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COMPARISON THEOREMS OF GAGA TYPE AND SERRE DUALITY

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Abstract

This thesis studies some foundational issues concerning algebraic varieties, more precisely projective ones.
The focus of the first part is on the comparison between some properties of algebraic varieties and morphisms among them and their counterparts when the algebraic variety is defined over the complex field. In particular we see as notions such as *separatedness* and *completeness* translate well known topological properties of the associated analytic space, namely being Hausdorff and being compact respectively. We will also see that an algebraic variety is irreducible if and only if the associated analytic space is connected. Accordingly, it will be proved that the projective space is both separated and complete. Thus, since these properties are inherited by subvarieties, all projective varieties are separated and complete. We will give a short treatment, without proofs, of the relative counterparts of these notions, namely *separated* morphism and of a *proper* morphism. Separated and proper morphisms correspond to proper morphisms between analytic spaces.
In the second part we will briefly introduce the mathematical structure of *scheme*, which conveniently generalizes the notion of variety, giving algebraic geometry a greater flexibility.
We will then develop the theory of cohomology of sheaves of modules on a given scheme and give explicit calculations of the cohomology on the projective space, by using Čech cohomology.
To conclude, we will give the proof of the Serre duality theorem, one of the most important theorems on coherent sheaves. The proof proceed by first considering the statement for coherent sheaves on the projective space then generalizing it to the case of an arbitrary projective scheme.

Chapter 1

Separatedness and Completeness

In this first chapter we study some topological properties of algebraic varieties. We inspect two properties, namely *separatedness* and *completeness* and then restrict our study to the affine complex space and to the projective one.

We work on the affine n-space over k, $\mathbb{A}^n$: the set of all n-tuples of elements in k, where k is an algebraically closed field.

1.1 Definitions and Examples

In analytic geometry, topological properties like being Hausdorff and being compact play an important role; however, in algebraic geometry such properties are not very meaningful. Any algebraic variety endowed with the Zariski topology is by definition quasi-compact and, while topological manifolds, and therefore differentiable and complex manifolds, have to be Hausdorff spaces, an algebraic variety is Hausdorff only if it consists of a finite set of points, hence a zero dimensional space. An affine variety is in fact defined as an irreducible closed subset of $\mathbb{A}^n$ and the only Hausdorff spaces

that are irreducible (i.e. can't be written as union of two closed subsets) are finite unions of points.
As a consequence, it makes sense to replace such properties with new ones.

First consider the equivalent characterizations.

Proposition 1.1.1. *Let X be a topological space. X is Hausdorff if and only if its diagonal $\Delta_X = \{(x, x) \in X \times X\}$ is closed.*

Proof. Suppose first that X is Hausdorff, we want to prove that Δ_X is closed in $X \times X$. Let $(x, y) \in (X \times X) \backslash \Delta_X$, then $x \neq y$ and, by definition, there exist two disjoint open neighborhoods $U, V \subseteq X$ such that $x \in U$, $y \in V$. Consider $U \times V$: this is an open neighborhood of (x, y) in the product topology which does not contain any point in Δ_X since U, V are disjoint, hence $X \times X \backslash \Delta_X$ is closed.
Now suppose that Δ_X is closed in $X \times X$ and let $x, y \in X$ be distinct. Then $(x, y) \in X \times X \backslash \Delta_X$ which is open by assumption: there exist an open set $W \subseteq X \times X \backslash \Delta_X$ containing (x, y). But, since a basis of product topology is given by $\{U \times V; U, V \subset X, \text{open}\}$, $W \supseteq U \times V$, with $x \in U$, $y \in V$ and U, V open in X and disjoint by definition of W. □

Proposition 1.1.2. *Let X be a topological space. X is quasi-compact if and only if, for any topological space Y, the projection map*

$$X \times Y \xrightarrow{\pi} Y$$

is closed.

Proof. Suppose X quasi-compact. To prove that the projection map is closed we need the following lemma:

Tube Lemma. *Let X, Y be topological spaces with X quasi-compact. If N is an open set containing $X \times \{\bar{y}\}$, $\bar{y} \in Y$, then there exists a tube $X \times V$, $V \subset Y$ open, such that*

$$X \times \{\bar{y}\} \subset X \times V \subset N.$$

Proof. For any $x \in X$ choose open sets $U_x \subseteq X$, $V_x \subseteq Y$ such that $(x, \bar{y}) \in U_x \times V_x \subseteq N$. By quasi-compactness of X there exist finite $\{U_x\}$ that cover X: $X = U_{x(1)} \cup \cdots \cup U_{x(n)}$. Define $V := V_{x(1)} \cap \cdots \cap V_{x(n)}$. V is open as a finite intersection of open sets, each of which contains $\bar{y}$, then

$$X \times \{\bar{y}\} \subset X \times V \subset N.$$

□

Take $C \subset X \times Y$ closed, $\bar{y} \notin \pi(C)$ and $N = X \times Y \backslash C$. By the Tube lemma there exists $V \subset Y$ open such that

$$(X \times Y \backslash C) \cap (X \times \{\bar{y}\}) \subset (X \times Y \backslash C) \cap (X \times V) \subset X \times Y \backslash C.$$

Then applying the projection π we have: $\bar{y} \in V \subseteq Y \backslash \pi(C)$, meaning that $Y \backslash \pi(C)$ is open, hence the projection is a closed map.
Conversely, suppose that X is not quasi-compact. We want to construct a topological space Y for which the assumption of π closed gives us a contradiction.
Let $\mathfrak{U} = \{U_\alpha\}$ be an open cover for X and define $\mathfrak{C} = \{X \backslash U | U \in \mathfrak{U}\}$. Elements in $\mathfrak{C}$ cannot be empty, otherwise there would exists $U \in \mathfrak{U}$ such that $U = X$, making X quasi-compact.
Let $y \notin X$, and define $Y = \{y\} \cup X$ with the induced topology. Consider then $\Delta = \{(x, x) \in X \times Y | x \in X\}$ and let $\bar{\Delta}$ be its closure in $X \times Y$. By assumption, the projection map π is closed: $\pi(\bar{\Delta})$ closed in Y. Note that $y \in \pi(\bar{\Delta})$: if $y \notin \pi(\bar{\Delta})$, then there exists an open neighborhood O of y in $Y \backslash \pi(\bar{\Delta})$, but this implies that O contains some points in X, hence in $\pi(\bar{\Delta})$. From this it follows that $\exists x \in X$ such that $(y, x) \in \bar{\Delta}$.
We now claim that open neighborhoods of y in Y are precisely subsets containing y and some elements of $\mathfrak{C}$. To prove this, assume by contradiction that there exists an open neighborhood of y, $A \ni y$ that contains no element

in $\mathfrak{C}$, then

$$
\begin{aligned}
&A \cap C = \emptyset, \forall C \in \mathfrak{C}, \\
&A \cap (X \backslash U) = \emptyset, \forall U \in \mathfrak{U}, \\
&A \cap X^C = \emptyset,
\end{aligned}
$$

which is impossible since it has to contain y. So $x \in C, \forall C \in \mathfrak{C}$ which contradicts the fact that x must be in X. $\square$

By replacing the standard topology with *Zariski topology*, defined by taking the open subsets to be the complements of algebraic sets, we may give, by analogy, the following definitions.

Definition 1.1. An algebraic variety X is *separated* if its diagonal $\Delta_X \subseteq X \times X$ is closed (w.r.t. the Zariski topology).

Note that the Zariski topology is much finer than the product topology. Therefore if X is a separated algebraic variety, its diagonal is not necessarily closed in the product topology. This allows X to be separated without being Hausdorff in the Zariski topology.

Definition 1.2. An algebraic variety X is *complete* if, for any algebraic variety Y, the projection morphism

$$X \times Y \xrightarrow{\pi} Y$$

is closed (w.r.t. the Zariski topology).

Because the Zariski topology is finer that the product topology, on $X \times Y$ closed subsets in the Zariski topology are way more than closed subsets in the product topology. Thus, completeness imposes a stronger condition than quasi-compactness. There are indeed quasi-compact algebraic varieties (all of them) which are not complete.

We give also the following criterion for separatedness.

Proposition 1.1.3. *An algebraic variety X is separated if and only if for any other algebraic variety Y and $Y \overset{f}{\underset{g}{\rightrightarrows}} X$, f, g morphisms of algebraic varieties, the set $\{y \in Y | f(y) = g(y)\}$ is closed.*

Proof. Suppose X separated. Consider $Y \xrightarrow{f \times g} X \times X$. Note that $\{y \in Y | f(y) = g(y)\} = (f \times g)^{-1}(\Delta_X)$ which has to be closed since Δ_X is closed and $f \times g$ is continuous.
Conversely, take $Y = X \times X$ and f, g as the two canonical projections. Δ_X is precisely the set of points where the two projection coincide, which is thus closed. □

These properties pass down to closed subset:

Remark 1. X separated, $Z \subseteq X$ closed $\Rightarrow$ Z separated.

Remark 2. X complete, $Z \subseteq X$ closed $\Rightarrow$ Z complete.

To see how varieties behave with respect to these properties we inspect some examples.

Example 1.1. Any affine variety is separated.
Consider first the affine space $\mathbb{A}^n$. The diagonal $\{(x_1, \ldots, x_n) = (y_1, \ldots, y_n)\}$ is given by the finite union of hyperplanes given by coordinate-wise equalities $x_i = y_i$, $i = 1, \ldots, n$, which is closed by Zariski.
In general, given an algebraic set X, the coordinate ring of the cartesian product $k[X \times X]$ is canonically isomorphic to $k[X] \otimes_k k[X]$, therefore the diagonal can be defined by the ideal $(f \otimes 1 - 1 \otimes f)$, $f \in k[X]$, and it is again closed in the Zariski topology.

Example 1.2. The affine line with doubled origin is not separated.
Let X be the union of two distinct copies of $\mathbb{A}^1$ where the points in $\mathbb{A}^1 \backslash \{0\}$ are identified by the identity map. Then there are two natural maps $\mathbb{A}^1 \hookrightarrow X$. The set of points in $\mathbb{A}^1$ that have same image under both maps is $\mathbb{A}^1 \backslash \{0\}$ which is clearly not closed in $\mathbb{A}^1$, then by the criterion for separatedness proved before, the affine line with doubled origin is not separated.

Example 1.3. Any affine variety of positive dimension is not complete.
We just prove this in the case $X = \mathbb{A}^1$. Consider the algebraic set $\{(x, y); xy - 1 = 0\} \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ (an hyperbola in $\mathbb{A}^1 \times \mathbb{A}^1$), this is sent by the projection map to $\mathbb{A}^1 \backslash \{0\}$ which is not closed.
For higher dimensions the proof is analogous.

We note that these examples reveal similarities between the properties just defined and those we were already familiar with. For instance, any affine variety is Hausdorff while the affine line with doubled origin is not. This will be motivated in the next section.

1.2 Complex Analytic Spaces

When we work over the complex field $\mathbb{C}$, separatedness and completeness can be viewed as substitutes of being Hausdorff and being compact respectively. More precisely we will prove that, if an algebraic variety is defined over $\mathbb{C}$ and we look at it as a complex analytic space with the standard topology, then separatedness is equal to Hausdorffness and completeness is equal to compactness.
At the end of the section we will also discuss about another interesting topological aspect that arise in the complex case, that is the equivalence between irreducibility and connectedness of the associated analytic space. In this section we will thus consider $k = \mathbb{C}$.

Let X be an algebraic variety over $\mathbb{C}$. We define the analytic (or standard) topology as the topology induced by the inclusion $X \hookrightarrow \mathbb{C}^n$, using the standard topology on $\mathbb{C}^n$. Because zero sets of polynomials are closed in $\mathbb{C}^n$, the standard topology is strictly finer than the Zariski topology. For instance, $\mathbb{Z} \subset \mathbb{C}$ is closed for the analytic topology, but not for the Zariski topology.
Furthermore, we note that any regular map is holomorphic, so we can endow any algebraic variety X with an analytic space structure and denote it with X^{an}.

Proposition 1.2.1. *Let $U \subseteq \mathbb{C}^n$ be non empty open dense in the Zariski topology. Then U has to be open dense in $\mathbb{C}^n$ with the standard topology.*

Remark 3. Note that this property is false if U is not open:
$\mathbb{Z} \subseteq \mathbb{C}$ is dense for the Zariski topology, but closed for the analytic one.

Proof of proposition 1.2.1. Assume by contradiction that U is not open dense in $\mathbb{C}^n$. Then there exists an open set O, which can be assumed to be an n-dimensional ball, such that $U \cap O = \emptyset$. In particular $O \subset U^c$ where U^c has to be closed, i.e. $U^c = V(I)$, where $V(I)$ denotes the zero set of the ideal $I \subseteq \mathbb{C}[z_1, \ldots, z_n]$. For any $f \in I$, f has to vanish on a n-dimensional subset in $\mathbb{C}^n$, so it has to be the zero polynomial. By the *Weak Nullstellensatz* [4], $U^c = \mathbb{C}^n$, which contradicts U being non-empty. □

Proposition 1.2.2. *Let X be an irreducible algebraic variety. If $U \subset X$ is Zariski open, then U it is dense in X.*

Proof. X being irreducible means that any two non empty open sets in X must have non empty intersection. But U not being dense implies that there exists another open subset, disjoint from U, giving a contradiction. □

Remark 4. $\mathbb{C}^n$ is irreducible: it is the zero set defined the zero ideal which is prime, hence it corresponds to an irreducible algebraic set by the $1-1$ correspondence of [1], I.1.4 . Therefore the previous lemmas implies that the Zariski closure in the affine complex space of a constructible set, i.e. a finite union of locally closed[1]sets, coincides with its closure in the standard topology.

Chevalley's theorem ([2])**.** *Let $f : X \to Y$ be any morphism of varieties. f maps constructible sets in X to constructible sets in Y.*

Proposition 1.2.3. *Let X be an algebraic variety over $\mathbb{C}$, then X is separated if and only if X^{an} is Hausdorff.*

[1]A set is *locally closed* if it is the intersection of an open set and a closed set, or equivalently it is open in its closure.

Proof. Consider the identity map of topological spaces

$$X^{\text{an}} \longrightarrow X.$$

The map above is continuous: a closed subset for the Zariski topology has to be closed for the analytic one, since the latter is finer.
Suppose that the diagonal Δ_X is closed in $X \times X$, then Δ_X^{an} has also to be closed in $(X \times X)^{\text{an}}$ by continuity, and for the standard topology we know that $(X \times X)^{\text{an}} = X^{\text{an}} \times X^{\text{an}}$, meaning that X^{an} is Hausdorff.
Conversely, let X^{an} be Hausdorff, so the diagonal is closed in the analytic topology. We need to prove that it is also closed in the Zariski topology. Consider its closure $\bar{\Delta}_X$ in X in the Zariski topology. As a consequence of Chevalley's theorem and remark 4, it has to coincide with the analytic closure, but by assumption Δ_X is closed in the analytic topology, therefore $\Delta_X = \bar{\Delta}_X$ equipped with both topologies. □

Proposition 1.2.4. *Let X be an algebraic variety over $\mathbb{C}$, then X is complete if and only if X^{an} is compact.*

Proof. Suppose that X^{an} is compact. Then a closed subset in $X \times Y$, for any algebraic variety Y, is also closed in $X^{\text{an}} \times Y^{\text{an}}$ and thus mapped into a closed subset in Y^{an} through the projection map, that is also closed in the Zariski topology by Chevalley's theorem and remark 4. The converse implication is a consequence of *Chow's Lemma* [2]:

Chow's Lemma. *Let X be a complete variety over an algebraically closed field k. Then there exists a closed subvariety Y of $\mathbb{P}_k^n$ for some n and a surjective birational morphism $Y \to X$.*

The morphism above is in particular a continuous map, thus it maps compact sets into compact sets and X^{an} is also compact. □

1.2.1 Connectedness

Let X be an algebraic variety over $\mathbb{C}$. It is clear that if it is connected for the standard topology then it is an irreducible algebraic variety, since the standard topology is finer than the Zariski topology. What we want to do in this section is to prove the converse: we will show that irreducibility implies connectedness in the standard topology. In order to do this, we require the following lemmas.

Lemma 1.2.5. *Let X, Y be algebraic varieties, with $Y \subsetneq X$ and X irreducible. If $X^{an} \backslash Y^{an}$ is connected then so it is X^{an}.*

Proof. Suppose by contradiction that $X^{\mathrm{an}} = M \sqcup N$, M, N disjoint non empty closed subsets. Then

$$X^{\mathrm{an}} \backslash Y^{\mathrm{an}} = (M \cap X^{\mathrm{an}} \backslash Y^{\mathrm{an}}) \sqcup (N \cap X^{\mathrm{an}} \backslash Y^{\mathrm{an}})$$

and since $X^{\mathrm{an}} \backslash Y^{\mathrm{an}}$ is connected by assumption, it must be equal either to $(M \cap X^{\mathrm{an}} \backslash Y^{\mathrm{an}})$ or $(N \cap X^{\mathrm{an}} \backslash Y^{\mathrm{an}})$, hence it must be contained either in M or N and also does its closure.
By propositions 1.2.1 and 1.2.2, $X \backslash Y$ is a Zariski open and therefore dense in X and X^{an}, moreover its closure coincides with the analytic closure, thus $X^{\mathrm{an}} = \overline{X \backslash Y} = \overline{X^{\mathrm{an}} \backslash Y^{\mathrm{an}}}$. This implies that one of M, N must be empty, contradicting what we have assumed before. □

Lemma 1.2.6. *If $U \subset \mathbb{C}^n$ open in the Zariski topology then U^{an} is connected.*

Proof. Let $V := \mathbb{C}^n \backslash U$, $x, y \in U^{\mathrm{an}}$ and L a line through x, y. L is not contained in any irreducible component of V, otherwise V would contain both x and y. Therefore $L \cap V$ is a finite set $\{y_1, \ldots, y_m\}$. Note that L^{an} is homeomorphic to $\mathbb{C}$, while $L^{\mathrm{an}} \cap U^{\mathrm{an}}$ is homeomorphic to $\mathbb{C} \backslash \{y_1, \ldots, y_m\}$ which is connected. Then $L^{\mathrm{an}} \cap U^{\mathrm{an}}$ is also connected and x, y are contained in the same connected component of U^{an}. Since x, y were chosen arbitrarily, U^{an} is connected. □

We will also need two additional analytic lemmas from [3], VII.2.4.

Lemma 1.2.7. *Let $S \subsetneq \mathbb{C}^n$ be an algebraic variety and g an analytic function on $\mathbb{C}^n \backslash S^{an}$. If g is bounded in a neighborhood of any point $s \in S^{an}$, then it can be extended to an analytic function on all $\mathbb{C}^n$, and the extension is unique.*

Lemma 1.2.8 (Liouville's theorem)**.** *Let f be an analytic function on $\mathbb{C}^n$. If there is a constant C such that*

$$|f(z)| < C|z|^k \qquad \textit{for } z = (z_1, \ldots, z_n) \textit{ where } |z| = \max |z_i|,$$

then f is a polynomial of degree $\leq k$.

Finally we are now able to prove the main theorem on the connectedness of an irreducible algebraic variety X, reducing to a simpler problem by assuming X affine. We recall first some properties of regular morphisms and field extensions. These will clear up the proof of the following lemma that will be fundamental in order to prove the main theorem.

