Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Corso di Laurea in Matematica

STABLE ISOMORPHISM VS ISOMORPHISM OF VECTOR BUNDLES:
AN APPLICATION TO QUANTUM SYSTEMS

Tesi di Laurea in Geometria Differenziale

Relatrice:
Prof.ssa
ALESSIA CATTABRIGA

Correlatore:
Chiar.mo Prof.
RALF MEYER

Presentata da:
CLARA PUNZI

VI Sessione
Anno Accademico 2017/2018

Abstract

La classificazione dei materiali sulla base delle fasi topologiche della materia porta allo studio di particolari fibrati vettoriali sul d-toro con alcune strutture aggiuntive. Solitamente, tale classificazione si fonda sulla nozione di isomorfismo tra fibrati vettoriali; tuttavia, quando il sistema soddisfa alcune assunzioni e ha dimensione abbastanza elevata, alcuni autori ritengono invece sufficiente utilizzare come relazione d′equivalenza quella meno fine di isomorfismo *stabile*. Scopo di questa tesi è fissare le condizioni per le quali la relazione di isomorfismo stabile può sostituire quella di isomorfismo senza generare inesattezze. Ciò nei particolari casi in cui il sistema fisico quantistico studiato non ha simmetrie oppure è dotato della simmetria discreta di inversione temporale.

Contents

Introduction

The dynamics of a quantum mechanical system has a mathematical formulation in terms of the Hamiltonian, that is, a self-adjoint element of the C^*-algebra $\mathscr{A}$ of all observable quantities. The possibility of continuously deforming Hamiltonians of different systems into each other leads to the classification of materials according to topological phases. Considering the spectral projection of the Hamiltonian at the Fermi level, this classification can be derived from the study of certain vector bundles over the d-torus, where d is the dimension of the material, with eventual extra structure encoding possible symmetries. Typically, it is based on the homotopy classes of these.

Moving from [EM, Paragraph 4.7], we argue in this thesis that a better behaved theory can be developed using the equivalence relation of stable isomorphism instead of that of isomorphism, which actually turns out to be a refinement of the former whose significance is not (very probably) physically relevant.

In Chapter 1, we are going to define some basic mathematical tools such as CW-complexes, bundles and vector bundles, we will recall their principal pro-perties and then prove some useful results. More sophisticated constructions will be described in Chapter 4 in order to take into account the presence of symmetries. Namely, we will describe how the action of the group $\mathbb{Z}_2$ on the base space of the vector bundle can endow it with specific extra structures called "Real" and "Quaternionic" vector bundles in [DG1, DG2].

The physical motivation is the content of Chapter 3, which also includes a description of the mathematical model of quantum mechanical systems and an introduction to topological phases, with a particular attention to insulators.

In Chapter 2 and Chapter 5, we will ilustrate the conditions under which stable isomorphism and isomorphism of vector bundles are equivalent, so that one can use the latter for the purpose of the classification of materials with the assurance of not losing information about the physical situation. In particular, in Chapter 2 we will outline the argument for the basic case of vector bundles without any additional structure, referring to [HUS], and we will generalize the same argument to the case of quantum systems with time-reversal symmetry in Chapter 5. As we will point out in the conclusion, the conditions on the dimension of the systems under which this argument holds is either not modified (in the case of the even time-reversal symmetry $\mathcal{T}+$) or slighlty modified (in the case of odd time-reversal symmetry $\mathcal{T}$-), as already underlined in [EM].

Chapter 1

The background of the non-equivariant problem

The properties of materials on which the classification of quantum mechanical systems is based are encoded by a complex vector bundle over the d-torus $\mathbb{T}^d$. A wide range of results can be applied in order to simplify this study when $\mathbb{T}^d$ is described as a CW-complex. This chapter contains an introduction of the main tools, namely, CW-complexes, bundles and vector bundles, which will be necesary to develop the topic of this thesis. More sophisticated tools will be instead presented in the following chapters.

References for Section 1.1 can be found in [HAT, MA1, HUS], whereas Sections 1.2 and 1.3 refer mainly to [HUS].

1.1 CW-complexes

Let $\mathbb{D}^n$ be the unit disc $\{\, x \in \mathbb{R}^n \,|\, |x| \leqslant 1 \,\}$ with boundary $\mathbb{S}^{n-1}$.

Definition 1.1. A **CW-complex** X is a topological Hausdorff space X which is a union of an increasing sequence of subspaces X^n constructed inductively in the following way:

1. Start with a discrete set X^0 whose elements are called 0-cells of X.

2. Let $\mathbb{S}^n_\alpha$, $\mathbb{D}^n_\alpha$ be copies of $\mathbb{S}^n$ and $\mathbb{D}^n$. Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-disks via attaching maps $\varphi_\alpha : \mathbb{S}^{n-1}_\alpha \to X^{n-1}$. Thus X^n is quotient space $X^{n-1} \cup_{\varphi_\alpha} \mathbb{D}^n_\alpha = (X^{n-1} \coprod_\alpha \mathbb{D}^n_\alpha)/\{\varphi_\alpha(x) \sim x \text{ for } x \in \partial\mathbb{D}^n_\alpha\}$. This construction is also described by the following pushout diagram (see Definition 1.4):

$$\begin{tikzcd} \coprod_\alpha \mathbb{S}^{n-1}_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, hook, "i"'] & X^{n-1} \arrow[d] \\ \coprod_\alpha \mathbb{D}^n_\alpha \arrow[r] & X^{n-1} \cup_{\varphi_\alpha} \mathbb{D}^n_\alpha = X^n \end{tikzcd}$$

3. Finally, set $X = \bigcup_n X^n$ with the **weak topology**, that is, a set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

The quotient of $\mathbb{D}^n_\alpha$ in X under the quotient map is denoted by σ^n_α and it is called an n-cell. So, each cell σ^n_α is associated to its characteristic map Φ_α, that is defined as the (continuous) composition $\mathbb{D}^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha \mathbb{D}^n_\alpha \to X^n \hookrightarrow X$. The subspace X^n is called the n-skeleton of X. A CW-complex is said to be finite-dimensional provided $X = X^n$ for some $n \in \mathbb{N}$ and the smallest such n is the dimension of X. One can then also describe the topology on X by saying that a set $A \subset X$ is open (or closed) iff $\Phi^{-1}_\alpha(A)$ is open (or closed) in $\mathbb{D}^n_\alpha$ for each characteristic map Φ_α.

The definition of the characteristic map facilitates the description of two important properties of CW-complexes, namely:

- The restiction $\Phi_\alpha(\mathrm{int}(\mathbb{D}^n_\alpha))$ of Φ_α to the interior of $\mathbb{D}^n_\alpha$ is a homeomorphism onto its image, that is, a cell $\sigma^n_\alpha \subset X$; the interiors of these cells are all disjoint and their union is X.

- For each cell σ^n_α, the image $\Phi_\alpha(\partial\mathbb{D}^n_\alpha)$ of the boundary of $\mathbb{D}^n_\alpha$ is contained in the union of a finite number of cells of dimension less than n.

Definition 1.2. More generally, let X be a CW-complex and let $A \subseteq X$ be a closed subspace such that $X - A$ is a disjoint union of open cells with attaching maps. Then (X, A) is called **relative CW-complex**. The relative

n-skeleton is indicated by $(X, A)^n$ and one says that (X, A) has dimension n if $X = X^n$ for $n \in \mathbb{N}$.

Definition 1.3. A **subcomplex** of a CW-complex X is a subspace $A \subset X$ which is the union of cells of X and such that the closure of each cell in A is contained in A. This also means that A is itself a CW-complex. Its topology is the one induced by X. The pair (X, A) can also be viewed as a relative CW-complex.

Remark 1. The mysterious letters *"CW"* refer to the following two properties satisfied by CW-complexes:

1. *Closure-finiteness*: the closure of each cell meets only finitely many other cells.

2. *Weak topology*: a set is closed if and only if it meets the closure of each cell in a closed set.

Given two CW-complexes X and Y, the product $X \times Y$ has a natural structure of CW-complex with cells $\sigma^n_\alpha \times \sigma^m_\beta$, where σ^n_α ranges over the cells of X and σ^m_β ranges over the cells of Y. This construction is well-defined since $\mathbb{D}^n \times \mathbb{D}^m \cong \mathbb{D}^{n+m}$.

Remark 2. The CW-complex natural topology on $X \times Y$ is in general finer than the product topology. However, this difference is small or even non-existent in most cases of interest. For example, [HAT, Theorem A.6] states that the two topologies coincide if either X or Y is compact or if both X and Y have countably many cells.

Example 1.1. Sphere $\mathbb{S}^n$.

- The sphere $\mathbb{S}^n$ has the structure of a CW-complex with just two cells, σ^0 and σ^n, where σ^0 is a point x_0 and the n-cell is attached by the constant map $\varphi : \mathbb{S}^{n-1} \to \sigma^0, x \mapsto x_0$.

- Alternatively, one can build $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^2, \ldots$ inductively. The feature of this construction is that each subsphere $\mathbb{S}^k$ for $k < n$ is now a subcomplex of $\mathbb{S}^n$, by regarding each $\mathbb{S}^k$ as being obtained inductively from the equatorial $\mathbb{S}^{k-1}$ by attaching two k-cells, the components of $\mathbb{S}^k - \mathbb{S}^{k-1}$.

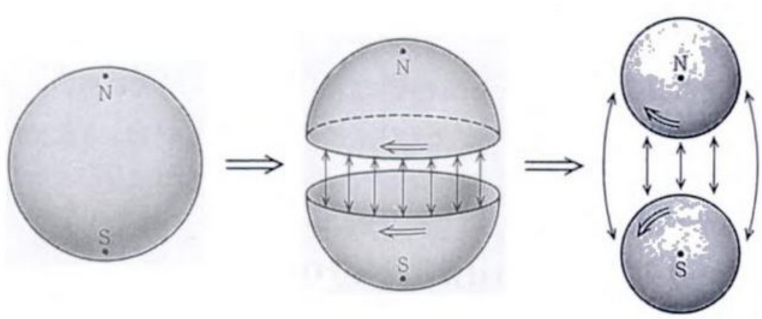

Figure 1.1: Inductive construction of the spheres $..., \mathbb{S}^1, \mathbb{S}^2, \mathbb{S}^3, ...$ (Image from [USM, p.108]).

Example 1.2. The **2-torus** $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ has a CW structure with one 0-cell σ^0, two 1-cells σ_a^1 and σ_b^1 and one 2-cell σ^2, as illustrated in the following picture:

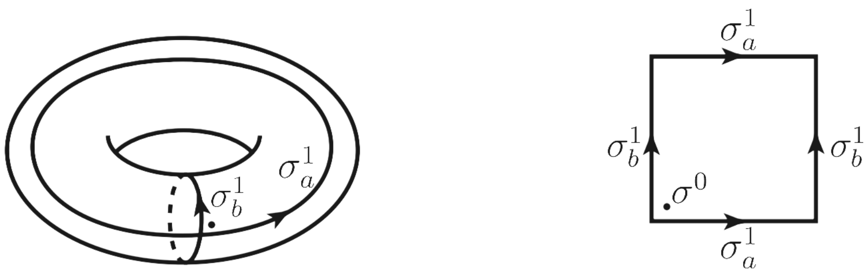

Figure 1.2: CW-complex decomposition of the 2-torus [HAT, p.5]

The **d-torus** $\mathbb{T}^d$ is defined as the product of d copies of the unit circle: $\mathbb{T}^d = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{d}$ and it has, consequently, a well defined CW-complex decomposition. In particular, $\mathbb{S}^1$ being compact, the product topology on $\mathbb{T}^d$ coincides with the natural topology defined by the CW-complex decomposition by Remark 2.

Before proving two basic results about CW-complexes, a reminder about the topological concept of *n-connectedness* is needed:

Definition 1.4. A topological space X is said to be n-**connected** for $n \in \mathbb{N}$ if $\pi_i(X) = 0$ for every $i \leqslant n$, where $\pi_i(X)$ denotes the i-th homotopy group and 0 the trivial group. Thus 0-connected means path-connected and 1-connected means simply-connected. Since n-connected spaces are 0-connected, the choice of a basepoint x_0 is not significant when X is a space that is n-connected for $n \geqslant 0$.

Definition 1.5. In category theory, given two morphism $f : Z \to X$ and $g : Z \to Y$, their **pushout** consists of an object P and two morphisms $i_1 : X \to P$ and $i_2 : Y \to P$ which make the diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \\ Y \arrow[r, "i_2"] & P \end{tikzcd}$$

commute and such that the set (P, i_1, i_2) is **universal** in the sense that, for any other pushout (Q, j_1, j_2) of f and g, there exists a unique $u : P \to Q$ also making the diagram commute:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \arrow[ddr, bend left, "j_1"] & \\ Y \arrow[r, "i_2"] \arrow[drr, bend right, "j_2"'] & P \arrow[dr, dashed, "!u"] & \\ & & Q \end{tikzcd}$$

Theorem 1.1. *Let X be a CW-complex and let $f : X^n \to Y$ be a map defined on the n-skeleton of X. Then f extends to a map $g : X^{n+1} \to Y$ if and only if $f \circ \varphi_\alpha$ is nullhomotopic for all $(n+1)$-cells σ_α^{n+1} with attaching map $\varphi_\alpha : \mathbb{S}^n \to X^n$.*

Proof. "$\Rightarrow$" Assume there exists a map $g : X^{n+1} \to Y$ such that the restriction $g|_{X^n}$ is f. Then we have the following commutative diagram:

$$\begin{tikzcd}
\coprod_\alpha \mathbb{S}^n_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, hook, "i"'] & X^n \arrow[d, "v"] \arrow[ddr, bend left, "f"] & \\
\coprod_\alpha \mathbb{D}^{n+1}_\alpha \arrow[r, "u"] \arrow[drr, dashed, bend right] & X^{n+1} \arrow[dr, "g"] & \\
& & Y
\end{tikzcd}$$

That is, $f \circ \varphi_\alpha = g \circ u \circ i$. In particular, $f \circ \varphi_\alpha$ is nullhomotopic for each α since it factors through the contractible space $\mathbb{D}^{n+1}$.

"$\Leftarrow$" Assume now that $f \circ \varphi_\alpha : \mathbb{S}^n_\alpha \to Y$ is nullhomotopic for all φ_α. Then it can be extended to a map $\overline{f}_\alpha : \mathbb{D}^{n+1}_\alpha \to Y$, since if $H_\alpha : \mathbb{S}^n_\alpha \times [0,1] \to X$ is a homotopy between $f \circ \varphi_\alpha$ and the constant map in $z_0 \in \mathbb{S}^n$, then for example the map $\overline{f}_\alpha(z) = \begin{cases} H_\alpha(\frac{z}{|z|}, 1 - |z|), \text{ if } z \neq 0 \\ z_0, \text{ if } z = 0 \end{cases}$ is a continuous extension of $f \circ \varphi_\alpha$. Therefore, it follows from the following diagram that $f \circ \varphi_\alpha = \overline{f}_\alpha \circ i$:

$$\begin{tikzcd}
\mathbb{S}^n_\alpha \arrow[r, "f \circ \varphi_\alpha"] \arrow[d, hook, "i"'] & Y \\
\mathbb{D}^{n+1}_\alpha \arrow[ur, "\overline{f}_\alpha"'] &
\end{tikzcd}$$

Writing down the pushout diagram

$$\begin{tikzcd}
\coprod_\alpha \mathbb{S}^n_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, hook, "i"'] & X^n \arrow[d, "v"] \arrow[ddr, bend left, "f"] & \\
\coprod_\alpha \mathbb{D}^{n+1}_\alpha \arrow[r, "u"] \arrow[drr, bend right, "\coprod_\alpha \overline{f}_\alpha"] & X^{n+1} \arrow[dr, dashed, "g"] & \\
& & Y
\end{tikzcd}$$

we can conclude by the universal property of the pushout that there exists a unique $g : X^{n+1} \to Y$ such that the diagram commutes. In particular, g is an extension of f. $\square$

This theorem applies as well to relative CW-complexes (X, A). As a consequence, it is not difficult to deduce the following:

Theorem 1.2. [HUS, Chaper 1, Theorem 1.2] *Let (X, A) be a relative CW-complex and let Y be a space that is n-connected in each dimension for which X has cells. Then each map $A \to Y$ extends to a map $X \to Y$.*

1.2 Bundles

Definition 1.6. A **bundle** ξ is a triple (E, p, X), where E and X are topological spaces and $p : E \to X$ is a continuous map. The space E is called the *total space*, X the *base space* and p the *projection* of the bundle. For each $x \in X$, the space $p^{-1}(x)$ is called the *fiber* of the bundle to $x \in X$. In case every fiber is homeomorphic to a space F, then F is said to be *the* fiber of the bundle ξ. In the following, the notation (E_ξ, p, X_ξ) to indicate the bundle $\xi = (E, p, X)$ will also be used.

Definition 1.7. A **global section**, or simply **section**, of a bundle is a continuous map $s : X \to E$ such that $p \circ s = \mathrm{id}_X$. This is equivalent to the requirement: $s(x) \in p^{-1}(x)$ for every $x \in X$. A **local section** is defined similarly but with domain restricted to an open subset U of X.

Example 1.3. The **product bundle** over B with fiber F is $(X \times F, \mathrm{pr}_1, X)$, where pr_1 is the projection over the first factor.

Proposition 1.3. *Every section s of a product bundle θ has the form $s(x) = (x, f(x))$ for a map $f : X \to F$ uniquely defined by s.*

Proof. Let $s : X \to X \times F$ be a map. Then $s(x) = (\tilde{s}(x), f(x))$ for maps $\tilde{s}, f$ uniquely defined by s. Since for a product bundle $(p \circ s)(x) = \tilde{s}(x)$, then s is a section of θ if and only if $\tilde{s}(x) = x$ if and only if $s(x) = (x, f(x))$ for each $x \in X$. $\square$

Given two bundles $\xi_1 = (E_1, p_1, X_1)$ and $\xi_2 = (E_2, p_2, X_2)$, it is possible to define a "fiber preserving" map, as explained in the following definition:

Definition 1.8. A **bundle morphism** $(u, f) : \xi_1 \to \xi_2$ is a pair of continuous maps $u : E_1 \to E_2$ and $f : X_1 \to X_2$ such that $p_1 \circ u = f \circ p_2$. A X-morphism between two bundles over the same base space X is then a continuous map $u : E_1 \to E_2$ such that $p_1 = p_2 \circ u$.

$$\begin{tikzcd} E_1 \arrow[r, "u"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd} \qquad \begin{tikzcd} E_1 \arrow[rr, "u"] \arrow[dr, "p_1"] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Remark 3. One can define a category **Bun** whose objects are all bundles ξ and whose morphisms are all bundle morphisms. Consequently, we say that (u, f) is a **bundle isomorphism** provided there exists a morphism $(\tilde{u}, \tilde{f}) : \xi_2 \to \xi_1$ with $\tilde{f} \circ f = \mathrm{id}_{B_1}$, $f \circ \tilde{f} = \mathrm{id}_{B_2}$, $\tilde{u} \circ u = \mathrm{id}_{E_1}$ and $u \circ \tilde{u} = \mathrm{id}_{E_2}$.

