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On the quantum Entanglement: a geometrical perspective.

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Abstract

Nella tesi viene affrontato il problema dell'entanglement da un punto di vista geometrico, usando sia la geometria differenziale che la geometria algebrica. Particolare attenzione viene data al problema della separabilità: ovvero il distinguere se uno stato è entangled o separabile. Nel primo capitolo si introduce il formalismo geometrico che verrà usato per analizzare la struttura della meccanica quantistica e dell'entanglement: vengono presentati elementi di geometria differenziale complessa, geometria proiettiva e geometria algebrica. Nel secondo capitolo, dopo un breve riepilogo sulla meccanica quantistica, vengono usati gli strumenti introdotti nel capitolo precedente per costruirne ed analizzarne la struttura differenziale. Nel terzo capitolo l'entanglement viene studiato con alcuni esempi ed applicazioni con metodo tradizionale, dopo di che anche gli aspetti geometrici vengono analizzati. Infine, nell'ultimo capitolo viene proposto un nuovo approccio di tipo algebrico derivato dalla dualità di Schur - Weyl.

Contents

Introduction

About thirty years passed from the birth of Quantum Mechanics to its almost complete mathematical formalization in 1932, year of the pubblication of Von Newmann's work "Mathematical Foundations of Quantum Mechanics" (originally titled "Mathematische Grundlagen der Quantenmechanik").

Three years later Einstein, Podolsky and Rosen, analysing the mathematical structure of this theory, came to the conclusion that Quantum Mechanics might not be a complete theory [11]. The main motivation that led them to this conclusion was that the quantum formalism permits to know exactly the state of a compound system leaving unknown the states of its single parts. Their paper was the starting point of a intense debate in the physicist world: While Einstein refused this conclusion and proposed the hidden variable theory, Schrödinger accepted this phenomenon and called it "Verschränkung" (translated in English by Schrödinger himself with the term Entanglement) [36]. This debate never came to an end and many results, both experimental and theoretical, were obtained from its start.

In this work we will analyse from a geometrical point of view the open problem of quantum entanglement, and in particular we will focus on the separability problem, which consists in understanding if a composite state is formed by individual subsystems's states.

Nowadays entanglement is no more a purely theoretical debate, it has become a resource: in fact entangled states have a great importance in Quantum Information Theory, Quantum Computation, Quantum Cryptography and many other fields. One of the first experimental results was obtained in 1982 [3] when couples of entangled photons were created for the first time, the idea to use quantum entanglement for communications was proposed in 1993 [6] using a sort of teleportation. Nowadays the reached frontier is the experiment performed in 2016 by a research group based in China, who put into orbit a communication satellite using entanglement to communicate over more then 1200 km [2]. A second perspective is quantum computation, born as an idea of Richard Feynman [13] in 1982. Now it has become one of the main frontier in computation: in 2001 the first quantum computer was born and it was formed by a system with seven qubits, this year Google Quantum Al Lab has announced a quantum processor with 72 qubits called "Bristlecone".

The geometrical approach to Quantum Mechanics is needed to explain many effects that arise in quantum theory and which have a geometrical explanation (as

the topological quantum effects); and one of the goals of this approach is standardizing the framework between most of the classical theories (such as the General Relativity) and Quantum Mechanics. It does not have big formal difficulties in the finite dimensional systems and it yields interesting results.

Without a geometrical frame many quantum effects cannot been explained so this perspective inevitably enriches our knowledge of the quantum world. The necessity of a geometrical and topological approach to quantum mechanics is evident in the treatment of topological phase of quantum matter, the quantum Hall effect [33], which is strictly correlated to the topological approach, and the well known Berry phase.

The state of the art of the geometrical description of quantum mechanics describes the Hilbert space of a system not to be the real arena of quantum mechanics, in fact a state is represented by a ray in the Hilbert space [12]; this evidence leads to a splitting of this space in equivalence classes that form a Projective Space. The geometrical structure of the Projective Spaces is well known both differential (it is a Kähler manifold) and algebraic points of view. The passage from the pure states to the mixed states is not trivial and it gives rise to a rank stratification of the set of Hermitian operators each of which is endowed with a submanifold structure inherited from the manifold of Hermitian operators [19].

The set of separable pure states is described by the Segre Embedding which rises only thanks to the Projective Structure [18] of the set of pure states; this description is particularly simple and characterized by a series of quadratic equations. The separability problem for the density states is not easy as the pure case: a complete description is known only for low dimensional systems [35] such as a system with two qubits or a system with a qubit and a qutrit. For higher dimensional system there are many characterizations provided by many criteria but no one is able to describe in a complete way the set of the separable mixed states, moreover it was proved that the separability problem is NP-hard [16].

In the first chapter we present the mathematical tools that are needed to understand this problem from this point of view. We introduce the main definitions that characterize complex manifolds in order to arrive at the description of Kähler manifolds that own a rich geometrical structure. Subsequently fibre bundles are introduced: they are object that locally, but not globally, can be considered Cartesian products of spaces. At the end of this first chapter we present some fundamental notions of algebraic geometry and projective geometry, that help to understand the entanglement in its mathematical construction.

In the second chapter, the mathematical elements of the first chapter are used to give a formal geometrical description of the objects which play a role in quantum mechanics. After a brief review of the traditional mathematical formalism of Quantum Mechanics, we show how a principal bundle structure on the original Hilbert space arises with the use of the postulates. This construction is followed by a differential description of the pure states space and the density states space which

gives rise to a peculiar structure of a convex set formed by sub-manifolds. In order to have a geometrical treatment we deal with the orbit method, which classifies the possible density matrices on the basis of the action of the unitary group and consequently on eigenvalues.

In the third chapter the leading argument is finally quantum entanglement. An introduction of this effect that addresses the nature, some application to entanglement and some elements about the separability criteria are followed by the geometrical description of the quantum entanglement, that this time uses also algebraic geometry and in particular the Segre Embedding, a map between Projective Spaces. A large part of this chapter is dedicated to the simplest system in which entanglement effects appear: a two qubits system, that is also one of the few systems that are completely solved. We present first a treatment of entanglement of the pure states and then the entanglement that arises in the case of mixed states.

In the last chapter we will draw some conclusions, by presenting also the outline of a new algebraic approach to the separability problem, which tackles this question via representation theory of the symmetric group and the Weyl-Schur duality. This approach needs to be further pursued before it can yield original results, but we hope it can provide new powerful techniques of investigation.

Chapter 1

The Geometrical Framework

The main aim of this chapter is to introduce the reader into the mathematical framework of this thesis: in the first part elements of complex differential geometry, symplectic spaces and Kähler manifolds will be introduced; in the second part the main topic will be fibre bundles with particular attention for principal bundles. A good review of preliminary notions, such as Riemannian geometry or calculus on manifold, is presented in [34], from which most of the content of this chapter come from. In the last part some algebraic geometry notion will be introduced to understand entanglement from this perspective, which we will discuss in chapter 3; [23] and [25] will be followed for this section.

1.1 Complex Manifolds

In order to analyse the complex differential structures let's recall the Cauchy Riemann conditions that characterize homomorphic functions.

Definition 1.1. *A complex valued function:*

$$f : \mathbb{C}^n \to \mathbb{C}; \tag{1.1}$$

$$x^\nu + iy^\nu \to f(x^\nu + iy^\nu) = u(x^\nu + iy^\nu) + iv(x^\nu + iy^\nu) \quad \{1 \leq \nu \leq n\}; \tag{1.2}$$

is holomorphic if:

$$\frac{\partial u}{\partial x^\nu} = \frac{\partial v}{\partial y^\nu}; \qquad \frac{\partial u}{\partial y^\nu} = -\frac{\partial v}{\partial x^\nu}. \tag{1.3}$$

This definition can be easily generalized to functions:

$$F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m; \tag{1.4}$$

by requiring that all $f_j : \mathbb{C}^n \to \mathbb{C}$ are holomorphic.

Now we can pass to the geometry formalism of complex structures.

1.1.1 First Definitions

Definition 1.2 (Complex manifold)**.** *M is an m-dimensional complex manifold if:*

1. *M is a topological space;*

2. *There exists a set* $\{(U_i, \phi_i)\}_i$ *of analytic charts:*

$$\phi_i : U_i \to \mathbb{C}^m; \tag{1.5}$$

$$p \longmapsto \phi(p) = \{z^\mu(p)\} \quad \{1 \leq \mu \leq m\}; \tag{1.6}$$

called atlas of the manifold: moreover the atlas form a complete overlap of the manifold:

$$\bigcup_i U_i = M. \tag{1.7}$$

3. *If there are two charts:* (U, ϕ) *and* (V, ψ) *with a non trivial intersection* $U \cup V \neq \emptyset$*; the compatibility function:*

$$\chi_{UV} = \phi^{-1} \circ \psi : M \to M; \tag{1.8}$$

is holomorphic.

Thus a complex manifold is locally homeomorphic to $\mathbb{C}^n$ and this leads to the identification of n as the complex dimension of the manifold; moreover the passage from a local system of coordinates to another is holomorphic and this ensures the preservation of the same complex structure of the manifold in all charts.

While the first example of complex manifold is $\mathbb{C}^n$, a more interesting example is the two dimensional sphere, $\mathbb{S}^2$.

Example 1. *A first example of a complex manifold is* $\mathbb{S}^2$*, which can be identified with* $\mathbb{C}^2 \cup \{+\infty\}$ *[34]. This object is first of all a differential manifold (see definition B.1); and two stereographic projections can be used as charts to give an atlas that covers* $\mathbb{S}^2$*. The first map is the projection from a point* $P(x, y, z) = (X_1, X_2) \in \mathbb{S}^2 \setminus \{NorthPole\}$*, and can be written as:*

$$(X_0, X_1) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right); \tag{1.9}$$

the second map can be built up in similar way considering the open set $\mathbb{S}^2 \setminus \{SouthPole\}$*:*

$$(Y_0, Y_1) = \left(\frac{x}{1+z}, \frac{-y}{1+z}\right). \tag{1.10}$$

Now it is time to see the complex structure of this manifold: on the open sets previously introduced we can define complex coordinates:

$$Z = X_0 + \imath X_1 \quad \overline{Z} = X_0 - \imath X_1 \tag{1.11}$$

$$W = Y_0 + \imath Y_1 \quad \overline{W} = Y_0 - \imath Y_1 \tag{1.12}$$

It can be seen that the change of coordinates is holomorphic:

$$W = \frac{x}{1+z} + \imath\frac{(-y)}{1+z} = \frac{x - \imath y}{1+z} = \frac{1-z}{1+z}(X_0 - \imath X_1) = \frac{1}{Z}; \tag{1.13}$$

that proves the holomorphic propriety on the intersection of the two open sets.

1.1.2 Almost complex structure

It has already been shown in the previous example that a complex manifold is also a differential manifold, in order to see it is enough to compare the definition 1.2 of a complex manifold and the definition B.1 of a real differential manifold. In practice this identification comes from the diffeomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ (it can be seen also in the example 1 where we use the differential structure of $\mathbb{S}^2$ to build up its structure as complex manifold). So let M be a complex manifold of $\dim_{\mathbb{C}} = m$ and p a point of it, with local coordinates:

$$\mathrm{p} = (z^1, \ldots, z^m) \text{ where } z^\mu = x^\mu + iy^\mu; \tag{1.14}$$

the tangent space in this point $T_{\mathrm{p}}M$ is spanned by:

$$\left\{ \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\mu} \right\}, \quad 1 \leq \mu \leq m; \tag{1.15}$$

therefore we have that the tangent space is a differential manifold.

It is possible to pass to complex coordinates also for the tangent space, that correspond to the $2m$ vectors:

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu} \right) \tag{1.16}$$

and its complex conjugates; the 2m vectors form a basis for the complex vector space $T_{\mathrm{p}}M^{\mathbb{C}}$ and it gives a complex structure to the tangent space.

The same holds true for the dual $T^*_{\mathrm{p}}M$, which is spanned by:

$$\{dx^\mu, dy^\mu\}, \quad 1 \leq \mu \leq m. \tag{1.17}$$

that with their complex conjugates form a basis for $T^*_{\mathrm{p}}M^{\mathbb{C}}$. A similar procedure can be repeated to define:

$$dz^\mu = dx^\mu + i dy^\mu. \tag{1.18}$$

That construction is equipped with the canonical duality relations:

$$\left\langle dz^\mu, \frac{\partial}{\partial z^\nu} \right\rangle = \delta^\mu_\nu; \tag{1.19}$$

$$\left\langle d\bar{z}^\mu, \frac{\partial}{\partial z^\nu} \right\rangle = 0. \tag{1.20}$$

Remark 1.1. *Let us remark that the tangent and the cotangent spaces of a complex manifold are constructed from its real structure, while its complex structure is inherited from the original complex manifold.*

Definition 1.3 (Almost Complex Structure)**.** *Let be M a complex manifold there is a map:*

$$J_{\mathrm{p}} : T_{\mathrm{p}}M \to T_{\mathrm{p}}M, \tag{1.21}$$

that acts as:

$$J_{\mathrm{p}}\left(\frac{\partial}{\partial x^{\mu}}\right) = \frac{\partial}{\partial y^{\mu}} \quad J_{\mathrm{p}}\left(\frac{\partial}{\partial y^{\mu}}\right) = -\frac{\partial}{\partial x^{\mu}}. \tag{1.22}$$

It is clear that:

$$J_{\mathrm{p}}^2 = -id_{T_{\mathrm{p}}M}; \tag{1.23}$$

and moreover it is independent on the chart taken.

It is possible to define a smooth tensor field that in every point takes the form of the map defined in 1.3, and this tensor J defines completely the complex structure of the manifold.

We can write the J_{p} in complex coordinates:

$$J_{\mathrm{p}} = dz^{\mu} \otimes \frac{\partial}{\partial z^{\mu}} - d\bar{z}^{\mu} \otimes \frac{\partial}{\partial \bar{z}^{\mu}}; \tag{1.24}$$

that leads to an important propriety of the action of the almost complex structure: the complex tangent space $T_{\mathrm{p}}M^{\mathbb{C}}$ is splitted in two parts:

$$T_{\mathrm{p}}M^{+} = \left\{ Z \in T_{\mathrm{p}}M^{\mathbb{C}} | J_{\mathrm{p}}Z = iZ \right\}, \tag{1.25}$$

$$T_{\mathrm{p}}M^{-} = \left\{ Z \in T_{\mathrm{p}}M^{\mathbb{C}} | J_{\mathrm{p}}Z = -iZ \right\}, \tag{1.26}$$

the two parts are disjoint:

$$T_{\mathrm{p}}M^{\mathbb{C}} = T_{\mathrm{p}}M^{+} \oplus T_{\mathrm{p}}M^{-}. \tag{1.27}$$

For the vectors fields the same splitting holds:

$$\chi(M)^{\mathbb{C}} = \chi(M)^{+} \oplus \chi(M)^{-}. \tag{1.28}$$

1.2 Complex differential forms

A complex manifold, as it was already shown, is a differential manifold, therefore we can define the space of the complex differential forms. Let be $\omega, \xi \in \Omega_{\mathrm{p}}^{q}(M)$ two differential form in the point $\mathrm{p} \in M$; a complex q-form can be defined as:

$$\eta = \omega + i\xi; \tag{1.29}$$

the vector space of the complex differential form is denoted as $\Omega_{\mathrm{p}}^{q}(M)^{\mathbb{C}}$. If now an almost complex manifold will be considered, it is useful to recall the splitting of $T_{\mathrm{p}}M^{\mathbb{C}}$:

$$T_{\mathrm{p}}M^{\mathbb{C}} = T_{\mathrm{p}}M^{+} \oplus T_{\mathrm{p}}M^{-}. \tag{1.30}$$

Now we are ready to see that this splitting leads to a different labelling of the vector space of the differential forms $\Omega^q_{\mathrm{p}}(M)^{\mathbb{C}}$:

Definition 1.4. *Let be M a complex manifold with $dim_{\mathbb{C}}(M) = m$. Let be $\eta \in \Omega^q_{\mathrm{p}}(M)^{\mathbb{C}}$ with $q \leq 2m$ and consider two indexes $r + s = q$. Let's define q vectors $X_i \in T_pM$ that can be in T_pM^+ or $T_{\mathrm{p}}M^-$. If the evaluation:*

$$\eta(X_1, \ldots, X_q) \neq 0, \tag{1.31}$$

only for r vectors in T_pM^+ and s vectors in T_pM^-, then η is called differential form of bidegree (r,s). The set of (r,s)-form at $p \in M$ is denoted by $\Omega^{r,s}_p(M)$, if a form can be defined on the whole manifold this space is denoted $\Omega^{r,s}(M)$.

The structure of $\Omega^q_{\mathrm{p}}(M)^{\mathbb{C}}$ derived form the labelling introduced in definition 1.4 is described in the following proposition.

Proposition 1.1. *If ξ and η are two complex differential form on a complex manifold M.*

- *If $\xi \in \Omega^{q,r}_p(M)$ then $\overline{\xi} \in \Omega^{r,q}_p(M)$.*
- *If we have two forms: $\xi \in \Omega^{q,r}_p(M)$ and $\eta \in \Omega^{q',r'}_p(M)$, then:*

$$\xi \wedge \eta = \Omega^{q+q',r+r'}_p(M). \tag{1.32}$$

- *A $\xi \in \Omega^q_p(M)$ can be written in a unique way as:*

$$\xi = \sum_{r+s=q} \xi^{r,s}; \tag{1.33}$$

where $\xi^(r,s) \in \Omega^{r,s}(M)$. Furthermore the following decomposition holds:

$$\Omega^q_p(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{r,s}_p(M). \tag{1.34}$$

1.2.1 Dolbeaut operators

Let us consider ω to be a (r,s)-form:

$$\omega = \omega_{\mu_1 \ldots \mu_r, \bar{\nu}_1 \ldots \bar{\nu}_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_s}; \tag{1.35}$$

the exterior derivative of this object can be calculated in local coordinates and it takes the form (here and in the following we use the notation in which repeated indices imply summation):

$$d\omega = \frac{1}{r!s!} \left(\frac{\partial}{\partial z^\lambda} \omega_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_s} dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda} \omega_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_s} d\bar{z}^\lambda \right) dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_s} \tag{1.36}$$

that is a mixture of a $(r+1,s)$ and a $(r,s+1)$ form. This splitting allows us to define two operators: the first maps a (r,s) form to a $(r+1,s)$ form, while the second maps a (r,s) form to a $(r,s+1)$ form.

Definition 1.5 (Doubeault Operators)**.** *The action of the external derivative can be separated into:*

$$d = \partial + \bar{\partial}; \tag{1.37}$$

and the actions of ∂ and $\overline{\partial}$ are given by:

$$\partial\omega = \frac{1}{r!}\left(\frac{\partial}{\partial z^{\lambda}}\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}dz^{\lambda}\right)dz^{\mu_1}\wedge\cdots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\cdots\wedge d\bar{z}^{\nu_s} \tag{1.38}$$

$$\overline{\partial}\omega = \frac{1}{s!}\left(\frac{\partial}{\partial \bar{z}^{\lambda}}\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}d\bar{z}^{\lambda}\right)dz^{\mu_1}\wedge\cdots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\cdots\wedge d\bar{z}^{\nu_s}; \tag{1.39}$$

This operators will be important when closed forms will be studied, in particular the Kähler one. The holomorphic forms can be now defined:

Definition 1.6. *An $\eta \in \Omega^{r,0}$ such that $\overline{\partial}\eta = 0$ is called holomorphic r-form.*

1.3 Hermitian and Kähler Manifolds

Now the metric structure of the complex manifold will be studied, let's start by recalling the definition of a Riemannian metric.

Definition 1.7 (Riemannian Metric)**.** *Let be M a differential manifold, a Riemannian metric g is a two form that at each point is a map:*

$$g_p : T_pM \times T_pM \to \mathbb{R} \tag{1.40}$$

with the proprieties:

- $g_p(X,Y) = g_p(Y,X)$ *where* $X, Y \in T_pM$*;*
- $g_p(X,X) \geq 0$*;*
- $g_p(X,X) = 0 \iff X = 0$*.*

Let M be a complex manifold endowed by a Riemannian metric g; if complex coordinates are used and by taking two vectors $Z, W \in T_{\mathrm{p}}M$, which can be expressed as in equation (1.11):

$$Z = X + iY \quad W = U + iV; \tag{1.41}$$

then we will have:

$$g_{\mathrm{p}}(Z,W) = g_{\mathrm{p}}(X,U) - g_{\mathrm{p}}(Y,V) + i\left[g_{\mathrm{p}}(X,V) + g_{\mathrm{p}}(Y,U)\right]; \tag{1.42}$$

and from this expression the components of the metric can be calculated in local coordinates:

$$g_{\mu\nu} = g_{\mathrm{p}}\left(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial z^{\nu}}\right) \quad g_{\bar{\mu}\bar{\nu}} = g_{\mathrm{p}}\left(\frac{\partial}{\partial \bar{z}^{\mu}}, \frac{\partial}{\partial \bar{z}^{\nu}}\right);$$
$$g_{\bar{\mu}\nu} = g_{\mathrm{p}}\left(\frac{\partial}{\partial \bar{z}^{\mu}}, \frac{\partial}{\partial z^{\nu}}\right) \quad g_{\mu\bar{\nu}} = g_{\mathrm{p}}\left(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial \bar{z}^{\nu}}\right).$$

Now it is the time to introduce a particular kind of metric, that is useful for our mathematical framework.

Definition 1.8 (Hermitian Metric)**.** *Let M be a complex manifold and g be a Riemannian metric on it that satisfies for all $X, Y \in T_pM$ and for all $p \in M$:*

$$g_p\left(J_pX, J_pY\right) = g_p\left(X, Y\right) \tag{1.43}$$

then g is called Hermitian Metric.

