

Theory of current-induced spin polarization in an electron gasCosimo Gorini,¹ Amin Maleki Sheikhabadi,² Ka Shen,³ Ilya V. Tokatly,^{4,5} Giovanni Vignale,⁶ and Roberto Raimondi²¹*Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany*²*CNISM and Dipartimento di Matematica e Fisica, Università Roma Tre, Via della Vasca Navale 84, 00146 Rome, Italy*³*Kavli Institute of NanoScience, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands*⁴*Nano-bio Spectroscopy group and ETSF Scientific Development Centre, Departamento Física de Materiales, Universidad del País Vasco, E-20018 San Sebastian, Spain*⁵*IKERBASQUE, Basque Foundation for Science, 48011 Bilbao, Spain*⁶*Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA*

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We derive the Bloch equations for the spin dynamics of a two-dimensional electron gas in the presence of spin-orbit coupling. For the latter we consider both the intrinsic mechanisms of structure inversion asymmetry (Rashba) and bulk inversion asymmetry (Dresselhaus), and the extrinsic ones arising from the scattering from impurities. The derivation is based on the SU(2) gauge-field formulation of the Rashba-Dresselhaus spin-orbit coupling. Our main result is the identification of a spin-generation torque arising from Elliott-Yafet scattering, which opposes a similar term arising from Dyakonov-Perel relaxation. Such a torque, which to the best of our knowledge has gone unnoticed so far, is of basic nature, i.e., should be effective whenever Elliott-Yafet processes are present in a system with intrinsic spin-orbit coupling, irrespective of further specific details. The spin-generation torque contributes to the current-induced spin polarization (CISP), also known as inverse spin-galvanic or Edelstein effect. As a result, the behavior of the CISP turns out to be more complex than one would surmise from consideration of the internal Rashba-Dresselhaus fields alone. In particular, the symmetry of the current-induced spin polarization does not necessarily coincide with that of the internal Rashba-Dresselhaus field, and an out-of-plane component of the CISP is generally predicted, as observed in recent experiments. We also discuss the extension to the three-dimensional electron gas, which may be relevant for the interpretation of experiments in thin films.

DOI: [10.1103/PhysRevB.95.205424](https://doi.org/10.1103/PhysRevB.95.205424)**I. INTRODUCTION**

Spin polarization by currents and its inverse effect are currently a subject of intensive theoretical and experimental investigation both in semiconducting and metallic systems. (For recent reviews, see Refs. [1–3].) Originally, the effect was proposed by Ivchenko and Pikus [4] and observed in tellurium [5]. Later, it was theoretically analyzed in a two-dimensional electron gas, either in the presence of Dresselhaus spin-orbit coupling by Ivchenko *et al.* [6] and Aronov and Lyanda-Geller [7], or Rashba spin-orbit coupling by Edelstein [8]. In non-magnetic conductors with chiral symmetry, a magnetoelectric effect was also investigated [9]. As highlighted in Ref. [1], the effect, which is the consequence of symmetry-allowed coupling between spin polarization and electrical current, may be referred to under different names. The generation of a current from a nonequilibrium spin polarization goes under the equivalent names of spin galvanic effect (SGE) or inverse Edelstein effect (IEE), while the reciprocal phenomenon of current-induced spin polarizations is referred to as inverse spin galvanic effect (ISGE) or Edelstein effect (EE).

The first experimental observation of the SGE/IEE was in quantum wells, by measuring the current produced by the absorption of polarized light [10–12]. More recently, it has been shown that a nonequilibrium spin polarization injected by spin-pumping into an Ag|Bi [13] or Fe|GaAs [14] interface also yields an electrical current. Notice that spin-to-charge conversion in this context is sometimes referred to as inverse Rashba-Edelstein effect [15]—for a recent theoretical discussion see Ref. [16]. The SGE/IEE has also been observed

at ferromagnet-topological insulator interfaces [17,18] and in LAO|STO systems [19]. In semiconducting structures, the reciprocal ISGE/EE is measured via optical detection of the current-induced spin polarization [20–25]. The ISGE/EE is also measured by analyzing the torques exercised, via exchange coupling, by the nonequilibrium polarization on the magnetization of an adjacent ferromagnetic system [26–32]. Recently, this has been extended also to antiferromagnets [33].

Although the key mechanism of the effect relies on the symmetry properties of gyrotropic media [63], most of the recent theoretical work has concentrated on models based on the Rashba and Dresselhaus spin-orbit coupling (respectively RSOC and DSOC in the following) in the presence of disorder scattering responsible for spin relaxation [15,34–51]. In a two-dimensional electron gas, as for instance the one studied in Ref. [25], spin-orbit coupling (SOC) is also due to the electric field of the scattering impurities and the interplay between intrinsic and extrinsic mechanisms becomes highly nontrivial [42,46,52,64]. In this paper, we analyze some aspects of this interplay focusing on the description of the ISGE/EE and SGE/IEE in a two-dimensional electron gas—we will, however, also discuss results valid in 3D gases.

The model Hamiltonian reads

$$H = \frac{p^2}{2m} + \alpha p_y \sigma_x - \alpha p_x \sigma_y + \beta p_x \sigma_x - \beta p_y \sigma_y + V(\mathbf{r}) - \frac{\lambda_0^2}{4} \boldsymbol{\sigma} \times \nabla V(\mathbf{r}) \cdot \mathbf{p}, \quad (1)$$

where \mathbf{p} , \mathbf{r} , and $\boldsymbol{\sigma}$ represent the momentum, coordinate, and spin (in units of $\hbar/2$) operators, respectively, for electrons in the two-dimensional electron gas. The latter lies in the (x, y) plane, while m is the effective mass and α and β are the RSOC and DSOC coupling constants, respectively. $V(\mathbf{r})$ is a random potential describing the scattering from the impurities. The potential is assumed to have zero average and second moment given by $\langle V(\mathbf{r})V(\mathbf{r}') \rangle = n_i v_0^2 \delta(\mathbf{r} - \mathbf{r}')$, where v_0 is the scattering amplitude from a single impurity and n_i is the impurity concentration. Finally, λ_0 is the effective Compton wavelength describing the strength of the extrinsic spin-orbit coupling.

The layout of the paper is as follows. The core message is summarized in Sec. II, where we immediately “get into the thick of it” by formulating the problem, providing its solution, and highlighting certain experimentally relevant consequences. We base our discussion on physical arguments only, and postpone the technical details substantiating our conclusions to Secs. III–V, which can thus be skipped by the reader not interested in the specifics of our derivation. More precisely, in Sec. III, we briefly review the SU(2) approach employed to build the kinetic equation in the presence of the RSOC and DSOC, and in Sec. IV, we derive the Bloch equations when only intrinsic SOC is present. Finally, in Sec. V, we present a rigorous derivation of the Bloch equations, including the corrections arising from extrinsic effects. Here we obtain the crucial new spin-generation torque arising from the Elliot-Yafet process, and discuss its implications for the ISGE/EE and SGE/IEE in specific experimental setups. Finer details concerning the calculation of the collision integral are provided in Appendices A and B. We adopt units such that $\hbar = c = 1$ throughout.

II. THE PROBLEM AND ITS SOLUTION

Consider an ensemble of carriers in a generic solid state environment, where the spin is not a conserved quantity. In a homogeneous sample, in the presence of an exchange/Zeeaman field $\mathbf{\Delta}$, the ensemble spin polarization \mathbf{S} will then obey the continuity (Bloch) equation

$$\partial_t S^a = -[\mathbf{\Delta} \times \mathbf{S}]^a + \mathcal{T}^a, \quad (2)$$

where here and throughout, latin superscripts stand for spin components $a = x, y, z$. The first term on the right-hand side (r.h.s.) describes precession around the field $\mathbf{\Delta}$, while \mathcal{T}^a is the a th component of the torque acting on the spin polarization, responsible for relaxation to equilibrium. In a simple isotropic medium, it reads

$$\mathcal{T}^a = -\frac{1}{\tau_s} (S^a - S_{\text{eq}}^a), \quad (3)$$

where τ_s is the spin relaxation time—of whatever origin—and the equilibrium spin density $\mathbf{S}_{\text{eq}} = \chi \mathbf{\Delta}$ is given in terms of the Pauli spin susceptibility $\chi = \frac{1}{4} \partial n / \partial \mu$, which at zero temperature reduces to $\chi = N_0/2$, with N_0 the density of states per spin at the Fermi energy.