Example 1.4. A bundle $\xi = (E, p, X)$ is said to be **trivial** with fiber F provided ξ is X-isomorphic to the product bundle $(X \times F, \mathrm{pr}_1, X)$.

Remark 4. Proposition 1.3 also holds for trivial bundles [HUS].

Similar definitions are used to descibe the local properties of bundles:

Definition 1.9. Two bundles ξ_1 and ξ_2 over X are **locally isomorphic** provided for each $x \in X$ there exists an open neighborhood U of x such that $\xi_1|_U$ and $\xi_2|_U$ are U-isomorphic. In particular, a bundle ξ over X is **locally trivial** with fibre F provided ξ is locally isomorphic with the product bundle $(X \times F, \mathrm{pr}_1, X)$.

A recurrent and often convenient way of building new bundles consists in pulling back along a map on the base space of a given bundle:

Definition 1.10. Let $\xi = (E, p, X)$ be a bundle with fibre F and let $f : X_1 \to X$ be a continuous map. The **induced bundle** or **pullback bundle** of ξ under f, denoted $f^*(\xi)$, has as base space the space X_1, as total space $E_1 = \{(x_1, y) \in X_1 \times E \mid f(x_1) = p(y)\}$ and as projection the map $(x_1, y) \overset{p_1}{\mapsto} b_1$. The **canonical morphism** of an induced bundle is defined as the pair $(f_\xi, f) : f^*(\xi) \to \xi$, where $f_\xi : E_1 \to E$ is the projection onto the second factor $f_\xi(x_1, y) = y$. The induced bundle has the same fibre F of the original bundle ξ.

Proposition 1.4. *Let $\xi = (E, p, X)$ be a bundle, let $f : X_1 \to X$ be a continuous map and let $(f_\xi, f) : f^*(\xi) \to \xi$ be the canonical morphism of the induced bundle.*

1. *If s is a section of ξ, then $\sigma : X_1 \to E_1$ defined by $\sigma(x_1) = (x_1, s(f(x_1))$ is a section of $f^*(\xi)$ with $f_\xi \circ \sigma = s \circ f$.*

2. *If f is a quotient map*[1] *and if σ is a section of $f^*(\xi)$ such that $f_\xi \circ \sigma$ is constant on all sets $f^{-1}(x)$ for $x \in X$, then there is a unique section s of ξ such that $s \circ f = f_\xi \circ \sigma$.*

Proof. 1. Using the definition of σ: $p_1(\sigma(x_1)) = p_1(x_1, s(f(x_1))) = x_1$. So $p_1 \circ \sigma = \mathrm{id}_{X_1}$, that is, σ is a section of $f^*(\xi)$.

$$\begin{tikzcd} E_1 \arrow[r, "f_\xi"] \arrow[d, shift left, "p_1"] & E \arrow[d, shift left, "p"] \\ X_1 \arrow[r, "f"] \arrow[u, dashed, shift left, "\sigma"] & X \arrow[u, shift left, "s"] \end{tikzcd}$$

Moreover, $f_\xi(\sigma(x_1)) = f_\xi(x_1, s(f(x_1))) = s(f(x_1))$.

2. Assume now that f is an identification and σ is a section as in the hypothesis. Then we have the following diagram:

$$\begin{tikzcd} E_1 \arrow[r, "f_\xi"] \arrow[d, shift left, "p_1"] & E \arrow[d, shift left, "p"] \\ X_1 \arrow[r, "f", "(su)"'] \arrow[u, shift left, "\sigma"] & X \arrow[u, dashed, shift left, "s"] \end{tikzcd}$$

Namely, one sets $s(f(x_1)) = (f_\xi \circ \sigma)(x_1)$ for every $x_1 \in X_1$. This is well-defined because f is surjective and $f_\xi \circ \sigma$ is constant on all sets $f^{-1}(x)$. Moreover, s is continuous: given an open set $A \subset E$, $(f_\xi \circ \sigma)^{-1}(A) = A'$

[1] $f : X \to Y$ between topological spaces (X, τ_X) and (Y, τ_Y) is a quotient map if it is surjective and if a subset U of Y is open if and only if $f^{-1}(U)$ is open in X.

is open in B_1 because $f_\xi \circ \sigma$ is continuous, being the composition of continuous functions; therefore, $s^{-1}(A) = f^{-1} \circ (f_\xi \circ \sigma)^{-1}(A) = f(A')$ is open in X since f is an identification map.

Hence, $s : X \to E$ is a well-defined continuous function. It is also a section of ξ since

$$(p \circ s \circ f)(x_1) = (p \circ f_\xi \circ \sigma)(x_1) = f((p_1 \circ \sigma)(x_1)) = f(x_1)$$

implies $p \circ s \circ f = f$, that is, $p \circ s = \mathrm{id}_X$. □

The following theorem describes a generalization of Theorem 1.1 to sections of locally trivial bundles (see Definition 1.9). This prolongation theorem is the fundamental step in the classification theory of vector bundles over CW-complexes, as it will be shown in Chapter 2 and 5.

Theorem 1.5. *Let $\xi = (E, p, X)$ be a locally trivial bundle with fibre F, where (X, A) is a relative CW-complex. Suppose that the space F is $(d-1)$-connected, where $d = \dim X$. Then all sections s of $\xi|_A$ prolong to a section s^* of ξ.*

Proof. In order to prove this theorem, we proceed by induction on the dimension d of X. If $d = 0$, then necessarily $A = X$ and there is nothing to show. Let then X be of dimension $d > 0$ and assume the statement true for all the spaces of lower dimension. This applies in particular to the $(d-1)$-skeleton of X. Thus there exists by inductive hyopothesis a section s' of $\xi|_{X^{d-1}}$ with $s'|_A = s$. Let C be a d-cell with characteristic map $\Phi_C : I^d \to X$, where the unit d-cube I^d replaces the d-disk $\mathbb{D}^d$ in view of the isomorphism $I^d \cong \mathbb{D}^d$, and consider the induced bundle $\Phi_C^*(\xi)$ with base space I^d. This inherits the local triviality from ξ (see [HUS, §2.6]). Moreover, since I^d is compact, it is possible to dissect it into equal d-cubes K of edge-lenght $1/k$ such that $\Phi_C^*(\xi)|_K$ is trivial.

By Proposition 1.4 (1), the section s' defines a section σ' of $\Phi_C^*(\xi)|_{\partial I^d}$ which can be assumed to be defined on the $(d-1)$-skeleton of I^d dissected into equal cubes K of edge-lenght $1/k$, that is, σ' is defined on K^{d-1} for each K. Since the induced bundle over each d-cube is trivial, σ' is given by a map $K^{d-1} \to F$ by Proposition 1.3. Applying the inductive hypothesis to the relative CW-complex (K, K^{d-1}), σ' can be extended to a section $K \to F$ because the fibre F is $(d-1)$-connected. Since this construction can be repeated fot all the d-cubes K, the prolongation over each of these leads to the definition of a section σ of $\varphi_C^*(\xi)$.

Using now Proposition 1.4 (2) together with the natural morphism $\varphi_C^*(\xi) \to \xi$, we get a section s_C of $\xi|_C$ such that $s_C|_{C\cap X^{d-1}} = s'|_{C\cap X^{d-1}}$. Finally, we define the desired section s^* of ξ that satisfies $s^*|_{X^{d-1}} = s'$ and $s^*|_C = s_C$. The continuity of s^* follows from the weak topology on X.
Suppose now that $\dim X = \infty$, which implies that F is connected for every $d < \infty$. In this case, given a section s of $\xi|_A$ and setting $s_{-1} = s$, we first construct inductively sections s_d of $\xi|_{X^d}$ such that $s_d|_{X^{d-1}} = s_{d-1}$. Then, we define a setion s^* of ξ by the requirement that $s^*|_{X^d} = s_d$. $\square$

1.3 Vector bundles

By adding particular requirements on the fibres of general bundles, it is possible to define different subcategories. A remarkable example is that of *vector bundles*, which are required to have an additional vector space structure on each fibre. More precisely:

Definition 1.11. Let $\mathbb{F}$ denote the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$. A **vector bundle** ξ over $\mathbb{F}$ is a bundle (E, p, X) together with the structure of a finite-dimensional vector space over $\mathbb{F}$ for each fibre $p^{-1}(x) =: E_x$ and such that the following local triviality condition is satisfied: for each point of X there exist a natural number k, an open neighborhood U and a U-isomorphism $\phi : U \times \mathbb{F}^k \to p^{-1}(U)$ called *local coordinate chart of* ξ such that:

- $(p \circ \phi)(x, v) = x$ for all $x \in U$ and for all $v \in \mathbb{F}$
- the restriction $\{x\} \times \mathbb{F}^k \xrightarrow{\phi} p^{-1}(x)$ is a vector space isomorphism for each $x \in U$, that is, the map $v \to \phi(x, v)$ is a linear isomorphism

between $\mathbb{F}^k$ and $p^{-1}(x)$.

Let k_x denote the dimension of the fibre $p^{-1}(x)$ for $x \in X$. Because of the local trivializations, k_x is constant on each connected component of X. If k_x is equal to a constant k on all X, then k is called the **rank** of the vector bundle and the writing ξ^k will be used to indicate that ξ has constant rank k. Vector bundles of rank 1 are called *line bundles*. Depending on $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, one talks about *real, complex* or *quaternionic vector bundles*, respectively.

Definition 1.12. Let $\xi = (E, p, X)$ be a vector bundle of rank k and let $U \subseteq X$ be an open subset. A **local frame of** E **over** U is an ordered k-tuple $(s_1, ..., s_k)$ of local sections over U such that $(s_1(x), ..., s_k(x))$ is a basis for the fibre E_x for each $x \in U$. It is called **global frame** if $U = X$.

The definitions of sections, morphisms, isomorphisms and induced bundles previously introduced for general bundles apply as well to vector bundles. But in this case they carry also additional properties, namely:

1. the projection p is an open map;
2. the sections of ξ form a module over the ring $C(X, \mathbb{F})$ of continuous $\mathbb{F}$-valued functions on the base space;
3. if (u, f) is a vector bundle morphism between $\xi_1 = (E_1, p_1, X_1)$ and $\xi_2 = (E_2, p_2, X_2)$, then the equivalence $p_2 \circ u = f \circ p_1$ holds and the restriction $u|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(f(x))$ must be linear for each $x \in X$. In particular, if $X_1 = X_2$, then $p_2 \circ u = p_1$ and $u|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ is linear for each $x \in X$;
4. given a vector bundle $\xi = (E, p, X)$ and a map $f : X_1 \to X$, the induced bundle $f^*(\xi) = (E_1, p_1, X_1)$ admits a unique vector bundle structure with $(f_\xi, f) : f^*(\xi) \to f$ being a morphism of vector bundles. Moreover, $f_\xi : p_1^{-1}(x_1) \to p^{-1}(x)$ is a vector space isomorphism for all $x_1 \in X_1$.

Proofs of these properties can be found, for example, in [HUS]. In particular, property (2) implies that the set of sections of ξ cannot be empty since it always contains the **zero section**, that is, the map $s : X \to E$ defined by $s(x) = 0 \in p^{-1}(x)$ for each $x \in X$, where 0 denotes the null-vector of $E_x = p^{-1}(x)$.

Example 1.5. The **product vector bundle** of rank k over $\mathbb{F}$ is $\theta = (X \times \mathbb{F}^k, \mathrm{pr}_1, X)$, where pr_1 is the projection on the first factor. The local trivialization is realized setting $U = X$ and $\phi =$id. A vector bundle ξ is said to be **trivial** if it is isomorphic to the product vector bundle θ.

Proposition 1.6. *A vector bundle is trivial if and only if it admits a global frame.*

Proof. See for example [LEE, Corollary 10.20]. □

Example 1.6. Let $\xi = (X \times \mathbb{F}^n, p, X)$ and $\eta = (X \times \mathbb{F}^m, p, X)$ be two trivial vector bundles. Then, a general vector bundle morphism $\xi \to \eta$ has the form $u(x, v) = (x, f(x, v))$ for a map $f : X \times \mathbb{F}^n \to \mathbb{F}^m$ linear in x (in analogy with Proposition 1.3). Denoting by $L(\mathbb{F}^n, \mathbb{F}^m)$ the space of linear transformations $\mathbb{F}^n \to \mathbb{F}^m$, one can also say that a map $f : X \times \mathbb{F}^n \to \mathbb{F}^m$ is continuous if and only if $x \mapsto f(x, \cdot)$ as a function $X \to L(\mathbb{F}^n, \mathbb{F}^m)$ is continuous.

Remark 5. The category of vector bundles is denoted by **VB**. Its objects are all vector bundles and its morphisms are all morphisms of vector bundles. The composition is thus the composition of morphisms of vector bundles. One can also define the subcategory $\mathbf{VB}_X$ of vector bundles over a common base space X. Given $k \geq 0$, $\mathbf{VB}^k$ denotes the full subcategory of vector bundles of rank k, while $\mathbf{VB}_X^k$=$\mathbf{VB}_X$∩$\mathbf{VB}^k$.

Operations on vector spaces can usually be extended to vector bundles, defining the same operations *fiberwise*. One particular example that will be very useful in the following discussion is the *Whitney sum*:

Definition 1.13. The **Whitney sum**, or **direct sum**, of two vector bundles $\xi_1 = (E_1, p_1, X)$ with fibre $\mathbb{F}^n$ and $\xi_2 = (E_2, p_2, X)$ with fibre $\mathbb{F}^m$ over a space X is denoted by $\xi_1 \oplus \xi_2 = (E_1 \times E_2, q, X)$ and it is defined fiberwise by

- $q^{-1}(x) = p_1^{-1}(x) \times p_2^{-1}(x)$
- $\phi_1 \oplus \phi_2 : U \times \mathbb{F}^{n+m} \to q^{-1}(U)$ is a local chart of $\xi_1 \oplus \xi_2$ if ϕ_1 and ϕ_2 are local charts of the underlying bundles.

Consider now an X-morphism of vector bundles $u : \xi \to \eta$, where $\xi = (E_\xi, p_\xi, X)$ and $\eta = (E_\eta, p_\eta, X)$ with the notation intoduced in Definition 1.6. One can define three bundles:

- **ker** u, that is, the subbundle of ξ with total space $\{y \in E_\xi \,|\, u(y) = 0 \text{ in } \eta \text{ over } p_\xi(y)\}$,
- **im** u, that is, the subbundle of η with total space $\{u(y) \in E_\eta \,|\, y \in E_\xi\}$,
- **coker** u, that is, the quotient bundle of η with total space $E_{\eta/\sim}$, where $\sim$ is the following equivalence relation: let $y_1, y_2 \in E_\eta$, then $y_1 \sim y_2$ iff $p_\eta(y_2) = p_\eta(y_1)$ and $y_1 - y_2 \in u(y)$ for some $y \in E_\xi$.

The question whether these bundles are also vector bundles is not always obvious, since it may happen that the property of local triviality is not satisfied. However, there is a case in which we can directly conclude with a poitive answer, namely, when the morphism u is of *constant rank*.

Definition 1.14. Let $u : \xi \to \eta$ be a vector bundle X-morphism. Then u is of **constant rank** k provided the linear map $u_x : p_\xi^{-1}(x) \to p_\eta^{-1}(x)$ is of rank[2] k for each $x \in X$.

Theorem 1.7. *Let $u : \xi^n \to \eta^m$ be an X-morphism of vector bundles of constant rank k, where $n = \mathrm{rank}(\xi)$ and $m = \mathrm{rank}(\eta)$. Then* $\ker u$, $\mathrm{im}\, u$ *and* $\mathrm{coker}\, u$ *are vector bundles over X.*

Proof. See [HUS, Chapter 3, Theorem 8.2]. □

Vector bundles are often given additional structures. For instance, they can be equipped with a *metric*. This is always possible when the bundle is defined over a paracompact space (e.g., on a CW-complex).

[2]The rank of a linear transformation $f : V \to W$ between finite dimensional vector spaces V and W is defined as $k = \dim(V) - \dim(\ker f) = \dim(\mathrm{im}\, f)$.

Definition 1.15. Let ξ be a vector bundle over X with fibre F. A **metric** β on ξ is a function $\beta : E_{\xi\oplus\xi} \to F$ such that, for each $x \in X$, $\beta|_{p^{-1}(x)\times p^{-1}(x)}$ is an inner product on $p^{-1}(x)$.

Example 1.7. Let θ^k be the product bundle of rank k over X. Then $\beta(x, v, v') = \langle v|v'\rangle$ is a metric with $\langle\cdot|\cdot\rangle$ the euclidian inner product.

Theorem 1.8. *Let* $0 \to \xi \overset{u}{\to} \eta \overset{v}{\to} \zeta \to 0$ *be a short exact sequence of vector bundles over* X*, that is:* u *is a monomorphism,* $\operatorname{im} u = \ker v$ *and* v *is an epimorphism. Let* β *be a metric on* η*. Then there exists an isomorphism* $w : \xi\oplus\zeta \to \eta$ *splitting the above exact sequence in the sense that the following diagram is commutative:*

$$\begin{tikzcd}
 & & \eta \arrow[dr, "v"] & & \\
0 \arrow[r] & \xi \arrow[ur, "u"] \arrow[dr, "i"'] & & \zeta \arrow[r] & 0 \\
 & & \xi \oplus \zeta \arrow[uu, dashed, "w", "\cong"'] \arrow[ur, "j"'] & &
\end{tikzcd}$$

Here, i *is the inclusion into the first factor and* j *is the projection onto the second factor.*

Proof. In order to prove this theorem, first of all we define two subsets $E_{\xi'}$ and $E_{\zeta'}$ of E_μ by:

- $E_{\xi'} := \operatorname{im} u \subset E_\mu$;
- $E_{\zeta'} = \{y' \in E_\mu : \beta(y, y') = 0 \text{ for all } y \in E_{\xi'} : p_\mu(y) = p_\mu(y')\} \subset E_\mu$; thus, ζ' is a subbundle of μ consisting of vector spaces.