Remark 5. Let X, Y be affine varieties and $f : X \to Y \subset \mathbb{A}^n_k$ be a regular morphism.
Then f induces ring homomorphism

$$f^* : k[Y] \to k[X]$$

where if $h \in k[Y]$ then $f^*(h)$ is given by $h \circ f$.
If we assume $f(X)$ to be dense in Y, then $f^* : k[Y] \hookrightarrow k[X]$ corresponds to an isomorphic inclusion: $f^*(h) = 0$ if and only if $h(f(x)) = 0$ for any $x \in X$, hence h vanishes on $\overline{f(X)} = Y$, i.e. $h = 0$ in $k[Y]$.
We can also extend $f^* : k[Y] \hookrightarrow k[X]$ in an obvious way to an isomorphic inclusion of the field of fractions: $f^* : k(Y) \hookrightarrow k(X)$.
Recall also the definition of the degree of f: if X, Y have the same dimension, the degree of the field extension $f^*(k(Y)) \subset k(X)$ is:

$$\deg f = [k(X) : f^*(k(Y))] .$$

If f is assumed to be finite[2], then its degree is finite and the primitive element theorem of Galois [9] implies that, when char$k$=0, any field extension of finite degree is simple, i.e. there exists an element $\alpha \in k(X)$ such that

[2]A morphism $f : X \to Y$ is said to be *finite* if $k[X]$ is integral over $k[Y]$, i.e. any element in $k[X]$ is the root of a monic polynomial over $k[Y]$.

$k(X) = f^*(k(Y))(\alpha)$. α is called a *primitive element.*
It is also a consequence of the primitive element theorem and proposition A.7 [3] that α can be chosen in $k[X]$.

Lemma 1.2.9. *Let X be an irreducible variety. Then there exists an open subset $U \subset X$ and a finite morphism $f : U \to V$, V Zariski open subset of the affine space $\mathbb{C}^n$ such that the following conditions hold:*

(a) U is isomorphic to a hypersurface $V(F) \subset V \times \mathbb{C}$, defined by $F = 0$, where $F(t) \in \mathbb{C}[z_1, \dots, z_n][t] \subset \mathbb{C}[V \times \mathbb{C}]$ is an irreducible polynomial over $\mathbb{C}[z_1, \dots, z_n]$ with leading coefficient 1, and f is induced by the projection $V \times \mathbb{C} \to V$.

(b) The continuous map $f : U^{an} \to V^{an}$ is an unramified cover[3].

Proof. By Noether normalization theorem [3] there exists a finite morphism $f : X \to \mathbb{C}^n$, with n=dim$X$, which is surjective by the properties of finite morphisms ([3], I.5.3).
Let $y \in \mathbb{C}^n$, and $\alpha \in \mathbb{C}[X]$ as in the previous remark that takes all distinct values at the points in $\{f^{-1}(y)\}$. Thus, $\mathbb{C}(X) = \mathbb{C}(z_1, \dots, z_n)(\alpha)$.
Let $F(t) \in \mathbb{C}[z_1, \dots, z_n][t]$ be the minimal polynomial of α, then, when replacing the coefficients of F with their values at y, it has m =deg$F$=deg$f$ distinct roots $\alpha(x_i)$, $i = 1, \dots, m$, which is a sufficient condition for f to be unramified at y.
Let V be an open neighborhood of y on which f is unramified and $U := f^{-1}(V)$. Then $f : U \to V$ is still a finite map.
Define $A' := \mathbb{C}[z_1, \dots, z_n][\alpha] = \mathbb{C}[z_1, \dots, z_n]/(F(t))$, then we can write $A' = \mathbb{C}[U']$, for some $U' \subset V \times \mathbb{C}$, defined by the equation $F(t) = 0$. □

Theorem 1.2.10. *If X is an irreducible algebraic variety over $\mathbb{C}$, then X^{an} is connected.*

[3] $f : X \to Y$ is a *cover* if for any $y \in Y$ there is an open neighborhood V s.t. $f^{-1}(V) = U_1 \sqcup \dots \sqcup U_N$, where U_j are disjoint and $f(U_j)$ is homeomorphic to V for any j.
f is said to be *unramified* at $y \in Y$ if the number of inverse images of y is equal to deg$f$.
f is *unramified* if it is unramified at any point in Y.

Proof. Consider $U, V, f : U \to V$ as in lemma above. By lemma 1.2.5 it is sufficient to prove that U^{an} is connected.

1. Suppose by contradiction that

$$U^{\text{an}} = M_1 \sqcup M_2, \qquad M_1, M_2 \text{ disjoint, non-empty closed subsets.}$$

Since finite morphisms are surjective and such that closed sets are mapped into closed sets and open sets are mapped into open sets ([3], I.5.3), $f(M_1), f(M_2)$ are both open and closed in V^{an}, that is connected by lemma 1.2.6. Thus

$$f(M_1) = f(M_2) = V^{\text{an}}.$$

2. Consider the restriction $f|_{M_1} : M_1 \to V^{\text{an}}$. This is still an unramified cover and let r be the number of inverse images in M_1 of any point $y \in V^{\text{an}}$. Since the same holds for M_2, we will have $r < m = \deg f = [\mathbb{C}(U) : f^*(\mathbb{C}(V))]$.

3. Let $y \in V^{\text{an}}$, and choose a neighborhood V_y of y for which $f^{-1}(V_y) = U_1 \sqcup \cdots \sqcup U_r$, where U_i are disjoint for $i \neq j$ and for all $i = 1, \dots r$, the restrictions $f|_{U_i} : U_i \to V_y$ are homeomorphisms. Denote by $f_i = f|_{U_i}$ such restrictions.

4. Recall what we have noted in the remark and let $\alpha \in \mathbb{C}[U]$ be integral over $\mathbb{C}[z_1, \dots, z_n]$, hence algebraic over $\mathbb{C}[z_1, \dots, z_n] \cong f^*(\mathbb{C}[z_1, \dots, z_n])$ by the isomorphic inclusion induced by f, and a primitive element of the field extension $\mathbb{C}(V) \subset \mathbb{C}(U)$, i.e. $\mathbb{C}(U) = \mathbb{C}(V)(\alpha)$.
Denote also by α_i the restrictions of α to U_i, $i = 1, \dots, r$, and by $g_1, \dots, g_r$ the elementary symmetric functions in $\alpha_1, \dots, \alpha_r$:

$$g_i := \sum_{1 \leq j_1 < \cdots < j_i \leq r} \alpha_{j_1} \dots \alpha_{j_i}.$$

5. $g_1, \dots, g_r$ are analytic functions in V^{an} :
Let $u_j = f^{-1}(y)$, $j = 1, \dots, m$. Local parameters at $u_j \in U$ are defined as regular functions at u_j in $\mathbb{C}(U)$ that form a basis for the tangent space at u_j, hence they have a simple zero at u_i. By point (a)

from the previous lemma, $U \subset V \times \mathbb{C}$ is defined by a polynomial $F(t) \in \mathbb{C}[z_1, \ldots, z_n][t]$ and, since local parameters at u_j are by definition locally invertible at $u_j \in U \subset V \times \mathbb{C}$, the implicit function theorem implies that they can be expressed as analytic functions in $f^*(z_1), \ldots, f^*(z_n)$ with $z_1, \ldots, z_n \in \mathbb{C}$.
Moreover, in a small neighborhood of y, because $\mathbb{C}(V) = \mathbb{C}(U)(\alpha)$, α is an analytic function in the local parameters at y.
It is a result from [3] II.2 that local parameters at a point x generate all regular function at that point in the local ring at x. Thus, by observing that the homeomorphisms f_i^{-1} defined in 3. induce ring homomorphisms $(f_i^{-1})^* : \mathbb{C}(U_i) \to \mathbb{C}(V_y)$, the functions $(f_i^{-1})(\alpha)$ are analytic on V_y in the coordinates $z_1, \ldots, z_n \in \mathbb{C}$.
Then by definition, $g_1, \ldots, g_r$ are also analytic functions in the coordinates $z_1, \ldots, z_n \in \mathbb{C}$, on V^{an}.

6. $g_1, \ldots, g_r$ are analytic functions in $\mathbb{C}^n$:
since $V \subset \mathbb{C}^n$ is open, $\mathbb{C}^n \backslash V =: S$ is an algebraic set. Let $s \in S^{\text{an}}$. α was chosen in 4. algebraic over $\mathbb{C}[z_1, \ldots, z_n] \cong f^*(\mathbb{C}[z_1, \ldots, , z_n])$, hence it satisfies an algebraic equation of degree $l \geq m$

$$\alpha^l + f^*(a_1)\alpha^{l-1} + \cdots + f^*(a_l) = 0, \qquad \text{with } a_i \in \mathbb{C}[z_1, \ldots, z_n]\,. \quad (1.1)$$

Note that $(f_i^{-1})(\alpha)$ are roots of this equation, therefore the g_i are bounded in any compact neighborhood of s and lemma 1.2.7 implies that g_i are analytic on the whole affine space.

7. $g_1, \ldots, g_r$ are actually polynomials:
set $|z| := \max_{i=1,\ldots,n} |z_i|$ for any $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.
For any $x \in M_1$, $\alpha(x)$ is the l-th root of an algebraic equation with coefficients $f^*(a_j)(x) = a_j(f(x))$, thus the following inequality[4] holds

$$|\alpha(x)| \leq 1 + max_{j=1,\ldots,l}|a_j(f(x))|$$

[4] This is the *Cauchy bound*: for any univariate polynomial $a_0 + a_i x + \cdots + a_n x^n$, $a_n \neq 0$, each root is bounded by $1 + \max\{\left|\frac{a_{n-1}}{a_n}\right|, \ldots, \left|\frac{a_0}{a_n}\right|\}$.

Since $a_j \in \mathbb{C}[z_1, \ldots, z_n]$, for any $\varepsilon > 0$ there exists a constant C such that

$$|\alpha(x)| < C|z|^k \qquad \text{for} |z| > \varepsilon,$$

where k denotes the maximum of the degrees of $a_j, j = 1, \ldots, l$.
Because we have noted that $(f_i^{-1})(\alpha)$ are roots of the equation (1.1), they satisfies the same inequality:

$$|(f_i^{-1})^*(\alpha)(z)| < C|z|^k \qquad \text{for all } i = 1, \ldots, r.$$

Thus

$$|g_i(z)| \leq \sum_{1 \leq j_1 < \cdots < j_i \leq r} |\max_{k=1,\ldots,i} \{\alpha_{j_k}\}|^i \leq C'|z|^{ik} \qquad i = 1, \ldots, r,$$

hence they have polynomial growth and by lemma 1.2.8 they are polynomials in $z_1, \ldots, z_n$.

8. What we have proved in 7. implies that there exists $p_1, \ldots, p_r \in \mathbb{C}[z_1, \ldots, z_n]$ that restricts to $g_1, \ldots, g_r$ in V_y. Furthermore, since $g_i, i = 1, \ldots, r$ are the elementary symmetric function in $\alpha_i, i = 1, \ldots, r$, they satisfies the following identity

$$\prod_{i=1}^{r} (\lambda - \alpha_i) = \lambda^n - g_1(\alpha_1, \ldots, \alpha_r)\lambda^{n-1} + \cdots + (-1)^r g_r(\alpha_1, \ldots, \alpha_n).$$

By taking $\lambda = \alpha$ and extending to all U, we get

$$\alpha^r - f^*(p_1)\alpha^{r-1} + \cdots + (-1)^r f^*(p_r) = 0. \tag{1.2}$$

9. We will see now that the equation (1.2) gives us a contradiction, concluding the proof.
In fact, what we have done until now can be repeated for M_2 : there exist two polynomials $P_1, P_2 \in \mathbb{C}[z_1, \ldots, z_n][t]$ of degree $< m$ such that $P_j(\alpha) = 0$ on $M_j, j = 1, 2$. Then

$$P_1(\alpha)P_2(\alpha) = 0$$

on $\mathbb{C}[U]$ that is an integral domain: $\mathbb{C}[U]$ is an integral domain if and only if U is irreducible ([4] 5.§1) which is true by assumption. This implies that α satisfies a polynomial equation of degree $< m$ which is impossible since the minimal polynomial of α has degree m.

□

1.3 The Projective Space

Now consider the projective space $\mathbb{P}^n$ over an algebraically closed field k. In this section we'll prove that $\mathbb{P}^n$, and all projective varieties, are both separate and complete.

Recall that $\mathbb{P}^n$ has an open covering $\mathfrak{U} = \bigcup_{i=0}^n U_i$ where

$$U_i = \{[x_0, \ldots, x_n] \in \mathbb{P}^n | x_i \neq 0\} \tag{1.3}$$

with isomorphisms $\varphi_i : U_i \to \mathbb{A}^n$.

Remark 6. $\mathbb{P}^n$ is an algebraic variety.
$\mathbb{P}^n$ is quasi-compact and the local rings $(U_i, \mathcal{O}_i)$ are clearly affine algebraic varieties, where $\mathcal{O}$ is the sheaf such that, for all i, $\mathcal{O}_i$ is defined by the isomorphism φ_i, hence it is isomorphic to the ring of polynomial equation in n coordinates and coefficients in k. What is left to prove is that $\mathcal{O}$ is well defined in the intersections, i.e. $\forall i, j$, $U_{ij} = U_i \cap U_j$ is open in both U_i and U_j (which is trivial in our case) and $\mathcal{O}_i | U_{ij} \cong \mathcal{O}_j | U_{ij}$.
Without loss of generality, suppose $i = 0$, $j = 1$,

$$U_0 = \{[x_0, \ldots, x_n] \in \mathbb{P}^n | x_0 \neq 0\}, \quad \mathcal{O}_0 = k\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right]$$

$$U_1 = \{[x_0, \ldots, x_n] \in \mathbb{P}^n | x_1 \neq 0\}, \quad \mathcal{O}_1 = k\left[\frac{x_0}{x_1}, \ldots, \frac{x_n}{x_1}\right].$$

Then

$$U_{0,1} = \{x \in U_0 | x_1 \neq 0\} \subseteq U_0,$$

and it corresponds to the inclusion of rings

$$k\left[\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right] \hookrightarrow k\left[\frac{x_1}{x_0},\dots,\frac{x_n}{x_0},\frac{x_0}{x_1}\right].$$

Similarly,

$$U_{1,0} = \{x \in U_1 | x_0 \neq 0\} \subseteq U_1,$$
$$k\left[\frac{x_0}{x_1},\dots,\frac{x_n}{x_1}\right] \hookrightarrow k\left[\frac{x_0}{x_1},\dots,\frac{x_n}{x_1},\frac{x_1}{x_0}\right].$$

Clearly $U_{0,1} = U_{1,0}$ and the isomorphism in $k(x_0,\dots,x_n)$ is given by the map

$$k\left[\frac{x_1}{x_0},\dots,\frac{x_n}{x_0},\frac{x_0}{x_1}\right] \xrightarrow{1-1} k\left[\frac{x_0}{x_1},\dots,\frac{x_n}{x_1},\frac{x_1}{x_0}\right]$$
$$\frac{x_0}{x_1} \mapsto \frac{x_1}{x_0}$$
$$\frac{x_i}{x_0} \mapsto \frac{x_i}{x_0}\left(\frac{x_1}{x_0}\right)^{-1}, \ i = 2,\dots,n.$$

This can be done $\forall\, i, j$.

To prove that all projective varieties are separated and complete, by remarks 1. and 2. it is sufficient to prove that $\mathbb{P}^n$ is separated and complete.

Remark 7. Separatedness of $\mathbb{P}^n$ follows automatically from the separatedness of $\mathbb{A}^n$: let $(x,y) \in \mathbb{P}^n \times \mathbb{P}^n$ be in the closure of the diagonal. $\mathbb{P}^n$ has an atlas consisting of $n+1$ charts isomorphic to $\mathbb{A}^n$ and, since x, y are in the closure of the diagonal, it is possible to find an open affine set $U \cong \mathbb{A}^n$, containing both x and y, and we have already proved that the affine space is separated, with diagonal defined by $\{x = y\}$, and this proves that the diagonal is closed in $\mathbb{P}^n$.

On the other hand, we observe that completeness of $\mathbb{P}^n$ follows from the main result from Elimination Theory:
Given r polynomials $f_1,\dots,f_r \in k[x_0,\dots,x_n,y_1,\dots,y_m]$, homogeneous in the variables $x_0,\dots,x_n$, there exist $g_1,\dots,g_s \in k[y_1,\dots,y_m]$ such that $\forall (a_1\dots,a_m) \in k^m$, for which $g_j(a_1,\dots,a_m) = 0$, $j = 1,\dots,s$, there exists $(b_0,\dots,b_n) \in k^{n+1}$ such that

$$f_i(b_0,\dots,b_n,a_1,\dots,a_m) = 0, \qquad \forall i = 1,\dots,r.$$

In other terms, this is equivalent to claim that the projection π of $\mathbb{P}^n \times \mathbb{A}^m$ onto $\mathbb{A}^m$ maps an algebraic set $V(f_1, \ldots, f_r) \subseteq \mathbb{P}^n \times \mathbb{A}^m$ to an algebraic set $V(g_1, \ldots, g_s) \subseteq \mathbb{A}^m$, hence the projection is closed.
This is the content of the Projective Extension Theorem, proved in [4] 8.§5.

We will give another proof instead. We do it by recalling the well known

Nakayama's Lemma. *Let M be a finitely generated R-module, and $A \subset R$ be an ideal such that $M = A \cdot M$. Then there is an element $f \in 1 + A$ which annihilates M.*

Proof. Let $m_1, \ldots, m_n$ be the generators of M as an R-module. By assumption

$$m_i = \sum_{j=1}^{n} a_{ij} \cdot m_j, \qquad \forall i = 1, \ldots, n, \quad \text{where } a_{ij} \in A.$$

Then

$$\sum_{j=1}^{n} (\delta_{ij} - a_{ij}) m_j = 0, \quad \text{where } \delta_{ij} \text{ is the Kronecker delta.}$$

By multiplying on the left by the adjoint of the matrix $(\delta_{ij} - a_{ij})_{ij}$, one finds that $f := det(\delta_{ij} - a_{ij})$ satisfies $f \cdot m_k = 0$, $\forall\, k$, and clearly $f \in 1 + A$. □

Theorem 1.3.1. *$\mathbb{P}^n$ is complete.*

Proof. Let Y be an algebraic variety and

$$\pi : \mathbb{P}^n \times Y \longrightarrow Y$$

be the projection map. By definition of algebraic variety, Y is a finite union of affine varieties, thus, in each of these, a subset is closed if and only if it is closed in Y and therefore we may assume that Y is affine with coordinate ring $R = k\,[Y]$.
Then $\mathbb{P}^n \times Y$ is covered by $n+1$ open $V_i = U_i \times Y$, with U_i as in (1.3), whose coordinate rings are $R\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$.
Let $Z \subseteq \mathbb{P}^n \times Y$ be closed, take $y \in Y \backslash \pi(Z)$, and let $\mathfrak{m} = I(y)$ be the

corresponding maximal ideal. We want to show that there is an open in $\mathbb{P}^n \times Y$ containing y.
For any i, $Z \cap V_i$ and $U_i \times \{y\}$ are closed in V_i and they have empty intersection:

$$(Z \cap V_i) \cap (U_i \times \{y\}) = ((Z \cap U_i) \times \pi(Z)) \cap (U_i \times \{y\}) = \emptyset.$$

Fix some i and denote $V = V_i$, with coordinate ring $R[X_1, \ldots, X_n]$ where $X_1 = \frac{x_0}{x_i}, \ldots, X_n = \frac{x_n}{x_i}$.
By taking the associated vanishing ideals, and applying *Weak Nullstellensatz*[4], we get the following equality

$$I(Z \cap V) + \mathfrak{m} \cdot R[X_1, \ldots, X_n] = R[X_1, \ldots, X_n].$$

This implies

$$a + \sum_j m_j g_j = 1, \tag{1.4}$$

where $a \in I(Z \cap V) \subseteq R[X_1, \ldots, X_n]$, $m_j \in \mathfrak{m}$ and $g_j \in R[X_1, \ldots, X_n]$.
Recall that the homogenization of a is a homogeneous polynomial $a' \in R[x_0, \ldots, x_n]$ of degree m such that

$$a'(x_0, \ldots, x_n) = x_i^m \cdot a(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}).$$

We note also that from a' we can construct another homogeneous polynomial $\tilde{a} \in R[x_0, \ldots, x_n]_N$ with the following property

$$\tilde{a}\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) \in I(Z \cap V_j), \qquad \text{for all } j. \tag{1.5}$$

Since a' is homogeneous, we can write

$$a'\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) = \frac{a'}{x_j^m},$$

which is clearly zero on $Z \cap V_j \cap V$.
Suppose that it is not zero on $Z \cap V_j$, then we can consider instead

$$\tilde{a} := a'\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) \cdot \frac{x_i}{x_j} = \frac{a'}{x_j^{m+i}} \cdot x_i = a\left(\frac{x_0}{x_i}, \ldots, \frac{x_0}{x_i}\right) \cdot \left(\frac{x_i}{x_j}\right)^{m+1}$$

which must be zero on both $Z \cap V_j \cap V$ and $Z \cap V_j \cap V^c$, hence on $Z \cap V_j$ as wanted.