In the presence of intrinsic SOC, a finite drift velocity \mathbf{v} of the ensemble is associated with a nonequilibrium spin polarization. Let us take a drift v_x in the x direction and RSOC for definiteness' sake. The nonequilibrium spin polarization

then reads

$$S^y = \chi B^y, \quad (4)$$

with

$$B^y = 2m\alpha v_x \quad (5)$$

an effective “drift field” felt by the moving ensemble. When the drift is caused by an electric field, $v_x = -e\tau E_x/m$, Eqs. (4) and (5) describe the usual ISGE/EE [7,8]. RSOC also leads to (anisotropic) Dyakonov-Perel (DP) spin relaxation

$$1/\tau_s \rightarrow \hat{\Gamma}_{\text{DP}} = \frac{1}{\tau_{\text{DP}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad (6)$$

with $1/\tau_{\text{DP}} = (2m\alpha)^2 D$ and $D = v_F^2 \tau / 2$ the diffusion constant. This suggests that we modify the Bloch equations (2) to

$$\partial_t S^a = -[\boldsymbol{\Omega} \times \mathbf{S}]^a + \mathcal{T}_{\text{int}}^a, \quad (7)$$

where $\boldsymbol{\Omega} = \mathbf{\Delta} + \mathbf{B}$ is the full effective exchange/Zeeaman field felt by the drifting carriers and (repeated indices are summed over, unless otherwise specified)

$$\mathcal{T}_{\text{int}}^a = -\hat{\Gamma}_{\text{DP}}^{ab} (S^b - \chi \Omega^b) \quad (8)$$

is the intrinsic torque, “intrinsic” meaning that spin-orbit effects from impurities are not yet included. This torque has a spin-relaxation component $-\hat{\Gamma}_{\text{DP}} \mathbf{S}$ and a spin-generation one $\hat{\Gamma}_{\text{DP}} \chi \boldsymbol{\Omega}$. The intuitive form of Eqs. (7) and (8) will be rigorously justified in Sec. IV, and holds for any kind of intrinsic SOC—e.g., RSOC + DSOC—with the appropriate form of $\hat{\Gamma}_{\text{DP}}$ and $\boldsymbol{\Omega}$. It shows that the spin polarization relaxes to a nonequilibrium steady-state value given by

$$\mathbf{S}_{\text{neq}} \equiv \chi \boldsymbol{\Omega} = \mathbf{S}_{\text{eq}} + \chi \mathbf{B}. \quad (9)$$

What happens to this intuitive picture once extrinsic SOC is taken into account? This is the central *problem* addressed in our work. While modifications to both the relaxation and the spin generation torques are clearly expected, their precise form is *a priori* far from obvious. This is because extrinsic SOC gives rise to several phenomena, such as side-jump, skew scattering, and Elliott-Yafet relaxation, which are not necessarily additive with respect to intrinsic SOC effects [46]. Let us start with the spin relaxation torque, which acquires a contribution due to Elliott-Yafet (EY) scattering

$$\hat{\Gamma}_{\text{DP}} \mathbf{S} \rightarrow [\hat{\Gamma}_{\text{DP}} + \hat{\Gamma}_{\text{EY}}] \mathbf{S} \equiv \hat{\Gamma} \mathbf{S}, \quad (10)$$

with $\hat{\Gamma}_{\text{EY}} \sim \lambda_0^4$. Unsurprisingly, spin-flip events at impurities, which are second order in the extrinsic SOC constant λ_0^2 , provide a parallel channel for relaxation. However, they also crucially affect the nonequilibrium steady-state value \mathbf{S}_{neq} the spins want to relax to. This is subtler, and highlights the difference between a true equilibrium state and a nonequilibrium steady state. Such state is determined by the spin generation torque, which extrinsic SOC modifies in two ways. First, via side-jump and skew scattering, which together add an extrinsic contribution $\theta_{\text{ext}}^{sH} \sim \lambda_0^2$ to the intrinsic spin Hall angle, $\theta_{\text{int}}^{sH} \sim (\alpha^2, \beta^2)$ (this can have the same or the opposite sign as the intrinsic angle) [65]. Second, via Elliott-Yafet relaxation, which yields a correction *opposite* to the nonequilibrium part

of the intrinsic spin generation term, i.e., the $\hat{\Gamma} \chi B$ part of $\hat{\Gamma} \chi \Omega$ in Eq. (8):

$$\begin{aligned} \hat{\Gamma}_{\text{DP}} \chi \Omega &\rightarrow [\hat{\Gamma}_{\text{DP}} + \hat{\Gamma}_{\text{EY}}] \chi \Delta \\ &+ \left[\hat{\Gamma}_{\text{DP}} + \hat{\Gamma}_{\text{DP}} \frac{\theta_{\text{ext}}^{sH}}{\theta_{\text{int}}^{sH}} - \hat{\Gamma}_{\text{EY}} \right] \chi \mathbf{B} \\ &\equiv \hat{\Gamma} \mathbf{S}_{\text{eq}} + \delta \hat{\Gamma} \chi \mathbf{B}, \end{aligned} \quad (11)$$

where

$$\hat{\Gamma} = \hat{\Gamma}_{\text{DP}} + \hat{\Gamma}_{\text{EY}} \quad (12)$$

and

$$\delta \hat{\Gamma} = \hat{\Gamma}_{\text{DP}} + \hat{\Gamma}_{\text{DP}} \frac{\theta_{\text{ext}}^{sH}}{\theta_{\text{int}}^{sH}} - \hat{\Gamma}_{\text{EY}}. \quad (13)$$

The full Bloch equations thus become

$$\partial_t S^a = -[\boldsymbol{\Omega} \times \mathbf{S}]^a - \hat{\Gamma}^{ab} (S^b - \chi \Delta^b) + \delta \hat{\Gamma}^{ab} \chi B^b. \quad (14)$$

This is the main result of our paper. It shows that, while intrinsic and extrinsic SOC act in parallel as far as relaxation to the equilibrium state is concerned—second term on the r.h.s. of Eq. (14)—they compete for the more interesting nonequilibrium contribution—the spin-generation torque, described by the third term on the r.h.s. of Eq. (14). In particular, the last term on the right-hand side of Eq. (13) describes an ‘‘Elliot-Yafet spin-generation torque,’’ which *opposes* the more familiar Dyakonov-Perel and spin Hall terms. Physically, this can be understood as follows. The nonequilibrium spin polarization appears when the drifted Fermi surface is split by intrinsic SOC, the outer surface having a larger spin content than the inner one. Such an effect is counteracted by Elliott-Yafet relaxation, which is proportional to the square of the momenta before and after scattering, and therefore more efficient for states on the outer surface and less efficient for states on the inner surface.

Equation (14) shows that the naive Bloch equation (8) is modified by extrinsic processes. While this fact had already been recognized in previous works (Refs. [15,46]) some terms (third order in SOC: first order in RSOC and second order in λ_0^2) of the diagrammatic expansion had been neglected leading to an incomplete form of $\delta \hat{\Gamma}$, in which the last term on the right-hand side of Eq. (13) was missing [66]. As a result, the numerical calculation of current-induced spin polarization must be reconsidered. Indeed, Eq. (14) implies that the competition between intrinsic and extrinsic torques can generate out-of-plane spin polarizations from in-plane spin-orbit fields. Note that such a mechanism is very basic in nature, in the sense that it does not require finer details such as band nonparabolicities or angle-dependent scattering [36] in order to be effective. As such, it may have important consequences both in the analysis of existing experimental data [20,22], and in the design of novel setups.