The action of the almost complex structure on a vector X creates another vector $J_{\mathrm{p}}X$, clearly the last one is orthogonal of the first one under a Hermitian metric:

$$g_{\mathrm{p}}\left(J_{\mathrm{p}}X, X\right) = g_{\mathrm{p}}\left(J_{\mathrm{p}}^2X, J_{\mathrm{p}}X\right) = -g_{\mathrm{p}}\left(J_{\mathrm{p}}X, X\right) = 0. \tag{1.44}$$

This construction is not so uncommon as can be thought, the following statement ensures that a Hermitian structure can be built over every complex manifold [34].

Theorem 1.1. *Any complex manifold admits a Hermitian metric.*

The component of a Hermitian metric can be calculated using the previous proprieties, and it can be seen that:

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial z^{\nu}}\right) = g\left(J\frac{\partial}{\partial z^{\mu}}, J\frac{\partial}{\partial z^{\nu}}\right) = -g\left(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial z^{\nu}}\right) = 0; \tag{1.45}$$

and consequently the structure of the metric is:

$$g = g_{\mu\bar{\nu}} dz^{\mu} \otimes d\bar{z}^{\nu} + g_{\bar{\mu}\nu} d\bar{z}^{\mu} \otimes dz^{\nu}. \tag{1.46}$$

Let us now introduce the Kähaler form on a Hermitian manifold (M, g). It is the tensor field Ω that acts on $T_{\mathrm{p}}M$:

$$\Omega_{\mathrm{p}}\left(X, Y\right) = g_{\mathrm{p}}(J_{\mathrm{p}}X, Y); \tag{1.47}$$

from its construction the skew-symmetry of Ω is trivial.

Definition 1.9 (Kähler form)**.** *A Kähler form Ω is a two-form associated to a Hermitian metric, with the proprieties:*

$$\Omega(X,Y) = g(JX,Y) = g(J^2X,JY) = -g(Jy,X) = -\Omega(Y,X) \tag{1.48}$$

$$\Omega(JX,JY) = g(J^2X,JY) = g(J^3X,J^2Y) = \Omega(X,Y) \tag{1.49}$$

It is possible to express the Kähaler form in the complex base in order to see better its components:

$$\Omega = g_{\mu\bar{\nu}}dz^\mu \otimes d\bar{z}^\nu - g_{\bar{\mu}\nu}d\bar{z}^\mu \otimes dz^\nu = g_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu = -J_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu \tag{1.50}$$

The Kähler form and the Riemannian metric can be combined and give rise a rich differential structure on a manifold called Kähler manifold.

Definition 1.10 (Kähler Manifold)**.** *A real manifold M with $\dim_{\mathbb{R}} M = 2m$ is called Kähler manifold, denoted by (M,J,g,Ω), if it is equipped with:*

1. *An almost complex structure J as in definition 1.3;*
2. *and a Hermitian Metric;*
3. *A Kähler form Ω which is $d\Omega = 0$, and satisfies:*

$$\Omega(X,JY) + \Omega(JX,Y) = 0. \tag{1.51}$$

It is possible to verify [9] that on a manifold a Kähler structure can be constructed using the following statement.

Theorem 1.2. *Let (M,g,J) be a Hermitian manifold such that:*

$$\nabla_\mu J = 0, \tag{1.52}$$

where ∇_μ is the Levi Civita connection associated to g. Then (M,g,J) is a Kähler manifold.

In a Kähler manifold $d\Omega = (\partial + \bar{\partial})\Omega = 0$ so it is possible to deconstruct this condition using the Dolbeaut operators:

$$(\partial + \bar{\partial})ig_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu = \tag{1.53}$$

$$i\partial_\lambda g_{\mu\bar{\nu}}dz^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu + i\partial_{\bar{\lambda}}g_{\mu\bar{\nu}}d\bar{z}^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu \tag{1.54}$$

$$\frac{1}{2}i\left(\partial_\lambda g_{\mu\bar{\nu}} - \partial_\mu g_{\lambda\bar{\nu}}\right)dz^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu + \frac{1}{2}i\left(\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} - \partial_{\bar{\nu}} g_{\lambda\bar{\mu}}\right)d\bar{z}^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu; \tag{1.55}$$

that leads to the conditions:

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial z^\lambda} = \frac{\partial g_{\lambda\bar{\nu}}}{\partial z^\mu} \qquad \frac{\partial g_{\mu\bar{\nu}}}{\partial \bar{z}^\lambda} = \frac{\partial g_{\bar{\lambda}\mu}}{\partial \bar{z}^\nu}. \tag{1.56}$$

Example 2 (Complex Euclid Space). *If we consider $\mathbb{C}^n$ as a complex manifold, its structure as differential manifold can be reconstructed with the usual identification:*

$$z^\mu = x^\mu + iy^\mu; \tag{1.57}$$

that leads to $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. On $\mathbb{R}^{2n}$ can be introduced the Euclidean metric (that is a Riemannian metric):

$$g\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\mu}\right) = g\left(\frac{\partial}{\partial y^\nu}, \frac{\partial}{\partial y^\mu}\right) = g_{\mu\nu}; \tag{1.58}$$

while:

$$g\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial y^\mu}\right) = 0. \tag{1.59}$$

The action of the almost complex structure on this manifold is easy to see:

$$J\left(\frac{\partial}{\partial x^\nu}\right) = \frac{\partial}{\partial y^\nu}; \tag{1.60}$$

and consequently g is a Hermitian metric. If we recall the complex coordinates, we can write the action of the Euclidean metric on the complex tangent space as:

$$g\left(\frac{\partial}{\partial z^\nu}, \frac{\partial}{\partial \bar{z}^\mu}\right) = g\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}\right) = \frac{1}{2} g_{\mu\nu}; \tag{1.61}$$

while:

$$g\left(\frac{\partial}{\partial z^\nu}, \frac{\partial}{\partial z^\mu}\right) = g\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) = 0. \tag{1.62}$$

Now the Kähler form can be calculated using the compatibility relations of definition 1.9:

$$\Omega = \frac{i}{2} dz^\mu \wedge dz^\mu = \frac{i}{2} dx^\mu \wedge dy^\mu; \tag{1.63}$$

Clearly $d\Omega = 0$; so $\mathbb{C}^n$, endowed by the Euclidean metric, is a Kähler manifold.

1.4 Fibre Bundle

In this section the main features of fibre bundles will be explained, following also [9] in addition to [34] that has been the main reference until this moment. For useful notions on Lie groups see the relative section of Appendix B, while for a more detailed treatment see [39] or [37].

1.4.1 Main Definitions

Roughly speaking a fibre bundle is a topological space that locally can be seen as a Cartesian product of two spaces, all the the product spaces are fibre bundles but these special cases are called trivial bundles.

Definition 1.11. *A fibre bundle (E, π, M, F, G) is a collection of object:*

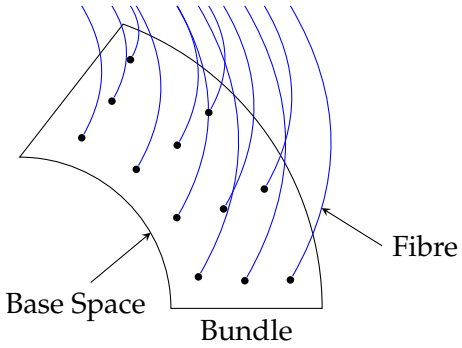


FIGURE 1.1: Fibre Bundle

1. *A manifold E called total space;*
2. *A manifold M called base space;*
3. *A manifold F called fibre;*
4. *A surjective map $\pi : E \to M$ called projection, the image of the inverse of $\pi^{-1}(p) = F_p \simeq F$, called fibre in the point p. (The relation $F_p \simeq F$ can be an homeomorphism or a diffeomorphism depending on the structure of the fibre bundle, where it will be not specified it will be assumed to be diffeomorphism).*
5. *A group G called structure group, that acts on F on the left (usually it will be taken to be a Lie Group, unless specified differently).*
6. *A set of couples $\{U_i, \phi_i\}$ formed respectively by open sets and maps (called local trivialization), such that:*

$$\phi_i : U_i \times F \to \pi^{-1}(U_i); \tag{1.64}$$

with the proprieties: $\pi \circ \phi_i(p, f) = p$. Note that the inverse ϕ_i^{-1} maps $\pi^{-1}(U_i)$ to the product $U_i \times F$ (also in this case ϕ_i can be a homeomorphism or a diffeomorphism, and this "choice" has to be coherent with the previous one).

7. *Let us denote $\phi_i(p, f) = \phi_{i,p}(f)$; the map:*

$$\phi_{i,p} : F \to F_p; \tag{1.65}$$

is a diffeomorphism. Let us consider two couples (U_i, ϕ_i) and (U_j, ϕ_j) with a non trivial intersection; we require the existence of:

$$G \ni t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F; \tag{1.66}$$

and so the two trivializations are related by:

$$t_{ij} : U_i \cap U_j \to G; \tag{1.67}$$

called transition function.

The transition functions have a group structure, and fulfil the so called compatibility relations:

- $t_{ii}(x) = id$ if $x \in U_i$
- $t_{ij}(x) = (t_{ji}(x))^{-1}$
- $t_{ij}(x)t_{jk}(x) = t_{ik}(x)$ if $x \in U_i \cap U_j \cap U_k$

The choice of the transition functions is not uniquely defined; if two different bundle E and E', with the same base space (M), same fibre (F), and same group structure (G); are considered, it is possible to take the couples (U_i, ϕ_i), related to E, and (U_i, ψ_i) related to E'; and construct the map:

$$\phi_i \circ \psi_i^{-1} : U_i \times F \to U_i \times F; \tag{1.68}$$

fixing a point $x \in U_i$ there is a homomorphism $\lambda_i(x)$ of bundles; which is also an element of the structure group G. The transition functions of the two bundles are related by the homomorphism:

$$t'_{ij} = \lambda_i^{-1} t_{ij} \lambda_j. \tag{1.69}$$

In order to study a bundle it is common to consider a map from the base space M and the total space E.

Definition 1.12 (Cross Section)**.** *Let be (E, M, π) (for brevity) a bundle. A cross section is a smooth map:*

$$s : M \to E; \tag{1.70}$$

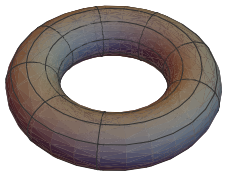
such that $\pi \circ s = id_M$.

In this map obviously $s(p) = s|_{\mathrm{p}}$ is an element of the local fibre $F_{\mathrm{p}} = \pi^{-1}(p)$, the set of all section of a bundle is $\Gamma(M, F)$. If we consider an open set U of M it can be possible to define a lots of local sections, but it is not common to have a section define over the whole bundle.

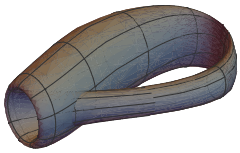
Example 3 (Tangent Bundle)**.** *An useful example of a fibre bundle is the tangent bundle, usually denoted by TM. Let us consider a differential manifold M, and an atlas $\{U_i, \phi_i\}$; if we consider a point p on M, its local coordinates in the open U_i will be given by $x^\mu = \phi_i(p)$; consequently a an element of $TU_i = \bigcup_{p \in U_i} T_pM$ is a couple (p, V), with $V \in T_pM$. The projection is given by:*

$$\pi(p, V) = p; \tag{1.71}$$

and does not depend on the choice of the local map, so it can be defined globally. An important remark has to be done: the tangent space T_pM takes now the role of the fibre of this bundle, while M is the base space; the usual local structure of the manifold ($M \simeq \mathbb{R}^n$) and of its tangent space ($T_pM \simeq \mathbb{R}^n$), allow us to identify $GL(n, \mathbb{R})$ as the Lie group involved in this construction.



(A) Torus.



(B) Klein Bottle.

FIGURE 1.2: Plots of example 4.

$$\begin{array}{ccc} G & \longrightarrow & E \\ & & \downarrow \\ & & M \end{array}$$

FIGURE 1.3: The usual structure of a principal Bundle

Example 4 (Klein Bottle)**.** *A more curious fibre bundle is the Klein Bottle [34]. By considering initially the manifold $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, the torus (figure 1.2a), looking on its product global structure we can see it as a trivial bundle; the structure group is the trivial group. The Klein Bottle (figure 1.2b) can be seen as a twisted torus: the base space and the fibre are both identified by $\mathbb{S}^1$ but the structure group is now defined by $\mathbb{Z}_2$.*

In a similar way the Möbius strip can be constructed.

1.4.2 Principal Bundle

Principal bundles are a special class of bundles with less object involved but they maintain the same rich structure.

Definition 1.13. *A principal Bundle, denoted by (E, π, M, G), is a structure:*

1. *A manifold E called total space;*
2. *A manifold M called base space;*
3. *A surjective and continuous function π:*

$$\pi : E \to M \tag{1.72}$$

   called projection.

4. *A lie Group G called fibre.*

It is called also a G bundle over M.

Recalling the definition 1.12, we can study the triviality (global product structure) of a principal bundle [34]:

Theorem 1.3. *A principal bundle is trivial $\iff$ $\exists$ a global section.*

1.5 Orbit Method

The action of a group is important, not only because it permits the construction of a bundle over a manifold; but also because it could create a stratification into equivalence classes. In this section elements of Orbit Method will be presented following [29]; for complements on Lie groups and Lie algebras see appendix B.

Let M be a differential manifold and G a Lie Group with a Lie algebra $\mathfrak{g}$ that corresponds to the T_eG the tangent space in the identity in G.

Definition 1.14 (Left Action)**.** *For any $g \in G$, $\tau(g) : M \to M$ is a diffeomorphism such that:*

$$\tau(g_1 g_2) = \tau(g_1) \circ \tau(g_2); \tag{1.73}$$

$$\tau(e)p = p \quad \forall p \in M. \tag{1.74}$$

Moreover the action is smooth.

Sometimes is more useful to define the action as:

$$\tau : G \times M \to M; \tag{1.75}$$

$$(g, p) \mapsto \tau(g)p. \tag{1.76}$$

Remark 1.2. *It will be denoted a general action of a group with $\tau(g)p = gp$.*

In the same way one can define a right action with the same proprieties. For our purposes we will denote:

$$L_g p = gp \qquad \text{Left Action;} \tag{1.77}$$

$$R_g p = pg^{-1} \qquad \text{Right Action.} \tag{1.78}$$

There are various types of action that can be classified in the following way:

Definition 1.15.

- *An action is called Transitive if $\forall p_1, p_2 \in M$, $\exists g \in G$ such that:*

$$gp_1 = p_2; \tag{1.79}$$

- *An action is called Free if $\forall g \neq e \in G$ if $\nexists p \in M$ such that:*

$$gp = p; \tag{1.80}$$

  that is the lack of any fixed point for every element of G different from the identity.

- *An action is called Effective if identity of the group is the only with a trivial action, that is $\forall p \in M$ than:*

$$gp = p \iff g = e \in G. \tag{1.81}$$

Remark 1.3. *It is important to emphasize that every free action is also effective, but the converse is not true: a simple example is given by the action of* $\mathbb{Z}_2 = \{1, -1\}$ *on the real number realized by the usual multiplication, this action is effective but not free since* $g = -1$ *has* $p = 0$ *as fixed point.*

Now let us have closer look at the action of a group on a manifold M. If the action is transitive then gp, by varying $g \in G$, becomes all possible points on M. In general the set of all possible points reached with the action of a group is called Orbit of the group:

Definition 1.16 (Orbit)**.** *Given a point* $p \in M$ *the orbit of p is defined as:*

$$O_p = \{q \in M | gp = q \forall g \in G\}. \tag{1.82}$$

Clearly $O_p \subseteq M$.

Definition 1.17 (Stabilizer)**.** *Let G be a Lie group that acts on a manifold M. The stabilizer of p,* $H(p)$ *is the set:*

$$H(p) = \{g \in G | \sigma(g, p) = p\}. \tag{1.83}$$

The stabilizer has many interesting proprieties:

- $H(p)$ is a subgroup of G;
- If G is a Lie Group, then $H(p)$ is a Lie subgroup;
- If the action of G is effective the stabilizer of all points are trivial.

Let us specify this construction for the action of G on its algebra $\mathfrak{g}$. The internal operation of $\mathfrak{g}$ is provided by the commutator of vectors:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{1.84}$$

$$(X, Y) \mapsto [X, Y] = XY - YX \tag{1.85}$$

The action of G on itself is transitive and can be made with L_g, R_g or an their combination called conjugation:

$$Cj_g = L_g \circ R_g : h \mapsto ghg^{-1}; \tag{1.86}$$

that leaves e in itself. This leads to an automorphism of the algebra.

Definition 1.18 (Adjoint Action)**.** *The adjoint action Ad is a map:*

$$Ad_g : \mathfrak{g} \to \mathfrak{g}; \tag{1.87}$$

and so we can define an homomorphism between G and $\mathcal{A}ut(\mathfrak{g})$ *(all the isomorphism from* $\mathfrak{g}$ *to it self) defined as:*

$$g \mapsto Ad_g. \tag{1.88}$$

$$\begin{array}{ccc} H(p) & \longrightarrow & G \\ & & \downarrow \\ & & O(p) \end{array}$$

FIGURE 1.4: The principal bundle structure of the group G.

This is called adjoint representation of the Lie group G on its algebra $\mathfrak{g}$.

The adjoint action can be differentiated (see Appendix B) and it is possible to define another action.

Definition 1.19. *The map:*

$$ad := (dAd)_e \tag{1.89}$$

is homomorphism from the Lie algebra $\mathfrak{g}$ *to the Lie algebra of the automorphism of* $\mathfrak{g}$.

Proposition 1.2. *Let* $X, Y \in \mathfrak{g}$, *then:*

$$ad(X)Y = [X, Y] \tag{1.90}$$

In an analogous way we can consider $\mathfrak{g}^*$ the dual of the algebra of G.

Definition 1.20. *A linear representation of* G *on* $\mathfrak{g}^*$ *is analogous to the adjoint and it is defined as the map:*

$$g \mapsto Ad_g^* = (Ad_g); \tag{1.91}$$

and considering a $\xi \in \mathfrak{g}^*$, *its action can be expressed as:*

$$\langle Ad_g^* \xi, X\rangle = \langle (Ad_g)\xi, X\rangle = \langle \xi, Ad_{g^{-1}} X\rangle; \tag{1.92}$$

where with $\langle \cdot, \cdot \rangle$ *is denoted the scalar product.*

Recalling the concepts of orbit and stabilizer it is possible to see the construction, for a $p \in \mathfrak{g}$:

$$O(p) \simeq G/H(p). \tag{1.93}$$

and the algebra of the stabilizer can be denoted with $\mathfrak{h}_\mathrm{p}$. This structure, considering the transitive action of G on the orbit, gives rise to a structure of a principal bundle with the total space G, the orbit $O(p)$ as the base space and with the fibre given by the stabilizer $H(p)$(see figure 1.4), in this case the projection is given by the coadjoint action:

$$\pi : g \mapsto Ad_g^* p \tag{1.94}$$

1.6 Projective Spaces

The projective space was introduced in the XVI century to explain the space seen by the human eye and introduced in mathematics in the first years of XVII century by Desargues.

Let's start with a formal description:

Definition 1.21. *A projective space $\mathbb{P}^n(K)$ over a field K, is the set in the one dimensional subspaces in the vector space K^{n+1}.*

We will deal mainly with complex projective spaces, so we will specify all the following for this structure.

Definition 1.22. *Let $V \simeq \mathbb{C}^n$ be a complex vector space, the associated projective space is:*

$$\mathbb{P}^{n-1}\mathbb{C} = (\mathbb{C}^n \setminus \{0\}) \,/\, \sim; \tag{1.95}$$

where the equivalence relation is defined by:

$$(x_0, \ldots, x_{n-1}) \sim (y_0, \ldots, y_{n-1}) \iff (x_0, \ldots x_{n-1}) = \lambda(y_0, \ldots, y_{n-1}); \tag{1.96}$$

in which $\lambda \in \mathbb{C}_0$ (the set of complex numbers without the 0).

A point in the projective space is then an equivalence class, sometimes denoted by $[x_0, \ldots, x_{n-1}]$. This description in a geometrical view is equivalent to consider any different straight line passing through the origin in the original space as a point of the new projective space.

There is another geometrical description for the projective space $\mathbb{P}^n\mathbb{C}$ as all the points of intersection of any straight lines in $\mathbb{C}^{n+1}$ including the point at infinity; in this way its straightforward to see that parallel lines have intersection.

Example 5. *While the projective space associated to $\mathbb{C}$ is trivial (equivalent to a single point), the first useful example for our purpose analyses $\mathbb{P}^1\mathbb{C}$ as all the equivalence classes in the form $[x_0, x_1]$.*

Now we can see that the projective space, by its construction, inherits a complex manifold structure. Let $z = [z_0, \ldots, z_{n-1}] \in \mathbb{P}^{n-1}\mathbb{C}$ the homogeneous coordinates; a chart $U_\mu \subset (C^n \setminus \{0\})$ in which $z^\mu \neq 0$ so we can use the inhomogeneous coordinate:

$$\phi^\nu_{(\mu)} = \frac{z^\nu}{z^\mu} \tag{1.97}$$

associated to the chart $(U_\mu, \phi_{(\mu)})$. The compatibility function between the two charts U_μ and U_λ in their intersection is:

$$\psi_{\mu\lambda} = \frac{z^\lambda}{z^\mu}\phi^\nu_{(\lambda)} \tag{1.98}$$

and it is holomorphic.

