ESSAYS ON DERIVATIVES PRICING IN INCOMPLETE MARKETS

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"Reality is frequently inaccurate."

— Douglas Adams
The Restaurant at the End of the Universe

CONTENTS

Contents v

List of Figures vii

List of Tables ix

1 Introduction 1

2 Time consistent pricing of options with embedded decisions 5
2.1 Introduction 6
2.2 Formal setup 9
2.3 From acceptance to pricing 11
2.4 Decisions 16
2.5 Time consistency 20
2.6 Discussion 24
2.7 Conclusion 25

3 Optimal discrete hedging of American options 27
3.1 Introduction 28
3.2 Theoretical fundament 31
3.3 Optimal hedging—the general formula 34
3.4 Optimal hedging of American options 37
3.5 Numerical demonstration 39
3.6 Conclusion 44

4 A note on utility indifference pricing 49
4.1 Introduction 50
4.2 Failure of the continuous trading assumption 52
4.3 Implications for indifference pricing 56
4.4 Implications for utility-based pricing 59
4.5 Discussion 60
4.6 Conclusion 62

5 The pricing efficiency of exchange-traded commodities 65
5.1 Introduction 66
5.2 Related literature 69
5.3 Data and methodology 71
5.4 Regression variables and hypotheses 76
5.5 Empirical analysis 83
5.6 Summary and conclusion 90

6 Addendum: Pricing and hedging the sandbox option 95

7 Conclusion 99

A Appendices to chapter 2 101
- A.1 Proofs 101
- A.2 Mathematical theorems 109

B Appendices to chapter 3 111
- B.1 Pointwise defined payoffs 111
- B.2 Proofs 111
- B.3 Results from Gerer and Dorfleitner (2016b) 113

Bibliography 115

LIST OF FIGURES

2.1 Example of a set of accepted premiums 14

3.1 Optimal hedging position 45
3.2 Optimal no-trading and rebalancing corridor 46
3.3 Worst-case exercise behavior 47
3.4 Absolute price reduction from optimal hedging 47
3.5 Absolute and relative price reduction from optimal hedging 48

4.1 Capped exponential utility function. 61

LIST OF TABLES

3.1 Requirements satisfied by relevant contributions 29

5.1 Frequency tables for categorical data 73
5.2 Descriptive statistics for metric variables 73
5.3 Results of PD analysis 83
5.4 Results of the quadratic and linear pricing methods 84
5.5 ANOVA results 86
5.6 OLS regression results 88
5.7 OLS regression results (4 November 2006 to 30 September 2010) 92
5.8 OLS regression results (1 October 2010 to 25 July 2012) 93

INTRODUCTION

1

FINANCIAL derivatives are contracts or products that derive their value from their so-called underlyings, which can be any kind of financially relevant quantity. The widespread use of derivatives in most areas of finance makes their pricing an important discipline of financial economics, relevant to financial institutions, investors, and regulators alike. The central question is "what is the price of a derivative?".

In an arbitrage-free and complete market, this question can be answered satisfyingly: there exists a unique no-arbitrage price for every derivative, such that trading at any other price can be turned into a risk-free profit. If the market is incomplete, the no-arbitrage price broadens into a no-arbitrage band bounded by the super- and sub-replication prices. In empirically supported market models, the sizes of these bands are often too large to be of practical use and fail to explain the empirically observed bid-ask spreads. While this justifies the existence of many derivatives, in that they significantly contribute to market completeness, it also demonstrates that no-arbitrage arguments cannot adequately answer the above question.

This dissertation comprises four essays (chapters 2 to 5) on the topic of derivatives pricing in incomplete markets, accompanied by an application of the proposed methods to so-called sandbox options (chapter 6) and is held together by this introduction and a conclusion. The first three essays take a theoretical perspective on the pricing of derivatives with embedded decisions and the associated aspect of dynamic hedging.

Fueled by the ambiguity introduced by market incompleteness, the field of derivatives valuation has fragmented into various directions concerned with different perspectives and interpretations of the initial question. Combined with the many different sources of market incompleteness and contract features, we are left with a plethora of—partly incompatible—approaches.

Particularly affected is the treatment of decisions embedded in derivatives contracts. Such decisions can originate from many different kinds of contract features and introduce varying degrees of complexity into the valuation problem. Starting with exercise rights granted to the holder, they range from the trivial European and the more complex American exercise decisions to contracts comprising a multitude of complex decisions by the holder, e.g. options on the proceeds of the holder's trading activity. Then, there are contracts—like callable convertible bonds or callable warrants—which grant exercise rights to both holder and writer. But what makes decisions by both parties a truly ubiquitous aspect of derivatives valuation are market frictions and the resulting need for the hedger to find a balance between risk reduction and transaction costs. This unavoidable source of market incompleteness implies that, in addition to the contract's original payoff, the possibility of a hedging activity and thus a continuum of hedging decisions need to be considered.

The valuation and hedging literature lacks methods to handle decisions in a consistent manner. Most contributions take ad-hoc approaches which are tied to certain types of exercise rights or even specific contracts and often to the market model. These approaches are not capable of solving problems that include a combination of decisions by both counterparties. Consequently, they fail even in seemingly simple and non-exotic cases. As an example, take the very limited number of articles discussing realistic hedging of American options[1]. The fact that none of these contributions sets out to solve the full problem indicates the conceptual limitations of existing approaches.

The first two essays aim to establish new methods for handling decisions embedded in derivative contracts that help to overcome the shortcomings of existing approaches. The first is included as chapter 2 and currently under review for publication in *Mathematical Finance*. It lays the foundation and derives a pricing principle for options with decisions. The second essay, included as chapter 3 and accepted for publication in *Review of Derivatives Research* subject to minor revisions, extends this principle to the problem of realistic hedging and applies it to American options. The third essay, included as chapter 4 and published in *International Journal of Theoretical and Applied Finance*, addresses problems with many utility functions that are used to derive prices in incomplete markets; problems encountered during the work on chapter 3.

Chapter 2 starts with a concise formalization of the required concepts: the decision behavior of both counterparties, payoffs that depend on these decisions, pricing functions acting on such payoffs and acceptance sets consisting of acceptable payoffs. At the heart of the proposed method lies a duality between pricing functions and acceptance sets that transforms the pricing problem into the language of acceptance.

This language is suitable for direct modeling of economic behavior and we use it to introduce *conservative acceptance*. This acceptance behavior captures the assumptions implicit in most of the existing literature and is enough to eliminate decisions from the pricing problem, resulting in a general pricing principle for options with decisions by both parties. Conservative acceptance should serve as a starting point in exploring alternatives and extensions.

Chapter 2 also includes a section motivating the need for time-consistent pricing functions and acceptance sets. It shows how time consistency further

[1] Refer to section 3.1 for an overview.

reduces the argumentative burden needed to eliminate decisions and derives a time-consistent pricing principle.

Chapter 3 considers the net payoff of a hedging option writer. This payoff is given by the sum of the option's premium, the negative of the option's payoff and the result of the writer's hedging activity subject to transactions costs. The framework of chapter 2 is then applied to this net payoff, which depends on a continuum of hedging decisions and on all decisions embedded in the original options contract. The result is a new general pricing and optimal hedging principle for options with decisions.

The second part of the chapter uses this result to derive a new and optimal solution to the problem of realistic hedging of American options. Numerical calculations for an American put option are performed, whose results are compared to classical delta hedging and analyzed along three dimensions: the optimal hedging strategy of the writer, the pessimal exercise strategy of the holder and the option's price. The results show clearly that delta hedging suffers from the possibility of exercise strategies that produce excessive transactions costs and that optimal hedging offers a significant improvement over delta hedging.

Chapter 4 reveals severe limitations to the practical applicability of two well-established parts of the pricing and hedging literature, namely *utility indifference pricing* and so-called *utility-based pricing*. In these strands of literature, utility functions are used to resolve the ambiguities caused by market incompleteness, and mathematical tractability is achieved through the assumption of continuous trading. However, the findings of chapter 4 show that this assumption is not justified when combined with one of many commonly used utility functions. The optimal behavior and thus prices and hedging ratios derived from such combinations are fundamentally different from the results obtained for discrete, i.e. practicable, trading strategies.

Chapter 4 focuses on hedging strategies involving the continuous rebalancing of short positions in the underlying, and proves that such strategies possess infinitely negative utility when approximated using practicable strategies. It demonstrates that the methods proposed by the literature on utility indifference and utility-based pricing cannot be applied by real-world hedgers, and thus are of questionable practical relevance. Affected are all HARA utility functions with exponents in $[-\infty,1)$, including the exponential CARA, and all CRRA utility functions, and therewith more than 55 published research articles and books on the topic.

The investigations in chapter 4 are inspired by particularities of the exponential utility function discovered during the preparation of the numerical calculations for chapter 3. However, it should be noted that the use of this utility function in the examples given in section 3.5, where the optimal strategy is *long*, poses no problems. Furthermore, the findings of chapter 4 do not negatively affect the results of chapter 3, because the latter does not use the assumption of continuous trading and is in no way limited to utility-based pricing functions. When working with discrete trading strategies, unrealistic utility functions yield unrealistic result,s that can be clearly recognized as such. It is only the assumption of continuous trading, which bears the risk of unintentionally deriving results which are disconnected from reality.

The fourth essay is included as chapter 5 and published in *Review of Managerial Science*. It takes an empirical perspective on the pricing of exchange-traded commodities (ETCs). ETCs are a very successful financial innovation, allowing investors to participate in the commodity markets with fewer hurdles compared to physically owning commodities or engaging in the commodities futures markets. Shares of an ETC can be created or redeemed on demand on the so-called *primary market* by a small group of *authorized parties*. Public trading of ETC shares takes place on the *secondary market*, usually on securities exchanges and over-the-counter markets.

Calculating an ETC's no-arbitrage price is simple: it equals the creation/redemption price, which is contractually specified in the ETC's prospectus. However, in reality, there are market imperfections like minimal creation/redemption block sizes, transaction costs and lack of competition among authorized parties. These limits to arbitrage between primary and secondary market cause the secondary marked to deviate from the no-arbitrage price.

Chapter 5 examines daily pricing data of 237 ETCs traded on the German market from 2006 to 2012 using different measures for price deviations and pricing efficiency. It is the first study to systematically explore the pricing efficiency of ETCs and its sample is unique in size and regional focus. It finds that, on average, ETCs trade at a premium over their fair price. Furthermore, nine hypotheses on factors that are expected to influence pricing efficiency are formulated and tested using regression analysis. Statistical evidence is found for seven of the nine hypotheses.

TIME CONSISTENT PRICING OF OPTIONS WITH EMBEDDED DECISIONS

2

(Joint work with Gregor Dorfleitner. Currently under review for publication in Mathematical Finance.*)*

Abstract

Many financial contracts are equipped with exercise rights or other features enabling the parties to actively shape the contract's payoff. These decisions pose a great challenge for the pricing and hedging of such contracts. Yet, the literature lacks a consistent way of dealing with these decisions, and instead only provides methods for specific contracts and not transferable to other models.

In this paper we present a framework that allows us to separate the treatment of the decisions from the pricing problem and derive a general pricing principle for the price of an option with decisions by both parties. To accomplish this we present a general version of the duality between acceptance sets and pricing functions, and use it to translate the pricing problem into the language of acceptance. Expressing certain aspects of economic behavior in this language is sufficient to fully eliminate the decisions from the problem.

Further, we demonstrate why time consistent pricing functions are crucial when dealing with options with embedded decisions and how the ad-hoc pricing functions used in many contributions can be derived if time consistency is added to our minimal set of assumptions.

2.1 Introduction

A theory of option pricing should be a rational theory that tries to explain more with less by formally deriving far reaching results from a small set of assumptions — a property that we call *scope of a theory*. A sharp distinction between assumptions and results as well as between motivation of the former and proof of the latter is crucial for the *quality of a theory*. The pursuit of a theory with such properties is witnessed by the literature from the past sixty years in this field, most notably by the titles of important papers such as "Rational theory of warrant pricing", Samuelson (1965) or "Theory of rational option pricing", and the use of Occam's razor to "analyze the assumptions to determine which ones are necessary" (Merton 1973b).

Usually, derivative pricing theories focus on the contingency of the contracts, while often not applying the same rigor to the decisions embedded in the contract. Of course, the holder's decision at expiry of European options is trivial, yet there exists many contracts that include complex decisions by either one or both of the parties (i.e warrants). Furthermore, every contract considered from the point of view of a hedger includes a vast number of non-trivial hedging decisions. Current option pricing literature does not handle decisions on the theory level but instead on an ad-hoc basis and — making things worse — in a way that is tightly entangled with the description and specifics of the model at hand.

In this paper we will present the foundation of a pricing theory that is powerful enough to abstract over the treatment of decisions in financial contracts and that can be derived from a small set of easily accessible assumptions and axioms. It is inspired by the current research on *acceptance sets* and their connection to coherent or convex risk or monetary utility functions (Artzner et al. 1999, 2007; Föllmer and Schied 2002). Even though our pricing functions are more general and not restricted to convex, coherent measures or monotone measures, we are able to derive this connection by explicitly employing the notion of an *acceptable opportunity* (a generalization of *arbitrage opportunities*, see Carr et al. 2001). Furthermore, we do not rely on a specific model for market price dynamics.

The literature on pricing of American options or warrants serves as an illustrative example of the problem we are trying to solve. The relevant publications can be classified into three stages. The first stage lies in the pre-risk neutral world, with its most prominent representatives being Chen (1970), McKean (1965), Merton (1973b), and Samuelson (1965). While still struggling from the lack of a meaningful notion of the price of a derivative security, these authors simply postulate the properties that the price of an American option should fulfill. Samuelson (1965) and McKean (1965) both postulate that the price cannot be lower than the values of either the exercised or continued versions of the option. McKean (ibid.) goes on to define the price as the smallest value that fulfills these properties. Samuelson (1965) fixes the price to be the maximum value of exercised and continued version of the option. Both authors then derive formulae for different stochastic market models. Chen (1970) defines the American option as a series of European-style compound options, for which the holder at each time period either receives the exercise price or the next option depending on which one has a greater value. In his seminal work, Merton (1973b) postulates the existence of a continuation region of a certain shape within which the price satisfies the Black-Scholes-Merton equation, and that its boundary be chosen in a price-maximizing way.

These theories can be considered to be of high quality, as the postulates and assumptions are clearly introduced as such at the beginning. The fact that formally complex properties of the price are simply postulated or defined as opposed to being consequences of simpler assumptions somehow limits their scope from a theoretical modeling perspective. This of course, does not apply to the derivation of the Black-Scholes-Merton equation. Instead, it applies to the situation concerning the theoretical treatment of the early exercise feature and the resulting decisions embedded in the contract.

Among the noteworthy contributions of the next stage are Brennan and Schwartz (1977), Cox et al. (1979), Geske and Johnson (1984), and Parkinson (1977). The aim of these articles is shifted towards providing usable algorithms to calculate the price of an American option. To this end they merely adopt the theoretical foundations of earlier contributions, thus inheriting their limited scope. Also, the quality — from the perspective of the treatment of decisions — suffers from an obscure use of assumptions, possibly even their complete absence (e.g. Brennan and Schwartz 1977).

The third stage concludes the development of a theory for the decisions embedded in American options, providing a basis for subsequent publications. Bensoussan (1984), like Samuelson (1965), but in a different framework, starts by postulating a type of complementarity problem for the price and then shows how the price is the solution to an optimal stopping problem. This is taken up by Karatzas (1988, 1989), who defines the price as the smallest amount of capital required to set up a super-replication strategy and is able to arrive at the same conclusion. In this setup, the explicit distinction between assumptions and theorems as well as between motivation and proofs warrants the quality. But more importantly, Karatzas (1989) finally fulfills the requirement of scope: His analysis opens the door to obtain all previously published results from one easily accessible definition of the price.[1]

The academic literature on American options is not an isolated case. The following shortcomings are equally applicable to the literature on all other types of options with embedded decisions (chooser, passport, shout, swing options etc.) as well as many textbooks on derivative pricing. The robustness and validity of the methods used is hard to verify, often difficult to follow with rigor, and different approaches are impossible to compare without further (mathematically involved) investigation. Contributions such as Myneni (1992), providing a detailed survey and proof of equivalence for the different approaches

[1] See Myneni (1992) for a detailed summary of the derivations and equivalences of the different formulations for the price of an American option.

to pricing an American option, show that these problems can be overcome with time for particular contracts in particular frameworks.

The much larger problem is the lack of any progress towards a coherent theory for options with embedded decisions. This results in a situation, in which the methods developed and the knowledge gained are not transferable; they do not actually deepen our understanding of the matter and provide no insight into the nature of the underlying problem. Instead, there remain many unanswered questions, most importantly: Are the various postulated methods consistent? Which assumptions about the decision making process are needed to derive the current results? Is the argumentation also valid in different models? Is an exclusion of the possibility of *clairvoyance* (as done by Karatzas 1988) actually necessary for deriving the pricing equations? Will the calculated price be correct if this assumption fails to hold?

While it could be the case that answering these questions was not the intention of earlier constributions, many articles still contain a considerable amount of argumentation motivating their numerous and, at times, quite obscure and complex assumptions. To answer these questions, we offer a framework in which all the postulated price properties can be (formally) derived from much simpler principles. This renders the mostly unsatisfactory explanations superfluous and dramatically reduces the argumentative burden. As an example we will demonstrate below, how the arbitrage-free price of an option in an incomplete market can be formally derived within our theory from a small set of assumption.

section 2.2 commences with the usual probabilistic setting of a filtered probability space. On top of that we will define *decision procedures*, which describe the choices made by the agent or counterparty in every possible evolution of the world. In the language of probability theory, a decision procedure is a stochastic process φ, where $\varphi_t(\omega)$ stands for the choice made at time t in the world state ω. With the concept of decision procedures we are able to describe options with embedded decisions. Traditionally, options are modeled by their cumulative discounted payoffs expressed as random variables. A natural generalization to options with embedded decisions is to consider an option's payoff f as a function that assigns every decision procedure φ a random variable $f(\varphi) : \omega \mapsto f(\varphi)(\omega)$ describing how much is paid out in each world state ω, if the agent and counterparty follow the decision procedure φ.