Thus $\tilde{a}$ satisfies (1.5) and we denote $A_N \subset S_N := R\left[x_0, \ldots, x_n\right]_N$ the vector space of such polynomials.

Multiplying (1.4) by x_i^N with N large enough gives

$$\tilde{a}(x_0, \ldots, x_n) + \sum_j m_j g_j' = x_i^N, \qquad g_j' \in S_N,$$
$$\Rightarrow x_i^N \in A_N + \mathfrak{m} \cdot S_N, \text{ for all } i.$$

By taking N even bigger, and repeating for each i, all monomials in $R\left[x_0, \ldots, x_n\right]$ of degree N are in $A_N + \mathfrak{m} \cdot S_N$, i.e.

$$S_N = A_N + \mathfrak{m} \cdot S_N. \tag{1.6}$$

Taking the quotient of (1.6) gives:

$$S_N / A_N = \mathfrak{m} \cdot S_N / A_N.$$

Then by *Nakayama's Lemma* there exists $f \in R + \mathfrak{m}$ such that

$$\begin{aligned} f \cdot S_N / A_N = 0 &\Leftrightarrow f \cdot S_N \subset A_N \\ &\Leftrightarrow f \cdot x_i^N \in A_N, \quad \forall i, \\ &\Leftrightarrow f \in I(Z \cap V_i), \quad \forall i, \end{aligned}$$

meaning that f vanishes on $\pi(Z)$, i.e. $V(f)^c$ is an open neighborhood of y contained in $Y \backslash \pi(Z)$, which is thus open. □

Note that $\mathbb{P}^n$ being complete reflects the fact that the projective space is compact over the complexes, accordingly to what we have said in the previous section.

1.4 Separated and Proper Morphisms

We now treat briefly the properties discussed in this chapter in terms of morphisms between varieties: we will define separated and proper morphisms

which are the counterparts of topological separatedness and completeness, respectively.
To be more precise, such morphisms are usually defined in the category of *schemes*, which is an enlargement of the category of algebraic varieties. Nonetheless, we will continue to work with algebraic varieties, and use the language of schemes starting from the next chapter.

Definition 1.3. Let $f : X \to Y$ be a morphism of varieties. f is *separated* if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$[5] is a closed immersion.
In this case we also say that X is *separated over* Y.

Remark 8. X is a separated algebraic variety over k if and only if the structure morphism $f : X \to \operatorname{Spec} k$ is separated.

Proof. Recall that we are considering k as an algebraically closed field. We will see in the next chapter that $\operatorname{Spec} k$ consists of a point, hence the fiber product over $\operatorname{Spec} k$ coincides with the cartesian product, meaning that the morphism being separated is equal to topological separatedness. □

Here we give some results about separatedness, without proofs.

Proposition 1.4.1 ([1]).

(a) Open and closed immersions are separated;

(b) composition of separated morphisms are separated;

(c) separatedness is stable under base change[6]*;*

(d) products of separated morphisms are separated;

(e) if $f : X \to Y$ and $g : Y \to Z$ are two morphisms and if $g \circ f$ is separated, then f is separated;

[5] It is a property of the category of schemes that the fiber product always exists.

[6] A property of morphisms of varieties is said to be stable under base change if for any morphism $X \to Y$ satisfying that property, all base changes $X \times_Y Y' \to Y'$ also have that property.

(f) *A morphism* $f : X \to Y$ *is separated if and only if* Y *can be covered by open subsets* V_i *such that* $f^{-1}(V_i) \to V_i$ *is separated for each* i.

Such properties can actually be extended to a whole class of schemes, called *'noetherian'*.

In ordinary topology, properness is a useful geometric property which is indeed a relative version of compactness. Here is a true fact about proper maps: suppose that X and Y are Hausdorff spaces and Y is locally compact (which again, we think of varieties as always being such), then a map $f : X \to Y$ is proper if and only if it is universally closed, i.e. for any topological space Z the map $f \times id_Z : X \times Z \to Y \times Z$ is closed. This gives sufficient motivation to make the following definition plausible:

Definition 1.4. Let $f : X \to Y$ be a morphism of varieties. f is *proper* if it is separated and universally closed. f is said to be *universally closed* if it is closed and, for any morphism $Y' \to Y$, the morphism $f' : X \times_Y Y' \to Y'$, obtained by base change, is also closed.

Remark 9. Actually, the definition of properness for a morphism of schemes requires to be of *'finite type'*, but we can omit this because we will see that the associated scheme to a variety is always of finite type.

Remark 10. X is a complete algebraic variety over k if and only if the structure morphism $f : X \to \mathrm{Spec}k$ is proper.

Similarly to separated morphisms, the following properties hold and can be extended to all noetherian schemes.

Proposition 1.4.2 ([1])**.**

(a) *Closed immersions are proper;*

(b) *compositions of proper morphisms are proper;*

(c) *proper morphisms are stable under base change;*

(d) *products of proper morphisms are proper;*

(e) *if $f : X \to Y$ and $g : Y \to Z$ are two morphisms and if $g \circ f$ is proper and g is separated, then f is proper;*

(f) *A morphism $f : X \to Y$ is proper if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is proper for each i.*

There is an important class of proper morphisms:

Definition 1.5. Let $f : X \to Y$ be a morphism of varieties. f is *projective* if it factors into a closed immersion $i : X \to \mathbb{P}^n_Y$, where $\mathbb{P}^n_Y := \mathbb{P}^n \times Y$ and the projection $\mathbb{P}^n_Y \to Y$.

Theorem 1.4.3. *A projective morphism of varieties is proper.*

Proof. To prove this, we will use the properties in the above proposition. By definition, a projective morphism $f : X \to Y$ is the composition of a closed immersion, that is proper by (a), and a projection $p : \mathbb{P}^n \times Y \to Y$. From (b) it is sufficient to prove that the latter is proper, but this follows by the completeness of the projective space, proved in the previous section. □

As we have seen for the topological properties corresponding to separatedness and properness, the latter also translate in a well known morphism between complex varieties, viewed as complex analytic spaces. This, as one can foresee, is the counterpart of the topological property of compactness: a proper map, i.e. a continuous map such that the inverse image of a compact is compact.

Let X, Y be complex algebraic varieties and $X^{\mathrm{an}}, Y^{\mathrm{an}}$ the associated analytic spaces. Let $f : X \to Y$ be a regular morphism, we denote with $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ the same map, viewed as a morphism between analytic spaces.

Proposition 1.4.4. *$f : X \to Y$ over $\mathbb{C}$ is separated if and only if f^{an} is separated.*

Proof. Let $\Delta : X \to X \times_Y Y$ and $\Delta^{\text{an}} : X^{\text{an}} \to X^{\text{an}} \times_{Y^{\text{an}}} Y^{\text{an}}$ be the diagonal immersions. It is a consequence of remark 4. that $\Delta(X)$ is closed in $X \times_Y Y$ if and only if $\Delta^{\text{an}}(X^{\text{an}})$ is closed in $X^{\text{an}} \times_{Y^{\text{an}}} Y^{\text{an}}$. □

Theorem 1.4.5. *$f : X \to Y$ over $\mathbb{C}$ is proper if and only if f^{an} is proper with respect to the analytic topology.*

Proof. Suppose f is proper. Since f^{an} being proper is a local property on Y^{an}, we may assume Y affine. Then, by Chow's Lemma, there exists a closed subvariety X' of $\mathbb{P}^n$ and a surjective birational morphism

$$g : X' \to X.$$

The morphism $(f \circ g)^{\text{an}} = f^{\text{an}} \circ g^{\text{an}}$ is projective, hence proper by theorem 1.4.3. g^{an} is surjective, thus by proposition 1.4.2 (e) f^{an} must be proper.
Conversely, suppose f^{an} is proper. By proposition 1.4.4, f is separated, so we need to prove that it is universally closed, and it suffices to prove that f is closed, since the morphism

$$f' : X \times_Y Y' \to Y'$$

is closed if $(f')^{\text{an}}$ is proper.
Let $T \subseteq X$ be closed. By Chevalley's Theorem, $f(T)$ is a constructible set and we have

$$f^{\text{an}}(\varphi^{-1}(T)) = \psi^{-1}(f(T)),$$

where φ, ψ are the canonical morphisms $\varphi : X^{\text{an}} \to X$, $\psi : Y^{\text{an}} \to Y$.
Since f^{an} is proper, $\psi^{-1}(f(T))$ must be closed in Y^{an}:

$$\psi^{-1}(\overline{f(T)}) = \psi^{-1}(f(T)).$$

This implies $\overline{f(T)} = f(T)$ i.e. f closed and, for what we observed above, proper. □

The most useful criteria to check separatedness and properness are the *valuative criteria.*

Intuitively, they enable one to reduce checking to curves or, more precisely, germs of curves. For instance, a complex analytic space X is Hausdorff if and only if any holomorphic map $f : D^* \to X$ has at most one extension to $\hat{f} : D \to X$ (here D^* denotes the punctured disc and $D := D^* \cup \{0\}$ the disc).

Similarly, X is compact if and only if any holomorphic map $f : D^* \to X$ has a unique extension $\hat{f} : D \to X$. However, such criteria are defined using schemes, therefore we will first define the structure of a scheme and then discuss it at a later time.

Chapter 2

Schemes and Sheaves of Modules

This chapter contains some basic definitions and results from the *Theory of schemes* which is the language that we will use to develop the topics that will be treated in the next chapters.
We will give the notions of *scheme*, *morphism of schemes*, *sheaf of modules* focusing mainly on *coherent sheaves.*

2.1 Schemes

Similarly to an algebraic variety, that is obtained by gluing together affine varieties, a scheme is something that locally looks like an 'affine scheme'. In this section we define affine schemes and construct their structure sheaf in order to define a scheme. We will also focus on an important class of schemes: the projective spectrum of a ring.

2.1.1 Affine Schemes

Let A be a commutative ring with unit.
We define the *Spectrum* of A, denoted by $\mathrm{Spec}A$, to be the set of all prime

ideals of A.
If $\mathfrak{a} \subseteq A$ is an ideal, then we define the subset

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}A | \mathfrak{a} \subseteq \mathfrak{p}\} \subseteq \mathrm{Spec}A.$$

Remark 11. Let $\mathfrak{a}, \mathfrak{b} \subseteq A$ be ideals. Then $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.
To see this, it suffices to note that the radical of an ideal is the intersection of all prime ideals containing it ([5] I.1.14).

We define the Zariski topology on $\mathrm{Spec}A$ by taking as closed subsets all subsets of the form $V(\mathfrak{a})$. It's easy to verify that

1. $V(\sum \mathfrak{a}_\lambda) = \bigcap V(\mathfrak{a}_\lambda)$;
2. $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Therefore $\{V(\mathfrak{a})\}$ defines a topology. To be more precise, the open sets are

$$D(\mathfrak{a}) = \mathrm{Spec}A \backslash V(\mathfrak{a})$$

for some ideal $\mathfrak{a}$. Let $f \in A$, $D(f) = \mathrm{Spec}A \backslash V((f)) = \{\mathfrak{p} \in \mathrm{Spec}A | f \notin \mathfrak{p}\}$. Then if $\mathfrak{a} \subseteq A$ is an ideal,

$$D(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}A | \mathfrak{a} \not\subseteq \mathfrak{p}\} = \bigcup_{f \in \mathfrak{a}} D(f).$$

So $\{D(f) | f \in A\}$ is a basis for the Zariski topology just defined.

$\mathrm{Spec}A$ will be our model of 'affine scheme'.
From a geometrical point of view, it is in fact very similar to an affine variety:

Remark 12. $\mathrm{Spec}A$ is Hausdorff if and only if A is zero dimensional:
Let $\mathfrak{p} \in \mathrm{Spec}A$. $V(\mathfrak{p}) = \{\mathfrak{p}' | \mathfrak{p}' \supseteq \mathfrak{p}\}$, then $\mathfrak{p}$ is a closed point if and only if it is maximal. Consequently $\mathrm{Spec}A$ is Hausdorff if and only if its Krull dimension[1] is 0.

[1]The *Krull dimension* of A is the supremum of all integers n such that there exists a chain of length n of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$, for any prime ideal $\mathfrak{p}$.

Remark 13. Spec$A$ is quasi-compact:
Consider a open covering of Spec$A$

$$\text{Spec}A = \bigcup_{f \in R \subseteq A} D(f).$$

since $D(1) = \text{Spec}A$, 1 has to be in R, hence it is a finite linear combination of elements is R: $1 = \sum_{i=1}^{N} c_i f_i$, $f_i \in R$, so we find a finite covering

$$\text{Spec}A = \bigcup_{i=1}^{N} D(f_i).$$

To make the definition more accurate, we endow X =Spec$A$ with a sheaf of rings $\mathcal{O}_X$, called its structure sheaf.
For any $U \subseteq X$ open, we define $\mathcal{O}_X(U)$ to be the set of functions

$$s : U \longrightarrow \amalg_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

where $A_{\mathfrak{p}}$ is the localization of A at $\mathfrak{p}$ w.r.t. the multiplicative system $A \backslash \mathfrak{p}$, such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ and s is locally the quotient of elements in A, i.e. $\forall \mathfrak{p} \in U$ there exists a open neighborhood V of $\mathfrak{p}$ where $\forall \mathfrak{q} \in V$, $s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}, a \in A, f \notin \mathfrak{q}$.
Take the identity to be the element which gives 1 in each $A_{\mathfrak{p}}$. Since sums and products of functions are again such, $\mathcal{O}_X(U)$ is a commutative ring with unit.
If $V \subseteq U$ are two open subsets, the projection

$$\amalg_{\mathfrak{p} \in U} A_{\mathfrak{p}} \longrightarrow \amalg_{\mathfrak{p} \in V} A_{\mathfrak{p}}$$

clearly restricts $\mathcal{O}_X(U)$ to $\mathcal{O}_X(V)$ and gives the structure of a presheaf.
Remark 14. Let $f, g \in A$.

$$D(f) \subseteq D(g) \Leftrightarrow V((f)) \supseteq V((g)) \overset{\text{Rk. 11}}{\Leftrightarrow} \sqrt{(f)} \subseteq \sqrt{(g)}$$

$$\Leftrightarrow f \in \sqrt{(g)} \Leftrightarrow f^m = h \cdot g, \text{ for some } m \Leftrightarrow A_g \subseteq A_f,$$

where A_f denotes the localization of A w.r.t. the multiplicative system $\{f, f^2, \dots\}$. This gives an inclusion map $A_g \longrightarrow A_f$.
Furthermore, if $\mathfrak{p} \in D(f)$, then $f \notin \mathfrak{p}$, so we get a natural map $A_f \longrightarrow A_\mathfrak{p}$, and the following commutative diagram

$$\begin{array}{ccc} A_g & \longrightarrow & A_f \\ & \searrow \quad \swarrow & \\ & A_\mathfrak{p} & \end{array}$$

Thus

$$\varinjlim_{f \notin \mathfrak{p}} A_f = A_\mathfrak{p}.$$

Lemma 2.1.1. *Suppose $D(f) = \bigcup_\alpha D(f_\alpha)$. If $g \in A_f$ has image 0 in all rings A_{f_α}, then $g = 0$.*

Proof. See Lemma 1 in [2] II.§1. □

Lemma 2.1.2. *Suppose $D((f)) = \bigcup_\alpha D(f_\alpha)$, and $g_\alpha \in A_{f_\alpha}$ are such that $\forall \alpha, \beta$, g_α and g_β have same image on $A_{f_\alpha f_\beta}$ (if not empty). Then there exists $g \in A_f$ that has image g_α in A_{f_α}, $\forall \alpha$.*

Proof. See Lemma 2 in [2] II.§1. □

The two lemmas above are sufficient to prove that $\mathcal{O}_X$ is a sheaf. To be more precise, $(X, \mathcal{O}_X)$ is a locally ringed space:

Proposition 2.1.3. *Let $f \in A$, then*

1. $\mathcal{O}_X(D(f)) \cong A_f$;
2. *The stalk of $\mathcal{O}_X$ at $\mathfrak{p}$ is $A_\mathfrak{p}$: $(\mathcal{O}_X)_\mathfrak{p} \cong A_\mathfrak{p}$.*

Proof. 1. Consider the map

$$A_f \longrightarrow \Pi_{\mathfrak{p} \in D(f)} A_\mathfrak{p}.$$

By Lemma 1 this map is injective, and by Lemma 2 it is surjective.

2. By definition,

$$(\mathcal{O}_X)_{\mathfrak{p}} := \varinjlim_{\mathfrak{p}\in D(f)} \mathcal{O}_X(D(f)) \overset{1.}{=} \varinjlim_{f\notin\mathfrak{p}} A_f = A_{\mathfrak{p}}.$$

□

Given this structure, we can now define a scheme.

Definition 2.1. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to the spectrum of some ring.

Definition 2.2. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that $\forall x \in X$ there exists an open neighborhood U that, together with $\mathcal{O}_X|_U$, is affine.

Example 2.1. Let k be a field, (0) is the only prime ideal on a field, hence Spec$k$ consists in only a point and its structure sheaf is k.
This also explains remarks 8 and 10 in section 1.4.

Example 2.2. Define the affine space over k as $\mathbb{A}^n_k :=$ Spec$k\,[x_1, \ldots, x_n]$. Its closed points, i.e. its maximal ideals, are in 1-1 correspondence with ordered n-tuples of elements of k by Weak Nullstellensatz [2], and therefore with the affine space $\mathbb{A}^n_k$ as a variety.