The differential structure of the projective space is deeper then the one seen so far, and as a matter of fact it is also a Kähler manifold. If we consider a chart $(U_\mu, \phi_{(\mu)})$ with the inhomogeneous coordinates, it's possible to define a function:

$$\mathcal{K}_\mu(p) = \sum_{\nu=1}^{m+1} \left| \frac{z^\nu}{z^\mu} \right|; \tag{1.99}$$

and if a point $p \in U_\mu \cap U_\alpha$ the two $\mathcal{K}$ functions are related each other by:

$$\mathcal{K}_\mu(p) = \left| \frac{z^\alpha}{z^\mu} \right|^2 \mathcal{K}_\alpha(p). \tag{1.100}$$

Taking the logarithm of the two sides we get:

$$\log(\mathcal{K}_\mu(p)) = \log(\mathcal{K}_\nu(p)) + \log\left(\frac{z^\mu}{z^\nu}\right) + \overline{\log\left(\frac{z^\mu}{z^\nu}\right)}. \tag{1.101}$$

For the holomorphic proprieties we know that:

$$\partial\bar{\partial}\mathcal{K}_\mu = \partial\bar{\partial}\mathcal{K}_\nu; \tag{1.102}$$

and so the closed two form Ω is:

$$\Omega = i\partial\bar{\partial}\log\left(\mathcal{K}_\mu\right); \tag{1.103}$$

and a Hermitian metric can be constructed with the Kähler form just defined considering two vectors $X, Y \in T_{\mathrm{p}}\mathbb{P}^{n-1}\mathbb{C}$ and defining a metric $g(X,Y) = \Omega(X, JY)$. The proof of the Hermitian propriety can be found in [34]. This is called Fubini-Study metric of $\mathbb{P}^{n-1}\mathbb{C}$.

1.7 Algrebric Geometry

Let be K a closed algebraic field (every not constant polynomial with coefficients in the field has a root in K). There are particular subsets on a projective spaces (definition 1.21) that have an important structure.

Definition 1.23 (Projective Varieties)**.** *A projective varieties $X \subset \mathbb{P}^n$ is the locus of a collection of homogeneous polynomials F_α.*

For locus we mean the set defined as all zeros of the set of F_α; this acquires a sophisticated structure when we find that the set of polynomial, and also the set of homogeneous polynomials, is a ring (a Field without the inverse for the multiplicative operation, e.g. $\mathbb{Z}$).

For the purpose of this thesis work, the algebraic fields that will be considered are $\mathbb{R}$ and $\mathbb{C}$; obviously dealing with $\mathbb{R}$ is not easy because is not algebraically closed. To understand these abstract construction we can study some examples.

Example 6 (Twisted Cubic)**.** *Let us consider the map:*

$$\phi : [x_0, x_1] \mapsto [x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3]; \tag{1.104}$$

clearly it is a map from $\mathbb{P}^1$ *to* $\mathbb{P}^3$*. If we are in* $\mathbb{P}^1\mathbb{C}$*, and choosing the inhomogeneous coordinates (imposing* $x_0 \neq 0$*) this map can be rewritten:*

$$\phi : t \mapsto (t, t^2, t^3). \tag{1.105}$$

To see how it can be a projective variety a set of polynomials has to be found. Let us consider $\mathbb{P}^3\mathbb{C}$*, a point of it has the coordinates* (z_0, z_1, z_2, z_3)*. The image of the map is given by the points with:*

$$\begin{aligned} z_0 &= x_0^3 \\ z_1 &= x_0^2 x_1 \\ z_2 &= x_0 x_1^2 \\ z_3 &= x_1^3; \end{aligned}$$

which are the locus of the polynomials:

$$\begin{aligned} P_0 &= z_0 z_2 - z_1^2; \\ P_1 &= z_1 z_3 - z_2^2; \\ P_2 &= z_0 z_3 - z_1 z_2; \end{aligned}$$

as it can be seen with a simple algebraic calculation. The locus is a curve which is called twisted cubic.

In order to visualize better this curve is convenient to pass from $\mathbb{C}$ *to* $\mathbb{R}$*. In figure 1.5 the real twisted cubic is plotted.*

Example 7 (Normal Rational Curve)**.** *A rational normal curve* $C \subset \mathbb{P}^n$ *is defined to be the image of the map:*

$$v_n : \mathbb{P}^1 \to \mathbb{P}^n \tag{1.106}$$

$$[Z_0, Z_1] \mapsto [Z_0^n, Z_0^{n-1} Z_1, \ldots, Z_0^1 Z_1^{n-1}, Z_1^n] = [Y_0, ..., Y_n] \tag{1.107}$$

C is the common zero locus of the polynomials:

$$Y_i Y_j - Y_{i-1} Y_{j+1}. \tag{1.108}$$

Clearly the twisted cubic represent the normal rational curve with d=3.

There is an important connection between an algebraic variety and a complex manifold: we have already shown that the complex projective space is a Kähler manifold, this imply that if X is a projective variety in $\mathbb{P}^n\mathbb{C}$ then it is also a compact Kähler manifold. In general it is not true that a Kähler manifold is a projective

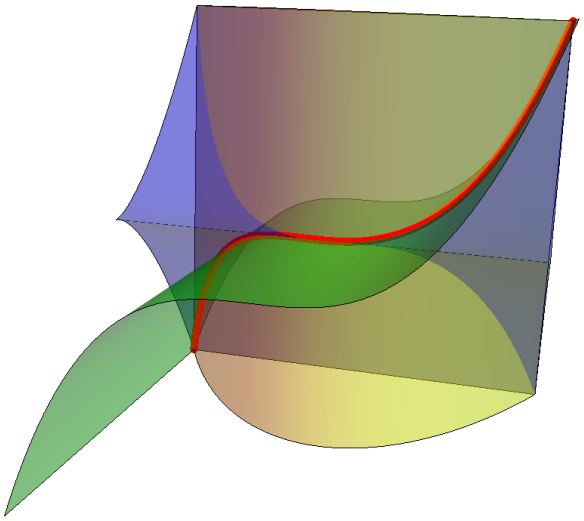


FIGURE 1.5: Twisted cubic for a real parameter: $z_2 - z_1^2 = 0$ in Yellow, $z_3 - z_1^3 = 0$ in Green, $z_2^3 - z_3^2 = 0$ in Blue and the twisted cubic in Red.

algebraic variety; there is a theorem that gives some condition to say if a Kähler manifold is a projective algebraic variety or not [24].

1.7.1 Determinantal Variety and Segre Varieties

Let us recall Example 7. There is a particular way to express the polynomials that describes the locus of a rational normal curve, which is important for our propose, it is the determinantal representation. The polynomials of that type of curve embedded in $\mathbb{P}^n$ can be recovered considering the minors of a matrix; fixing an integer k ($0 < k < n$) it is possible to define a matrix $(k+1) \times (n-k+1)$ of the type:

$$\begin{pmatrix} Y_0 & Y_1 & \dots & Y_k \\ Y_1 & Y_2 & \dots & Y_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n-k} & Y_{n-k+1} & \dots & Y_n \end{pmatrix} \tag{1.109}$$

where the Y_i are the homogeneous coordinates of the Projective space $\mathbb{P}^n$.

Example 8 (Determinantal Representation of Twisted Cubic)**.** *Let us reconsider the examples 6 and 7. The twisted cubic is the map:*

$$v_3 : \mathbb{P}^1 \to \mathbb{P}^3. \tag{1.110}$$

We can choose k uniquely equal to 1 or 2; this leads to the matrices:

$$\begin{pmatrix} Y_0 & Y_1 \\ Y_1 & Y_2 \\ Y_2 & Y_3 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} Y_0 & Y_1 & Y_2 \\ Y_1 & Y_2 & Y_3 \end{pmatrix}; \tag{1.111}$$

the two matrices has the very same three minors that leads to the polynomials:

$$\begin{aligned} P_0 &= Y_0 Y_2 - Y_1^2; \\ P_1 &= Y_1 Y_3 - Y_2^2; \\ P_2 &= Y_0 Y_3 - Y_1 Y_2. \end{aligned}$$

that are the same of the example 6.

Sometimes it is possible to generalize this representation for a variety $X \subset \mathbb{P}^n$, if it can be done then X is called a determinantal variety. For this propose let $A = \mathbb{P}^{nm-1}$ the projective space associated to the space M_{nm} of $n \times m$ matrices. Fixing an integer r that represents the rank of a matrix in M_{nm} we can define a projective variety, A_r, as the subset of all the matrices of rank r or less; it can be seen, imposing that all the r-minors be zero, as a subset of a projective variety defined by homogeneous polynomial and so A_r needs to be a projective variety.

One of the first examples that can be made are the Segre varieties: which are the image of the so called Segre map:

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}; \tag{1.112}$$

that is defined by the equation:

$$\mathbb{P}^n \times \mathbb{P}^m \ni ([X], [Y]) \overset{\sigma}{\mapsto} [\dots, X_i Y_j, \dots] \in \mathbb{P}^{(n+1)(m+1)-1}. \tag{1.113}$$

Remark 1.4. *It is important to emphasise that if we consider the projective space over a numerical field as $\mathbb{C}$ or $\mathbb{R}$, the Segre map become, using the complex field for example:*

$$\mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{(n+1)(m+1)-1}; \tag{1.114}$$

where $\mathbb{P}^n\mathbb{C}$ is the complex projective space associated to the complex space $\mathbb{C}^{n+1}$, $\mathbb{P}^m\mathbb{C}$, in analogy, is the complex projective space associated to the complex space $\mathbb{C}^{m+1}$ and $\mathbb{P}^{(n+1)(m+1)-1}\mathbb{C}$ is associated to $\mathbb{C}^{(n+1)(m+1)}$. The fundamental remark is that:

$$\mathbb{C}^{(n+1)} \otimes \mathbb{C}^{(m+1)} \simeq \mathbb{C}^{(n+1)(m+1)}; \tag{1.115}$$

and this propriety has a big consequence when we deal with quantum mechanics that will be treated in the chapter 3.

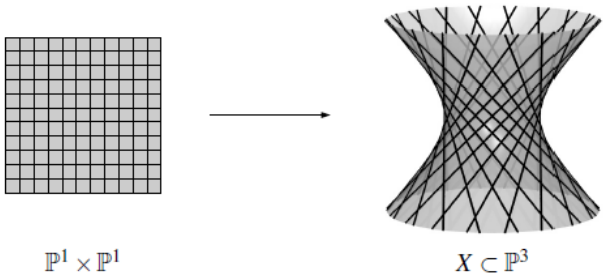


FIGURE 1.6: Plot of Segre embedding in a real projective space.

The Segre variety is the common zero locus of polynomial of the type:

$$Z_{i,j}Z_{k,l} - Z_{i,l}Z_{k,j} \tag{1.116}$$

where $Z_{i,j} = X_i Y_j$. This is enough to introduce the following proposition [25]:

Proposition 1.3 (Segre Embedding). *The product $\mathbb{P}^m(K) \times \mathbb{P}^n(K)$ is a projective algebraic variety and can be realized as the locus in $\mathbb{P}^{mn+m+n}(K)$ of the 2x2 minors of:*

$$\begin{pmatrix} Z_{00} & \dots & Z_{0n} \\ \vdots & \ddots & \vdots \\ Z_{m0} & \dots & Z_{mn} \end{pmatrix}; \tag{1.117}$$

vanishes. It is important to see $\mathbb{P}^{mn+m+n}(K)$ as the projective space of every $(m+1) \times (n+1)$ matrix, this leads to an embedding that maintains the matrix structure.

The collection of polynomials that define this projective variety are described by the determinant map of the all possible minor of size 2 and so represents all possible matrix of rank 1. The total number of this kind of minors is:

$$\binom{m}{2}\binom{n}{2}. \tag{1.118}$$

Example 9. *The first example of this construction is:*

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \tag{1.119}$$

$$([x_0, x_1], [y_0, y_1]) \mapsto [x_0y_0, x_0y_1, x_1y_0, x_1y_1]; \tag{1.120}$$

the equation in $\mathbb{P}^3$ is:

$$Z_{00}Z_{11} - Z_{10}Z_{01} = 0. \tag{1.121}$$

In order to see this variety it is convenient to choose the field of real numbers, so let us consider:

$$\sigma : \mathbb{P}^1\mathbb{R} \times \mathbb{P}^1\mathbb{R} \to \mathbb{P}^3\mathbb{R}; \tag{1.122}$$

the homogeneous coordinates of $\mathbb{P}^3\mathbb{R}$, that are $[Z_0, Z_1, Z_2, Z_3]$, can be parametrized in the

following way:

$$\begin{aligned} Z_0 &= X + U; \\ Z_1 &= X - U; \\ Z_2 &= V + Y; \\ Z^3 &= V - Y. \end{aligned}$$

Then equation (1.121) *leads to:*

$$Z_0 Z_3 - Z_1 Z_2 = X^2 + Y^2 - U^2 - V^2 = 0; \tag{1.123}$$

and going to the inhomogeneous coordinates, "fixing" V=1, the equation becomes:

$$X^2 + Y^2 - U^2 = 1; \tag{1.124}$$

that is the equation of a hyperboloid, as shown in figure 1.6. It is important to emphasise that the projection of the original projective spaces onto the Segre Varieties leads to linear subspaces (that holds also in general, not only in this example as can be seen in [5]).

In the following chapters these elements will be used to analyse the differential structure of quantum mechanics.

Chapter 2

Quantum Mechanics

In this first chapter we will first review some fundamental concepts of quantum mechanics with particular attention to the reformulation of its postulates using the density matrix; then we will discuss the space from a geometric point of view, first the pure state space, seen as a Khäler manifold, and second the space of mixed states: $\mathcal{D}(\mathcal{H})$. We will follow [17] and [22] for first section, while for the second and the third ones [12], [18] and [5].

2.1 Brief Review

In this first section the traditional mathematical formalism of quantum mechanics will be reviewed in order to define all the structure that will be seen with a geometrical view in the following sections.

2.1.1 Hilbert Space

Quantum mechanics, in the usual formulation, is based on five postulates that characterize the correspondence between mathematical framework and physical meaning. The arena of quantum mechanics is the Hilbert space:

Definition 2.1 (Hilbert space)**.** *A Hilbert Space $\mathcal{H}$ is an inner product space that is also a complete metric space with respect to the distance function induced by the inner product.*

The main Hilbert space used in quantum mechanics is $\mathcal{L}^2$ the square integrable function space defined on a domain $\Omega \subset \mathbb{R}^N$ whose scalar product is:

$$(\psi(x), \phi(x)) = \int_\Omega \psi^*(x)\phi(x)d^n x.$$

In many examples we will see the case of $\mathcal{H} = \mathbb{C}^n$ as an n-dimensional Hilbert space with the usual scalar product. The usual formalism in quantum mechanics is due to Dirac bracket in which: $|\psi\rangle \in \mathcal{H}$ and $\langle\psi| \in \mathcal{H}^*$ ($\mathcal{H}^*$ represent the dual space of the original Hilbert space). In this case the scalar product and the norm are given by:

$$\langle\psi|\phi\rangle\,; \quad \langle\psi|\psi\rangle = |\psi|^2 \tag{2.1}$$

2.1.2 Postulates

In this section we presents the five postulates in the Schrödinger picture as it is presented in [17] and [22]; but there isn't a unique choice of the postulates that characterize the quantum theory.

Postulate 1 (Physical State)**.** *A physical state is represented by a unit vector on a Hilbert space. If $|\psi\rangle$,$|\phi\rangle \in \mathcal{H}$ are two vectors and $|\psi\rangle = c\,|\phi\rangle$ for some $c \in \mathbb{C}_0$, then the two states are equal.*

Due to the structure of the Hilbert space as a vector space we know that if $|\psi\rangle$ and $|\phi\rangle$ are vectors in the Hilbert space also $\lambda_1\,|\psi\rangle + \lambda_2\,|\phi\rangle$ is a vector in the Hilbert space ($\lambda_1, \lambda_2 \in \mathbb{C}$).

It is clear that the first postulate reduces the set of the possible space from the Hilbert space $\mathcal{H}$ to a subset of normalized vectors with phase invariance; this propriety has a fundamental importance in the context of the next sections.

Postulate 2 (Measurement Process)**.** *An observable $\hat{O}$ of a quantum system is a Hermitian operator defined on $\mathcal{H}$. The possible outcomes are the eigenvalues λ_i of the operator. If the state is $|\psi\rangle$ we decompose:*

$$|\psi\rangle = \sum c_i\,|\boldsymbol{e}_i\rangle\,, \tag{2.2}$$

with the condition $\sum|c_i|^2 = 1$ and where the vectors $|\boldsymbol{e}_i\rangle$ are the eigenvector of the observable. The measure λ_i has the probability $P_{\lambda_i} = |c_i|^2$. After the measure the wave function collapse in the correspondent state (Destructive Measure).

The second postulate in combination with the spectral theorem for Hermitian operators allows us to find an orthonormal base $\{\boldsymbol{e}_i\}_i$ formed only by eigenvectors of the Operator: in practical way it makes us able to decompose the action of $\hat{O}$ in couples of subspaces (represented by a projector) and eigenvalues $(\hat{P}_{\lambda_i}, \lambda_i)$.

The mean value of a measure is given by the sum of all the measures weighted on the probability:

$$\langle\hat{O}\rangle = \sum \lambda_i P_{\lambda_i} = \sum \lambda_i |c_i|^2 = \langle\psi|\hat{O}|\psi\rangle\,. \tag{2.3}$$

Postulate 3 (Time Evolution)**.** *The time evolution of a state of a quantum system (in absence of measurement process) is given by the Schrödinger equation:*

$$i\hbar\frac{d\,|\psi\rangle}{dt} = \hat{H}\,|\psi\rangle\,; \tag{2.4}$$

in which $\hat{H}$ is the Hamiltonian of the system.

The usual situation is a time independent Hamiltonian (observable of Energy) $\hat{H}$, in this situation the solution of (2.4) is easy to find:

$$|\psi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}}\,|\psi_0\rangle\,; \tag{2.5}$$

where $|\psi_0\rangle$ is the initial state of the system defined by the fourth postulate. We notice that the probability is conserved: in fact the evolution operator is unitary. The scalar product is conserved:

$$\langle\psi(t)|\phi(t)\rangle = \langle\psi_0|e^{\frac{i\hat{H}t}{\hbar}}e^{\frac{-i\hat{H}t}{\hbar}}|\phi_0\rangle = \langle\psi_0|\phi_0\rangle\,. \tag{2.6}$$

Postulate 4 (Preparation of a System)**.** *A quantum system is said prepared if a complete commutating set of observables are diagonalized.*

The last postulate links respectively Bosons and Fermions with their spin values.

Postulate 5 (Spin)**.** *Bosons are identified with a symmetric wave-function and an integer value of spin. Fermions are identified with an anti-symmetric wave-function and a half-integer value of spin.*

2.1.3 Density Matrix and reformulation of Postulates

The formulation of quantum mechanics in the Hilbert space has some limits: experimentally either it is impossible to build up a complete set of observables or we cannot observe all particles of a system; in this way we will have some probabilities $p_1, ..., p_n$ ($\sum_i p_i = 1$ for a statistical reason) that the system is in a particular normalized state $|i\rangle$, such a statistical mixture can build up in the following way:

$$\rho = \sum_i p_i\,|i\rangle\,\langle i| \tag{2.7}$$

where no additional proprieties, a part from normalization ($\langle i|i\rangle = 1$), are required for the set of the possible state.

Remark 2.1. *It is important to remark the fact that no orthogonality between the states is required. Nevertheless, being ρ a Hermitian operator it can be diagonalized and rewritten as sum of orthogonal projectors.*

Is trivial to observe that a general density state ρ cannot be written as a vector state of an Hilbert space as we can see in the following example.

Example 10. *Lets consider a density matrix:*

$$\rho = \frac{1}{2}\,|1\rangle\,\langle 1| + \frac{1}{2}\,|2\rangle\,\langle 2|\,;$$

where in this case $\langle i|j\rangle = \delta_{i,j}$. *If there were a corresponding vector state* $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$ *that leads to* ρ *means that:*

$$\begin{aligned}\tilde{\rho} =& |\psi\rangle\langle\psi| \\ =&(c_1|1\rangle + c_2|2\rangle)(c_1^*\langle 1| + c_2^*\langle 2|) \\ =&|c_1|^2|1\rangle\langle 1| + c_1c_2^*|1\rangle\langle 2| + c_2c_1^*|2\rangle\langle 1| + |c_2||2\rangle\langle 2| \\ \neq&\rho;\end{aligned}$$

for any choice of c_1 *and* c_2.

So the set of the density states $\mathcal{D}(\mathcal{H})$ is bigger than the Hilbert space of the system and it is natural to rewrite the postulates in order to make them consistent with the new mathematical framework [22]. The states that can be written as elements of the original Hilbert space and so in the form $\rho = |\psi\rangle\langle\psi|$ are called pure state.

Postulate 6. *A physical state of a system is represented by a density matrix* ρ*: i.e. a semi-positive Hermitian matrix with unitary trace. A pure state of a system is represented by a density matrix* ρ *with the further request:* $\rho^2 = \rho$.

A pure state is represented by a projector, as it can be easily seen by looking at equation (2.7). The state now is an operator, so the natural way to see its time evolution is similar to the Heisenberg picture of evolutions of observables.

Postulate 7 (Time Evolution)**.** *The time evolution of a quantum state is given by the equation:*

$$\left[\rho, \hat{H}\right] = i\hbar\frac{d\rho}{dt} \tag{2.8}$$

whose solution is:

$$\rho(t) = e^{-i\hat{H}t/\hbar}\rho_0 e^{i\hat{H}t/\hbar} \tag{2.9}$$

where ρ_0 *is the density matrix at initial time* ($t = 0$).