At the core of our formalism lies a duality between acceptance sets and pricing functions derived in section 2.3. Acceptance sets are collections of payoffs that are accepted by the agent as *zero cost investments*, i.e. option contracts he or she would enter without any additional payments. Pricing functions convert future random payoffs into prices known today. The same concept applies to payoffs describing options with decisions. However, special care needs to be taken when the payoff depends on past decisions, in which case the price inherits this dependence. Our first major result is a characterization of the essential properties required to derive a bijection between acceptance sets and pricing functions.

In section 2.4 we introduce *conservative acceptance* to eliminate decisions from the pricing problem, and give an example deriving the arbitrage-free price. section 2.5 discusses why time consistency is essential in pricing options with decisions, provides a characterization of time consistent acceptance sets and derives the the price of a general option with decisions. section 2.6 concludes with a general discussion of the results.

2.2 Formal setup

The theory is formulated from the perspective of a single market participant, that we will refer to as *agent*, engaging in financial activities and entering contracts with other agents, called her counterparty.

Assumption 2.1 (Probabilistic world)**.** *All possible evolutions of the world, their physical probabilities and the time-dependence of information about the evolution are described by a filtered probability space* $(\Omega, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, \mathbb{P})$*, where all points of time are given by the totally ordered set* $\mathcal{T}$*.*

Definition 2.1 (Random variables)**.** *Let* L_t^G *represent all* $\mathcal{F}_t$*-measurable random variables into the set* $G \subseteq \overline{\mathbb{R}}$*. We will use the abbreviations* $L_t \equiv L_t^{\langle-\infty,\infty\rangle}$*,* $L_t^- \equiv L_t^{[-\infty,\infty\rangle}$*,* $L_t^+ \equiv L_t^{\langle-\infty,\infty]}$ *and* $L_t^\pm \equiv L_t^{[-\infty,\infty]}$*. We will employ the convention* $\infty - \infty \equiv \infty$*.*

Define also the set of positive t-premiums $V_t \equiv \left\{x \in L_t^+ \,\middle|\, 0 \overset{a.s.}{<} x\right\}$.

The values ∞ and $-\infty$ represent values higher or lower than any possible real value. Remark 2.3 will provide the rationale for the inclusion of these values.

2.2.1 *Decisions*

We assume decisions happen at predetermined times. These times are then used to identify a decision (e.g. when describing a payoff's dependence on a decision). This does not prevent us from handling more complex decision for which the point of time can also be chosen by the agent, for example in contracts with the so called *American* exercise feature.

The set of times at which decision are made is called $\mathbf{T}_d \subseteq \mathcal{T}$. At each point of time $t \in \mathbf{T}_d$ there can be exactly one decision by either the agent or the counterparty.

Remark 2.1. This poses no limitation because multiple decisions by one agent can be combined into one choice tuple and decisions by different agents cannot be effectively simultaneous in practice: The agent can either react to the counterparty's decision or not, implying that a chronological order always exists.

Decisions to be made by the agent happen at times $\mathbf{T}_a$ and decisions by the counterparty at times $\mathbf{T}_c \equiv \mathbf{T}_d \setminus \mathbf{T}_a$.

The indexed sets D_t contain all possible choices at time t.

The decision behavior of the agents will be modeled by *decision procedures*, describing how the choices for a subset of decisions depend on the world state.

The set of *decision procedures* for decisions at times $\mathbf{T} \subseteq \mathbf{T}_d$ is abbreviated by $\Phi_{\mathbf{T}}$ and defined as the set of stochastic processes taking at time t values in D_t:

$$\Phi_{\mathbf{T}} \equiv \left\{ \varphi : \mathbf{T} \times \Omega \to \bigcup_{t \in \mathbf{T}} D_t \,\middle|\, \varphi_t : \Omega \to D_t, \text{ for all } t \in \mathbf{T} \right\}$$

We will use the abbreviation $\Phi \equiv \Phi_{\mathbf{T}_d}$.

2.2.2 *Options*

Before we can actually describe options (by their payoffs), we need one more assumption:

Assumption 2.2 (Cash-flows)**.** *The timing of discounted payments is irrelevant, i.e. cash-flows are fully determined by their cumulative discounted values.*

A sufficient condition in most theories for this assumption to hold is the existence of a risk-free investment instrument. The complexity added by the use of random processes, i.e. cash flows with timing information, instead of random variables could not be justified within the goals of this work.

By Assumption 2.2 the payoff of an option with embedded decisions can be described by a function specifying the cumulative discounted cash-flow to be *received by the agent* for any possible combination of choices and world states. Additionally, we need to be able to restrict our considerations to payoffs that depend only on a subset of decisions. This is provided by the following definition.

Definition 2.2 (Payoffs)**.** *Define $\mathcal{X}_{\mathbf{T}}^t$ as the set of $\mathcal{F}_t$-measurable payoffs that depend only on decisions made at times $\mathbf{T} \subseteq \mathcal{T}$:*

$$\mathcal{X}_{\mathbf{T}}^t \equiv \left\{ f : \Phi \to L_t^{\pm} \,\middle|\, f(\psi) \overset{B}{=} f(\varphi), \textit{ if } B \in \mathcal{F}_t \textit{ and } \psi_t \overset{B}{=} \varphi_t \textit{ for all } t \in \mathbf{T} \cap \mathbf{T}_d \right\}$$

Putting a set $B \in \mathcal{F}_\infty$ above a comparison operator means conditionally *almost surely equal: $x \overset{B}{=} y \Leftrightarrow \mathbb{P}\big(\{x = y\} \big| B\big) = 1$, with $\{x = y\} \equiv \left\{\omega \in \Omega \,\middle|\, x(\omega) = y(\omega)\right\}$.*

We will use the abbreviations $\mathcal{X}_{\mathbf{T}} \equiv \mathcal{X}_{\mathbf{T}}^\infty$ and $\mathcal{X} \equiv \mathcal{X}_{\mathcal{T}}$.

In other words, the values of a payoff in the set $\mathcal{X}_{\mathbf{T}}$ are almost surely determined by decisions at times $\mathbf{T}$. Making different choices at times outside $\mathbf{T}$ does not lead to different cash flows. As an important example, consider the set $\mathcal{X}_{[t,\infty\rangle}$ containing all payoffs with no decisions before t.

Some further remarks on payoffs and their definition are as follows:

Remark 2.2. This definition gives an indirect description of payoffs. It allows payoffs (as functions from Φ to $L_\infty^{\pm}$) to show arbitrarily complex and non-local dependence on the decision procedure, only to restrict this freedom at the same time. A more straightforward approach would be to define the cash flow described by payoff f for a given decision procedure φ in the world state ω

by something like $f(\varphi(\omega))(\omega)$. However, this ω-by-ω definition would be too limiting. As an example, take *stochastic integrals*, which are used extensively in the financial modeling of portfolios, trading gains and hedging. They cannot be defined in a pathwise manner (as the limits involved diverge for almost any ω) and thus cannot be handled by this naive approach. Definition 2.2 alleviates this problem and Corollary A.4 shows that the stochastic integral is in fact included in the definition.

Remark 2.3. The definition uses the set $L_t^{\pm}$ and thus allows $\pm\infty$ in payoffs. This is required to assure that the price of an option can again be treated as a payoff and to enable iterative application of pricing functions. Both aspects will be of importance when dealing with time consistency (section 2.5). The same could be achieved with a restriction to bounded payoffs. However, this limitation would exclude too many important applications of practical relevance.

Remark 2.4. If a random variable $x \in L_\infty^{\pm}$ is used in the context of payoffs, it is understood as the corresponding constant payoff given by $\psi \mapsto x$, which is an element of $\mathcal{X}_\emptyset$, and vice versa.

Remark 2.5. If not stated differently, all operators, relations and also suprema and infima used on payoffs are the pointwise versions of their $L_\infty^{\pm}$, $\mathbb{P}$-almost sure variants: $f \mathrm{R} g \Leftrightarrow \forall \varphi \in \Phi : f(\varphi) \overset{\text{a.s.}}{\mathrm{R}} g(\varphi)$

Finally we need a way to consider the *effective* payoff that results if an agent or counterparty follows a decision procedure for a certain subset of decisions. These decisions can be considered *fixed* and the effective payoff does not depend on them anymore. We introduce the following notation:

Definition 2.3 (Effective payoff)**.** *For any payoff $f \in \mathcal{X}$ and decision procedure $\varphi \in \Phi_{\mathbf{T}}$ define the* effective payoff, *$f[\varphi] \in \mathcal{X}_{\mathcal{T}\setminus\mathbf{T}}$ by*

$$f[\varphi](\psi) \equiv f(\varphi \mathbb{1}_{\mathbf{T}} + \psi \mathbb{1}_{\mathbf{T}_d\setminus\mathbf{T}}), \textit{ for all } \psi \in \Phi.$$

2.3 From acceptance to pricing

2.3.1 *Acceptable opportunities*

In their ground-breaking work Artzner et al. (1999) “provide [...] a definition of risks [...] and present and justify a unified framework for the analysis, construction and implementation of measures of risk”. They make the “acceptable future random net worths” the center of attention and postulate four economically motivated axioms for acceptability, leading to coherent risk measures, which posses a general representation using “generalized scenarios”. Their framework enjoys great popularity and was generalized to convex (Föllmer and Schied 2002) and dynamic risk-measures(Artzner et al. 2007; Cheridito et al. 2006; Cheridito and Kupper 2011).

Carr et al. (2001) use this notion of acceptability to address the shortcomings of many pricing theories, which either require the existence of complete markets (a questionable requirement) or are unable to predict the smallness of observed spreads, "by expanding the role played by arbitrage opportunities to acceptable opportunities".

The problem with these contributions is that it is hard to see which axioms are actually needed to ascertain the connection between acceptance sets and pricing functions. Acceptance sets are introduced as black boxes lacking any inner structure. The desired structure is then superimposed using axioms which can be too restrictive (as in the case with coherent risk-measures, that had to be generalized to convex risk-measures by Carr et al. 2001; Föllmer and Schied 2002).

Our approach is slightly different. We begin one step earlier by formalizing the *acceptable opportunity* directly and motivate its properties. The properties of the acceptance set can then be formally derived. This is not only more natural, but also reduces the number of axioms needed to derive the desired properties to one. Properties and connections that finally lead to a better understanding of what prices and risk premiums are and how they can be modeled and calculated.

First, two remarks about our terminology must be made:

Remark 2.6. We will use *premium* to describe the discounted net amount that is paid by the agent upon entering the option's contract. We will often use the set of positive t-premiums V_t from Definition 2.1.

Remark 2.7. Payoffs and the options they describe should be understood in the sense of *zero cost investments* or *opportunities*, i.e. for the question of acceptance an option's premium is understood to be already included in the payoff (which is possible due to Assumption 2.2).

The introduction of acceptable opportunities is based on the following assumption:

Assumption 2.3 (Acceptable opportunity)**.** *For every payoff with no past decisions it can be answered in every world and at every point of time using only information available at that time, whether the agent accepts it (making it an* acceptable opportunity*) or not.*

Thus, for each time t and option f there exists an event $\alpha_t(f) \in \mathcal{F}_t$, which encodes the acceptability of f at time t. We impose only property upon α_t:

Axiom 2.1. *For all $B \in \mathcal{F}_t$ and $f \in \mathcal{X}_{[t,\infty\rangle}$ the following holds:*

$$\mathbb{P}\big(\alpha_t(f)\big|B\big) = 1 \Longleftrightarrow \Big(\mathbb{P}\big(\alpha_t(g)\big|B\big) = 1, \text{ if } g \overset{B}{=} f + x \text{ for some } x \in V_t\Big)$$

The "$\Longrightarrow$" directionof this axiom is economically uncontroversial: If an option f is accepted in an event B, then any option is accepted whose payoff in the event B is higher than f's by a positive premium. The other direction is simply there to exclude the pathological case in which an unacceptable opportunity can become acceptable by adding an arbitrarily small premium. As we shall see in Example 2.2, this property is especially important in the context of options with decisions.

Now we can define acceptance sets and derive their properties. We will call any set of payoffs $\mathcal{A} \subseteq \mathcal{X}_{[t,\infty\rangle}$ a *t-acceptance set*.

An acceptance set should contain all acceptable payoffs. Thus, from a given function α_t we can derive the corresponding t-acceptance set

$$\mathcal{A} \equiv \left\{ f \in \mathcal{X}_{[t,\infty\rangle} \;\middle|\; \mathbb{P}\big(\alpha_t(f)\big) = 1 \right\}. \tag{2.1}$$

In the following sections we will work directly with acceptance sets, or more specifically *proper* acceptance sets, which have the same properties that Axiom 2.1 induces in $\mathcal{A}$ (cf. Corollary 2.1):

Definition 2.4 (Proper acceptance sets)**.** *A t-acceptance set $\mathcal{A}$ is called* proper *if it is t-compatible (see below) and*

$$\mathcal{A} = \left\{ f \in \mathcal{X}_{[t,\infty\rangle} \;\middle|\; \left\{ f + x \;\middle|\; x \in V_t \right\} \subseteq \mathcal{A} \right\}. \tag{2.2}$$

Properness uses the following definition of t-compatibility:

Definition 2.5 (t-compatibility)**.** *A non empty set X of functions from some set G into $L_t^\pm$ is t-*compatible*, if for all $\{x_n\} \subseteq X$ and mutually disjoint $\{B_n\} \subseteq \mathcal{F}_t$ with $\mathbb{P}(\bigcup_n B_n) = 1$ it holds $\sum_n^\infty x_n \mathbb{1}_{B_n} \in X$, where $\mathbb{1}_{B_n}$ is the indicator function of the set $G \times B_n$.*

Corollary 2.1. *If α_t satisfies Axiom 2.1 and $\mathcal{A}$ from eq. (2.1) is not empty, then $\mathcal{A}$ is a proper acceptance set.*

Proof. See Appendix A.1.1. □

2.3.2 *The price of an option*

Formally, we will call any function $\pi : \mathcal{X}_{[t,\infty\rangle} \to L_t^\pm$, from the set of options with no decisions before t to the set of t-measurable random variables, a *t-pricing function*. It is our aim to construct a t-pricing function for a given t-acceptance set. The intuitive characterization of the price of an option could be summarized as the highest premium the agent would accept to pay for entering the contract. More precisely this describes the agent's *bid* price. If an option's *ask* price is wanted, it is given by the negative of the bid price of the reversed option.

One possible formalization of this description of the bid price is given by:

Definition 2.6 (Associated pricing function)**.** *For any t-acceptance set $\mathcal{A}$ define its t-pricing function $P[\mathcal{A}]$ by*

$$P[\mathcal{A}](f) \equiv \sup\left\{ x \in L_t^- \;\middle|\; f - x \in \mathcal{A} \right\} \text{ for all } f \in \mathcal{X}_{[t,\infty\rangle},$$

where sup *stands for the* essential supremum*.*

It is easy to see that this definition actually specifies a t-pricing function, as the supremum always exists in $L_t^\pm$ (Theorem A.1). But instead of the *highest acceptable premium*, which does not exist in general, this definition uses the supremum. Random variables and thus premiums are not totally ordered and the supremum of such sets can be far away from its elements. Figure 2.1 illustrates how the supremum of the set of acceptable premiums is in general *not acceptable*. However, Corollary 2.2 will ensure that for proper $\mathcal{A}$ the use of the supremum is justified.

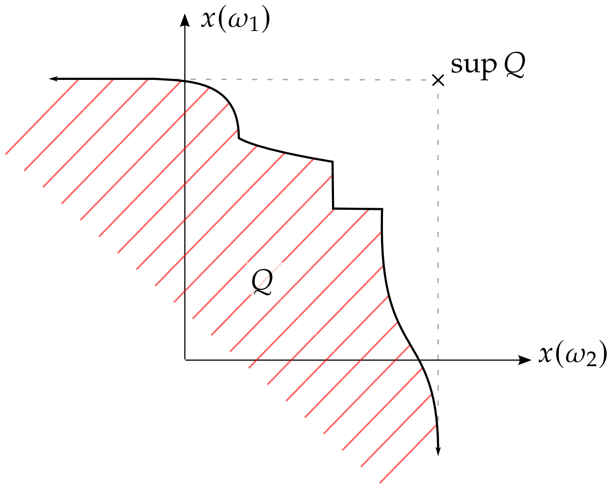

Figure 2.1: Example of a set of accepted premiums $Q = \left\{x \in L_t^- \mid f - x \in \mathcal{A}\right\}$ taking into account eq. (2.2) from Definition 2.4 in a probability space with $\mathcal{F}_t = \mathcal{P}(\{\omega_1, \omega_2\})$ and no decisions. A premium x can be visualized by a 2D-point $(x(\omega_1), x(\omega_2))$.

In section 2.5 we will use the term normalized pricing function:

Definition 2.7 (Normalized pricing function)**.** *A pricing function π is called* normalized*, if $\pi(0) = 0$. Every pricing function π with $|\pi(0)| < \infty$ has a* normalized version *$x \mapsto \pi(x) - \pi(0)$.*

We do not require pricing functions to be normalized, as this would unnecessarily restrict the following results and make them more complicated without providing additional value.

2.3.3 *The duality*

In the this section we present an important connection between pricing functions and acceptance sets. Similar relationships have been derived in previous literature (Artzner et al. 1999; Cheridito et al. 2006; Detlefsen and Scandolo 2005; Föllmer and Schied 2002). However, these relationships are formulated only for specialized versions of risk measures and acceptance sets.

Our work uses acceptance sets as the starting point of a pricing theory for options with decision and therefore deeply relies on this relationship, whose generality will carry over to the generality of our results. Thus, our focus lies in finding the most general relationship that still permits a sensible definition of the price of an option, i.e. solving the problem of the existence of the maximum and the associated question of validity of Definition 2.6 from the last section.

The relationship we will use, is a duality between proper acceptance sets and cash invariant pricing functions:

Definition 2.8 (Cash invariance)**.** *A t-pricing function π is called* cash invariant *if for each $f \in \mathcal{X}_{[t,\infty\rangle}$ and $x \in L_t^+$ it holds: $\pi(f + x) = \pi(f) + x$*

Cash invariance ensures that adding a premium to any option simply increases its price by that amount. This clearly is a desirable property for any theory and, as Theorem 2.1 will show, holds for $P[\mathcal{A}]$ if $\mathcal{A}$ is proper.