For each $x \in X$, let $g : E_\mu \to E_{\xi'}$ be the projection of the fibre $E_{\mu,x}$ of μ onto the fibre $E_{\xi',x}$ of ξ'. This map is continuous, in fact, locally: let $U \subset X$ be an open neighborhood of $x \in X$ and let $e_1, \dots, e_n$ be an orthonormal basis of sections of $U \times F^n$, where F^n is the fibre of ξ'. Suppose $u : U \times F^m \to U \times F^n$ is a X-monomorphism (F^m denotes the fibre of ξ) and $\beta(x, y, y')$ is a metric on $U \times F^n$. Then $g : U \times F^m \to U \times F^n$ is given by $g(x, y) =$

$(x, \sum_{1\leqslant n\leqslant m} \beta(x, y, u(e_i))e_i)$, which is continuous. Since $g : \mu \to \xi'$ is a X-epimorphism, $\ker g$ is a vector bundle by Theorem 1.7. In particular, $\zeta' = \ker g$ is a vector bundle. Moreover, $v|_{\zeta'} : \zeta' \to \zeta$ is an X-isomorphism since it is an isomorphism on the fibres, and thus invertible.

Finally, we define an isomorphism $w : \xi \oplus \zeta \to \mu$ setting:

- $w|_\xi$ equal to the isomorphism $u : \xi \to \xi' \subset \mu$;
- $w|_\zeta$ equal to the isomorphism $(v|_{\zeta'})^{-1} : \zeta \to \zeta' \subset \mu$. □

Chapter 2

Stability properties of vector bundles

In order to classify vector bundles, one needs to find an appropriate equivalence relation. As usual, this is given by isomorphism, that is, two vector bundles are said to be in the same equivalence class if there exists an isomorphism between them. However, there exist cases under which a weaker condition, namely, *stable isomorphism*, is sufficient, in the sense that in such cases being stably isomorphic implies being isomorphic, too. In the first section of this chapter we will point out some homotopy properties of vetor bundles in order to use them to prove the main result about stable isomorphism in the second section. Both sections refer to [HUS].

2.1 Homotopy properties of vector bundles

By Definition 1.8, an X-isomorphism of vector bundles over a space X is a morphism $u : \xi_1 \to \xi_2$ such that there exists a morphism $v : \xi_2 \to \xi_1$ with $v \circ u = \mathrm{id}_{\xi_1}$ and $u \circ v = \mathrm{id}_{\xi_2}$. The following theorem provides a criterion to determine whether a X-morphism is an X-isomorphism.

Theorem 2.1. *Let $u : \xi_1 \to \xi_2$ be a X-morphism of vector bundles. Then u is an X-isomorphism if and only if $u|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ is a vector* *space isomorphism for each $x \in X$.*

Proof. $\Rightarrow$ This is trivial, since if u is an isomorphism, that is, it admits an inverse v, then the inverse of $u|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ is just the restriction of v to the fiber $p_2^{-1}(x)$, and this is an isomorphism of vector spaces by the definition of morphisms of vector bundles (see p.16).

$\Leftarrow$ Let $v : \xi_2 \to \xi_1$ be the map such that $v|_{p_2^{-1}(x)}$ is the inverse of the restricted linear transformation $u|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$. In order to show that v is the inverse of u, it is sufficient to show that v is continuous.

Let U be an arbitrary open subset of X and let ϕ_1 and ϕ_2 be local coordinate charts of ξ_1 and ξ_2, respectively. It suffices to prove that $v|_U$ is continuous. First, one can note that the composition $\phi_2^{-1} \circ u \circ \phi_1 : U \times \mathbb{F}^n \to U \times \mathbb{F}^m$ has the form $(x, v) \mapsto (x, f_x(x, v))$ by Example 1.6, where $x \mapsto f_x = f(x, \cdot)$ is a map $U \to L(\mathbb{F}^n, \mathbb{F}^m)$. Therefore, taking the inverse: $\phi_1^{-1} \circ u^{-1} \circ \phi_2$ has the form $(x, v) \mapsto (x, f_x^{-1}(x, v))$, where again $x \mapsto f_x^{-1} = f^{-1}(x, \cdot)$ is a map $U \to L(\mathbb{F}^m, \mathbb{F}^n)$. Since these two maps are continuous being the composition of continuous functions, the restriction $v : p_2^{-1}(U) \to p_1^{-1}(U)$ is also continuous by Example 1.6. $\square$

The next few theorems describe some homotopy properties of vector bundles that will be necessary to understand and prove many results about the stability of vector bundles that we will discuss in the next section.

Lemma 2.2. *Let ξ be a vector bundle over $X \times I$. Then there exists an open covering $\{U_j\}_{j \in J}$ of X such that $\xi|_{U_j \times I}$ is trivial.*

Proof. For every $(x, t) \in X \times I$ there exist open neighborhoods U_x of x in X and U_t of t in I such that $\xi|_{U_x \times U_t}$ is trivial. Since $I = [0, 1]$ is compact, we can choose a finite a sequence $0 = t_0 < t_1 < ... < t_n = 1$ and open neighborhoods U_i of x in X such that $\xi|_{U_i \times [t_{i-1}, t_i]}$ is trivial for $1 \leqslant i \leqslant n$.

Define $U_J = \bigcap\limits_{1 \leqslant j \leqslant n} U_i$. Now one can show that, given a vector bundle ξ over $X = X_1 \cup X_2$, where $X_1 = A \times [a, b]$ and $X_2 = A \times [b, c]$, $a \leqslant b \leqslant c$, if $\xi|_{X_1}$ and

$\xi|_{X_2}$ are trivial, then ξ is trivial [HUS, Chapter 3, Lemma 4.1]. Therefore, applying this recursively, we can conclude that $\xi|_{U_j \times I}$ is trivial. Repeting this procedure for each $x \in X$, we get an open covering $\{U_j\}_{j\in J}$ of X, such that $\xi|_{U_j \times I}$ is trivial for all $j \in J$. □

Theorem 2.3. *Let $r : X \times I \to X \times I$ be defined by $r(x,t) = (x,1)$ and let $\xi = (E, p, X \times I)$ be a vector bundle over $X \times I$, with X paracompact. Then there is a map $u : E \to E$ such that $(u, r) : \xi \to \xi$ is a morphism of vector bundles and u is an isomorphism on each fibre.*

Proof. See [HUS, Chapter 3, Theorem 4.3]. □

Corollary 2.4. *With the notation in* Theorem 2.3 *there exists, after restriction, an isomorphism $(u, r) : \xi|_{X\times\{0\}} \to \xi|_{X\times\{1\}}$.*

Proof. Set $\xi_1 = \xi|_{X\times\{0\}}$, $\xi_2 = \xi|_{X\times\{1\}}$ and notice that $r \equiv \text{id}$ on $X \times \{0\} \cong X \times \{1\} \cong X$. The previous theorem guarantees the existence of a map $u : E \to E$ such that $(u, r) : \xi \to \xi$ is a morphism of vector bundles and u is an isomorphism on each fibre. Moreover, $(u, r) : \xi|_{X\times\{0\}} \to \xi|_{X\times\{1\}}$ is an X-morphism and so, by Theorem 2.1 $(u, r) : \xi_1 \to \xi_2$ is an isomorphism. □

For the sake of completeness, we will state two other important applications of Theorem 2.3 in homotopy thoery. Proofs can be found in [HUS, §3.4].

Theorem 2.5. *Let $f, g : X_1 \to X_2$ two homotopic maps, where X_1 is a paracompact space and let ξ be a vector bundle over X_2. Then the induced bundles $f^*(\xi)$ and $g^*(\xi)$ are isomorphic.*

Corollary 2.6. *Every vector bundle over a contractible space is trivial.*

2.2 Stability

Throughout this section, θ^k denotes the trivial bundle of rank k and X denotes an n-dimensional CW-complex. As before, $\mathbb{F}$ can be either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Let c be $\dim_{\mathbb{R}}\mathbb{F}$. Moreover, the symbol $\lceil x \rceil$ indicates the *ceiling part* of x, that is $\lceil x \rceil = \min\{l \in \mathbb{N} \,|\, l \geq x\}$.

Definition 2.1. Two vector bundles ξ and η over X are called **stably equivalent**, short **s-equivalent**, provided there exists a natural number n such that $\xi \oplus \theta^n \cong \eta \oplus \theta^n$. Stable classes of vector bundles over X form a ring over finite-dimensional spaces with the Whitney sum $\oplus$ inducing the addition operation and the tensor product $\otimes$ the multiplication operation.

The aim of this section is to prove the following theorem:

Theorem 2.7. *Let $m = \lceil \frac{n+2}{c} - 1 \rceil$. If ξ_1^k and ξ_2^k are two vector bundles of rank k over X such that $m \leq k$ and $\xi_1 \oplus \theta^l$ and $\xi_2 \oplus \theta^l$ are isomorphic for some $l \in \mathbb{N}$, then ξ_1 and ξ_2 are isomorphic.*

The proof of this theorem will be split into four steps. First of all, we will mention a proposition whose content is of particular interest also independently of the previous theorem, since it shows how it is possible to decompose high-dimensional vector bundles in the Withney sum of a trivial bundle and another bundle of lower dimension.

Proposition 2.8. *If ξ is a vector bundle of rank k and $d \leq ck - 1$, then ξ is isomorphic to $\eta \oplus \theta^1$ for some vector bundle η of rank $k - 1$.*

Proof. Let ξ_0 denote the subbundle of nonzero vectors, i.e., $\xi_0 = (E_0, p, X_0)$, $X_0 = \{x \in X \,|\, p^{-1}(x) \neq 0\}$. Since 0 is not in E any more, the fibre of E_0 is $\mathbb{F}^k \setminus \{0\}$, and this is homotopy equivalent to the sphere $\mathbb{S}^{ck-1}$. From the theory of homotopy groups, we know that $\pi_i(\mathbb{S}^n) = 0$ for each $i \leqslant d - 1$, that is, $\mathbb{S}^{ck-1}$ is $(ck - 2)$-connected. This means in particular that $\mathbb{F}^k \setminus \{0\}$ satisfies the condition in the hypothesis of Theorem 1.5: let $m \leqslant d$, then $m - 1 \leqslant d - 1 \leqslant ck - 2$, so $\mathbb{F}^k \setminus \{0\}$ is $(m-1)$-connected. Therefore, we have a section s^* of ξ_0. Moreover, this can be seen as a nowhere vanishing section of ξ.
I define now the X-morphism $u : \theta^1 \to \xi$ by $u(b, a) = a\, s(b)$ for every $(b, a) \in E_{\theta^1} = X \times \mathbb{F}$. Since every section is a monomorphism, i.e., an injective

X-morphism in the category $\mathbf{VB}_X$, u is a well-defined monomorphism too. Set $\eta = \operatorname{coker} u$. This is a vector bundle by Theorem 1.7. Finally, X being a CW-complex and so a paracompact space, it is possible to define a metric on ξ and to apply Theorem 1.8. Hence, there exists an isomorphism between ξ and $\theta^1 \oplus \eta$. □

Proceeding by induction, the previous proposition generalizes to the following corollary:

Corollary 2.9. *Let $m = \lceil \frac{d+1}{c} - 1 \rceil$. Then each vector bundle ξ^k of rank k is isomorphic to $\eta^m \oplus \theta^{k-m}$ for some vector bundle η of rank m.*

Proof. By induction on $k \geq m$:

- if $k = m$, then $\xi^k = \eta^k \oplus \theta^0$ is trivially true;
- assume the statement for all $k = m + j$, $j \in \mathbb{N}$ and set $k = m + j + 1$. Now we have the following inequalities:
$$k = m + j + 1 \geqslant \tfrac{d+1}{c} - 1 + j + 1 = \tfrac{d+1}{c} + j$$
$$\Rightarrow \; d \leqslant c(k - j) - 1 \leqslant ck - 1$$
and, by Proposition 2.8, ξ is isomorphic to $\eta_1 \oplus \theta^1$ for a vector bundle η_1 of rank $k - 1$. Now we can apply the inductive hypothesis many times to η and get
$$\xi^k \cong \eta_1^{k-1} \oplus \theta^1 \cong (\eta^m \oplus \theta^{k-1-m}) \oplus \theta^1 \cong \eta^m \oplus \theta^{k-m}.$$

□

Remark 6. Taking into account the possible values of c, the constant m is also consequently determined, specifically:

- $m = d$ in the case of real bundles, i.e., $\xi^k \cong \eta^n \oplus \theta^{k-n}$;
- $m = \lceil \frac{d-1}{2} \rceil$ for complex bundles, in particular, $m = 0$ for vector bundles over base spaces X of dimension 1 and $m = 1$ for those over 2- and 3-dimensional spaces X;
- $m = \lceil \frac{d-3}{4} \rceil$ for quaternionic bundles; in particular, $m = 0$ every time the base space X has dimension less than or equal to 3.

Remark 7. In reference to the dimension d of the base space, Corollary 2.9 says that a vector bundle over a point is trivial since $d = 0 \Rightarrow m = 0$ in all cases. In particular it has a basis.

Lemma 2.10. *If $u, v : \theta^1 \to \xi^k$ are two X-monomorphisms of vector bundles with $d \leqslant ck - 2$ then* $\operatorname{coker} u$ *and* $\operatorname{coker} v$ *are isomorphic over X.*

Proof. As in the proof of Proposition 2.8 we can assume to have a nowhere vanishing section of ξ_0 that determines the monomorphism $u : \theta^1 \to \xi$. Similarly, a homotopy of monomorphisms u and v is then determined by a section s of $\xi_0 \times I = (\xi \times I)_0$ over $X \times \{0, 1\}$, where $s|_{X\times 0} = u$ and $s|_{X\times 1} = v$.
The estimate on the dimension of X shows that $\dim(X \times I) = d+1 \leqslant ck-1$.
In order to apply Theorem 1.5, one should check that the fibre $F^k \setminus \{0\}$ is $(m-1)$-connected for all $m \leqslant n$. In the situation of the theorem,

$$m \leqslant d < d+1 < ck-1$$

from the previous estimate. Therefore, the section s of $(\xi \times I)_0$ prolongs to a section s^* of $\xi \times I$ over $X \times I$ with $s^*|_{(\xi\times I)_0} = s$.
The section s^* determines a monomorphism $w : \theta^1 \to \xi \times I$ given by

$$w(x,t,y) = y\, s^*(x,t) \quad \text{for} \quad (x,t,y) \in X \times I \times F$$

such that:

$$w|_{X\times\{0\}}(x,0,y) = y\, s^*|_{X\times\{0\}}(x,0) = y\, u(x) \implies \operatorname{coker} w|_{X\times\{0\}} \cong \operatorname{coker} u$$
$$w|_{X\times\{1\}}(x,1,y) = y\, s^*|_{X\times\{1\}}(x,1) = y\, v(x) \implies \operatorname{coker} w|_{X\times\{1\}} \cong \operatorname{coker} v$$

Applying Corollary 2.4 to $\operatorname{coker} w$, which is a vector bundle because of Theorem 1.7, there exists an isomorphism between $\operatorname{coker} w|_{X\times\{0\}}$ and $\operatorname{coker} w|_{X\times\{1\}}$, that is $\operatorname{coker} u \cong \operatorname{coker} v$. □

Finally, we can complete the proof of Theorem 2.7:

Proof of Theorem 2.7. Proceeding by induction on $l \geqslant 1$:

- $l = 1$. Since $k \geqslant m \geqslant \frac{d+2}{c} - 1$ implies $d \leqslant c(k-1) - 2$, it is easy to verify that the inequality $d \leqslant ck - 2$ hold. Therefore, we can apply Proposition 2.8 and Lemma 2.10 and obtain:

$$\xi_1 \overset{\text{Prop.2.8}}{\cong} \operatorname{coker} u \oplus \theta^1 \overset{\text{Lem.2.10}}{\cong} \operatorname{coker} v \oplus \theta^1 \overset{\text{Prop.2.8}}{\cong} \xi_2$$

where $u, v : \theta^1 \to \xi_i^k, i \in \{1, 2\}$, are monomorphisms.

- Suppose the theorem is true for l. Then, for $l + 1$:
$$(\xi_1^k \oplus \theta^1) \oplus \theta^l \overset{\text{hyp}}{\cong} (\xi_2^k \oplus \theta^1) \oplus \theta^l$$
The terms inside the brackets are vector bundles of rank $k + 1$ and Proposition 2.8 shows that they can be seen as the cokernels of monomorphisms $\theta^1 \to \xi^{k+1}$. Therefore, they are isomorphic if $d \leqslant c(k + 1) - 2$ by Lemma 2.10, but this is true by hypothesis. Hence $\xi_1^k \oplus \theta^1 \cong \xi_2^k \oplus \theta^1$ and by the induction hypothesis $\xi_1^k \cong \xi_2^k$. □

Remark 8. Theorem 2.7 can be seen as a proper generalization of the uniqueness theorem for vector spaces, which says that two vector spaces whose basis have the same number of elements are isomorphic. Given two vector bundles, we can then say that they are isomorphic provided their direct sums with a trivial bundle of dimension $l \in \mathbb{N}$ are isomorphic.

Chapter 3

Quantum mechanical systems

In this chapter we are going to introduce the mathematical model for quantum mechanical systems and we will explain how this helps to study the topological properties of materials, referring to [ME1, ME2, DG1, AKH] in the first section and to [BEL, EM, KEL, ME2] in the second section.

