $\Phi = -\Lambda \approx 0$  and where the slow-motion approximation is valid,  $Q$  and  $y$  have power-law fall-offs:

$$Q = \frac{(2l-1)!!}{r^l} i\omega F_\omega, \quad y = -\frac{(l-1)(2l-1)!!}{i\omega r^{l+2}} F_\omega \quad \text{for } M \ll r \ll \lambda \quad \text{and } r > R. \quad (67a)$$

Here  $F_\omega$  is the same  $l$ -pole moment used elsewhere in this paper, and  $(2l-1)!! \equiv (2l-1)(2l-3)\cdots 1$ . These power-law fall-offs are the asymptotic solutions of equations (66b, c). They can also be derived, including the precise coefficients involving  $l$ ,  $\omega$  and  $F_\omega$ , by solving the non-slow-motion, Fourier-decomposed Regge–Wheeler equation (equation 50b with  $\rho = \mu = 0$  and  $r \gg 2M$ ), matching to (51c) and (52) to obtain

$$Q = F_\omega \omega^{l+2} r h_l^{(2)}(\omega r), \quad y = -F_\omega \omega^l r^{-2} \partial_r [r^2 h_l^{(2)}(\omega r)] \quad (68)$$

for all  $r \gg M$  and  $r > R$ , where  $h_l^{(2)}$  is the spherical Hankel function, and by then expanding these solutions in powers of  $\omega r$  in the near zone  $\omega r \ll 1$ .

By virtue of the smallness of  $Q$  in the slow-motion approximation, the no-torque-at-surface boundary condition (51b) reduces to

$$\mu Y' \rightarrow 0 \quad \text{as} \quad r \rightarrow R_-; \quad (67b)$$

but the smoothness boundary conditions (51a) and (53a) at the star's centre remain unchanged.

$$Y \sim r^{l-1}, \quad y \sim r^{l-1}, \quad Q \sim r^{l+1} \quad \text{as} \quad r \rightarrow 0. \quad (67c)$$

The eigenvalue problem in the slow-motion approximation consists of the coupled equations (66a, b) for  $Y$  and  $y$  (not  $Y$  and  $Q$  as previously!), which must be solved subject to the boundary conditions (67a, b, c). The resulting eigenfunctions and eigenfrequencies will be real (no damping in slow-motion approximation!) and discrete. They can be derived from (66a, b), (67) by standard techniques, including the following action principle:

Define  $\Omega^2 \equiv B/A$  where  $B$  and  $A$  are the integrals (58b, c) with  $R_\infty = \infty$  and with the surface term removed and with  $Q$  set to zero. Choose a trial function  $Y$  which satisfies the boundary conditions (67), and from it compute  $y$  by integrating (66b) subject to the boundary conditions (67). Then insert  $Y$  and  $y$  into  $\Omega^2 \equiv B/A$  and ask whether  $\delta \Omega^2 = 0$  for arbitrary variations  $\delta Y$ . If  $\delta \Omega^2 = 0$ , then the trial function  $Y$  and the computed function  $y$  are eigenfunctions, and their value of  $\Omega$  is the corresponding eigenfrequency  $\omega$ .

After the slow-motion eigenvalue problem has been solved, one can use the energy method to compute the tiny imaginary part  $i/2\tau$  of  $\omega$ , which the slow-motion approximation ignores. Specifically,  $\tau$  will be given by equation (61), where the star's pulsation energy  $\bar{E}_{\text{star}}$  is (62a) with  $R_\infty = \infty$  and  $Q = 0$ ; and where the energy flux  $\bar{S}^r$  is given by the second line of (62b), with  $|Q|^2$  replaced by its wave-zone value  $|\omega^{l+1} F_\omega|^2$  (equation 51c) and  $F_\omega$  evaluated from the near-zone expression (67a) for the eigenfunction  $y$ .

#### 4.6 WEAK-FIELD METHOD

For a torsionally oscillating star with weak internal gravity,

$$\mu/\rho \leq P/\rho \sim \Lambda \sim \Phi \sim M/R \ll 1 \quad (69)$$

(e.g. a white dwarf), the slow-motion approximation is automatically valid, and the slow-motion equations simplify. Most importantly, the fact that  $y \sim (M/R) Y \ll Y$  (equation 65b) enables the equation of motion of the matter, (66a), to decouple from all gravitational fields

$$(\mu r^4 Y')' - (l+2)(l-1)r^2 \mu Y = -\omega^2 \rho r^4 Y. \quad (70)$$

This equation, together with the boundary conditions (67b, c)

$$\mu Y' \rightarrow 0 \quad \text{as} \quad r \rightarrow R_-, \quad Y \sim r^{l-1} \quad \text{as} \quad r \rightarrow 0, \quad (71)$$

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forms a Sturm–Liouville eigenvalue problem, which is well known and widely studied in the geophysics literature (e.g. Alterman, Jarosch & Pekeris 1959), and which can be solved by standard techniques. Once it has been solved, the Fourier amplitude of the  $l$ -pole moment can be computed from

$$F_\omega = \frac{-16\pi i\omega}{(l-1)(2l+1)!!} \int_0^R r^{l+3} \rho Y dr. \quad (72)$$