2.1.2 Projective schemes

Here we consider an important class of schemes: the *Projective spectrum* of a graded ring.
Let S be a graded ring and $S_+ = \bigoplus_{d>0} S_d$ its *irrelevant ideal*, we denote Proj$S$ to be the set of all homogeneous prime ideals which do not contain all of S_+, and define the Zariski topology on Proj$S$ by taking as closed sets, those of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Proj}S | \mathfrak{p} \supseteq \mathfrak{a}\},\ \mathfrak{a} \text{ homogeneous ideal in } S.$$

Analogously to the affine case, we can define a sheaf of rings on $X = \text{Proj}S$: take T as the multiplicative system consisting of all homogeneous elements of $S\backslash\mathfrak{p}$, with $\mathfrak{p}$ prime ideal, and denote by $S_{(\mathfrak{p})}$ the subring of $T^{-1}S$ consisting of all fractions $\frac{f}{g}$, with f, g homogeneous of the same degree. We define the structure sheaf $\mathcal{O}_X$ by taking, for any open set U, $\mathcal{O}_X(U)$ as the set of functions

$$s : U \to \amalg_{\mathfrak{p}\in U} S_{(\mathfrak{p})}$$

such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and s is locally a quotient.
By repeating what we've done for the affine case, it can be proved that this is indeed a sheaf and Proj$S$ is a locally ringed space: for any $\mathfrak{p} \in \text{Proj}S$, the stalk at $\mathfrak{p}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.
We can say more:

Proposition 2.1.4. Proj$S$ *is a scheme.*

Proof. Proj$S$ is already a locally ringed space, therefore it is sufficient to prove that it can be covered by open affine schemes. Note that, since elements in Proj$S$ cannot contain all of S_+, a cover is given by open sets

$$D_+(f) = \text{Proj}S\backslash V((f)) = \{\mathfrak{p} \in \text{Proj}S | f \notin \mathfrak{p}\}$$

with $f \in S_+$ homogeneous.
We want to show that there is an isomorphism of ringed space

$$(\varphi, \varphi^\#) : (D_+(f), \mathcal{O}_{\text{Proj}S}(D_+(f))) \to (\text{Spec}S_{(f)}, \mathcal{O}_{\text{Spec}S_{(f)}})$$

where $S_{(f)}$ is the subring of S_f of elements of degree 0 and the map φ is defined by

$$\varphi(\mathfrak{a}) = \mathfrak{a}S_f \cap S_{(f)},\ \mathfrak{a} \subseteq S \text{ homogeneous ideal},$$

where $\mathfrak{a}S_f$ denotes the smallest ideal in S_f containing $\varphi(\mathfrak{a})$. Then for any $\mathfrak{p} \in D_+(f)$, $\varphi(\mathfrak{p}) \in \text{Spec}S_{(f)}$ and it is bijective by localization properties. By noticing also that $\forall\mathfrak{a} \subseteq S$ homogeneous, $\mathfrak{p} \supseteq \mathfrak{a} \Leftrightarrow \varphi(\mathfrak{p}) \supseteq \varphi(\mathfrak{a})$ this is indeed a homeomorphism.
Furthermore, if $\mathfrak{p} \in D_+(f)$ then $S_{(\mathfrak{p})}$ and $(S_{(f)})_{\varphi(\mathfrak{p})}$ are isomorphic as local rings. This induces a morphism from the structure sheaf on Spec$S_{(f)}$ to the direct image of the structure sheaf on Proj$S$ restricted to $D_+(f)$

$$\varphi^\# : \mathcal{O}_{\text{Spec}S_{(f)}} \to \varphi_*(\mathcal{O}_{\text{Proj}S}|_{D_+(f)})$$

which is an isomorphism. □

Example 2.3. Define the projective n-space over k as $\mathbb{P}_k^n$:=Proj$k\left[x_0,\ldots,x_n\right]$. The set of closed points is exactly the projective space, as a variety: closed points in Proj$k\left[x_0,\ldots,x_n\right]$ are homogeneous maximal ideals $\neq (x_0,\ldots,x_n)$ which are in a 1-1 correspondence with points in $\mathbb{P}^n$:

$$(a_i x_j - a_j x_i | i,j = 0,\ldots,n) \overset{1-1}{\leftrightarrow} (a_0,\ldots,a_n) \in \mathbb{P}^n.$$

By putting this together with the definition of affine space as a scheme, we observe that the topological space of a scheme has more points than the corresponding variety. This suggests that the notion of schemes generalizes the notion of variety. We will see in the next section that this is actually true.

2.2 Varieties as Schemes

The category of schemes is in fact an enlargement of the category of varieties. To prove this, we give first the following definitions.

Definition 2.3. Let S be a fixed scheme. A scheme over S is a scheme X, together with a morphism $X \to S$.
An S-morphism from X to Y, where X, Y are schemes over S, is a morphism $X \to Y$ compatible with the given morphisms to S.

We denote $\mathfrak{Sch}(k)$ the category of schemes over k (meaning over Spec$k$), or k-schemes, together with Spec$k$-morphisms, and $\mathfrak{Var}(k)$ the category of varieties where morphisms are regular maps.

Proposition 2.2.1. *Let X be a scheme, $Z \subset X$ irreducible closed subset. There exists only one point $z \in Z$ such that $Z = \overline{\{z\}}$. z is called a generic point.*

Proof. If $U \subset X$ is an open affine scheme such that $Z \cap U \neq \emptyset$ then

1. any point dense in Z must be in $Z \cap U$: any non empty open subset has to contain it;

2. if $z \in Z \cap U$ and $\overline{\{z\}} \supset Z \cap U$ then z is dense in Z, otherwise Z wouldn't be irreducible.

So we can assume $Z = V(\mathfrak{a})$, for some ideal $\mathfrak{a}$. But closed irreducible subset in a scheme are of the form $V(\mathfrak{p})$, $\mathfrak{p}$ prime, and $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. □

Proposition 2.2.2. *Let k be an algebraically closed field. There is a natural fully faithful functor*

$$t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k),$$

i.e. $\forall X, Y \in \mathfrak{Var}(k)$

$$t_{X,Y} : Hom_{\mathfrak{Var}(k)}(X, Y) \to Hom_{\mathfrak{Sch}(k)}(t(X), t(Y))$$

is bijective.

Proof. First, we define t as a functor of topological spaces. Let X be a topological space, we define

$$t(X) := \{Y \subseteq X | Y \text{ closed and irreducible}\}$$

and a topology on $t(X)$, by taking as closed sets $t(Y) \subseteq t(X)$, with $Y \subseteq X$ closed.
Let $f : X_1 \to X_2$ be a continuous map between topological spaces, then we get a map

$$\begin{aligned} t(f) : & t(X_1) \to t(X_2) \\ & Y \mapsto \overline{f(Y)} \end{aligned} \tag{2.1}$$

which is continuous, hence t is indeed a functor of topological spaces.
Define

$$\alpha : X \to t(X) \tag{2.2}$$

$$P \mapsto \overline{\{P\}} \tag{2.3}$$

continuous map. From proposition 2.2.1 there is a 1-1 correspondence between points of X and closed irreducible subsets in X, i.e. points of $t(X)$. Furthermore this induces a bijection between open sets in X and open sets in $t(X)$ which allows us to define $t(f)^{\#}$ between sheaves.

Take X as an algebraic variety $(V, \mathcal{O}_V)$, we prove that $(t(V), \alpha_*(\mathcal{O}_V))$ is a k-scheme.

Since any variety is covered by affine varieties, we can assume V affine, with coordinate ring $A := k\,[V]$. Consider the following morphism of locally ringed spaces

$$\begin{aligned}\beta :&(V, \mathcal{O}_V) \to (\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})\\ &P \mapsto \mathfrak{m}_P\end{aligned}$$

where $\mathfrak{m}_P$ is the maximal ideal corresponding to the point P. By Weak Nullstellensatz there is a bijection of V onto the set of closed points in $\mathrm{Spec}A$. So β gives a homeomorphisms onto its image.

$\forall U \subseteq \mathrm{Spec}A$ open, define the ring homomorphism

$$\mathcal{O}_{\mathrm{Spec}A}(U) \to \beta_*(\mathcal{O}_V)(U) = \mathcal{O}_V(\beta^{-1}(U))$$

that takes a section $s \in \mathcal{O}_{\mathrm{Spec}A}(U)$ and defines a regular map from $\beta^{-1}(U)$ to k as follows: for any $P \in \beta^{-1}(U)$, take the image of s in the stalk $\mathcal{O}_{\mathrm{Spec}A,\beta(P)} \cong A_{\mathfrak{m}_P}$ and pass to the quotient $A_{\mathfrak{m}_P} \backslash \mathfrak{m}_P \cong k$. s is regular by definition of section in $\mathcal{O}_{\mathrm{Spec}A}(U)$, and this gives the isomorphisms of rings $\mathcal{O}_{\mathrm{Spec}A}(U) \cong \mathcal{O}_V(\beta^{-1}(U))$.

By recalling that prime ideals of A are in 1-1 correspondence with irreducible closed subsets of V, we have that $(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$ is isomorphic to $(t(V), \alpha_*(\mathcal{O}_V))$, so the latter is indeed an affine scheme.

It is in fact a scheme over $\mathrm{Spec}k$: because $(\mathrm{Spec}k, \mathcal{O}_{\mathrm{Spec}k})$ consists of a point with structure sheaf given by the field k, as we have seen in example 2.1, it is sufficient to give an homomorphism of rings $k \to \mathcal{O}_V$ that maps any $\lambda \in k$ to the corresponding constant function. Therefore $(t(V), \alpha_*(\mathcal{O}_V))$ is a scheme over $\mathrm{Spec}k$.

What is left to prove is the bijection of t when restricted to morphisms. We note the following first:

1. V affine variety, $t(V)$ =Spec$A$, $A = k\,[V]$. $\mathfrak{p} \in t(V)$ is a closed point if and only if it is a maximal ideal, hence the residue field $k(\mathfrak{p}) := A_{\mathfrak{p}} \backslash \mathfrak{p} A_{\mathfrak{p}}$ is k;

2. Let $f : X \to Y$ be a morphism of schemes and $\mathfrak{p} \in X$ such that $k(\mathfrak{p}) = k$. The ring morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a map between residue fields $k(f(\mathfrak{p})) \to k(\mathfrak{p})$ which are both extensions of k and gives the following inclusions

$$k \hookrightarrow k(f(\mathfrak{p})) \hookrightarrow k(\mathfrak{p}) = k$$

   $\Rightarrow k(f(\mathfrak{p})) = k$.

Now fix V, W affine and consider $F : V \to W$, $G : V \to W$ regular maps. Recall from (2.1) that $t(F)$ maps irreducible closed subsets of V to their closure in $t(W)$. To prove injectivity suppose $t(F) = t(G)$, by 1. and 2. we see that a morphism of schemes maps closed points to closed points, and these correspond to the points of the varieties, so we have that $\forall P \in V$

$$F(P) = \overline{F(P)} = t(F)(P) = t(G)(P) = \overline{G(P)} = G(P),$$

thus $F = G$ as regular maps.
To prove surjectivity, instead, let

$$(\varphi, \varphi^{\#}) : (t(V), \alpha_*\mathcal{O}_V) \to (t(W), \alpha_*\mathcal{O}_W)$$

be a morphism of schemes. From what we have proved above, we have the following isomorphisms:
$(t(V), \alpha_*\mathcal{O}_V) \cong (\mathrm{Spec}A_V, \mathcal{O}_{\mathrm{Spec}A_V})$,
$(t(W), \alpha_*\mathcal{O}_W) \cong (\mathrm{Spec}A_W, \mathcal{O}_{\mathrm{Spec}A_W})$,
which give the commutative diagram

$$\begin{array}{ccc} (t(V), \alpha_* \mathcal{O}_V) & \xrightarrow{(\varphi, \varphi^\#)} & (t(W), \alpha_* \mathcal{O}_W) \\ \downarrow & & \downarrow \\ (\text{Spec}A_V, \mathcal{O}_{\text{Spec}A_V}) & \xrightarrow{(\psi, \psi^\#)} & (\text{Spec}A_W, \mathcal{O}_{\text{Spec}A_W}) \end{array}$$

By restricting $\psi^\#$ on global sections we get $F^\# : A_W \to A_V$ which induces a morphism of varieties $F : V \to W$. To conclude the proof we need to prove that this F is such that $\varphi = t(F)$, meaning that if Y is a closed irreducible subset of V then $\varphi(Y)$ is the closure of $F(Y)$ in $t(W)$.
Since closed irreducible subsets correspond to prime ideals, Y has to correspond to a prime ideal $\mathfrak{p}$ $\in$SpecA_V. $\psi(\mathfrak{p}) = (F^\#)^{-1}(\mathfrak{p})$, or by the commutativity of the diagram above, $\varphi(Y) = Z((F^\#)^{-1}(\mathfrak{p}))$ where $Z(\cdot)$ denotes the zero set operator. Now take a closed set in W containing $F(Y)$, where $Y = Z(\mathfrak{p})$. By applying the vanishing ideal operator and the the zero set operator we get that such closed set has to contain the zero set of $(F^\#)^{-1}(\mathfrak{p})$, hence $Z((F^\#)^{-1}(\mathfrak{p}))$ is exactly the closure of $F(Y)$ and this concludes the proof. □

To be more precise, we want to identify the image of the functor t in $\mathfrak{Sch}(k)$. To do so, we need to distinguish some classes of schemes.

Definition 2.4. A scheme X is *integral* if for any open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Definition 2.5. A scheme X is *locally noetherian* if it can be covered by open affine subsets SpecA_i, A_i noetherian ring. X is *noetherian* if it is locally noetherian and quasi-compact.

Definition 2.6. A morphism $f : X \to Y$ of schemes is *of finite type* if there exists a covering of Y by open affine subsets V_i =SpecB_i such that, for each i, $f^{-1}(V_i)$ can be covered by a finite number of open affine subsets U_{ij} =SpecA_{ij}, A_{ij} finitely generated B_i-algebra.

Because every finitely generated k-algebra is noetherian, if follows immediately that every k-scheme of finite type is noetherian.

Recall the definition 1.5 of projective morphism between varieties. Here's the analogue in terms of schemes:

Definition 2.7. Let Y be a scheme. We denote by $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{Spec\mathbb{Z}} Y$ the projective n-space over Y. A morphism $f : X \to Y$ of schemes is *projective* if it factors into a closed immersion $i : X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$.
A morphism $f : X \to Y$ is *quasi-projective* if it factors into an open immersion $j : X \to X'$ and a projective morphism $g : X' \to Y$.

Remark 15. Let V be a variety over k algebraically closed field. $t(V)$ is an integral noetherian scheme of finite type over k.

Proof. Since V can be covered by a finite number of open affine varieties V_i, then, by the proof of Proposition 2.2.2. $t(V)$ can be covered by a finite number of open affine schemes SpecA_i where each A_i is the coordinate rings of the affine variety V_i, which is an integral domain and a finitely generated k-algebra, hence noetherian. □

Proposition 2.2.3 ([1])**.** *Let k be an algebraically closed field. the image of the functor*

$$t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$$

is the set of quasi-projective integral schemes of finite type over k. The image of projective varieties is the set of projective integral schemes of finite type over k.

Schemes are indeed a generalization of varieties. Therefore from now on we will work on schemes, but clearly what we will do will also hold for algebraic varieties. In particular, we are now able to give the *Valuative criteria* mentioned at the end of chapter 1.
We first redefine separated and proper morphisms in terms of schemes.

Definition 2.8. Let $f : X \to Y$ be a morphism of schemes. f is *separated* if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion.

Definition 2.9. Let $f : X \to Y$ be a morphism of schemes. f is *proper* if it is separated, of finite type and universally closed.

Proposition 2.2.4 ([1])**.** *Any morphism of affine schemes is separated.*

Theorem 2.2.5 ([1])**.** *A projective morphism of noetherian schemes is proper. A quasi-projective morphism of noetherian schemes is of finite type and separated.*

2.2.1 Valuative Criteria of Separatedness and Properness

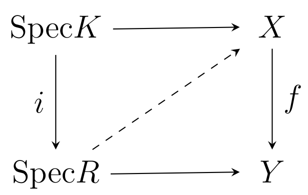
Definition 2.10. Let B be an integral domain, Q its field of fractions. B is a *valuation ring* if, for each $x \neq 0$, $x \in Q$, either $x \in B$ or $x^{-1} \in B$ (or both).

Example 2.4.

- Any field $\mathbb{F}$ is a valuation ring;
- if $\mathbb{F}$ is a field, the ring of formal power series $\mathbb{F}[[X]]$ is a valuation ring.

Let $f : X \to Y$ a morphism of schemes with X noetherian.

Valuative criterion of Separatedness. *f is separated if and only if the following condition holds. For any field K and for any valuation ring R with quotient field K, given a morphism of* Spec$K$ *to X and a morphism of* Spec$R$ *to Y that make the following diagram commutative*

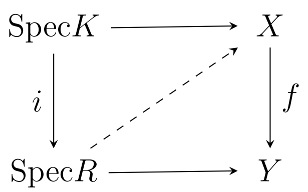


where i is the morphism induced by the inclusion $R \subseteq K$, there is at most one morphism Spec$R \to X$ *making the whole diagram commutative.*

As we have anticipated at the end of chapter 1, we can think of Spec$R$ as the germ of a curve, and Spec$K$ as the germ minus the origin. Thus the criterion says that X is separated over Y if and only if, given a map from a germ of a curve to Y and a lift outside the origin to X, there is at most one way to lift the map from the entire germ.

Example 2.5. Let X be the affine line with double origin over k, hence Y =Spec$k$.
Take Spec$R$ to be the germ of the affine line at the origin, R is the localization of $k[X]$ at the maximal ideal, and consider the map of the germ minus the origin to X. The map from the entire germ can be extended over the origin in two different ways (mapping the origin to one of the two origins in X) and thus fails the valuative criterion for separatedness. As we have already proved in the first chapter, the affine line with double origin is indeed not separated.

Valuative criterion of Properness. *f is proper if and only if the following condition holds. For any field K and for any valuation ring R with quotient field K, given a morphism of* Spec$K$ *to X and a morphism of* Spec$R$ *to Y that make the following diagram commutative*



where i is the morphism induced by the inclusion $R \subseteq K$, there is a unique morphism Spec$R \to X$ *making the whole diagram commutative.*

On the other hand, X is proper over Y if and only if, given a map from a germ of a curve to Y and a lift outside the origin to X, there is exactly one way to lift the map from the entire germ.

Example 2.6. We can use the valuative criterion of properness to prove that the projective space over k is complete.

Consider $X = \mathbb{P}^n_k \to Y =$Spec$k$.
Given Spec$K \to X$ and Spec$R \to$Spec$k$, they induce respectively the morphisms $k\left[x_0, \dots, x_n\right]_{d \geq 0} \to K$ and $k \to R$.
Thus, any solution of a homogeneous polynomial is given by $(X_0, \dots, X_n)$, $X_i \in K$ such that they cannot be all zero and, by rescaling the coordinates such that they all belong to R and one of them is a unit in R, we may assume that all $X_i \in R$ where X_0 is a unit. We can then extend the morphism Spec$K \to X$ to Spec$R \to X$ by defining the induced morphism that maps the coordinates x_i/x_0 in the affine subset $D_+(x_0) \cong$Spec$k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right]$ to $X_i/X_0 \in R$.
Because the projective space is separated, the extension is unique and thus the valuative criterion of properness is satisfied.

Summing up, a morphism of schemes f is separated if and only if it satisfies the existence part of the valuative criteria, while it is proper if and only if it satisfies the uniqueness part.

2.3 Sheaves of Modules on a given scheme

Fix $(X, \mathcal{O}_X)$ ringed space.

Definition 2.11. A *sheaf of* $\mathcal{O}_X-$*modules*, or simply an $\mathcal{O}_X-$*module*, is a sheaf $\mathcal{F}$ of abelian groups on X such that, $\forall\, U \subseteq X$ open, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)-$module and, $\forall\, V \subseteq U$ open, the restriction map

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

is compatible with the module structures via the ring homomorphisms

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V),$$

i.e. the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

is commutative.