3.1 The single-particle model

Consider a single particle, e.g., an electron, moving in a crystal extended infinitely in all directions in a d-dimensional space in the field generated by all other electrons and nuclei. Typically, this is modelled as a vector ψ in the Hilbert space $L^2(\mathbb{R}^d, \mathrm{d}x) \otimes \mathbb{C}^2$, that is, the tensor product of the Hilbert space of square Lebesgue-integrable functions[1] on $\mathbb{R}^d$ (where d is usually 1, 2, or 3) and $\mathbb{C}^2$, which is introduced in order to take into account the spin (up or down) of the particle. This space can also be described

[1]Given a measure space (X, Ω, μ), define $\mathcal{L}^2(X, \Omega, \mu) = \{f : X \to \mathbb{C}$ measurable $: (\int_X |f(x)|^2 \mathrm{d}\mu(x))^{1/2} < \infty\}$, where $|\cdot|$ means the complex modulus. The space $L^2(X, \omega, \mu)$ of square integrable functions is thus defined as the quotient of $\mathcal{L}^2(X, \Omega, \mu)$ by the equivalence relation that identifies functions equal almost everywhere, i.e. $f \sim g$ if and only if $\mu(\{x \in X : |f(x) - g(x)| \neq 0\}) = 0$. $L^2(X, \mu)$ is an Hilbert space for the inner product $\langle f|g\rangle = \int_X \overline{f(x)} g(x) \mathrm{d}\mu(x)$. Given $N \in \mathbb{N}$, the space $L^2(\mathbb{R}^d, \mathbb{C}^N)$ is defined analogously, but with measurable $\mathbb{C}^N$-valued functions $f : \mathbb{R}^d \to \mathbb{C}^N$.

as the space $L^2(\mathbb{R}^d, \mathbb{C}^2)$ of square integrable functions with values in $\mathbb{C}^2$ [ME2]. In the absence of disorder, the atomic nuclei are in their equilibrium positions and the system is translation-invariant with respect to a certain lattice in $\mathbb{R}^d$ that can be chosen to be the standard lattice $\mathbb{Z}^d$. Moreover, as a measure space, $\mathbb{R}^d$ can be decomposed as $\mathbb{R}^d = \mathbb{Z}^d \times [0,1)^d$, with the Lebesgue measure corresponding to the product measure of the Lebesgue measure on $[0,1)^d$ and the counting measure on $\mathbb{Z}^d$. We may also replace $[0,1)^d$ by $[0,1]^d$ because they differ by a set of measure 0. Therefore, the approximation of the background space can be further simplified by means of the following isomorphisms:

$$L^2(\mathbb{R}^d, \mathrm{d}x) \cong \ell^2(\mathbb{Z}^d) \otimes L^2([0,1]^d, \mathrm{d}x)$$

where $\ell^2(\mathbb{Z}^d)$ is the space of square summable sequences[2] on $\mathbb{Z}^d$. This leads to $L^2(\mathbb{R}^d, \mathrm{d}x) \otimes \mathbb{C}^2 \cong \ell^2(\mathbb{Z}^d) \otimes L^2([0,1]^d, \mathrm{d}x) \otimes \mathbb{C}^2$. According to [ME1], the result of this tensor product is equivalent to space of square summable sequences on $\mathbb{Z}^d$ with values in $\mathcal{K} := L^2([0,1]^d, \mathbb{C}^2)$, the space of square integrable functions on $[0,1]^d$ with values in $\mathbb{C}^2$. In particular, the Hilbert space $\mathcal{K}$ describes all possible states of the electron inside a unit cell of the crystal, that is, the cube $[0,1]^d$.

Moreover, since most of the states in this cube have an energy that is too high or too low to be physically relevant, it is reasonable to replace it with a finite-dimensional subspace $\mathbb{C}^N$. As a consequence of this observation, one usually defines the single particle Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \tag{3.1}$$

where $\mathbb{C}^N$ takes account of the internal degrees of freedom of the system. In particular, within this approximation, the Hamiltonian is a bounded operator $H \in \mathbb{B}(\ell^2(\mathbb{Z}^d, \mathbb{C}^N))$[3], that is, there exists a constant $M > 0$ such that

[2]This is a particular case of L^2-space for $\Omega = \mathcal{P}(X)$ and μ being the counting measure. In other words, $\ell^2(X) = \{f : X \to \mathbb{C} : \sum_{x \in X} |f(x)|^2 < \infty\}$. It is an Hilbert space for the inner product $\langle f | g \rangle = \sum_{x \in X} \overline{f(x)} g(x)$. Analogously, $\ell^2(\mathbb{Z}^d, \mathbb{C}^N) = \{f : \mathbb{Z}^d \to \mathbb{C}^N,\ f = (f_1, \ldots, f_N) : \sum_{x \in \mathbb{Z}^d} |f_i(x)|^2 < \infty \text{ for all } i = 1, \ldots, N\}$ [ME1].

[3]Given an Hilbert space $\mathcal{H}$, $\mathbb{B}(\mathcal{H})$ denotes the space of bounded linear operators on X.

$\|H(\psi)\| \leqslant M\|\psi\|$ for every $\psi \in \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$.

The dynamics of a quantum mechanical system is decribed by the **Hamiltonian** H, that is, a self-adjoint operator on $\mathcal{H}$ such that, if $\psi_t \in \mathcal{H}$ is the state of the system at the time $t \in \mathbb{R}$, then it satisfies the differential equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H(\psi_t), \tag{3.2}$$

where typically H is the Schrödinger operator

$$H(\psi)(x) = -\frac{\hbar^2}{2\mu}\sum_{j=1}^{3}\frac{\partial^2\psi}{\partial x_j^2}(x) + V(x)\psi(x) \tag{3.3}$$

with μ the reduced mass of the particle, $\hbar$ the Planck constant and $V : \mathbb{R}^3 \to \mathbb{R}$ the external potential. In this description, given $\psi \in \mathcal{H}$, the inner product $\langle\psi, H(\psi)\rangle \in \mathbb{R}$ corresponds to the expectation value of the measurement of H when the system is in the state ψ [ME1].

Within the approximation (3.1), the Hamiltonian H can also be represented as a matrix $(H^{k,m})_{k,m\in\mathbb{Z}^d}$ whose blocks are the unique matrices $H^{k,m} \in \mathbb{M}_N(\mathbb{C})$ such that $(H\psi)(k) = \sum_{m\in\mathbb{Z}^d} H^{k,m}\psi(m)$ for all $k \in \mathbb{Z}^d$, $\psi : \mathbb{Z}^d \to \mathbb{C}^N$ with finite support and where $H^{k,m}\psi(m)$ means matrix-vector multiplication. From the discussion above, H is also supposed to satisfy the following two important properties:

1. H is *translation invariant*: let $n, m \in \mathbb{Z}^d$ and define the translation (unitary) operator $T_n : \mathcal{H} \to \mathcal{H}$ by $T_n(\psi(m)) = \psi(m-n)$. Thus, translation invariant means that H satisfies $T_nH = HT_n$ or, in matrix representation: $H^{k+n,m+n} = H^{k,m}$ for all $k, m, n \in \mathbb{Z}^d$. This property can also be expressed by saying that H is $\mathbb{Z}^d$-*periodic*.

2. H is a *controlled operator*, that is, it has finite propagation in the sense that there exists $R > 0$ such that if $\psi_1, \psi_2 : \mathbb{Z}^d \to \mathbb{C}^N$ have disjoint supports of distance at least R, then $\langle\psi_1, H(\psi_2)\rangle = 0$. Equivalently: $H_{k,m} = 0$ for $\|k - m\| \geqslant R$.

These two assumptions yield a description of H in terms of a matrix with just a finite number of non-zero blocks $\tilde{H}^n$, $n \in \mathbb{Z}^d$, where the index of the block has been rewritten as $\tilde{H}^{k-m} := H^{k,m}$.

Given a Hamiltonian in this form, [DG1] and [ME1] show that it is possible to compute its spectrum using the **Bloch-Floquet transform**[4] $\mathscr{F}_{\mathcal{B}}$, which is a generalization of the discrete Fourier transform. The result is that the spectrum of H is the disjoint union of at most N compact intervals $[a_i, b_i] \subseteq \mathbb{R}$, $i \in \{1, \dots, N\}$:

$$\sigma(H) = \coprod_{i=1,\dots,N} [a_i, b_i]. \tag{3.4}$$

Each interval is called in physics **energy band** and its elements, which also correspond to the eigenvalues of $\hat{H} := \mathscr{F}_{\mathcal{B}} H \mathscr{F}_{\mathcal{B}}^{-1} \in L^2(\mathbb{T}^d, \mathbb{C}^N)$, are interpreted as the allowed energies of the system. Finally, one also defines the **Brillouin zone** $\mathbb{B} := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ as a particular unit cell in the *reciprocal lattice*, i.e., the Fourier Transform of the direct (standard) lattice, such that inside it the energy and momentum of an electron in the crystal vary continuously (without quantum jumps, see Section 3.2). Clearly, $\mathbb{B} \cong \mathbb{T}^d$, as for example through the isomorphism $\mathbb{B} \ni (\kappa_1, ..., \kappa_d) \mapsto (t_1, ..., t_d) = t \in \mathbb{T}^d$ given by $t_j(\kappa) = (\cos \kappa_j, \sin \kappa_j)$, $j = 1, ..., d$.

The classification of materials in physics is based on the distribution of these eigenvalues with respect to the **Fermi level** E_F, which is defined as the work required to add an electron to the body. Since this is a difference of potentials, one can assume without loss of generality that $E_F = 0$ and therefore all negative energies $E_n < 0$ correspond to the filled states, whereas positive energies $E_n > 0$ correspond to the empty states. Materials can thus be basically classified with regards to their electrical behaviour as follows:

- if the Fermi level E_F belongs to the spectrum of the Hamiltonian H of the system, it means that there are states with appropriate energy for the electron to move in the crystal; the material is classified as a **conductor**;

[4] Given $\psi \in \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ and $k \in \mathbb{Z}^d$, the Bloch-Floquet transform $\mathscr{F}_{\mathcal{B}}$ is defined as $(\mathscr{F}_{\mathcal{B}}\psi)(x, \kappa) := \sum_{n \in \mathbb{Z}^d} e^{\mathrm{i}\kappa n} \psi(x - n)$. The result lies in the space $L^2(\mathbb{T}^d, \mathbb{C}^N)$.

- if the Fermi level E_F does not belong to the spectrum of H but lies in a *band gap*, which is an energy range where no electron state can exist, then all empty states are above E_F and therefore there is a finite energy cost to excite the system above its ground state. In this case the material is called **insulator**.

3.2 Topological phases and Bloch bundles

From the point of view of topology, the question one can ask is whether or not the Hamiltonians of two different quantum systems can be continuously transformed into each other. In case of a positive answer, one says that the two systems are **topologically equivalent**. In other word, quantum systems are classified on the basis of homotopies between their Hamiltonians.

Following from the discussion in the previous section, one can deduce that in a system without constraints on the Hamiltonian, like a conductor, all quantum systems are topologically equivalent, since in this situation any two Hamiltonians H_1 and H_2 are linearly connnected along the path $(1-t)H_1 + tH_2$, which is continuous because the Hilbert space $\mathcal{H}$ in which they lie is convex.

On the other hand, in the case of systems with an energy gap around the Fermi level E_F and with particular symmetries, the concept of *topolocial equivalence* needs to be redefined. Two systems are now said to be topologically equivalent if their Hamiltonians can be continuously deformed into each other *without ever closing the gap* and *preserving the symmetries*. Elements of an equivalence class are said to have the same **topological phase**.

But in order to give sense to this discussion, the first question to answer is about fixing a suitable background space and topology with respect to which continuous paths should be considered.

One possible way to answer is to describe insulators by Hamiltonians on some Hilbert space $\mathcal{H}$ and then consider continuous paths in $\mathbb{B}(\mathcal{H})$, i.e., the space of bounded linear operator on $\mathcal{H}$. However, for non-trivial systems, it turns out that any choice of topology on $\mathcal{H}$ is not completly satisfactory for the purpose of the topological classification [KEL].

When dealing with more sophisticated systems, a different mathematical approch becomes necessary (see for example [BEL, EM, KEL, DG1, DG2]). We will sketch the idea of this new approach. A physical system can be modelled as a C^*-algebra[5] $\mathscr{A}$ endowed with the norm topology, with the Hamiltonian being a self-adjoint invertible element $H \in \mathscr{A}$, that is, such that $H^* = H$. In general, the set of insulators with observable algebra $\mathscr{A}$ corresponds to the set of self-adjoint invertible elements of $\mathscr{A}$ and it is denoted by $\mathscr{A}^{\text{s.a.}}_{\text{inv}}$ ("s.a." stays for self-adjoint, "inv" for invertible). The topological phases are thus classified by $\pi_0(\mathscr{A}^{\text{s.a.}}_{\text{inv}})$, that is, the homotopy classes in $\mathscr{A}^{\text{s.a.}}_{\text{inv}}$. As shown in [KEL], this can be turned into a group.

Within this approach, symmetries are automorphisms of order two of the C^*-algebra. The topological phases of insulators with symmetries are classified in this case by the homotopy classes of elements belonging to just a subspace of $\mathscr{A}^{\text{s.a.}}_{\text{inv}}$ that is invariant under such symmetries. I will define and discuss the properties of some of these symmetries in Chapter 5.

Now, if H describes an insulator, it must have a spectral gap $S = (E_-, E_+) \subset \mathbb{R}\setminus\sigma(H)$ enclosing the Fermi level and, therefore, $\chi_{(-\infty,E_F]} \in C(\sigma(H))$, where χ is the characteristic function on the subset $(-\infty, E_F]$ of the spectrum of H and $C(\sigma(H))$ is the space of continuous function on the spectrum of H. The *spectral projection* P_S of the operator H with respect to E_F in the C^*-algebra $\mathscr{A}$ is defined as:

$$P_S(H) := \chi_{(-\infty,E_F]}(H). \tag{3.5}$$

The assumption $S \nsubseteq \sigma(H)$ assures that the map $z \mapsto P_S(H)(z)$, $z \in \mathbb{T}^d$ is continuous. Therefore, the dimension of the range

$$\operatorname{im} P_S(H)(z) := \{v \in \mathbb{C}^N \mid P_S(H)(z)v = v\}$$

[5] A C^*-algebra is a Banach algebra $\mathscr{A}$ with a conjugate-linear involution $* : \mathscr{A} \to \mathscr{A}$ that satisfies $(x \cdot y)^* = y^* \cdot x^*$ for all $x, y \in \mathscr{A}$ and $\|x^*x\| = \|x\|^2$ for all $x \in \mathscr{A}$ [ME1]. An involution over a set X is a function $f : X \to X$ such that $f(f(x)) = x$ for every $x \in X$ (see also Definition 4.1).

is constant over $\mathbb{T}^d$. Hence, it is possible to define a vector bundle $\xi = (E, p, \mathbb{B})$ on the Brillouin zone $\mathbb{B} \cong \mathbb{T}^d$ setting as total space

$$E := \coprod_{z \in \mathbb{T}^d} \operatorname{im} P_S(H)(z).$$

Thus, $\xi = (E, p, \mathbb{T}^d)$ is a vector bundle with fibers $\operatorname{im} P_S(H)(z)$ and rank equal to $k = \mathrm{Tr}_{\mathbb{C}^N} P_S(H)(z)$. It is also called **Bloch bundle**[6].

Finally, since the spectrum is an algebraic invariant, one can associate each gap to the equivalence class of its gap projection. And since $\mathscr{A}$ can also be assumed to be separable, the number of equivalence classes of projections in $\mathscr{A}$ is at most countable. After some functorial machinery [KEL], this set of equivalence classes can be endowed with the structure of an abelian group. This is also the norm-closed subalgebra $\mathscr{A}_{\mathrm{inv}}^{\mathrm{s.a.}}$ of $\mathscr{A}$ where the classification of insulators has to be considered.

Remark 9. This argument shows that there is a correspondence between isomorphism classes of gapped periodic systems and elements of $\mathbf{VB}^k_{\mathbb{T}^d}$. This is actually an application of the *Serre-Swan Theorem*[7], which states that there exists a continuous bijection between isomorphism classes of vector bundles over a topological space X and projections in $\mathbb{M}_N(C(X))$. Any topologically protected characteristic of such systems are translated into additional structures of the corresponding vector bundle.

Remark 10. In absence of any other constraint on the system, one can describe the d-torus $\mathbb{T}^d$ as a d-dimensional CW-complex and use the results in the previous chapter to show the needlessness of a classification of Bloch bundles (and thus of materials) up to isomorphism rather than up to stable isomorphism. Concretely, in the case of a 3-dimensional crystal, $n = 3$ and, consequently, with the notation of Theorem 2.7,

$$m = \begin{cases} 4 & \text{if } F = \mathbb{R} \\ 2 & \text{if } F = \mathbb{C} \\ 1 & \text{if } F = \mathbb{H} \end{cases} \quad \text{where } F \text{ is intended as the fibre of the bundle.}$$

[6]Named after the Swiss physicist Felix Bloch (1905–1983).

[7]Named after the French mathematician Jean-Pierre Serre (1926) and the american mathematican Richard Swan (1933).

Hence, given two vector bundles of this type, the property of being s-isomorphic agrees with that of being isomorphic when the dimension of the bundles is at least 4 in case of real vector bundles, at least 2 for complex vector bundles and for quaternionic vector bundles of all dimensions different from 0.

Remark 11. This modeling of crystals through complex vector bundles on the torus applies in particular to **topological insulators**. These form a class of materials that are insulating in the bulk but can let current flow on the boundary. Despite this flow, the property of being an insulator is a consequence of an energy gap at the Fermi level E_F in the spectrum of the Hamiltonian H of the system, that means, we can construct the Bloch bundle and classify topological insulators on the basis of other possible symmetries of the system.

Example 3.1 (Topological insulator of class **A**)**.** Quantum systems which are not subject to any symmetry are denoted with **A** in the *Cartan classification scheme* of symmetric spaces. The study of these systems leads to complex vector bundles without extra structure, which can be classified, up to isomorphism, by the sets of equivalence classes $\mathrm{Vec}^k_{\mathbb{C}}(X)$ of complex vector bundles of the corresponding dimension m over the base space $X = \mathbb{T}^d$.

Chapter 4

The equivariant problem

In order to give this discussion a more realistic physical significance, it is necessary to add more structure to the Bloch bundle introduced in the previous chapter. First of all, the base space $X = \mathbb{T}^d$ has to be endowed with an involution τ. Secondly, one can consider possible symmetry constraints acting on the quantum system and encode them through unitary or antiunitary operators that commute or anticommute with the Hamiltonian H of the system. This will lead to the definition of "Real" and "Quaternionic" vector bundles [AT, DU, DG1, DG2]. Starting from this chapter, we will describe and discuss some of these situations.

4.1 Involution spaces and general G-spaces

In order to study quantum systems with symmetries, the first step is to slightly modify the Bloch bundle in such a way as to take into account the presence of an involution on the base space X. This involution is lifted to the total space E of the bundle through the projection p. The result of this construction is a particular case of a G-vector bundle.

The content of this section refers to [SHA, BL, MA2].

Definition 4.1. An **involution** $\tau : X \to X$ over a topological space X is a homeomorphism of period 2, i.e., $\tau^2 = 1$. A topological space X together with an involution τ is called **involutive space**. The *fixed point* set of (X, τ) is defined as

$$X^\tau := \{x \in X \mid \tau(x) = x\}.$$

Example 4.1. A basic example of involution is the complex conjugation $\tau : \mathbb{C} \to \mathbb{C}$, $\tau(x) := \overline{x}$.

An involutive space can also be seen as a *G-space* for $G = \mathbb{Z}_2 = \{\pm 1\} \cong \{\tau, \mathrm{id}_X\}$:

Definition 4.2. A **topological group** G is a group endowed with a topology such that both the addition and the inverse operations are continuous with respect to this topology. A **G-space** X is a topological space X equipped with a continuous action $G \times X \to X$ of a topological group G such that both

$$(1) \quad ex = x \qquad\qquad (2) \quad g(g'(x)) = (gg')(x)$$

hold for any $x \in X$, $g \in G$ and with e being the neutral element of G.