The inverse duality operation, which derives an acceptance set from a pricing function, is the literal translation of "the agent accepts any option he would pay a non-negative premium for":

Definition 2.9 (Dual acceptance set)**.** *For any t-pricing function π define its* dual *t-acceptance set*

$$A[\pi] \equiv \{f \in \mathcal{X}_{[t,\infty\rangle} \mid 0 \leq \pi(f)\}.$$

Now, the complete duality can be formally stated:

Theorem 2.1 (Duality)**.** *For any t, a bijection between the set of cash invariant t-pricing functions and the set of proper t-acceptance sets exists. The bijection and its inverse are given by Definitions 2.6 and 2.9. I.e. for any cash invariant t-pricing function π: (1) $A[\pi]$ is a proper t-acceptance set, and (2) $P[A[\pi]] = \pi$ and for any proper t-acceptance set $\mathcal{A}$: (3) $P[\mathcal{A}]$ is a cash invariant t-pricing function, and (4) $A[P[\mathcal{A}]] = \mathcal{A}$.*

Proof. See Appendix A.1.2. The key step, which exploits the inner structure of acceptance sets and enables us to arrive at these new results, is found in Lemma A.3. □

With this theorem we can answer the question of the previous section:

Corollary 2.2. *For any proper acceptance set $\mathcal{A}$ and option $f \in \mathcal{X}_{[t,\infty\rangle}$ it holds:*

$$P[\mathcal{A}](f) = \max\{x \in L_t^- \mid f - x \in \mathcal{A}\}, \text{ if } P[\mathcal{A}](f) < \infty$$

Proof. See Appendix A.1.3 □

But beyond this, the duality proves that our formalization is consistent with our intuition. And more importantly, it provides the justification for using the two notions, acceptance sets and pricing functions, interchangeably in developing an axiomatic option pricing theory. Properties that are best expressed for one of the two can be easily translated for the other.

Furthermore, it can be used to prove the important property of *locality* for any cash invariant pricing function.

Corollary 2.3. *Any cash invariant t-pricing function, π, is also* local*, i.e.* $\pi(f) \overset{B}{=} \pi(g)$*, if* $f \overset{B}{=} g$ *and* $B \in \mathcal{F}_t$*.*

Proof. See Appendix A.1.4. □

2.4 Decisions

The aim of this work is the development of a pricing theory for options with embedded decisions. The duality result from the previous section enables us to develop our theory in terms of the more directly accessible language of acceptance sets. The derivation of the pricing function then merely becomes a mechanical exercise.

2.4.1 The counterparty's decisions

What is the agent's price of a contract whose payoff depends on a decision by the counterparty? First we give an intuitive answer to the question, when does the agent accept such a contract, and then derive the pricing function using the duality from the last section.

The agent has neither previous knowledge about nor influence on the counterparty's decision. The counterparty has to be considered completely free in its choice. This suggests the following acceptance set, which is employed implicitly in most of the option pricing literature: The agent accepts an option if and only if the option is acceptable for any possible behavior of the counterparty. It is important to understand, that this presumes nothing about the counterparty's actual behavior.

To formalize this we need a given set of *admissible decision procedures S* for decisions by the counterparty, which, at this point, does not need to be specified further:

Definition 2.10. *For a given t-acceptance set $\mathcal{A}$ and a set of admissible decision procedures S define the* conservative acceptance set for counterparty decisions*:*

$$\mathcal{A}^{\forall S} \equiv \left\{ f \in \mathcal{X}_{[t,\infty\rangle} \,\middle|\, \forall \varphi \in S : f\left[\varphi\right] \in \mathcal{A} \right\}$$

The agent's price corresponding to this acceptance set is given by the lowest price attainable by any decision procedure in S:

Theorem 2.2. *If $\mathcal{A}$ is a proper t-acceptance set with pricing function π, then $\mathcal{A}^{\forall S}$ also is a proper t-acceptance set and its pricing function is given by*

$$P[\mathcal{A}^{\forall S}](f) = \inf_{\varphi \in S} \pi\left(f\left[\varphi\right]\right) \text{ for all } f \in \mathcal{X}_{[t,\infty\rangle}$$

Furthermore, the agent's price for any actual decisions procedure followed by the counterparty is in general equal to or higher than this price. The difference adds to the agent's profit. However, the counterparty can make this profit arbitrarily small (if the infimum is finite).

Proof. See Appendix appendix A.1.6. □

This type of acceptance is called *conservative* because it involves no estimate of the counterparty's behavior.

For more complex and realistic problems this kind of acceptance can be too limiting. In these cases non-conservative acceptance sets, that by definition cannot insure against every possible behavior of the counterparty, are needed. They are of interest if the counterparty is somehow limited in his or her actions, acts upon a different maxim (like maximization of another objective function, like a utility function or the value of some larger portfolio) or in cases of market access or information asymmetries, e.g. retail banking customers.

The concepts introduced in this section could be extended by a probabilistic description of the counterparty's behavior used to formulate non-conservative acceptance, where the effective payoffs are accepted in some statistical sense.

A less intrusive way to introduce non-conservative acceptance is to declare certain procedures by the counterparty that are theoretically admissible as practically impossible. Formally, this can be achieved by simply restricting the set of admissible decision procedures to a subset of $S' \subset S$. In this case the acceptance set would be $\mathcal{A}^{\forall S'}$ and Theorem 2.2 applies analogously. An important application of such non-conservative acceptance arises for problems of aligned interest between agent and counterparty as in the case of the following example.

Example 2.1. A minority share of common stock issued by a company can be understood as a call option on the company's assets. The "payoff" crucially depends on decisions by the counterparty, i.e. the majority owner and management of the issuing company. In this example, the boundaries of the applicability of conservative acceptance become obvious: even if prohibited by law, management of the company could deliberately steer into bankruptcy. By the admissibility of this procedure, the conservative stock price as given by Theorem 2.2 would be zero. The situation is more accurately described by a smaller set of admissible procedures incorporating the fact that such behavior would hurt the counterparty's own interests.

The remainder of this work is limited to conservative acceptance, which is the predominant—albeit implicitly used—method to handle decision in the current pricing literature. While beyond the scope of this article, one of the motivations for this work is to enable the development of non-conservative models.

2.4.2 *The agent's decisions*

The agent's decisions can be handled analogously by first formulating the conditions for acceptance and then deriving the price. As it is now her decision, the only rational conduct is to accept the option if and only if there exists at least one decision procedure that makes the option acceptable.

Unlike in the counterparty case, a naive translation of this behavior would not in general produce a proper acceptance set—as demonstrated by Example 2.2. This can be cured by loosening the requirement of acceptance of $f[\varphi]$ to the acceptance of $f[\varphi] + x$ for any positive premium x:

Definition 2.11. *For a given t-acceptance set $\mathcal{A}$ and set of admissible decision procedures S define the* conservative acceptance set of agent decisions*:*

$$\mathcal{A}^{\exists S} \equiv \left\{ f \in \mathcal{X}_{[t,\infty\rangle} \;\middle|\; \forall x \in V_t, \exists \varphi \in S : f[\varphi] + x \in \mathcal{A} \right\}$$

In contrast to the counterparty case, we also need to place a restriction on the set S in order to be able to derive an analogous dual pricing function. S has to be t-compatible (Definition 2.5). If this is the case, then the agent's price is given by the highest effective price attainable by any decision procedure:

Theorem 2.3. *If S is t-compatible and $\mathcal{A}$ is a proper t-acceptance set with pricing function π, then $\mathcal{A}^{\exists S}$ is also a proper t-acceptance set and its pricing function is given by:*

$$P[\mathcal{A}^{\exists S}](f) = \sup_{\varphi \in S} \pi\left(f[\varphi]\right) \text{ for all } f \in \mathcal{X}_{[t,\infty\rangle}$$

Furthermore, while this price will in general be higher than the price for any actual decision procedure followed by the agent, she can make this loss arbitrarily small (if the supremum is finite).

Proof. See Appendix appendix A.1.7. □

Example 2.2. This demonstrates why a simpler definition of $\mathcal{A}^{\exists S}$, analogous to $\mathcal{A}^{\forall S}$'s definition, does not ensure properness.

Define a simple $(t = 0)$-acceptance set $\mathcal{A} \equiv \left\{ f \;\middle|\; f \geq 0 \right\}$, an option $f : \varphi \mapsto \varphi$ with $\varphi \in S \equiv \langle -\infty, 0\rangle$ paying an arbitrary negative number of the agent's choosing at time $t = 0$.

It is clear to see, that no decision procedure exists, such that the payoff becomes non-negative and thus accepted. Consequently, f would not be in the following alternative version of $\mathcal{A}^{\exists S}$: $f \notin \mathcal{B} \equiv \left\{ f \in \mathcal{X}_{[t,\infty\rangle} \;\middle|\; \exists \varphi \in S : f[\varphi] \in \mathcal{A} \right\}$. However, if we add any positive value x ($\in V_0$) to f, then there exists a decision procedure, which makes $f + x$ acceptable (of course $\varphi = -x$) and thus $f + x \in \mathcal{B}$. This is a violation of eq. (2.2) from Definition 2.4 and thus $\mathcal{B}$ is not proper.

An undesirable consequence of this fact is that f is not acceptable, yet it has price zero and thus violates Theorem 2.1.4:

$$P[\mathcal{B}](f) = \sup\left\{ x \in L_0^- \;\middle|\; \exists \varphi \in S : \varphi \geq x \right\} = \sup\left\{ x \in L_0^- \;\middle|\; 0 > x \right\} = 0$$

2.4.3 *Arbitrage-free pricing*

In this section, we present a derivation of the arbitrage-free price in incomplete markets (which includes the complete market as a special case) for options without decisions using our framework and the above introduced concepts.

The discounted price processes of the market's assets are modeled by an N-dimensional semi-martingale $X = (X_t)_{t \geq 0}$. Furthermore, this example only makes sense in an arbitrage-free market or, more precisely, in a market with *no free lunch with vanishing risk*. By the *fundamental theorem of asset pricing*, derived for general processes (with unbounded jumps) by Delbaen and Schachermayer (1998), absence of arbitrage is equivalent to the existence of a sigma-martingale measure:

$$\mathcal{M} \equiv \left\{ Q \sim \mathbb{P} \;\middle|\; X \text{ is a } Q \text{ sigma-martingale} \right\} \neq \emptyset.$$

The agent is allowed to hedge against her risk exposure by continuously trading in the markets. To translate this into our framework, we need to model the proceeds of the hedging activity. Let the decision procedure φ describe the number of shares of each asset held in the agent's hedging portfolio at different points of time (formally, $\mathbf{T}_a \equiv [0, \infty\rangle$ and $D_t \equiv \mathbb{R}^N$ for any t, see subsection 2.2.1). The payoff describing the proceeds of a strategy is defined as the stochastic integral with respect to X (the $\cdot$ denotes scalar product between two vectors):

$$H(\varphi) = \int_0^\infty \varphi_t \cdot \mathrm{d}X_t$$

A strategy is *admissible*, i.e. in $S \subseteq \Phi_{[0,\infty\rangle}$, if this stochastic integral is well defined and bounded from below. The last requirement excludes so-called doubling strategies and the possibility of infinite wealth generation by trading (see Delbaen and Schachermayer 1994, and references therein).

The agent is infinitely risk averse and will only accept investments that almost surely do not loose any money. Using Definition 2.11, her acceptance set is thus given by $\left\{ f \middle| f \geq 0 \right\}^{\exists S}$. Because she will not only receive the option's actual payoff, but also the proceeds of her hedging strategy, this set equals the sum of the acceptance set of actual option payoffs, $\mathcal{A}$, and H, $\left\{ f + H \;\middle|\; f \in \mathcal{A} \right\}$, and leads to the following price:

Theorem 2.4. *In an arbitrage-free market ($\mathcal{M} \neq \emptyset$) the bid price of an upper-bounded option $f \in L_\infty$ for an agent with $\left\{ f + H \;\middle|\; f \in \mathcal{A} \right\} = \left\{ f \middle| f \geq 0 \right\}^{\exists S}$ is given by*

$$P[\mathcal{A}](f) = \inf_{Q \in \mathcal{M}} \mathbb{E}_Q \left[f \right],$$

also known as the super-replication price.

Proof. See Appendix A.1.8. □

2.4.4 Pricing options with decisions

The last section was an example that used the new formalism to derive a well known result. In this section we will demonstrate how to use Theorems 2.2 and 2.3 to solve the problem of pricing options with decisions by reducing it to the classical pricing of options without decision.

Our starting point is classical option pricing. We presume the existence of a t-pricing function π^0—with dual $\mathcal{A}^0$—defined for options without decisions, i.e. for $\mathbf{T}_d = \varnothing$ in our formalism. The aim is to develop a theory with an even number of decisions, $n \in \mathbb{N}$, taking place at times given by an increasing sequence $(\tau_i)_{i \le n}$ alternating between counterparty, $\mathbf{T}_a \equiv \{\tau_1, \tau_3, \dots, \tau_{n-1}\}$, and agent, $\mathbf{T}_c \equiv \{\tau_2, \tau_4, \dots, \tau_n\}$. As in the previous sections, we first construct the acceptance set and then derive the price.

To build the full acceptance set from $\mathcal{A}^0$, we will handle one the decision at a time, each with its own set of admissible decision procedures denoted by $S_i \subseteq \Phi_{\{\tau_i\}}$ for each $i \le n$. It is clear that the temporal order of a contract's decisions plays an important role in accepting or pricing it. While later decisions can react to decisions made earlier, earlier decisions are fixed and have to take different possible later decisions into account. As a consequence, the reasoning behind the acceptance sets introduced in subsections 2.4.1 and 2.4.2 can only be meaningfully applied to the earliest decision in the contract.

This suggests the following recursive extension of the acceptance set $\mathcal{A}^0$. Let $\mathcal{D}_i$ be the acceptance set for options with decisions at or after time τ_i. For an agent's decision at τ_i an option is in $\mathcal{D}_i$ if and only if it is in $\mathcal{D}_{i+1}^{\exists S_i}$. For a counterparty decision we have $\mathcal{D}_i = \mathcal{D}_{i+1}^{\forall S_i}$. This scheme stops at $\mathcal{D}_{n+1} \equiv \mathcal{A}^0$ and the full acceptance set is given by $\mathcal{D}_1$, which expands to

$$\mathcal{D}_1 = \mathcal{A}^{0,\exists S_n \, \forall S_{n-1} \dots \exists S_2 \, \forall S_1}. \tag{2.3}$$

Equipped with this full acceptance set we can now calculate prices for options with decisions:

Theorem 2.5. *If S_i is t-compatible (Definition 2.5) for every i with $\tau_i \in \mathbf{T}_a$ and $\mathcal{A}^0 \subseteq \mathcal{X}_\varnothing$ is a proper t-acceptance set, then $\mathcal{A}^{0,\exists S_n \, \forall S_{n-1} \dots \exists S_2 \, \forall S_1}$ also is a proper t-acceptance set. Its dual pricing function is given by*

$$\pi(f) = \inf_{a_1 \in S_1} \sup_{a_2 \in S_2} \dots \inf_{a_{n-1} \in S_{n-1}} \sup_{a_n \in S_n} \pi^0\big(f[a_1][a_2]\dots[a_n]\big)$$

for $f \in \mathcal{X}_{[t,\infty\rangle}$.

Proof. See Appendix A.1.9. □

As motivated above, the order of decisions is important. Using an order different from that in eq. (2.3) can lead to a different price. For example, it is worth a non-negative premium to be able to react to the counterparty's decision, due to $\sup_x \inf_y f(x,y) \le \inf_y \sup_x f(x,y)$ for any $f : X \times Y \mapsto Z$.

2.5 Time consistency

So far, the acceptance and thus price of an option is based on *today*'s assessment of effective payoffs corresponding to different *future* decision procedures. In general, there is no guarantee that this assessment remains valid over time.

2.5.1 Problems with time inconsistency

A family of pricing functions at different times exhibiting this inconsistency is called *time inconsistent*. Using a time inconsistent family of pricing functions introduces an ambiguity into the handling of options with decisions; the result will depend on the point of time at which the effects of different choices are assessed. There is, however, no valid argument to choose one point of time over the other. We will describe the shortcomings that come with each possibility, and thereby make a strong point against the use of time inconsistent methods.

Consider the case, in which the agent buys at time t_1 an option $f \in \mathcal{X}_{\{t_2\}}$ with one decision by the agent at time t_2 for a price $p_a = \sup_{\varphi \in S} \pi\left(f\left[\varphi\right]\right)$ calculated using Theorem 2.5. p_a is as high as possible, with the restriction that at least one decision procedure φ^* for her exists that makes the option acceptable *today* (at time t_1). However, when the time comes for her to actually face the decision, the only rational choice is to choose a φ that maximizes the option's present (i.e. time t_2) value. For time inconsistent pricing functions, this procedure will generally be different from φ^*. The problem is that the effective option resulting from this behavior has a t_1-price p'_a lower than p_a, due to $\pi_t\left(f\left[\varphi\right]\right) \le \sup_{\varphi \in S} \pi\left(f\left[\varphi\right]\right)$ and in retrospect the agent paid too much.

Now consider the case with a counterparty decision at t_2. The agent accepts the price $p_c = \inf_{\varphi \in S} \pi\left(f\left[\varphi\right]\right)$ for any possible future behavior of the counterparty. However, at time t_2, the worst case behavior of the counterparty would be to follow the decision procedure ψ that minimizes the option's present (i.e. time t_2) value. For time inconsistent pricing functions, the resulting effective option $f\left[\psi\right]$ has a time t_1-price higher than the infimum p_c. The agent did not pay her highest price and could have offered a more competitive bid price.

On the other hand, if the agent based her prices on φ and ψ from above, the resulting prices p'_a and p'_c would have the following deficiencies: She could force herself at time t_1 to stick to the decision procedure φ^* and consequently offer the more competitive bid price $p_a \ge p'_a$. In the counterparty case, she paid too much, because there are adverse decision procedures by the counterparty that only justify the lower price $p_c \le p'_c$ and thus leave her with a strictly unacceptable position at time t_1.

These shortcomings strongly suggest the use of time consistent families of pricing functions.

2.5.2 Time consistent acceptance and pricing

In this section we extend our framework to express time consistency of pricing functions and acceptance sets and derive a duality between the two, required to obtain the time consistent price of a general option with decisions in the next section. To do this conveniently we formalize the concept of a *family*:

Definition 2.12 (Families). *A time indexed set $\{x_t\}_t$, also written as $x_\cdot$, is called* pricing family *or* acceptance family, *if every x_t is a cash invariant t-pricing function or proper t-acceptance set, respectively.*

The duality from Theorem 2.1 extends to families. $\mathcal{A}_.$'s dual is written as $P[\mathcal{A}_.]$ and $\pi_.$'s dual is written as $A[\pi_.]$.