(This equation can be derived by setting  $\Phi = \Lambda = 0$  in (66c), multiplying by  $r^{l+1}$ , integrating from  $r = 0$  to  $r = \infty$ , using the asymptotic form (67a) of  $Q$  to evaluate the surface terms, and using the equation of motion (70) to rewrite the integral.) The imaginary part  $i/2\tau$  of the eigenfrequency can then be evaluated using the energy method (equations 61, 62, 51c)

$$\tau = \bar{E}_{\text{star}} / \bar{S}^r, \quad (73a)$$

$$\bar{E}_{\text{star}} = \frac{\pi l(l+1)}{(2l+1)} \int_0^R [\omega^2 \rho r^4 Y^2 + \mu r^4 Y'^2 + (l+2)(l-1)\mu r^2 Y^2] dr, \quad (73b)$$

$$\bar{S}^r = \frac{(l-1)l(l+1)(l+2)}{8(2l+1)} \omega^{2l+2} |F_\omega|^2. \quad (73c)$$

Notice that, aside from an angular factor,  $\rho i\omega Yr$  is the density of momentum, i.e. of mass current; consequently  $F_\omega$  (equation 72) is proportional to  $\int r^l \times$  (mass current density)  $\times$  (angular factor)  $d\text{vol}$ ; i.e. in the language of Thorne (1980, especially equation 5.27b)  $F_\omega$  is the Fourier amplitude of the star's current  $l$ -pole moment.

For the special case of a star with uniform density  $\rho$  and radially constant shear modulus  $\mu$  the eigenequation (70) reduces to the spherical Bessel equation for  $rY$ ; and consequently

$$Y \propto r^{-1} j_l(kr), \quad k \equiv (\rho/\mu)^{1/2} \omega, \quad (74)$$

where  $j_l$  is the spherical Bessel function. The eigenfrequencies are fixed by the no-torque-at-surface boundary condition  $Y'(R) = 0$  (equation 71). Straightforward calculations using standard Bessel-function identities then yield the following formulas for the star's oscillations and gravitational waves, in terms of the star's radius  $R$ , mass  $M = 4\pi\rho R^3/3$ , shear-wave velocity  $v_s = (\mu/\rho)^{1/2}$ , and amplitude of oscillations

$$\beta \equiv (\text{maximum value of angular displacement function } Y \text{ inside star}). \quad (75)$$

The  $n$ th normal mode (of given angular quantum number  $l$ ) has eigenfrequency and wave-number

$$\omega_n = (v_s/R)x_n, \quad k_n = x_n/R. \quad (76a)$$

The angular displacement of the star's crystal is

$$\begin{aligned} \delta\phi &\equiv \xi^\phi \equiv Yb^\phi \cos \omega_n t \\ &= \beta \frac{j_l(k_n r)}{\alpha k_n r} \frac{\partial_\vartheta P_l(\cos \vartheta)}{\sin \vartheta} \cos \omega_n t \\ &= -3\beta \frac{j_2(k_n r)}{\alpha k_n r} \cos \vartheta \cos \omega_n t \quad \text{if } l = 2. \end{aligned} \quad (76b)$$

The star's energy of oscillation is

$$\bar{E}_{\text{star}} = E_n M v_s^2 \beta^2. \quad (76c)$$

Table 1. Constants governing quadrupole ( $l = 2$ ) torsional oscillations of a star with uniform  $\rho$  and  $\mu$  and with weak gravity.

$n$	$x_n$	$E_n$	$G_n$	$L_n$	$D_n$
1	2.5011	2.030	2.001	15.03	0.3379
2	7.1360	1.461	-1.062	34.45	0.3026
3	10.515	0.7171	0.7274	35.09	0.2148
4	13.772	0.4267	-0.5568	35.28	0.1666
5	16.983	0.2833	0.4519	35.35	0.1361

$$\alpha = 0.10403$$

The gravitational wave field has as its only non-zero components in an orthonormal, spherical basis

$$\begin{aligned} h_{\vartheta\phi}^{\text{TT}} &= G_n (M/r) v_s^{l+1} \beta \sin \vartheta b_{\vartheta, \varphi}^{\phi} \cos [\omega_n (t - r_*) + (l+1)\pi/2] \\ &= 3G_n (M/r) v_s^3 \beta \sin^2 \vartheta \sin [\omega_n (t - r_*)] \quad \text{if } l = 2. \end{aligned} \quad (76d)$$

The power carried off by the waves is

$$\begin{aligned} \bar{S}^r &= -\frac{d\bar{E}_{\text{star}}}{dt} = L_n (M/R)^2 v_s^{2l+4} \beta^2 \\ &= L_n (M/R)^2 v_s^8 \beta^2 \quad \text{if } l = 2. \end{aligned} \quad (76e)$$

This power loss causes the energy to decay by  $1/e$  in a number of oscillations given by

$$\begin{aligned} \omega_n \tau_n &= D_n (M/R)^{-1} v_s^{-(2l+1)} \\ &= D_n (M/R)^{-1} v_s^{-5} \quad \text{if } l = 2. \end{aligned} \quad (76f)$$

Here the constants  $\alpha$ ,  $x_n$ ,  $E_n$ ,  $G_n$ ,  $L_n$ ,  $D_n$  are given by

$$\begin{aligned} \alpha &\equiv j_l(x_1)/x_1, \\ x_n &\equiv n\text{th root of } \partial_x [j_l(x)/x] = 0, \\ E_n &\equiv \frac{3l(l+1)}{4(2l+1)} \left[ \frac{j_l(x_n)}{\alpha} \right]^2 \left[ 1 - \frac{(l+2)(l-1)}{x_n^2} \right], \\ G_n &\equiv \frac{12}{\alpha(l-1)(2l+1)!!} x_n^{l-1} j_{l+1}(x_n), \\ L_n &\equiv \frac{18l(l+1)(l+2)}{\alpha^2(l-1)(2l+1)[(2l+1)!!]^2} [x_n^l j_{l+1}(x_n)]^2, \\ D_n &\equiv E_n x_n / L_n. \end{aligned} \quad (76g)$$

and are tabulated in Table 1 for  $l = 2$ .