Postulate 8 (Measurement Process)**.** *The expectation value of an observable* $\hat{O}$ *is:*

$$\mathrm{Tr}\left[\rho\hat{O}\right] = \langle\hat{O}\rangle\,. \tag{2.10}$$

This is consequently from (2.3) and the cyclic propriety of the trace.

The last axiom describes the collapse of the wave function:

Postulate 9 (Wave Function collapse)**.** *If at given moment the quantum system is described by the density matrix* ρ *and it is performed a measurement of the observable* $\hat{O}$ *with the outcome* o*, the density matrix will become* ρ'*:*

$$\rho' = \frac{1}{\mathrm{Tr}[P_o\rho P_o]}P_o\rho P_o; \tag{2.11}$$

where P_o *is the projector to the eigenspace of* $\mathcal{H}$ *with eigenvalue* $o \in \mathbb{R}$ *of the observable* $\hat{O}$.

$$\begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathcal{H} \simeq \mathbb{C}^n \\ & & \downarrow \\ U(1) & \longrightarrow & \mathbb{S}^{2n-1} \\ & & \downarrow \\ & & \mathbb{P}\mathcal{H} \simeq \mathbb{P}^{n-1}\mathbb{C} \end{array}$$

FIGURE 2.1: Fibration of the Hilbert space due to the equivalence classes.

Is straightforward to see that the only possible outcomes of a measurement of the observable $\hat{O}$ are the ones with $P_o \rho P_o \neq 0$.

2.2 Geometrization of Hilbert Space

Now we will use the differential geometry notions of the chapter 1 to describe quantum mechanics. The quantum systems that will be analysed are finite dimensional in order to avoid the difficulties due to the passage to the infinite dimensional Hilbert spaces; after this requirement we are able to identify the Hilbert space of the system with $\mathbb{C}^n$.

Until now we have used the bracket notation for quantum mechanics, in which a vector is denoted by $|a\rangle$ while the same object in differential geometry formalism is denoted by a^μ, we will pass from a notation to another when its convenient to understand the concept.

We will start by studying the geometrization of the Hilbert space $\mathcal{H}$ that can be done by using the first postulate: this construction gives rise to a principal fibration.

2.2.1 Fibration

After a complete measurement of the system the state remains undefined for a real scalar and for a phase: a measurement identify only a ray in the Hilbert space.

The real scalar, identified as the norm, has to be fixed to one for the probabilistic meaning of the scalar product (see postulate 2).

In the geometrical formalism this leads to an equivalence class of vectors in the Hilbert space of the system, that takes the form of a principal bundle, as in the definition 1.13, of the type in the figure 2.1 with base $\mathbb{P}\mathcal{H}$, and fibre $\mathbb{C}_0 = \mathbb{R}^+ \times U(1)$.

The final space in which the quantum mechanics of pure states is played is a projective Hilbert space: a projective space (definition 1.22) with a Hermitian inner product inherited from the original Hilbert space.

Now we are ready to identify an element of $\mathbb{P}\mathcal{H}$ with an element of $\mathcal{D}_1^1(\mathcal{H})$, the space of rank one projectors of trace one (a projector with rank one needs to have trace one):

$$[|\psi\rangle] \iff \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = \rho_\psi \tag{2.12}$$

This identification will be further examined in the next section.

To understand better this fibration is convenient to do an simple example that involves the typical object of quantum information: the qubit.

Example 11. *The choice to consider a qubit isn't casual: it is the one of the elementary systems with an important role in quantum information and the study of entanglement, beside this fact it is the simplest quantum system.*

Let's consider only the spin freedom of an electron (the momentum and any other "coordinates" will be neglected), so the correspondent Hilbert space will be isomorphic to $\mathbb{C}^2$*. Now, removing the origin, it is possible to consider the fibre* $\mathbb{C}_0 \simeq \mathbb{R}^+ \times U(1)$ *as we have just seen in the general situation. In the qubit cases this is an example of Hopf fibration (see figure 2.2).*

$$\begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{C}^2 \\ & & \downarrow \\ U(1) \simeq \mathbb{S}^1 & \longrightarrow & \mathbb{S}^3 \\ & & \downarrow \\ & & \mathbb{P}\mathcal{H} \simeq \mathbb{S}^2 \end{array}$$

FIGURE 2.2: Fibration of the Hilbert space of a qubit an example of Hopf fibration.

Remark 2.2. *In general this procedure leads to a particular results called Adam's theorem, [1], which states that the only possible Hopf fibrations are:*

- $\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \to \mathbb{S}^1$
- $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ *The qubit case.*
- $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$
- $\mathbb{S}^8 \hookrightarrow \mathbb{S}^{15} \to \mathbb{S}^8$

We can now try to visualize geometrically this situation, this is the so called Block Sphere [5], which is plotted in figure 2.3. The block sphere will be used as the optimal choice of coordinates for a qubit in following chapters.

If the generators of $\mathfrak{su}(2)$ *(see Appendix C) will be used as a basis of* $\mathcal{D}(\mathbb{C}^2)$*, that are:*

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.13}$$

The most general density matrix can be written in the form:

$$\rho = \frac{1}{2}\mathbb{1} + \vec{\tau} \cdot \vec{\sigma} = \begin{pmatrix} \frac{1}{2} + z & x - iy \\ x + iy & \frac{1}{2} - z \end{pmatrix}, \tag{2.14}$$

with the condition:

$$\det[\rho] \geq 0 \Leftrightarrow x^2 + y^2 + z^2 \leq \frac{1}{4}; \tag{2.15}$$

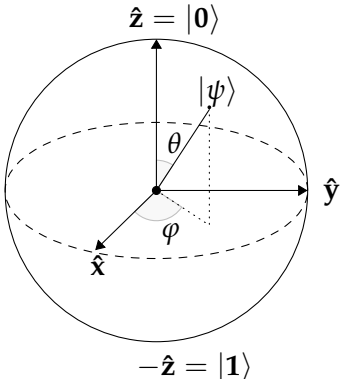


FIGURE 2.3: Rappresentation of the phase space of a qubit: The Block Sphere

that guarantees the non-negativity.

The pure states that are points in $\mathbb{P}\mathbf{C}^1$ *that need to have rank one, therefore the determinant of* ρ *has to be zero; this condition leads to the equation of pure states:*

$$x^2 + y^2 + z^2 = \frac{1}{4}; \tag{2.16}$$

it represents the surface of $\mathbb{S}^2$ *as we have seen with the fibration method. Now we can consider this surface and using the constraint given by equation* (2.16) *it is possible to parametrize the density matrix* (2.14) *using the standard spherical coordinates:*

$$\rho = \begin{pmatrix} \frac{1}{2} + z & x - iy \\ x + iy & \frac{1}{2} - z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + \cos(\theta) & \sin(\theta) e^{-i\phi} \\ \sin(\theta) e^{i\phi} & 1 - \cos(\theta) \end{pmatrix}. \tag{2.17}$$

We want to emphasize that while the pure states are on the boundary of the three dimensional ball and the internal points are the mixed states for the qubit case, the boundary are not only the pure states for higher dimensional systems [18].

2.2.2 Kähler structure of the projective Hilbert space

Now the main goal is to build up a differential structure for the projective Hilbert space, some notions of differential structures on the projective spaces are already been introduced in chapter 1. Let $\mathcal{H}$ be a n dimensional Hilbert space, with a Hermitian product $\langle x, y \rangle_{\mathcal{H}}$ (the convention that we will use is the same of the previous section: linear in the second argument and anti-linear in the first one), and let us consider the unitary group $U(H)$ (all operator $A \in GL(\mathcal{H})$ such that $AA^\dagger = \mathbb{1}$) and its action on the Hilbert space that preserves the Hermitian product:

$$\langle Ax, Ay \rangle = \langle x, y \rangle . \tag{2.18}$$

The realification of such Hilbert space is the starting point of the geometrical approach for quantum mechanics. Consider $\mathcal{H}_{\mathbb{R}}$ (see appendix A.5 for the realification procedure) and see it as a Kähler manifold $(\mathcal{H}_{\mathbb{R}}, J, g, \omega)$ where:

- J is the complex structure:

$$J : T\mathcal{H}_{\mathbb{R}} \to T\mathcal{H}_{\mathbb{R}}; \tag{2.19}$$

- g is a Riemannian metric:

$$g = \Re\left(\langle \cdot, \cdot \rangle\right); \tag{2.20}$$

- ω is a symplettic form:

$$\omega = \Im\left(\langle \cdot, \cdot \rangle\right); \tag{2.21}$$

which fulfil the relation of the definition 1.9, as can be seen in [18]. Its tangent bundle is: $T\mathcal{H}_{\mathbb{R}} = \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}}$, so the structures are so identified:

$$J(X) = iX, \tag{2.22}$$

$$\langle X, Y \rangle = g(X, Y) + i\omega(X, Y). \tag{2.23}$$

It is useful to define also the inverses of the previous tensors: $G = g^{-1}$ is the controvariant metric tensor and $\Omega = \omega^{-1}$ is the Poisson tensor; their combination form another Hermitian product:

$$\langle \phi, \psi \rangle_{\mathcal{H}^*} = G(\phi, \psi) + i\Omega(\phi, \psi); \tag{2.24}$$

where ϕ and ψ are elements of the dual of the real Hilbert space (with J^* as complex structure).

If $\mathcal{H}^*_{\mathbb{R}}$ (the dual space) is identified with $\mathcal{H}_{\mathbb{R}}$, we can introduce two brackets for smooth functions on $\mathcal{H}_{\mathbb{R}}$:

- Riemann-Jordan bracket: $\{r, s\}_g = G(dr, ds)$;
- Poissson bracket: $\{r, s\}_\omega = \Omega(dr, ds)$;

They can be combined and extended to complex functions in order to form the total bracket:

$$\{r, s\}_{\mathcal{H}} = \langle dr, ds \rangle_{\mathcal{H}^*} = \{r, s\}_g + i\{r, s\}_\omega. \tag{2.25}$$

Choosing a basis for $\mathcal{H}$ induces the identification $\mathcal{H} \simeq \mathbb{C}^n$ and consequently for all structures:

$$\langle a, b \rangle = \bar{a}_k b_k; \tag{2.26}$$

with the Einstein convention of the repeated indexes. The groups acting on the Hilbert space become groups of matrices:

- $U(\mathcal{H}) \simeq U(n)$;
- $\mathfrak{u}(\mathcal{H}) \simeq \mathfrak{u}(n)$;
- $\mathfrak{u}^*(\mathcal{H}) \simeq \mathfrak{u}^*(n)$.

The global coordinates, chosen a basis $\{e_k\}_k$ in $\mathcal{H}$, are (q_k, p_k), they are defined by:

$$\langle e_k, x\rangle_{\mathcal{H}} = q_k + ip_k; \tag{2.27}$$

and so the complex structure becomes:

$$J = \partial_{p_k} \otimes dq_k - \partial_{q_k} \otimes dp_k \tag{2.28}$$

where e_k becomes ∂_{q_k} and ie_k becomes ∂_{p_k}.

Introducing the notation $a \wedge b = a \otimes b - b \otimes a$ and $a \vee b = a \otimes b + b \otimes a$, the Rimannian tensor and the sympletic tensor can be written as (for the definition of $\vee$ see Appendix A):

$$g = \frac{1}{2}\left(dq_k \vee dq_k + dp_k \vee dp_k\right); \tag{2.29}$$

$$\omega = dq_k \wedge dp_k. \tag{2.30}$$

Using the same procedure for the inverses, we find:

$$G = \frac{1}{2}\left(\partial_{q_k} \vee \partial_{q_k} + \partial_{p_k} \vee \partial_{p_k}\right), \tag{2.31}$$

$$\Omega = \partial_{q_k} \wedge \partial p_k. \tag{2.32}$$

Using complex coordinates, the total Hermitian products can be rewritten as:

$$g + i\omega = d\bar{z}_k \otimes dz_k; \tag{2.33}$$

$$G + i\Omega = 4\partial_{z_k} \otimes \partial_{\bar{z}_k}. \tag{2.34}$$

Let us back to linear operators, for each $A \in \mathfrak{gl}(\mathcal{H})$ it is possible to define a quadratic form f_A using the Hermitian product of the Hilbert space:

$$f_A = \langle x, Ay\rangle_{\mathcal{H}}; \tag{2.35}$$

it is easy to see that this function is real only if $A \in u^*(\mathcal{H})$ is a Hermitian operator. We identify $u^*(\mathcal{H})$ as the Hermitian operators on $\mathcal{H}$ and of skew-Hermitian operators with $\mathfrak{u}(\mathcal{H})$; this can be done trough the relation:

$$\langle A, B\rangle = \frac{i}{2}\operatorname{Tr}[AB]. \tag{2.36}$$

(In appendix B is presented the relation between Lie algebras and Lie groups that gives rise to the previous relation.)

In order to pass from the algebra to the dual is needed to define an isomorphism: $T \mapsto iT$. Due to this isomorphism, also the dual becomes an algebra with the bracket:

$$[A, B] = -i(AB - BA), \tag{2.37}$$

and with the scalar product derived from equation (2.36):

$$\langle A, B\rangle_{u^*(\mathcal{H})} = \frac{1}{2}\,\mathrm{Tr}[AB]. \tag{2.38}$$

We can also consider another bracket: the anticommutator; or the Jordan-Riemann bracket, denoted with $[\cdot,\cdot]_+$, this and the usual commutator are compatible with the scalar product, i.e.:

$$\langle [A, B], C\rangle_{\mathfrak{u}^*} = \langle A, [B, C]\rangle_{\mathfrak{u}^*}\,, \tag{2.39}$$

$$\langle [A, B]_+, C\rangle_{\mathfrak{u}^*} = \langle A, [B, C]_+\rangle_{\mathfrak{u}^*}\,, \tag{2.40}$$

that leads to the clear identification of the skew-Hermitian operator as the covectors of the Hermitian operators manifold. This construction induce the adjoint and coadjoint action (see definition 1.18): $Ad_U T = UTU^\dagger$.

2.2.3 Momentum Map

Now we can study the passage form the realified Hilbert space to the projective Hilbert space. Let be f a smooth function on $\mathcal{H}_{\mathbb{R}}$, to it can be associated two vector fields:

- Gradient: $grad_f$; that fulfils:

$$g(\cdot, grad_f) = df \;\text{ or }\; grad_f = G(df, \cdot); \tag{2.41}$$

- Hamiltonian: Ham_f; that fulfils:

$$\omega(\cdot, Ham_f) = df \;\text{ or }\; Ham_f = \Omega(df, \cdot); \tag{2.42}$$

in this way an $A \in \mathfrak{gl}(\mathcal{H})$ induces a vector field $\hat{A}$ on the Hilbert space.

Proposition 2.1. *If A is a Hermitian operator the gradient and the Hamiltonian vector field associated to it are:*

$$grad_{f_A} = \hat{A} \tag{2.43}$$

$$Ham_{f_A} = i\hat{A} \tag{2.44}$$

where f_A is the quadratic function defined before.

Proposition 2.2. *For two operators $A, B \in \mathfrak{gl}(\mathcal{H})$, the following relations hold:*

$$\{f_A, f_B\}_{\mathcal{H}} = f_{2AB}; \tag{2.45}$$

$$\{f_A, f_B\}_g = f_{[A,b]_+}; \tag{2.46}$$

$$\{f_A, f_B\}_\omega = f_{[A,B]}. \tag{2.47}$$

The action of $U(\mathcal{H})$ is unitary and Hamiltonian [18] and defines the so called momentum map:

$$\mu : \mathcal{H}_{\mathbb{R}} \to \mathfrak{u}^*(\mathcal{H}). \tag{2.48}$$

Let us recall the associated vector field to $iA \in \mathfrak{u}(\mathcal{H})$ (with $A \in \mathfrak{u}^*(\mathcal{H})$):

$$\left.\frac{d}{dt}\right|_{t=0} e^{-\frac{t}{i}A}(x) = iA(x). \tag{2.49}$$

The Hamiltonian vector field associated to f_A is iA and consequently the momentum map can be defined as:

$$\langle \mu(x), -iA\rangle = f_A(x) = \frac{1}{2}\langle x, Ax\rangle_{\mathcal{H}} \tag{2.50}$$

$$= \frac{i}{2}\operatorname{Tr}\left[\mu(x)(-i)A\right] = \frac{1}{2}\operatorname{Tr}\left[\mu(x)A\right] \tag{2.51}$$

$$\Rightarrow \langle x, Ax\rangle_{\mathcal{H}} = \operatorname{Tr}\left[\mu(x)A\right] \Rightarrow \mu(x) = |x\rangle\langle x|\,; \tag{2.52}$$

the last expression is written in the Dirac notation. Before studying the structure of the image of the momentum map is important to focus the attention to the dual forms of the metric and the Hermitian product, f_A is the pullback by μ of iA since:

$$f_A = \mu^*((-i)A) = (-i)A \circ \mu, \tag{2.53}$$

and it leads to the cotangent bundle $T^*\mathfrak{u}^*(\mathcal{H})$, in this way G and Ω can be related to the controvariant tensors of $\mathfrak{u}^*(\mathcal{H})$:

- R the Riemann- Jordan:

$$R(C)(-iA, -iB) = \langle C, [A, B]_+\rangle_{\mathfrak{u}^*} = \frac{1}{2}\operatorname{Tr}\left[C[A, B]_+\right]; \tag{2.54}$$

- Λ the sympletic:

$$\Lambda(C)(-iA, -iB) = \langle C, [A, B]\rangle_{\mathfrak{u}^*} = \frac{1}{2}\operatorname{Tr}\left[C[A, B]\right]; \tag{2.55}$$

combining together these two tensors, we get the dual of the Hermitian product trough the momentum map:

$$\mu^*(G + i\Omega) = R + i\Lambda. \tag{2.56}$$

If we consider an element $|x\rangle \in \mathcal{H}_0$, μ maps it to $|x\rangle\langle x|$; so the image of this map in $\mathfrak{u}^*(\mathcal{H})$ correspond to the non negative Hermitian operators of rank one denoted by $\mathcal{P}^1(\mathcal{H})$. If we consider $\mathbb{P}(\mathcal{H})$ instead of the entire Hilbert space the image becomes only the non negatively Hermitian operators of rank 1 with trace 1 denoted by $\mathcal{S} = \{x \in \mathcal{H} |\, \langle x, x\rangle_{\mathcal{H}} = 1\}$. The action of the unitary group $U(\mathcal{H})$ on

this space is the coadjoint one:

$$(U, \rho) \mapsto U\rho U^{\dagger}; \tag{2.57}$$

that foliates $\mathcal{P}^1(\mathcal{H})$ into:

$$\mathcal{D}_r^1(\mathcal{H}) = \{|x\rangle \langle x| \, | \|x\| = r\}; \tag{2.58}$$

with the obvious condition $r > 0$. Now we will see that $\mathcal{D}_1^1$ is a Kähler manifold, and that $x \in S_{\mathcal{H}}$ is mapped to $\xi \in \mathcal{D}_1^1$.

Let us consider a coadjoint orbit O in the space of the Hermitian operators. The vectors form the tangent space of O (the $U(\mathcal{H})$) orbit). The Poisson structure previously defined:

$$\Lambda(C)(-iA, -iB) = \langle C, [A, B] \rangle_{\mathfrak{u}_{\mathcal{H}}^*} = \langle [C, A], B \rangle \,, \tag{2.59}$$

induces a map from one forms to the vector fields:

$$\tilde{\Lambda}_C(-iA) = [C, A]; \tag{2.60}$$

and we will denote the inverse of this with $\eta_C([C, A]) = -iA$. This inverse tensor leads to the definition of a 2-form:

$$\eta_C \left([A, C], [B, C]\right) = \langle [A, C], B \rangle = - \langle C, [A, B] \rangle \,. \tag{2.61}$$

Proposition 2.3. *The restriction of η_C on the orbit O of the unitary group has the property:*

$$\eta_C^O \left([A, C], [B, C]\right) = \langle [A, C], B \rangle = - \langle C, [A, B] \rangle \,. \tag{2.62}$$

More over the same construction can applied to the Riemann tensor and it defines a partial tensor σ with the property:

$$\sigma_{\xi} \left([A, C]_+, [B, C]_+\right) = \langle [A, C]_+, B \rangle = \langle C, [A, B]_+ \rangle \,. \tag{2.63}$$

Now we can combine the restrictions of σ and η on $\mathcal{D}_r^1$ to induce on it a Kähler structure. The proofs of the following results can be found in [18].

Proposition 2.4. *The restriction of σ on the unitary orbit $\mathcal{D}_1^1$ through $\xi = |x\rangle \langle x|$ (with $\langle x|x\rangle = 1$) is the scalar product on $\mathfrak{u}^*(\mathcal{H})$.*

Definition 2.2. *Let be $\xi \in \mathcal{D}_1^1$ a rank one projectors, and A an Hermitian operator of $\mathfrak{u}^*(\mathcal{H})$ then $J_{\xi}(A)$ is a (1,1) tensor defined as:*

$$J_{\xi}(A) = [A, \xi]. \tag{2.64}$$

Proposition 2.5. *The tensor $J_{\xi}(A)$:*

- *fulfil* $J^3 = -J$;
- *induce a complex structure on* $\mathcal{D}_1^1$;
- *is compatible with a Kähler structure in the definition 1.9:* $(\mathcal{D}_r^1(\mathcal{H}), J, \sigma, \eta)$.

It is important to emphasize that this structure come from the original Kähler structure of the realified Hilbert space: the symplectic structure originates from the reduction of the analogous structure $(\mathcal{H}_{\mathbb{R}}, \omega)$; and the same holds for the Riemannian structure.