The nontrivial modification of the ISGE/EE arising from Eq. (14) implies a corresponding modification of the SGE/IEE, so as to fulfill Onsager relations. To be explicit, in the scenario reciprocal to the one considered in Eqs. (4) and (5), the charge current J_x generated by a nonequilibrium spin polarization

$S^y - \chi \Delta^y$ acquires the correction

$$\delta J_x = \frac{2e\alpha\tau}{\tau_{\text{EY}}} (S^y - \chi \Delta^y). \quad (15)$$

This ensures reciprocity between the spin response to an electric field E_x and the charge response to a time-dependent magnetic field $-\Delta^y(t)$ [15]. A microscopic derivation of (15) in a more general context is discussed in Ref. [16] and will not be pursued here.

III. THE SU(2) APPROACH FOR INTRINSIC SOC

A convenient way to deal with the RSOC and DSOC of Hamiltonian of Eq.(1) is the SU(2) approach, where the SOC is described in terms of a spin-dependent gauge field [43]. This formalism, introduced in the context of quark-gluon kinetic theory [53,54], was recently also extended to superconducting structures with SOC [55,56]. For a recent pedagogical introduction, see Ref. [57]. Here we limit ourselves to recall the key aspects of the approach to make this presentation self-contained. Neglecting for the time being the extrinsic SOC, the RSOC and DSOC of (1) can be written in the form of a spin-dependent vector potential and the Hamiltonian reads

$$H = \frac{(\mathbf{p} + e\mathcal{A}^a \sigma^a / 2)^2}{2m} - \frac{e\Psi^a \sigma^a}{2} + V(\mathbf{r}), \quad (16)$$

where terms $\mathcal{O}(\mathcal{A}^2)$ are dropped, as they are second-order in $\mathcal{A}/p_F \ll 1$ [67]. The only nonzero components of \mathcal{A}^a are

$$e\mathcal{A}_x^x = 2m\beta, \quad e\mathcal{A}_x^y = -2m\alpha, \quad e\mathcal{A}_y^x = 2m\alpha, \quad e\mathcal{A}_y^y = -2m\beta. \quad (17)$$

Relations (17) follow by comparing (16) with (1). In the Hamiltonian, we have also included a Zeeman term

$$H_Z = -\frac{\Delta^a \sigma^a}{2} \equiv -\frac{e\Psi^a \sigma^a}{2}, \quad (18)$$

which can be seen as a spin-dependent scalar potential. In the above $\Delta = g_L \mu_B B_{\text{exter}}$ with g_L the gyromagnetic factor, μ_B the Bohr magneton and B_{exter} the external magnetic field. In this way the theory can be written in terms of a SU(2) gauge theory of electrons coupled to a d -potential gauge field (Ψ, \mathcal{A}) , where each component of the d vector is expanded in Pauli matrices. Notice that in this description the standard scalar and vector potentials can be included as the identity σ^0 components. For the sake of generality, we formulate the theory in d dimensions. Whereas our first motivation is the description of the spin dynamics in a 2DEG, our conclusions apply also to the three-dimensional electron gas. This is specially relevant in experimental situations where one deals with semiconducting thin films. In the following, we make use of the compact (relativistic) space-time notations for the potentials

$$\mathcal{A}^\mu = (\Psi, \mathcal{A}), \quad \mathcal{A}_\mu = (-\Psi, \mathcal{A}), \quad (19)$$

the coordinate and momentum

$$x^\mu = (t, \mathbf{r}), \quad x_\mu = (-t, \mathbf{r}), \quad p^\mu = (\epsilon, \mathbf{p}), \quad p_\mu = (-\epsilon, \mathbf{p}) \quad (20)$$

and the corresponding derivatives

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}, \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \partial_p^\mu \equiv \frac{\partial}{\partial p_\mu}, \quad \partial_{p,\mu} \equiv \frac{\partial}{\partial p^\mu}. \quad (21)$$

In this way the product $p^\mu x_\mu = -\epsilon t + \mathbf{p} \cdot \mathbf{r}$ has the correct Lorentz metrics. We also introduce mixed Wigner coordinates given by the center-of-mass coordinates (t, \mathbf{r}) and energy-momentum variables (ϵ, \mathbf{p}) , which are the Fourier-transformed variables of the relative coordinates.

According to the analysis of Ref. [43], a semiclassical Boltzmann kinetic equation can be derived from a microscopic Keldysh formulation in the presence of non-Abelian gauge fields. The starting point is the left-right subtracted Dyson equation

$$[\check{G}_0^{-1}(x_1, x_3) \otimes \check{G}(x_3, x_2)] = [\check{\Sigma}(x_1, x_3) \otimes \check{G}(x_3, x_2)], \quad (22)$$

where we have used space-time coordinates $x_1 \equiv (t_1, \mathbf{r}_1)$, etc., and quantities with a “check” ($\check{G}_0^{-1}, \check{G}, \check{\Sigma}$) are matrices in Keldysh space [58]. In Eq. (22), the symbol \otimes implies integration over x_3 and matrix multiplication both in Keldysh and spin spaces. Furthermore,

$$\check{G}_0^{-1}(x_1, x_3) = (i\partial_t - H)\delta(x_1 - x_3), \quad (23)$$

where H is the Hamiltonian operator (16), while the self-energy $\check{\Sigma}$ appearing in the collision kernel [right-hand side of (22)] will be specified later.

The key step, with respect to the standard way of obtaining semiclassical transport theories à la Boltzmann from their microscopic counterparts, is the introduction of a locally covariant Green function $\check{G}(x_1, x_2)$ (to be defined in the following). From the Wigner transformed covariant Green function $\check{G}(p, x)$, one can define the SU(2) covariant distribution function to be determined by the kinetic equation. The introduction of the covariant Green function in the presence of non-Abelian gauge fields generalizes the well known shift in the momentum dependence of the Green function when one wants to make it gauge invariant under U(1) gauge transformations [59,60]. In the SU(2) case, as shown in Ref. [43], such a shift, due to the noncommutative nature of the symmetry group, can be carried out in terms of Wilson lines of the gauge field, whose definition is recalled in Appendix A. For our purposes, under the assumption that the spin-orbit energy scale is small compared to the Fermi energy, it is enough to perform the shift to lowest order in the gauge field and, as shown in Appendix A, obtain

$$\check{\Phi}(p, x) = \Phi(p, x) - \frac{1}{2}\{e\mathcal{A}^\mu, \partial_{\mu,p}\Phi(p, x)\}, \quad (24)$$

where $\Phi(p, x)$ is any quantity in the Wigner representation to which the shift can be applied. The inverse transformation reads

$$\Phi(p, x) = \check{\Phi}(p, x) + \frac{1}{2}\{e\mathcal{A}^\mu, \partial_{\mu,p}\check{\Phi}(p, x)\}. \quad (25)$$

We stress that in obtaining Eqs. (24) and (25) terms $O(\mathcal{A}^\mu \partial_{p,\mu})^2$ have been neglected in the above, and will be throughout the paper.

In order to obtain the SU(2) Boltzmann equation from the quantum kinetic equation, we apply the transformation (24) to the Eq. (22) and to the matrix Keldysh Green function

$$\check{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} \rightarrow \check{\check{G}} = \begin{pmatrix} \check{G}^R & \check{G}^K \\ 0 & \check{G}^A \end{pmatrix}, \quad (26)$$

where $G^{R,A,K}$ denotes, respectively, the retarded, advanced and Keldysh Green’s function [58]. As a result, we get

$$\check{G}^R - \check{G}^A = -2\pi i\delta(\epsilon - \epsilon_p), \quad (27)$$

$$\check{G}^K = -2\pi i\delta(\epsilon - \epsilon_p)[1 - 2f(\mathbf{p}, x)], \quad (28)$$

where $\epsilon_p = p^2/2m - \mu$ measures the energy with respect to the chemical potential μ . Notice that the SU(2)-shifted spectral density ($\sim \check{G}^R - \check{G}^A$) has no spin structure: the latter is all in the distribution function $f(\mathbf{p}, x)$. The fact that the locally covariant $\check{G}^{R,A}$ do not depend on the gauge fields, i.e., on the RSOC and DSOC, is the great advantage of the approach as will appear later.