### 5 Dipole torsional oscillations

We now turn attention to dipole torsional oscillations, i.e. oscillations with  $l = 1$  (and, with only trivial loss of generality,  $m = 0$ ). For  $l \geq 2$  we used our gauge freedom to annul  $h_{\vartheta\phi}$ . For  $l = 1$   $h_{\vartheta\phi}$  vanishes identically in all gauges because its angular dependence is  $\sin^2 \vartheta b_{\vartheta, \varphi}^{\phi} \equiv 0$ . Thus, we can use our gauge freedom instead to annul  $h_{r\vartheta}$  (i.e. to set  $Q = 0$ ),

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thereby leaving us with only one non-zero metric perturbation

$$h_{r\phi} = h_{\phi r} = -r^2 \dot{y} b_\phi = r^2 \dot{y} \sin^2 \vartheta \quad (77)$$

(*cf.* equation 6b). The displacement function is defined as for  $l \geq 2$

$$\xi^r = \xi^\vartheta = 0, \quad \xi^\phi = Y b^\phi = -Y \quad (78)$$

(*cf.* equation 8b); and the Ricci tensor and stress-energy tensor then also have the same forms as for  $l \geq 2$  (equations 7 and 17 with specialization to  $l = 1$  and  $Q = 0$ ).

Our equations of motion (19) for  $l \geq 2$  were derived using the Einstein field equation  $[\delta R_{\vartheta\phi} - 8\pi\delta(T_{\vartheta\phi} - 1/2 Tg_{\vartheta\phi})]/[\sin^2 \vartheta b^\phi_{,\vartheta}]$  (i.e.  $\epsilon_{\vartheta\phi} = 0$ ; equation A3). Because this equation is invalid for  $l = 1$  (it involves dividing by  $\sin^2 \vartheta b^\phi_{,\vartheta} \equiv 0$ ), we cannot obtain the correct  $l = 1$  equations by simply setting  $Q = 0$  and  $l = 1$  in (19). Rather, we must derive our equations of motion directly from the Einstein equations (A.1)–(A.5), with the omission of the  $\epsilon_{\vartheta\phi}$  equation. The result is

$$(\rho + P) e^{-2\Phi} (\ddot{Y} - \ddot{y}) = r^{-4} e^{-\Phi - \Lambda} (\mu r^4 e^{\Phi - \Lambda} Y')', \quad (79a)$$

$$(\rho + P) e^{-2\Phi} (Y - y) = -(16\pi)^{-1} r^{-4} e^{-\Phi - \Lambda} (r^4 e^{-\Phi - \Lambda} y')'. \quad (79b)$$

A third Einstein equation, which is related to these two by the Bianchi identities, is

$$\ddot{y}' = -16\pi\mu e^{2\Phi} Y'. \quad (80)$$

The equations of motion (79) are derivable from the action principle (37), (38) in which the Lagrangian density  $\mathcal{L}$  is specialized to  $l = 1$  and  $Q \equiv 0$ . The corresponding  $l \geq 2$  conservation law  $S^\alpha_{,\alpha} = 0$  is also valid for  $l = 1$ , with the  $S^\alpha$  of equations (40) specialized to  $l = 1$  and  $Q \equiv 0$ ; and the proof in Section 3.5 that if  $\mu \geq 0$  then the star is stable, which is based on the conservation law  $S^\alpha_{,\alpha} = 0$ , remains valid for  $l = 1$ .

Equation (80) implies that  $\dot{y}$  is time-independent outside the star; and equation (79b) says that its radial dependence there is  $\dot{y} = A + B/r^3$  (recall that  $\Phi + \Lambda = 0$  in vacuum). The constant  $A$  is physically unacceptable, while the term  $B/r^3$  describes the dragging of inertial frames by the star's constant angular momentum (see, e.g. Hartle 1967). With only trivial loss of generality we shall set the star's angular momentum to zero (i.e. we shall refuse to consider purely stationary, rotational perturbations), thereby enforcing  $y \equiv 0$  everywhere outside the star. As a result, our oscillating star not only will produce no gravitational waves (a consequence of the dipole angular dependence of our perturbations); it will not have any gravitational perturbations whatsoever outside itself.