Definition 2.12. A *morphism of* $\mathcal{O}_X-$*modules* is a morphism

$$\mathcal{F} \to \mathcal{G}$$

such that $\forall U \subseteq X$ open

$$\mathcal{F}(U) \to \mathcal{G}(U)$$

is a homomorphism of $\mathcal{O}_X(U)$-modules.

We will consider $(X, \mathcal{O}_X)$ scheme, using the definitions above we will thus get sheaves of modules on a given scheme and morphisms of such.

2.3.1 Locally Free Sheaves

Definition 2.13. An $\mathcal{O}_X-$module $\mathcal{F}$ is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U-$module, i.e. there exists a basis $\{s_1, \ldots, s_r\} \subseteq \mathcal{F}(U)$. r is said to be the *rank*[2] of $\mathcal{F}$ on U.

Remark 16. if X is connected, then the rank of a locally free $\mathcal{O}_X-$module is the same on every open set.

Definition 2.14. A locally free sheaf of rank 1 is called an *invertible sheaf.*

Remark 17. Locally free sheaves are equivalent to the notion of vector bundles. In particular, an invertible sheaf is a vector bundle of rank one, i.e a line bundle.

[2] r might also be infinite.

2.3.2 Coherent Sheaves

Let A be a commutative ring with unit. By repeating the construction of the structure sheaf of the affine scheme X =Spec$A$, we can define on it a sheaf $\tilde{M}$, associated to an $A-$module M, instead of A :
For any $\mathfrak{p} \in$ Spec$A$, let $M_{\mathfrak{p}}$ be the localization of M at $\mathfrak{p}$.
$\forall U \subseteq$ Spec$A$ open, we define $\tilde{M}(U)$ as the set of functions

$$s : U \to \amalg_{\mathfrak{p}\in U} M_{\mathfrak{p}}$$

s.t. $\forall \mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ and it is locally a quotient.
Clearly this is again a sheaf, using the obvious restriction maps.

$M_{\mathfrak{p}}$ is naturally an $A_{\mathfrak{p}}-$module, hence $\tilde{M}(U)$ is an $\mathcal{O}_X(U)-$module, meaning that $\tilde{M}$ is an $\mathcal{O}_X-$module. Moreover, by repeating exactly what we have done when constructing $\mathcal{O}_X$ and replacing A with M,

1. For any $f \in A$, $\tilde{M}(D(f)) \cong M_f$.

2. For any $\mathfrak{p} \in$ Spec$A$, $(\tilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$.

3. $\Gamma(X, \tilde{M}) = M$, where Γ is the global section functor.

Definition 2.15. Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X-$modules $\mathcal{F}$ is *quasi-coherent* if X can be covered by open subsets $U_i =$ SpecA_i, such that for each i there exists an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. If we can take the modules M_i to be finitely generated, we say $\mathcal{F}$ is *coherent*.

Example 2.7. The structure sheaf $\mathcal{O}_X$ is trivially quasi-coherent, and in fact coherent.

Proposition 2.3.1. *An $\mathcal{O}_X-$module $\mathcal{F}$ is quasi-coherent if and only if for every open affine subset $U =$ Spec$A$ of X, there is an $A-$module M such that $\mathcal{F}|_U \cong \tilde{M}$. If X is noetherian, then $\mathcal{F}$ is coherent if and only if the same is true and M can be taken to be a finitely generated $A-$module.*

Proposition 2.3.2. *Let $f : X \to Y$ be a morphism of noetherian schemes. If $\mathcal{G}$ is a coherent sheaf on Y, then $f^*\mathcal{G}$ is coherent on X.*

Proof. By the previous proposition, we may assume both X and Y affine, where $X =$Spec$A$, $Y =$Spec$B$.
Because $\mathcal{G}$ is coherent on Y, it is of the form $\tilde{M}$, where M is a finitely generated B-module.
Then, by definition $f^*(\mathcal{G})$ is the tensor product

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

and, for any $U \subset X$ open,

$$f^*\mathcal{G}(U) = \tilde{M}(f(U)) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X(U) = (M \otimes_B A)^{\sim}(U),$$

since localization commutes with the tensor product. Therefore $f^*\mathcal{G}$ is of the form $(M \otimes_B A)^{\sim}$ where $M \otimes_B A$ is a finitely generated A-module, proving that $f^*\mathcal{G}$ is coherent. □

On the other hand, if $\mathcal{F}$ is a coherent sheaf on X, it is not true in general that the direct image $f_*\mathcal{F}$ is coherent on Y. It is true when the morphism f is finite[3].

Proposition 2.3.3. *Let $f : X \to Y$ be a finite morphism of noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on F, then $f_*\mathcal{F}$ is coherent on Y.*

Proof. By assumption, Y is covered by affine subsets $V_i =$SpecB_i and $U_i := f^{-1}(V_i)$ is equal to SpecA_i, where each A_i is a finitely generated B_i-algebra.
Moreover, because $\mathcal{F}$ is coherent, $\mathcal{F}|_{U_i} \cong \tilde{M_i}$, where each M_i is a finitely generated A_i-module, hence a finitely generated B_i-module.
Then, for any i, $f_*\mathcal{F}|_{V_i}$ is of the form $\tilde{M_i}$, where M_i is considered as a B_i-module. □

[3]A morphism $f : X \to Y$ of schemes is *finite* if there is a covering of Y by open affine subsets $V_i =$SpecB_i, such that each $f^{-1}(V_i)$ is also affine and equal to SpecA_i, where A_i is a B_i-algebra which is a finitely generated B_i-module.

We are also interested in the projective case. What we have done while constructing $\tilde{M}$, can also be done on Proj$S$, with S graded ring. As $\tilde{M}$ was our model to check if a sheaf was either quasi-coherent or not, the same works for the projective case.

2.3.3 Twisted Sheaves

Let's focus on the projective case and recall the construction of $\tilde{M}$: let $S = \bigoplus_{d\in\mathbb{Z}} S_d$ be a graded ring, $X =$Proj$S$.
Take $M := S_d$, which is clearly an S-module, and define $\tilde{M}$ as usual. We get a coherent sheaf and we denote it with $\mathcal{O}_X(d)$.

Definition 2.16. We call $\mathcal{O}_X(1)$ the *twisting sheaf of Serre.* For any $\mathcal{O}_X-$module $\mathcal{F}$ we also denote by $\mathcal{F}(d)$ the *twisted sheaf* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$.

Here are some properties:

Proposition 2.3.4.

(a) $\mathcal{O}_X(n)$ *is an invertible sheaf on* X*;*

(b) $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$*. This implies*

$$\mathcal{O}_X(n) \cong \bigotimes_{i=1}^{n} \mathcal{O}_X(1).$$

We also give an example to better understand the twisting sheaf of Serre.

Example 2.8. Let $S = A\left[x_0, \dots, x_n\right]$, $X =$Proj$S = \mathbb{P}^n_A$. By proposition 2.3.4 (a), $\mathcal{O}_X(d)$ is an invertible sheaf for each d, so we can think of it as a line bundle.
Let $\{U_i\}_{i=0,\dots,n}$ be the standard open covering of $\mathbb{P}^n_A$. Transition functions are defined by

$$g_{ij} = \left(\frac{x_i}{x_j}\right)^d,$$

and sections on an open set U are given by sections

$$s_i \in \mathcal{O}_X(U \cap U_i)$$

such that

$$s_i = g_{ij} s_j \qquad \text{on } U \cap U_i \cap U_j.$$

Consider $h(x_0, \dots, x_n) \in S$, homogeneous of degree d. The identification

$$s_i = \frac{h}{x_i^d} = h\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

allows us to identify the vector space of global sections of the twisted sheaf $\mathcal{O}_X(d)$ with the vector space of homogeneous polynomials of degree d.
In particular, when $d = 1$, $\mathcal{O}_X(1)$ can be read as the sheaf of 'coordinates' for $\mathbb{P}^n_A$, since the x_i are literally the coordinates for the projective $n-$space.

This gives motivation to the importance of the twisting sheaf: when S is a polynomial ring, it recovers the algebraic information about the grading of S that was lost when we considered fractions of degree 0 while constructing $\mathcal{O}_X$, where $X = \text{Proj}S$.

Definition 2.17. We define the *graded S-module associated to $\mathcal{O}_X$* to be

$$\Gamma_*(\mathcal{O}_X) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n)),$$

and we give it the structure of graded S-module as follows.
Any $s \in S_d$ determines in a natural way a global section $s \in \Gamma(X, \mathcal{O}_X(d))$. For any $t \in \Gamma(X, \mathcal{O}_X(n))$ we define the product $s \cdot t \in \Gamma(X, \mathcal{O}_X(n+d))$ to be the tensor product $s \otimes t$ by using the natural map $\mathcal{O}_X(d) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(n+d)$, given by proposition 2.3.4 (b).

Proposition 2.3.5. *Let A be a ring, $S = A[x_0, \dots, x_r]$, $X =$Proj$S = \mathbb{P}^r_A$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.*

Proof. Recall that, while proving that Proj$S$ is a scheme, we have observed that an open affine covering of Proj$S$ is given by $\{D_+(f); f \in S_+ \text{ homogeneous}\}$, and since S_+ is generated by x_i, $i = 0, \dots, r$,

$$\mathfrak{U} = \{U_i\}_{i=0,\dots,r}; \qquad U_i := D_+(x_i)$$

is an open affine cover for X. Take $t \in \Gamma(X, \mathcal{O}_X(n)) = \mathcal{O}_X(n)(X)$, and call t_i the restrictions to the opens of the cover $\mathfrak{U}$:

$$t_i = t|_{U_i} \in \mathcal{O}_X(n)(U_i),$$

such that $t_i = t_j$ on $U_i \cap U_j$ where

$$\begin{aligned} D_+(x_i) \cap D_+(x_j) &= \{\mathfrak{p} \in \text{Proj}S | x_i \notin \mathfrak{p} \text{ and } x_j \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \text{Proj}S | x_i x_j \notin \mathfrak{p}\} \\ &= D_+(x_i x_j). \end{aligned}$$

By the interpretation of the twisting sheaf given in the previous example, t_i is a homogeneous element of degree n in the localization S_{x_i}.
Restricting t_i to $D_+(x_i x_j)$ is equal to taking its image through the natural map

$$S_{x_i} \to S_{x_i x_j}.$$

Summing over all $n \in \mathbb{Z}$, we get that elements in $\Gamma_*(\mathcal{O}_X)$ can be identified with $(r+1)-$tuples $(t_0, \dots, t_r)$ such that $t_i \in S_{x_i}$ for all i, and they agree on intersections, so we can conclude that

$$\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^{r} S_{x_i}.$$

Note that x_i are non zero divisors in S, so the maps

$$S \to S_{x_i}, \qquad S_{x_i} \to S_{x_i x_j}$$

are all injective and clearly all these localizations are subrings of $S_{x_0, \dots, x_r}$.
Let $g \in S_{x_0, \dots, x_r}$, it can be uniquely written as

$$x_0^{m_0} \dots x_r^{m_r} f(x_0, \dots, x_r), \quad m_k \in \mathbb{Z}, f \in S, \text{ not divisible by any } x_i.$$

$g \in S_{x_i}$ if and only if the only variable that might appear at the denominator is x_i, i.e. if each $m_k \geq 0 \ \forall k \neq i$.
Then $g \in S_{x_i x_j}$ if and only if $m_k \geq 0 \ \forall k \neq i$ and $\forall k \neq j$, hence for all k. Therefore $S_{x_i x_j} = S$ and also $\bigcap_{i=0}^{r} S_{x_i} = S$. □

Thus, we will only consider S as a polynomial ring.

Finally, we give the following result that allows us to write coherent sheaves in terms of twisted sheaves. This will be very useful in the last chapter.

Definition 2.18. Let $\mathcal{F}$ be an $\mathcal{O}_X-$module. $\mathcal{F}$ is *generated by global sections* if there is a family of global sections $\{s_i \in \Gamma(X, \mathcal{F})\}_{i \in I}$, such that for any $x \in X$ the images of s_i in the stalk $\mathcal{F}_x$ generate it as an $\mathcal{O}_x-$module.

Remark 18. Any coherent sheaf $\mathcal{F}$ on an affine scheme X =Spec$A$, with A noetherian, is generated by a finite number of global sections: by prop. 2.3.1 $\mathcal{F}$ is of the form $\tilde{M}$, where M is a finitely generated $A-$module and $\Gamma(X, \tilde{M}) = M$. So it is sufficient to take $\{s_i\}_{i=1,\ldots,N}$ as the generators of M as an $A-$module.

Let X be a projective scheme over a noetherian ring A. Then, by definition, there is a closed immersion $i : X \to \mathbb{P}^r_A$, for some r. Let $\mathcal{O}(1)$ be the twisting sheaf on the projective space. We denote by $\mathcal{O}_X(1) := i^*(\mathcal{O}(1))$ the inverse image of $\mathcal{O}(1)$, that is still a coherent sheaf by proposition 2.3.2.

Theorem 2.3.6. *Let X be a projective scheme over a noetherian ring A, and $\mathcal{F}$ a coherent sheaf on X. There is an integer n_0 such that, for any $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.*

Proof. Without loss of generality we can assume $X = \mathbb{P}^r_A$. In fact a closed immersion is a finite morphism (a closed immersion is such that the preimage of any open affine Spec$B$ is still an open affine Spec$A$ and the map induced on the structure sheaves is surjective, meaning that A is a finitely generated $B-$module), therefore $i_*\mathcal{F}$ is coherent on the projective space by prop. 2.3.3, and global sections of $(i_*\mathcal{F})(n) = i_*(\mathcal{F}(n))$ are the same of $\mathcal{F}(n)$.
An open cover of X is given by $\{D_+(x_i)\}_{i=1,\ldots,r}$, and by coherence of $\mathcal{F}$, for each i, there is a finitely generated B_i-module M_i such that $\mathcal{F}|_{D_+(x_i)} \cong \tilde{M}_i$, where $B_i = A\left[x_0/x_i, \ldots, x_n/x_i\right]$.
For any i, let s_{ij} be the generators of M_i, that are finite by assumption. It is a

consequence of the sheaf axioms and the definition of localization that there in an integer n such that $x_i^n s_{ij}$ extends to a global section $t_{ij} \in \Gamma(X, \mathcal{F}(n))$. Choose n that works for all i, j. Then $\mathcal{F}(n)$ corresponds to a B_i-module on $D_+(x_i)$ with generators $x_i^n s_{ij}$ which is isomorphic to M_i because of the isomorphism induced by $\times x_i^n$. Therefore the global sections t_{ij} generate all $\mathcal{F}(n)$. □

Corollary 2.3.7. *Any coherent sheaf $\mathcal{F}$ on X can be written as a quotient of a sheaf $\mathcal{E}$, where $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}_X(-q)$, $q >> 0$.*

Proof. Let $q >> 0$, $\mathcal{F}(q)$ is generated by a finite number of global section by the theorem that we have just proved. Therefore we have a surjection

$$\bigoplus_{i=1}^N \mathcal{O}_X \to \mathcal{F}(q) \to 0,$$

then, tensoring with $\mathcal{O}_X(-q)$, gives

$$\bigoplus_{i=1}^N \mathcal{O}_X(-q) \to \mathcal{F} \to 0.$$

□

Chapter 3

Cohomology

3.1 Sheaf Cohomology

In this chapter, we will discuss the cohomology of sheaves on a separated and noetherian scheme, focusing on coherent sheaves. We will introduce both the derived functor approach of Grothendieck and Čech cohomology and use the latter to calculate explicitly the cohomology of the twisted sheaves $\mathcal{O}(d)$ on a projective space $\mathbb{P}^n$, defined in the previous chapter.

Recall first some basic definitions from homological algebra.
Let $\mathfrak{A}$ be an abelian category, that is an additive category in which there exist well-behaved kernels and cokernels for each morphism, so that the notion of exact sequence makes sense in $\mathfrak{A}$.

Definition 3.1. A *cochain complex* $A^\bullet$ in an abelian category $\mathfrak{A}$ is a collection of objects A^i of $\mathfrak{A}$, $i \in \mathbb{Z}$, together with morphisms $d^i : A^i \to A^{i+1}$ such that $d^i \circ d^{i+1} = 0$ for all i. d^i are called *coboundary maps.*

Definition 3.2. To any complex $A^\bullet$ we can associate the *cohomology objects*

$$h^i(A^\bullet) := \frac{\ker d^i}{\mathrm{im} d^{i-1}}.$$

Definition 3.3. A *morphism of complexes* $f : A^\bullet \to B^\bullet$ is a collection of maps $f^i : A^i \to B^i$ that commutes with the coboundary maps.

Any such morphism induces a morphism on the cohomology objects: $h^\bullet(f) : h^\bullet(A^\bullet) \to h^\bullet(B^\bullet)$.

Definition 3.4. Two morphisms of complexes $f, g : A^\bullet \to B^\bullet$ are *homotopic* if there exists $k : A^\bullet \to B^\bullet$ of degree -1, i.e. $k^i : A^i \to B^{i-1}$ for each i, such that $f - g = dk + kd$.

Remark 19. If f, g are homotopic then they induce the same morphism on cohomology, i.e. $h^\bullet(f) = h^\bullet(g)$.

Theorem 3.1.1 (Snake lemma)**.** *Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a short exact sequence of complexes in $\mathfrak{A}$*. Then the induced sequence in cohomology*

$$0 \to h^0(A^\bullet) \to h^0(B^\bullet) \to h^0(C^\bullet) \to h^1(A^\bullet) \to \dots$$

is exact.

3.1.1 Derived Functors Cohomology

Let $\mathfrak{Ab}$ be the category of abelian groups, and $\mathfrak{A}$ any abelian category. Fix an object A in $\mathfrak{A}$, the functor

$$\begin{aligned} \mathrm{Hom}(\cdot, A) : &\mathfrak{A} \to \mathfrak{Ab} \\ &B \mapsto \mathrm{Hom}(B, A) \end{aligned}$$

is a contravariant left exact functor.

Definition 3.5. An object I in $\mathfrak{A}$ is *injective* if the functor $\mathrm{Hom}(\cdot, I)$ is exact.

Definition 3.6. An abelian category $\mathfrak{A}$ *has enough injectives* if each object in $\mathfrak{A}$ can be embedded in an injective object, i.e. it is isomorphic to a subobject of an injective object in $\mathfrak{A}$.

Lemma 3.1.2. *If $\mathfrak{A}$ has enough injectives then any object A in $\mathfrak{A}$ admits an injective resolution, which is a long exact sequence*

$$0 \to A \to I^0 \to I^1 \to \dots$$

where each I^j is injective.

Proof. Embed A in I^0. Then embed the cokernel of the inclusion $\varepsilon : A \to I^0$ in an injective I^1 and take $I^0 \to I^1$ to be the composition $I^0 \to \mathrm{coker}\varepsilon \to I^1$, and so on. □

Suppose $\mathfrak{A}$ is an abelian category with enough injectives and let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor. Then, for any object A in $\mathfrak{A}$ choose an injective resolution

$$0 \to A \to I^0 \to I^1 \to \dots$$

If we apply the functor F on the complex obtained forgetting about A we still get a complex $F(I^\bullet)$.

Definition 3.7. We define R^iF, $i \geq 0$, to be the right derived functors of F, where

$$R^iF(A) := h^i(F(I^\bullet)).$$

Remark 20. Let F be a left exact functor, $A \in \mathfrak{A}$ and $0 \to A \to I^0 \to I^1 \to \dots$ an injective resolution. By left exactness of F, $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact, therefore

$$R^0F(A) = \mathrm{ker}F(d^0) : F(I^0) \to F(I^1) = \mathrm{im}F(\varepsilon) : F(A) \to F(I^0) = F(A).$$

For any short exact sequence $0 \to A \to B \to C \to 0$, then the long exact sequence in cohomology is

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \dots$$

Thus the right derived functors R^iF 'measure' how far is F from being exact.