In the simplest case, the acceptance family is time independent. As an example take subsection 2.4.3, where a payoff without decisions is accepted, if and only if it is non-negative. This behavior is clearly independent of time and consequently all acceptance sets of the corresponding family would be given by $\{f|f \geq 0\}$ and contain the same payoffs. In general, acceptance families exhibit more complex time dependency. For example, it could be argued that an option's acceptance in all possible states at a future time implies its acceptance today, i.e. $\mathcal{A}_s \subseteq \mathcal{A}_t$, for all $s \geq t$, which is called *weak acceptance consistency* (Artzner et al. 2007).

So far, pricing functions were only defined for options with no past decisions (cf. subsection 2.3.2). However, working with pricing families to express time consistency requires an extension to general payoffs. The price of an option with past decisions does also depend on these past decisions, i.e. is again a payoff in the sense of Definition 2.2. The following lemma shows how this extension can be defined formally and that it works as expected when based on cash invariant pricing functions:

Lemma 2.1 (Extended cash invariant pricing function)**.** *Let $\varphi\big|_{\langle -\infty,t\rangle}$ denote the restriction of φ to past decisions. Then for any $f \in \mathcal{X}$ and cash invariant t-pricing function π, the mapping $\varphi \mapsto \pi\left(f\left[\varphi\big|_{\langle -\infty,t\rangle}\right]\right)$ is a payoff with no present or future decisions, i.e. element of $\mathcal{X}^t_{\langle -\infty,t\rangle}$.*

From here on we identify $\pi(f)$ with this payoff, giving rise to an extended version of cash invariance:

$$\pi(f+g) = \pi(f) + g, \text{ for any } g \in \mathcal{X}^t_{\langle -\infty,t\rangle} \text{ with } g > -\infty \tag{2.4}$$

Proof. See Appendix A.1.11. □

A notion of time consistency we can use within our minimalist setting is the so called *recursiveness*. The price of an option equals the price of any other option paying the first option's normalized price (c.f. Definition 2.7) at some future point of time:

Definition 2.13 (Time consistent pricing family)**.** *A pricing family $\pi_.$ is called* time consistent *if for all $s \geq t$ and $f \in \mathcal{X}$:*

$$\pi_t(\pi_s(f) - \pi_s(0)) = \pi_t(f)$$

There is another, more intuitive characterization of time consistency: If at some future point of time one option costs more than another option in every world state, the same should be true today (see also Detlefsen and Scandolo 2005, and the references therein). These two definitions are equivalent if we impose more restrictions on the pricing family:

Corollary 2.4 (Alternative time consistency formulation). *A pricing family $\pi_.$ with $|\pi_.(0)| < \infty$ consisting of monotone pricing functions, i.e. for all t and $f, g \in \mathcal{X}$*

$$f \geq g \Longrightarrow \pi_t(f) \geq \pi_t(g),$$

is time consistent if and only if for all $s \geq t$ and $f, g \in \mathcal{X}$

$$\pi_s(f) \geq \pi_s(g) \Longrightarrow \pi_t(f) \geq \pi_t(g). \tag{2.5}$$

Proof. See Appendix A.1.10. □

A time consistent acceptance family can be characterized as follows: The agent accepts an option if and only if she accepts any option, which pays the first option's normalized price calculated at some future time. This definition is equivalent to Definition 2.13:

Theorem 2.6 (Time consistent acceptance family). *$\pi_.$ is time consistent, if and only if its dual acceptance family, $\mathcal{A}_.$, is time consistent, which is defined by:*

$$f \in \mathcal{A}_t \Longleftrightarrow \pi_s(f) - \pi_s(0) \in \mathcal{A}_t, \text{ for all } s \geq t. \tag{2.6}$$

Proof. See Appendix A.1.12. □

2.5.3 *Application to options with decisions*

A time consistent theory does not suffer from the problems outlined in subsection 2.5.1. As we will demonstrate, there is no ambiguity in handling decisions and as a consequence, pricing options with decisions requires far less argumentation compared to the general (time inconsistent) case from subsection 2.4.4.

We will work in the same setting as in subsection 2.4.4, consisting of n decisions at times $\mathbf{T}_d = (\tau_i)_{i \leq n}$ with the corresponding admissible decision procedures $S_i \subseteq \Phi_{\{\tau_i\}}$ for each i, a proper acceptance family $\mathcal{A}^0_. \subseteq \mathcal{X}_\varnothing$ for options without decisions and its dual pricing family $\pi^0_.$. In contrast to subsection 2.4.4, the complex construction of the full acceptance family, which includes imposing the specific order of decision elimination, will not be necessary. It suffices to specify how options with *present time* decisions are handled by the full acceptance family $\mathcal{A}_.$. Following the reasoning from subsections 2.4.1 and 2.4.2, we assume the following conditions for all $i \leq n$:

$$\begin{aligned} \mathcal{A}_{\tau_i} &= \mathcal{A}^{\exists S_i}_{\tau_i} \text{ and } S_i \text{ is } \tau_i\text{-compatible} && \text{if } \tau_i \in \mathbf{T}_a, \\ \mathcal{A}_{\tau_i} &= \mathcal{A}^{\forall S_i}_{\tau_i} && \text{if } \tau_i \in \mathbf{T}_c \end{aligned} \tag{2.7}$$

This fully determines $\mathcal{A}_.$ and thus the price of an option with decisions:

Theorem 2.7 (Time consistent conservative price). *If $\mathcal{A}^0_.$ is time consistent, it has only one extension $\mathcal{A}_.$, i.e. $\mathcal{A}_. \cap \mathcal{X}_\varnothing = \mathcal{A}^0_.$ that is also time consistent and satisfies eq. (2.7). If $\pi^0_.$ is normalized, then $\mathcal{A}_.$'s dual pricing family is given by*

$$\pi_t(f) = \pi^0_t\left(\inf_{a_i \in S_i} \pi^0_{\tau_i}\left(\sup_{a_{i+1} \in S_{i+1}} \pi^0_{\tau_{i+1}}\left(\dots \inf_{a_{n-1} \in S_{n-1}} \pi^0_{\tau_{n-1}}\left(\sup_{a_n \in S_n} \pi^0_{\tau_n}(f[a_i]\dots[a_n])\right)\right)\right)\right),$$

for all $f \in \mathcal{X}$ and t with $\tau_{i-1} < t \leq \tau_i$ and $\tau_i \in \mathbf{T}_c$. If $\tau_i < t \leq \tau_{i+1}$, then $\pi_t(f)$ is given by the same expression but without the first infimum.

Proof. See Appendix A.1.13. □

Compared to Theorem 2.5, this pricing function is more suitable for actual calculation. The recursive structure can in many interesting cases be used to separate the n optimization problems, therefore removing the n-exponent in the time complexity dependence on the size of S_i and enabling numeric calculations. Classic examples would be the valuation of American options (as limiting case of the Bermudan option) or options on trading gains.

2.6 Discussion

To illustrate how our results can be applied and to round up the discussion initiated in section 2.1, we will interpret two common approaches to the pricing of American options within our framework.

One approach lies in the derivation of an optimal stopping problem from economically justifiable principles. For example, Karatzas (1988) defines the price of an American option as the minimal initial capital required to set up a portfolio never worth less than the exercise value, $g_\cdot$, and then shows that it is given by

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}\left[g_\tau\right], \text{ where } \mathcal{S} \text{ denotes the set of stopping times.} \tag{2.8}$$

Despite its advantages, this approach still suffers from the same problem of a very indirect treatment of the decision. The American exercise feature is deeply intermingled with the pricing model and thus this method of handling decisions in an option contract is neither extensible nor transferable to other problems.

These shortcomings are overcome in our framework. In the most straightforward formulation the decision of the holder lies in choosing the (stochastic) exercise time and the option's payoff is given by $f : \varphi \mapsto g_{\varphi_0}$. For an infinitely risk averse agent that uses conservative acceptance and is continuously hedging in an arbitrage free market, Theorems 2.2 and 2.4 yield eq. (2.8).[2] This modular approach provides the flexibility to experiment with different acceptance mechanism and other than the arbitrage-free pricing functions, like e.g. good-deal bounds.

Another popular approach is based on dynamic programming. Early, noteworthy contributions were Chen (1970) or the binomial method from Cox et al. (1979). The American exercise feature is approximated by a finite number of exercise dates. Following an ad-hoc argument, the price at each decision is defined as the maximum of the exercise and continuation value, which in turn

[2] While the theorem gives an infimum, following our sign conventions in subsections 2.2.2 and 2.3.2, the price for the writer is given by the negative price of $-f$, and thus by the infimum-supremum duality equals the supremum.

depends on the price at the next decision. This approach bears no resemblance to the above approach and it does not answer the question as to why it gives the right answer either.

This can be directly translated into our framework by setting $D_t = \{1, 0\}$, where 1 stands for "exercise" and 0 for "do not exercise". The payoff then takes the form $f : \varphi \mapsto g_{\min\{t \,|\, \varphi_t = 1\}}$ and the dynamic programming problem follows directly from Theorem 2.7. In addition to conservative acceptance, it requires the pricing function π^0 to be time consistent, a crucial but not usually explicitly mentioned property.

The same result can be derived from our formulation of the first approach. The bijection between stopping times φ_0 that take only finitely many values and the set of discrete-time adapted processes taking values in $\{0, 1\}$, leads to the above formulation and makes the relationship and compatibility of the two approaches obvious.

2.7 Conclusion

In this article we close the argumentative gap between pricing theories for classical payoffs and a theory for options with decisions.

This is accomplished in two steps. The first step lies in the derivation of a duality between between *cash invariant* pricing functions and so called *proper* acceptance. Proper acceptance sets unite some meaningful properties, but are more general than coherent or convex acceptance sets. This duality effectively shows that acceptance sets and pricing functions contain the same information.

In the second step, it allows us to solve the problem of pricing options with decisions. As acceptance sets are best suited to model agent behavior, we provide a characterization of acceptance of options with decisions corresponding to what is implicitly performed in the option pricing literature, and which we call *conservative acceptance*. Theorem 2.5 and Theorem 2.7 then translate these acceptance sets via the duality into pricing functions for options with decisions.

Conservative acceptance, however, fails to capture more complex real world situations. For example, alignment of interests between agent and counterparty (as in Example 2.1) demands a different kind of acceptance. Our contribution consists of a concise framework and a precise characterization of the status quo, i.e. conservative acceptance, implicitly assumed in today's pricing theories. Our aim is to enable further research and the development of non-conservative pricing theories.

As demonstrated in the previous section, our results enable a consistent and modular treatment of the decisions within option contracts. Instead of constantly reinventing the wheel, it suffices to motivate how options with decisions are accepted and then use the duality and our formalism to determine the price. This minimizes the argumentative burden and provides a basis to investigate new ways of treating decisions. Furthermore, the modular nature ensures that different approaches can be transferred to a wide variety of pricing methods and models.

One underdeveloped area that will significantly benefit from these methods is the pricing and hedging of options with complex decisions for *both* holder and writer. In this area, in which existing approaches fail, our framework proves most fruitful and can be used to generate new insights. Gerer and Dorfleitner (2016a) apply our results to the problem of pricing and realistically hedging American options featuring a complex interplay between exercise and hedging decisions. Based on the results provided in this article, they provide the optimal solution to the full problem, which is — to our knowledge — the first of its kind.

OPTIMAL DISCRETE HEDGING OF AMERICAN OPTIONS
USING AN INTEGRATED APPROACH TO OPTIONS WITH COMPLEX EMBEDDED DECISIONS

3

(Joint work with Gregor Dorfleitner. Accepted for publication in Review of Derivatives Research *subject to minor revisions.)*

Abstract

In order to solve the problem of optimal discrete hedging of American options, this paper utilizes an integrated approach in which the writer's decisions (including hedging decisions) and the holder's decisions are treated on equal footing. From basic principles expressed in the language of acceptance sets we derive a general pricing and hedging formula and apply it to American options. The result combines the important aspects of the problem into one price. It finds the optimal compromise between risk reduction and transaction costs, i.e. optimally placed rebalancing times. Moreover, it accounts for the interplay between the early exercise and hedging decisions.

We then perform a numerical calculation to compare the price of an agent who has exponential preferences and uses our method of optimal hedging against a delta hedger. The results show that the optimal hedging strategy is influenced by the early exercise boundary and that the worst case holder behavior for a sub-optimal hedger significantly deviates from the classical Black-Scholes exercise boundary.

3.1 Introduction

Whenever an option writer hedges an option, their net payoff is given by the option's premium minus the tracking error of the hedging activity. For European options and in a complete market, there is one hedging strategy that will turn the random future tracking error into a constant known at inception, rendering the pricing problem trivial. In reality, markets are neither complete nor friction-less and there are options and other claims whose payoff can be modified by the holder. Thus, in practice, the tracking error is random and can depend on a time-continuum of decisions by both the writer (deciding whether to change the current hedging position) and the holder (e.g. in the case of the American option, deciding whether to exercise or not).

While both aspects enjoy extensive treatment in scientific publications, most contributions only look at one aspect in isolation from the other, i.e. they focus either on realistic (discrete) hedging or on exercise features. The common ad-hoc approach to decisions embedded in option contracts is stretched over its limit, when applied to a complex combination of decisions by both counterparties.

This issue concerns, among other fields, the literature on the realistic hedging of American options, which despite its practical relevance comprises only a handful of contributions. These contributions provide important groundwork, and satisfy many requirements that are in our view desirable for a solution of the problem. This paper's aim is to improve on existing work by combining all these requirements. Table 3.1 contains the requirements numbered from one to eight in the column headers and provides an overview of the requirements satisfied by each contribution.

Due to the limited number of contributions we also include two notable, newer contributions on optimal hedging of European options, that therefore violate requirement 1.

Contributions failing the second requirement assume some externally given exercise strategy of the holder. As an example take Constantinides and Zariphopoulou (2001), who state that "[the holder's exercise time is given by some predetermined stopping time τ, which] may be the optimal exercise time [...] in the absence of transaction costs", or Coleman et al. (2007) claiming that "[t]he holder will choose an exercise strategy to maximize the option value to him; hedging decisions of the writer are irrelevant to his exercise decision." This view neglects the existence of a possible exercise strategy of the holder which is more expensive to hedge and which would thus lead to too low selling prices and insufficient insurance against all possible holder behaviors.

The third requirement is violated if rehedging times are somehow restricted by an external mechanism e.g. by a predetermined number of hedges, or if rebalancing is only allowed after some risk measure exceeds a certain threshold (e.g. as in Ahn and Wilmott 2009). Such a restriction is always sub-optimal, because it ignores the fact that at any given point of time, rebalancing is optimal if and only if the reduction in risk does outweigh the cost associated with rebalancing. This kind of sub-optimality implies unrealistic behavior: if a

	American exercise (1)	Worst-case holder strategy (2)	No restriction of hedging times (3)	Transaction costs (4)	Weighing of risk/cost (5)	Not only super-hedging (6)	Risk & costs influence price (7)	Time consistency (8)
Chalasani and Jha (2001)	✓	✓	✓	✓				✓
Bouchard and Temam (2005)	✓	✓	✓	✓				✓
Tokarz and Zastawniak (2006)	✓	✓	✓	✓				✓
Coleman et al. (2007)	✓					✓		
De Vallière et al. (2008)	✓	✓	✓	✓				✓
Roux and Zastawniak (2009)	✓	✓	✓	✓				✓
Roux and Zastawniak (2014)	✓	✓	✓	✓				✓
Roux and Zastawniak (2016)	✓	✓	✓	✓				✓
Constantinides and Zariphopoulou (2001)	✓		✓	✓	✓	✓	✓	
Constantinides and Perrakis (2007)	✓	✓	✓	✓	✓	✓	✓	n.a.
Bayraktar et al. (2015)	✓	✓	✓					✓
Ahn and Wilmott (2009)		n.a.				✓		
Gobet and Landon (2014)		n.a.			✓	✓		

Table 3.1: Requirements satisfied by relevant contributions

trade leads to a more favorable hedging position *now*, why should any past consideration stop the hedger from executing it?

These considerations also originate the need to include transaction costs (requirement 4)—be it traditional transaction fees, spreads or opportunity costs. Without transaction costs, the optimal hedging strategy always consists of quasi-continuous rebalancing, i.e. rebalancing to the optimal hedging position as fast as practically possible, which is simply unrealistic.

As mentioned above, deriving an optimal hedging strategy means weighing risk against transaction costs (requirement 5).

Requirement 6 might seem superfluous as it could be argued that the derivation of a super-hedging price actually satisfies requirement 5. However, the underlying risk-measure is too risk-averse to be realistic. Option writers *do* accept the risk of hedging losses, hence requirement 6.

Requirement 7 demands the remaining risk and transaction costs to actually contribute to the final option price. This requirement is for example missed by

Gobet and Landon (2014). They minimize the product of the number of hedges and the quadratic variance of the hedging error, two quantities undoubtedly influencing the bottom line of a real-world hedging activity. However, their combination into a product is completely arbitrary and does not translate into a consistent option price, for there will always exist hedging strategies that do not optimize the above product and still lead to a lower selling price.

Time consistency (requirement 8), which is neglected in many contributions, is an important property, especially in the context of pricing options with complex decisions. Results obtained by time-inconsistent methods will either assume sub-optimal future choices or do not give the optimal solution from today's perspective (see Gerer and Dorfleitner 2016b, for more details on the relation of decisions and time consistency in option pricing).

The article of Constantinides and Perrakis (2007) actually satisfies nearly all of our requirements, yet their contribution consists of the derivation of stochastic bounds on option prices for utility maximizing agents and thus has a different focus.

The isolated treatment of the two decisions by the holder and the writer completely ignores one of the core aspects of realistically hedged options, namely their interplay. Questions about the possibility of a holder's strategy that explicitly exploit e.g. a fixed number of hedges by the writer or makes the original hedging strategy inhibitingly expensive due to transactions costs are not even considered.

This paper solves the problem of realistically pricing and hedging an American option. It is based on the insight that a realistic hedging theory is always a theory of hedging with transaction costs: it is not the physical impossibility of continuous trading that needs to be addressed. With today's trend to sub-millisecond order execution, continuous hedging could be approximated sufficiently if there were no transaction costs. Instead, we allow continuous trading (in the limit), but acknowledge that there is some kind of cost associated with each trade. This cost is weighed against the remaining hedging risk. Expressing both risk and transaction costs monetarily, gives rise to an optimization problem whose solution is a finite number of optimally placed hedging trades and a consistent option price.