2.3 Geometrical Structure of Mixed States

As it was shown in the first part of this chapter there is a necessity to consider the density states instead of the pure ones. The non-negatively Hermitian operators (the set $\mathcal{P}(\mathcal{H})$) on $\mathfrak{gl}(\mathcal{H})$ can be written in the form:

$$\rho = T^\dagger T; \tag{2.65}$$

in which $T \in \mathfrak{gl}(\mathcal{H})$. The set of density states $\mathcal{D}(\mathcal{H})$ are embedded in the previous set by the equation $\mathrm{Tr}[\rho] = 1$, imposed for probabilistic reasons. Thus:

$$\mathcal{D}(\mathcal{H}) \hookrightarrow \mathcal{P}(\mathcal{H}) \hookrightarrow \mathfrak{u}^*(\mathcal{H}). \tag{2.66}$$

The rank of these operators are important, so it is useful to decompose the positive operators into the spaces $\mathcal{P}^k$ of operators of fixed rank k:

$$\mathcal{P}^k(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) | Rank(\rho) = k\}; \tag{2.67}$$

and consequently from the definition of the density states:

$$\mathcal{D}^k(\mathcal{H}) = \mathcal{D}(\mathcal{H}) \cap \mathcal{P}^k(\mathcal{H}). \tag{2.68}$$

The set of the density states take the form of a convex cone in the space of Hermitian operators, so every density state can be written as a convex combination of rank 1 projectors: for these reasons the pure states are the extremal points of $\mathcal{D}(\mathcal{H})$.

The structure of $\mathcal{D}(\mathcal{H})$ will be investigated through the coadjoint orbits of the unitary group and their classifications; extending this analysis to the set of the Hermitian operators $\mathfrak{u}^*(\mathcal{H})$ it is possible to give and calculate explicitly the metric tensor, the symplectic tensor and the complex structure [18].

Now the foliation due to the coadjoint action of the unitary group reappears also in the the set of the density matrix:

Proposition 2.6. *Let $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$ with $\mathcal{H} \simeq \mathbb{C}^n$; then ρ_1 and ρ_2 are unitary equivalent:*

$$\rho_1 = T\rho_2 T^\dagger, \tag{2.69}$$

with $T \in U(n)$ iff ρ_1 and ρ_2 has the same spectrum with the same molteplicity.

The Proposition 2.6 has many consequences:

- the orbit O_ρ is determined by its spectrum;
- every orbit can be seen as an equivalence class, represented by a diagonal matrix;
- if ρ_1, ρ_2 are two density matrices in the same orbit then the following relation hold true:
$$\mathrm{Tr}[\rho_1^p] = \mathrm{Tr}[\rho_2^p] \quad \forall p \in \mathbb{N}. \tag{2.70}$$

It is common to order the eigenvalues that define the orbit in decreasing order of values:

$$\lambda_i > \lambda_j \quad i < j; \tag{2.71}$$

in order to have a unique representation of every orbit; therefore $U(n)$ foliates $\mathcal{D}(\mathcal{H})$ in infinitely many different orbits (called also strata). The structure of these strata are related to the group structure of the unitary group and the orbit theory [29].

Proposition 2.7. *If G is a Lie compact group acting on an Hausdorff space X, and $H(p)$ is the stabilizer of $p \in X$, then there exists a homomorphism:*

$$\phi : G/H(p) \to O_p. \tag{2.72}$$

Using this proposition one can prove the following theorem [18].

Theorem 2.1. *Let $\rho \in \mathcal{D}(\mathcal{H})$ with r eigenvalues λ_i with multiplicity n_i ($\sum_{i=1}^r \lambda_i n_i = 1$), its orbit is homomorphic to the manifold:*

$$U(n)/\left[U(n_1) \times ... \times U(n_r)\right]; \tag{2.73}$$

with dimension $n^2 - \sum_{i=1}^r n_i$.

The first consequence is that the Hibert projective space (pure states) is homeomorphic to $U(n)/\left[U(1) \times U(n-1)\right]$.

Now it is convenient to study the geometry of $\mathfrak{u}^*(\mathcal{H})$ in order to understand better the forms of these orbits. The set $\mathfrak{u}^*_{k_+,k_-}(\mathcal{H})$ will be defined as the Hermitian matrix with k_+ positive eigenvalues and k_- negative eigenvalues, obviously we have the relation:

$$rank(A) = k_+(A) + k_-(A), \tag{2.74}$$

and also is clear that:

$$\mathcal{P}^k(\mathcal{H}) = \mathfrak{u}^*_{k,0}(\mathcal{H}). \tag{2.75}$$

After the usual passage from $\mathcal{H}$ to $\mathbb{C}^n$, we define a diagonal marix:

$$\mathrm{Y}_{k+,k-} = diag(\underbrace{+1,\ldots,+1}_{k_+ times}, \underbrace{-1,\ldots,-1}_{k_-\ \text{times}}); \tag{2.76}$$

and with it we can build up a pseudo-Hermitian product in $\mathbb{C}^n$ denoted by $\langle\cdot,\cdot\rangle_{k+,k-}$: so taking two vectors $a, b \in \mathbb{C}^n$:

$$\langle a, b\rangle = \bar{a}_i \left[\mathrm{Y}_{k+,k-}\right]^{ij} b_j; \tag{2.77}$$

after these definitions we can reconsider the ideas presented at the beginning of this section.

Proposition 2.8. *Every $A = (a_{ij}) \in \mathfrak{u}^*_{k_+,k_-}(n)$ can be written as $A = T^\dagger \mathrm{Y}_{k_+,k_-} T$, with $T \in GL(n,\mathbb{C})$.*

Now we can built up in details the smooth structures, of $\mathcal{P}(\mathcal{H})$, the following theorem characterizes the various $\mathfrak{u}^*_{k_+,k_-}(n)$.

Theorem 2.2. *The collection of subspaces:*

$$\mathfrak{u}^*_{k_+,k_-}(n); \tag{2.78}$$

with $k_+ + k_- \leq n$, is the family of the orbits of the smooth adjoint action of GL. So every $\mathfrak{u}^*_{k_+,k_-}$ is a connected submanifold with elements B of tangent space of the form:

$$\langle Bx, y\rangle_{\mathcal{H}} = 0 \ \ \forall x, y \in ker(\xi); \tag{2.79}$$

for a $\xi \in \mathfrak{u}^*_{k_+,k_-}$ and $B \in T_\xi \mathfrak{u}^*_{k_+,k_-}$.

For a particular choice of k_- and k_+, the same of the equation (2.75) (that is $k_- = 0$); the following theorem can be proved [18] which ensures the smoothness of $\mathcal{P}(\mathcal{H})$:

Theorem 2.3. *The following conditions are equivalent:*

- $\mathfrak{u}^*_{k+,k-} = \mathcal{P}^k(\mathcal{H})$;
- $\mathfrak{u}^*_{k+,k-} \subset \mathcal{P}(\mathcal{H})$;
- $k_- = 0$;
- $\mathfrak{u}^*_{k+,k-}$ *intersects* $\mathcal{P}(\mathcal{H})$.

The proprieties of curves on the positive Hermitian operators can be useful to describe the border of $\mathcal{D}(\mathcal{H})$.

Definition 2.3 (Border of $\mathcal{D}(\mathcal{H})$)**.** *If $\mathcal{H}$ is a n-dimensional Hilbert space, the border of $\mathcal{D}(\mathcal{H})$ is:*

$$\partial\mathcal{D}(\mathcal{H}) = \bigcup_{k<n} \mathcal{D}^k(\mathcal{H}); \tag{2.80}$$

Let us start with a theorem that characterize some proprieties of the smooth curves in this particular manifold.

Theorem 2.4. *Let be γ a smooth curve on $\mathcal{P}(\mathcal{H}) \subset \mathfrak{u}^*(\mathcal{H})$ if:*

$$\gamma \subset \mathcal{P}^k(\mathcal{H}), \tag{2.81}$$

then:

$$\dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{P}^k\mathcal{H}. \tag{2.82}$$

So all curves on $\mathcal{P}(\mathcal{H})$ cannot cross any $\mathcal{P}^k\mathcal{H}$ in a transversal way. Moreover $\mathcal{D}(\mathcal{H})$ is the level set of the function trace with the value 1.

The trace function leads to the identification of the topological structure of $\mathcal{P}(\mathcal{H})$ thanks to the invariance for homoteties (multiplication by a positive real number):

$$\mathcal{P}^k(\mathcal{H}) \simeq \mathbb{R} \times \mathcal{D}^k(\mathcal{H}). \tag{2.83}$$

So we are ready to describe in its entirety the smooth structure of the strata in the density states set [18].

Theorem 2.5. *Every $\mathcal{D}^h\mathcal{H}$ are smooth submanifolds of $\mathfrak{u}^*(\mathcal{H})$, moreover if $\gamma \subset \mathcal{D}(\mathcal{H})$ is a smooth curve and if:*

$$\gamma \subset \mathcal{D}^k\mathcal{H}, \tag{2.84}$$

then

$$\dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{D}^k\mathcal{H}; \tag{2.85}$$

and the stratification is maximal (the orbits cannot intersect each other and smooth curves of matrices cannot pass from a orbit to another).

The last element is the characterization of the border of $\mathcal{D}(\mathcal{H})$ built in Definition 2.3.

Theorem 2.6. *The border of $\mathcal{D}(\mathcal{H})$ is not a sub-manifold of $\mathfrak{u}^*(\mathcal{H})$ if the dimension n is greater then 2.*

This describe exhaustively the differential structure of the density states, all the proofs can be found in [12], [18] and [19].

Chapter 3

Entaglement and its differential structure

Entanglement is one of the most important non classical effect of quantum mechanics: analysed as a consequence of the mathematical formulation of quantum mechanics by Einstein, Podolski and Rosen in [11] and called originally "Verschränkung" by Schrödinger in his paper [36]. This effect have given rise to a debate between who, as Einstein, believed in the non completeness of quantum mechanics and therefore in the theory of the hidden variables and the supporters of the usual formulation of quantum theory. This debate had a theoretical breakthrough in 1964 when Bell published [4]; between that time and now no experiment have violated the inequalities derived from Bell's theorem, but despite of these progresses the debate is still open.

In this chapter we will introduce Entanglement following [28] and after it we will recall elements from the last section of [18] and from [19] in order to introduce the geometrical structure of entanglement.

3.1 Introduction to Entanglement

The Entanglement is a non classical effect that comes out when a quantum composite system is considered: while the composite classical systems have a phase space that is the Cartesian product of the single system phase space, the structure of quantum mechanics needs a composite Hilbert space that is the tensor product of the single Hilbert space. It is clear that this different construction leads to the possibility that a generic state of a quantum system cannot be written as a product state: no-product states are entangled states.

Remark 3.1. *A composite classical system has as dimension the sum of the dimensions of the single parts while in the quantum case the dimension is the product of the dimensions of the single parts.*

3.1.1 Entanglement and Pure States

Let $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$ be a finite dimensional Hilbert space, then a pure state in the composite space can be written in the form[28]:

$$|\psi\rangle = \sum_{\boldsymbol{i}_n} c_{\boldsymbol{i}_n} |\boldsymbol{i}_n\rangle ; \tag{3.1}$$

where $\boldsymbol{i}_n = i_1, \ldots, i_n$ is a multiindex, and the $|\boldsymbol{i}_n\rangle = |i_1\rangle \otimes \cdots \otimes |i_n\rangle$ is a single product state: the composite state is a linear combination of product states, following the proprieties of the tensor product of vector spaces (see appendix A).

The pure states of a composite quantum system can be divided in two classes: separable states and entangled states. It is not simple to discern if a state is separable or entangled and it is called separability problem. It has been proved that the separability problem has a NP complexity in [20] and [16]. In order to face up this problem there are two main approaches [28]: the algorithmic approach, that comes from the quantum information world, and the criterion approach.

In order to analyse this problem let's start enunciating some important definitions.

Definition 3.1 (Separable Pure States)**.** *A pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is called separable iff it can be written in the form:*

$$|\psi\rangle = |\psi\rangle_1 \otimes \cdots \otimes |\psi\rangle_n ; \tag{3.2}$$

in which the states $|\psi\rangle_i$ belong respectively to each $\mathcal{H}_i$ (one of the Hilbert spaces in the decomposition of the composite system).

The other type of states are the entangled ones, and they are defined in contraposition with the separable states.

Definition 3.2 (Entangled Pure States)**.** *The states in $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ which are not separable are called entangled.*

In general a composite state cannot be written as a product of states of the single partitions; these states, as it has just been seen, are the entangled ones. The experimental construction of a composite entangled state can be made using a projective measurement (postulate 2) on the whole system or leaving the system to evolve with a non separable Hamiltonian (usually an Hamiltonian with interactions) [28].

A first example in which entanglement appear can be build up considering a composite system consisting of two two-level system.

Example 12. *Let us consider a bipartite system:*

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B; \tag{3.3}$$

with dimension $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 2$ (the dimension of the composite system in this case is $\dim_{\mathbb{C}} \mathcal{H} = 4$). Following the definition of tensor product (appendix A), it is possible

to choose a basis $\{|0\rangle_A, |1\rangle_A\}$ *for* $\mathcal{H}_A$ *and* $\{|0\rangle_B, |1\rangle_B\}$ *for* $\mathcal{H}_B$*, and consequently to give a natural basis to* $\mathcal{H}$ *(called also computational basis), that is:*

$$\Big\{ |0\rangle_A \otimes |0\rangle_B, |0\rangle_A \otimes |1\rangle_B, |1\rangle_A \otimes |0\rangle_B, |1\rangle_A \otimes |1\rangle_B \Big\}. \tag{3.4}$$

It is possible to use the action of the unitary group (in this case $U(4)$*, that preserve the Hermitian product) to change the base of* $\mathcal{H}$ *from the computational basis to the so called Bell states basis, or simply Bell basis:*

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle_1 \otimes |0\rangle_2 \pm |1\rangle_1 \otimes |1\rangle_2 \Big), \tag{3.5}$$

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle_1 \otimes |1\rangle_2 \pm |1\rangle_1 \otimes |0\rangle_2 \Big); \tag{3.6}$$

these are entangled pure states and it is easy to verify that are linearly independent. Moreover the vectors of the Bell basis have an even more interesting propriety: they are the pure states that maximize entanglement [28], in fact if the composite system stays in one of these states and a projective measurement operation is performed on one of the two subsystems, the result state found in the other system could be with equal probability the state or $|0\rangle$ *or* $|1\rangle$*; in this case we can say that we know all about the composite system, but nothing about the subsystems: it is a classical ensemble of two states with equal weight. Bell states can be generalized also for higher dimensional Hilbert spaces, i.e. if we consider a bipartite system of the type:* $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = d$*; and so one of the maximally entangled pure states is in the form [28]:*

$$|\phi^{+}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle . \tag{3.7}$$

A last remark has to be done in this example: the passage from the natural basis given by the tensor product to the maximally entangled Bell basis is an unitary operation that in general is an element of the group $U(\mathcal{H})$*; so it is easy to see that this group does not preserve the entanglement of a state, we will discuss later the operation that preserves the separability.*

We want now to define the notion of a reduced state, and in particular what is a partial trace. If ρ represents a density matrix of a composite system, we can define an operation called partial trace that is a map from the set of the density matrix ($\mathcal{D}(\bigotimes_i \mathcal{H}_i)$) of the composite system to the set of the density matrix of a subsystem ($\mathcal{D}(\mathcal{H}_j)$). To simplify the notation, we give the following definition for a bipartite system.

Definition 3.3 (Partial Trace)**.** *Let* ρ *be a density matrix of a bipartite system* $\mathcal{H}_A \otimes \mathcal{H}_B$*, the partial trace over the subsystem A is a map:*

$$\mathrm{Tr}_A : D(\mathcal{H}_A \otimes \mathcal{H}_B) \to D(\mathcal{H}_B). \tag{3.8}$$

that corresponds at choosing a basis of $\mathcal{H}_A$ *and performing the trace with this basis on* ρ*. The result of this operation is a density matrix* ρ_B *called reduced state. Obviously it can be*

defined analogously also Tr_B *that maps* ρ *to* $D(\mathcal{H}_B)$.

Remark 3.2. *It is important to emphasize that the trace is an operation independent from the choice of the basis [30].*

For the multipartite systems the various trace operations that can be defined on (depending on the number of the subsystems) are interchangeable, or rather commutes [10]:

$$Tr_i[\mathrm{Tr}_j[\rho]] = \mathrm{Tr}_j[\mathrm{Tr}_i[\rho]]; \tag{3.9}$$

where Tr_i means the trace over the i-th subsystem.

The relation between a pure separable state and the reduced states can be explained by the following proposition, that has an important role also in the open system.

Proposition 3.1. *If* ρ *represents a density matrix of a pure state of the composite system its any reduced state is in general a mixed state. If the two subsystems stay both in a pure state then the state of a composite system is pure and separable.*

For composite systems there is an important theorem that describes the structure of the pure states, it is the Schmidt's decomposition theorem [28].

Theorem 3.1 (Schmidt's decomposition)**.** *Every pure state of a bipartite system can be written in the form:*

$$|\phi\rangle = \sum_{i=1}^{M} \sqrt{q_i}\, |e_i\rangle \otimes |f_i\rangle\,; \tag{3.10}$$

where $|e_i\rangle$ *and* $|f_i\rangle$ *are elements of two orthonormal basis respectively of the first and the second Hilbert space; moreover the following inequality holds:* $M \leq \min(\dim \mathcal{H}_1, \dim \mathcal{H}_2)$.

This theorem restricts a lot the possible forms of the states: given the equation (3.1), which describes a pure state of a multipartite system, in a bipartite system the statement can be reduced to:

$$|\psi\rangle = \sum_{i_1} \sum_{i_2} c_{i_1 i_2}\, |i_1\rangle \otimes |i_2\rangle\,; \tag{3.11}$$

comparing this equation with the equation (3.10) it is possible to note some differences: first, the Schmidt theorem reduce the number of sums from two to one, second, the coefficients that characterize the state are listed in a vector (Schmidt coefficients vector) instead of in a matrix as in (3.1).

The coefficients q_i are called Schmidt's coefficients and they fulfil the following two conditions:

- $\sum_i q_i = 1$;
- $q_i \geq 0$.

These conditions are due to the constraints of the states of the two subsystems: the normalization (the trace of any density state has to be equal to one) and the positivity of the density matrices.

This theorem has a corollary which can be useful for the entanglement detection, in order to reach this result it is useful to give the following definition.

Definition 3.4. *The number s of the non-zero Schmidt's coefficients q_i of a given state is called Schmidt's rank of the state.*

And after the definition we are now able to enunciate the corollary.

Corollary 3.1.

- *The Schmidt rank s is equal to the usual rank of the reduced density matrices.*
- *If $s > 1$ the state is entangled.*

This great theorem has a big limit: it cannot be generalized to multipartite systems, even if there is a similar statement for three qubits systems [28].

The second point of the previous corollary allow us to determine if a state is entangled or not and it is linked with the concept of Reduced Density matrix, previously introduced. Recalling the previous chapter a pure state has a density matrix in the form:

$$\rho = |\psi\rangle \langle\psi| ; \tag{3.12}$$

where $|\psi\rangle \in \mathcal{H}$ with $\langle\psi|\psi\rangle = 1$. Computing the partial trace on subsystem A:

$$\sum_i \langle e_i| \, |\psi\rangle \langle\psi| \, |e_i\rangle = \mathrm{Tr}_A \left[|\psi\rangle \langle\psi|\right] = \rho_B \in \mathcal{D}(\mathcal{H}_B). \tag{3.13}$$

If the result is a pure state then the original state is a separable pure state (consequence of theorem 3.1).

Nowadays entanglement has importance also in a number of applications, in particular in the fields of quantum information and quantum cryptography. Let us introduce an application in quantum teleportation.

Example 13 (Quantum Teleportation of a qubit)**.** *Let us consider two experimental physicists that, for sake of simplicity, we will call respectively Alice and Bob. Alice wants to send the state of a unknown qubit:*

$$|\chi\rangle = a_0 \, |0\rangle_{A1} + a_1 \, |1\rangle_{A1} ; \tag{3.14}$$

to Bob. Let us suppose that Alice and Bob share a entangled two qubits system that is in one of the Bell's states: $|\psi^+\rangle$. The state of the composite system formed by the unknown

qubit and the two entangled qubits is:

$$
\begin{aligned}
&|\chi\rangle \otimes |\psi^+\rangle = \\
&= (a_0 |0\rangle_{A1} + a_1 |1\rangle_{A1}) \otimes \frac{1}{\sqrt{2}} (|0\rangle_{A2} |1\rangle_B + |1\rangle_{A2} |0\rangle_B) = \\
&= \frac{1}{\sqrt{2}} \Big(a_0 |0\rangle_{A1} |0\rangle_{A2} |1\rangle_B + a_0 |0\rangle_{A1} |1\rangle_{A2} |0\rangle_B \\
&\qquad + a_1 |1\rangle_{A1} |0\rangle_{A2} |1\rangle_B + a_1 |1\rangle_{A1} |1\rangle_{A2} |0\rangle_B \Big) = \\
&= |\psi^+\rangle (a_0 |0\rangle_B + a_1 |1\rangle_B) + |\psi^-\rangle (a_0 |0\rangle_B - a_1 |1\rangle_B) \\
&\qquad + |\phi^+\rangle (a_1 |0\rangle_B + a_0 |1\rangle_B) + |\phi^-\rangle (a_1 |0\rangle_B - a_0 |1\rangle_B)
\end{aligned}
$$

in which from the second line and on the tensor product symbol has been omitted to get a simpler notation.