Moreover right derived functors are independent of the choice of the injective resolutions. This is because of the following

Theorem 3.1.3 ([10]). *Let* $0 \to B \to I^\bullet$ *be an injective resolution and* $0 \to A \to J^\bullet$ *an arbitrary resolution. Then any morphism* $f : A \to B$ *induces a unique morphism of complexes* $f^\bullet : J^\bullet \to I^\bullet$*, up to homotopy.*

The theorem above implies that any two injective resolutions of the same object, $0 \to A \to I^\bullet$, $0 \to A \to J^\bullet$, are homotopic equivalent, i.e. there exist $f : I^\bullet \to J^\bullet$ and $g : J^\bullet \to I^\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to the respective identity maps of $I^\bullet$ and $J^\bullet$.
Thus, by remark 19

$$h^i(F(I^\bullet)) \cong h^i(F(J^\bullet)),$$

for each i, meaning that $R^iF(A)$ is well defined.

Actually we could say more: sometimes it is more useful to use resolutions which are not necessarily injective.

We will in fact consider *acyclic resolutions.*

Definition 3.8. A object A of $\mathfrak{A}$ is $F-$*acyclic* if $R^iF(A) = 0$ for $i > 0$.
A resolution of A, $0 \to A \to C^\bullet$ is $F-$*acyclic* if each C^i is $F-$acyclic.

Proposition 3.1.4. *Let* $F : \mathfrak{A} \to \mathfrak{B}$ *be a left exact functor,* I *injective in* $\mathfrak{A}$*, then* I *is* $F-$*acyclic.*

Proof. It suffices to consider the injective resolution $0 \to I \to I \to 0$ and compute $R^iF(I)$. □

Proposition 3.1.5 ([10]). *If* $0 \to A \to J^\bullet$ *is an* $F-$*acyclic resolution, then there exists a natural isomorphism* $R^iF(A) \cong h^i(F(J^\bullet))$.

Let X be a topological space, and denote with
$\mathfrak{Ab}$ the category of abelian groups,
$\mathfrak{Ab}(X)$ the category of sheaves of abelian groups on X, which are both abelian

categories.
Consider the global section functor

$$\begin{aligned}\Gamma(X,\cdot):&\mathfrak{Ab}(X)\to\mathfrak{Ab}\\ &\mathcal{F}\mapsto\mathcal{F}(X).\end{aligned}$$

Proposition 3.1.6. $\Gamma(X,\cdot)$ *is a covariant left exact functor, i.e. for any short exact sequence of sheaves*

$$0\to\mathcal{F}\xrightarrow{\varphi}\mathcal{G}\xrightarrow{\psi}\mathcal{H}\to 0,$$

the sequence

$$0\to\Gamma(X,\mathcal{F})\to\Gamma(X,\mathcal{G})\to\Gamma(X,\mathcal{H})$$

is exact.

Proof. Let $U\subseteq X$ open, and consider

$$0\to\mathcal{F}(U)\xrightarrow{\varphi(U)}\mathcal{G}(U)\xrightarrow{\psi(U)}\mathcal{H}(U).$$

φ being an injective morphism of sheaves implies that $\varphi(U)$ is injective, therefore it is sufficient to prove that im$\varphi(U)=$ker$\psi(U)$.
By definition of exactness for a sequence of sheaves, for any $p\in U$, the sequence induced at the level of stalks is exact:

$$0\to\mathcal{F}_p\xrightarrow{\varphi_p}\mathcal{G}_p\xrightarrow{\psi_p}\mathcal{H}_p\to 0. \tag{3.1}$$

Take a section $s\in\Gamma(U,\mathcal{F})$, for each $p\in U$

$$(\psi(U)(\varphi(U)(s)))_p=\psi_p(\varphi_p(s))=0$$

by exactness of (3.1). Hence $\psi(U)(\varphi(U)(s))=0$ and im$\varphi(U)\subseteq$ker$\psi(U)$.
Now take $v\in$ker$\psi(U)$, by exactness of (3.1), $\forall p\in U$, there exists $s_p\in\mathcal{F}_p$ such that $\varphi(s_p)=v_p\in\mathcal{G}_p$.
Then, by definition of stalks, there exists a covering $\{U_i\}$ of U and $s_i\in\mathcal{F}(U_i)$ such that

$$\varphi(s_i)=v|_{U_i}. \tag{3.2}$$

Thus

$$\varphi(s_i|_{U_i\cap U_j}) = v|_{U_i\cap U_j} = \varphi(s_j|_{U_i\cap U_j}), \qquad \text{if } U_i \cap U_j \text{ not empty.}$$

Therefore

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$$

by injectivity of φ.
Using the fact that $\mathcal{F}$ is a sheaf, we get that there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i, and together with (3.2), we have that $\varphi(U)(s) = v$, i.e. ker$\psi(U) \subseteq$im$\varphi(U)$. □

Proposition 3.1.7. *The category* $\mathfrak{Ab}(X)$ *has enough injectives.*

Proof. See [1] III, corollary 2.3. □

Thus, for what we have said at the beginning of the subsection, the right derived functors of the global section functor are well defined and we can give the following definition.

Definition 3.9. For any sheaf $\mathcal{F}$ of abelian groups on X, the *cohomology groups* of $\mathcal{F}$ are the groups

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F}).$$

3.1.2 Čech Cohomology

Most of the times, cohomology defined using derived functors is impossible to calculate, in these cases, we will use instead Čech cohomology. We will see in fact that the two definitions agree when we consider coherent sheaves on separated and noetherian schemes.

Let X be a topological space, and let $\mathcal{F}$ be a sheaf of abelian groups. Suppose that an open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of X is given.

Definition 3.10. For $q = 0, 1, 2, \dots$ define the q-th *cochain group* of $\mathcal{F}$ with respect to $\mathfrak{U}$ as

$$C^q(\mathfrak{U}, \mathcal{F}) := \prod_{i_0,\dots,i_q \in I} \mathcal{F}(U_{i_0,\dots,i_q}),$$

where $U_{i_0,\dots,i_q} = U_{i_0} \cap \dots \cap U_{i_q}$. Elements in $C^q(\mathfrak{U}, \mathcal{F})$ are called *q-cochains.* The q-th *coboundary operator* is defined as follows

$$\begin{aligned} \partial : & C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F}) \\ (\partial f)_{i_0,\dots,i_{q+1}} := & \sum_{j=0}^{q+1} (-1)^j f_{i_0,\dots,\hat{i}_j,\dots,i_{q+1}}|_{U_{i_0,\dots,i_{q+1}}}. \end{aligned}$$

Lemma 3.1.8. *Let ∂ be the coboundary operator defined above. Then $\partial^2 = 0$.*

Proof. It is sufficient to note that, when applying ∂^2, we omit each couple of indices twice, with opposite signs. □

Therefore $C^\bullet(\mathfrak{U}, \mathcal{F})$ is a cochain complex, with coboundary map ∂, so we can define the cohomology objects.

Definition 3.11. The group

$$\check{H}^q(\mathfrak{U}, \mathcal{F}) := h^q(C^\bullet(\mathfrak{U}, \mathcal{F})) = \frac{\ker\partial : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})}{\mathrm{im}\partial : C^{q-1}(\mathfrak{U}, \mathcal{F}) \to C^q(\mathfrak{U}, \mathcal{F})}$$

is the $q-$th *cohomology group* of $\mathcal{F}$ with respect to the covering $\mathfrak{U}$.

Proposition 3.1.9. *The group $\check{H}^0(\mathfrak{U}, \mathcal{F})$ is independent of the covering $\mathfrak{U}$ and*

$$\check{H}^0(X, \mathcal{F}) := \Gamma(X, \mathcal{F}).$$

Proof. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be any open covering of X.

$$\check{H}^0(\mathfrak{U}, \mathcal{F}) := \ker\partial : C^0(\mathfrak{U}, \mathcal{F}) \to C^1(\mathfrak{U}, \mathcal{F}).$$

By definition of cochains, any element α in $C^0(\mathfrak{U}, \mathcal{F})$ is given by $\{\alpha_i \in \mathcal{F}(U_i)\}$.
Then, for any $i < j$, $(\partial\alpha)_{ij} = \alpha_j - \alpha_i$.
Note that $\alpha \in \ker\partial$ iff $\alpha_i = \alpha_j$ in $U_i \cap U_j$.
Then by the sheaf axioms $\alpha \in \ker\partial : C^0(\mathfrak{U}, \mathcal{F}) \to C^1(\mathfrak{U}, \mathcal{F})$ iff $\alpha \in \Gamma(X, \mathcal{F})$. □

However, for higher values of q, the cohomology groups may depend on the covering.
Let $\mathfrak{U} = \{U_i\}_{i\in I}$, $\mathfrak{V} = \{V_j\}_{j\in J}$ be two covering of X. $\mathfrak{V}$ is said to be *finer* than $\mathfrak{U}$, and we denoted it with $\mathfrak{V} < \mathfrak{U}$, if there is a map $\tau : J \to I$ such that $V_j \subset U_{\tau(j)}$ for every $j \in J$.
From τ we can define a mapping on the cohomology groups for each q:

$$\tau^{\mathfrak{U}}_{\mathfrak{V}} : C^q(\mathfrak{U}, \mathcal{F}) \to C^q(\mathfrak{V}, \mathcal{F})$$
$$\tau^{\mathfrak{U}}_{\mathfrak{V}}(f_{i_0,\ldots,i_q}) = g_{j_0,\ldots,j_q},$$

where $g_{j_0,\ldots,j_q} = f_{\tau(j_0),\ldots,\tau(j_q)}$ for $j_0, \ldots, j_q \in J$.
This mapping commutes with ∂, thus it induces a morphism of the cohomology groups $\check{H}^q(\mathfrak{U}, \mathcal{F}) \to \check{H}^q(\mathfrak{V}, \mathcal{F})$ and we denote it also by $\tau^{\mathfrak{U}}_{\mathfrak{V}}$.
It can also be proved (see [12]) that this map is independent of the choice of τ, thus the direct limit

$$\varinjlim_{\mathfrak{U}} \check{H}^q(\mathfrak{U}, \mathcal{F})$$

is well defined.

Definition 3.12. We define

$$\check{H}^q(X, \mathcal{F}) := \varinjlim_{\mathfrak{U}} \check{H}^q(\mathfrak{U}, \mathcal{F})$$

to be the $q-$th *Čech cohomology group* of the topological space X with coefficients in the sheaf $\mathcal{F}$.

In certain cases, we can calculate the cohomology groups using only one covering of X.

Definition 3.13. Let $\mathcal{F}$ be a sheaf of abelian groups on X, a *Leray cover* of X is a cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X such that for every non empty finite set $\{i_1, \ldots, i_n\} \subset I$, and for all $q > 0$, $\check{H}^q(U_{i_1,\ldots,i_n}, \mathcal{F}) = 0$. Moreover, we say that $\mathcal{F}$ is *acyclic* over $\mathfrak{U}$.

Theorem 3.1.10 (Leray). *Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X and let $\mathfrak{U}$ be a Leray cover for X. Then*

$$\check{H}^q(\mathfrak{U}, \mathcal{F}) = \check{H}^q(X, \mathcal{F}) = H^q(X, \mathcal{F})$$

for any q, where $H^\bullet(X, \mathcal{F})$ denotes the derived functor cohomology.

Remark 21. Note that the last equality is a consequence of proposition 3.1.5, since $\mathcal{F}$ being acyclic over $\mathfrak{U}$ means that the resolution of $\mathcal{F}$ in the category $\mathfrak{Ab}(X)$ is $\Gamma(X, \cdot)-$acyclic.

Theorem 3.1.11 ([6]). *Let X be a separated and noetherian scheme, and $\mathcal{F}$ a quasi-coherent sheaf on X. Then any cover of X consisting of open affine schemes is a Leray cover.*

As a consequence, the definitions of cohomology of (quasi-)coherent sheaves on separated and noetherian schemes given in the last two subsections are equivalent, when we consider an open affine cover. From now on we will thus consider only separated and noetherian schemes so that there will be no ambiguity when talking about cohomology.

3.1.3 Cohomology on an Affine Scheme

Before focusing on the projective space, we compute first the cohomology of a coherent sheaf on an affine scheme.
We will need the following definition.

Definition 3.14. A sheaf $\mathcal{F}$ on a topological space X is *flasque* if for any inclusion $V \subset U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Proposition 3.1.12. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of sheaves on a topological space X.*

(a) If $\mathcal{F}$ is flasque, then for any open set $U \subseteq X$, the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to H(U) \to 0$$

is short exact.

(b) $\mathcal{F}, \mathcal{G}$ *flasque* $\Rightarrow \mathcal{H}$ *flasque.*

Proof. (a) We already know that the global section functor is left exact, so it is sufficient to prove that $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.
Let $s \in \mathcal{H}(U)$, define

$$T := \{(V,t); V \subseteq U \text{ open}, t \in \mathcal{G}(V) \text{ such that } t \text{ is mapped to } s|_V in \mathcal{H}\}.$$

T is not empty by exactness of the sequence of sheaves. We define a partial ordering on T:

$$(V,t) < (V',t') \text{ iff } V \subseteq V' \text{ and } t'|_V = t.$$

If $\{(V_\alpha, t_\alpha) | \alpha \in A\}$ is a totally ordered subset of T, then $V := \bigcup_{\alpha \in A} V_\alpha$ is an open containing all V_α and there exists one $t \in \mathcal{G}(V)$ such that $t|_{V_\alpha} = t_\alpha$ by the sheaf axioms.
Thus, by Zorn lemma, there exists (V,t) maximal in T.
Let $x \in U$, $W \subset U$ a small neighborhood of x, and $t' \in \mathcal{G}(W)$ mapping to $s|_W$ in $\mathcal{H}$.

$$t'|_{W \cap V} - t|_{W \cap V} \text{ maps to } 0 \text{ in} \mathcal{H},$$

then, again by exactness, it must come from some $r \in \mathcal{F}(W \cap V)$.
$\mathcal{F}$ being flasque implies that $\exists\, r' \in \mathcal{F}(W)$ such that $r'|_{W \cap V} = r$.
Take t' as the image of such r', then t, t' restrict to the same section on $W \cap V$ and there exists $\tilde{t} \in \mathcal{G}(W \cup V)$ such that

$$\tilde{t}|_W = t' \text{ and } \tilde{t}|_V = t.$$

By maximality, $x \in W \cap V = V$, hence $x \in V$ and $U = V$, which proves surjectivity.

(b) It follows directly from (a).

□

Lemma 3.1.13. *Let* $(X, \mathcal{O}_X)$ *be a locally ringed space. Any injective*[1] $\mathcal{O}_X$*-module is flasque.*

[1] An injective sheaf is an injective object in the category of abelian sheaves on a topological space.

Proof. $\forall\ U \subseteq X$ open, let $\mathcal{O}_U$ denote the sheaf obtained restricting $\mathcal{O}_X$ on U and extending to 0 outside U, and let $\mathcal{I}$ be an injective $\mathcal{O}_X-$ module. For any $V \subset U$ open set, we have the inclusion of $\mathcal{O}_X-$modules

$$0 \to \mathcal{O}_V \to \mathcal{O}_U.$$

Since $\mathcal{I}$ is injective, applying the contravariant functor $\mathrm{Hom}(\cdot, \mathcal{I})$ gives us the surjection

$$\begin{array}{ccccc} \mathrm{Hom}(\mathcal{O}_U, \mathcal{I}) & \to & \mathrm{Hom}(\mathcal{O}_V, \mathcal{I}) & \to & 0. \\ \| & & \| & & \\ \mathcal{I}(U) & & \mathcal{I}(V) & & \end{array}$$

□

Proposition 3.1.14. *If $\mathcal{F}$ is flasque, then $H^i(X, \mathcal{F}) = 0\ \forall i > 0$.*

Proof. $\mathfrak{Ab}(X)$ has enough injectives, so we can embed $\mathcal{F}$ in an injective object $\mathcal{I}$ of $\mathfrak{Ab}(X)$ and get the following short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{F}/\mathcal{I} \to 0.$$

$\mathcal{F}$ is flasque by assumption and $\mathcal{I}$ is flasque by the previous lemma. Then by proposition 3.1.12 (b) the quotient $\mathcal{F}/\mathcal{I}$ must also be flasque, and by (a) we have the following short exact sequence:

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{F}/\mathcal{I}) \to 0. \tag{3.3}$$

$\mathcal{I}$ being injective implies that $H^i(X, \mathcal{I}) = 0$ for all $i > 0$ by proposition 3.1.4. Look at the long exact sequence in cohomology:

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{I}) \to H^0(X, \mathcal{F}/\mathcal{I}) \to H^1(X, \mathcal{F}) \to 0 \to \dots$$

$$\dots \to 0 \to H^{i-1}(X, \mathcal{F}/\mathcal{I}) \to H^i(X, \mathcal{F}) \to 0 \to \dots$$

We get: $H^1(X, \mathcal{F}) = 0$ by (3.3) and $H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{F}/\mathcal{I})$ for all $i \geq 2$. But $\mathcal{F}/\mathcal{I}$ is also flasque, so by induction we have that $H^i(X, \mathcal{F}) = 0\ \forall i > 0$. □

Remark 22. This shows that flasque sheaves are $\Gamma(X, \cdot)-$acyclic, therefore by prop. 3.1.5 we can use flasque resolutions to compute cohomology.

Theorem 3.1.15. *Let X =Spec$A$, with A noetherian. For any (quasi-) coherent sheaf $\mathcal{F}$ on X, $H^i(X,\mathcal{F}) = 0 \ \forall i > 0$.*

Proof. Let $M = \Gamma(X,\mathcal{F})$, and take the injective resolution $M \to I^\bullet$ in the category of $A-$modules. We get an exact sequence of sheaves

$$0 \to \tilde{M} \to \tilde{I}^\bullet$$

on X, where $\mathcal{F} = \tilde{M}$, and each $\tilde{I}^i$ is flasque by the previous lemma. Applying Γ allows us to recover $0 \to M \to I^\bullet$, hence $H^0(X,\mathcal{F}) = \mathcal{F}(X) = M$ and $H^i(X,\mathcal{F}) = 0$ since $\tilde{I}^i$ are flasque for all $i > 0$. □

Clearly the theorem above holds for any noetherian affine scheme. The converse is also true:

Theorem 3.1.16 ([1])**.** *Let X be a noetherian scheme. X is affine if and only if $H^i(X,\mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$ and all $i > 0$.*

For arbitrary sheaves on a noetherian topological space, we also give the following result, due to Grothendieck.

Theorem 3.1.17 ([1] Vanishing theorem of Grothendieck)**.** *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, $H^i(X,\mathcal{F}) = 0$.*

3.2 Cohomology on the Projective Space

Let $S = A\left[x_0, \ldots, x_r\right]$, with A noetherian, and $X = \mathbb{P}^r_A$.
Take $\mathcal{F} := \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$. This is a coherent sheaf and it is a result from [1] III.2.9 that sheaf cohomology on a noetherian scheme commutes with infinite direct sums, hence

$$H^i(X,\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} H^i(X,\mathcal{O}_X(n)). \tag{3.4}$$

We have already seen in chapter 2 that an open affine covering for X is given by open sets $U_i := D_+(x_i)$, $i = 0, \ldots, r$. Then by theorem 3.1.11, $\mathfrak{U} = \{U_i\}$

is a Leray cover for X and we can use Čech cohomology to compute (3.4).