Our approach is motivated by the understanding of a "price" as an intrinsically one-dimensional quantity, which does not leave much conceptual freedom. The results from Gerer and Dorfleitner (2016b) imply, that under mild assumptions, the decisions can be *formally eliminated* from the problem in a consequent manner without the need to resort to external concepts and without any further motivating argument. Our analysis differs from others in that we seek to calculate the agent's indifference or reservation price in an uncompromising fashion.

In section 3.2, we summarize the formalism and results from Gerer and Dorfleitner (ibid.) used in this paper. In section 3.3, a general duality between pricing functions and acceptance sets for payoffs with decisions is applied to derive a general pricing and hedging principle for options with decision by both parties. Section 3.4 specializes this principle to a formula for American options, which is then numerically solved in section 3.5. Conclusions are given in section 3.6.

3.2 Theoretical fundament

The theory is formulated from the perspective of a single market participant, that we will refer to as *agent*, engaging in financial activities and entering contracts with other agents, collectively called her *counterparty*.

All possible evolutions of the world, their physical probabilities and the time-dependence of information are described by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, \mathbb{P})$, where all points of time are given by the totally ordered set $\mathcal{T}$.

Let L_t^G represent all $\mathcal{F}_t$-measurable random variables into the set $G \subseteq \overline{\mathbb{R}}$. We will use the abbreviations $L_t \equiv L_t^{\langle-\infty,\infty\rangle}$, $L_t^- \equiv L_t^{[-\infty,\infty\rangle}$, $L_t^+ \equiv L_t^{\langle-\infty,\infty]}$ and $L_t^\pm \equiv L_t^{[-\infty,\infty]}$, and employ the convention $\infty - \infty = \infty$ on $\overline{\mathbb{R}}$.

We assume decisions happen at predetermined times, $\mathbf{T}_d \subseteq \mathcal{T}$. As we will see later, this does not prevent us from describing more complex decision for which the point of time can also be chosen by the agent, like options with *American* exercise. At each point of time $t \in \mathbf{T}_d$ there can be exactly one decision by either the agent or the counterparty. This poses no limitation, as instantaneous decisions by the same agent can be merged into a tuple of decisions and it ensures that there is always a well-defined order between decisions by different agents, even if the physical time between them can be infinitesimally short. Decisions to be made by the agent happen at times $\mathbf{T}_a$ and decisions by the counterparty at times $\mathbf{T}_c \equiv \mathbf{T}_d \setminus \mathbf{T}_a$.

The decision behavior of the agents will be modeled by *decision procedures*, describing how the choices for a subset of decisions depend on the world state. The set of *decision procedures* for decisions at times $\mathbf{T} \subseteq \mathbf{T}_d$ is abbreviated by $\Phi_\mathbf{T}$ and defined as the set of stochastic processes whose values at time t are elements of D_t:

$$\Phi_\mathbf{T} \equiv \left\{ \varphi : \mathbf{T} \times \Omega \to \bigcup_{t\in\mathbf{T}} D_t \,\middle|\, \varphi_t : \Omega \to D_t, \text{ for all } t \in \mathbf{T} \right\} \tag{3.1}$$

D_t contains all possible choices at time t. We will use the abbreviation $\Phi \equiv \Phi_{\mathbf{T}_d}$.

The payoff of an option with embedded decisions is described by specifying the cumulative discounted cash-flow to be *received by the agent* for any possible combination of choices and world states:

Definition 3.1 (Payoffs). *Define $\mathcal{X}_\mathbf{T}^t$ as the set of $\mathcal{F}_t$-measurable payoffs that only depend on decisions made at times $\mathbf{T} \subseteq \mathcal{T}$:*

$$\mathcal{X}_\mathbf{T}^t \equiv \left\{ f : \Phi \to L_t^\pm \,\middle|\, f(\psi) \overset{B}{=} f(\varphi), \text{ if } B \in \mathcal{F}_t \text{ and } \psi_t \overset{B}{=} \varphi_t \text{ for all } t \in \mathbf{T} \cap \mathbf{T}_d \right\}$$

Putting a set $B \in \mathcal{F}_\infty$ above a comparison operator means conditionally *almost surely equal: $x \overset{B}{=} y \Leftrightarrow \mathbb{P}\big(\{x = y\}\big|B\big) = 1$, with $\{x = y\} \equiv \big\{\omega \in \Omega \,\big|\, x(\omega) = y(\omega)\big\}$.*

We will use the abbreviations $\mathcal{X}_{\mathbf{T}} \equiv \mathcal{X}^{\infty}_{\mathbf{T}}$ and $\mathcal{X} \equiv \mathcal{X}_{\mathcal{T}}$.

Remark 3.1. If a random variable $x \in L^{\pm}_{\infty}$ is used in the context of payoffs, it is understood as the corresponding constant payoff given by $\psi \mapsto x$, which is an element of $\mathcal{X}_{\varnothing}$, and vice versa.

If not stated differently, all operators, relations and also suprema/infima used on payoffs are the pointwise versions of their $L^{\pm}_{\infty}$, $\mathbb{P}$-almost sure, variants: $f \mathrm{R} g \Leftrightarrow \forall \varphi \in \Phi : f(\varphi) \overset{\text{a.s.}}{\mathrm{R}} g(\varphi)$

Furthermore, we provide an operation to produce the *effective* payoff, that results if an agent or counterparty follows a decision procedure for a certain subset of decisions. These decisions can be considered *fixed* and the effective payoff does not depend on them anymore:

Definition 3.2 (Effective payoff). *For any payoff $f \in \mathcal{X}$ and decision procedure $\varphi \in \Phi_{\mathbf{T}}$ define the* effective payoff, *$f\left[\varphi\right] \in \mathcal{X}_{\mathcal{T}\setminus\mathbf{T}}$ by*

$$f\left[\varphi\right](\psi) \equiv f(\varphi \mathbb{1}_{\mathbf{T}} + \psi \mathbb{1}_{\mathbf{T}_d \setminus \mathbf{T}}), \text{ for all } \psi \in \Phi.$$

The framework aims to provide the tools to build and analyze pricing theories for options with decisions. This is achieved by providing a minimal characterization of acceptance sets and pricing functions, proving their equivalence, and thus making these concepts usable interchangeably.

A t-acceptance set contains the agent's *acceptable opportunities* at time t, i.e. payoffs without decisions before time t that she accepts as zero-cost investments. We require the following property to ensure that an acceptance set can serve as modeling tool for pricing theories:

Definition 3.3 (Proper acceptance sets). *A t-acceptance set $\mathcal{A}$ is called* proper *if it is t-compatible (see below) and $\mathcal{A} = \big\{f \in \mathcal{X}_{[t,\infty\rangle} \,\big|\, \big\{f + x \,\big|\, x \in V_t\big\} \subseteq \mathcal{A}\big\}$, where V_t is the set of positive t-*premiums $V_t \equiv \left\{x \in L^{+}_{t} \,\middle|\, 0 \overset{a.s.}{<} x\right\}$.

Definition 3.4 (t-compatibility). *A non empty set X of functions from some set G into $L^{\pm}_{t}$ is t-*compatible, *if for all $\{x_n\} \subseteq X$ and mutually disjoint $\{B_n\} \subseteq \mathcal{F}_t$ with $\mathbb{P}(\bigcup_n B_n) = 1$ it holds $\sum_n^\infty x_n \mathbb{1}_{B_n} \in X$, where $\mathbb{1}_{B_n}$ is the indicator function of the set $G \times B_n$.*

Remark 3.2. In this framework, the result of a pricing function is understood as the highest premium the agent would accept to pay for entering the contract, or her *bid* price. If a contract's *ask* price is wanted, it can be calculated by the negative of the bid price of the reversed contract.

A *t-pricing function* is any function $\pi : \mathcal{X}_{[t,\infty\rangle} \to L^{\pm}_{t}$.

Remark 3.3. For a general option $f \in \mathcal{X}$—possibly including decisions before t—and a t-pricing function π, we use $\pi\left(f\right)$ to denote $\varphi \mapsto \pi\left(f\left[\varphi|_{\langle-\infty,t\rangle}\right]\right)$, which is a payoff in the sense of Definition 3.1.

For pricing functions, the property corresponding to properness is *cash invariance*.

Definition 3.5 (Cash invariance)**.** *A t-pricing function π is called* cash invariant *if for any payoff $f \in \mathcal{X}$ and payoff $g \in \mathcal{X}^t_{\langle -\infty, t\rangle}$ with no present or future decision and $g > -\infty$, it holds $\pi\big(f + g\big) = \pi\big(f\big) + g$.*

We will also use the term normalized pricing function:

Definition 3.6 (Normalized pricing function)**.** *A pricing function π is called* normalized, *if $\pi(0) = 0$. Every pricing function π with $|\pi(0)| < \infty$ has a* normalized version *$x \mapsto \pi(x) - \pi(0)$.*

As proved in Gerer and Dorfleitner (2016b, Theorem 3.1) there exists a one-to-one correspondence between the set of cash invariant t-pricing functions and the set of proper t-acceptance sets. The *duality operations* are given in the following definition and correctly replicate the description of pricing functions in Remark 3.2:

Definition 3.7 (Duality operations)**.** *For any t-acceptance set $\mathcal{A}$ define its* dual *t-pricing function $P[\mathcal{A}]$ by $P[\mathcal{A}](f) \equiv \sup\big\{x \in L^-_t \;\big|\; f - x \in \mathcal{A}\big\}$ for all $f \in \mathcal{X}_{[t,\infty\rangle}$. (The* sup *operator denotes the* essential supremum.*)*

For any t-pricing function π define its dual *t-acceptance set $A[\pi] \equiv \big\{f \in \mathcal{X}_{[t,\infty\rangle} \;\big|\; 0 \le \pi\big(f\big)\big\}$.*

This duality enables us to develop our pricing theory for options with decisions in terms of the more directly accessible language of acceptance. Specifically, we will use *conservative acceptance*, which is employed implicitly in most of the option pricing literature. For a given t-acceptance set $\mathcal{A}$ and a set of admissible decision procedures S, $\mathcal{A}^{\forall S}$ and $\mathcal{A}^{\exists S}$ represent the *conservative acceptance sets for decisions by the counterparty* or *the agent*, respectively. $\mathcal{A}^{\forall S}$ includes an option if and only if for every possible decision procedure by the counterparty the resulting effective option is acceptable:

$$\mathcal{A}^{\forall S} \equiv \big\{f \in \mathcal{X}_{[t,\infty\rangle} \;\big|\; \qquad \forall \varphi \in S : f\big[\varphi\big] \in \mathcal{A}\big\} \tag{3.2}$$

It is important to understand, that this presumes nothing about the counterparty's actual behavior. For her own decisions, $\mathcal{A}^{\exists S}$ contains an option if and only if there always is at least one decision procedure the agent could follow to make the effective option almost acceptable, as indicated by the $+x$:[1]

$$\mathcal{A}^{\exists S} \equiv \big\{f \in \mathcal{X}_{[t,\infty\rangle} \;\big|\; \forall x \in V_t, \exists \varphi \in S : f\big[\varphi\big] + x \in \mathcal{A}\big\} \tag{3.3}$$

In order to talk about acceptance sets and pricing functions at different times, we introduce *acceptance families* as time indexed families of proper acceptance

[1] See Gerer and Dorfleitner (2016b, Example 4.2) for why this definition needs a more complicated form than eq. (3.2)

sets $\{\mathcal{A}_t\}_t$ written as $\mathcal{A}_.$ and *pricing families*, as time indexed families of cash invariant pricing functions written as $\pi_.$.

An especially important property of such families is time consistency (requirement 8 in Table 3.1). By Theorem B.3, the following definitions are equivalent:

Definition 3.8 (Time consistency)**.** *A proper acceptance family $\mathcal{A}_.$ with a normalized dual $\pi_.$ is called* time consistent *if $f \in \mathcal{A}_t \iff \pi_s(f) \in \mathcal{A}_t$, for all $s \geq t$.*

A normalized pricing family $\pi_.$ is called time consistent *if for all $s \geq t$ and $f \in \mathcal{X}$: $\pi_t(\pi_s(f)) = \pi_t(f)$.*

3.3 Optimal hedging—the general formula

We will treat the hedging activity as decisions within our theory of options with decisions. Between rehedges there can be further decisions for both the agent and its counterparty.

This approach will produce the pricing function for the hedging agent as well as the optimal hedging ratios, i.e. optimally placed rehedgings, without the need to formulate an exogenous optimization problem. Instead these results are direct consequences of the construction and imposed properties of the acceptance set and their relation to pricing functions.

We impose no practical limitation on the number of rehedgings the agent can perform. For formal reasons, we approximate the set of hedging decisions using a finite, increasing sequence $(\tau_i)_{i \leq n}$, where the hedging position is closed on the last date τ_n.

Remark 3.4. While this introduces a dependency of the results on the particular choice of this sequence, the time intervals can be made smaller than any physical time scale of our world and thus its practical influence eliminated. Numerical calculations typically will be feasible only for much larger intervals.

The actual number of performed rehedgings will usually be smaller than n, as at each time τ_i the agent can decide not to rebalance her hedging portfolio.

The discounted price processes of the assets available for hedging are modeled by an N-dimensional adapted process $X = (X_t)_t$ with finite components. A hedging decision consists of choosing the amount of shares to hold from each asset, which we will model using N-dimensional vectors, i.e. $D_{\tau_i} \subseteq \mathbb{R}^N$ for all $i \leq n$.

Given a decision procedure $\varphi \in \Phi$, the future cash-flow of a hedging activity started with an initial position $\varphi_{\tau_{i-1}}$ at a time $t \in (\tau_{i-1}, \tau_i]$ is given by

$$H_t(\varphi) \equiv \sum_{j \geq i}^{n+1} \varphi_{\tau_{j-1}} \cdot \left(X_{\tau_j} - X_{\max\{\tau_{j-1}, t\}} \right) - C_j(\varphi). \tag{3.4}$$

The $\cdot$ marks scalar product between two vectors. The first term calculates the gains from market price movements, and $C_j(\varphi)$ stands for the finite, $\mathcal{F}_{\tau_j}$-measurable transaction costs associated with changing the portfolio from $\varphi_{\tau_{j-1}}$ to φ_{τ_j} at time τ_j. C_{n+1} corresponds to the special case of liquidating the last position φ_{τ_n} at τ_{n+1}. It is easy to see that H_t depends on decisions at time τ_{i-1} and later, i.e. $H_t \in \mathcal{X}_{\{\tau_j\}_{i-1}^{n}}$.

In order to derive the general pricing function of the hedging agent for options with decisions, we employ the method of Gerer and Dorfleitner (2016b, subsection 4.3) to derive the super-replication price for continuous trading. We start with the agent's *internal acceptance family*, $\mathcal{A}_\cdot$, containing payoffs or zero-cost investments she accepts "as is", i.e. payoffs that cannot be modified by her beyond the decisions contained in the payoff, especially not be hedged against. $\mathcal{A}_\cdot$ needs to capture the agent's risk aversion, business model and regulatory requirements. In this section we treat it as given, for it is completely independent from the aspect of hedging; a separation of concerns made possible by the development of the proposed framework.

Next, this acceptance family is transformed into the agent's *external acceptance family*, $\mathcal{B}_\cdot$, by subtraction of any modifications, which are not part of the original contract specifications. In the current setting this means subtracting the proceeds of her hedging activity:

$$\mathcal{B}_t(\varphi) \equiv \left\{ f \;\middle|\; f + H_t\left[\varphi\big|_{\langle -\infty, t\rangle}\right] \in \mathcal{A}_t \right\} \tag{3.5}$$

$\mathcal{B}_t$ depends on the decision procedure φ, because H_t depends on past decisions, more specifically on the most recent hedging decision. Making this dependency explicit ensures, that $\mathcal{B}_t(\varphi)$ itself contains only options with no past decisions.

Equation (3.5) can also be read in the following way: The agent accepts a contract with another party, if and only if, she *internally* accepts the contract's payoff plus the result of her hedging activity.

Through the duality we know that these acceptance sets uniquely define the hedging agent's prices – denoted by $\eta_\cdot$, which can be calculated using the duality operation from Definition 3.7:

$$\eta_t(f)(\varphi) \equiv P[\mathcal{B}_t(\varphi)](f)(\varphi) \tag{3.6}$$

for any option with decisions $f \in \mathcal{X}$ and decision procedure $\varphi \in \Phi$.

Of course, $\eta_\cdot$ can also be expressed using $\mathcal{A}_\cdot$'s dual pricing family $\pi_\cdot = P[\mathcal{A}_\cdot]$:

Lemma 3.1. *$\eta_t(f) = \pi_t(f + H_t)$ for all t and $f \in \mathcal{X}$.*

Proof. See Appendix B.2.1. □

This is in agreement with the expected result that the price of an option is given by the internal price of the hedged option.

Remark 3.5 (Normalization). The above construction of $\eta_\cdot$ will in general not yield normalized pricing functions, i.e. the price of the zero payoff is different from zero: $\eta_t(0) = \pi_t(H_t) \neq 0$. Depending on the specific nature of $\mathcal{A}_\cdot$ and $\pi_\cdot$, it is possible that the agent assigns a positive net present value to the proceeds of the trading activity H_i, i.e. $\pi_{\tau_i}(H_i) \geq 0$. This can happen for example, if $\pi_t(X_s) > X_t$ (for $s > t$), i.e. buying or selling the market assets represents an acceptable or even arbitrage opportunity.

If the market is arbitrage free, or the agent's transaction costs destroy any acceptable or arbitrage opportunity, then $\pi_{\tau_i}\big(H_{\tau_i}\big)(\varphi)$ is zero, if $\varphi_{\tau_{i-1}} = 0$ and even negative for $\varphi_{\tau_{i-1}} \neq 0$, due to the unavoidable costs for closing the current position eventually.

Economically meaningful prices are obtained from the normalized version:

$$f \mapsto \eta_t\big(f\big) - \eta_t(0)$$

This calibration ensures that the price of any sure payoff equals the payoff itself ($\eta_t\big(g\big) = g$ if $g \in L_t^+$, which follows from cash invariance), and it is plausible with the two cases described above: If the agent would pay a positive amount $\eta_t(0) > 0$ for the situation he already is in, normalization decreases all bid prices by that amount, which could be interpreted as compensation for giving up the current favorable position upon entering the new contract. In other words, normalization erases the additional value $\eta_t\big(f\big)$ assigns to the possibility of trading in the market, which the agent can also do without f and whose value is thus given by $\eta_t(0)$.