The eigenvalue problem for normal-mode oscillations with  $l = 1$  consists of the coupled differential equations

$$(\mu r^4 e^{\Phi - \Lambda} Y')' = -\omega^2 r^4 e^{\Lambda - \Phi} (\rho + P) (Y - y), \quad (81a)$$

$$(r^4 e^{-\Phi - \Lambda} y')' = -16\pi r^4 e^{\Lambda - \Phi} (\rho + P) (Y - y) \quad (81b)$$

(equations 79 with  $Y \propto e^{i\omega t}$  and  $y \propto e^{i\omega t}$ ), together with the boundary conditions of smoothness and zero torque at the origin, the surface, and solid–fluid interfaces

$$Y \sim \text{constant} + O(r^2), \quad y \sim \text{constant} + O(r^2) \text{ near } r = 0, \quad (82a)$$

$$y, y' \quad \text{and} \quad \mu Y' \rightarrow 0 \quad \text{as} \quad r \rightarrow R_-, \quad (82b)$$

$$\mu Y' \rightarrow 0 \quad \text{at solid–fluid interfaces.} \quad (82c)$$

[Equation (82a) rules out the divergent solutions  $Y \sim y^{-3}$  and  $y \sim r^{-3}$ ; equation (82b) follows from  $y \equiv 0$  outside the star and integrations of (79b) through the star's surface, and from (80) or (51b); equation (82c) follows from (53b) or from integrations of (80) through the interfaces.] The oscillation frequencies  $\omega$  and eigenfunctions  $Y, y$  will be real since there is no gravitational radiation and no energy loss.

Note that the eigenequations (79) for  $l = 1$  are identical to those of the  $l \geq 2$  slow-motion approximation (equations 66a, b). Here the absence of retardation of the gravitational field  $y$  is due to its  $l = 1$  angular dependence, which forbids gravitational radiation. There the absence of retardation and of waves was due to the slow-motion assumption. Here, as there, an action principle for the eigenvalue problem is given by  $\delta\Omega^2 = 0$ , where  $\Omega^2 \equiv A/B$  with  $A$  and  $B$  given by expressions (58b, c) with  $R_\infty = R_-$ , the surface term removed,  $Q$  set to zero, and  $l$  set to one. For  $l = 1$  this action principle does not require slow motion, and a slow-motion assumption produces no simplifications.

For a star with weak internal gravity the dipole eigenvalue problem (81), (82) simplifies to (70), (71) specialized to  $l = 1$ . When the star is homogeneous with  $\rho$  and  $\mu$  constant, that eigenvalue problem has the analytic solution (74), (75), (76a, b, c, g) specialized to  $l = 1$ . [For  $l = 1$  the gravitational-wave related equations (72), (73a, c), (76d, e, f) are irrelevant and incorrect.]

## 6 Concluding remarks

It should be straightforward to use the formalisms of this paper to evaluate numerically the characteristics of normal-mode torsional oscillations of neutron star models. Such calculations should be performed both to improve the approximate formulas given in the introduction of this paper (equations 3 and 4) and to discover quantitatively how the physical properties of neutron star matter influence a star's normal-mode frequencies, damping times and gravity-wave strengths.

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#### Appendix A: Equations of motion

In deriving the equations of motion (19a, b, c) we shall denote by

$$\epsilon_{\alpha\beta} \equiv [\delta R_{\alpha\beta} - 8\pi\delta(T_{\alpha\beta} - 1/2 Tg_{\alpha\beta})]/f_{\alpha\beta}(r, \vartheta), \quad \epsilon_T \equiv T_{\phi;\alpha}^\alpha/f_T(r, \vartheta) \quad (\text{A.1})$$

the expressions obtained by combining equations (5), (7), (18), (6) and (17) and dividing by the functions

$$f_{r\phi} = -1/2 r^2 e^{-2\Lambda} b_\phi, \quad f_{\vartheta\phi} = 1/2 r^2 e^{-2\Phi} \sin^2 \vartheta b_{\phi,\vartheta}, \\ f_{r\phi} = -1/2 r e^{\Lambda-3\Phi} b_\phi, \quad f_T = r^2 e^{-2\Phi} b_\phi.$$

These expressions are:

$$\epsilon_{r\phi} = \partial_t \left\{ \frac{-e^{\Phi+\Lambda}}{r^4} (r^4 e^{-\Phi-\Lambda} y')' + e^{2\Lambda} \left[ 16\pi(\rho+P) + \frac{(l+2)(l-1)}{r^2} \right] y \right. \\ \left. - 16\pi(\rho+P)e^{2\Lambda} Y + \frac{e^{\Phi+\Lambda}}{r^4} (r^3 e^{-2\Phi} Q)' \right\}; \quad (\text{A.2})$$

$$\epsilon_{\vartheta\phi} = \ddot{y} - \frac{e^{\Phi-\Lambda}}{r^2} (rQ)' + 16\pi\mu e^{2\Phi} Y; \quad (\text{A.3})$$

$$\epsilon_{r\phi} = \ddot{Q} + e^{2\Phi} \left[ 16\pi\mu + \frac{(l+2)(l-1)}{r^2} \right] Q - r e^{\Phi-\Lambda} \ddot{y}' - 16\pi\mu e^{3\Phi-\Lambda} r Y'; \quad (\text{A.4})$$

$$\epsilon_T = (\rho+P)\ddot{Y} - r^{-4} e^{\Phi-\Lambda} (\mu r^4 e^{\Phi-\Lambda} Y')' + (l+2)(l-1)r^{-2} e^{2\Phi} \mu Y \\ - (\rho+P)\ddot{y} + r^{-4} e^{\Phi-\Lambda} (\mu r^3 Q)'. \quad (\text{A.5})$$

Note that  $\epsilon_T$  can be expressed as the following combination of the  $\epsilon_{\alpha\beta}$  (Bianchi identity):

$$\epsilon_T = (16\pi)^{-1} [(l+2)(l-1)r^{-2}\epsilon_{\vartheta\phi} - e^{-2\Lambda}\dot{\epsilon}_{t\phi} + r^{-4}e^{\Phi-\Lambda}(r^3e^{-2\Phi}\epsilon_{r\phi})']. \quad (\text{A.6})$$

The perturbed Einstein field equations are  $\epsilon_{\alpha\beta} = 0$ ; the law of conservation of energy–momentum for the perturbed system,  $\delta T^{\alpha\beta}{}_{;\beta} = 0$ , reduces to the single equation  $\epsilon_T = 0$ .