Obviously Bob can observe only the qubit labelled by B, while Alice can observe both the qubits labelled by A_1 and A_2 individually or considering them as a composite system. Alice can now perform a projective operation on her system ($\mathcal{H}_{A1} \otimes \mathcal{H}_{A2}$), and consequently her system collapses in one of the Bell's states. The situations that can rise now are four:

- *Collapsed state is $|\psi^+\rangle$; Bob receives $a_0 |0\rangle_B + a_1 |1\rangle_B$;*
- *Collapsed state is $|\psi^-\rangle$; Bob receives $a_0 |0\rangle_B - a_1 |1\rangle_B$;*
- *Collapsed state is $|\phi^+\rangle$; Bob receives $a_1 |0\rangle_B + a_0 |1\rangle_B$;*
- *Collapsed state is $|\phi^-\rangle$; Bob receives $a_1 |0\rangle_B - a_0 |1\rangle_B$;*

consequently Alice can send with classical method only two bits (two bits can represent the number from zero to three) that are in bijection with the possible results of Alice's measurement previously listed. After the receiving of that information Bob can perform a unitary operation (usually turning on a external field) on his qubit to restore the initial unknown state.

There are some highlights:

- *Alice does not need to know the original state $|\chi\rangle$: she cannot perform any measurement without losing the information of the state that she wants to send;*
- *after the teleportation the entangled qubits couple becomes the two qubits in Alice's laboratory: Bob's qubit is now un-entangled; it means that in this way the entanglement is transported but not created.*

3.1.2 Density Matrix and Entanglement

Let us now focus on the set of the density matrices $\mathcal{D}(\mathcal{H})$, the definition of Entanglement is not the same of the pure states.

Definition 3.5 (Separable Mixed State)**.** *A separable $\rho \in \mathcal{D}(\mathcal{H})$ of a n dimensional quantum system is called separable if can be written as a convex combination of product states [28]:*

$$\rho = \sum_{i}^{M} p_i \rho_1^i \otimes \cdots \otimes \rho_n^i; \tag{3.15}$$

where the single $\rho_j^i \in \mathcal{D}(\mathcal{H}_j)$ and $0 \leq p_i \leq 1$ with $\sum_i p_i = 1$.

It is important to underline that now product states are separable, but it is not true that all separable states are product states. It can be proved that the two definitions of separable states (definition 3.1 and definition 3.5) are equivalent for the pure states [5] [28].

Let us focus only on the finite dimensional bipartite system (in which it is possible to create a isomorphism between the Hilbert spaces and the complex $\mathbb{C}^n$): $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$; with the respectively dimension N, N_A and N_B. A bipartite separable state is:

$$\rho = \sum_{i=1}^{M} p_i \rho_A^i \otimes \rho_B^i; \tag{3.16}$$

where $\rho_j^i \in \mathcal{D}(\mathcal{H}_j)$, but it is important to underline that this decomposition is not unique as it was explained in the Chapter 2.

Another important highlight is that the separability problem differs a lot from the eigenproblem: there is the possibility that ρ has entangled eigenvectors (non-product eigenstates), while ρ is separable; the following example shows it.

Example 14. *Let us define a density matrix on a 2 qubits system, we use the spectral decomposition with two entangled projectors:*

$$\rho = \frac{1}{2} |\phi^+\rangle \langle\phi^+| + \frac{1}{2} |\phi^-\rangle \langle\phi^-| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{3.17}$$

from the eigenproblem entangled eigenvectors can be found, but making a different choice of basis (as we have done in the matrix representation), ρ can be written as sum of separable projectors, and so in the form given by equation (3.15)*:*

$$\rho = \frac{1}{2} |00\rangle \langle 00| + \frac{1}{2} |11\rangle \langle 11|. \tag{3.18}$$

The Caratheodory theorem limits the minimum value of the cardinality (the minimum number of addenda needed to reconstruct the state, M in equation (3.15)) of a separable state: $M \leq (N_A)^2 (N_B)^2$ (this can be seen in [38] and [27]). In specific cases of particular systems there are other bound on cardinality that will be cited in the various situations.

We will see in the section 3.3 that the set of separable density matrices (the set of all the ρ in the form (3.15)) is convex, compact and invariant under the action of

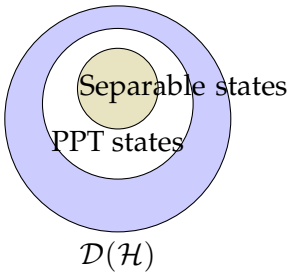


FIGURE 3.1: Structure of the set of the density states respect the PPT states and the separable states.

unitary matrices that can be written in the form:

$$U_A \otimes U_B; \tag{3.19}$$

with $U_i \in U(\mathcal{H}_i)$, these operations are called separable operations (often called SO).

The state of art provide a lot of criteria for deciding if a state is entangled or not; in general they are categorized in three classes [5]:

- Sufficient and necessary conditions but with a not easy usage;
- sufficient but not necessary with a easy usage;
- only necessary with an easy usage;

but there not exist any criterion that solve completely this problem.

A classical example of criterion is the PPT criterion [35].

Theorem 3.2. *If $\rho^{T_A} \ngeq 0$ then ρ is entangled.*

The operation T_A is called partial transpose of the subsystem A and acts in general on a separable bipartite state in the following way:

$$\rho^{T_A} = (T \otimes \mathbb{1})(\rho) = \sum_i p_i (\rho_A^i)^{T_A} \otimes \rho_i^B. \tag{3.20}$$

This operation divides $\mathcal{D}(\mathcal{H})$ in the PPT states and the non PPT states, as can be seen in figure 3.1. It can be also proved that in global systems with $\dim_{\mathcal{H}} \leq 6$ this condition is both sufficient and necessary, so Theorem 3.2 become an iff [26]. This last statement has for us an important consequence: for a couple of two qubits or a qubit and a qutrit, this characterizes completely the separable states.

3.2 Multipartite Entanglement

Let us now go back to multipartite systems with a fixed decomposition:

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n; \tag{3.21}$$

in literature there is a distinction between the states that fulfil the definition 3.5 that from now we will call full separable states (a more restrictive condition), and the partial separable states [28].

3.2.1 Full Separability

The definition of full separability is the same of definition 3.5; also in this case the Caratheodory theorem fixes a superior limit for $M \leq (\dim(\mathcal{H}))^2$.

The structure of the set of the full separable entangled states is peculiar, it is the convex envelope of the separable pure states for definition and it is a closed set (is a set with a boundary) over the trace norm (called also Frobenious norm that was introduced in chapter 2); and last the set is invariant under separable operators actions (SO are maps that send a separable state to another separable state).

For the pure states of a multipartite system definition 3.1 remains true but more complications rise up from the structure of the system: the Schmidt's decomposition is not still valid [28], only a few states admit a sort of generalization of Schmidt decomposition.

The most useful condition remains the one defined by the reduced matrix[28].

Proposition 3.2. *A pure state is fully separable if and only if the reduced density matrices of elementary subsystems are pure.*

For the density states there are not generalizations of the PPT condition (Theorem 3.2); a lot of criteria exist but they don't give any complete characterization of entanglement and it makes the problem still open.

3.2.2 Partial Separability

There are many generalizations of separability which have a lot of applications, the simpler is the notion of separability with respect to partitions. If we consider a multipartite system it is possible to create a disjoint partition of $J = \{1, ..., n\}$:

$$P = \{J_1, ...J_k\}, \tag{3.22}$$

where J is the set of the index of the Hilbert spaces involved in the multipartite system.

Definition 3.6 (Separability Respect a Partition)**.** *A density state ρ of a multipartite system is called separable with respect to a partition P such that:*

$$\bigcup_{j=1}^{k} J_j = J, \tag{3.23}$$

iff:

$$\rho = \sum_{i}^{N} p_i \rho_1^i \otimes \cdots \otimes \rho_k^i, \quad \textit{where} \quad \rho_j \in \mathcal{D}(\bigotimes_{l \in I_j} \mathcal{H}_l). \tag{3.24}$$

The number of possible partitions in a multipartite system rises exponentially with respect to the cardinality of J. One of the most notable types of the partial separability are the semiseparable states, that are the states which are separable under all $(1, n-1)$ possible partitions, that are partitions of the type:

$$\mathcal{H}_i \otimes \left(\bigotimes_{j \neq i} \mathcal{H}_j \right). \tag{3.25}$$

This does not guarantee the full separability of the state [28], an example can be found considering a 3-qubit system in the teleportation configuration (see example 13).

Let us change the point of view: in n-partite systems it could be interesting to impose that the maximum entangled particles are in a fixed number (for example s); this condition creates a stratification of the entanglement [28]: a mixture of states is created which is separable only with respect to some fixed partition, where the set of index has the fixed cardinality s.

3.3 Differential Structure of Entanglement

In this section we will present a sort of generalization of the last part of [18] in order to consider also multipartite systems; we will focus only on the full separability that has a rich geometrical structure. The missing proofs in this section can be found in the aforementioned paper or in [19]. We will use the geometrical methods introduced in chapter 1, and also the application on quantum mechanics that are presented in chapter 2.

Let's consider a quantum composite system with the fixed decomposition:

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n; \tag{3.26}$$

associated to it there is a canonical Segre embedding (see section 1.7.1):

$$\mathcal{S}\mathrm{eg} : \mathbb{P}\mathcal{H}_1 \times \cdots \times \mathbb{P}\mathcal{H}_n \to \mathbb{P}\left(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n\right); \tag{3.27}$$

$$\left[\ldots, x_{1i}, \ldots\right], \ldots, \left[\ldots, x_{nj}, \ldots\right] \mapsto \left[\ldots, \prod_{j=1}^{n} x_{ij}, \ldots\right]; \tag{3.28}$$

that can be expressed also in coordinate free form:

$$\mathcal{S}\mathrm{eg} : \mathbb{P}\mathcal{H}_1 \times \cdots \times \mathbb{P}\mathcal{H}_n \to \mathbb{P}\left(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n\right); \tag{3.29}$$

$$[x_1], \ldots, [x_n] \mapsto [v_1 \otimes \cdots \otimes v_n]. \tag{3.30}$$

The elements of the imagine of this map are called pure separable states with respect to the fixed decomposition presented in equation (3.26); it's trivial to see that

this construction is analogous to the definition 3.1. This embedding solve completely the separability problem for the pure states, it is enough to check the value of the minors of the associated matrix as consequence of the proposition 1.3.

Example 15. *Let us consider a composite system formed by two qutrits:*

$$\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3 = \mathbb{C}^9; \tag{3.31}$$

and let us consider the following state:

$$|\psi\rangle = \frac{\sqrt{2}}{2} |1\rangle \otimes |3\rangle + \frac{\sqrt{2}}{2} |2\rangle \otimes |1\rangle ; \tag{3.32}$$

it is easy to check with the Corollary 3.1 that $|\psi\rangle$ is entangled, if its density state is built up:

$$\rho_\psi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{3.33}$$

and performing the partial trace over the first subsystem the reduced density matrix that we obtain is:

$$\mathrm{Tr}_A[\rho_\psi] = \rho_{\psi_B} = \frac{1}{2} |3\rangle \langle 3| + \frac{1}{2} |1\rangle \langle 1| ; \tag{3.34}$$

and now we have the certainty that $|psi\rangle$ is entangled. Now let us to use the criterion given by the Segre embedding; $|\psi\rangle$ in coordinates of the computational basis is given by:

$$\langle\psi| = \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{3.35}$$

that it can be written in the matrix form as:

$$\langle\psi| \mapsto \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.36}$$

If we compute all the nine possible 2-minors of (3.36) *we find all equal to zero except for the one:*

$$\det \left[\begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \right] = -\frac{1}{2}, \tag{3.37}$$

that confirm the fact that $|\psi\rangle$ is entangled.

This embedding is in relation to the representation of the unitary groups of the single parts of the total system:

$$U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_n) \ni (\sigma_1, \ldots, \sigma_n) \mapsto \sigma_1 \otimes \cdots \otimes \sigma_n \in U(\mathcal{H}) \tag{3.38}$$

with the scalar product defined in equation (A.3); in this way we can generalize the Segre embedding to:

$$\mathcal{S}\text{eg} : u^*(\mathcal{H}_1) \times \cdots \times u^*(\mathcal{H}_n) \to u^*(\mathcal{H}); \tag{3.39}$$

with a particular attention to the set of the density states, and we provide the following proposition:

Proposition 3.3. *The Segre immersion maps positive definite operators of rank k_i into positive definite operators of the composite systems, it acts as:*

$$\mathcal{P}^{k_1}(\mathcal{H}_1) \times \cdots \times \mathcal{P}^{k_n}(\mathcal{H}_n) \to \mathcal{P}^{\prod_i k_i}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) \tag{3.40}$$

and it is the density states of the same rank acts as:

$$\mathcal{D}^{k_1}(\mathcal{H}_1) \times \cdots \times \mathcal{D}^{k_n}(\mathcal{H}_n) \to \mathcal{D}^{\prod_i k_i}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) \tag{3.41}$$

Proof. Let $A_i \in \mathcal{P}^{k_i}(\mathcal{H}_i)$ with spectrum:

$$A_i \left|j\right\rangle_i = \lambda_{ij} \left|j\right\rangle_i . \tag{3.42}$$

A particular attention should be put on indexes: i is the index of the particular Hilbert Space, while j is the index of the basis element. We can construct the computational basis of the total Hilbert space from such class of states, namely: $\left|j_1\right\rangle_1 \otimes \cdots \otimes \left|j_n\right\rangle_n$, moreover this vectors are also eigenvectors of the tensor product operator $\bigotimes_i A_i$; the spectrum of the composite operator is then:

$$\bigotimes_i A_i \left|j_1\right\rangle_1 \otimes \cdots \otimes \left|j_n\right\rangle_n = \left(\prod_{i=1} \lambda_{ij}\right) \left|j_1\right\rangle_1 \otimes \cdots \otimes \left|j_n\right\rangle_n . \tag{3.43}$$

Obviously the newly built operator is not negative (the eigenvalues are the product of not negative numbers) and of rank $\prod_{i_1} k_i$. If all of A_i are with trace 1 then:

$$\sum_i \lambda_{ij} = 1 \Rightarrow \sum_{j_1,\ldots,j_n} \lambda_{j1} \ldots \lambda_{jn} = \prod_{j=1} \sum_{ji} \lambda_i = 1 \tag{3.44}$$

and so $\bigotimes_i A_i \in \mathcal{D}^{\prod_i k_i}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$. □

Now it is possible to define the various state sets: we will denote the image of the Segre embedding, namely $\mathcal{S}\text{eg}\left(\mathcal{D}^{k_1}(\mathcal{H}_1) \times \cdots \times \mathcal{D}^{k_n}(\mathcal{H}_n)\right)$ with $S^{k_1,\ldots,k_n}(\mathcal{H})$.

This embedding, when we deal with rank fixed density states, leads to the complete description of the product states of the composite system.

Example 16. *Let us consider in the Example 15 a compound system formed by two qutrits. The possible generalized Segre embeddings which give rise to the product states are:*

$\mathcal{S}\text{eg}(\mathcal{D}^1 \times \mathcal{D}^1)$	$\mathcal{S}\text{eg}(\mathcal{D}^1 \times \mathcal{D}^2)$	$\mathcal{S}\text{eg}(\mathcal{D}^1 \times \mathcal{D}^3)$
$\mathcal{S}\text{eg}(\mathcal{D}^2 \times \mathcal{D}^1)$	$\mathcal{S}\text{eg}(\mathcal{D}^2 \times \mathcal{D}^2)$	$\mathcal{S}\text{eg}(\mathcal{D}^2 \times \mathcal{D}^3)$
$\mathcal{S}\text{eg}(\mathcal{D}^3 \times \mathcal{D}^1)$	$\mathcal{S}\text{eg}(\mathcal{D}^3 \times \mathcal{D}^2)$	$\mathcal{S}\text{eg}(\mathcal{D}^3 \times \mathcal{D}^3)$

This construction leads to an important remark: if the rank of a density matrix of the compound system is 5,7 or 8 this means that it cannot be a product state because the aforementioned embeddings do not have image in these strata. This is a result that comes out directly from the geometrical structure of the Segre embedding.

The sets of the pure separable state, rank one projectors acting on $\mathcal{H}$ will be denoted simply as $S^1(\mathcal{H})$. The set of separable states are denoted $S(\mathcal{H})$ and correspond to the convex envelop of $\mathcal{S}\text{eg}\left(\mathcal{D}(\mathcal{H}_1) \times \cdots \times \mathcal{D}(\mathcal{H}_n)\right)$ (see of Preposition 3.4).

Definition 3.7. *The set of the entangled density states are defined by the quotient:*

$$\mathcal{E}(\mathcal{H}_1 \otimes \ldots \mathcal{H}_n) = \mathcal{D}(\mathcal{H}_1 \otimes \ldots \mathcal{H}_n)/S(\mathcal{H}_1 \otimes \ldots \mathcal{H}_n). \tag{3.45}$$

Proposition 3.4.

- $S(\mathcal{H})$ *is the convex envelop of* $S^1(\mathcal{H})$*;*
- $S^1(\mathcal{H})$ *is the set of the extremal points of* $S(\mathcal{H})$*;*
- $S^1(\mathcal{H})$ *is invariant under the action of:*

$$U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_n) \tag{3.46}$$

with the adjoint action:

$$(T_1, \ldots, T_n) \circ A \circ (T_1, \ldots, T_n)^\dagger \tag{3.47}$$

Proof. Let's start showing that the envelope of $S^1(\mathcal{H})$ contains the image of the Segre embedding:

$$\mathcal{S}\text{eg}\left(\mathcal{D}(\mathcal{H}_1) \times \cdots \times \mathcal{D}(\mathcal{H}_n)\right) \subseteq S(\mathcal{H}). \tag{3.48}$$

If $\mathcal{D}^1(\mathcal{H}_i)$ is the set of the extremal points of $\mathcal{D}(\mathcal{H}_i)$ for all subsystem, and $A_i \in \mathcal{D}(\mathcal{H})$ then we can write:

$$A_i = t_s^i \rho_s^i \quad \rho_s^i \in \mathcal{D}^1(\mathcal{H}_i); \tag{3.49}$$

while a generic product state is:

$$\bigotimes_i A_i = \sum_{\{s_i\}} t_{s_1}^1 \ldots t_{s_n}^n \underbrace{\rho_{s_1}^1 \otimes \cdots \otimes \rho_{s_n}^n}_{\mathcal{D}^1(\mathcal{H})}, \tag{3.50}$$

and it is extremal for $\mathcal{D}(\mathcal{H})$. It cannot be written as trivial linear combination of elements of $\mathcal{D}(\mathcal{H})$ but of a smaller set: $S(\mathcal{H})$.

The invariance is trivial. □

Now it is clear how the definition of separable states given in the previous proposition is coherent with the definition 3.5, but we have some new elements now: the set of extremal points of $\mathcal{D}(\mathcal{H})$ is isomorphic to the Segre embedding of the projective Hilbert spaces of the subsystems. There is a remark, the last part of the previous theorem ensure that entanglement is invariant under separable unitary operation of the two subsystems, moreover it can be proved (see [19]) that this invariance is true in bipartite case also for the action of $GL(\mathcal{H}_1) \times GL(\mathcal{H}_2)$. The preservation of the separability over the action (adjoint action in our case) of a Lie group brings us to recall the Orbit Method. This approach will be used in the next section in the qubit example.

We have now proved that the separable states are a convex set, the next step is to deal with the closeness.

Proposition 3.5. *Let V an n-dimensional real vector space and x a convex combination of point of V; then $x = \sum_i a_i x_i$ is a convex combination of at most n+1 point of x_i.*

And at last the compactness:

Proposition 3.6. *Let E a compact subset of a real vector space V; then the convex hull of E is compact.*

We have shown that the set of the separable state (the convex hull of the pure separable states) is a convex, closed and compact subset of $\mathfrak{u}^*(\mathcal{H})$; as already anticipated in the previous sections.

3.4 An Important Example: Two Qubits

As it can be seen in the previous sections the most simple system in which the phenomena of entanglement can be studied is the two qubits system. For this example we will follow mainly [5] and [8].

3.4.1 Pure States

The identification of $\mathbb{C}^2 \otimes \mathbb{C}^2$ as the total Hilbert space is straightforward, and the corresponding space of pure state can be identified as $\mathbb{P}^3\mathbb{C}$, with a generic point $[Z_0, Z_1, Z_2, Z_3]$. This gives rise to a fibration similar to the one presented in the previous chapter (see figure 3.2). The set of separable pure states as we have already seen is provided by the Segre Embedding:

$$\sigma : \mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C} \to \mathbb{P}^3\mathbb{C}; \tag{3.51}$$

$$\begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{C}^2 \otimes \mathbb{C}^2 \\ & & \downarrow \\ U(1) \simeq \mathbb{S}^1 & \longrightarrow & \mathbb{S}^7 \\ & & \downarrow \\ & & \mathbb{P}^3\mathbb{C} \end{array}$$

FIGURE 3.2: Fibration of the qubit case for pure states.

which provides the equation of the states in a polynomial form:

$$Z_0 Z_3 - Z_1 Z_2 = 0; \tag{3.52}$$

therefore, considering that the real dimension of $\mathbb{C}^3\mathbb{P}$ is 6 (see figure 3.2); equation (3.52) reduces the dimensionality of the space of the pure separable states to 4. This can be seen in many different ways; this method can be generalized but for this propose we will focus only on two qubits systems. Let us focus on the Schmidt decomposition of the general pure state (see theorem 3.1):

$$|\psi\rangle = \sum_{i=1}^{2} \sqrt{q_i}\, |\phi^i\rangle_A\, |\chi^i\rangle_B\,; \tag{3.53}$$

the Schmidt coefficients determine uniquely the state $|\psi\rangle$; so in our case the Schmidt vector can be:

1. (a, b) with generic $a \neq b$;
2. $(1, 0)$;
3. $(\frac{1}{2}, \frac{1}{2})$;

these three cases correspond respectively to a general pure state, a pure separable, and to a maximum entangled state. Now from here we can compute the different orbits on $\mathbb{P}^3\mathbb{C}$.