Definition 4.3. A map $f : X \to Y$ between G-spaces X and Y is said to be **equivariant** or a **G-map** provided

$$f(gx) = gf(x) \tag{4.1}$$

for all $g \in G$ and for all $x \in X$. A **homotopy** between G-maps $f_0, f_1 : X \longrightarrow Y$ is a continuous map $F : X \times [0,1] \to Y$ such that $F(t, -) = f_t(-)$ for $t = 0, 1$ and F is a G-map as well with G acting trivially on $I = [0,1]$. We write $[X, Y]_G$ for the set of equivalence classes of equivariantly homotopic maps between G-spaces X and Y.

From now on, all spaces will be assumed to be *weak Hausdorff* (i.e., the diagonal $X \subset X \times X$ is a closed subset) and *compactly generated* (i.e., a subspace is closed if and only if its intersection with all Hausdorff compact subspaces is closed). Moreover, in order to assure that the product $X \times Y$ between spaces with such properties is again compactly generated, we endow it with the following topology: let $\{K_i\}$ denote the family of compact subsets of $X \times Y$, then we say that a subset $A \subset X \times Y$ is closed if and only if $A \cap K_i$

is closed in K_i for each i. With this topology the product $X \times Y$ results to be compactly generated because it can be shown that the closed subsets of this topology and those of the induced topology are the same.

Remark 12. Let $G\mathscr{U}$ be the category of G-spaces with G-maps as morphisms. Then G acts diagonally on the cartesian products of spaces endowed with the topology introduced above and acts by conjugation on the space $\mathrm{Map}(X,Y) := \{f : X \to Y \text{ continuous}\}$ endowed with compact-open topology[1], where X and Y are G-spaces, that is:

$$(g \cdot f)(x) = gf(g^{-1}x).$$

As a consequence, the G-homeomorphism:

$$\mathrm{Map}_G(X \times Y, Z) \cong \mathrm{Map}_G(X, \mathrm{Map}(Y,Z))$$

holds for any G-spaces X, Y and Z [MA2]. Here, $\mathrm{Map}_G(X,Y)$ denotes the space of G-equivariant maps $X \to Y$ in the subspace topology of all maps $X \to Y$, which are instead the elements of $\mathrm{Map}(X,Y)$.

Let H be a closed subgroup of G. The space of fixed-points with respect to H is defined as $X^H := \{x \in X \,|\, hx = x \text{ for all } h \in H\}$. In particular, it is easy to see that $x \in X^H$ if $H \subseteq G_x$, where $G_x = \{h \in H \,|\, hx = x\}$ is the isotropy group of x. Let also G/H denote the orbit space, i.e. $G/H = \{gH : g \in G\}$.

Remark 13. Endowing X with the trivial G-action, that is, $g \cdot x = x$ for all $g \in G$ and all $x \in X$, a fundamental property is the existence of the G-homeomorphisms

$$\mathrm{Map}_G(G/H \times X, Y) \cong \mathrm{Map}_H(X,Y) \cong \mathrm{Map}(X, Y^H) \tag{4.2}$$

obtained by sending $\phi \in \mathrm{Map}_G(G/H \times X, Y)$ to $\phi' \in \mathrm{Map}(X, Y^H)$ given by $\phi'(x) = \phi(eH, x)$, with e being the neutral element of G; vice versa one recovers ϕ from ϕ' through $\phi(gH, x) = g\,\phi'(x)$. This trick will be very useful in the following discussion because it allows to go back and forth between the equivariant and non-equivariant theories.

[1]The compact-open topology on the space $\mathrm{Map}(X,Y)$ has a basis consisting of sets of maps taking a finite number of compact sets $K_i \subset X$ to open sets $U_i \subset Y$ [HAT].

4.2 G-CW-complexes

I have shown in the previous chapter that a periodic quantum system can be represented by a vector bundle on the d-torus $\mathbb{T}^d$. Example 1.2 illustrates that $\mathbb{T}^d$ has a well defined CW-complex decomposition. In order to include in this description an involution τ, or equivalently an action of the group $\mathbb{Z}_2$ on $\mathbb{T}^d$, a slightly more general form of CW-complex must be used (see also [KEN, MA2, SHA]):

Definition 4.4. Let G be a discrete group and let H be a closed subgroup. A **G-CW-complex** X is a union of sub-G-spaces X^n constructed inductively as follows:

- X^0 is the disjoint union of orbits $G/H = \{gH \,|\, g \in G\}$; the orbits gH form a partition of G and this is the reason why they act as points in the equivariant theory.
- X^n is constructed from the $(n-1)$-skeleton by attaching equivariant n-cells σ_α^n of the form $G/H \times \mathbb{D}^n$ via equivariant attaching maps $\varphi_\alpha : G/H_\alpha \times \mathbb{S}^{n-1} \to X^{n-1}$; the group G acts on $\mathbb{D}^n$, and therefore on $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$, trivially, i.e., as the identity, while it acts by left multiplication on G/H_α.
- Set $X = \bigcup\limits_n X^n$ with the weak topology. The G-CW-complex X has dimension n for the largest $n \in \mathbb{N}$ such that $X = X^n$.

This construction is also decribed by the following pushout diagram:

$$\begin{tikzcd}
\coprod_\alpha G/H_\alpha \times \mathbb{S}^{n-1} \arrow[r, "\varphi_\alpha"] \arrow[d, hook] & X^{n-1} \arrow[d] \\
\coprod_\alpha G/H_\alpha \times \mathbb{D}^n \arrow[r] & X^n = X^{n-1} \cup_{\varphi_\alpha} (G/H_\alpha \times \mathbb{D}^n)
\end{tikzcd}$$

More general **relative G-CW-complexes** (X, A) for a G-space A are defined analogously to the nonequivariant case, that is, the 0-skeleton is replaced by the union of A and orbits G/H. Similarly, a G-CW-complex

(Y, B) is a **subcomplex** of (X, A) provided B is a closed G-subspace of A and $Y^n = Y \cap X^n$ in the CW decomposition.

Remark 14. Unlike the non-equivariant case, the product of two G-CW-complexes is not always well defined, since problems may arise when looking at the cell structure of the product of two orbits $G/H \times G/K$, as discussed in [SHA]. However, if G is a disrete group, then the product of two orbits is a disjoint union of orbits. Hence, the product of two G-CW-complexes is a $G \times G$-CW-complex.

Remark 15. The G-homeomorphism (4.2) applies to the equivariant attaching maps φ_α. This means that a G-map $\varphi : G/H \times \mathbb{S}^n \to X$ can be reduced to a non-equivariant map $\varphi' : \mathbb{S}^n \to X^H$.

Fix now $G = \mathbb{Z}_2 = \{\pm 1\}$. This group has only two subgroups: the unit $\{+1\}$ and the full group $\mathbb{Z}_2$ itself. Therefore, there are only two possible types of $\mathbb{Z}_2$-cells for each dimension n, namely:

- *fixed* cells $\sigma^n := \{+1\} \times \mathbb{D}^n \cong \mathbb{D}^n$ with trivial G-action; the $\mathbb{Z}_2$-boundary of a fixed cell is $\partial\sigma^n = \{+1\} \times \partial\mathbb{D}^n \cong \partial\mathbb{D}^n \cong \mathbb{S}^{n-1}$;
- *free* cells $\tilde{\sigma}^n := \mathbb{Z}_2 \times \mathbb{D}^n$ having trivial action on $\mathbb{D}^n$ and action by permutation on $\mathbb{Z}_2$; the $\mathbb{Z}_2$-boundary of such a cell is $\partial\tilde{\sigma}^n := \mathbb{Z}_2 \times \mathbb{S}^{n-1}$.

Hence a $\mathbb{Z}_2$-CW-complex has the following structure:

- $X^0 = (\coprod_{i=1}^{N_0} \sigma_i^0) \amalg (\coprod_{i=1}^{\tilde{N}_0} \tilde{\sigma}_i^0)$,
  where N_0 is the number of fixed 0-cells $\sigma^0 \cong \{*\}$, while $\tilde{N}_0$ is the number of free 0-cells $\tilde{\sigma}^0 \cong \mathbb{Z}_2$. Notice that both N_0 and $\tilde{N}_0$ can also be equal to ∞;
- $X^n := X^{n-1} \bigcup_{\varphi_i} (\coprod_{i=1}^{N_n} \sigma_i^n) \bigcup_{\tilde{\varphi}_i} (\coprod_{i=1}^{\tilde{N}_n} \tilde{\sigma}_i^n)$,
  where $\varphi_i : \{+1\} \times \mathbb{S}^{n-1} \to X^{n-1}$ are the $\mathbb{Z}_2$-attaching maps of the fixed cells, whereas $\tilde{\varphi}_i : \mathbb{Z}_2 \times \mathbb{S}^{n-1} \to X^{n-1}$ are those of the free cells. The symbols $\bigcup_{\varphi_i}$ and $\bigcup_{\tilde{\varphi}_i}$ denote the union modulo the identification $\varphi_i(x) \sim x$ for all $x \in \partial\sigma_i^n$ resp. $\tilde{\varphi}_i(x) \sim x$ for all $x \in \partial\tilde{\sigma}_i^n$.

4.3 “Real” vector bundles

As previously pointed out, the presence of symmetries in a quantum system is translated to the existence of automorphisms of the total space of the Bloch bundle. The case of time-reversal symmetry, which we will discuss in the rest of this thesis, gives rise to the interesting classes of “Real” and “Quaternionic” vector bundles as previously reported by [DG1, DG2, AT].

Definition 4.5. A **“Real” vector bundle**, or **$\mathscr{R}$-vector bundle**, over the involutive space (X,τ) is a complex vector bundle $p: E \to X$ endowed with a topological homeomorphism $\tau_E : E \to E$ such that:

- $\tau_E^2 = \mathrm{id}_E$, that is, τ_E is an involution;
- the projection p is *real*, in the sense that it commutes with the involutions on X and E: $p \circ \tau_E = \tau \circ p$;
- τ_E is anti-linear on each fiber, i.e., $\tau_E(\lambda z) = \overline{\lambda}\tau_E(z)$ for every $\lambda \in \mathbb{C}, z \in E$.

Remark 16. The terminology of *“Real”* vector bundles comes from the fact that this notion can be seen as an extension of that of real vector bundle as explained in [AT], namely, a space E with involution τ_E is isomorphic to the complexification $E_{\mathbb{R}}^C$ of the real subspace $E_{\mathbb{R}} = \{z \in E \,|\, \tau_E(z) = z\}$. We write $\mathbb{R}$-$\mathbf{VB}_X$ for the category of $\mathbb{R}$-vector bundles over X and $\mathscr{R}$-$\mathbf{VB}_X$ for that of $\mathscr{R}$-vector bundles over the same space. Then one has the following inverse relations:

1. $\mathscr{R}$-$\mathbf{VB}_X \to \mathbb{R}$-$\mathbf{VB}_X$ given by $E \mapsto E_{\mathbb{R}}$
2. $\mathbb{R}$-$\mathbf{VB}_X \to \mathscr{R}$-$\mathbf{VB}_X$ given by $E \mapsto E \otimes_{\mathbb{R}} \mathbb{C} =: E^C \cong E \oplus iE$

One can also prove that $\dim_{\mathbb{R}} E = \dim_{\mathbb{C}} E^C$ (see [AT]).

An $\mathscr{R}$-morphism u of $\mathscr{R}$-vector bundles ξ_1 and ξ_2 over the same involutive space (X,τ) is a vector bundle morphism commuting with the involutions, i.e., $u \circ \tau_{E_1} = \tau_{E_2} \circ u$. The set of isomorphism classes of $\mathscr{R}$-vector bundles of rank k over (X, τ) is denoted by $\mathrm{Vec}^k_{\mathscr{R}}(X, \tau)$; here, k is intended as the (constant) real dimension of the fibre of the bundle. The previous remark suggests then the isomorphism:

$$\mathrm{Vec}^k_{\mathbb{R}}(X) \cong \mathrm{Vec}^k_{\mathscr{R}}(X, \mathrm{id}_X) \quad \text{for all } k \in \mathbb{N}. \tag{4.3}$$

The process of forgetting the "Real" structure and considering just the complex bundle over X gives a map

$$j : \mathrm{Vec}^k_{\mathscr{R}}(X, \tau) \to \mathrm{Vec}^k_{\mathbb{C}}(X) \tag{4.4}$$

that is in general neither injective nor surjective.

Example 4.2. The set $\mathrm{Vec}^k_{\mathscr{R}}(X, \tau)$ cannot be empty since it contains at least the "Real" product vector bundle $X \times \mathbb{C}^m \to X$ endowed with the product $\mathscr{R}$-structure $\tau_0(x, v) = (\tau(x), \overline{v})$ given by the complex conjugation $v \mapsto \overline{v}$. An $\mathscr{R}$-vector bundle is **$\mathscr{R}$-trivial** if it is isomorphic to the product $\mathscr{R}$-bundle in the category $\mathscr{R}$-$\mathbf{VB}_X$.

Consider now the set $\Gamma(E)$ of all the sections of an $\mathscr{R}$-bundle over the involutive space (X, τ). This is a module over $C(X)$ as pointed out in Section 1.3. Moreover, it inherits from the $\mathscr{R}$-structure of (E, τ_E) an anti-linear involution $\tau' : \Gamma(E) \to \Gamma(E)$ defined as the composition

$$\tau'(s) := \tau_E \circ s \circ \tau \tag{4.5}$$

Elements of the set $\Gamma(E)' = \{s \in \Gamma(E) \,|\, \tau'(s) = s\}$ of fixed points with respect to τ' are said to be $\mathscr{R}$-sections of the bundle. In the case of complex vector bundles, the presence of a frame of complex sections is equivalent to the triviality of the vector bundle, as already mentioned in Proposition 1.6. A similar result holds for "Real" vector bundles [DG1, Theorem 4.8]:

Proposition 4.1 ($\mathscr{R}$-triviality)**.** *An $\mathscr{R}$-vector bundle is trivial if and only if it admits a global $\mathscr{R}$-frame, that is, a global frame composed of $\mathscr{R}$-sections.*

The following important result mentioned in [DG1] is a generalization of Theorem 1.5 to the case of $\mathscr{R}$-vector bundles and $\mathscr{R}$-sections.

Lemma 4.2 (Extension Lemma)**.** *Let (X,τ) be an involutive space, where (X,τ) is a $\mathbb{Z}_2$-CW-complex. Let $\xi = (E,p,X)$ be an $\mathscr{R}$-bundle over (X,τ) with τ_E the lift of τ and let also $Y \subset X$ be a closed subset such that $\tau(Y) = Y$. Then each $\mathscr{R}$-section $s : Y \to E|_Y$ prolongs to an $\mathscr{R}$-section $\tilde{s} : X \to E$, that is, a global $\mathscr{R}$-section.*

Proof. Theorem 1.5 guarantees already the existence of a section $s^* : X \to E$. In order to get an $\mathscr{R}$-section it is sufficient to exhibit a section $\tilde{s}$ that satisfies $\tau'(\tilde{s}) = \tilde{s}$. This can be easily found defining $\tilde{s} = \frac{1}{2}(s^* + \tau'(s^*))$, in fact, by (4.5):

$$\tau'(\tilde{s}) = \tau_E \circ \tilde{s} \circ \tau = \frac{1}{2}(\tau_E \circ (s^* + \tau'(s^*)) \circ \tau) = \frac{1}{2}(\tau_E \circ s^* \circ \tau + \tau_E \circ \tau_E \circ s^* \circ \tau \circ \tau)$$
$$= \frac{1}{2}(\tau_E \circ s^* \circ \tau + s^*) = \frac{1}{2}(\tau'(s^*) + s^*) = \tilde{s}. \quad \square$$

The definition of morphism between two complex vector bundles ξ and ξ' over the same space X can be reformulated in terms of the **homomorphism bundle**, i.e., the complex vector bundle $\mathrm{Hom}_{\mathbb{C}}(E,E') \to X$. This has a "Real" structure given by $\overline{\phi_x}(v) = \overline{\phi_x(\overline{v})}$, where $\phi_x \in \mathrm{Hom}_{\mathbb{C}}(E_x, E'_x)$ and $\overline{\phi}_x \in \mathrm{Hom}_{\mathbb{C}}(E_{\overline{x}}, E'_{\overline{x}})$. Hence a morphism $u : E \to E'$ is nothing but an $\mathscr{R}$-section of $\mathrm{Hom}_{\mathbb{C}}(E,E')$.

By definition, an $\mathscr{R}$-vector bundle is locally trivial in the category $\mathbb{C}$-$\mathbf{VB}_X$ of complex vector bundles. Actually, its local triviality in the category $\mathscr{R}$-$\mathbf{VB}_X$ of "Real" vector bundles can be proved as well.

Theorem 4.3 (Local $\mathscr{R}$-triviality)**.** *Let $\xi = (E,p,X)$ be an $\mathscr{R}$-vector bundle over the $\mathbb{Z}_2$-CW-complex X. Then ξ is locally trivial, meaning that for all $x \in X$ there exists a τ-invariant neighborhood $\mathcal{U}$ of x and an $\mathscr{R}$-isomorphism $h : p^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{C}^k$, where $k \in \mathbb{N}$ and the product bundle $\mathcal{U} \times \mathbb{C}^k \to \mathcal{U}$ is endowed with the trivial $\mathscr{R}$-structure given by the complex conjugation. Moreover,*

- *if $x = \tau(x)$, then $\mathcal{U}$ can be choosen to be connected;*
- *if $x \neq \tau(x)$, then $\mathcal{U}$ can be choosen as the disjoint union of two open sets $\mathcal{U} := \mathcal{U}' \cup \mathcal{U}''$ such that $x \in \mathcal{U}'$ and $\tau : \mathcal{U}' \to \mathcal{U}''$ is a homeomorphism.*

Proof. There are two different cases to be considered:

- $x \in X$ is a real point, i.e., $\tau(x) = x$ and $E_x \cong \mathbb{C}^k$.
  By the Extension Lemma 4.2, there exists a neighborhood $\mathcal{U}$ of x such that $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{C}^k$.