The equations of motion (19a, b, c) used in the text are the following combinations of field equations:

equation (19a), initial-value equation for  $y$ :

$$\int \epsilon_{t\phi} dt = 0; \quad (\text{A.7})$$

equation (19b), wave-equation for  $Y$ :

$$e^{-2\Phi} [\epsilon_T + (\rho + P)\epsilon_{\vartheta\phi}] = 0; \quad (\text{A.8})$$

equation (19c), wave-equation for  $Q$ :

$$e^{-2\Phi} [\epsilon_{r\phi} + re^{\Phi-\Lambda}\epsilon_{\vartheta\phi}'] = 0. \quad (\text{A.9})$$

We must show that our equations of motion (19) are complete, i.e. that all physically acceptable solutions of (19) also satisfy the full set of perturbed Einstein equations  $\epsilon_{\alpha\beta} = 0$  and the equation of energy–momentum conservation  $\epsilon_T = 0$ . To prove this, we combine the equations of motion (19) with the Bianchi identity (A.6) to obtain the Sturm–Liouville equation

$$r^{-4}e^{3\Lambda} (r^4e^{-\Phi-\Lambda}\epsilon_{\vartheta\phi}')' - e^{2\Lambda} [16\pi(\rho + P) + (l+2)(l-1)r^{-2}]\epsilon_{\vartheta\phi} = 0 \quad (\text{A.10})$$

for  $\epsilon_{\vartheta\phi}$ . This equation, together with (19), leads to perturbation functions  $Y$ ,  $Q$ ,  $y$  which satisfy the physical boundary conditions (26a, d, e) only if  $\epsilon_{\vartheta\phi} \sim r^{l-1}$  near  $r=0$  and  $\epsilon_{\vartheta\phi} \sim r^{-l-2}$  near  $r=\infty$ . However, the signs of the terms in (A.10) make it impossible for these two asymptotic formulae to join on to each other except in the case  $\epsilon_{\vartheta\phi} = 0$ . From this we conclude that our equations of motion and boundary conditions imply  $\epsilon_{\vartheta\phi} = 0$ ; this, together with the equations of motion themselves, implies trivially that all the  $\epsilon_{\alpha\beta}$  and  $\epsilon_T$  vanish (*cf.* equations A.6–A.9). QED.

### Appendix B: Foundations for action principles

Friedman & Schutz (1975; their section II) have given an elegant formulation of the general theory of action principles for systems which can radiate waves to infinity. Unfortunately, their analysis was not carried far enough to embrace the Detweiler–Ipser (1973) type of action principle for normal-mode pulsations, which we use in Sections 3.4 and 4.3. In this appendix we extend the Friedman–Schutz analysis to encompass such action principles, and we use it to derive various results presented in the text of the paper. The general theory is presented with full-left margins; the application to torsionally oscillating stars is presented indented.

Consider a system described by functions  $Z_A$  ( $A = 1, 2, \dots, n$ ) in a spacetime with coordinates  $x^\alpha$  ( $\alpha = 0, 1, 2, \dots, m$ ). Assume that the equations of motion for  $Z_A$  are derivable from an action principle

$$\delta \int_{\Omega} \mathcal{L} dx^0 \dots dx^m = 0, \quad (\text{B.1})$$

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where  $\delta Z_A = 0$  on  $\partial\Omega$ . Assume that the Lagrangian density  $\mathcal{L}$  is quadratic and symmetric, i.e.  $\mathcal{L} = L(Z, Z)$  with

$$L(Z^\dagger, Z) \equiv \sum_{k=1}^p \sum_{l=1}^p \Lambda^{A B \alpha_1 \dots \alpha_k \beta_1 \dots \beta_l} Z_{A, \alpha_1 \dots \alpha_k}^\dagger Z_{B, \beta_1 \dots \beta_l} \quad (\text{B.2})$$

where  $\Lambda^{A B \alpha_1 \dots \alpha_k \beta_1 \dots \beta_l}$  is a function of the coordinates  $x^\mu$  which is completely symmetric in the indices  $\alpha_1 \dots \alpha_k$ , completely symmetric in the  $\beta_1 \dots \beta_l$  and also symmetric under interchange of  $A \alpha_1 \dots \alpha_k$  with  $B \beta_1 \dots \beta_l$  (so  $L(Z^\dagger, Z) = L(Z, Z^\dagger)$ ). The  $Z_A^\dagger$  are a set of functions which have no special relationship to the  $Z_A$ , and  $Z_{A, \alpha_1 \dots \alpha_k}^\dagger \equiv \partial^k Z_A^\dagger / \partial x^{\alpha_1} \dots \partial x^{\alpha_k}$ . The quantity  $p$  is the maximum number of derivatives that appear in the Lagrangian, and there is an implied summation over repeated function indices  $A, B$  as well as coordinate indices  $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l$ . Define

$$L^A(Z) \equiv \sum_l (-1)^l \partial_{\alpha_1} \dots \partial_{\alpha_l} \frac{\partial L(Z^\dagger, Z)}{\partial Z_{A, \alpha_1 \dots \alpha_l}^\dagger} \quad (\text{B.3})$$