To do this let consider a bigger system in order to have a generalized treatment:

$$\mathcal{H}_N \otimes \mathcal{H}_K = \mathcal{H}_{NK}; \tag{3.54}$$

of the respectively dimensions: N, K and NK (for the sake of simplicity we assume that $N \leq K$); therefore the correspondent Schmidt vector in this case has N components (N-1 independent components):

$$\vec{q} = (\underbrace{q_1, \ldots, q_1}_{n_1\text{-times}}\, \underbrace{q_2, \ldots, q_2}_{n_2\text{-times}}, \ldots, \underbrace{q_j, \ldots, q_j}_{n_j\text{-times}}, \underbrace{0, \ldots, 0}_{n_0\text{-times}}); \tag{3.55}$$

where n_k is the multiplicity of the value of q_k, and the following relation holds true:

$$\sum_{j=0}^{J} n_j = N. \tag{3.56}$$

The group that leaves the entanglement invariant (it preserves the separability) is a subgroup of SU(NK) that acts as:

$$|\phi\rangle' = (U_N \otimes U_K)\,|\phi\rangle\,; \tag{3.57}$$

with $U_N \in SU(N)$ and $U_K \in SU(K)$. The local orbit generated by that action has a structure of a principal bundle (that is not trivial in general); it has as base space:

$$U(N) \Big/ [U(n_0) \times \cdots \times U(n_j)]; \tag{3.58}$$

and as fibre:

$$U(N) \Big/ [U(n_0) \times U(1)] \tag{3.59}$$

and the corresponding dimension of the local orbit is computable from the equation:

$$2N^2 - 1 - 2n_0^2 - \sum_{i=1}^{j} n_i^2. \tag{3.60}$$

If we come back to our case of 2 qubits we have three cases:

1. (a, b) with $0 \neq a \neq b \neq \frac{1}{2}$ the local orbit has the local structure:

$$\left(U(2) \Big/ [U(1) \times U(1)]\right) \times \left(U(2) \Big/ U(1)\right); \tag{3.61}$$

   with the consequent real dimension 5. As last remark we want to emphasise that this fibration can be written in the form:

$$F^2 \times \mathbb{P}^3\mathbb{R}, \tag{3.62}$$

   where F^2 is a flag manifold (see appendix C).

2. The situation $(1, 0)$ corresponds to the separable pure state the local orbit has the structure:

$$\left(U(2) \Big/ [U(1) \times U(1)]\right) \times \left(U(2) \Big/ [U(1) \times U(1)]\right); \tag{3.63}$$

   with the consequent real dimension 4. This fibration can be written in the form:

$$\mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C}; \tag{3.64}$$

   that is clearly the Segre variety.

3. The last situation the Schmidt's vector is:

$$\left(\frac{1}{2},\frac{1}{2}\right), \tag{3.65}$$

and represent the maximum entangled pure state (the Bell states), the consequent form of the local orbit is:

$$\left(U(2)\Big/U(2)\right)\times\left(U(2)\Big/[U(1)]\right); \tag{3.66}$$

that is isomorphic to a point for the base space and $\mathbb{RC}^3$ in the fibre (this appends only in this particular system), this can be seen better with the identification of:

$$\mathbb{P}^3\mathbb{R}\simeq SU(2)/\mathbb{Z}_2 \tag{3.67}$$

as another form of this fibration; this is important because the sets of the type $SU(N)/\mathbb{Z}_N$ are a Lagrangian and a minimal submanifold (see Appendix B).

A general treatment of the foliation of the pure state set based on the Schmidt theorem (theorem 3.1)will be presented in the next remark.

Remark 3.3 (General Foliation of Pure State)**.** *In this remark we will show briefly the previous procedure on $\mathbb{P}^{N^2-1}\mathbb{C}$: the space of the pure states of a Hilbert space: $\mathbb{C}^N\otimes\mathbb{C}^N$.*

- *For a generic state the orbit is:*

$$\mathcal{O}=\left(U(N)\Big/U^N(1)\right)\times\left(U(N)\Big/U(1)\right)=F^N\times\left(U(N)\Big/U(1)\right); \tag{3.68}$$

where F^N is a flag manifold; the dimension of this orbit is N^2-N-1.

- *The separable states give rise to the Segre embedding one more time:*

$$\begin{aligned}\mathcal{O}_{sep}=&\left(U(N)\Big/U(N-1)\times U(1)\right)\times\left(U(N)\Big/U(N-1)\times U(1)\right) &(3.69)\\ =&\mathbb{P}^{N-1}\mathbb{C}\times\mathbb{P}^{N-1}\mathbb{C}; &(3.70)\end{aligned}$$

which has as real dimension: $4(N-1)$.

- *The last is the sub-set of the maximally entangled states:*

$$\mathcal{O}=\left(U(N)\Big/U(N)\right)\times\left(U(N)\Big/U(1)\right)=SU(N)/\mathbb{Z}_N; \tag{3.71}$$

with dimension N^2-1, the half of the total dimensionality of the total pure states.

This create a singular foliation (a foliation in which there exist leaves with measure zero of various topology and dimensions, for more details see appendix A) of $\mathbb{P}^{N-1}\mathbb{C}$.

Now another way to study the pure states is passing to the density matrix formalism: in this case a state is represented by a rank-one projector. If a state $|\psi\rangle$,

represented by $[Z_0, Z_1, Z_2, Z_3]$, has a density matrix $\rho = |\psi\rangle \langle\psi|$ of the form:

$$\rho = \begin{pmatrix} z_0^2 & z_0 z_1^* & z_0 z_2^* & z_0 z_3^* \\ z_1 z_0^* & z_1^2 & z_1 z_2^* & z_1 z_3^* \\ z_2 z_0^* & z_1^* z_2 & z_2^2 & z_2 z_3^* \\ z_3 z_0^* & z_3 z_1^* & z_3 z_2^* & z_3^2 \end{pmatrix}; \tag{3.72}$$

If we choose to denote:

$$\Gamma_{ij} = (\Gamma_{00}, \Gamma_{01}, \Gamma_{10}, \Gamma_{11}) = (Z_0, Z_1, Z_2, Z_3); \tag{3.73}$$

the density matrix can be seen as a 4-index object:

$$\rho_{(ij)(kl)} = \frac{1}{N} \Gamma_{ij} \Gamma_{kl}^*; \tag{3.74}$$

Now it is easy to compute the partial trace, this operation is a good tool to determine if a state is entangled or not: if the reduced density matrix is of a pure state than the original state is separable, otherwise it is entangled.

Remark 3.4. *The Schmidt's coefficients plays a central role because they are the eigenvalues of the reduced density matrix indeed.*

Let's see two different cases:

- We are looking for the reduced density matrix with eigenvalues (1,0): so computing the partial trace:

$$\mathrm{Tr}_A \, \rho = \sum_k \rho_{(kj)(kl)} = \sum_k \Gamma_{kj} \Gamma_{kl}^* \tag{3.75}$$

- We are looking for the reduced density matrix with eigenvalues (1/2,1/2): so computing the partial trace:

$$\mathrm{Tr}_A \, \rho = \sum_k \rho_{(kj)(kl)} = \sum_k \Gamma_{kj} \Gamma_{kl}^* = \frac{1}{2} \delta_{j,l}; \tag{3.76}$$

this means that Γ_{ij} is a unitary matrix and in the coordinates of $\mathbb{P}^3\mathbb{C}$ it is represented by $[a, b, -b^*, a^*]$. This is the case of the Lagrangian sub-manifold (see appendix B).

3.4.2 Mixed States

Let us start using the orbit method to classifying the all possible structures of the density matrix due to the action of the unitary group $U(4)$.

Remark 3.5. *We recall from the previous sections that $U(4)$ do not preserve entanglement. The biggest group that preserves entanglement is the subgroup of $GL(4)$ formed by*

$$\begin{array}{ccc} U^4(1) & \longrightarrow & U(4) \\ & & \downarrow \\ & & U(4)/U^4(1) \end{array}$$

(A) Fibration for a system of two qubit a situation of the type: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$.

$$\begin{array}{ccc} U^2(1) & \to & U(4)/U(2) \\ & & \downarrow \\ & & U(4)/[U^2(1) \times U(2)] \end{array}$$

(B) Fibration for a system of two qubit for a situation of the type: $\lambda_1 \neq \lambda_2 \neq \lambda_3 = \lambda_4$.

$$\begin{array}{ccc} U^2(2) & \longrightarrow & U(4) \\ & & \downarrow \\ & & U(4)/[U(2) \times U(2)] \end{array}$$

(C) Fibration for a system of two qubit for a situation of the type: $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$.

$$\begin{array}{ccc} U^2(1) & \to & U(4)/U(3) \\ & & \downarrow \\ & & U(4)/[U(3) \times U(1)] \end{array}$$

(D) Fibration for a system of two qubit for a situation of the type: $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$.

FIGURE 3.3: Fibrations of a mixed states for a system with two qubits.

separable matrices of the form:

$$A \otimes B; \tag{3.77}$$

with $A, B \in GL(2)$.

A 4x4 density matrix (Hermitian, semi-positive, with trace one) can be written in the form:

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}; \tag{3.78}$$

using the adjoint action of $U(4)$; we can divide the orbits in the following cases [8]:

1. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$. In this case the possible fibration is pictured in figure 3.3a. The dimension of this sub-manifold is easy to calculate knowing that:

$$M_1 = U(4)/[U(1)]^4; \tag{3.79}$$

and so $\dim_{\mathbb{R}} M_1 = 12$.

2. $\lambda_1 \neq \lambda_2 \neq \lambda_3 = \lambda_4$. In this case (fibration in figure 3.3b) the manifold take the structure:

$$M_2 = U(4)/[U^2(1) \times U(2)]; \tag{3.80}$$

and so its dimension is $\dim_{\mathbb{R}} M_2 = 10$.

3. $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$. The corresponding fibration is pictured in 3.3c. The manifold has the structure:

$$M_3 = U(4)/[U(2) \times U(2)]; \tag{3.81}$$

and so its dimension is $\dim_{\mathbb{R}} M_2 = 8$.

4. $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$. The manifold take the form:

$$M_4 = U(4)/[U(3) \times U(1)] \simeq \mathbb{P}^3\mathbb{C}; \tag{3.82}$$

and so its dimension is $\dim_{\mathbb{R}} M_2 = 6$. The fibration is pictured in 3.3d. This case is fundamental because, if $\lambda_2 = \lambda_3 = \lambda_4 = 0$, we are in the case of the pure states: rank one projector; the final result is in fact the same of the pure state already treated.

5. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{4}$. In this last case the orbit has the structure:

$$M_4 = U(4)/[U(4)]; \tag{3.83}$$

and so its dimension is $\dim_{\mathbb{R}} M_2 = 0$ and it is isomorphic to a point.

Now let us consider the general structure of a mixed state of a 2×2 using the base of $SU(2) \otimes SU(2)$ (similar to the structure of the density matrix of a qbit):

$$\rho = \frac{1}{4}\left[\mathbb{1}_4 + \sum_{k=1}^{3} \epsilon_{k0}^A \sigma_k^A \otimes \mathbb{1}_2 + \sum_{s=1}^{3} \epsilon_{0s}^B \mathbb{1} \otimes \sigma_s^B + \sum_{k=1}^{3}\sum_{s=1}^{3} \epsilon_{ks} \sigma_k^A \otimes \sigma_s^B\right]. \tag{3.84}$$

This can be extended to a bipartite system whose components have respectively dimension N and K with the basis of $SU(N) \otimes SU(K)$: it is called Fano's form.

The coefficients ϵ_{ks} represent the interaction between the two subsystems, in particular it can be shown [5] that if they are all zero than ρ is separable; the other way is not true as we will see. Now lets see the Fano's form of a separable state: recalling the definition 3.5:

$$\rho = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B \tag{3.85}$$

this is a element of a subset of the manifold $u^*(\mathbb{C}^4)$ that has the scalar product:

$$\langle A, B \rangle_{u^*} = \mathrm{Tr}\,[AB] = \mathrm{Tr}_A[\mathrm{Tr}_B[AB]] = \mathrm{Tr}_B[\mathrm{Tr}_A[AB]]. \tag{3.86}$$

We can compute the coefficients using this method:

$$\mathrm{Tr}[\rho \mathbb{1}] = 1; \tag{3.87}$$

that is trivial. The second step is compute the coefficients in front of the part of the basis like $\sigma_k^A \otimes \mathbb{1}$, in the following we denote with $\tau_i j$ a sort of generalized Block

vector:

$$\mathrm{Tr}[\rho(\sigma_k^A \otimes \mathbb{1})] = \tag{3.88}$$
$$= Tr_B[\sum_i \lambda_i \rho_i^B Tr_A[\rho_i^A \sigma_k^A]] \tag{3.89}$$
$$= Tr_B[\sum_i \lambda_i \rho_i^B Tr_A[(\frac{1}{2}\mathbb{1}_2 + \sum_l \tau_{il}^A \sigma_{il}^A)\sigma_k^A]] \tag{3.90}$$
$$= Tr_B[\sum_i \lambda_i \rho_i^B Tr_A[(\frac{1}{2}\sigma_k^A + \sum_l \tau_{il}^A(\delta_{k,l}\mathbb{1}_2 + i\epsilon_{kls}\sigma_s))]] \tag{3.91}$$
$$= Tr_B[\sum_i \lambda_i \tau_{ik}^A \rho_i^B] \tag{3.92}$$
$$= \sum_i \lambda_i \tau_{ik}^A; \tag{3.93}$$

where we have used the relations presented in the sections C.1 and C.2. The same calculation holds for the part of the basis of the form: $\mathbb{1} \otimes \sigma_k^B$:

$$\mathrm{Tr}[\rho(\mathbb{1} \otimes \sigma_k^B)] = \sum_i \lambda_i \tau_{ik}^B. \tag{3.94}$$

The last calculations involve the part of the basis in the form: $\sigma_k^A \otimes \sigma_s^B$, but also these can be calculated in a similar way:

$$\mathrm{Tr}[\rho(\sigma_k^A \otimes \sigma_s^B)] = \sum_i \lambda_i \,\mathrm{Tr}_A[\rho_i^A \sigma_k^A]\,\mathrm{Tr}_B[\rho_i^B \sigma_s^B]; \tag{3.95}$$

but we know already the results of the single partial traces, and that leads to:

$$\mathrm{Tr}[\rho(\sigma_k^A \otimes \sigma_s^B)] = \sum_i \lambda_i \tau_{ik}^A \tau_{is}^B. \tag{3.96}$$

In conclusion the form of a separable state is:

$$\rho = \sum_i \frac{\lambda_i}{4}\left[\mathbb{1} + \sum_{k=1}^{3} \tau_{ik}^A \sigma_k^A \otimes \mathbb{1} + \sum_{s=1}^{3} \tau_{is}^B \mathbb{1} \otimes \sigma_s^B + \sum_k^3 \sum_s^3 \tau_{ik}^A \tau_{is}^B \sigma_k \otimes \sigma_s\right]. \tag{3.97}$$

The analysis now consist in confronting equations (3.84) and (3.97) in order to see how the coefficients change in the two different forms.

The coefficient in front of the identity leads to the trivial relation of $\sum_i \lambda_i = 1$, which derives from the definition of density matrix. The other relations are:

$$\sum_i \lambda_i \tau_{ik}^A = \epsilon_{k0}^A; \tag{3.98}$$
$$\sum_i \lambda_i \tau_{is}^B = \epsilon_{0s}^B; \tag{3.99}$$
$$\sum_i \lambda_i \tau_{ik}^A \tau_{is}^B = \epsilon_{ks}; \tag{3.100}$$

these relations give in the parameter manifold the equations of pure states. It is

important to remind that the minimal number of λ_i is bounded as it can be seen in the initial part of this chapter.

Let's now recall Theorem 3.2 that in our case characterize completely the set of separable states in the cases of systems with two qubits or with a qubit and a qutrit. The application of the partial transpose (3.20) to a general density matrix written in the Fano's form (3.84) leads to the following expression:

$$\rho^{T_A} = \frac{1}{4}\left[\mathbb{1}_4 - \sum_{k=1}^{3} \epsilon_{k0}^{A}\sigma_k^A \otimes \mathbb{1}_2 + \sum_{s=1}^{3} \epsilon_{0s}^{B}\mathbb{1} \otimes \sigma_s^B - \sum_{k=1}^{3}\sum_{s=1}^{3} \epsilon_{ks}\sigma_k^A \otimes \sigma_s^B\right]. \tag{3.101}$$

Remark 3.6. *Acting with $T \otimes \mathbb{1}$ or with $\mathbb{1} \otimes T$ on a density matrix is the same for our propose: if H is a block matrix with the components:*

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.102}$$

the two operations leads respectively to:

$$H^{T_A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \text{ and } H^{T_B} = \begin{pmatrix} A^T & B^T \\ C^T & D^T \end{pmatrix}; \tag{3.103}$$

but hold also the relation: $H^{T_B} = (H^{T_A})^T$ and so the two results has the same eigenvalues and so:

$$H^{T_B} > 0 \iff H^{T_A} > 0. \tag{3.104}$$

Chapter 4

Conclusions, New Perspectives and Open Problems

In this chapter an new algebraic approach to the separability problem will be presented and the conclusion of this work will be drawn. For the mathematical notions of the first section the main references are [15] and [14].

4.1 An algebraic approach

In this section we would like to outline some new elements for the separability problem. The main idea is to link the representation theory of the symmetric group $\mathfrak{S}_n$ (it is often used when dealing with many particle symmetries) to the index symmetries of a tensor; the mathematical framework that is needed to understand this approach is presented in Appendix A.4.

Let us consider a vector space V with a finite dimension, with the tensor product it is possible to built up a space:

$$\underbrace{V \otimes \cdots \otimes V}_{k}; \tag{4.1}$$

of rank k tensors. It is well know that a rank 2 tensor $t_{\mu\nu} \in V \otimes V$ can be decomposed into symmetric and skew-symmetric part:

$$t_{\mu\nu} = \frac{1}{2} \underbrace{\left(t_{\mu\nu} + t_{\nu\mu}\right)}_{\text{symmetric part}} + \frac{1}{2} \underbrace{\left(t_{\mu\nu} - t_{\nu\mu}\right)}_{\text{skew-symmetric part}} \quad ; \tag{4.2}$$

this symmetry splits the space of rank two tensor in two submodules (see A):

$$V \otimes V = S^2(V) \oplus \bigwedge^2(V). \tag{4.3}$$

This decomposition can be extended to higher dimensional tensors, but the tensor space is not more splitted into symmetric and skew-symmetric parts only, the index permutations are linked with the so called Young Symmetrizer, that gives rise to the so called Schur functors and the IRR's of the symmetric group [15].

The tensor product between Schur functors is ruled by its relation to the Young Tableaux and in literature it is a well studied subject. If we consider for example the tensor product of two symmetric tensors of rank 2 in a vector space V the problem is to compute:

$$S^2(V) \otimes S^2(V); \tag{4.4}$$

this can be computed with the Pieri formula [15]:

$$S^2(V) \otimes S^2(V) = \bigoplus_{0 \leq a \leq 2} \mathbb{S}_{(2+a,2-a)}(V) = S^4(V) \oplus \mathbb{S}_{(3,1)}(V) \oplus \mathbb{S}_{(2,2)}(V). \tag{4.5}$$

This constrains all the linear combination of the tensor product of symmetric matrices over V, that are a subspace of the symmetric matrices over $V^{\otimes 2}$. Let us consider $S^2(V^{\otimes 2})$, also this subspace can be decomposed in a similar way [15]:

$$S^2(V \otimes V) = \bigoplus_{\lambda} \mathbb{S}_\lambda V \otimes \mathbb{S}_\lambda V; \tag{4.6}$$

where λ is any partition of 2: $(1,1)$ and $(2,0)$; so it can be expressed as:

$$S^2(V^{\otimes 2}) = S^4(V) \oplus \mathbb{S}_{(3,1)}(V) \oplus \mathbb{S}_{(2,1,1)}(V) \oplus \mathbb{S}_{(2,2)}(V)^{\oplus 2} \oplus \bigwedge^4 V. \tag{4.7}$$

It is evident that an element of $S^2(V) \otimes S^2(V)$ is also an element of $S^2(V^{\otimes 2})$ but the opposite doesn't hold.

4.1.1 The application in the separability problem.

Let us consider a bipartite compound system:

$$\mathcal{H} = \mathcal{H}'_A \otimes \mathcal{H}'_B; \tag{4.8}$$

if the two subsystems are equal it is possible to write $\mathcal{H} = \mathcal{H}'^{\otimes 2}$. A density matrix of this system is formally an element of:

$$\mathcal{H} \otimes \mathcal{H}^* = (\mathcal{H}')^{\otimes 2} \otimes (\mathcal{H}'^*)^{\otimes 2}; \tag{4.9}$$

where with $\mathcal{H}'^*$ is denoted the dual space of the original Hilbert space. We know that the density states set is a subset of the set of the Hermitian operators, that we have denoted as $\mathfrak{u}^*(\mathcal{H})$.

The connection between the separability problem, which was introduced in the previous chapter, and the representation theory is to combine the action of the symmetric group with the complex conjugation operators; in order to expand this construction from the symmetric part to the Hermitian part.

This approach is strictly related to the contemporary research in algebraic geometry [7], [31] and in tensor networks [21]: it is related with an attempt to generalize the singular value decomposition (which is also fundamental for the Schmidt

Decomposition). As we have already shown a density matrix of a compound system can be seen as high rank tensor because is a element of the set of matrices of a tensor product space.