For an initial unfavorable position $\varphi_{\tau_{i-1}} \neq 0$, $\eta_t(0)\big[\varphi\big]$ would be a negative number representing the negative of the cost associated with optimally closing that position. In this case normalization would increase the agent's naked bid price $\eta_t\big(f\big)$, because entering and optimally hedging f would spare her the cost of closing her current position. □

In addition to the hedging decisions, we explicitly add times for general counterparty decisions, which will then be used for the early exercise decision in the next section. Between any two hedging times τ_i and τ_{i+1} there is a decision of the counterparty located at s_i:

$$\tau_i < s_i < \tau_{i+1}$$

Furthermore, for each decision we need a set of admissible decision procedures, denoted by $R_i \subseteq \Phi_{\{\tau_i\}}$ and $S_i \subseteq \Phi_{\{s_i\}}$ for all $i \leq n$.

The general result needs the following axioms. The first assures that hedging decision cannot see in the future. The additional requirement of t-compatibility derives from the technical differences between conservative acceptance for the agent and the counterparty (cf. Theorems B.1 and B.2):

Axiom 3.1. *For all $i \leq n$, R_i contains only $\mathcal{F}_t$-adapted decision procedures and is τ_i-compatible (cf. Definition 3.4).*

The second axiom specifies how decisions are treated. We use conservative acceptance (as defined in eqs. (3.2) and (3.3)) at the time of each particular decision. This, together with time consistency, will be enough to eliminate all future decisions from the pricing problem.

Axiom 3.2 (Conservative acceptance)**.** *$\mathcal{A}_{\tau_i} = \mathcal{A}_{\tau_i}^{\exists R_i}$ and $\mathcal{A}_{s_i} = \mathcal{A}_{s_i}^{\forall S_i}$ for all $i \leq n$.*

Feeding all this into our formalism yields the pricing formula for an optimally hedged option:

Theorem 3.1 (Optimal hedging)**.** *Let a $\mathcal{A}_.$ be a time consistent internal acceptance family with a normalized dual pricing family $\pi_.$ and assume Axioms 3.1 and 3.2. At the end of the hedging activity, the price of an option $f \in \mathcal{X}$ can be calculated directly by*

$$\eta_{\tau_{n+1}}(f) = \pi_{\tau_{n+1}}(f) - C_{n+1}, \tag{3.7}$$

and earlier prices for $i \leq n$ can be calculated recursively:

$$\eta_{\tau_i}(f) = \sup_{\varphi \in R_i} \pi_{\tau_i}\left(\inf_{\psi \in S_i} \pi_{s_i}\left(\eta_{\tau_{i+1}}(f)[\varphi][\psi] + \varphi_{\tau_i} \cdot (X_{\tau_{i+1}} - X_{\tau_i})\right)\right) - C_i[\varphi] \tag{3.8}$$

Proof. See Appendix B.2.2. ◻

For a hedging strategy a and decision procedure of the counterparty b, the net payoff of the whole hedging activity is given by the difference of the realized tracking error and the option's upfront premium:

$$\delta \equiv (f + H_t)[a][b] - \eta_t(f)$$

Conservative acceptance ensures, that for any decision procedure b and $\epsilon \in V_t$ there exists a hedging strategy a such that $\delta + \epsilon$ is acceptable.

3.4 Optimal hedging of American options

In this section we specialize the results from the previous section in order to price and hedge an American option from the perspective of the writer. The holder of an *American option with discounted exercise value process g* has the right to exercise it at any time t prior or equal to the expiration date T and receive the amount g_t.

This exercise right gives rise to infinitely many decisions: At every instant the holder can decide to exercise or not to exercise. To model these decision we use the finite set of decision times $\{s_i\}$ introduced in the last section, with $s_n \equiv T$. Following the reasoning from Remark 3.4 this implies no loss of generality. We define

$$D_{s_i} \equiv \{1, 0\}, \text{ where 1 stands for "exercise" and 0 for "do not exercise"}, \tag{3.9}$$

for all i. The payoff of an American option that has not been exercised before time s_i is then for any $\varphi \in \Phi$ and $\omega \in \Omega$ given by:

$$f_i(\varphi) \equiv g_{t^*} \text{ with the stopping time } t^*(\omega) = \min\left\{t \in \mathbf{T}_a \;\middle|\; t \geq s_i \wedge \varphi_t(\omega) = 1\right\} \tag{3.10}$$

This definition shows the expected behavior, if the current decision is fixed using a constant decision procedure:

$$f_i\left[s_i \mapsto 1\right] = g_{s_i} \text{ for exercise, and } f_i\left[s_i \mapsto 0\right] = f_{i+1} \text{ for continuation.} \tag{3.11}$$

Basic theory of stochastic processes ascertains that f is measurable if g is progressively measurable and φ is adapted. And thus due to Corollary B.1 and f_i's pointwise definition it is a payoff according to our Definition 3.1, or more precisely $f_i \in \mathcal{X}_{\{s_j\}_{j=i}^n}$, as it only depends on decision at times $\{s_i, \dots, s_n\}$.

While we restrict the admissible hedging procedures through Axiom 3.1 from the last section, the only restriction placed on exercise procedures is their being adapted:

$$S_i \equiv \Phi_{\{s_i\}} \cap \left\{\varphi \;\middle|\; \varphi \text{ is adapted to } \mathcal{F}\right\}, \text{ for all } i. \tag{3.12}$$

So far the optimizations in Theorem 3.1 have to be performed over random variables making a direct numerical implementation infeasible. However, as they are limited to "current time" decisions, they can be simplified. We give sufficient (but not necessary) conditions under which the essential supremum over the set of procedures can be simplified to a pointwise supremum directly over the set of choices:

Lemma 3.2 (Countable present time decisions)**.** *If D_t or Ω is countable and $S \equiv \Phi_{\{t\}} \cap \left\{\varphi \;\middle|\; \varphi \text{ is adapted to } \mathcal{F}\right\}$, then for every cash invariant t-pricing function π and $f \in \mathcal{X}$ it holds:*

$$\sup_{\varphi \in S} \pi\left(f\left[\varphi\right]\right) = \sup_{a \in D_t} \pi\left(f\left[t \mapsto a\right]\right)$$

Proof. See Appendix B.2.3. ◻

As the payoffs involved do not have any decisions besides the hedging and exercise decisions, we can give a result that reduces the pricing problem to classical option pricing theory for options without decisions. To make this explicit we will use $\mathcal{A}^0_\cdot$ and $\pi^0_\cdot$ to denote the restrictions of $\mathcal{A}_\cdot$ and $\pi_\cdot$ to options without decisions. Formally, we define $\mathcal{A}^0_\cdot \equiv \mathcal{A}_\cdot \cap \mathcal{X}_\varnothing$ and $\pi^0_\cdot = P[\mathcal{A}^0_\cdot]$. Lemma B.1 proves the expected connection between $\pi^0_\cdot$ and $\pi_\cdot$.

Furthermore, we assume for all i that the agent's hedging decision at τ_{i+1} happens instantly after the exercise decision s_i. Formally, these two times collapse for quantities that do not depend on the decision at s_i, i.e.

$$g_{s_i} = g_{\tau_{i+1}}, \text{ and } \pi_{s_i}(f) = \pi_{\tau_{i+1}}(f), \text{ if } f \in \mathcal{X}_{[\tau_{i+1}, \infty\rangle}. \tag{3.13}$$

Now we can derive the main result:

Theorem 3.2. *Given X, C, $\mathcal{A}_\cdot$, $\{R_i\}$ and $\eta_\cdot$ as in Theorem 3.1, $\{S_i\}$, f, $\mathcal{A}^0_\cdot$ and $\pi^0_\cdot$ as defined above, we define p_i as the* ask *price (cf. Remark 3.2) at time τ_i of an optimally hedged American option:*

$$p_i \equiv -\eta_{\tau_i}\left(-f_i\right), \text{ for all } i \le n.$$

The price after expiration is given by

$$p_{n+1} = C_{n+1}, \tag{3.14}$$

and earlier prices for $i \leq n$ can be calculated recursively:

$$p_i = \inf_{\varphi \in R_i} -\pi^0_{\tau_i}\left(-\max\left\{g_{\tau_{i+1}} - \eta_{\tau_{i+1}}(0), p_{i+1}\right\}[\varphi] + \varphi_{\tau_i} \cdot \left(X_{\tau_{i+1}} - X_{\tau_i}\right)\right) + C_i[\varphi] \tag{3.15}$$

Proof. See Appendix B.2.4. □

As expected, the writer chooses the most favorable hedge and it is most expensive for her if the holder exercises as soon as the payoff exceeds the price of the continued option.

The terms C_{n+1} and $-\eta_{\tau_i}(0) = -\pi^0_{\tau_i}(H_i)$ occurring above could be identified with the cost of optimally closing the current hedging position. As already discussed in Remark 3.5, they are non-negative in a market without acceptable opportunity and thus add to the payment of g_{τ_i} faced by the hedging option writer upon exercise by the holder.

Their appearance is a consequence of the fact that $\eta_.$ is not normalized (in general) and it can be trivially checked, that normalizing the result—i.e. calculating $p_i + \eta_{\tau_i}(0)$—would remove both terms, whilst introducing a similar term in the continuation value. We did not state the normalized result, as it would complicate the recursive calculations, which are more naturally expressed in unnormalized values.

3.5 Numerical demonstration

In this section we produce numerical results from Theorem 3.2. We will use a simple market model consisting of a riskless money market account with interest rate r and a single stock whose discounted price process $X_.$ is a geometric Brownian motion with drift $\mu > r$ and volatility σ. Besides analytical tractability and intuitiveness, it reveals interesting features of the pricing and hedging problem. It should be noted that our hedging formula can be applied to any model for the price process X.

We are pricing an American put with strike K, i.e. a payoff f as defined in eq. (3.10) with discounted exercise value written as $g_t(X_t) = X_t - Ke^{-rt}$.

The transaction costs consist of a fixed component k_0 and a component proportional to the transaction value (with factor k_1). Its discounted value is calculated as follows:

$$C_i(\varphi) \equiv c_i(\varphi_{\tau_i} - \varphi_{\tau_{i-1}}, X_{\tau_i}) \text{ with } c_i(q, x) = e^{-r\tau_i} k_0 \, \mathbb{1}_{q \neq 0} + k_1 x \, |q| \tag{3.16}$$

3.5.1 *Selecting a pricing function*

Before we can actually implement a numerical program, we need to devise the agent's internal pricing family for options without decisions, $\pi^0_.$

Let us first state the requirements to be met by $\pi^0_\cdot$. There is, of course *cash invariance* (requirement I), the basic property imposed by our formalism, and *time consistency* (requirement II) upon which the results of the previous section rely.

In addition to these two requirements concerning $\pi^0_\cdot$ directly, we place three further restrictions on the resulting external pricing family $\eta_\cdot$ (cf. Lemma 3.1). To ensure consistency with existing results we require that without transaction costs and in the limit of infinitely many hedging times the well-known arbitrage-free prices, i.e. *risk-neutral expectation values*, are obtained for continuously replicable payoffs (requirement III).

As noted in Remark 3.5, $\eta_\cdot$ is not normalized. Thus, $\eta_t(0) = \pi_t(H_t)$ can be negative due to transaction costs for closing the current position or positive, if trading in the market constitutes an acceptable opportunity for the agent. While these effects can be handled satisfactory by normalizing the result, normalization is only meaningful if $|\eta_t(0)| < \infty$, or informally stated, if the agent *cannot extract infinite wealth from his trading activity* (requirement IV).

Besides these theoretical requirements, for the purpose of this demonstration we need *readily implementable, numerical algorithms* (requirement V).

To satisfy requirement V we exclude all pricing functions or risk-measures whose calculation relies on Monte Carlo methods. We are aware of the existence of Monte Carlo methods suitable for American options, but extending and implementing them for our problem — while deemed possible — would go beyond the scope of this paper. This excludes all candidates containing the value-at-risk and its variants or derivatives like the expected shortfall or conditional value-at-risk, most of which also violate requirement II, time consistency (cf. Cheridito and Stadje 2009).

We use

$$\pi^0_t(f) \equiv \frac{-1}{\gamma} \ln\Big(\mathbb{E}\left[e^{-\gamma f} \middle| \mathcal{F}_t\right]\Big), \tag{3.17}$$

for some positive degree of risk aversion γ. This function is the indifference price of the exponential utility function, also known as the negative of the conditional entropic risk measure and has gained much attention in the field of utility indifference pricing, among others.

For the remainder of this paper we use the pricing family $\eta_\cdot$ as defined in eq. (3.6) for an acceptance family $\mathcal{A}_\cdot$ satisfying Axiom 3.2 and $\mathcal{A}_\cdot \cap \mathcal{X}_\varnothing = A\left[\pi^0\right]$ with $\pi^0_\cdot$ as defined above in eq. (3.17). We also define the normalized pricing family $\overline{\eta}_\cdot$:

$$\overline{\eta}_t(f) \equiv \eta_t(f) - \eta_t(0), \text{ for all } f \in \mathcal{X}$$

It is well-known (see e.g. Cheridito and Kupper 2009, eq. 3.3) that $\pi^0_\cdot$ is cash invariant (requirement I) and time consistent (requirement II). It also satisfies requirement V, because PDE discretization methods for the calculation of conditional expectations are widely-used and can be applied directly to solve eq. (3.15) from Theorem 3.2.

We do not present a formal proof of requirement III for $\overline{\eta}_\cdot$, but instead point out two supporting facts. First, for continuous trading strategies without transaction costs it has been shown that $\overline{\eta}_.$ yields the risk-neutral expectation value (cf. Davis et al. 1993, Theorem 1, or for a more recent presentation Becherer 2003, eq. 3.8, who calls this *elementary no-arbitrage consistency*). Secondly, we confirmed by numerical calculations that making the time between two rehedges short enough will result in prices sufficiently close to the Black-Scholes price and optimal strategies coinciding with the Black-Scholes delta.

Again without a formal proof, requirement IV follows from another well-known result (see e.g. Henderson and Hobson 2002b, eq. 2): In the case of continuous trading without transaction costs, it holds that $\eta_t(0) < \infty$ and the optimal strategy is given by:

$$Z_t(X_t) \equiv \frac{\mu - r}{\gamma \sigma^2 X_t} \tag{3.18}$$

Due to the monotonicity of the supremum in Theorem 3.1 and the monotonicity of π^0, restricting to discrete strategies and introducing transaction costs results in even smaller prices.

3.5.2 *Translating Theorem 3.2 into a computational procedure*

In order to perform the hedging optimization numerically we only consider a finite number of different hedging positions, i.e. D_{τ_i} finite for all $i \leq n$. Then Lemma 3.2 and the Markov property of $X_.$ enable us to write Theorem 3.2 in a form suitable for numerical calculation. Using the ordinary functions $z_i(h, X_{\tau_i}) \equiv -\eta_{\tau_i}(0)\left[\tau_{i-1} \mapsto h\right]$ for the price of the zero claim and

$$v_i(h, X_{\tau_i}) \equiv \max\left\{g_{\tau_i}(X_{\tau_i}) + z_i(h, X_{\tau_i}), -\eta_{\tau_i}\left(-f_i\right)\left[\tau_{i-1} \mapsto h\right]\right\}$$

for the price of the optimally hedged option, both with a current hedging position h, we get:

$$\begin{aligned}
v_{n+1}(h,x) &= \max\left\{g_{\tau_{n+1}}(x), 0\right\} + c_{n+1}(h,x)\\
z_{n+1}(h,x) &= c_{n+1}(h,x)\\
\mathrm{Cont}_i(h,x,b,q) &\equiv \frac{1}{\gamma} \ln\, \mathbb{E}_t\left[e^{\gamma\left(b_{i+1}(q, X_{\tau_{i+1}}) - q X_{\tau_{i+1}}\right)}\middle| X_{\tau_i} = x\right] + q\,x + c_i(h-q, x)\\
q^*_{\tau_i}(h,x) &\equiv \arg\min_{q \in D_{\tau_i}} \mathrm{Cont}_i(h,x,v,q) &(3.19)\\
v_i(h,x) &= \max\left\{g_{\tau_i}(x) + z_i(h,x), \mathrm{Cont}_i(h,x,v,q^*(h,x))\right\} &(3.20)\\
z_i(h,x) &= \min_{q \in D_{\tau_i}} \mathrm{Cont}_i(h,x,z,q)
\end{aligned}$$

We are going to compare the price obtained under the optimal hedging strategy q^* with classical delta hedging. Let d represent the ask price of an agent who rebalances daily to the optimal continuous, zero-transaction costs strategy given by the sum of the option's Black-Scholes delta, $\Delta_t(x) = \frac{\partial}{\partial x}\mathrm{BS}_t(x)$, and the utility optimizing strategy from eq. (3.18), $Z_t(x)$:

$$\begin{aligned} d_{n+1}(h,x) &= v_{n+1}(h,x) \\ d_i(h,x) &= \max\Big\{g_{\tau_i}(x) + z_i(h,x), \mathrm{Cont}_i(h,x,d,\Delta_{\lfloor\tau_i\rfloor}(x) + Z_{\lfloor\tau_i\rfloor}(x))\Big\} \end{aligned} \tag{3.21}$$

$\lfloor\tau_i\rfloor$ rounds down τ_i to the beginning of the most recent day.

Our numerical program written in C++ is a direct translation of the above equations. The conditional expectations are calculated using a finite difference method with optimal spatial finite difference weights à la Ito and Toivanen (2009) and Crank-Nicolson time-stepping with Rannacher startup (Giles and Carter 2006; Rannacher 1984). The expectation values in $\mathrm{Cont}_i(h,x,b,q)$ are calculated for all pairs $(b,q) \in \{v,z,d\} \times D_{\tau_i}$ in parallel, achieving quasi-linear speedup on multi-core CPUs.

We will write normalized prices as $\overline{v} \equiv v - z$ and $\overline{d} \equiv d - z$.

3.5.3 *Numerical results*

We now present the results of the calculation using the following specifications.

The American put has a strike price $K = 100$, volatility $\sigma = 50\%$ p.a., drift $\mu = 10\%$ p.a., risk-free rate $r = 5\%$ p.a., where one year consists of 252 business days. We assume very moderate transaction costs, with a fixed component of $k_0 = 0.001$ and proportional component of $k_1 = 0.025\%$. The writer's coefficient of constant absolute risk aversion is $\gamma = 0.001$.