Then the Euler–Lagrange equations are  $L^A(Z) = 0$ , and the integration-by-parts identity used in deriving these Euler–Lagrange equations from the action principle (B.1) is

$$Z_A^\dagger L^A(Z) = L(Z^\dagger, Z) - \partial_\mu Q^\mu(Z^\dagger, Z) \quad (\text{B.4})$$

The  $Q^\mu$  are determined only up to a divergence-free vector. Two versions of  $Q^\mu$ , which differ from each other by a divergence-free vector, are

$$Q^\mu(Z^\dagger, Z) = \sum_{k,l} (-1)^l Z_{A, \beta_1 \dots \beta_k}^\dagger \partial_{\alpha_1} \dots \partial_{\alpha_l} \frac{\partial L(Z^\dagger, Z)}{\partial Z_{A, \alpha_1 \dots \alpha_l \beta_1 \dots \beta_k}^\dagger}; \quad (\text{B.5a})$$

and

$$Q^0(Z^\dagger, Z) = \sum_{j=1}^p \sum_{k=0}^p \sum_{l=1}^j (-1)^{l-1} \binom{j+k}{j} Z_{A, a_1 \dots a_k}^\dagger \left( \frac{\partial L}{\partial Z_{A, a_1 \dots a_k}^\dagger} \right)^{(j-l)}, \quad (\text{B.5b})$$

$$Q^b(Z^\dagger, Z) = \sum_{j=0}^p \sum_{k=1}^p \sum_{l=1}^k (-1)^{j+k-l} \binom{j+k}{j} Z_{A, a_1 \dots a_{l-1}}^\dagger \left( \frac{\partial L}{\partial Z_{A, a_1 \dots a_{k-1} b}^\dagger} \right)^{(j)}_{, a_l \dots a_{k-1}} \quad (\text{B.5c})$$

The second version (equations B5b,c) has the virtue that the time component  $Q^0(Z^\dagger, Z)$  contains the lowest possible number of spatial derivatives of the  $Z_A$ ; it is the version which we use in our analysis of torsional oscillations of stars. In the second version the Latin letters  $b$  and  $a_1, \dots, a_p$  denote spatial tensorial indices and run from 1 to  $m$ ;

$$\binom{j+k}{j}$$

is the binomial coefficient; and superscripts in parentheses denote time derivatives as in the text:  $Z_{A, a_1 \dots a_k}^\dagger{}^{(j-l)} = Z_{A, a_1 \dots a_k}^\dagger$  with  $j-l$  zeros. It is imperative when using equations (B.3), (B.5a–c) and others below that  $L(Z^\dagger, Z)$  be properly symmetrized (including making a careful distinction, e.g., between  $Z_{A,10}$  and  $Z_{A,01}$  and symmetrizing  $L$  in them), cf. discussion following equation (B.2). Failure to symmetrize will produce in (B.3), (B.5a–c) multiple counting of second and higher-order derivative terms.

For our torsionally oscillating star the coordinates are  $x^0 = t, x^1 = r$ ; the functions  $Z_A$  are  $Y, y, Q$ ; the Lagrangian density  $\mathcal{L}$  is equation (37). From the Lagrangian density we can

read off

$$\begin{aligned}
 L(Z^\dagger, Z) = & \frac{2\pi l(l+1)}{(2l+1)} \left[ (\rho + p)r^4 e^{\Lambda-\Phi} (\dot{Y}^\dagger - \dot{y}^\dagger)(\dot{Y} - \dot{y}) \right. \\
 & + \frac{r^4 e^{-\Phi-\Lambda}}{16\pi} (y^{\dagger'} - e^{\Lambda-\Phi} \dot{Q}^\dagger/r)(y' - e^{\Lambda-\Phi} \dot{Q}/r) + \frac{(l+2)(l-1)}{16\pi} r^2 e^{\Lambda-\Phi} y^\dagger \dot{y} \\
 & - \mu r^4 e^{\Phi-\Lambda} (Y^{\dagger'} - e^{\Lambda-\Phi} \dot{Q}^\dagger/r)(Y' - e^{\Lambda-\Phi} \dot{Q}/r) - \mu(l+2)(l-1)r^2 e^{\Phi+\Lambda} Y^\dagger Y \\
 & \left. - (1/16\pi)(l+2)(l-1)e^{\Lambda-\Phi} \dot{Q}^\dagger \dot{Q} \right]. \tag{B.6}
 \end{aligned}$$

The Euler–Lagrange expressions  $L^A(Z)$ , obtained from expression (B.3) (in which one must take careful account of the symmetry properties of  $L$ ) or by varying the action, are the following:

$$\begin{aligned}
 L^Y(Z) &= \frac{-2\pi l(l+1)}{(2l+1)} r^4 e^{\Lambda-\Phi} \epsilon_T, \\
 L^y(Z) &= \frac{-l(l+1)}{8(2l+1)} r^4 e^{-\Phi-\Lambda} \epsilon_{t\phi}, \\
 L^Q(Z) &= \frac{-l(l+1)}{8(2l+1)} r^2 e^{\Lambda-3\Phi} \epsilon_{r\phi}, \tag{B.7}
 \end{aligned}$$