Finally, the equation for $\check{\check{G}}$ reads

$$V^\mu[\check{\partial}_\mu \check{\check{G}} + \frac{1}{2}\{e\mathcal{F}_{\mu\nu}, \partial_p^\nu \check{\check{G}}\}] = [\check{\Sigma}, \check{\check{G}}], \quad (29)$$

where $V^\mu = (1, \mathbf{p}/m)$ is the d -current operator and we have introduced the covariant derivative

$$\check{\partial}_\mu \check{\check{G}} = \partial_\mu \check{\check{G}} + i[e\mathcal{A}_\mu, \check{\check{G}}] \quad (30)$$

and the field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[e\mathcal{A}_\mu, \mathcal{A}_\nu]. \quad (31)$$

An intuitive way to understand Eq. (29) is by noticing that the combination $V^\mu \partial_\mu$ is the ordinary hydrodynamical derivative entering the Boltzmann equation, $\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{r}}$, written in compact d -vector notation. Furthermore, in the case of the Abelian U(1) electromagnetic gauge field, the combination $V^\mu F_{\mu\nu} \partial_p^\nu$ yields the familiar Lorentz force. Equation (29) represents its extension to the SU(2) scenario, as will become clear in the following. The right-hand side of Eq. (29) follows by applying the covariant transformation to the Keldysh collision kernel, $I_K = -i[\Sigma, G]$ and taking advantage of the unitarity of the Wilson line as shown in Appendix A [cf. Eq.(A6)]. By taking the Keldysh component of (29) and separating time and space components, we get

$$\left(\check{\partial}_t + \frac{\mathbf{p}}{m} \cdot \check{\nabla}_{\mathbf{r}}\right)f(\mathbf{p}, \mathbf{r}, t) - \frac{1}{2}\{\mathbf{F} \cdot \nabla_{\mathbf{p}}, f(\mathbf{p}, \mathbf{r}, t)\} = I, \quad (32)$$

where \mathbf{F} is the spin-dependent force due to the SU(2) gauge fields

$$F_i = e\mathcal{F}_{0i} + e\frac{P_k}{m}\mathcal{F}_{ki} = e\mathcal{E}_i + e\epsilon_{ikj}\frac{P_k}{m}\mathcal{B}_j. \quad (33)$$

Here, $\mathcal{E}_i = \mathcal{F}_{0i}$ and $\mathcal{B}_i = \frac{1}{2}\epsilon_{ijk}\mathcal{F}_{jk}$ are the SU(2) electric and magnetic fields, respectively.

The Boltzmann collision integral in Eq. (32) is given by

$$I \equiv \int \frac{d\epsilon}{2\pi i} I_K = - \int \frac{d\epsilon}{2\pi} [\check{\Sigma}, \check{\check{G}}_p]^K. \quad (34)$$

To the level of the self-consistent Born approximation, the self-energy due to the disorder potential is shown in Fig. 1 and reads [68]

$$\check{\Sigma}_0(p, x) = n_i v_0^2 \sum_{\mathbf{p}'} \check{G}(p', x) \quad (35)$$

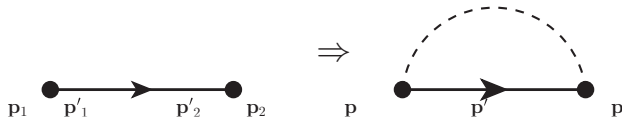


FIG. 1. Self-energy diagram to second order in the impurity potential (black dot vertex). The diagram on the left is before the impurity average, which is carried in the diagram on the right as a dashed line connecting the two impurity insertions.

and yields the familiar Boltzmann collision integral for impurity scattering

$$I[f] = -2\pi n_i v_0^2 \sum_{\mathbf{p}'} \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}) (f(\mathbf{p}, \mathbf{r}, t) - f(\mathbf{p}', \mathbf{r}, t)). \quad (36)$$

In the next section, we show how the Bloch equations can be obtained starting from the generalized Boltzmann equation (32).

IV. THE “INTRINSIC” BLOCH EQUATIONS

As shown in Ref. [43], the spin density and spin current density defined by

$$\begin{aligned} S^a(\mathbf{r}, t) &= \sum_{\mathbf{p}} \text{Tr} \left[f(\mathbf{p}, \mathbf{r}, t) \frac{\sigma^a}{2} \right], \\ J_i^a(\mathbf{r}, t) &= \sum_{\mathbf{p}} \frac{p_i}{m} \text{Tr} \left[f(\mathbf{p}, \mathbf{r}, t) \frac{\sigma^a}{2} \right], \end{aligned} \quad (37)$$

obey a continuity-like equation:

$$\tilde{\partial}_t S^a + \tilde{\partial}_i J_i^a = 0. \quad (38)$$

The above result can be derived from Eq. (32) by taking the Pauli matrix component σ^a and integrating over the momentum. After making explicit the covariant derivatives according to (30), the continuity-like equation (38) becomes

$$\partial_t S^a + \epsilon_{abc} e \Psi^b S^c + \nabla_i J_i^a - \epsilon_{abc} e \mathcal{A}_i^b J_i^c = 0. \quad (39)$$

Here, ϵ_{abc} is the fully antisymmetric Ricci tensor. The second term in Eq. (39) is the standard *precession* term due to the Zeeman term (18). The last term of (39) can be made explicit by providing the expression for the spin current J_i^a , where the lower (upper) index indicates the space (spin) component. In Ref. [43] [cf. therein Eq. (68)] the expression of J_i^a was derived via a microscopic theory in the diffusive regime. The expression reads

$$J_i^a = v_i S^a - D(\nabla_i S^a - \epsilon_{abc} e \mathcal{A}_i^b S^c) - \frac{e\tau n}{4m} (\mathcal{E}_i^a + \epsilon_{ijk} v_j \mathcal{B}_k^a), \quad (40)$$

where $v_i = -\frac{e\tau}{m} E_i$ is the average drift velocity of electrons driven by the external electric field. All the terms in Eq. (40) have a specific physical origin. The first is a *drift* term, containing the spin density S^a carried by the electrons drifted by the electric field E_i . The second is a *diffusion* term that contains two contributions: (i) the standard diffusion current proportional to $\nabla_i S^a$, and (ii) the contribution originating from the gauge-field part of the covariant derivative (30) acting on

the spin density. The third term corresponds the SU(2) *drift* current driven by the spin-dependent force of Eq. (33). In particular, the second contribution in this term yields the spin *Hall* coupling due to the SU(2) magnetic field \mathcal{B}_k^a .