We use 8 hedging and exercise decision times per day, i.e. $\tau_{i+1} - \tau_i = 1/8$ day. There are 150 allowed hedging positions, $D_{\tau_i} = \{0, \delta, 2\delta, \dots, 5\}$ with $\delta = 5/149 \approx 0.0336$. Using a higher numbers of daily decisions and possible hedging positions does not significantly change the calculated values.

The results comprise three aspects: the writer behavior, the holder behavior and the option price.

The writer's optimal hedging position is given by $q_t^*(x,h)$ from eq. (3.19) and depends on current stock price x and hedging position h. Figure 3.1 plots two examples for fixed values of h. Instantly after rebalancing to this optimal position, $h' \equiv q_t^*(x,h)$, the current stock price x will lie in one of possibly several plateaus where $x \mapsto q_t^*(x,h')$ is constant with value h'. As soon as the stock price leaves this plateau, it is again optimal to rebalance.

The information contained in q_t^* can be completely described by two corridors, the no-trading and the rebalancing corridor. They are depicted in Figure 3.2 for two different times t and have the following interpretation: it is optimal to rebalance to the nearest point of the rebalancing corridor, but only if the stock price leaves the no-trading corridor. The spikes visible in Figure 3.2 in both corridors occur in the vicinity of the optimal exercise boundary of the holder. An effect, of course only revealed by solving the full optimization problem.

Based on experiments with different values of k_0 and k_1, we make the following numerical observations. The rebalancing corridor is always fully contained within the no-trading corridor, its width is monotonically increasing in k_1, the proportional component of the transaction costs, and it collapses for $k_1 = 0$. The space between the two corridors exhibits an analogous relationship with k_0, the fixed component. The delta hedging position lies within the no-trading corridor and for stock prices above the exercise boundary also within the rebalancing corridor. Without transaction costs both corridors collapse to the delta hedging strategy.

The exercise behavior of the holder is characterized by the exercise boundary, which separates the continuation regions from the their complement, the exercise regions. With conservative acceptance the writer is insured against the worst possible or *pessimal* exercise strategy. The corresponding continuation regions consist of states where the maximum in eq. (3.20) or eq. (3.21) equals the continuation value.

Figure 3.3 shows the holder's pessimal exercise boundaries against different writers. Against the optimal hedger the continuation region is only slightly larger than in the Black-Scholes case. The most striking finding is that the holder could in fact harm the delta hedger. The boundary against the delta hedger clearly exhibits a daily recurring pattern lining up with her daily rebalancings. As expected, the continuation region is much larger than in the optimal case, which can be explained by the fact that continuing the claim means more hedging costs for the (sub-optimal) delta hedger. This holds true as long as the delta hedger actually rebalances and thus incurs transaction costs. If the delta hedger's current position equals the next delta hedging position, there are no transactions costs and thus at these points (also marked in Figure 3.3) the extended continuation region stops.

This shows that for a delta hedger the common assumption of a Black-Scholes exercise boundary will result in an underestimation of risk caused by the interplay of American exercise feature, discrete hedging and transaction costs.

Holder and writer behavior are by-products of the main result: the price. Figures 3.4 and 3.5 provide two different perspectives on the put's normalized ask prices of an optimally hedging writer, $\overline{v}$, and a delta hedging writer, $\overline{d}$. Figure 3.4 shows how the absolute difference $\overline{v} - \overline{d}$ depends on the current hedging position for fixed stock prices. As is expected, once it is optimal to rebalance, the additional transaction costs of both optimal and delta hedging will be equal, making the difference between the two independent of the current hedging position.

For a fixed current hedging position and varying stock prices the difference between both prices and the Black-Scholes price are reflected in Figure 3.5. It also contains the relative price reduction to be achieved by optimal hedging, calculated as $\overline{d}/\overline{v} - 1$.

Both figures demonstrate that the writer can always offer a more competitive price or make a sure profit by optimal hedging instead of delta hedging. This profit increases with decreasing moneyness. For example, 63 days before expiration at a stock price of 145.3, the optimal-hedging price is $\overline{v} \approx 0.787$, which is 10.3 % lower than the delta-hedging price.

Last, reducing the risk aversion increases both the optimal-hedging and the delta-hedging price. However, the effect on the delta-hedging price is weaker and consequently, the profit from optimal hedging will be larger for a less risk-averse agent. In the above example, if we change γ to $\gamma/2 = 0.0005$, $\overline{v}$ will be 17.3 % lower than $\overline{d}$.

3.6 Conclusion

Applying the methods of Gerer and Dorfleitner (2016b) to the problem of hedging options with decisions allows us to derive a general hedging principle in a rigorous but straight forward manner, starting from a small set of clearly stated assumptions. This principle is then further specialized to a formula for realistically hedging American options; a formula that is not conjectured, but formally derived and proved in non-preexisting fashion.

To demonstrate how to turn this completely model-independent formula into actual numbers, we fix a market model, a pricing function and transaction costs and perform numerical calculations. The results of these numerical experiments show that when compared to the delta hedger, the optimal hedger can offer a significantly better price or make a sure profit. Further, they reveal that indeed there is a complex interaction between hedging decisions and the early exercise decisions.

In addition to the conceptual and theoretical advantages demonstrated by our holistic approach to decisions embedded in option contracts, these results prove the usefulness of our method in realistic applications.

We leave it to further research to apply our methods more realistic models than the above example and to overcome the challenge of a numerical implementation of our results for these models.

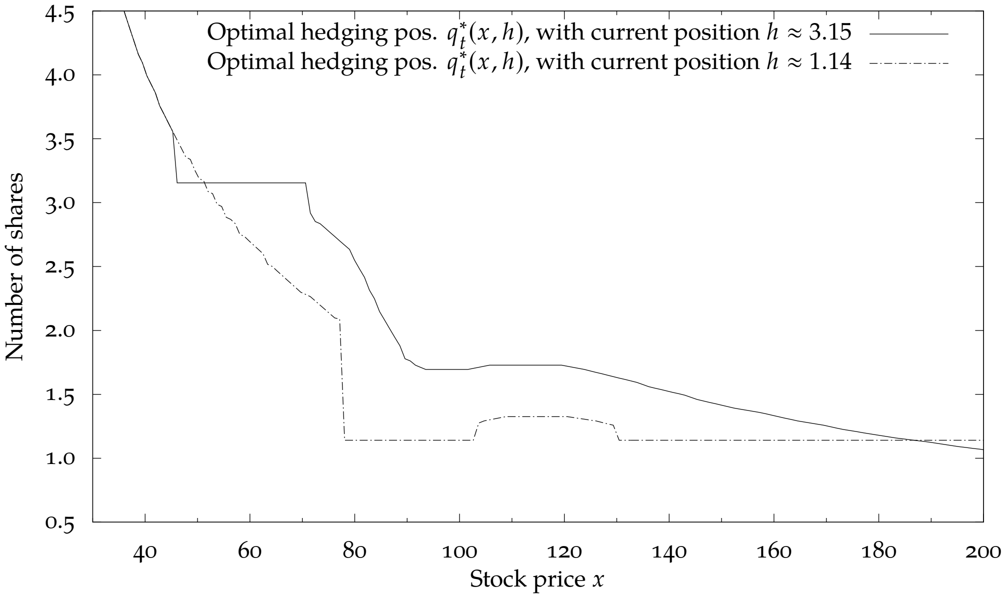

Figure 3.1: Optimal hedging position at time $t = T - 11$ days, $q_t^*(x,h)$, for different stock prices x and two different current positions h. It is optimal not to rebalance if the stock prices stay in regions with $q_t^*(x,h) = h$.

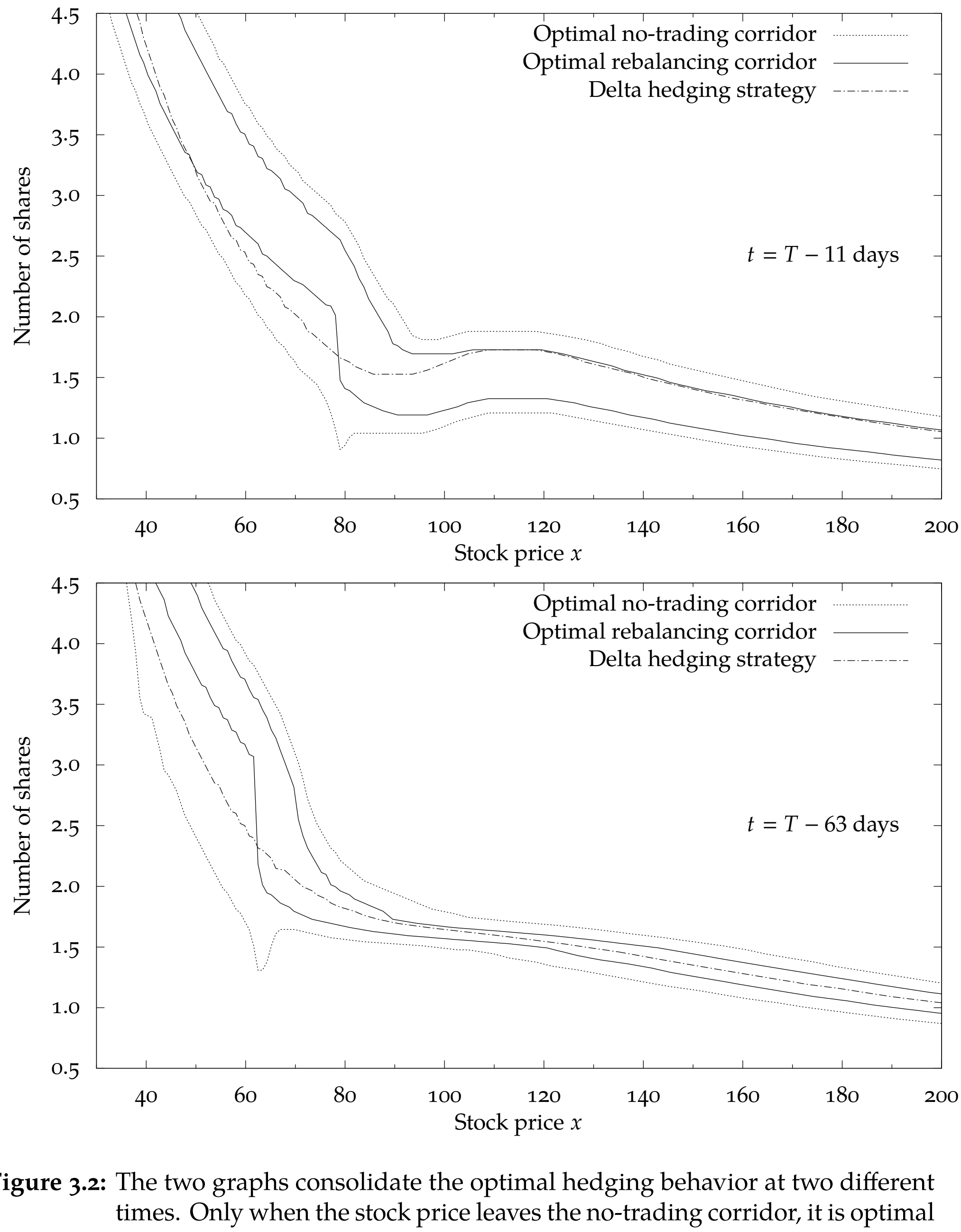

Figure 3.2: The two graphs consolidate the optimal hedging behavior at two different times. Only when the stock price leaves the no-trading corridor, it is optimal to rebalance. The new optimal position is then given by the nearest point of the rebalancing corridor. For comparison the delta hedging strategy, $\Delta_t(x) + Z_t(x)$, is also shown.

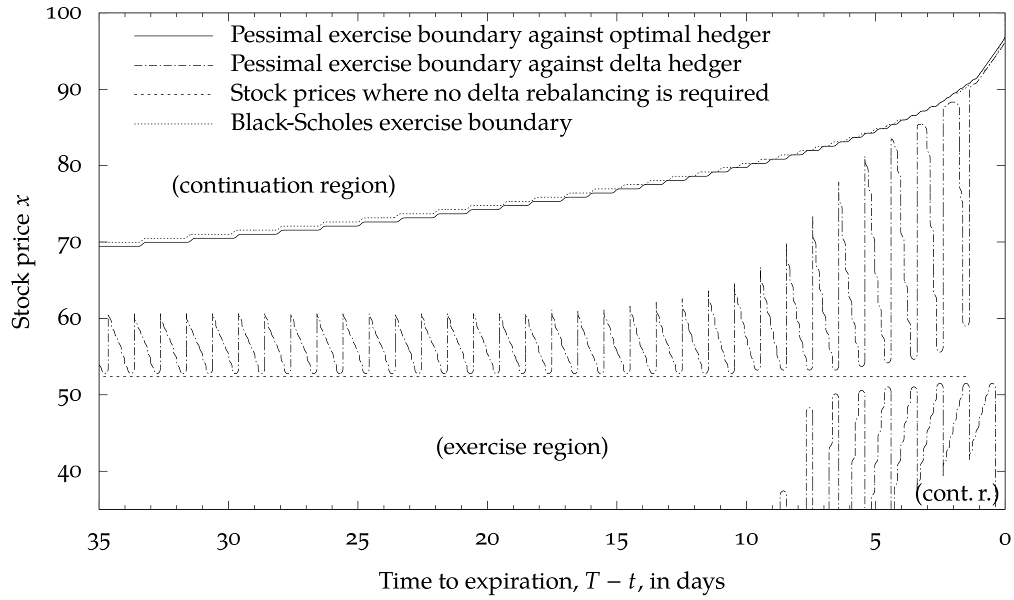

Figure 3.3: Worst-case exercise behavior of the holder against optimal and delta hedging writer with hedging position $h \approx 3.02$, described by boundaries separating exercise and continuation regions. We marked stock prices x that satisfy $\Delta_t(x) + Z_t(x) = h$, i.e. where the delta hedger does not need to rehedge (cf. subsection 3.5.3). For comparison the Black-Scholes exercise boundary is also shown.

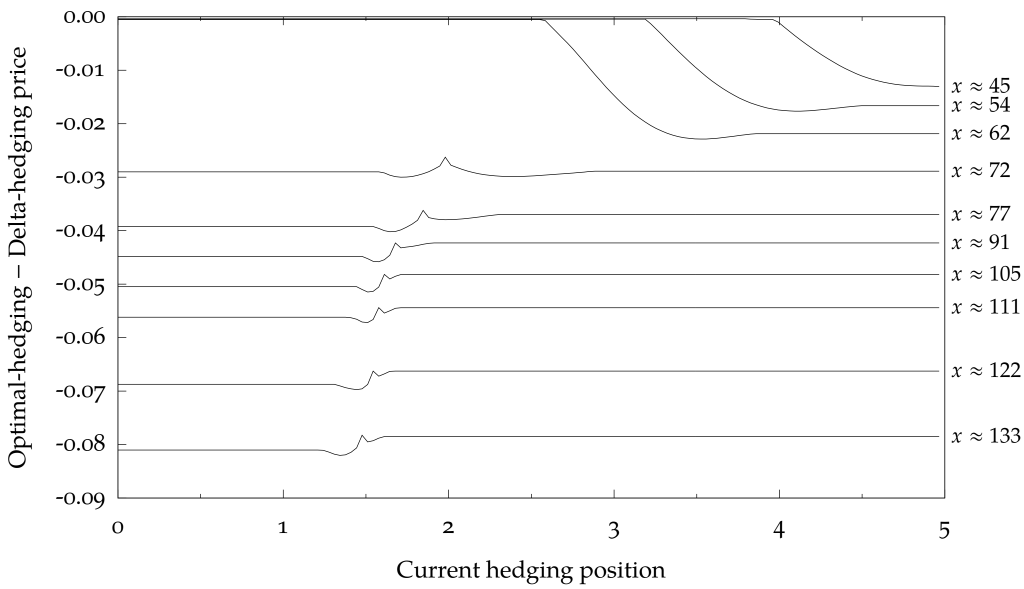

Figure 3.4: Price reduction achieved by optimal hedging compared to delta hedging, calculated at $t = T - 63$ days. Each line corresponds to a fixed stock price x and varying current hedging positions.

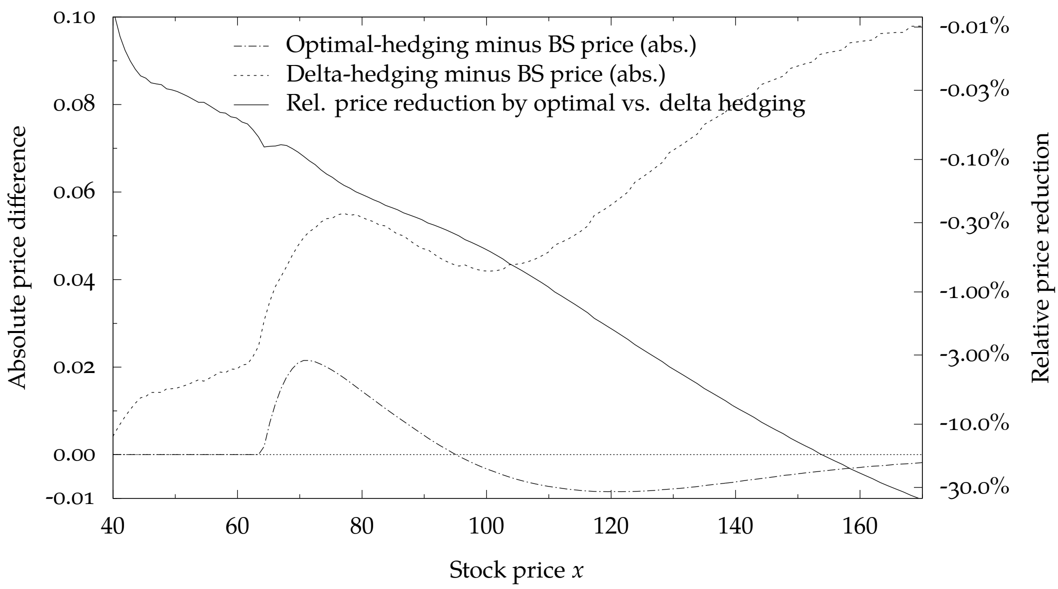

Figure 3.5: Comparison of normalized optimal-hedging, delta-hedging and Black-Scholes price at time $t = T - 63$ days with current hedging position $h \approx 4.83$ for different stock prices x.