Example 17. *For a bipartite system as the two qubits system the density matrices can be seen as a multi-index object, fixing a base we obtain:*

$$\rho = \rho_{ijks} \left|e_i\right\rangle \left\langle e_j\right| \otimes \left|f_k\right\rangle \left\langle f_s\right|, \tag{4.10}$$

which linearity is insured by its nature of Hermitian operator.

The state of art of this research is focusing on Unitarily Decomposable Tensors that are a particular type of tensor that can be written with orthogonal vectors with respect to a Hermitian product [7].

Definition 4.1. *Let $\{V_i\}_i$ be a set of vector spaces over the field of complex number $\mathbb{C}$ with an Hermitian product. An element σ of the tensor product space:*

$$\bigotimes_i V_i, \tag{4.11}$$

is called Unitarily Decompostable (or udeco) if it can be written in the form:

$$\sigma = \sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{id}; \tag{4.12}$$

where an orthogonal relations holds:

$$\langle v_{sj} | v_{kj} \rangle_{V_j} = \delta_{k,s} \quad \forall j. \tag{4.13}$$

We want to emphasise that the difficulties of this approach are due to the lack of literature on this subject since it is also a new mathematical framework.

4.2 Conclusions

In this work we have seen how quantum theory can be approached with a geometrical perspective, in particular we have focused on the quantum phenomena of Entanglement in both traditional and geometrical framework in order to give the most complete global view of the subject.

In the last two chapters it is highlighted how the geometrical approach cannot be avoided if we want a complete description of entanglement: the main result is the characterization of structure of the set of separable states in the Projective Hilbert Space of a compound system as an algebraic variety using the Segre embedding. The use of the orbit method in combination with the Schmidt decomposition divide not only the separable and the entangled pure states but characterize in a certain way also its measure.

Despite this elaborate construction, the description of the separable mixed states set isn't complete, the problems without solution are still many: there are a lot of criteria that characterize either sufficient or necessary conditions, and so they describe either smaller or bigger sets. The numerical approaches is followed when we are interested to an approximation of the set because for now the separability problem is a NP-hard problem and so these methods are usually time consuming and expensive for higher dimensional systems.

The geometrical approach to the separability problem (which is introduced in chapter 3) could be helpful also if we want to generalize the Proposition 3.2 to the mixed states: there is a relation between the rank of the reduced density matrix and the original density state due to the separability propriety? This fact becomes evident looking at the definition 3.5 of a separable state , if a density state:

$$\rho = \sum_i \xi_i \rho_i^A \otimes \rho_i^B, \tag{4.14}$$

is separable then the whole orbit of ρ, which is generated by the action of the group $U(n) \times U(m) \subset U(nm)$, is composed by separable states, $U(n)$ and $U(m)$ diagonalize the respective reduced density matrices, and the ranks are related to the Determinantal variety introduced in chapter 1.

Finally the algebraic approach introduced in this last chapter could be an interesting point of view which, however, is still in an early phase of development.

Appendix A

Some complements on linear Algebra

A.1 Tensor Product

Let be W_1 and W_2 two vectors spaces over a field K; the tensor product $W_1 \otimes W_2$ is another vector space:

$$W_1 \times W_2 \to W_1 \otimes W_2 \tag{A.1}$$

$$(w_1, w_2) \mapsto w_1 \otimes w_2 \tag{A.2}$$

If W_1 and W_2 have for basis respectively $\{e_i\}_i$ and $\{f_j\}_j$, the basis of $W_1 \otimes W_2$ is $\{e_i \otimes f_j\}_{i,j}$. A similar construction can be made for an arbitrary number of vector spaces: $W_1 \otimes W_2 \otimes \cdots \otimes W_n$. It is possible to prove a series of proprieties of this construction for example uniqueness or induction to isomorphic spaces, we refer to the appendix B of [15]. The main proprieties of tensor product of spaces are:

- Commutativity: $W_1 \otimes W_2 \simeq W_2 \otimes W_1$
- Distributivity: $(W_1 \oplus W_2) \otimes V \simeq (W_1 \otimes V) \oplus (W_2 \otimes V)$
- Associativity: $W_1 \otimes W_2 \otimes W_3 \simeq (W_1 \otimes W_2) \otimes W_3) \simeq (W_1 \otimes W_2) \otimes W_3$

If in every vector spaces W_i a scalar or a Hermitian product is defined then the tensor product space $W_1 \otimes W_2 \otimes \cdots \otimes W_n$ inherits a scalar or a Hermitian product (we will use the bracket notation):

$$\langle w_1 \otimes \cdots \otimes w_n | v_1 \otimes \cdots \otimes v_n \rangle_{W_1 \otimes \cdots \otimes W_n} = \langle w_1 | v_1 \rangle_{W_1} \cdots \langle w_n | v_n \rangle_{W_n} . \tag{A.3}$$

In the case of $W_i = W$ for all i, we will denote:

$$\underbrace{W \otimes \cdots \otimes W}_{\text{n-times}} = W^{\otimes n}. \tag{A.4}$$

A.2 Exterior Powers

The exterior product of a vector space V is usually denoted by:

$$\bigwedge^{n} V \text{ or } Alt^{n}(V). \tag{A.5}$$

This set can be construct as the quotient of $V^{\otimes n}$ from the subspaces generating by $v_1 \otimes \cdots \otimes v_n$ where exist at least two $v_i = v_j$ with $i \neq j$. So there is a projection π such that:

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n. \tag{A.6}$$

In a similar way the exterior product can be also constructed as a subspace of $V^{\otimes n}$; the imbedding of this space is given by:

$$\mathfrak{i}(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}; \tag{A.7}$$

where $\mathfrak{S}_n$ is the permutation (symmetric) group of order n, and σ one of its elements.

A.3 Symmetric Powers

The symmetric power of a vector space V, usually denoted by $Sim^d V$ or $S^d V$; can be constructed as the quotient of $V^{\otimes d}$ by the subspace generated by:

$$v_1 \otimes \cdots \otimes v_d - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}; \tag{A.8}$$

where $\sigma(\cdot)$ permutes two successive factors. The projection π can be build up in the following way:

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \vee \cdots \vee v_n. \tag{A.9}$$

In analogy to the exterior powers we can define this set as a subspace of $V^{\otimes n}$ with the following imbedding:

$$\mathfrak{i}(v_1 \vee \cdots \vee v_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}; \tag{A.10}$$

where $\mathfrak{S}_n$ is the permutation (symmetric) group of order n, and σ one of its elements.

A.4 Other Multi-linear spaces

The symmetric product and the exterior product are particular cases of a deep decomposition linked to the partition of a integer. Let us consider a the tensor product

of d identical spaces:

$$\underbrace{W \otimes \cdots \otimes W}_{d} = W^{\otimes d}; \tag{A.11}$$

which has $(\dim W)^d$ as dimension. The space $W^{\otimes d}$ can be split into irreducible representation (IRR's) of the symmetric group $\mathfrak{S}_d$ which dimension is related to the Young Tableaux [15].

A.4.1 Young Tableaux and Young Symmetrizers

Before defining the Young Tableaux it is useful to explain what a partition of an integer is; let us consider a $d \in \mathbb{N}$, there are several way to build up it as a sum:

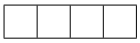
$$d = d \tag{A.12}$$
$$d = (d-1) + 1 \tag{A.13}$$
$$\vdots \tag{A.14}$$
$$d = 1 + \cdots + 1 \tag{A.15}$$

a partition λ of d is a non-increasing order vector $(\lambda_1, \ldots, \lambda_s)$ such that $\sum_i \lambda_i = d$; a convenient way to represent a partition is the Young frame.

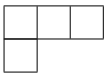
Definition A.1 (Young Frame)**.** *A young tableaux of dimension d is a collection of d boxes left-justified that represent a particular partition of d.*

Example 18. *Let us consider d=4, the possible partitions and the related frames are:*

$4 = 4$

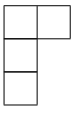


$= 3 + 1$



$= 2 + 2$

$= 2 + 1 + 1$



$= 1 + 1 + 1 + 1.$

When the boxes of a Young frame are being numbered the frame is called a Young Tableaux; for our propose it is interesting only the standard (or canonical) labelling that consists in labelling the boxes with the numbers $\{1, \ldots, d\}$ one time for each number in an increasing way.

Example 19. *Let us consider example 18, the associated Young tableaux with the canonical labelling are:*

$4 = 4$ | 1 2 3 4

$= 3 + 1$ | 1 2 3 / 4

$= 2 + 2$ | 1 2 / 3 4

$= 2 + 1 + 1$ | 1 2 / 3 / 4

$= 1 + 1 + 1 + 1.$ | 1 / 2 / 3 / 4

Given a partition and consequently a standard Young tableaux it is possible to construct two subgroups of the symmetric group $\mathfrak{S}_d$:

$$P = P_\lambda = \{g \in \mathfrak{S}_d : g \quad \text{preserves each row}\}, \tag{A.16}$$

and

$$Q = Q_\lambda = \{g \in \mathfrak{S}_d : g \quad \text{preserves each column}\}, \tag{A.17}$$

with these two subgroups we can define the Young Symmetrizers that are operators that act on $W^{\otimes d}$.

Definition A.2 (Young Symmetrizers)**.** *Let us consider an integer d and a partition λ of it. To the subgroups P and Q of $\mathfrak{S}_d$ are associated the two operators a_λ and b_λ such that:*

$$a_\lambda = \sum_{g \in P} e_g, \qquad b_\lambda = \sum_{g \in Q} sgn(g) e_g, \tag{A.18}$$

where e_g represents an index permutation of an element of $W^{\otimes d}$ with respect to the correspondent element of the symmetric group. A Young Symmetrizer is an operator c_λ defined as:

$$c_\lambda = a_\lambda b_\lambda = \left(\sum_{g \in P} e_g \right) \left(\sum_{g \in Q} sgn(g) e_g \right). \tag{A.19}$$

The Young Symmetrizers are important because their action split $W^{\otimes d}$ with respect to the symmetries of indexes; we have already treated the most common situations: $S^d(W)$ and $\wedge^d(W)$.

Example 20. *Let us still consider the situation of Example 18. The submodule $S^4(W)$ is associated to the young tableaux:*

1 2 3 4

that leads to the following Young symmetrizer:

$$c_\lambda(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = \sum_{g \in \mathfrak{S}_4} v_{g(1)} \otimes v_{g(2)} \otimes v_{g(3)} \otimes v_{g(4)}. \tag{A.20}$$

A similar procedure can be made for $\wedge^4(W)$ that is associated to the Young Tableaux:

and leads to skew-symmetry.

Remark A.1. *It is now quite clear that in a Young Tableaux the row represents the symmetry while the column the skew-symmetry.*

The image of the Young Symmetrizers are IRR's of the symmetric group $\mathfrak{S}_d$ labelled by the partitions λ, and they are represented by: $\mathbb{S}_\lambda$.

Definition A.3 (Schur Functor)**.** *A Schur functor $\mathbb{S}_\lambda$ associated to a partition λ of a integer d is:*

$$\mathbb{S}_\lambda(W) = c_\lambda(W^{\otimes d}) \tag{A.21}$$

For a deeper treatment of the Young Tableaux than the one presented here and its relation to representation theory please refer to [15].

A.5 Realification

The realification, or decomplexification, of a vector space V defined on the field of complex number is a procedure that leads to another vector space $V_{\mathbb{R}}$ defined on the field of real numbers.

Considering V and retaining the multiplication only over $\mathbb{R}$, than we get a vector field over $\mathbb{R}$ that is the realification of V.

Theorem A.1. *1. Let be $\{|j\rangle\}_j$ a basis of V over $\mathbb{C}$, then a basis of $V_{\mathbb{R}}$ over $\mathbb{R}$ is*

$$\{|j\rangle\}_j \cup \{\imath\,|j\rangle\}_j. \tag{A.22}$$

2. If $A = B + \imath B$ is a matrix in $\mathfrak{gl}(V)$ in the basis $\{|j\rangle\}_j$, where B and C are real matrices, then the realification of A is:

$$A_{\mathbb{R}} = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}; \tag{A.23}$$

in the base (A.22)

Now it is useful to recall how the set of operators acting on the original space becomes in the realified space:

- Unitary matrices ($A^\dagger = A^{-1}$) become orthogonal matrices ($A^T = A^{-1}$);
- Hermitian matrices ($A = A^\dagger$) become symmetric matrices ($A = A^T$);

Appendix B

Complements on Differential Geometry, Lie Groups and Lie Algebras

In this appendix will be presented complements on Differential Geometry, Lie Groups and Lie Algebras needed to understand the mathematical framework of this thesis.

B.1 Basic Differential Geometry

For this section the main reference is [34], but we will take elements also from [39].

Definition B.1 (Differential Manifold)**.** *M is a m dimensional differentiable ($\mathbb{C}^\infty$) manifold if:*

- *M is a Topological Space;*
- *Exist a family of $\{(U_i, \phi_i)\}$ (called atlas) such that $\{U_i\}$ cover M:*

$$\bigcup_i U_i = M; \tag{B.1}$$

and $\{\phi_i\}$ are homeomorphism such that:

$$\phi_i : U_i \to \mathbb{R}^m. \tag{B.2}$$

- *If two open sets has a non trivial overlap:*

$$U_i \cap U_j \neq \emptyset; \tag{B.3}$$

than the map $\psi_{ij} = \phi_i \circ \phi_j^{-1}$ is differentiable.

In order to characterize the set inside a manifold is fundamental introduce the concept of differential of a map.

Definition B.2 (Differential Map)**.** *Let be M and N two differential manifold and consider a map:*

$$f : M \to N; \tag{B.4}$$

then f induces a map:

$$df : T_pM \to T_{f(p)}N. \tag{B.5}$$

The differential of a map can be constructed in coordinates in the point $p \in M$ as:

$$df(x^\mu) = \frac{\partial f(x^\mu)}{\partial x^\nu} dx^\nu. \tag{B.6}$$

So now it is possible to define a sub manifold, an immersion and a embedding.

Definition B.3 (Submanifold, and Embedding)**.** *Let M and N be two differential manifolds with respectively dimensions $\dim(M) = m \leq \dim(N) = n$, and consider a map:*

$$f : M \to N; \tag{B.7}$$

then:

- *The map f is a immersion of M in N if df is injective.*
- *The map f is a embedding if it is injective and an immersion. In this case $f(M)$ (the image of f in N) is a sub-manifold of N.*

Related to this definition there are two theorems that regard the inverse of a function and the submersions.

Theorem B.1 (Local Inversion)**.** *Let M and N be two manifolds and f a map such that:*

$$f : M \to N, \tag{B.8}$$

then $df_p \neq 0$ with $p \in M$ iff exist an open set $U \subset M$ and an open set $V \subset N$ such that:

$$U \underset{F^{-1}}{\overset{F}{\rightleftharpoons}} V; \tag{B.9}$$

so that f is a local diffeomorphism.

Theorem B.2. *Let M and N be two manifolds with respectively dimension m and n (with $m \geq n$); if f is a map between M and N, then its differential df_p is surjective in p iff it exists an open set $U \subset M$ and a open set $V \subset N$ in which the coordinates in V are the projection of the first n coordinates of M.*

Moreover $q \in N$ is a regular value of f if for all $p \in f^{-1}(q)$, df_p is surjective; then $f^{-1}(q)$ is a submanifold and $T_p f^{-1}(q) = ker(df_p)$.

B.2 Lie Groups and Lie Algebras: some important results

In this section will be presented some important notions on Lie groups, Lie algebras and their relations. The first thing to define is:

Definition B.4 (Lie Group)**.** *G is a Lie Group if it is a Group and a differential manifold such that:*

- *The group operation is differentiable:*

$$G \times G \in G \tag{B.10}$$
$$(g, h) \mapsto gh; \tag{B.11}$$

- *The inverse operation is differentiable:*

$$G \to G \tag{B.12}$$
$$g \mapsto g^{-1}. \tag{B.13}$$

We will deal with Lie groups with representation in $\mathbb{C}^n$ or $\mathbb{R}^m$; this leads to identify the group as a matrix group; for this reason is important the following theorem due to Von Neumann [37].

Theorem B.3 (Von Neumann)**.** *If a set G is defined in $M_n\mathbb{R}$ (the vector space of matrix $n \times n$ with values in $\mathbb{R}$) or in $GL(n, \mathbb{R})$ through polynomial equations in the matrix elements then G is a Lie closed subgroup of $GL(n, \mathbb{R})$.*

The other main structure in this section is the Lie algebra:

Definition B.5 (Lie Algebra)**.** *Let $\mathfrak{g}$ be a vector space over field $\mathbb{R}$, $\mathfrak{g}$ is a Lie algebra if exist a linear bracket operation:*

$$\exists[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}; \tag{B.14}$$

which fulfil the following proprieties:

- *Antisymmetry propriety:* $[X, Y] = -[Y, X]$;
- *Jacobi identity:* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The relation between Lie groups and Lie algebra will be clear after the following proposition that uses the differential structure of a Lie group to identify its algebra.

Proposition B.1. *Let be G a Lie group, the Lie algebra associated to G is isomorphic to the tangent space of G at the identity:*

$$\mathcal{L}ie(G) \simeq T_eG. \tag{B.15}$$

An important remark has to be done: a Lie algebra needs a closed antisymmetric bracket operation defined on it (see definition B.5), this structure is the usual defined on the tangent space. The passage from the algebra to the associated group is not easy and needs more requests, the statement that describes this is Ado's Theorem [39].

In order to describe the structure of the orbits the following theorem is important:

Theorem B.4. *Let G be a Lie group, then:*

- *if $H \subseteq G$ is a closed Lie subgroup then there is a differential structure C^∞ on G/H such that:*

$$G \times G/H \to G/H \tag{B.16}$$
$$(g, g'h) \mapsto gg'h, \tag{B.17}$$

  is a differentiable map, and its action is transitive (see definition 1.15).

- *if S is a differential manifold, and the action of G on S:*

$$G \times S \to S; \tag{B.18}$$

  is transitive then there is a diffeomorphism:

$$G/Stab_{s_0} \to S. \tag{B.19}$$

This theorem is useful to build up the quotients of a Lie Group and has important consequences in the orbit method used in this thesis.

B.3 Other differential structures

In this section will be described other differential structures used in this thesis: the flag manifold and the Lagrangian submanifolds.

B.3.1 Flag Manifold

Let us start by defining what a flag is in linear algebra.

Definition B.6. *Let V be a vector space over a field F, a flag is an ordered sequence of subspaces:*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V. \tag{B.20}$$

A flag is called complete if the dimension of $\dim V_j = j$ *for all j.*

Any incomplete flag manifold can be completed, and if can be done in different ways. Now it can be defined the flag manifold [5].

Definition B.7 (Flag Manifold)**.** *The space of all flags of a given kind is called flag manifold, denoted by:*

$$F^{(N)}_{d_1,\ldots,d_r}. \tag{B.21}$$

Some example of flag manifolds are:

- the projective space is the flag manifold in which are present only one dimensional subspaces;

- the Grasmannian of order k is the flag manifold in which all subspaces have dimension k.

In this thesis we deal only with flag manifolds of the Unitary group, ans so we define:

$$F^N = U(N) \Big/ \underbrace{U(1) \times \cdots \times U(1)}_{N-times} = U(N) \Big/ U^N(1). \tag{B.22}$$

B.3.2 Lagrangian submanifolds

In this subsection we define a Lagrangian submanifold [32]:

Definition B.8. *Let M be a symplectic manifold and let ω be its symplectic form. A Lagrangian submanifold of M is a submanifold L such that:*

- *the restriction of ω to L is vanishing;*
- *the dimension of L is half of the dimension of M.*

Appendix C

$\mathfrak{su}(2)$ and $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$

In this appendix we list the generators of $\mathfrak{su}$ and useful relation that concern these algebras.

C.1 Basis of $\mathfrak{su}(2)$: sigma matrices

The generators of the algebra of $SU(2)$ are the sigma matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{C.1}$$

that have the usual commutation relations:

$$[\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k. \tag{C.2}$$

C.2 Basis of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$

The tensor product of two $\mathfrak{su}(2)$ algebras has the following generators:

$$\mathbb{1} \otimes \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$\mathbb{1} \otimes \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \mathbb{1} \otimes \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\sigma_1 \otimes \mathbb{1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad \sigma_2 \otimes \mathbb{1} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix};$$

$$\sigma_3 \otimes \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad \sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

$$\sigma_1 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; \quad \sigma_1 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};$$

$$\sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \quad \sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix};$$

$$\sigma_2 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; \quad \sigma_3 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

$$\sigma_3 \otimes \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}; \quad \sigma_3 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

The commutation relations of these generators, that rise up the structure constant are:

$$\left[\mathbb{1} \otimes \sigma_i, \mathbb{1} \otimes \sigma_j\right] = 2i \left(\mathbb{1} \otimes \epsilon^{ijk} \sigma_k\right); \tag{C.3}$$

$$\left[\sigma_i \otimes \mathbb{1}, \sigma_j \otimes \mathbb{1}\right] = 2i \left(\epsilon^{ijk} \sigma_k \otimes \mathbb{1}\right); \tag{C.4}$$

$$\left[\sigma_i \otimes \sigma_k, \sigma_j \otimes \sigma_s\right] = 2i\delta_{i,j}\epsilon_{ksm} \left(\mathbb{1} \otimes \sigma_m\right) + 2i\epsilon_{ijl}\delta_{k,s} \left(\sigma_l \otimes \mathbb{1}\right) + \tag{C.5}$$

$$-4\epsilon_{ijl}\epsilon_{ksm} \left(\sigma_l \otimes \sigma_m\right); \tag{C.6}$$

List of Figures

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