Because of non-Abelian nature of the SU(2) gauge group the corresponding magnetic and electric fields can be nonzero even for spatially homogeneous potentials provided their components are not commuting. In this important special case, the SU(2) magnetic and electric fields are determined by the commutator term in Eq. (31) (cf. also Eqs. (25)–(30) in Ref. [43])

$$\epsilon_{ijk} \mathcal{B}_k^a = -\epsilon_{abc} e \mathcal{A}_i^b \mathcal{A}_j^c, \quad (41)$$

$$\mathcal{E}_i^a = \epsilon_{abc} e \mathcal{A}_i^b \Psi^c. \quad (42)$$

Using this representation for the fields and recalling the Einstein relation $\frac{\tau n}{m} = D \frac{\partial n}{\partial \mu} \equiv 4D\chi$, one can combine the gauge potential-dependent terms in Eq. (40) into a single item, and rewrite the expression for the spin current in the following compact form:

$$J_i^a = v_i S^a - D \nabla_i S^a + D \epsilon_{abc} e \mathcal{A}_i^b (S^c - \chi \Omega^c), \quad (43)$$

where Ω is the total magnetic field introduced in Sec. II:

$$\Omega^a = e \Psi^a - e \mathcal{A}_k^a v_k \equiv \Delta^a + B^a. \quad (44)$$

Here, Δ is the usual Zeeman field defined after Eq. (18) and B represents the internal SOC field induced by the electric current (electric field):

$$B^a = -e \mathcal{A}_k^a v_k = \frac{e\tau}{m} e \mathcal{A}_k^a E_k. \quad (45)$$

Now the Bloch equation describing the global spin dynamics in the presence of intrinsic SOC can be derived by assuming a homogeneous spin density ($\nabla_i S = 0$) and substituting the spin current of Eq. (43) into Eq. (39). The resulting equation reads

$$\partial_t S^a = -(\Omega \times S)^a - \hat{\Gamma}_{\text{DP}}^{ab} (S^b - \chi \Omega^b), \quad (46)$$

where Ω^a is given by Eq. (44) and the DP relaxation tensor $\hat{\Gamma}_{\text{DP}}^{ab}$ is defined as follows:

$$\hat{\Gamma}_{\text{DP}}^{ab} = e^2 D \epsilon_{afc} \epsilon_{bdc} \mathcal{A}_i^d \mathcal{A}_i^f = e^2 D (\delta^{ab} \mathcal{A}_i^c \mathcal{A}_i^c - \mathcal{A}_i^a \mathcal{A}_i^b). \quad (47)$$

Equation (46) generalizes Eq. (7) of Sec. II to the case of arbitrary intrinsic SOC. It is worth noticing that in the present formalism the DP relaxation arises as the *second-order* covariant derivative (the covariant Laplacian). One needs to act twice with the gauge field to get the quadratic dependence on the SOC in the spin relaxation matrix.

The second term on the right-hand side of the Bloch equation (46) corresponds to the intrinsic torque T_{int}^a for generic SOC. The part of T_{int}^a proportional to the internal SO field B (45) can be recognized as the spin generation torque

$$T_{\text{int,sg}}^a = \hat{\Gamma}_{\text{DP}}^{ab} B^b = e^4 D \frac{\tau}{m} \mathcal{A}_i^b (\mathcal{A}_i^a \mathcal{A}_k^a - \mathcal{A}_k^b \mathcal{A}_i^a) E_k. \quad (48)$$

The intrinsic spin generation torque $T_{\text{int,sg}}^a$ is given by the covariant divergence of the spin Hall current, that is the very last term proportional to \mathcal{B}_k in Eq. (40). Therefore the spin generation torque vanishes for the configurations of the gauge potentials with vanishing SU(2) magnetic field.

These configurations correspond to a so called pure gauge SOC for which different space components of the SU(2) potential are commuting and the intrinsic spin Hall effect is absent. Our results imply that in this situation the current-induced spin polarization is also absent.

It is instructive to write explicitly the above general formulas for the specific form of the vector potential of Eq. (17), which corresponds to the Rashba-Dresselhaus SOC. In this case, the SU(2) magnetic field has only one nonzero component

$$e\mathcal{B}_z^z = -e\mathcal{F}_{xy}^a = (2m\beta)^2 - (2m\alpha)^2. \quad (49)$$

As the SU(2) magnetic field determines the spin Hall coupling, it can be expressed in terms of the spin Hall (SH) angle for the intrinsic SOC defined by

$$\theta_{\text{SH}}^{\text{int}} = m\tau(\beta^2 - \alpha^2) = \frac{e\tau\mathcal{B}_z^z}{4m}. \quad (50)$$

The expression for the spin Hall angle has a suggestive interpretation by recalling the classical Hall effect where the coupling between the mutually orthogonal charge currents is given by the product of the cyclotron frequency and the scattering time $\omega_c\tau = eB_{\text{exter}}\tau/m$. In the present case to get the spin Hall angle (50), one needs to combine the SU(2) cyclotron frequency $e\mathcal{B}_z^z/(4m)$ with the scattering time τ . An intuitive way to understand the origin of the factor of 4 in the denominator of the SU(2) cyclotron frequency is the following. Let us imagine that spin-up and spin-down particles undergo the ordinary Hall effect in opposite directions with a spin-dependent magnetic field, $j_y^\uparrow = (\tau/m)B^\uparrow j_x^\uparrow$ and $j_y^\downarrow = -(\tau/m)B^\downarrow j_x^\downarrow$. By defining the spin current as $J_y^z = (j_y^\uparrow - j_y^\downarrow)/2$ and identifying $B^\uparrow = -B^\downarrow = \mathcal{B}_z^z/2$, one immediately finds the ‘‘SU(2)’’ cyclotron frequency $e\mathcal{B}_z^z/(4m)$.

By introducing further an in-plane Zeeman field $e\Psi^x \equiv \Delta^x$ and $e\Psi^y \equiv \Delta^y$, we find that the only nonzero components of the SU(2) electric field are

$$e\mathcal{E}_x^z = \Delta_x 2m\alpha + \Delta_y 2m\beta, \quad (51)$$

$$e\mathcal{E}_y^z = \Delta_x 2m\beta + \Delta_y 2m\alpha. \quad (52)$$

In this case, the total magnetic field $\mathbf{\Omega}$ of Eq. (44) also has only in-plane components

$$\Omega^x = \Delta^x + B^x, \quad (53)$$

$$\Omega^y = \Delta^y + B^y \quad (54)$$

with the internal SO field \mathbf{B} (45) of the form

$$B^x = 2e\tau(\beta E_x + \alpha E_y), \quad (55)$$

$$B^y = -2e\tau(\alpha E_x + \beta E_y). \quad (56)$$

The general DP relaxation matrix $\hat{\Gamma}_{\text{DP}}$ of Eq. (47) entering Eq. (46) simplifies as follows:

$$\hat{\Gamma}_{\text{DP}} = \begin{pmatrix} \tau_\alpha^{-1} + \tau_\beta^{-1} & 2\tau_{\alpha\beta}^{-1} & 0 \\ 2\tau_{\alpha\beta}^{-1} & \tau_\alpha^{-1} + \tau_\beta^{-1} & 0 \\ 0 & 0 & 2(\tau_\alpha^{-1} + \tau_\beta^{-1}) \end{pmatrix}, \quad (57)$$

where $\tau_\alpha^{-1} = (2m\alpha)^2 D$, $\tau_\beta^{-1} = (2m\beta)^2 D$, and $\tau_{\alpha\beta}^{-1} = (2m)^2 \alpha\beta D$. Notice that for $\beta = 0$, the matrix $\hat{\Gamma}_{\text{DP}}$ becomes diagonal, and τ_α reduces to the Dyakonov-Perel relaxation time introduced in Eq. (6). Finally, the spin generation torque reads

$$\begin{aligned} \mathcal{T}_{\text{int,sg}} &\equiv \hat{\Gamma}_{\text{DP}} \chi \mathbf{B} \\ &= -2m\theta_{\text{SH}}^{\text{int}} (-2eN_0 D) \begin{pmatrix} -\alpha E_y + \beta E_x \\ -\beta E_y + \alpha E_x \\ 0 \end{pmatrix}. \end{aligned} \quad (58)$$

The above equation generalizes the spin generation torque introduced in Eq. (8) to the case of RSOC and DSOC for arbitrary direction of the electric field [69]. In agreement with the general discussion after Eq. (48), the spin generation torque is proportional to the spin Hall angle. Therefore it vanishes for SOC giving $\theta_{\text{SH}}^{\text{int}} = 0$, which in the present case corresponds to the compensated RSOC and DSOC with $\alpha = \pm\beta$.