where  $\epsilon_T$ ,  $\epsilon_{r\phi}$  and  $\epsilon_{t\phi}$  are the Einstein field-equation expressions given in Appendix A. One of us (BLS) originally derived the Lagrangian density  $\mathcal{L}$  by constructing the expression  $Z_A^\dagger L^A(Z)$ , by adding a perfect divergence (equation B.4) and by then setting  $Z_A^\dagger = Z_A$  (method of Chandrasekhar 1964a, b; Detweiler & Ipser 1973). For the quantities  $Q^\mu$  which appear in the divergence, we shall use expressions (B.5b, c) because they lead to a  $Q^0$  (and subsequently  $S^0$ ) which contain only first derivatives of  $Y$ ,  $y$  and  $Q$ :

$$\begin{aligned}
 Q^0(Z^\dagger, Z) = & \frac{2\pi l(l+1)}{(2l+1)} \left[ (\rho + p)r^4 e^{\Lambda-\Phi} (Y^\dagger - y^\dagger)(\dot{Y} - \dot{y}) \right. \\
 & + \frac{r^4 e^{-\Phi-\Lambda}}{16\pi} (y^{\dagger'} - e^{\Lambda-\Phi} \dot{Q}^\dagger/r)(y' - e^{\Lambda-\Phi} \dot{Q}/r) \\
 & \left. + \frac{(l+2)(l-1)}{16\pi} r^2 e^{\Lambda-\Phi} y^\dagger \dot{y} \right], \tag{B.8a}
 \end{aligned}$$

$$\begin{aligned}
 Q^r(Z^\dagger, Z) = & \frac{-2\pi l(l+1)}{(2l+1)} \left[ \frac{r^4 e^{-\Phi-\Lambda}}{16\pi} y^\dagger (\dot{y}' - e^{\Lambda-\Phi} \dot{Q}/r) \right. \\
 & \left. + \mu r^4 e^{\Phi-\Lambda} Y^\dagger (Y' - e^{\Lambda-\Phi} \dot{Q}/r) \right]. \tag{B.8b}
 \end{aligned}$$

Whenever the Lagrangian is stationary in the sense that

$$\left[ \partial L(Z^\dagger, Z) / \partial x^0 \right]_{Z^\dagger, Z \text{ held fixed}} = 0, \tag{B.9a}$$

the Euler–Lagrange equations enforce a law of energy conservation:

$$L^A(Z) = 0 \text{ and (B.9a) imply that } \partial_\mu S^\mu = 0, \tag{B.10a}$$

where

$$S^\mu \equiv 2 Q^\mu(\dot{Z}, Z) - \delta_0^\mu L(Z, Z). \tag{B.10b}$$

The arbitrariness in  $Q^\mu$  (freedom to add any divergence-free vector) produces a corresponding arbitrariness in  $S^\mu$ .

Our Lagrangian (B.6) is stationary. From our chosen form (B.8) for  $Q^\mu$  and expression (B.6) for  $L$  we derive expressions (40a, b) for our energy density,  $S^0$  and energy flux,  $S^r$ . Had we chosen any other  $Q^\mu$ , the resulting energy density,  $S^0$  would not have been equal to the Lagrangian with sign reversal of the potential energy terms.

We now turn attention to functions  $Z_A$  with exponential and sinusoidal time dependence  $Z_A(x^\alpha) \equiv z_A(x^j)e^{i\omega t}$  ( $j = 1, 2, \dots, m$ ;  $\omega$  a complex frequency) and we decompose  $L$ ,  $L^A$ , and  $Q^\mu$  into powers of  $\omega$ :

$$\begin{aligned} L(z^\dagger e^{-i\omega t}, z e^{i\omega t}) &= \omega^n L_n(z^\dagger, z), & L^A(z e^{i\omega t}) &= \omega^n L_n^A(z) e^{i\omega t}, \\ Q^\mu(z^\dagger e^{-i\omega t}, z e^{i\omega t}) &= \omega^n Q_n^\mu(z^\dagger, z), \end{aligned} \quad (\text{B.11})$$

where there is an implied summation over the integer  $n$ . In our discussion we shall require that  $L$  be stationary (equation B.9a); this guarantees the existence of solutions with  $e^{i\omega t}$  time dependence. We shall also require that  $L_n$  contain only even powers of  $\omega$

$$L_n(z^\dagger, z) = 0 \quad \text{for } n \text{ odd}; \quad (\text{B.9b})$$

this, together with symmetry of  $L$  [ $L(Z^\dagger, Z) = L(Z, Z^\dagger)$ ] and definition (B.11) of  $L_n$ , implies that  $L_n$  is symmetric

$$L_n(z^\dagger, z) = L_n(z, z^\dagger). \quad (\text{B.12})$$

Note that the fundamental identity (B.4) implies that

$$z_A^\dagger L_n^A(z) = L_n(z^\dagger, z) - \partial_j Q_n^j(z^\dagger, z). \quad (\text{B.13})$$

For  $Z = z e^{i\omega t}$  our equations of motion  $L^A(Z) = 0$  reduce to the eigenequation

$$\begin{aligned} \omega^n L_n^A(z) = 0 &\text{ if and only if } z \text{ is an eigenfunction and } \omega \text{ is its eigenvalue;} \\ &\text{i.e. if and only if } z e^{i\omega t} \text{ is a normal mode.} \end{aligned} \quad (\text{B.14})$$