- $x \neq \tau(x) = \overline{x}$.
  A generic complex isomorphism $E_x \cong \mathbb{C}^k$ induces a complex isomorphism $E_{\overline{x}} \cong \mathbb{C}^k$. Define $Y := \{x, \overline{x}\}$. Then $E|_Y \cong Y \times \mathbb{C}^k$. Again, the Extension Lemma 4.2 gives an isomorphim $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{C}^k$, where $\mathcal{U}$ is a τ-invariant open neighborhood of Y. $\square$

4.4 “Quaternionic” vector bundles

Definition 4.6. A **“Quaternionic” vector bundle** or **$\mathscr{Q}$-vector bundle** is a complex vector bundle $p : E \to X$ over an involutive space (X, τ) endowed with a topological homeomorphism $\tau_E : E \to E$ such that:

1. $\tau_E^2|_{E_x} = -\mathrm{id}_{E_x}$ for all $x \in X$, that is, it is an anti-involution;
2. the projection p is equivariant, in the sense that it commutes with the involution on X: $p \circ \tau_E = \tau \circ p$
3. τ_E is anti-linear on each fibre, i.e., $\tau_E(\lambda z) = \overline{\lambda}\tau_E(z)$ for every $\lambda \in \mathbb{C}$, $z \in E$.

A **$\mathscr{Q}$-morphism** u between two $\mathscr{Q}$-vector bundles ξ_1 and ξ_2 over the same involutive space (X, τ) is a vector bundles morphism commuting with the involutions, i.e., $u \circ \tau_{E_1} = \tau_{E_2} \circ u$. The set of isomorphism classes of $\mathscr{Q}$-vector bundles of rank k' over (X, τ) is denoted by $\mathrm{Vec}_{\mathscr{Q}}^{k'}(X, \tau)$; here, k' is intended as the complex dimension of the fibre of the vector bundle.

The restriction of the anti-involution to a fibre E_x over a fixed point $x \in X^\tau$ endows E_x with a **quaternionic structure**[2]. In addition, a complex

[2]A complex vector space $\mathcal{V}$ has a quaternionic structure if there is an anti-linear map $\mathrm{J} : \mathcal{V} \to \mathcal{V}$ such that $J^2 = -\mathrm{id}$.

vector space $\mathcal{V}$ can be endowed with a quaternionic structure if and only if its complex dimension n is even: $k' = 2k$, $k \in \mathbb{N}$ [DG2]. In this case, the pair $(\mathcal{V}, J)$ results to be isomorphic to the space $\mathbb{H}^k = (\mathbb{C} \oplus \mathbb{C}\,\mathrm{j})^k$ with the left multiplication by j, or equivalently, to the space $\mathbb{C}^{2k}$ endowed with the standard quaternionic structure $v \mapsto Q\,\overline{v}$, where Q is the real matrix:

$$Q := \left(\begin{array}{cc|c|cc} 0 & -1 & & & \\ 1 & 0 & & & \\ \hline & & \ddots & & \\ & & & \ddots & \\ \hline & & & 0 & -1 \\ & & & 1 & 0 \end{array}\right)$$

Here $\mathbb{H}$ denotes the skew-field of **quaternions**, that is,

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\,\mathrm{i} \oplus \mathbb{R}\,\mathrm{j} \oplus \mathbb{R}\,\mathrm{k}, \qquad \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$$

with respect to the basis $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$. Or:

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\,\mathrm{j} = (\mathbb{R} \oplus \mathbb{R}\,\mathrm{i}) \oplus (\mathbb{R} \oplus \mathbb{R}\,\mathrm{i})\,\mathrm{j}, \qquad \mathrm{ij} = \mathrm{k}$$

with respect to the basis $(1, \mathrm{j})$.

Example 4.3. The set $\mathrm{Vec}^{2k}_{\mathscr{Q}}(X, \tau)$ cannot be empty since it contains at least the “Quaternionic” product vector bundle $X \times \mathbb{C}^{2k} \to X$ endowed with the product $\mathscr{Q}$-structure $\tau_0(x, v) = (\tau(x), Q\,\overline{v})$ where $\overline{v}$ denotes the complex conjugation. A $\mathscr{Q}$-vector bundle is **$\mathscr{Q}$-trivial** if it is isomorphic to the product $\mathscr{Q}$-bundle in the category $\mathscr{Q}$-$\mathbf{VB}_X$.

Proposition 4.4. *Let (X, τ) be an involutive and path-connected space. If $X^\tau \neq \emptyset$, then every $\mathscr{Q}$-vector bundle over (X, τ) has even rank.*

Proof. In order to support a quaternionic structure, the fibres E_x must have an even complex dimension. The path-connectedness of X ensures, in addition, that this dimension is constant. Hence, any such $\mathscr{Q}$-vector bundle has even rank. □

In analogy with the case of “Real” vector bundles, the following propositions hold (proofs and further references can be found in [DG2]):

Proposition 4.5. *Let $Vec^k_{\mathbb{H}}(X)$ be the set of isomorphism classes of vector bundles over X with fibre $\mathbb{H}^k$. Then*

$$\mathrm{Vec}^k_{\mathbb{H}}(X) \cong \mathrm{Vec}^{2k}_{\mathscr{Q}}(X, \mathrm{id}_X) \qquad \forall k \in \mathbb{N}. \tag{4.6}$$

Proposition 4.6. *The process of forgetting the "Quaternionic" structure defines a map*

$$j : \mathrm{Vec}^k_{\mathscr{Q}}(X, \tau) \to \mathrm{Vec}^k_{\mathbb{C}}(X) \tag{4.7}$$

such that $j([0]) \to [0]$, where $[0]$ denotes the trivial class in the appropriate category.

Let $\Gamma(E)$ be the set of sections of a $\mathscr{Q}$-vector bundle $\xi = (E, p, X)$ over the involutive space (X, τ). As for any vector bundle, it has the structure of a module over the algebra $C(X)$ and it also inherits from the "Quaternionic" structure of the bundle an anti-linear anti-involution

$$\tau' : \Gamma(E) \to \Gamma(E), \qquad \tau'(s) := \tau_E \circ s \circ \tau. \tag{4.8}$$

In order to discuss the stability of "Quaternionic" vector bundles, it is first necessary to indroduce the notion of "Quaternionic" *pairs* of sections.

Definition 4.7. Let $\mathcal{U} \subset X$ be a τ-invariant open set and let $s_1 \in \Gamma(E)$ be a section of a $\mathscr{Q}$-vector bundle ξ such that $s_1(x) \neq 0$ for every $x \in \mathcal{U}$. Define

$$s_2 := \tau'(s_1). \tag{4.9}$$

The pair (s_1, s_2) is said to be a **"Quaternionic"** (or **Kramer**, [DG2]) **pair** over $\mathcal{U}$. It has the following properties:

1. $\tau'(s_2) = -s_1$, since $\tau'(s_2) = \tau_E \circ s_2 \circ \tau = \tau_E^2 \circ s_1 \circ \tau^2 = -s_1$;

2. $s_2(x) \neq 0$ for every $x \in \mathcal{U}$ since τ' is a homeomorphism;

3. s_1 and s_2 are linearly independent: suppose $s_1 = \lambda\, s_2$ for $\lambda \in \mathbb{C}$, then $s_2 \overset{\mathrm{def}}{=} \tau'(s_1) = \tau'(\lambda\, s_2) \overset{(1)}{=} -\overline{\lambda}\, s_1 = -|\lambda|^2 s_2$ can be satisfied if and only if $\lambda = 0$, that is, if and only if s_1 and s_2 are linearly independent.

Given this definition, one can prove the following fundamental theorem concerning the extension of $\mathscr{Q}$-pairs of sections.

Lemma 4.7 (Extension Lemma for $\mathscr{Q}$-pairs)**.** *Let (X, τ) be an involutive space such that X admits a $\mathbb{Z}_2$-CW-decomposition and let $\xi = (E, p, X)$ be a $\mathscr{Q}$-vector bundle over (X, τ) with anti-involution τ_E. Let $Y \subseteq X$ be a closed subset such that $\tau(Y) = Y$. Then each $\mathscr{Q}$-pair $(\overline{s}_1, \overline{s}_2)$ of $\xi|_{p^{-1}(Y)}$ extends to a $\mathscr{Q}$-pair (s_1, s_2) of ξ.*

Proof. Consider the $\mathscr{Q}$-pair $\{\overline{s}_1, \overline{s}_2\}$, where by definition $s_2 = \tau'(\overline{s}_1)$. Using Theorem 1.5, there exists a section s_1 of ξ that extends $\overline{s}_1$. Now set $s_2 := \tau'(s_1)$: since $s_2|_Y = \overline{s}_2$, then (s_1, s_2) results to be a $\mathscr{Q}$-pair extending $(\overline{s}_1, \overline{s}_2)$. □

Chapter 5

Topological quantum systems with time-reversal symmetry

In this final chapter we will introduce the concept and role of symmetries in physics referring to [RS, AKH] and we will focus from the second section on to one of the three fundamental discrete symmetries, namely, the *time-reversal symmetry*. The content of the last two sections is the core of this thesis, since I will give proofs of the conditions under which isomorphism and stable isomorphism of vector bundles with time-reversal symmetries are equivalent, see also [DG1, DG2, EM]. This can also be seen as an "equivariant" generalization of the results in Chapter 2.

5.1 Symmetries

In physics, a symmetry transformation consists in a change of point of view that does not alter the results of possible experiments. In particular, in quantum physics a symmetry is intended as a transformation that preserves transition probabilities between states, since these determine the probability of obtaining an expectation value after measurements [MUK].

The mathematical formulation of symmetries is given in terms of operators on the Hilbert space $\mathcal{H}$ describing all the physical states and it is the main content of Wigner's Theorem[1]. Before formulating this theorem, I will briefly recall that an operator U on a Hilbert space $\mathcal{H}$ is said to be:

- **unitary** if $U : \mathcal{H} \to \mathcal{H}$ is bijective and such that $\langle U\varphi, U\psi\rangle = \langle \varphi, \psi\rangle$; consequently, unitary operators are linear;

- **anti-unitary** if $U : \mathcal{H} \to \mathcal{H}$ is bijective and such that $\langle U\varphi, U\psi\rangle = \overline{\langle \varphi, \psi\rangle} = \langle \psi, \varphi\rangle$; anti-unitary operators are anti-linear, i.e., $U(\lambda v) = \overline{\lambda} U(v)$.

Moreover, an operator U on $\mathcal{H}$ is said to **commute**, resp. **anti-commute**, with the Hamiltonian H if $HU = UH$, resp. $UH = -HU$.

Remark 17. It can be easily seen from the definitions above that:

- the product of two anti-unitary operators is unitary

- the product of a unitary operator with an anti-unitary operator is again anti-unitary.

A physical quantity is said to be *conserved* if its operator Q commutes with H, that is, $QH = HQ$ or, briefly, $[Q, H] = 0$, where $[\cdot, \cdot]$ denote the commutator bracket[2]. In the case of continuous symmetries, *Noether's Theorem*[3] states that every such symmetry has a corresponding conservation law. In the scenario of quantum physics, a wider range of symmetries is admitted, e.g., discrete symmetries. Therefore, Noether's Theorem is not sufficient anymore. A new description of symmetries is needed:

Theorem 5.1 (Wigner, 1931). *Any symmetry transformation can be represented on the Hilbert space of physical states by an operator U that is either linear and unitary or anti-linear and anti-unitary.*

Proof. See for example [MUK]. □

[1] Named after Hungarian-American physicist Eugen Wigner, 1902–1995.

[2] Given two observables A and B, one of which bounded, their commutator is defined as $[A, B] = AB - BA$.

[3] Named after the German mathematician Emmy Noether, 1882–1935.

In addition, symmetries can commute or anti-commute with the Hamiltonian H of the quantum system.

In a more recent work (see [AZ]) it has been pointed out that there exists a one-to-one correspondence between the symmetry type of single particle Hamiltonians of gapped quantum systems and the set of symmetric spaces previously introduced by Cartan[4]. In particular, only ten different classes can be identified: this is the reason why this classification is called *"Ten-Fold Way"*.

In order to get all these classes, it is sufficient to consider the three *Fundamental (discrete) symmetries*, namely:

- the **time-reversal symmetry** $\mathcal{T}$, which is represented by an anti-unitary operator U_T that commutes with H and such that $U_T^2 = \pm 1$
- the **particle-hole symmetry** $\mathcal{P}$, which is represented by an anti-unitary operator U_P that anti-commutes with H and such that $U_P^2 = \pm 1$
- the **chiral symmetry** $\mathcal{C}$, which is represented by a unitary operator U_C that anti-commutes with H and such that $U_C^2 = 1$.

These three symmetries are not independent. In fact, whenever a system has both $\mathcal{T}$ and $\mathcal{P}$, then it has also $\mathcal{C}$, whereas if it only has either $\mathcal{T}$ or $\mathcal{P}$ but not both, then it cannot have $\mathcal{C}$. On the other side, the absence of both $\mathcal{T}$ and $\mathcal{P}$ does not influence the presence or absence of $\mathcal{C}$. The following table summarize the ten possible combinations [RS]:

[4]Named after the French mathematician Élie Cartan, 1869-1951. The classification scheme dates back to year 1926.

Cartan label	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{C}$
A (unitary)			
AI (orthogonal)	+1		
AII (symplectic)	-1		
AIII (ch.unit.)			+1
BDI (ch.orth.)	+1	+1	+1
CII (ch.sympl.)	-1	-1	+1
D		+1	
C		-1	
DIII	-1	+1	+1
CII	+1	-1	+1

Table 5.1: ***The Ten-Fold Way***: *listed are the ten classes of single particle Hamiltonians H classified by their behaviour with respect to the three fundamental symmetries: time-reversal $\mathcal{T}$, particle-hole $\mathcal{P}$ and chiral $\mathcal{C}$. The column "Cartan label" is the name of the symmetric space according to Cartan's classification scheme. The three last columns describe the presence/absence of such a symmetry and their square.*

5.2 Time-reversal symmetry

Definition 5.1. The **time-reversal** involution $\tau_1 : \mathbb{S}^d \to \mathbb{S}^d$ on the sphere is defined as

$$\tau_1(z_0, z_1, ..., z_d) := (z_0, -z_1, ..., -z_d). \tag{5.1}$$

Periodic quantum systems are described by Bloch bundles over the involutive space $(\mathbb{T}^d, \tau)$, where the time-reversal involution τ is the diagonal map

$$\tau := \tau_1 \times \cdots \times \tau_1 \tag{5.2}$$

with τ_1 being the time-reversal involution over the 1-sphere given by (5.1).

In order to describe the $\mathbb{Z}_2$-CW decomposition of $(\mathbb{T}^d, \tau)$ it is just necessary to find a proper $\mathbb{Z}_2$-CW decomposition of $(\mathbb{S}^1, \tau)$ in view of Remark 14. This is described as follows. The only fixed points of τ_1 on the unit circle are $(\pm 1, 0)$. These correspond to fixed 0-cells $\sigma^0_\pm$ in the $\mathbb{Z}_2$-CW decomposition of $\mathbb{S}^1$ and together they form its 0-skeleton X^0. Let $\ell_\pm := \{(z_0, z_1) \in \mathbb{S}^1 \mid \pm z_0 \geqslant 0\}$ be the upper and lower hemispheres. The pair $\{\ell_+, \ell_-\}$ can be seen as a free 1-cell $\tilde{\sigma}^1$ since ℓ_+ and ℓ_- are conjugate to each other. The result of attaching $\tilde{\sigma}^1$ to X^0 is therefore the full space $(\mathbb{S}^1, \tau_1)$.

The $\mathbb{Z}_2$-CW decomposition of $(\mathbb{T}^d, \tau)$ can now be derived easily:

- The 0-skeleton is composed of $N_0 = 2^d$ fixed 0-cells
$$\sigma^0_{\epsilon_1 \ldots \epsilon_d} := (\sigma^0_{\epsilon_1}, \ldots, \sigma^0_{\epsilon_d}) = \sigma^0_{\epsilon_1} \times \cdots \times \sigma^0_{\epsilon_d} \quad ,$$
where $\epsilon_i \in \{+, -\}$ for each $i = 1, \ldots, d$.

- The 1-skeleton is composed of a total of $\tilde{N}_1 = d\, 2^{d-1}$ free 1-cells defined as
$$\tilde{\sigma}^1_{\epsilon_1 \ldots \underline{\epsilon_j} \ldots \epsilon_d} := (\sigma^0_{\epsilon_1}, \ldots, \sigma^0_{\epsilon_{j-1}}, \tilde{\sigma}^1, \sigma^0_{\epsilon_{j+1}}, \ldots, \sigma^0_{\epsilon_d})$$
where the free 1-cell $\tilde{\sigma}^1$ of $(\mathbb{S}^1, \tau)$ replaces the fixed 0-cell $\sigma^0_{\epsilon_j}$ for each $j = 1, \ldots, d$. There are no fixed 1-cells in this dimension.

- In order to define the 2-cells, one can notice that for each pair $i < j$ there are two cells
$$\tilde{\sigma}^2_{\epsilon_1 \ldots \underline{\epsilon_i} \ldots \underline{\epsilon_j} \ldots \epsilon_d} := (\sigma^0_{\epsilon_1}, \ldots, \sigma^0_{\epsilon_{i-1}}, \tilde{\sigma}^1, \sigma^0_{\epsilon_{i+1}}, \ldots, \sigma^0_{\epsilon_{j-1}}, \tilde{\sigma}^1, \sigma^0_{\epsilon_{j+1}}, \ldots, \sigma^0_{\epsilon_d})$$
$$\tilde{\sigma}'^2_{\epsilon_1 \ldots \underline{\epsilon_i} \ldots \underline{\epsilon_j} \ldots \epsilon_d} := (\sigma^0_{\epsilon_1}, \ldots, \sigma^0_{\epsilon_{i-1}}, \tilde{\sigma}^1, \sigma^0_{\epsilon_{i+1}}, \ldots, \sigma^0_{\epsilon_{j-1}}, \tau(\tilde{\sigma}^1), \sigma^0_{\epsilon_{j+1}}, \ldots, \sigma^0_{\epsilon_d})$$
with different behaviors under the $\mathbb{Z}_2$-action of τ. Counting all possible combinations, one finds that there are in total $\tilde{N}_2 = d\,(d-1)\, 2^{d-2}$ free 2-cells (and again no fixed 2-cells).

- The decomposition proceeds analogously for higher dimensional cells, so that one obtains that $\mathbb{T}^d$ has no fixed n-cells for $n \geqslant 1$ but only free n-cells in number equal to $\tilde{N}_n = \binom{d}{n} 2^{d-n}$ for each $1 \leqslant n \leqslant d$.

Let then $(X, \tau) = (\mathbb{T}^d, \tau)$ as an involutive space, where $X = \mathbb{B} \cong \mathbb{T}^d$ is the Brillouin zone introduced in Section 3.1. Consider also again the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ equipped with an anti-linear involution $C : \mathcal{H} \to \mathcal{H}$ induced by the complex conjugation.