The meaning of Eq. (46) is that, under stationary conditions, $\mathbf{S} = \chi \mathbf{\Omega}$, provided the spin Hall angle is nonzero. This implies that the spin polarization follows the total magnetic field and (for an energy-independent scattering time [43]) there can be no out-of-plane spin polarization since $\mathbf{\Omega}$ lays in the xy plane. This is no longer the case when one considers the extrinsic SOC as will be shown in the following section.

V. THE EFFECTS OF EXTRINSIC SOC

The interplay of intrinsic and extrinsic SOC was investigated previously in Refs. [42,48,52,57]. According to the analysis therein Eq. (46) acquires two modifications. The first, to order λ_0^2 , is an additional contribution to the spin Hall coupling in the third term in expression (40) for the spin current. This arises from the inclusion of side-jump and skew-scattering effects due to the extrinsic SOC and leads to a renormalization of the spin Hall angle in the expression of the spin generation torque in Eq. (58):

$$\theta_{\text{SH}}^{\text{int}} \rightarrow \theta_{\text{SH}} = \theta_{\text{SH}}^{\text{int}} + \theta_{\text{SH}}^{\text{ext}}. \quad (59)$$

The second term, which arises to order λ_0^4 , is an additional contribution to the spin relaxation matrix (the EY spin relaxation). In fact, as discussed in Sec. II, there exists, to the same order λ_0^4 , a third *new* contribution, which will be derived in detail in the following.

To see how the new contribution arises, we focus on the term of order λ_0^4 in the self-energy, whose Feynman diagram is shown in Fig. 2 and whose expression reads

$$\check{\Sigma}_{\text{EY}}(\mathbf{p}) = n_i \sum_{\mathbf{p}'} \hat{V}_{\mathbf{p},\mathbf{p}'} \check{G}_{\mathbf{p}'} \hat{V}_{\mathbf{p}',\mathbf{p}}, \quad (60)$$

where $\hat{V}_{\mathbf{p},\mathbf{p}'}$ is the spin-dependent part of the impurity scattering amplitude

$$\hat{V}_{\mathbf{p},\mathbf{p}'} = i v_0 (\lambda_0/2)^2 (\mathbf{p} \times \mathbf{p}') \cdot \boldsymbol{\sigma}. \quad (61)$$

Shifting the self-energy of Eq. (60) according to SU(2) shifts (24) and (25) yields the locally covariant EY self-energy:

$$\check{\Sigma}_{\text{EY}} = \check{\Sigma}_{\text{EY}}^{(0)} + \check{\Sigma}_{\text{EY}}^{(1)}. \quad (62)$$

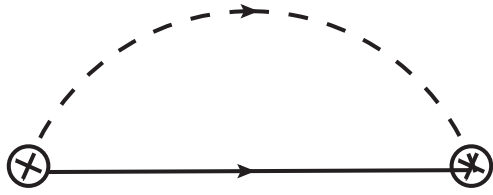


FIG. 2. Self-energy diagram in second order in the spin-orbit impurity potential, shown as a crossed empty dot, contributing to the Elliott-Yafet spin relaxation.

In Eq. (62), we separated the term responsible for the EY relaxation

$$\tilde{\Sigma}_{\text{EY}}^{(0)} = n_i \sum_{\mathbf{p}'} \hat{V}_{\mathbf{p},\mathbf{p}'} \tilde{G}_{\mathbf{p}'} \hat{V}_{\mathbf{p}',\mathbf{p}} \quad (63)$$

from that giving rise to the new contribution

$$\begin{aligned} \tilde{\Sigma}_{\text{EY}}^{(1)} = & \frac{n_i}{2} \sum_{\mathbf{p}'} (\hat{V}_{\mathbf{p},\mathbf{p}'} \{ \mathcal{A}_k, \partial_{p'_k} \tilde{G}_{\mathbf{p}'} \} \hat{V}_{\mathbf{p}',\mathbf{p}} \\ & - \{ \mathcal{A}_k, \partial_{p_k} \hat{V}_{\mathbf{p},\mathbf{p}'} \} \tilde{G}_{\mathbf{p}'} \hat{V}_{\mathbf{p}',\mathbf{p}}). \end{aligned} \quad (64)$$

In the last equation, the summation over the repeated index k is understood. Correspondingly, the Keldysh collision kernel acquires two contributions to order λ_0^4 and reads

$$\delta I_K \equiv -i[\tilde{\Sigma}_{\text{EY}}, \tilde{G}] = -i[\tilde{\Sigma}_{\text{EY}}^{(0)}, \tilde{G}] - i[\tilde{\Sigma}_{\text{EY}}^{(1)}, \tilde{G}]. \quad (65)$$

The first term on the right-hand side gives rise to the EY spin relaxation [52], and contributes an extra relaxation channel on the right-hand side of Eq. (46),

$$-\frac{1}{\tau_{\text{EY}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d-2 \end{pmatrix} (\mathbf{S} - \chi \mathbf{\Delta}) \equiv -\hat{\Gamma}_{\text{EY}} (\mathbf{S} - \chi \mathbf{\Delta}), \quad (66)$$

where we have introduced the dimensionality-dependent EY spin relaxation rate given by

$$\frac{1}{\tau_{\text{EY}}} = \frac{4(d-1)}{d^2} \frac{1}{\tau} \left(\frac{\lambda_0 p_F}{2} \right)^4. \quad (67)$$

In the above $d = 2$ and 3 , is the dimensionality of the space where particles move. The $d = 2$ case corresponds to the two-dimensional electron gas case, where we have concentrated our attention until now. The z component of the spin is a constant of the motion and does not undergo relaxation in this case. However, the peculiarity of the new term we are going to derive appears also, and more remarkably, in the $d = 3$ case. For this reason, we keep the dependence on the dimensionality from now on.

The Keldysh (K) component of the second term in (65) reads

$$\begin{aligned} \delta I_K^K &= -i(\tilde{G}^R - \tilde{G}^A) \tilde{\Sigma}_{\text{EY}}^{(1),K} \\ &\quad - i(\tilde{\Sigma}_{\text{EY}}^{(1),R} \tilde{G}^K - \tilde{G}^K \tilde{\Sigma}_{\text{EY}}^{(1),A}) \\ &\equiv \delta I^{(1)} + \delta I^{(2)}, \end{aligned} \quad (68)$$

having used that $\tilde{G}^{R,A} \sim \sigma^0$. In order to obtain the Bloch equation, we need to sum over the momentum as done for obtaining the continuity equation (38). The summation over momentum of the Boltzmann collision integral [70] is obtained as

$$\sum_{\mathbf{p}} \int \frac{d\epsilon}{2\pi i} \delta I_K^K \equiv \Delta I^{(1)} + \Delta I^{(2)}. \quad (69)$$

By replacing \tilde{G}^R , \tilde{G}^A , and \tilde{G}^K with the expressions (27)–(28), one obtains

$$\begin{aligned} \Delta I^{(1)} = & \frac{n_i \pi}{2} \sum_{\mathbf{p}, \mathbf{p}'} \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}) (\partial_{p'_k} \hat{V}_{\mathbf{p},\mathbf{p}'} \{ \mathcal{A}_k, (1 - 2f_{\mathbf{p}'}) \} \hat{V}_{\mathbf{p}',\mathbf{p}} \\ & + \partial_{p_k} \{ \mathcal{A}_k, \hat{V}_{\mathbf{p},\mathbf{p}'} (1 - 2f_{\mathbf{p}'}) \} \hat{V}_{\mathbf{p}',\mathbf{p}}) \end{aligned} \quad (70)$$

and

$$\begin{aligned} \Delta I^{(2)} = & -\frac{n_i \pi}{2} \sum_{\mathbf{p}, \mathbf{p}'} \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}) \frac{1}{2} \{ (\partial_{p'_k} \hat{V}_{\mathbf{p},\mathbf{p}'} 2\mathcal{A}_k \hat{V}_{\mathbf{p}',\mathbf{p}} \\ & + \partial_{p_k} \{ \mathcal{A}_k, \hat{V}_{\mathbf{p},\mathbf{p}'} \} \hat{V}_{\mathbf{p}',\mathbf{p}} \}, (1 - 2f_{\mathbf{p}'}) \}. \end{aligned} \quad (71)$$

In both of the above equations, the first term, after the delta function, has been obtained by an integration by parts with respect to the momentum \mathbf{p}' . As a result, the derivatives with respect to p'_k and p_k act on the $\hat{V}_{\mathbf{p},\mathbf{p}'}$ factors only. In Eq. (70), the dependence on the directions of the momentum \mathbf{p} is restricted to the $\hat{V}_{\mathbf{p},\mathbf{p}'}$ factors only, so that one can perform at once the integration over the solid angle of \mathbf{p} and then take the derivative with respect to \mathbf{p}' . Appendix B provides some useful identities [see Eqs. (B2)–(B4)] on how to carry out these operations. Notice also that the second term in round brackets of Eq. (70) vanishes, because the derivative with respect to \mathbf{p} yields a linear dependence on \mathbf{p} so that the solid angle integral gives zero.