We shall be interested in normal modes which are defined on a compact region  $\mathcal{V}$  of space (not spacetime). Then the identity (B.13) together with the symmetry condition (B.12) implies the following action principle: Define  $\omega(z)$  by  $I(\omega, z) = 0$  where

$$\begin{aligned} I(\omega, z) &= \omega^n \int_{\mathcal{V}} L_n(z, z) d^m x - \omega^n \int_{\partial \mathcal{V}} Q_n^j(z, z) d^{m-1} \Sigma_j \\ &= \int_{\mathcal{V}} L(z e^{-i\omega t}, z e^{i\omega t}) d^m x - \int_{\partial \mathcal{V}} Q^j(z e^{-i\omega t}, z e^{i\omega t}) d^{m-1} \Sigma_j \\ &= \omega^n \int_{\mathcal{V}} z_A L_n^A(z) d^m x. \end{aligned} \quad (\text{B.15})$$

In general there will be several roots  $\omega(z)$ . Consider each root in turn. The eigenfunctions  $z_A$  are those for which  $\omega(z)$  is stationary under small perturbations  $\delta z_A$ , with

$$\omega^n \int_{\partial \mathcal{V}} [Q_n^j(z, \delta z) - Q_n^j(\delta z, z)] d^{m-1} \Sigma_j = 0. \quad (\text{B.16})$$

Note that constraint (B.16) corresponds to certain combinations of the  $z_A$  and their derivatives being held fixed on  $\partial \mathcal{V}$ . This constraint ensures that the Euler–Lagrange equations associated with the action principle are  $\omega^n L_n^A(z) = 0$  (equation B.14).

For our torsionally oscillating star we choose  $\mathcal{V}$  to be the interior of a sphere,  $r \leq R_\infty$  with boundary  $R_\infty$  far out in the radiation zone. Then the function  $I(\omega, z)$  is easily evaluated from equations (B.6) and (B.8b) for  $L$  and  $Q^r$

$$I(\omega, z) = \frac{2\pi l(l+1)}{(2l+1)} (\omega^2 A - B), \tag{B.17}$$

where  $A$  and  $B$  are expressions (58b, c); and the constraint (B.16) on  $Q, y, Y$  is easily evaluated from (B.8b)

$$\omega^2 \frac{2\pi l(l+1)}{(2l+1)} \frac{r^2}{16\pi} y^2 \delta(\chi/y) = 0 \quad \text{at} \quad r = R_\infty. \tag{B.18}$$

Here  $\chi = -r^2 e^{-\Phi-\Lambda} (y' - e^{\Lambda-\Phi} Q/r)$  (equation 57b). The action principle thus consists of extremizing  $\omega^2 = B/A$  with respect to variations of  $Q, y, Y$ , with  $\chi/y$  held fixed at  $R_\infty$ .

The initial-value equation (19a) for  $y$  is one of the Euler–Lagrange equations of this action principle. Because it is independent of  $\omega$ , (19a) can be imposed as a constraint on all trial functions before the action is varied. The normal modes obviously will still give stationary  $\omega$ , and one can verify that this procedure does not introduce any spurious solutions – only the normal modes give stationary  $\omega$ . This is the version of the action principle presented in the text (Section 4.3).

Assume that  $L(Z^\dagger, Z)$  is ‘real’ in the sense that  $L(Z^{\dagger*}, Z^*) = [L(Z^\dagger, Z)]^*$ , where  $*$  denotes complex conjugation. Then if  $Z = z e^{i\omega t}$  is a solution of the Euler–Lagrange equations,  $Z^* = z^* e^{-i\omega^* t}$  will also be a solution; and from the complex solution  $z e^{i\omega t}$  we can build a real solution

$$Z = 1/2 (z e^{i\omega t} + z^* e^{-i\omega^* t}), \quad \omega \equiv \sigma + i/2\tau. \tag{B.19}$$

If we insert this real solution into expression (B.10b) for  $S^\mu$  we obtain

$$S^\mu = \bar{S}^\mu e^{-t/\tau} + \tilde{S}^\mu \cos(2\sigma t + \vartheta^\mu) e^{-t/\tau}, \tag{B.20}$$

where

$$\bar{S}^0 = \{\text{Im} [\omega^* Q^0 (z^* e^{-i\omega^* t}, z e^{i\omega t})] - 1/2 \text{Re} [L(z^* e^{-i\omega^* t}, z e^{i\omega t})]\} e^{t/\tau}, \tag{B.21a}$$

$$\bar{S}^j = \text{Im} [\omega^* Q^j (z^* e^{-i\omega^* t}, z e^{i\omega t})] e^{t/\tau}, \tag{B.21b}$$

and where  $\tilde{S}^\mu$  is not of interest to us. In the law of energy conservation  $\partial_\mu S^\mu = 0$ , the pure exponential terms and the sinusoidal terms must be conserved separately. It is the pure exponential terms that interest us; for them, energy conservation says

$$(1/\tau) \bar{S}^0 = \bar{S}^j{}_{,j}; \tag{B.22}$$

integration over the spatial region  $\mathcal{V}$  implies

$$(1/\tau) \int_{\mathcal{V}} \bar{S}^0 d^m x = \int_{\partial \mathcal{V}} \bar{S}^j d^{m-1} \Sigma_j. \tag{B.23}$$

For our torsional oscillations  $\bar{S}^0$  and  $\bar{S}^r$ , as computed from equations (B.21), (B.6) and (B.8), are the expressions given in equations (62), where  $\bar{E}_{\text{star}} = \int \bar{S}^0 dr$ .