Definition 5.2. A topological quantum system $\mathbb{B} \ni z \mapsto H(z)$ has a **time-reversal symmetry** ($\mathcal{T}\varepsilon$) of parity $\varepsilon \in \{\pm\}$ if there is a continuous unitaryvalued map $z \mapsto U_T(z)$ such that:

1. $U_T(z)H(z)U_T^*(z) = CH(\tau(z))C$

2. $CU_T(\tau(z))C = \varepsilon U_T^*(z)$

Systems having *even* time-reversal symmetry $\mathcal{T}+$ are in the class **AI** of Cartan's classification scheme, whereas those with *odd* time-reversal symmetry $\mathcal{T}-$ are in the class **AII**.

Consider a periodic topological quantum system in class **AI** or **AII**. In order to study the topological properties of one these systems, one needs to build the Bloch bundle $\xi = (E, p, \mathbb{T}^d)$ over the involutive space $(\mathbb{T}^d, \tau)$. The time-reversal involution on the base space is transfered through the Fermi projection to the bundle ξ, which in this way gets as additional structure an anti-linear homeomorphism $\tau_E : E_z \to E_{\tau(z)}$. In particular, when $\tau = \mathcal{T}+$, ξ results to be a “Real” vector bundle, since in this case $\tau_E^2 = +1$. On the other hand, if $\tau = \mathcal{T}-$, then $\tau_E^2 = -1$; the resultant Bloch bundle is in this case a “Quaternionic” vector bundle over $(\mathbb{T}^d, \tau)$.

5.3 Stability properties of “Real” vector bundles

Thanks to the particular CW-decomposition of $(\mathbb{T}^d, \tau)$, it is possible to generalize the results in Chapter 2 about the equivalence between stable isomorphism and isomorphism to the case of $\mathscr{R}$-vector bundles ξ endowed with an involution τ_E inherited from the time-reversal symmetry $\mathcal{T}+$ acting on the base space $\mathbb{T}^d$. The first step consists in finding conditions under which there exists a global $\mathscr{R}$-section s of ξ, that is, an $\mathscr{R}$-section s such that $s(x) \neq 0$ for every $x \in X$. The existence of such a section will be in fact necessary in the proof of Theorem 5.2, where it will allow to split off a trivial $\mathscr{R}$-vector bundle of rank 1 as a direct summand.

Theorem 5.2 (Existence of a nowhere vanishing global $\mathscr{R}$-section). *Let (X,τ) be an involutive space such that X has a finite $\mathbb{Z}_2$-CW-complex decomposition of dimension d and let d_τ be the dimension of the fixed-point subcomplex. Let $\xi = (E,p,X)$ be a k-dimensinal $\mathscr{R}$-vector bundle over (X,τ). If $k > \max\{d_\tau, d/2\}$, then ξ has a nowhere vanishing global $\mathscr{R}$-section.*

Remark 18. Consider the zero section $s_0(x) = 0 \in E_x$ for all $x \in X$. This is in particular an $\mathscr{R}$-section since, for every $x \in X$:

$$\tau'(s_0)(x) = (\tau_E \circ s_0 \circ \tau)(x) = \tau_E(s_0(\tau(x))) = \tau_E(0) = 0 = s_0(x) \tag{5.3}$$

by the definition of s_0 and the anti-linearity of τ_E.

Let $\xi_0 \subset \xi$ be the subbundle of non-zero vectors. Since ξ is a complex vector bundle, each fibre $E_{0,x}$ of ξ_0 is isomorphic to $\mathbb{C}^k \setminus \{0\}$ and therefore $2(k-1)$-connected, just as in Chapter 2. With the additional hypothesis $k > d/2$, which implies $d \leqslant 2k-1$, each $E_{0,x}$ is $(j-1)$-connected for each $j \leqslant d$. The involution τ_E restricted to ξ_0 turns it into an $\mathscr{R}$-bundle as well. Hence, an $\mathscr{R}$-section of ξ_0 can be seen as a nowhere-vanishing $\mathscr{R}$-section of ξ.

Proof. Firstly, suppose that X has no fixed cells in dimensions greater than 0. Then the maximum between $d_\tau = 0$ and $d/2$ is $d/2$ and therefore the hypothesis becomes $k > d/2$. The steps of the proof under this assumption follow mostly those of the proof of Theorem 1.5. As in that case, the claim will be proved by induction on the dimension of the skeleton of X.

- If $d = 0$, then X^0 is just a collection of fixed points $\{x_j\}_{j=1}^{N_0}$ and conjugated pairs $\{(x_j, \tau(x_j))\}_{j=1}^{\tilde{N}_0}$. A global section s' can be defined setting

$$\begin{cases} s'(x_j) \in E_x^{\tau_E} \cap E_{0,x} \cong \mathbb{R}^m \setminus \{0\} & \text{for fixed points} \\ s'(x_j) \in E_x^{\tau_E} \cong \mathbb{C}^m \setminus \{0\} & \text{for free pairs} \end{cases}$$

together with the constraint $s'(\tau(x_j)) := \tau_E(s'(x_j))$ that makes s' an $\mathscr{R}$-section: $\tau'(s') = \tau_E \circ s' \circ \tau = \tau_E \circ \tau_E \circ s' = s'$.

- Assume the claim true for the $\mathbb{Z}_2$-CW-subcomplexes X^{j-1}, $1 \leqslant j \leqslant d$ and let $\tilde{\sigma}^j \cong \mathbb{Z}_2 \times \mathbb{D}^j$ be a free j-cell with equivariant attaching map $\varphi : \mathbb{Z}_2 \times \mathbb{D}^j \to X$. Since $j \geqslant 1$, there are no fixed cells by hypothesis. Consider also the induced bundle $\varphi^*(\xi_0)$ over $\mathbb{Z}_2 \times \mathbb{D}^j$, that also has an $\mathscr{R}$-structure since the map φ is equivariant, then $\varphi^*(\xi_0)$ is locally trivial by Theorem 4.3.
  By the inductive hypothesis there exists a global $\mathscr{R}$-section s' of $\xi|_{X^{j-1}}$ that defines an $\mathscr{R}$-section σ' of $\varphi^*(\xi_0)|_{\mathbb{Z}_2 \times \partial\mathbb{D}^j}$ by $\sigma' := s' \circ \varphi$. Since $\mathbb{D}^j$ is contractible, $\varphi^*(\xi_0)$ is trivial, that is, it is isomorphic to the trivial $\mathscr{R}$-vector bundle over $\mathbb{Z}_2 \times \mathbb{D}^j$. So σ' can be identified with an equivariant map $\mathbb{Z}_2 \times \partial\mathbb{D}^j \to \mathbb{C}^k \setminus \{0\}$. The restriction of this map $\{1\} \times \partial\mathbb{D}^j \to \mathbb{C}^k \setminus \{0\}$ extends to a map $\{1\} \times \mathbb{D}^j \to \mathbb{C}^k \setminus \{0\}$ since $j-1 \leqslant d-1 \leqslant 2k-2 = 2(k-1)$ and $\pi_{j-1}(\mathbb{C}^k \setminus \{0\}) \cong 0$ by the previous remark. This can be even further extended, because of the equivariant constraint, to a map $\sigma : \mathbb{Z}_2 \times \mathbb{D}^j \to \mathbb{C}^k \setminus \{0\}$, which is indeed an $\mathscr{R}$-section of $\varphi^*(\xi_0)$. Using the natural isomorphism of induced bundles $(\varphi_\xi, \varphi) : \varphi^*(\xi_0) \to \xi_0$, it is possible to define an $\mathscr{R}$-section s'' of $\xi_0|_{\overline{\tilde{\sigma}^j}}$ by $\varphi_\xi \circ \sigma = s'' \circ \varphi$ which satisfies $s''|_{X^{j-1} \cap \overline{\tilde{\sigma}^j}} = s'$.
  Finally, one defines a global $\mathscr{R}$-section s of $\xi_0|_{X^{j-1} \cup \tilde{\sigma}^j}$ by the requirements $s|_{X^{j-1}} = s'$ and $s|_{\tilde{\sigma}^j} = s''$. The section s is continuous by the weak topology property of vector bundles.
  This procedure can be iterated for all free j-cells and since there are no fixed cells, the claim results to be true on X^j.

Suppose now that $d_\tau > 0$ and consider the fixed-point subcomplex X^τ. Because of the isomorphism (4.3), the fibres over the points of X^τ are real vector spaces. By Theorem 2.8 a nowhere vanishing section s' exists if $d_\tau \leqslant k-1$, which results to be true because of the hypothesis $k > d_\tau$. Moreover, s' is an $\mathscr{R}$-section, that is, $\tau_E \circ s' \circ \tau|_{X^\tau} = \tau_E \circ s' = s'$. In fact, if there was $x \in X^\tau$ such that $s'(x) \neq \tau_E(s'(x))$, then $p(s'(x)) \neq p(\tau_E(s'(x)))$, but since p commutes with involutions of X and E, $p(\tau_E(s'(x))) = \tau|_{X^\tau}(p(s'(x))) = p(s'(x))$.

Therefore, the previous assumption leads to a contradiction. This also means that s' is an $\mathscr{R}$-section.
Finally, consider the relative $\mathbb{Z}$-CW-complex (X, X^τ). Since X^τ is a closed subset of X and the involution τ acts trivially on the fixed-point subcomplex, the condition $\tau(X^\tau) = X^\tau$ is satisfied. Hence, one can apply Theorem 4.2 and extend s' to an $\mathscr{R}$-section $s : X \to E$ such that $s|_{X^\tau} = s'$. This concludes the proof. □

Proposition 5.3. *Let (X, τ) be an involutive space such that X has a finite $\mathbb{Z}_2$-CW decomposition of dimension d and let d_τ be the dimension of the fixed-point subcomplex. Then each $\mathscr{R}$-vector bundle ξ of rank k over (X, τ) with $k > \max\{d_\tau, d/2\}$ splits as $\xi \cong \theta_{\mathscr{R}}^{k-m} \oplus \eta$, where η is an $\mathscr{R}$-vector bundle over (X, τ) of rank m, $m := \lceil \frac{d-1}{2} \rceil$ and $\theta_{\mathscr{R}}^{k-m}$ denotes the trivial $\mathscr{R}$-vector bundle of rank $k - m$.*

Proof. Let s be a global $\mathscr{R}$-section, which exists by the previous theorem. This defines a monomorphism $u : \theta_{\mathscr{R}}^1 \to \xi$ by $u(x, a) = a\, s(x)$ for eve-ry $(x, a) \in X \times \mathbb{C}$ (see the proof of Proposition 2.8). The map u is equivariant:

$$(u \circ \tau)(x, a) = u(\tau(x), \overline{a}) = \overline{a}\, s(\tau(x)) = \tau_E(a\, s(x)) = (\tau_E \circ u)(x, a)$$
$$\Rightarrow u \circ \tau = \tau_E \circ u.$$

Let η_1 denote $\operatorname{coker} u$. Then η_1 is a vector bundle of dimension $k - 1$ by Theorem 1.7 and it is endowed with the same $\mathscr{R}$-structure as ξ because of the equivariance of u. Since the base space X is paracompact, we can apply Theorem 1.8 to the short exact sequence $0 \to \theta_{\mathscr{R}}^1 \xrightarrow{u} \xi \xrightarrow{v} \eta_1 \to 0$ and thus get an isomorphism of $\mathscr{R}$-vector bundles $\xi \cong \theta_{\mathscr{R}}^1 \oplus \eta_1$. This is $\mathscr{R}$-compatible because such isomorphism is defined through equivariant maps (see also the proof of Theorem 1.8). Using now the same induction of the proof of Corollary 2.9 applied to the particular case of $c = 2$, the claim follows directly. □

The following Lemma 5.4 and Theorem 5.5 show that, in the case of vector bundles endowed with a "real" structure, the condition on the rank that ensures the equivalence between isomorphism and stable isomorphism of vector bundles belonging to this class is not affected by the presence of the even time-reversal symmetry $\mathcal{T}+$.

Lemma 5.4. *Let* (X, τ) *be an involutive space such that* X *has a finite* $\mathbb{Z}_2$-*CW decomposition of dimension* d *and let* d_τ *be the dimension of the fixed-point subcomplex. Let* $u, v : \theta^1_{\mathscr{R}} \to \xi$ *be monomorphisms of* $\mathscr{R}$-*vector bundles with* $k > \max\{d_\tau, d/2\}$. *Then* $\operatorname{coker} u$ *and* $\operatorname{coker} v$ *are isomorphic over* X.

Proof. As shown in Theorem 5.3, a monomorphism of vector bundles is determined by an $\mathscr{R}$-section of the subbundle of non-zero vectors ξ_0. Therefore, a homotopy between monomorphisms is also completely determined by an $\mathscr{R}$-section s of $\xi_0 \times [0,1] = (\xi \times [0,1])_0$ over $X \times \{0,1\}$ such that $s|_{X\times\{0\}} = u$ and $s|_{X\times\{1\}} = v$. The interval $I = [0,1]$ is here naturally endowed with the trivial action, that is, $\tau(t) = t$ for every $t \in I$. In other words, it is first necessary to find a nowhere vanishing $\mathscr{R}$-section of $\xi \times I$ over $X \times I$.

Consider now the fixed-point subcomplex $(X \times I)^\tau = X^\tau \times I$. As pointed out in the proof of the Theorem 5.2, the fibres over points of $X^\tau \times I$ are real vector spaces and, as a consequence, a nowhere vanishing $\mathscr{R}$-section s_1 of $X^\tau \times I$ is a nowhere vanishing real section. This exists if $d_\tau + 1 \leqslant (k+1) - 1 = k$ by Theorem 2.8, where we have also considered the fact that $\xi \times I$ has rank $(k+1)$ over the $(d_\tau + 1)$-dimensional CW-complex $X^\tau \times I$. This condition is guaranteed by the hypotheis $k > d_\tau$.

In order to prolong s_1 to a global section defined on the whole $X \times I$, consider the relative $\mathbb{Z}_2$-CW-complex $(X \times I, X^\tau \times I)$. Since the involution τ acts trivially on the closed fixed-point subcomplex, $\tau(X^\tau \times I) = X^\tau \times I$ and, by Theorem 4.2, there exists a global $\mathscr{R}$-section $s : X \times I \to E \times I$ which extends s_1. Moreover, s determines a monomorphism $w : \theta^1_{\mathscr{R}} \to \xi \times I$ given by $w(x,t,y) = y\, s(x,t)$ for every $(x,t,y) \in X \times I \times \mathbb{C} \setminus \{0\}$, which results to be equivariant since s is. Hence, $\operatorname{coker} w$ is endowed with an $\mathscr{R}$-structure.

Finally, one can conclude as in the proof of Lemma 2.10 that $\operatorname{coker} u$ and $\operatorname{coker} v$ are isomorphic in view of the isomorphisms: $\operatorname{coker} w|_{X\times\{0\}} \cong \operatorname{coker} u$ and $\operatorname{coker} w|_{X\times\{1\}} \cong \operatorname{coker} v$. □

Theorem 5.5. *Let (X, τ) be an involutive space such that X has a finite $\mathbb{Z}_2$-CW decomposition of dimension d, let d_τ be the dimension of the fixed-point subcomplex X^τ and set $m := \lceil \max\{d_\tau, d/2\} \rceil$. If ξ_1 and ξ_2 are two k-dimensional $\mathscr{R}$-vector bundles such that $k \geqslant m$ and $\xi_1 \oplus \theta^l_{\mathscr{R}} \cong \xi_2 \oplus \theta^l_{\mathscr{R}}$ for some $l \geqslant 1$, then ξ_1 and ξ_2 are isomorphic.*

Proof. The proof of this theorem proceeds by induction on $l \geqslant 1$ exactly as that of Theorem 2.7. By hypothesis, $k > m \geqslant \max\{d_\tau, d/2\}$. Hence, we can apply both Proposition 5.3 and Lemma 5.4.
The base case $l = 1$ goes as follows:

$$\xi_1^k \overset{\text{Prop.5.3}}{\cong} \operatorname{coker} u \oplus \theta^1_{\mathscr{Q}} \overset{\text{Lem.5.4}}{\cong} \operatorname{coker} v \oplus \theta^1_{\mathscr{Q}} \overset{\text{Prop.5.3}}{\cong} \xi_2^k \tag{5.4}$$

After that, the inductive step is the same as in the non-equivariant case, that is, Theorem 2.7. □

The involutive space over which the Bloch bundle is built is $(\mathbb{T}^d, \tau)$. In section 1.2, we have described a $\mathbb{Z}_2$-CW-complex decomposition of $(\mathbb{T}^d, \tau)$ such that there are fixed cells only dimension 0, that is, $d_\tau = 0$ and $\max\{d_\tau, d/2\} = d/2$. Therefore, given a quantum mechanical system with even time-reversal symmetry $\mathcal{T}+$, the classification up to isomorphism can be safely replaced by the weaker classification up to stable isomorphism any time the dimension of the corresponding Bloch bundle k is greater than 1 for 1- and 2-dimensional systems ($m = 1$) and greater than 2 for 3-dimensional systems like crystals, since in this case $m = 2$. These estimates are exactly the same found in Theorem 2.7 for complex vector bundles (case $c = 2$). In other words, the condition on the rank of $\mathscr{R}$-vector bundles under which stable isomorphism becomes isomorphism is the same as for complex vector bundles.

5.4 Stability properties of “Quaternionic” vector bundles

As for the case of “Real” vector bundles, we are now going to prove a series of propositions which will lead to the main result about the equivalence between isomorphism and stable isomorphism of vector bundles endowed, this time, with the odd time-reversal symmetry $\mathcal{T}-$.

Proposition 5.6 (Existence of a global $\mathscr{Q}$-pair of sections)**.** *Let (X, τ) be an involutive space such that X has a finite $\mathbb{Z}_2$-CW-complex decomposition of dimension d and let d_τ be the dimension of the fixed-point subcomplex. Let $\xi = (E, p, X)$ be a $\mathscr{Q}$-vector bundle over (X, τ) of rank $2k$. If $k > \max\{\frac{d_\tau}{4}, \frac{d+2}{4}\}$, there exists a pair of sections $(s_1, s_2) \in \Gamma(E)$ which is a global $\mathscr{Q}$-pair.*

Proof. The proof of this theorem follows the same steps of Theorem 5.2, with the only difference that in this case, instead of “Real” sections, $\mathscr{Q}$-pairs of sections are needed. In the following, we will sketch the proof focusing on the new parts.