By reasoning in the same way, one sees that the first term in round brackets within the anticommutator of Eq. (71) also vanishes. In the second term, one can make at once the integration over the solid angle of \mathbf{p}' , again by using the results of Appendix B. As a result, after working out the Pauli algebra, one gets

$$\begin{aligned} \Delta I = & \pi n_i v_0^2 \left(\frac{\lambda_0}{2} \right)^4 \frac{d-1}{d} \sum_{\mathbf{p}, \mathbf{p}'} \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}) p^2 \\ & \times \left[\frac{d-2}{d-1} (\sigma^k \{ e\mathcal{A}_k, p'_i f(\epsilon_{\mathbf{p}'} \}) \} \sigma^l \right. \\ & + \sigma^l \{ e\mathcal{A}_k, p'_i f(\epsilon_{\mathbf{p}'} \}) \} \sigma^k - \sigma^i \{ e\mathcal{A}_k, p'_k f(\epsilon_{\mathbf{p}'} \}) \} \sigma^i \\ & \left. + 2p^2 \{ e\mathcal{A}_k, p_k f(\epsilon_{\mathbf{p}} \}) \} \right]. \end{aligned} \quad (72)$$

In Eq. (72), the summation over repeated indices runs over x, y, z for $d = 3$. For $d = 2$, the last two lines of Eq. (72) survive and only the $i = z$ term remains. Then the sum over momentum of the Boltzmann collision integral is [71]

$$\begin{aligned} \Delta I^a = & \frac{1}{2} \text{Tr}[\sigma^a \Delta I] = \frac{1}{\tau} \left(\frac{\lambda_0}{2} \right)^4 p_F^2 \frac{(d-1)}{2d} \sum_{\mathbf{p}} f^0(\epsilon_{\mathbf{p}}) p_i \\ & \times \left(e\mathcal{A}_i^a + \frac{d-2}{d-1} (e\mathcal{A}_a^i + e\mathcal{A}_n^a \delta_{ai}) \right). \end{aligned} \quad (73)$$

This is *zero* as long as $f^0(\epsilon_p)$ is isotropic, which is the case in a homogeneous system at equilibrium. Things change as soon as an electric field is switched on and carriers have a finite drift velocity $\mathbf{v} = -e\tau\mathbf{E}/m$. We then have the spin generation torque due to the interplay of intrinsic and extrinsic SOC, $\mathcal{T}_{\text{ext,sg}}^a \equiv \Delta I^a$,

$$\begin{aligned} \mathcal{T}_{\text{ext,sg}}^a &= -\frac{N_0}{2\tau_{\text{EY}}}\left(e\mathcal{A}_i^a + \frac{d-2}{d-1}(e\mathcal{A}_a^i + e\mathcal{A}_n^n\delta_{ai})\right)v_i \\ &= C_i^a v_i, \end{aligned} \quad (74)$$

where we have introduced the extrinsic SOC torque tensor C_i^a . In $d = 3$, it is instructive to represent this tensor as follows

$$\begin{aligned} C_i^a &= -\frac{eN_0}{2\tau_{\text{EY}}}\left[\mathcal{A}_n^n\delta_{ia} + \frac{3}{2}\left(\frac{1}{2}(\mathcal{A}_i^a + \mathcal{A}_a^i) - \frac{1}{3}\mathcal{A}_n^n\delta_{ia}\right) \right. \\ &\quad \left. + \frac{1}{2}(\mathcal{A}_i^a - \mathcal{A}_a^i)\right] \end{aligned} \quad (75)$$

by separating explicitly all irreducible tensor parts—the unit, the traceless symmetric, and antisymmetric contributions. Comparing this with the similar representation for the plain \mathcal{A}_i^a , we see the symmetric (“Dresselhaus”) part has a contribution three times as large relative to the antisymmetric (“Rashba”) part. Hence Eq. (75) shows that the value at which the spin polarization would like to relax to by EY processes has a form different from the SOC internal field defined in Eq. (45) due to DP processes. The latter has the same structure as the first term in the brackets of Eq. (74) but with an opposite sign. Three-dimensional motion adds an entirely new term to the internal SOC field induced by the electric field. Although when going to $d = 3$ the linear DSOC may not be appropriate anymore, the overall message is that the interplay of extrinsic SOC and SU(2) intrinsic SOC is extremely rich. The exploration of the consequences of this are however beyond the scope of the present paper.

For $d = 2$, only the first term in the brackets of Eq. (74) survives, so that, by considering $a = y$ for RSOC ($e\mathcal{A}_x^y = -2m\alpha$), we have

$$\mathcal{T}_{\text{ext,sg}}^y = -\frac{1}{\tau_{\text{EY}}}N_0\alpha m v_x. \quad (76)$$

Hence the spin generation torque due to the interplay of RSOC and extrinsic SOC has the opposite sign with respect to the corresponding term originating by the Dyakonov-Perel precessional relaxation $\mathcal{T}_{\text{int,sg}}^y = 1/\tau_{\text{DP}}(N_0\alpha m v_x)$. We name this new term the Elliott-Yafet torque (EYT).

We can then write the Bloch equation in the final form:

$$\partial_t \mathbf{S} = -\boldsymbol{\Omega} \times \mathbf{S} - (\hat{\Gamma}_{\text{DP}} + \hat{\Gamma}_{\text{EY}})(\mathbf{S} - \chi \boldsymbol{\Delta}) + \mathcal{T}_{\text{sg}}, \quad (77)$$

where the spin generation torque \mathcal{T}_{sg} , in the presence of extrinsic SOC is given by

$$\mathcal{T}_{\text{sg}} = \mathcal{T}_{\text{int,sg}} + \delta\mathcal{T}_{\text{int,sg}} + \mathcal{T}_{\text{ext,sg}}, \quad (78)$$

where

$$\mathcal{T}_{\text{int,sg}} = \hat{\Gamma}_{\text{DP}}\chi\mathbf{B}, \quad (79)$$

$$\delta\mathcal{T}_{\text{int,sg}} = \frac{\theta_{\text{SH}}^{\text{ext}}}{\theta_{\text{SH}}^{\text{int}}}\hat{\Gamma}_{\text{DP}}\chi\mathbf{B}, \quad (80)$$

$$\mathcal{T}_{\text{ext,sg}} = -\hat{\Gamma}_{\text{EY}}\chi\mathbf{B}. \quad (81)$$

Hence the extrinsic SOC yields two additional spin generation torques (80) and (81) associated to spin Hall effect (to order λ_0^2) and Elliott-Yafet processes (to order λ_0^4), respectively. The second torque has the same form but opposite sign of the intrinsic torque, indicating that the EY spin-relaxation is detrimental to the ISGE/EE as anticipated in Sec. II. The Bloch equations (77) together with the expressions of the various torques (78)–(81), the DP ($\hat{\Gamma}_{\text{DP}}$) and EY ($\hat{\Gamma}_{\text{EY}}$) spin relaxation matrices (57) and (66) and the definition of the total magnetic field $\boldsymbol{\Omega}$ (53)–(54) are the main result of this paper. In accordance with the experimental observations of Ref. [25], Eq. (77) shows that, in general, the static nonequilibrium spin polarization will not be aligned along the internal effective magnetic field $\boldsymbol{\Omega}$.