Note also that restricting to $D_+(x_{j_0}\dots x_{j_q})$ is equal to take the image in the localization

$$S_{x_{j_0}\dots\hat{x}_{j_k}\dots x_{j_q}} \to S_{x_{j_0}\dots x_{j_q}}.$$

Thus

$$\mathcal{F}(U_{j_0\dots j_q}) \cong S_{x_{j_0}\dots x_{j_q}}$$

and the Čech complex is given by

$$C^\bullet(\mathfrak{U},\mathcal{F}) : \prod S_{x_{j_0}} \xrightarrow{\partial^0} \prod S_{x_{j_0}x_{j_1}} \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{r-1}} S_{x_0\dots x_r} \xrightarrow{\partial^r} 0.$$

1. If $i > r$,
   then $H^i(X,\mathcal{F}) = 0$ by the Vanishing theorem of Grothendieck, in fact the complex vanishes above degree r. Thus $H^i(X,\mathcal{O}_X) = 0$.

2. If $i = 0$,
   we have that $H^0(X,\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} H^0(X,\mathcal{O}_X(n)) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X,\mathcal{O}_X(n)) = \Gamma_*(\mathcal{O}_X) \cong S$ by proposition 2.3.5 and $H^0(X,\mathcal{O}_X(n)) = \Gamma(X,\mathcal{O}_X(n)) = S_n$.

3. If $i = r$,

   $$H^r(X,\mathcal{F}) = \frac{\ker\partial^r}{\mathrm{im}\partial^{r-1}} = \frac{S_{x_0\dots x_r}}{\mathrm{im}\partial^{r-1}},$$

   where

   $$\partial^{r-1} : \prod_{j=0}^{r} S_{x_0\dots\hat{x}_j\dots x_r} \to S_{x_0\dots x_r}.$$

   Elements in $S_{x_0\dots x_r}$ are of the form

   $$x_0^{m_0}\dots x_r^{m_r} f(x_0,\dots,x_r), \qquad m_j \in \mathbb{Z}, f \in S,$$

   and they belong to $\mathrm{im}\partial^{r-1}$ if at least one x_j is not appearing in the denominator, i.e. $m_j \geq 0$ for some j.
   Thus $H^r(X,\mathcal{F})$ is an $A-$module with a basis given by monomials $x_0^{m_0}\dots x_r^{m_r}$ such that $m_j < 0$ for all $j = 0,\dots,r$, and those of degree n are form a basis for $H^r(X,\mathcal{O}_X(n))$.

4. If $0 < i < r$,

we want to show that $H^i(X, \mathcal{F}) = 0$, and consequently $H^i(X, \mathcal{O}_X) = 0$. We will prove this by induction on r.

For $r = 1$, there is nothing to prove, so let $r > 1$.

If we localize the complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ by inverting x_r, we get the complex corresponding to $\mathcal{F}|_{U_r}$ with respect to the open affine covering $\mathfrak{U}_r := \{U_i \cap U_r\}$ of $U_r = D_+(x_r)$ which is affine, thus by theorem 3.1.16, $H^i(U_r, \mathcal{F}|_{U_r}) = 0$.

Since localization is an exact functor, it commutes with cohomology, i.e $H^i(X, \mathcal{F})_{x_r} = 0$, hence every element in $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

Thus proving that $H^i(X, \mathcal{F}) = 0$ is equal to prove that the multiplication by any power of x_r, hence by x_r itself, is injective.

To do so, consider the short exact sequence

$$0 \to S(-1) \xrightarrow{\times x_r} S \to S/(x_r) \to 0.$$

$\{x_r = 0\}$ defines an hyperplane $H \cong \mathbb{P}_A^{r-1}$, so the sequence above gives the short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

Twisting for each $n \in \mathbb{Z}$ and taking the sum, we get the short exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0, \tag{3.5}$$

and the induced long exact sequence in cohomology

$$0 \to \mathcal{F}(-1)(X) \to \mathcal{F}(X) \to \mathcal{F}_H(X) \to H^1(X, \mathcal{F}(-1)) \to$$

$$\to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}_H) \to H^2(X, \mathcal{F}(-1)) \to \dots$$

By (3.5), $H^1(X, \mathcal{F}(-1)) = 0$ and, since $H \cong \mathbb{P}_A^{r-1}$, we can apply the induction hypothesis on $\mathcal{F}_H$ that gives

$$H^i(X, \mathcal{F}_H) = 0 \text{ for all } 0 < i < r - 1.$$

Then from the long exact sequence we get the isomorphisms

$$H^i(X, \mathcal{F}(-1)) \cong H^i(X, \mathcal{F}) \text{ for all } 0 < i < r-1$$

which imply that the multiplication $\times x_r$ is bijective for all $0 < i < r-1$, and for $i = r-1$, we have

$$0 \to H^{r-1}(X, \mathcal{F}(-1)) \to H^{r-1}(X, \mathcal{F}) \to H^{r-1}(X, \mathcal{F}_H)$$

which implies that the multiplication $\times x_r$ is injective, as wanted.

All these computations prove the following

Theorem 3.2.1. *Let $X = \mathbb{P}^r_A$, with A noetherian. Then*

(a) $H^i(X, \mathcal{O}_X(n)) = 0$ *for all* $0 < i < r$;

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ *for all* $i > r$;

(c) $H^r(X, \mathcal{O}_X(-r-1)) \cong A$;

(d) The natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of $A-$modules.

Proof. (a), (b) follows by the previous computation.

(c) By the computation above, recall that $H^r(X, \mathcal{F})$ has a basis given by monomials

$$x_0^{m_0} \dots x_r^{m_r} \text{ such that } m_j < 0 \text{ for all } j = 0, \dots, r.$$

The grading of each of these monomials is $\sum_{j=0}^r m_j$ and $\sum_{j=0}^r m_j = -r-1$ if and only if $m_j = -1$ for each j, since the m_j must all be strictly negative.
Thus $H^r(X, \mathcal{O}_X(-r-1))$ has only one generator, namely $x_0^{-1} \dots x_r^{-1}$ and we get the desired isomorphism of A-modules.

(d) Note first that for $n < 0$, $H^0(X, \mathcal{O}_X(n)) = 0$ and $H^r(X, \mathcal{O}_X(-n-r-1)) = 0$ because n being negative implies that $-n-r-1 > -r-1$ and there are no monomials with all negative exponents of degree strictly bigger that $-r-1$. So (d) is trivial if $n < 0$.
Assume $n \geq 0$, $H^0(X, \mathcal{O}_X(n))$ has a basis given by monomials

$$x_0^{l_0} \dots x_r^{l_r} \text{ such that } l_j \geq 0 \text{ and } \sum_{j=0}^{r} l_j = n.$$

Then the pairing is given by

$$\begin{aligned} H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) &\to H^r(X, \mathcal{O}_X(-r-1)) \\ (x_0^{l_0} \dots x_r^{l_r}, x_0^{m_0} \dots x_r^{m_r}) &\mapsto x_0^{l_0+m_0} \dots x_r^{l_r+m_r} \end{aligned}$$

with $l_j \geq 0; \sum_{j=0}^{r} l_j = n$ and $m_j < 0; \sum_{j=0}^{r} m_j = -n-r-1$, where the right hand side is always zero unless it is $x_0^{-1} \dots x_r^{-1}$ for what we have seen in (c).
Then, by defining the dual of $x_0^{l_0} \dots x_r^{l_r}$ as the multiplication by $x_0^{-l_0-1} \dots x_r^{-l_r-1}$ in $\mathrm{Hom}(H^r(X, \mathcal{O}_X(-n-r-1)), H^r(X, \mathcal{O}_X(-r-1)))$, we get the isomorphism $H^0(X, \mathcal{O}_X(n)) \cong \mathrm{Hom}(H^r(X, \mathcal{O}_X(-n-r-1)), H^r(X, \mathcal{O}_X(-r-1)))$, i.e. a perfect pairing.

□

The computations above, can also be generalized to any projective scheme over a noetherian ring.

Theorem 3.2.2. *Let A be a noetherian ring, and X a projective scheme with closed immersion $j : X \hookrightarrow \mathbb{P}^r_A$ where $\mathcal{O}_X(1)$ is the inverse image through j of the twisting sheaf of Serre on the projective space. Then if $\mathcal{F}$ is a coherent sheaf on X, for any $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated $A-$module and there exists an integer n_0, depending on $\mathcal{F}$, such that*

$$H^i(X, \mathcal{F}(n)) = 0 \qquad \text{for } n \geq n_0,\ i > 0.$$

Proof. We have already seen that $\mathcal{F}$ coherent implies that $j_*\mathcal{F}$ is also coherent in $\mathbb{P}^r_A$. Moreover their cohomology is the same: $H^i(X, \mathcal{F}) = H^i(\mathbb{P}^r_A, j_*\mathcal{F})$. Indeed, by remark 22, cohomology can be computed using flasque resolutions and, if $\mathcal{J}^\bullet$ is a flasque resolution of $\mathcal{F}$ on X, then clearly $j_*\mathcal{J}^\bullet$ is still flasque on $\mathbb{P}^r_A$ and for any i

$$\Gamma(\mathbb{P}^r_A, j_*\mathcal{J}^i) = j_*\mathcal{J}^i(\mathbb{P}^r_A) = \mathcal{I}^i(j^{-1}\mathbb{P}^r_A) = \mathcal{J}^i(X) = \Gamma(X, \mathcal{J}^i).$$

Therefore we may assume $X = \mathbb{P}^r_A$.
If $\mathcal{F} = \mathcal{O}_X$, the proof follows directly by the computation on the projective space, recalling that, when $i = r$, $H^r(X, \mathcal{O}_X(n))$ is generated by monomials whose variables have all negative powers, so it suffices to choose n positive.
When $\mathcal{F}$ is an arbitrary coherent sheaf we prove the theorem using descending induction. If $i > r$, then $H^i(X, \mathcal{F}) = 0$ by the Vanishing theorem of Grothendieck.
Let $i \leq r$, we use corollary 2.3.7 and write $\mathcal{F}$ as a quotient of a finite direct sum of twisted sheaves $\mathcal{E} = \bigoplus \mathcal{O}_X(q_k)$, for some integers q_k. Let $\mathcal{R}$ be the kernel of the projection $\mathcal{E} \to \mathcal{F}$, $\mathcal{R}$ is still a coherent sheaf and we have the following short exact sequence:

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0,$$

that induces the long exact sequence in cohomology

$$\cdots \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{R}) \to \dots,$$

where $H^i(X, \mathcal{E})$ is a finitely generated $A-module$, because finite sum of such, and $H^{i+1}(X, \mathcal{R})$ is a finitely generated $A-$module by inductive hypothesis. Hence $H^i(X, \mathcal{F})$ is also a finitely generated $A-$module.
By twisting for some $n >> 0$, the induced long exact sequence in cohomology becomes

$$\cdots \to H^i(X, \mathcal{E}(n)) \to H^i(X, \mathcal{F}(n)) \to H^{i+1}(X, \mathcal{R}(n)) \to \dots.$$

For each i, $H^i(X, \mathcal{E}(n)) = 0$ since the same holds for $\mathcal{O}_X(n + q_k)$, and $H^{i+1}(X, \mathcal{R}(n)) = 0$ by the inductive hypothesis. Therefore $H^i(X, \mathcal{F}(n)) = 0$. □

Chapter 4

Serre Duality

4.0.1 Ext Groups and Sheaves

Let $(X, \mathcal{O}_X)$ be a noetherian scheme.
Denote by $\mathfrak{Mod}(X)$ the category of $\mathcal{O}_X-$modules. For any two objects $\mathcal{F}, \mathcal{G}$ in $\mathfrak{Mod}(X)$, let $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ be the group of $\mathcal{O}_X-$module homomorphisms.
For any $U \subset X$ open, $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U-$module and the presheaf $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf that we denote it by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, which is also an $\mathcal{O}_X-$module.
Fix $\mathcal{F}$ as above and consider the covariant left exact functors

$$\mathrm{Hom}(\mathcal{F}, \cdot) : \mathfrak{Mod}(X) \to \mathfrak{Ab},$$

$$\mathcal{H}om(\mathcal{F}, \cdot) : \mathfrak{Mod}(X) \to \mathfrak{Mod}(X).$$

Since they are covariant left exact functors and $\mathfrak{Mod}(X)$ has enough injectives by [1], III.2.2, their right derived functors are well defined.

Definition 4.1. Let

$$\mathrm{Ext}^i(\mathcal{F}, \cdot) := R^i\mathrm{Hom}(\mathcal{F}, \cdot),$$

$$\mathcal{E}xt^i(\mathcal{F}, \cdot) := R^i\mathcal{H}om(\mathcal{F}, \cdot)$$

For any i and $\mathcal{G}$ in $\mathfrak{Mod}(X)$, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ is called *ext group* and $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is called *ext sheaf.*

Remark 23. From what we have seen in chapter 3.1

$$\mathrm{Ext}^0 = \mathrm{Hom}, \quad \mathcal{E}xt^0 = \mathcal{H}om. \tag{4.1}$$

Proposition 4.0.3. *Let $\mathcal{G} \in \mathfrak{Mod}(X)$. Then*

(a) $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$;

(b) $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$ *for all* $i > 0$;

(c) $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$, *for all* $i \geq 0$.

Proof. Note that $\mathcal{H}om(\mathcal{O}_X, \cdot)$ is the identity functor. Then by (4.1) $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{H}om(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$ and for $i > 0$ its right derived functors are zero by the exactness of the identity functor.
On the other hand, $\mathrm{Hom}(\mathcal{O}_X, \cdot)$ is $\Gamma(X, \cdot)$, thus $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) = R^i\mathrm{Hom}(\mathcal{O}_X, \mathcal{G}) = R^i\Gamma(X, \mathcal{G}) = H^i(X, \mathcal{G})$ for any i. □

Proposition 4.0.4 ([1])**.** *Let $\mathcal{L}$ be a locally free sheaf of finite rank and $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ be its dual. Then for any $\mathcal{F}, \mathcal{G} \in \mathfrak{Mod}(X)$,*

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}),$$

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee.$$

Proposition 4.0.5 ([1])**.** *Let $\mathcal{F}$ be a coherent sheaf on X and $\mathcal{G}$ be any $\mathcal{O}_X-$module. Then for any closed point $x \in X$*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x).$$

Proposition 4.0.6. *Let X be a projective noetherian scheme, $\mathcal{F}$ a locally free sheaf on X, $\mathcal{G}$ a coherent sheaf on X. Then there exist an integer n_0, depending on $\mathcal{F}$ and $\mathcal{G}$, such that, for any $n \geq n_0$,*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

Proof. For $i = 0$ the proof is immediate for any n:

$$\Gamma(X, \mathcal{E}xt^0(\mathcal{F}, \mathcal{G}(n))) = \Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}(n))) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}(n)) = \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}(n)).$$

Let $i > 0$ and consider first the case $\mathcal{F} = \mathcal{O}_X$.
By proposition 4.0.3 (c), $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}(n)) \cong H^i(X, \mathcal{G}(n))$. Thus, for $n \geq n_0$ and $i > 0$

$$H^i(X, \mathcal{G}(n)) = 0$$

by theorem 3.2.2.
On the other hand, $\Gamma(X, \mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}(n))) = 0$ for $i > 0$ by proposition 4.0.3(b). Thus the proposition is proved when $\mathcal{F} = \mathcal{O}_X$.
When $\mathcal{F}$ is an arbitrary locally free O_X-module, because $\mathcal{O}_X$ is coherent, $\mathcal{F}$ is also coherent, hence of finite rank, therefore, by proposition 4.0.4, $\mathrm{Ext}(\mathcal{F}, \mathcal{G}(n)) \cong \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}^\vee \otimes \mathcal{G}(n))$ and similarly for $\mathcal{E}xt$, so we can reduce to the previous case. □

4.0.2 The Canonical Sheaf

Recall the notion of derivation.

Definition 4.2. Let A be a commutative unitary ring. Let B be an $A-$module, and M a $B-$module. An $A-$derivation is a function

$$d : B \to M$$

such that, for any $b, b' \in B$, $a \in A$,

1. $d(b + b') = db + db'$;
2. $d(bb') = (db)b' + b(db')$[1];
3. $da = 0$.

An $A-$ derivation $d : B \to M$ is said to satisfy the *universal property* if for any $B-$module M' and for any $A-$derivation $d' : B \to M'$ there exists one and only one $B-$module homomorphism $f : M \to M'$ making the diagram

[1]This is the *Leibniz rule*.

$$\begin{array}{ccc} B & \xrightarrow{\quad d \quad} & M \\ & {}_{d'}\searrow \quad \swarrow_{f} & \\ & M' & \end{array}$$

commutative.

Definition 4.3. The *module of relative differential forms* of B over A, denoted by $\Omega_{B/A}$, is a $B-$module, together with an $A-$derivation $d : B \to \Omega_{B/A}$ satisfying the universal property.

Remark 24. One way to construct $\Omega_{B/A}$ is to take the $B-$module generated by the symbols db, $b \in B$ and quotient out by equivalence relation defined by properties 1,2 and 3 in the definition 4.2. Define then the derivation by sending b to db.

The notion of module of relative differential forms can be generalized to sheaves of rings: let X be a topological space and consider $\mathcal{A}, \mathcal{B}$ sheaves of rings and $\mathcal{A} \to \mathcal{B}$ a morphism of sheaves of rings.
Then $\mathcal{B}$ is an $\mathcal{A}-$module and we define a presheaf $\tilde{\Omega}_{\mathcal{B}/\mathcal{A}}$ by

$$U \mapsto \tilde{\Omega}_{\mathcal{B}/\mathcal{A}},$$

with restriction maps $\tilde{\Omega}_{\mathcal{B}/\mathcal{A}}(U) \to \tilde{\Omega}_{\mathcal{B}/\mathcal{A}}(V)$ defined by taking the restriction $\mathcal{B}(U) \to \mathcal{B}(V)$, when $V \subset U$, which is an $\mathcal{A}(U)-$derivation on $\tilde{\Omega}_{\mathcal{B}/\mathcal{A}}(V)$.
Then sheafify[2] $\tilde{\Omega}_{\mathcal{B}/\mathcal{A}}$ and denote by $\Omega_{\mathcal{B}/\mathcal{A}}$ the sheafification.
When we are considering a scheme $(X, \mathcal{O}_X)$ over k, as we have seen in chapter 2, we have a morphism $X \to Y$ where $Y = \mathrm{Spec}k$ =point, with structure sheaf k.
Then $\Omega_{\mathcal{O}_X/k}$ is a quasi-coherent (the sheafification has the same construction of $\tilde{M}$ defined in chapter 2) $\mathcal{O}_X-$module and we define

$$\Omega_X := \Omega_{\mathcal{O}_X/k}.$$

[2]Let $\mathcal{F}$ be a presheaf on a topological space X, and denote by $\mathcal{F}^+(U)$, $U \subset X$ open, the collection of functions $s : U \to \amalg_{P \in U} \mathcal{F}_P$ such that, for any $P \in U$, $s(P) \in \mathcal{F}_P$ and there exists an open neighborhood of P, $V \subset U$, and $t \in \mathcal{F}(V)$ such that $t_q = s(q)\ \forall q \in V$. $\mathcal{F}^+$ is a sheaf and it is called the *sheafification* of $\mathcal{F}$.

Example 4.1. If X is the affine space $\mathbb{A}_k^n$, then Ω_X is generated by $dx_1, \dots, dx_n$, where $x_1, \dots, x_n$ are the affine coordinates.

Definition 4.4. The *canonical sheaf* on a scheme X over k is the $n-$th exterior algebra

$$\omega_X := \Lambda^n \Omega_X,$$

where $n = \dim X$.