As in Remark 18, observe that the *zero* section $s_0(x) = 0 \in E_x$ for all $x \in X$ is τ_E-invariant and therefore the subbundle $\xi|_0$ of non-zero vectors can be endowed through τ_E with a “Quaternionic” structure over (X, τ). The fibres of ξ_0 are isomorphic to $\mathbb{C}^{2k} \setminus \{0\}$, which is $2(2k-1)$-connected, i.e., $\pi_j(E_x) \cong 0$ for every $0 \leqslant j \leqslant 2(2k-1)$. Given a $\mathscr{Q}$-pair for ξ_0, this can also be seen as a global $\mathscr{Q}$-pair for the original bundle ξ. The claim is that this pair always exists when $k > \max\{\frac{d_\tau}{4}, \frac{d+2}{4}\}$.
First, suppose that $d_\tau = 0$ and proceede by induction on the dimension of the skeleton:

- In the base case of X^0, one defines a pair of vectors $(s'_1(x_j), s'_2(x_j)) \in E_{0,x} \times E_{0,x}$ with $s'_2(x_j) := \tau'(s'_1)(x_j)$ for fixed points $x_j = \tau(x_j)$ and for $j = 1, ..., N_0$; whereas for free pairs $\{x_j, \tau(x_j)\}$ one sets $s'_1 :=$

$(s'_1(x_j), s'_1(\tau(x_j))) \in E_{0,x} \times E_{0,\tau(x)}$ and $s'_2 := \tau'(s'_1)$. In both cases the result is a $\mathscr{Q}$-pair of $E_{0,x} \cup E_{0,\tau(x)}$.

- Assume the claim is true for the $\mathbb{Z}_2$-CW-subcomplex X^{j-1}, $1 \leqslant j \leqslant d$ and let $\tilde{\sigma}^j \cong \mathbb{Z}_2 \times \mathbb{D}^j$ be a free j-cell with equivariant characteristic map $\varphi : \mathbb{Z}_2 \times \mathbb{D}^j \to X$. The induced bundle $\varphi^*(\xi_0)$ over $\mathbb{Z}_2 \times \mathbb{D}^j$ has a $\mathscr{Q}$-structure since the map φ is equivariant. By the inductive hypothesis there is a $\mathscr{Q}$-pair of sections (s'_1, s'_2) of $\xi|_{X^{j-1}}$ which can be used to define a $\mathscr{Q}$-pair (σ'_1, σ'_2) on $\varphi^*(\xi_0)$ by $\sigma'_j := s'_j \circ \varphi$. Since the $\mathbb{D}^j$ is contractible, $\varphi^*(\xi_0)$ is trivial and therefore (σ'_1, σ'_2) can be identified with a pair of linearly independent equivariant maps $\mathbb{Z}_2 \times \partial\mathbb{D}^j \to \mathbb{C}^{2k} \setminus \{0\}$. Considering the inequalities $j - 1 \leqslant d - 1 \leqslant 4k - 4 = 2(2k - 2)$, the following hold:
  - $\pi_{j-1}(\mathbb{C}^{2k} \setminus \{0\}) \cong 0$ implies that the restriction of σ'_1 to $\{1\} \times \partial\mathbb{D}^j \to \mathbb{C}^{2k} \setminus \{0\}$ extends to a map $\{1\} \times \mathbb{D}^j \to \mathbb{C}^{2k} \setminus \{0\}$;
  - the fact that σ'_1 and σ'_2 are linearly independent implies that σ'_2 can be seen as a map $\mathbb{Z}_2 \times \partial\mathbb{D}^j \to (\mathbb{C}^{2k}/_{\langle\sigma'_1\rangle}) \setminus \{0\} \cong \mathbb{C}^{2k-1} \setminus \{0\}$ and since also $\pi_{j-1}(\mathbb{C}^{2k-1} \setminus \{0\}) \cong 0$, the restriction of σ'_2 to $\{1\} \times \partial\mathbb{D}^j \to \mathbb{C}^{2k-1} \setminus \{0\}$ extends to a map $\{1\} \times \mathbb{D}^j \to \mathbb{C}^{2k-1} \setminus \{0\}$ as well; by construction, this is also linearly independent with the prolongation of σ'_1.
  - impose the equivariant constraints $\begin{cases} \sigma'_2(-1, x) := (\tau_0\sigma_1)(1, x) \\ \sigma'_1(-1, x) := -(\tau_0\sigma_2)(1, x) \end{cases}$ in order to get a $\mathscr{Q}$-pair of sections (σ_1, σ_2) of $\varphi^*(\xi_0)$;
  - using the natural isomorphism of induced bundles $(\varphi_\xi, \varphi) : \varphi^*(\xi_0) \to \xi_0$, define a $\mathscr{Q}$-pair of sections (s''_1, s''_2) of $\xi_0|_{\overline{\tilde{\sigma}^j}}$ by $\varphi_\xi \circ \sigma_i = s''_i \circ \phi$ for $i = 1, 2$, which satisfies $s''_i|_{X^{j-1} \cap \overline{\tilde{\sigma}^j}} = s'_i$;
  - finally, define a global $\mathscr{Q}$-pair (s_1, s_2) of $\xi|_{X^{j-1} \cup \sigma^j}$ by the requirements that $s_i|_{X^{j-1}} \equiv s'_i$ and $s_i|_{\tilde{\sigma}^j} \equiv s''_j$. These sections are continuous by the weak topology property of CW-complexes.

The iteration of this procedure on all other free j-cells yields the claim on X^j; by induction it is also valid on $X^d = X$.

Suppose now that $d_\tau > 0$ and consider the fixed-point subcomplex X^τ. The fibres over its points are quaternionic vector bundles of dimension k in view of Proposition 4.5. By Theorem 2.8, a nowhere vanishing section $s'_1 : X^\tau \to E$ exists if $d_\tau \leqslant 4k-1$, condition that is guaranteed by the hypothesis $k > d_\tau/4$. Define also $s'_2 := \tau'(s'_1)$. Thus, (s'_1, s'_2) is a $\mathscr{Q}$-pair over X^τ.
Finally, consider the relative $\mathbb{Z}_2$-CW-complex (X, X^τ). Since $\tau(X^\tau) = X^\tau$, by Lemma 4.7 (s'_1, s'_2) extends to a $\mathscr{Q}$-pair (s_1, s_2) defined on the whole X. □

Proposition 5.7. *Let (X,τ) be an involutive space such that X has a finite $\mathbb{Z}_2$-CW decomposition of dimension d and $d_\tau = \dim X^\tau$. Then each $\mathscr{Q}$-vector bundle of rank $2k$ over (X,τ) with $k > \max\{\frac{d_\tau}{4}, \frac{d+2}{4}\}$ splits as $\xi \cong \theta_{\mathscr{Q}}^{2(k-m)} \oplus \eta$, where η is a $\mathscr{Q}$-vector bundle over (X,τ), $m := \lceil \frac{d+2}{4} \rceil$ and $\theta_{\mathscr{Q}}^{m-k}$ denotes the trivial $\mathscr{Q}$-vector bundle over $X \times \mathbb{C}^{2(k-m)}$ of rank $k-m$.*

Proof. Let $(s_1, s_2) \subset \Gamma(E)$ be a global $\mathscr{Q}$-pair of sections, which exists by Proposition 5.6. This defines a monomorphism $u : \theta_{\mathscr{Q}}^1 \to \xi$ by $u(x, (a_1, a_2)) := a_1\, s_1(x) + a_2\, s_2(x)$ for every $(x,a) \in X \times \mathbb{C}^2$; u is equivariant:

$$(u \circ \tau)(x, (a_1, a_2)) = u(\tau(x), (\overline{a_1}, \overline{a_2}) = \overline{a_1}\, s_1(\tau(x)) + \overline{a_2}\, s_2(\tau(x))$$
$$= \tau_E(a_1\, s_1(x)) + \tau_E(a_2\, s_2(x)) = \tau_E(a_1\, s_1(x) + a_2\, s_2(x)) = (\tau_E \circ u)(x, (a_1, a_2))$$
$$\Rightarrow u \circ \tau = \tau_E \circ u$$

Now denote by η_1 the $\operatorname{coker} u$, which is a vector bundle of dimension $2k-2$ by Theorem 1.7 and it is endowed with the same $\mathscr{Q}$-structure as ξ because of the equivariance of u. Since the base space X is paracompact, we can apply Theorem 1.8 to the short exact sequence $0 \to \theta_{\mathscr{Q}}^1 \xrightarrow{u} \xi \xrightarrow{v} \eta_1 \to 0$ in order to get an isomorphism of $\mathscr{Q}$-vector bundles $\xi \cong \theta_{\mathscr{Q}}^1 \oplus \eta_1$. Repeating the argument for η_1 and by iterating this procedure one gets the claim. □

Lemma 5.8. *Let (X,τ) be an involutive space such that X has a finite $\mathbb{Z}_2$-CW decomposition of dimension d and $d_\tau = \dim X^\tau$. Let ξ be a $\mathscr{Q}$-vector* *bundle of dimension $2k$ over (X,τ) and let $u,v:\theta^1_{\mathscr{Q}}\to\xi$ be monomorphisms of $\mathscr{Q}$-vector bundles with $k>\frac{d_\tau-3}{4}$. Then* $\operatorname{coker} u$ *and* $\operatorname{coker} v$ *are isomorphic over X.*

Proof. This proof is analogous to Lemma 5.4 for the case of $\mathscr{Q}$-pairs of sections. In particular, a homotopy between u and v can now be assumed to be completely determined by a global $\mathscr{Q}$-pair of sections (s_1,s_2) of the $\mathscr{Q}$-vector bundle $(\xi\times I)$ over $X\times I$ ($I=[0,1]$ endowed with the trivial action), such that

$$\begin{cases}(a_1s_1(x,0)+a_2s_2(x,0))|_{X\times\{0\}}=u(x,a_1,a_2)\\(a_1s_1(x,1)+a_2s_2(x,1))|_{X\times\{1\}}=v(x,a_1,a_2).\end{cases}$$

The requirement on these sections of being nowhere vanishing can now be skipped since it is implicit in the definition of a $\mathscr{Q}$-pair.

First, we consider the fixed-point subcomplex $(X\times I)^\tau=X^\tau\times I$. Proposition 4.5 shows that $\mathrm{Vec}^{2k+1}_{\mathscr{Q}}(X^\tau\times I,\mathrm{id})\cong\mathrm{Vec}^{(2k+1)/2}_{\mathbb{H}}(X^\tau\times I)$. Therefore, a nowhere vanishing $\mathscr{Q}$-section s_1^* of $X^\tau\times I$ is also a non-equivariant section of the quaternionic vector bundle over $X^\tau\times I$ with half dimension. In view of Theorem 2.8, this exists if $d_\tau+1\leqslant 4(k+1)-1$, that is, if $d_\tau\leqslant 2(2k+1)$ and this is guaranteed by the hypothesis on k. Define also $s_2^*:=\tau'(s_1^*)$. Hence, (s_1^*,s_2^*) is $\mathscr{Q}$-pair over the subcomplex $X^\tau\times I$. We need now to extend it to a $\mathscr{Q}$-pair (s_1,s_2) over the whole $\mathbb{Z}_2$-CW-complex $X\times I$.

In order to do that, we consider the relative $\mathbb{Z}_2$-CW-complex $(X\times I, X^\tau\times I)$. Since τ is the trivial involution, the equivalence $\tau(X^\tau\times I)=X^\tau\times I$ is satisfied and we can use the extension Lemma 4.7 to get the desired global $\mathscr{Q}$-pair (s_1,s_2).

Finally, we can define an equivariant monomorphism $w:\theta^1_{\mathscr{Q}}\to\xi\times I$ given by $w(x,a_1,a_2,t)=a_1\,s_1^*(x,t)+a_2\,s_2^*(x,t)$ for every $(x,a_1,a_2,t)\in X\times\mathbb{C}^2\setminus\{0\}\times I$. Hence, $\operatorname{coker} w$ is endowed with a $\mathscr{Q}$-structure and the claim follows from Lemma 2.10. □

Theorem 5.9. *Let $m:=\lceil\max\{\frac{d_\tau}{2},\frac{d+2}{2}\}\rceil$. If ξ_1 and ξ_2 are two $2k$-dimensional $\mathscr{Q}$-vector bundles such that $2k>m$ and $\xi_1\oplus\theta^l_{\mathscr{Q}}\cong\xi_2\oplus\theta^l_{\mathscr{Q}}$ for some $l\geqslant 1$,* *then ξ_1 and ξ_2 are isomorphic.*

Proof. From the hypothesis $m := \lceil \max\{\frac{d_\tau}{2}, \frac{d+2}{2}\} \rceil$, it follows that $k > \max\{\frac{d_\tau}{4}, \frac{d+2}{4}\}$ and thus we can apply both Proposition 5.7 and Lemma 5.8. Therefore, by induction on $l \geqslant 1$:

- For the base case $l = 1$:

$$\xi_1^{2k} \overset{\text{Prop.5.7}}{\cong} \operatorname{coker} u \oplus \theta^1_{\mathscr{Q}} \overset{\text{Lem.5.8}}{\cong} \operatorname{coker} v \oplus \theta^1_{\mathscr{Q}} \overset{\text{Prop.5.7}}{\cong} \xi_2^{2k}$$

- The inductive step proceeds as in the proof of Theorem 2.7. □

The involutive space $(\mathbb{T}^d, \tau)$ of the Bloch bundle has fixed cells only dimension 0, which means that $d_\tau = 0$. Hence, the value of the constant m in the last theorem becomes $m = \lceil \frac{d+2}{2} \rceil$. The physical meaning of this discussion is that, given a quantum mechanical system with odd time-reversal symmetry $\mathcal{T}-$, the classification up to isomorphism is equivalent to the classification up to stable isomorphism whenever the corresponding Bloch bundle ξ has dimension $k' := 2k \geqslant \lceil \frac{d+2}{2} \rceil$. This is also the same estimate for a non-equivariant complex vector bundle over a base space X of two higher dimensions. For example, k' must be greater than 2 for 1 and 2-dimensional systems and greater than 3 for 3-dimensional systems.

Conclusions

The results obtained in Section 5.3 prove that the presence of the even time-reversal symmetry on a quantum mechanical system does not alter the stability properties of the corresponding Bloch bundle. This implies that Theorem 2.7 is sufficient to deduce the condition on the rank k of an $\mathscr{R}$-vector bundle under which isomorphism and stable isomorphism are equivalent.

For quantum systems with odd time-reverse symmetry, Theorem 5.9 states that in this case the lower bound on the rank of stable $\mathscr{Q}$-vector bundle has to be slightly higher, specifically, it agreed with the lower bound on the rank of a complex vector bundle without symmetries over a base space X of two higher dimensions.

The following table synthesizes these results for the case of low dimensional systems:

	$d = 1$	$d = 2$	$d = 3$	$d = 4$
complex vector bundles (no symmetries)	1	1	2	2
$\mathscr{R}$-vector bundles ($\mathcal{T}+$ symmetry)	1	1	2	2
$\mathscr{Q}$-vector bundles ($\mathcal{T}-$ symmetry)	2	2	3	3

Table 5.2: *The table indicates the values of the constant m such that stably isomorphic Bloch bundles (with or without symmetries) of rank $k \geqslant m$ are also isomorphic. In the first line, d is the dimension of the base space X.*

As introduced in Section 5.1, quantum systems can be endowed with other types of symmetries. An interesting development of this thesis would be to extend the results of Chapter 5 about the stability properties of particular vector bundles to all the other classes of symmetries.

Bibliography

[AKH] A. Akhmerov, *Topology in Condensed Matter*, TU Delft OpenCourseWare (2003).

[AZ] A. Altland and M. R. Zirnbauer *Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures*, Phys. Rev. B **55** (1997), 1142–1161.

[AT] M. F. Atiyah, *K-theory and reality*, Q. J. Math. **17** (1966), 367–386.

[BEL] J. Bellissard, D.J.L. Herrmann and M. Zarrouati, *Hull of aperiodic solids and gap labeling theorems* in *Directions in mathematical quasicrystals*, CRM monograph series **13** (2000), 207–258.

[BL] A. Blumberg, *Equivariant homotopy theory*, Lecture Notes in Algebraic Topology, The University of Texas at Austin (2017).

[BR1] G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, Springer-Verlag (1997).

[BR2] G. E. Bredon, *Equivariant Cohomology Theories*, Lecture Notes in Mathematics, Springer-Verlag (1967).

[DG1] G. De Nittis and K. Gomi, *Classification of "Real" Bloch-bundles: topological quantum systems of type* **AI**, J. Geom. Phys. **86** (2014), 303–338.

[DG2] G. De Nittis and K. Gomi, *Classification of "Quaternionic" Bloch-bundles: topological quantum systems of type* **AII**, Comm. Math. Phys. **339** (2015), 1–55.

[DU] J. L. Dupont, *Symplectic Bundles and KR-Theory*, Math. Scand. **24** (1969), 27-30.

[EM] E. E. Ewert and R. Meyer, *Coarse geometry and topological phases*, Commun. Math. Phys, (2019) 1-30. *Doi*: https://doi.org/10.1007/s00220-019-03303-z

[HAT] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002).

[HUS] D. Husemöller, *Fibre bundles*, 2nd ed., Graduate Texts in Mathematics, Springer-Verlag (1975).

[KEL] J. Kellendonk, *On the C^*-Algebraic Approach to Topological Phases for Insulators*, Ann. Henri Poincaré **18** (2017), 2251–2300.

[KEN] R. Kennedy, *Homotopy theory of topological insulators*, Ph.D Thesis, Universität zu Köln (2014).

[LEE] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag (2003).

[MA1] J. P. May, *A Concise Course in Algebraic Topology*, The University of Chicago Press (1999).

[MA2] J. P. May, et al., *Equivariant Homotopy and Cohomology Theory*, NSF-CBMS Regional Conf. Ser. in Math. **91** (1996).

[ME1] R. Meyer, *Lecture Note on Functional analysis*, Universität Göttingen (2018).

[ME2] R. Meyer, *Modelling topological phases of matter*, Universität Göttingen (2019)

[MUK] A. K. Mukhopadhyay, *Wigner's Symmetry Representation Theorem*, Reasonance **19** (2014), 900–916.

[MUT] T. Muthukumar, *Bloch-Floquet Transform*, Lecture Notes in Mathematics, Indian Institute of Technology Kanpur (2014).

[RS] S. Ryu, et al., *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, New J. Phys. **12** (2010), 065010.

[SHA] J. M. Shah, *Equivariant Algebraic Topology*, Preprint (2010).

[USM] K. Ueno, K. Shiga and S. Morita, *A Mathematical Gift, I: The interplay between topology, functions, geometry, and algebra*, AMS (2003).