VI. CONCLUSIONS

In this paper, we have considered the phenomenon of spin orientation by current by analyzing the interplay of intrinsic (Rashba and Dresselhaus) and extrinsic spin-orbit coupling. We have derived the Bloch equation governing the spin dynamics by identifying the various relaxation and spin generation torques. Whereas in the presence of purely intrinsic spin-orbit coupling the spin polarization follows the internal effective magnetic field, this no longer happens when the extrinsic spin-orbit is present. The precise relation between the spin polarization and the Rashba-Dresselhaus internal field depends on the relative magnitude of the Dyakonov-Perel and Elliott-Yafet spin relaxation rates, as well as on the spin Hall angle in the system. These observations may be very useful in analyzing existing experiments on the ISGE/EE, and in suggesting new ones.

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APPENDIX A: THE COVARIANT GREEN FUNCTION IN TERMS OF WILSON LINES

The *locally* covariant Green function is defined as

$$\check{G}(x_1, x_2) = U_\Gamma(x, x_1)\check{G}(x_1, x_2)U_\Gamma(x_2, x), \quad (A1)$$

where

$$U_\Gamma(x, x_1) = \mathcal{P} \exp\left(-i \int_{x_1}^x eA^\mu(y)dy_\mu\right). \quad (A2)$$

The line integral of the gauge field is referred to as the Wilson line. In Eq. (A2), \mathcal{P} is a path-ordering operator. Since the Wilson line transforms covariantly under a gauge transformation $O(x)$,

$$U_\Gamma(x, x_1) \rightarrow O(x)U_\Gamma(x, x_1)O^\dagger(x_1), \quad (A3)$$

one easily sees that the covariant Green function \check{G} transforms in a locally covariant way

$$\check{G}(x_1, x_2) \rightarrow O(x) \check{G}(x_1, x_2) O^\dagger(x). \quad (\text{A4})$$

To the lowest order in the gauge field, one may expand the exponential of the Wilson line and, after Fourier transforming with respect of the relative coordinate, obtain Eqs. (24) and (25) of the main text.

The Wilson line is unitary in the sense that

$$U_\Gamma(x, x_1) U_\Gamma(x_1, x) = 1. \quad (\text{A5})$$

The unitarity of the Wilson line allows to express the covariant transformation of a convolution product of non covariant objects in terms of the convolution of the covariant transformed objects. In particular, the covariant transformation of the Keldysh collision integral gives

$$\begin{aligned} U_\Gamma(x, x_1) [\check{\Sigma}(x_1, x_3) \otimes \check{G}(x_3, x_2)] U_\Gamma(x_2, x) \\ = [\check{\Sigma}(x_1, x_3) \otimes \check{G}(x_3, x_2)] \end{aligned} \quad (\text{A6})$$

after using the unitarity of the Wilson line by inserting

$$U_\Gamma(x_3, x) U_\Gamma(x, x_3) = 1$$

between the self-energy and the Green function.

APPENDIX B: AN IDENTITY CONCERNING ANGULAR INTEGRATION

In the text, we need to perform the integration over the solid angle of \mathbf{p} :

$$\int \left(\frac{\sin(\theta_{\mathbf{p}}) d\theta_{\mathbf{p}}}{2} \right)^{d-2} \frac{d\phi_{\mathbf{p}}}{2\pi} \hat{V}_{\mathbf{p}, \mathbf{p}'} \dots \hat{V}_{\mathbf{p}', \mathbf{p}} \equiv \langle \hat{V}_{\mathbf{p}, \mathbf{p}'} \dots \hat{V}_{\mathbf{p}', \mathbf{p}} \rangle. \quad (\text{B1})$$

In the above the dots indicate any operator acting on the spin indices, but not depending on the momenta \mathbf{p} and \mathbf{p}' . By writing explicitly the cross products in the $\hat{V}_{\mathbf{p}, \mathbf{p}'}$ factors one has

$$\begin{aligned} & -v_0^2 \left(\frac{\lambda_0}{2} \right)^4 \left\langle \sum_{ijklmn} \epsilon_{ijk} \epsilon_{lmn} p_i p'_j \sigma^k \dots p_l p'_m \sigma^n \right\rangle \\ & = -v_0^2 \left(\frac{\lambda_0}{2} \right)^4 \sum_{ijklmn} \epsilon_{ijk} \epsilon_{lmn} \langle p_i p_l \rangle p'_j p'_m \sigma^k \dots \sigma^n \\ & = -v_0^2 \left(\frac{\lambda_0}{2} \right)^4 \frac{p^2}{d} \sum_{ijklmn} \epsilon_{ijk} \epsilon_{lmn} \delta_{il} p'_j p'_m \sigma^k \dots \sigma^n \\ & = -v_0^2 \left(\frac{\lambda_0}{2} \right)^4 \frac{p^2}{d} (p'^2 \sigma^i \dots \sigma^i (d-2) \mathbf{p}' \cdot \boldsymbol{\sigma} \dots \mathbf{p}' \cdot \boldsymbol{\sigma}), \end{aligned} \quad (\text{B2})$$

where in $d = 3$, it is understood a summation over $i = x, y, z$ and in $d = 2$ $i = z$. If the dots are replaced by the identity in the spin space,

$$\langle \hat{V}_{\mathbf{p}, \mathbf{p}'} \dots \hat{V}_{\mathbf{p}', \mathbf{p}} \rangle = -v_0^2 \left(\frac{\lambda_0}{2} \right)^4 \frac{2p^2 p'^2}{d} \sigma^0. \quad (\text{B3})$$

Then the derivative with respect to p'_k yields

$$\begin{aligned} & \partial_{p'_k} \langle \hat{V}_{\mathbf{p}, \mathbf{p}'} \dots \hat{V}_{\mathbf{p}', \mathbf{p}} \rangle \\ & = -v_0^2 \left(\frac{\lambda_0}{2} \right)^4 \frac{p^2}{d} (2p'_k \sigma^i \dots \sigma^i \\ & \quad - \sigma^k \dots \mathbf{p}' \cdot \boldsymbol{\sigma} - \mathbf{p}' \cdot \boldsymbol{\sigma} \dots \sigma^k). \end{aligned} \quad (\text{B4})$$

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- [63] See Ref. [1]. A system is called gyrotropic if in its point symmetry group some components of polar vectors and components of axial vectors transform according to the same representation.
- [64] Interesting interplay effects are also predicted in graphene doped with SOC impurities [61].
- [65] The precise form of $\theta_{\text{extr}}^{sH}$ depends on various microscopic details, but is inconsequential for our discussion and thus will not be specified. Explicit microscopic expressions, however, can be found in Ref. [46].
- [66] Note that the terms here taken into account are beyond those considered in Ref. [62], where, e.g., $\mathcal{O}(\lambda_0^4)$ corrections are neglected, and Ref. [38], where purely orbital mechanisms, not involving the spin of the carriers, are discussed.
- [67] Furthermore, for homogeneous spin-orbit fields, as in the present case, such terms give rise to a constant offset, which can be absorbed in the chemical potential.
- [68] Since the scattering is elastic $p^\mu = (\epsilon, \mathbf{p})$ and $p'^\mu = (\epsilon, \mathbf{p}')$, i.e., the energy does not change.
- [69] Notice that $\mathcal{T}_{\text{int,sg}}$ corresponds to γ in the notations of Ref. [25].
- [70] The Boltzmann collision integral is the Keldysh collision integral (65) integrated over the energy
- [71] The corrections to the charge collision integral ΔI^0 , relevant for the reciprocal SGE/IEE case in, e.g., spin pumping setups, require considering higher-order terms and are discussed in Ref. [16].