When X is the projective space over the field k, we find that

$$\omega_X \cong \mathcal{O}_X(-n-1). \tag{4.2}$$

This is a consequence of the following theorem.

Theorem 4.0.7. *Let $X = \mathbb{P}_k^n$. Then there is an exact sequence of sheaves on X*

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0. \tag{4.3}$$

Proof. Denote
$S := k[x_0, \dots, x_n]$, $E := S(-1)^{n+1}$, the set of $n+1-$tuples with degree $d+1$, if they have degree d in S.
A basis for E is given by $e_0 = (1, \dots, 0), \dots, e_n = (0, \dots, 1)$, each with degree 1.
Define then the homomorphism of graded $S-$modules

$$\begin{aligned} \varphi : & E \to S \\ & e_i \mapsto x_i \qquad i = 0, \dots, n. \end{aligned}$$

Let $M := \ker\varphi$, then we have the exact sequence

$$0 \to M \to E \to S,$$

which gives the exact sequence of sheaves

$$0 \to \tilde{M} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X.$$

Note also that φ is surjective in degree ≥ 1, thus $\mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X$ is surjective and

$$0 \to \tilde{M} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0$$

is exact, and it suffices to prove that $\tilde{M} \cong \Omega_X$.
Localize both E, S at x_i for some i, then $E_{x_i} \to S_{x_i}$ is surjective, and

$$e_j - \frac{x_j}{x_i} e_i \mapsto x_j - \frac{x_j}{x_i} x_i = 0, \qquad j \neq i$$

while if $j = i$, then $e_j - \frac{x_j}{x_i} e_i = 0$.
Thus M_{x_i} is a free $S_{x_i}-$module of rank n with a basis

$$\{\frac{e_j}{x_i} - \frac{x_j}{x_i^2} e_i;\ j \neq i\}.$$

Recall also that, by the construction of the sheaf $\tilde{M}$ from M, if $U_i = D_+(x_i)$,

$$\tilde{M}|_{U_i} \cong M_{x_i},$$

i.e. it is a free $\mathcal{O}_{U_i}-$module generated by sections $\frac{e_j}{x_i} - \frac{x_j}{x_i^2} e_i;\ j \neq i$.
On the other hand, $U_i = D_+(x_i) \cong \text{Spec} k\left[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}\right]$, hence $\Omega_X|_{U_i}$ is a $\mathcal{O}_{U_i}-$module generated by $d\left(\frac{x_0}{x_i}\right), \dots, d\left(\frac{x_n}{x_i}\right)$.
Define

$$\begin{aligned} \psi_i : &\Omega_X|_{U_i} \to \tilde{M}|_{U_i} \\ &d\left(\frac{x_j}{x_i}\right) \mapsto \frac{1}{x_i^2}(x_i e_j - x_j e_i). \end{aligned}$$

This is clearly an isomorphisms: the basis of $\Omega_X|_{U_i}$ is mapped to the basis of $\tilde{M}_{U_i}$. In the intersections $U_i \cap U_j$,

$$\frac{x_k}{x_i} = \frac{x_k}{x_j} \frac{x_j}{x_i},$$

and by the Leibniz rule

$$d\left(\frac{x_k}{x_i}\right) - \frac{x_k}{x_j} d\left(\frac{x_j}{x_i}\right) = d\left(\frac{x_k}{x_j}\right) \frac{x_j}{x_i}, \text{ on } \Omega_X|_{U_i \cap U_j}.$$

Applying ψ_i, ψ_j to the left hand side and to the right one respectively, gives us the same result, extending the isomorphism to all X:

$$\begin{aligned}\psi_i\left(d\left(\frac{x_k}{x_i}\right)-\frac{x_k}{x_j}d\left(\frac{x_j}{x_i}\right)\right) &= \frac{1}{x_i^2}(x_ie_k - x_ke_i) - \frac{x_k}{x_i^2x_j}(x_ie_j - x_je_i)\\ &= \frac{1}{x_i^2x_j}(x_ix_je_k - x_jx_ke_i - x_ix_ke_j + x_jx_ke_i)\\ &= \frac{1}{x_ix_j}(x_je_k - x_ke_j).\end{aligned}$$

$$\begin{aligned}\psi_j\left(d\left(\frac{x_k}{x_j}\right)\frac{x_j}{x_i}\right) &= \frac{1}{x_j^2}(x_je_k - x_ke_j)\frac{x_j}{x_i}\\ &= \frac{1}{x_ix_j}(x_je_k - x_ke_j).\end{aligned}$$

□

Therefore, taking the $n-$th exterior product of the short exact sequence (4.3) allows us to write (4.2).

4.0.3 $\delta-$Functors

Definition 4.5. Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories. A (covariant) $\delta-$*functor* from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $T = (T^i)_{i\geq 0}$, together with morphisms $\delta^i : T^i(A'') \to T^{i+1}(A')$, for any $i \geq 0$ and any short exact sequence $0 \to A' \to A \to A'' \to 0$ of objects in $\mathfrak{A}$, such that

1. there is a long exact sequence

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^0} T^1(A') \to T^1(A) \to \dots$$

2. If $0 \to B' \to B \to B'' \to 0$ is another short exact sequence, then the diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta^i} & T^{i+1}(B') \end{array}$$

is commutative.

Remark 25. Given any functor F between abelian categories, the collection of right derived functors $R^iF(\cdot)$, when they are well defined, is a $\delta-$functor.

Definition 4.6. Let T as above. It is said to be *universal* if for any other $\delta-$functor $T' = (T'^i)_{i\geq 0} : \mathfrak{A} \to \mathfrak{B}$, and given any morphism of functors $f^0 : T^0 \to T'^0$, there exists a unique sequence of morphisms $f^i : T^i \to T'^i$, $i \geq 0$, starting with the given f^0, which commutes with δ^i for any short exact sequence.

Definition 4.7. An additive functor $F : \mathfrak{A} \to \mathfrak{B}$ is *effaceable* if, for any object $A \in \mathfrak{A}$, there is a monomorphism $u : A \to M$ for some M, such that $F(u) = 0$.

Theorem 4.0.8 ([8]). *Let $T = (T^i)_{i\geq 0}$ be a covariant $\delta-$functor. If T^i is effaceable for any $i > 0$, T is universal.*

4.1 The Serre Duality Theorem

In this last section we will prove the most important result of this thesis, namely the *Serre duality theorem* for the cohomology of coherent sheaves on a projective scheme. We will consider first the case of the projective space $\mathbb{P}^n_k$ and then generalize it for an arbitrary projective scheme.

Theorem 4.1.1 (Serre duality for $\mathbb{P}^n_k$). *Let $X = \mathbb{P}^n_k$,*

(a) There is a canonical isomorphism $H^n(X, \omega_X) \cong k$.

(b) For any coherent sheaf $\mathcal{F}$ on X,

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \cong k$$

is a perfect pairing of finite dimensional vector spaces over k.

(c) For any $i \geq 0$, there exists a natural isomorphisms of $k-$modules

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \to H^{n-i}(X, \mathcal{F})' := \mathrm{Hom}(H^{n-i}(X, \mathcal{F}), k).$$

Proof. (a) We know from (4.2) that when $X = \mathbb{P}^n_k$, $\omega_X \cong \mathcal{O}_X(-n-1)$, and by theorem 3.2.1 (c), $H^n(X, \mathcal{O}_X(-n-1)) \cong k$.

(b) Let $\varphi \in \mathrm{Hom}(\mathcal{F}, \omega_X)$. φ induces a morphism $H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$, and thus define the pairing $\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \cong k$ by (a).
To see that this pairing is perfect, we want to show that $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})'$.
Suppose first that $\mathcal{F} = \mathcal{O}_X(q)$ for some $q \in \mathbb{Z}$.
Then

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_X(q), \omega_X) &\cong \mathrm{Hom}(\mathcal{O}_X(q), \mathcal{O}_X(-n-1)) \text{ by (4.2)}\\ &\cong \Gamma(X, \mathcal{O}_X(-q-n-1))\\ &\cong H^0(X, \mathcal{O}_X(-q-n-1)).\end{aligned}$$

The natural pairing of theorem 3.2.1 (d) induces the isomorphism of $k-$modules

$$H^0(X, \mathcal{O}_X(-q-n-1)) \cong H^n(X, \mathcal{O}_X(q))',$$

thus $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})'$ when $\mathcal{F} = \mathcal{O}_X(q)$.
This also hold if $\mathcal{F} = \bigoplus_{i=1}^N \mathcal{O}(q_i)$, since cohomology commutes with direct sums.
Finally, if $\mathcal{F}$ is an arbitrary coherent sheaf, we recall that corollary 2.3.7 allows us to write $\mathcal{F}$ as a quotient of a sheaf $\mathcal{E}$, where $\mathcal{E}$ is a finite direct sum of twisted sheaves $\mathcal{O}(-q)$, $q >> 0$.
Equivalently, $\mathcal{F}$ is the cokernel of a morphism of sheaves $\mathcal{E}_1 \to \mathcal{E}_2$, where $\mathcal{E}_i = \bigoplus \mathcal{O}(-q_i), i = 1, 2$. Thus we get the exact sequence

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{F} \to 0,$$

and since $\mathrm{Hom}(\cdot, \omega_X)$ and $H^n(X, \cdot)'$ are contravariant left exact functors, the sequence above induces the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \omega_X) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_2, \omega_X) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_1, \omega_X) \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & H^n(X, \mathcal{F})' & \longrightarrow & H^n(X, \mathcal{E}_2)' & \longrightarrow & H^n(X, \mathcal{E}_1)' \end{array}$$

Then by the 5-lemma, $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})'$.

(c) For $i = 0$, we get exactly (b).
For $i > 0$, we observe that $(\mathrm{Ext}^i(\cdot, \omega_X))_{i\geq 0}$ and $(H^{n-i}(X, \cdot))_{i\geq 0}$ are both $\delta-$functors, therefore, if we prove that they are both universal, then they have to be isomorphic for each $i > 0$. Then, using theorem 4.0.8, it is sufficient to prove that they are both effaceable functors, for each $i > 0$, and $\mathcal{F}$ in the category of coherent sheaves.
To do so, we use again the fact that $\mathcal{F}$ is the quotient of $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}_X(-q)$. Then

$$\begin{aligned} \mathrm{Ext}^i(\mathcal{E}, \omega_X) &= \bigoplus \mathrm{Ext}^i(\mathcal{O}_X(-q), \omega_X) \\ &\cong \bigoplus H^i(X, \omega_X(q)) \text{ by proposition 4.0.3 (c)} \\ &\cong \bigoplus H^i(X, \mathcal{O}(q-n-1)) \\ &= 0 \text{ by theorem 3.2.1 (a).} \end{aligned}$$

On the other hand,

$$\begin{aligned} H^{n-i}(X, \mathcal{E})' &= \bigoplus H^{n-i}(X, \mathcal{O}_X(-q))' \\ &= 0 \text{ by theorem 3.2.1 (a) if } n-i < n, \text{ i.e. } i > 0. \end{aligned}$$

□

In order to generalize the Serre duality theorem to an arbitrary projective scheme, we need to replace the canonical sheaf with the notion of *dualizing* *sheaf.*

Moreover, we will also give some definitions and results of commutative algebra that will be used in the proof.

Definition 4.8. Let X be a proper scheme over k, with $\dim X = n$. A *dualizing sheaf* for X is a coherent sheaf ω_X° on X, together with a *trace* morphism

$$t : H^n(X, \omega_X^\circ) \to k$$

such that, for any $\mathcal{F}$ coherent sheaf on X, the pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ),$$

followed by t, gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})'$$

Lemma 4.1.2. *Let X be a proper scheme over k. If there exists a dualizing sheaf for X, then it is unique, up to isomorphisms.*

Proposition 4.1.3. *Any projective scheme over a field k has a dualizing sheaf.*

To be more precise, the dualizing sheaf of the proposition above, is

$$\omega_X^\circ := \mathcal{E}xt_P^r(\mathcal{O}_X, \omega_P),$$

where $X \hookrightarrow \mathbb{P}_k^N =: P$ is a closed immersion making X projective and $r = N - \dim X$.

Definition 4.9. Let $(A, \mathfrak{m})$ be a local ring. A is *Cohen-Macaulay* if its *depth* is equal to its Krull dimension.

The *depth* of a local ring A is the maximum length of a *regular sequence* in $\mathfrak{m}$, that is a sequence of elements $x_1, \dots, x_r \in \mathfrak{m}$ such that, for any $i = 1, \dots, r$, x_i is a non-zero divisor of $A/(x_1, \dots, x_{i-1})$.

It follows by the definition that the depth of A is always less or equal than its Krull dimension.

Definition 4.10. Let A be a ring. An $A-$module P is said to be *projective* if the functor $\mathrm{Hom}(P, \cdot) : A - \mathfrak{Mod} \to \mathfrak{Ab}$ is exact, where $A - \mathfrak{Mod}$ denotes the category of $A-$modules.

Proposition 4.1.4 ([13])**.** *Let M be an $A-$module. Then*

$$\mathrm{pd}M \leq n \text{ if and only if } \mathrm{Ext}^i(M, N) = 0 \text{ for all } i > n \text{ and all } A-\text{modules } N,$$

where pd$M$ *denotes the projective dimension of M, that is the least length of a projective resolution*[3] *of M.*

Proposition 4.1.5 ([13])**.** *If A is a regular local ring of dimension n and M is a finitely generated $A-$module, then*

$$\mathrm{pd}M + \mathrm{depth}M = n.$$

Theorem 4.1.6 (Serre duality for a projective scheme)**.** *Let X be a projective scheme of dimension n over an algebraically closed field k. Let ω_X° be a dualizing sheaf on X and $\mathcal{O}_X(1)$ the inverse image of the twisting sheaf on the projective space through the closed immersion $j : X \to \mathbb{P}_k^N$. Then:*

(a) For any $i \geq 0$ and $\mathcal{F}$ coherent sheaf on X, there exist natural functorial maps

$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})'.$$

(b) The following are equivalent:

(i) X is Cohen-Macaulay[4] *and equidimensional*[5]*.*

[3]The definition of a *projective resolution* is obtained from the definition of an injective resolution by replacing injective objects with projective ones.

[4]A scheme X is *Cohen-Macaulay* if it is locally noetherian and its local ring A at any point is Cohen-Macaulay.

[5]X is *equidimensional* if all its irreducible components have the same dimension.

(ii) For any locally free sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$, $q >> 0$.

(iii) θ^i from (a) are isomorphisms.

Proof. (a) Observe that $(\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ))_{i\geq 0}$ and $(H^{n-i}(X, \mathcal{F})')_{i\geq 0}$ are both $\delta-$functors and θ^0 is the map given by the definition of dualizing sheaf. Thus, by definition of universal $\delta-$functor, we need to prove that $(\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ))_{i\geq 0}$ is universal, and it is sufficient to show that $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ)$ is effaceable for $i > 0$ by theorem 4.0.8.
Recall that $\mathcal{F}$ can be written as a quotient of $\mathcal{E} = \bigoplus \mathcal{O}_X(-q)$, $q >> 0$. Then $\mathrm{Ext}^i(\mathcal{E}, \omega_X^\circ) \cong \bigoplus H^i(X, \omega_X^\circ(q))$ which is 0 for $i > 0$ and $q >> 0$ by theorem 3.2.2.

(b)

(i)⇒(ii) Since X is Cohen-Macaulay and equidimensional by assumption, for any closed point $x \in X$ and $\mathcal{F}$ locally free sheaf on X,

$$\mathrm{depth}\mathcal{F}_x = n.$$

Let $P := \mathbb{P}_k^N$, and consider $j : X \hookrightarrow P$. Set $A := \mathcal{O}_{P,j^*(x)}$, where $j^*(x) = x$, j induces a surjective morphism of local rings $A \to \mathcal{O}_{X,x}$, hence $\mathrm{depth}_A\mathcal{F}_x = \mathrm{depth}\mathcal{F}_x = n$. Because P is non singular, A is regular: $\dim A = \dim P = N$ and, by proposition 4.1.5, $\mathrm{pd}_A\mathcal{F}_x = N - n$.
Then, by propositions 4.0.5 and 4.1.4, for any $\mathcal{O}_P-$module $\mathcal{G}$,

$$\mathcal{E}xt_P^i(\mathcal{F}, \mathcal{G})_x = \mathrm{Ext}_A^i(\mathcal{F}_x, \mathcal{G}_x) = 0 \qquad , \forall i > N - n. \tag{4.4}$$

From the Serre duality theorem proved for P, we find that, for $q >> 0$,

$$H^i(X, \mathcal{F}(-q))' \cong \mathrm{Ext}_P^{N-i}(\mathcal{F}, \omega_P(q))$$

where $\mathrm{Ext}_P^{N-i}(\mathcal{F}, \omega_P(q)) \cong \Gamma(X, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P(q)))$ by proposition 4.0.6, and the latter is 0 if $i < n$ by (4.4).

(ii)⇒(i) Take $\mathcal{F} = \mathcal{O}_X$, and fix $i > N-n$. We claim that $\mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P(q)) = 0$:

$$\begin{aligned}\Gamma(P, \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P(q))) &\cong \mathrm{Ext}^{N-i}_P(\mathcal{O}_X, \omega_P(q)) \text{ by proposition 4.0.6} \\ &\cong H^i(X, \mathcal{O}_X(-q))'\end{aligned}$$

by the Serre duality theorem for the projective space, and it is zero by assumption.

Then, by proposition 4.0.5, $\mathrm{Ext}^i_A(\mathcal{O}_{X,x}, A) = 0$ for all $i > N-n$ which implies $\mathrm{pd}_A(\mathcal{O}_{X,x}) \leq N-n$ and $\mathrm{depth}\mathcal{O}_{X,x} \geq n$ by proposition 4.1.4. and 4.1.5. Therefore X is Cohen-Macaulay and equidimensional.

(ii)⇒(iii) We have already proved in (a) that $(\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ))_{i\geq 0}$ is a universal $\delta-$functor, thus, if we prove that $(H^{n-i}(X, \mathcal{F})')_{i\geq 0}$ is also universal, θ^i are isomorphisms. We use again the fact that it is sufficient to prove that $H^{n-i}(X, \mathcal{F})'$ is an effaceable functor for $i > 0$ and this is true by assumption since, as usual, we can write $\mathcal{F}$ as a quotient of $\mathcal{E} = \bigoplus \mathcal{O}_X(-q)$, $q >> 0$.

(iii)⇒(ii) By assumption, for any locally free sheaf $\mathcal{F}$, $q >> 0$, $H^i(X, \mathcal{F}(-q))' \cong \mathrm{Ext}^{n-i}(\mathcal{F}(-q), \omega_X^\circ)$. Consider the right hand side:

$$\begin{aligned}\mathrm{Ext}^{n-i}(\mathcal{F}(-q), \omega_X^\circ) &\cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ(q)) \text{ by prop. 4.0.4} \\ &\cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ(q)) \text{ by prop. 4.0.3(c)}\end{aligned}$$

that is 0 when $i < n$ by the generalization of the computation of cohomology on a projective scheme.

□

Corollary 4.1.7. *Let X be a projective Cohen-Macaulay scheme over k, of equidimension n. Then for any $\mathcal{F}$ locally free sheaf on X there are natural isomorphisms*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)' \qquad \forall i \geq 0.$$

Proof.

$$\begin{aligned} H^i(X, \mathcal{F}) &\cong \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X^\circ)' \text{ by the previous theorem} \\ &\cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ)' \text{ by proposition 4.0.4} \\ &\cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)' \text{ by proposition 4.0.3 (c).} \end{aligned}$$

□

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