4

A NOTE ON UTILITY INDIFFERENCE PRICING

(Joint work with Gregor Dorfleitner. Electronic version of an article published in International Journal of Theoretical and Applied Finance ***19****(6), 2016, p. 1650037,* `doi:10.1142/S0219024916500370` *© World Scientific Publishing Company.* `http://www.worldscientific.com/worldscinet/ijtaf`*.)*

Abstract

Utility-based valuation methods are enjoying growing popularity among researchers as a means to overcome the challenges in contingent claim pricing posed by the many sources of market incompleteness. However, we show that under the most common utility functions (including CARA and CRRA), any realistic and actually practicable hedging strategy involving a possible short position has infinitely negative utility. We then demonstrate for utility *indifference prices* (and also for the related so-called *utility-based (marginal) prices*) how this problem leads to a severe divergence between results obtained under the assumption of continuous trading and realistic results. The combination of continuous trading and common utility functions is thus not justified in these methods, raising the question of whether and how results obtained under such assumptions could be put to real-world use.

4.1 Introduction

In recent years, utility indifference pricing, initially proposed by Hodges and Neuberger (1989) and refined by Davis et al. (1993), has gained much attention in the literature on pricing and hedging contingent claims (see Henderson and Hobson (2009) for a survey). Utility indifference pricing employs the tools of continuous time finance, combining the ideas of Black and Scholes (1973) and utility theory. However, as we will demonstrate, the assumptions used by many contributions miss an important step in the link between the idealistic model and reality. Their results cannot be applied practically and it is unclear what insight they could provide for practical problems.

There are several sub-strands of literature which are affected by our findings, namely indifference pricing and hedging in incomplete markets in general, including Barrieu and El Karoui (2009), Biagini and Frittelli (2005), Collin-Dufresne and Hugonnier (2007, 2013), Duffie et al. (1997), Frei and Schweizer (2010), Grasselli and Hurd (2007), Henderson and Hobson (2011), Henderson et al. (2014), Hu et al. (2005), Kramkov and Sîrbu (2006), Malamud et al. (2013), Mania and Schweizer (2005), Musiela and Zariphopoulou (2004a,b), Rheinländer and Steiger (2010), and Svensson and Werner (1993), and the duality methods for the underlying optimization problem, including Becherer (2004), Delbaen et al. (2002), Frittelli (2000a,b), İlhan et al. (2005), İlhan and Sircar (2006), Kabanov and Stricker (2002), Kallsen and Rheinländer (2011), Monoyios (2006), and Rouge and El Karoui (2000). Also contributions on trading restrictions and substitute hedging, like basis risk, basket/index options, and real options, including Becherer (2003), Davis (2006), Frei and Schweizer (2008), Henderson (2002, 2007), Henderson and Hobson (2002a,b), Karatzas and Kou (1996), and Monoyios (2004b) are subject to our results, as well as studies on employee stock options (Grasselli and Henderson 2009; Henderson 2005; Leung and Sircar 2009a,b; Rogers and Scheinkman 2007) and on transaction costs in derivatives pricing (Barles and Soner 1998; Constantinides and Zariphopoulou 1999; Davis et al. 1993; Davis and Yoshikawa 2015; Davis and Zariphopoulou 1995; Hodges and Neuberger 1989; Mohamed 1994; Monoyios 2003, 2004a).

Inspired by indifference pricing, and likewise affected, are contributions on so-called *utility-based prices*, also known as *neutral or shadow prices*, and their marginal version. Contributions not already mentioned above include Hugonnier et al. (2005), Kallsen and Kühn (2004, 2006), Kramkov and Hugonnier (2004), and Owen and Žitković (2009).

The assumption of continuous trading allowed Black and Scholes (1973) to derive a unique option price based on arbitrage arguments that make no assumptions about the market participants' preferences. This ground-breaking idea was developed further towards the concept of a complete market, in which any derivative can be perfectly replicated by continuously trading in the underlyings and thus uniquely priced by arbitrage arguments (see e.g. Delbaen and Schachermayer 1994).

Of course, in practice the time between two hedges is finite. Let us assume that for a *practicable trading strategy*, this time cannot be shorter than some $\delta > 0$. The idealistic assumption of continuous trading is justified by the fact that a continuous strategy H can be approximated to arbitrary accuracy through practicable strategies H_δ with smaller and smaller δ. The proceeds of the self-financing strategy H in a market with discounted price process S are given by the stochastic integral $H \cdot S$, whose mere definition (see e.g. Bichteler 2002) guarantees the existence of H_δ with $H_\delta \cdot S \to H \cdot S$ as δ tends to zero. However, stated differently, for very small values of δ the results do not depend on the particular value of δ and its influence can be neglected. Therefore, working in the limit $\delta \to 0$ is a justified means to achieving clearer and more general results.

Another classical problem in quantitative finance is that of optimal intertemporal portfolio selection. What is optimal is determined by the investor's preferences — even in complete markets — and is usually modelled using von Neumann–Morgenstern utility. Following the same rationale as above, the assumption of continuous trading is still very appealing and thus in widespread use in the field initiated by Merton (1969, 1971)[1], who pioneered in solving the problem for continuous consumption and trading.

These two areas of research have long been considered as being quite distinct from each other. However, in reality markets are not complete and the unique price broadens into an entire range of arbitrage-free prices. It has been shown in many cases, Davis and Clark (1994) and Soner et al. (1995) for proportional transaction costs and Cvitanić et al. (1999) for unbounded stochastic volatility that this range includes the trivial buy-and-hold strategy that dominates the claim. These examples show that arbitrage-free pricing fails to explain the substantially lower prices observed in options markets and a different pricing methodology is needed.

Acknowledging that pricing of unhedgeable risk has to take the agent's preferences into account, Hodges and Neuberger (1989) proposed combining the two above methods. After Davis et al. (1993) gave a rigorous treatment of their idea, it slowly gathered traction and matured into what is known today as *(utility) indifference pricing and hedging*. The agent following a hedging strategy H assigns the expected utility

$$U(X;H) := \mathbb{E}[u(H \cdot S + X)]$$

to every final wealth X, where u is her utility function. Being able to choose hedging strategies from a given set of admissible strategies $\mathcal{H}$ the highest achievable utility is:

$$\overline{U}(X;\mathcal{H}) := \sup_{H \in \mathcal{H}} U(X;H).$$

Most of our results do not depend on a specific definition of $\mathcal{H}$. We will indicate whenever a specific definition is required.

[1] See Merton (1973a) and Sethi and Taksar (1988) for errata.

The *indifference buy price* $p = p(X, W; \mathcal{H}) \in \mathbb{R}$ for a claim X is defined such that the hedging agent with initial random endowment W is indifferent between doing nothing and buying the claim for that price:

$$\overline{U}(W + X - p; \mathcal{H}) = \overline{U}(W; \mathcal{H}). \tag{4.1}$$

A number $b \in \mathbb{R}$ is in the set of *utility-based prices*, $B(X, W; \mathcal{H})$, if it is optimal not to trade the claim at price b:

$$\overline{U}(W; \mathcal{H}) \geq \overline{U}(W + q(X - b); \mathcal{H}), \quad \text{for all } q \in \mathbb{R}. \tag{4.2}$$

One typically looks at $B(X, x + qX; \mathcal{H})$ with initial wealth $x \in \mathbb{R}$ and initial quantity $q \in \mathbb{R}$ in the claim X. Elements of $B(X, x; \mathcal{H})$ are called *marginal prices* of X.

The contribution of this work is to raise and answer the question of whether the assumption of continuous trading is still justified in this setting. As we will argue in the next sections, the answer is *no* if u is exponential or u' approaches ∞ at some finite wealth. This concerns, for example, large part of HARA utility functions (including the logarithmic limiting case) and thus a vast amount of published literature on the topic. The results raise doubts about the use of such utility functions for indifference pricing and related fields in general.

In section 4.2 we show that the assumption fails in cases where the optimal strategy includes shorting one or more assets. Interestingly, this is also what differentiates Merton's portfolio problem from indifference pricing: when solving the problem of "lifetime portfolio selection", negative excess returns or negative equity risk premiums for risky assets would be considered implausible. Thus, the optimal strategy is *long*. However, in indifference pricing the optimal hedging strategy against a shorted put is *short*—in accordance with intuition and other theories. Even though, both areas exhibit this problem on the formal level, it only negatively impacts the practical relevance of the more recent theory of indifference pricing.

Sections 4.3 and 4.4 use the example of a dynamically replicable claim to demonstrate how indifference pricing and utility-based pricing, respectively, are affected by the results in section 4.2. Section 4.5 discusses some failed resolutions attempts and gives a first characterization of utility functions not suffering from these difficulties. We conclude with section 4.6.

4.2 Failure of the continuous trading assumption

The discounted price process of the risky asset is given by S, a real valued semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. A trading strategy H is an element of $L(S)$, the set of predictable S-integrable processes. The self-financing result of H is given by the stochastic integral $H \cdot S$. Payoffs of claims are modeled by $\mathcal{F}$-measurable random variables. Comparison operators on random variables are understood in the $\mathbb{P}$-almost sure sense. We will need the set of *practicable* hedging strategies $\mathcal{P}$, where rebalancing happens at a finite number of fixed times and positions have to be closed if the wealth at the time of trade drops below a certain negative threshold.

Definition 4.1 (Practicable hedging strategies). *A strategy $H \in L(S)$ is in $\mathcal{P}$, if there exists an increasing finite sequence of fixed times $\{t_i\} \subset [0, \infty)$ and an $M \in \mathbb{R}$, such that*

$$H = \sum_i h_i \, \mathbb{1}_{(t_i, t_{i+1}]},$$

where for any i, h_i is an $\mathcal{F}_{t_i}$-measurable random variable, and

$$H \cdot S^{t_i} \leq -M \Rightarrow h_i = 0 \tag{4.3}$$

holds almost surely. ($S_t^{t_i} := S_{\min(t_i, t)}$ is the price process stopped at t_i.)

By keeping in mind that the above definition includes strategies of several billion possible trades per second, it is obvious that our definition of *practicable strategies* is in no way a restriction of what is actually possible in practice.

It is also this concept of practicable strategies that is used in academic publications and text books to motivate the definition and use of continuous-time concepts.

Remark 4.1. We leave it to future research to extend the following arguments to handle a more general definition of $\mathcal{P}$, where trades happen at stopping times and the minimal time between two trades is finite.

The restriction in eq. (4.3) was introduced to exclude so-called doubling or martingale strategies. It is far less restrictive than margin requirements found in practice which are constantly monitored and enforced and usually limit the position size to a multiple of the available collateral.

Remark 4.2. Pricing theories, where continuous trading is allowed, exclude such unwanted strategies by the definition of $\mathcal{H}$, which usually requires $H \cdot S^t$ to be bounded from below for all t. For practicable strategies, however, this is too restrictive, as it would disallow any short position. Only the (unrealistic) assumption of continuous trading makes it possible to limit the unbounded risk of a short position.

Furthermore, we define the set $\mathcal{S}$ of practicable hedging strategies that include a possible *short position*.

Definition 4.2. $\mathcal{S} = \{H \in \mathcal{P} \mid \exists\, t : \mathbb{P}(H_t < 0) > 0\}$

For the rest of this section, we make the following assumption about the utility function and market model which covers many common settings and is enough to prove the failure without much effort.

Assumption 4.1. *We assume the utility function, $u : \mathbb{R} \to \overline{\mathbb{R}}$, is a concave and non-decreasing function. Furthermore, we assume one of the following cases:*

Case 1. *(a)* $u(x) = -e^{-\gamma x}$ *for some* $\gamma > 0$ *and*
(b) the distribution of the stock price has heavy right tails, i.e.

$$\lim_{x\to\infty} e^{\lambda x}\mathbb{P}(S_t > x|\mathcal{F}_s) = \infty, \textit{for any } \lambda > 0 \textit{ and } t > s.$$

Case 2. *(a)* u *satisfies the Inada condition*

$$\lim_{x\downarrow B} u'(x) = \infty \textit{ for some } B \in \mathbb{R}$$

and (b) $S_t - S_s$ *is unbounded for any* $t > s > 0$.

Remark 4.3 (Case 1). Condition (b) is even satisfied for log-normally distributed S_t. As a consequence, practically all continuous-time models and models for stock returns used in literature possess heavy tails in the sense of Case 1(b).

Empirical evidence suggests (Ibragimov et al. 2015) that the distributions of the *logarithms* of returns are heavy tailed, a requirement going much further than the above condition.

Remark 4.4 (Case 2). Utility functions covered by Case 2(a) include, among others, all utility functions with hyperbolic absolute risk aversion (HARA) and exponent $\gamma \in (-\infty, 1)$:

$$u(x) = \frac{1}{\gamma}((x - B)^{\gamma} - 1). \tag{4.4}$$

In the limit $\gamma \to 0$, this also includes the logarithmic utility function $u(x) = \ln(x - B)$. Setting $B = 0$ yields utility functions with constant relative risk aversion (CRRA), also called isoelastic utility functions.

All of these functions have domains $D \subseteq [B, \infty)$. We will use their *unique* concave extensions given by $u = -\infty$ on $\mathbb{R}\backslash D$. Without extension, the original utility functions could not be used at all due to the possibility of unbounded losses incurred by short positions in the underlying. Of course, such an extension does not alter any pre-existing results.

The core of the problem is that under Assumption 4.1, any practicable strategy with possible short positions has infinitely negative utility.

Theorem 4.1. *Consider a payoff* X*, bounded from above, and a strategy* $H \in \mathcal{S}$. *If* $U(X; H) = \mathbb{E}[u(H\cdot S + X)]$ *exists, then its value is* $-\infty$.

Proof. The expectation of a random variable Y taking values in $\overline{\mathbb{R}}$ is defined (e.g. by Doob 1994, VI.4.) as $\mathbb{E}[Y^+]-\mathbb{E}[Y^-]$, if one of these non-negative expectations is finite, where $Y^{\pm} := \max(\pm Y, 0)$. Thus, it suffices to show $\mathbb{E}[u(H\cdot S + X)^-] = \infty$.

Without loss of generality, we assume the strategy is short on $(0, \tau]$, with $H_\tau = -h$ for some real $h > 0$. Other cases can be handled by taking expectations conditioned on the instant of time of opening the first short position.

Now take an arbitrary real number $x > M/h + S_0$ with M from Definition 4.1 and prove:

$$\begin{aligned}\mathbb{E}[u(H\cdot S + X)^-] &\geq \mathbb{E}\left[\mathbb{1}_{S_\tau > x}\, u(h(S_0 - S_\tau) + X)^-\right] \\ &\geq \mathbb{E}\left[\mathbb{1}_{S_\tau > x}\, u(h(S_0 - x) + \sup X)^-\right] \\ &= u\big(h(S_0 - x) + \sup X\big)^- \mathbb{P}(S_\tau > x). \end{aligned} \tag{4.5}$$

The first inequality uses the fact that $Y \geq \mathbb{1}_{S_\tau > x}\, Y$ for any positive Y. Furthermore, it applies the restriction from eq. (4.3): If $S_\tau > x$, then $H\cdot S^\tau = h(S_0 - S_\tau) < -M$ and thus all hedging positions after τ are zero. The second inequality uses the monotonicity of u^- together with $X \leq \sup X$ and $h(S_0 - S_\tau) < h(S_0 - x)$, if $S_\tau > x$.

Taking the limit $x \to \infty$ of eq. (4.5) in *Case 1* yields due to $\sup X < \infty$:

$$\mathbb{E}[u(H\cdot S + X)^-] \geq e^{-\gamma(hS_0 + \sup X)} \lim_{x\to\infty} e^{\gamma h x}\mathbb{P}(S_\tau > x) = \infty.$$

In *Case 2* due to $\sup X < \infty$, we can choose an x such that $h(S_0 - x) + \sup X < B$ and thus $u(h(S_0 - x) + \sup X)^- = \infty$ by u's concavity (see also Remark 4.4). The unboundedness of S_τ implies $\mathbb{P}(S_\tau > x) > 0$ and consequently eq. (4.5) proves $\mathbb{E}[u(H\cdot S + X)^-] = \infty$. □

Remark 4.5. Theorem 4.1 considers only X that are bounded from above, which includes, for example, the common cases of long or short put options and short call options.

A more general version would complicate the proof. In particular applications, however, the requirement of boundedness can be significantly relaxed without much effort. Theorem 4.1 holds as long as X does not exhibit asymptotic long exposure in the asset S, ensuring that X does not effectively neutralize a short hedging position for large values of S.

An immediate consequence of Theorem 4.1 is that short strategies are practically forbidden for the agent and excluded from the utility optimization over any set $\mathcal{H}$ of hedging strategies, that is

$$\overline{U}(X;\mathcal{H}) = \overline{U}(X;\mathcal{H} \setminus \mathcal{S}). \tag{4.6}$$

Theorem 4.1 contradicts the fact that short positions are used in practice and consequently Assumption 4.1 has to be rejected.

Furthermore, it implies that the assumption of continuous trading is not justified: Consider a continuous strategy $H \in L(S)$ that has a positive probability of a short position. Any sensible practicable approximation $K \in \mathcal{P}$ of this strategy will then be an element of $\mathcal{S}$ and thus have infinitely negative utility.

This affects, for example, the standard approximation for strategies with continuous paths, as used in Monoyios (2004b):

$$K\cdot S = \sum_i H_{t_i}\big(S_{t_{i+1}} - S_{t_i}\big), \text{ for some } (t_i) \in \mathbb{R}^n .$$

In fact, approximating the optimal continuous-time strategy has strictly lower utility than the zero strategy:

$$U(X;0) > U(X;K), \quad (= -\infty, \text{ by Theorem 4.1}).$$

Or stated differently: the agent will always prefer *not to hedge at all.*

In summary, results obtained under Assumption 4.1 and the assumption of continuous trading are disconnected from what is possible in reality.

4.3 Implications for indifference pricing

In this section, we give simple examples that explicitly demonstrate how things can go wrong under Assumption 4.1 and why neither the price nor the hedging strategy derived under the continuous trading assumption have any meaning for an agent limited to practicable strategies.

For demonstration purposes, we price and hedge dynamically replicable contingent claims. It is well known, that under the assumption of continuous trading (with a suitable $\mathcal{H}$) the utility indifference price of a replicable claim equals the arbitrage-free price for a broad range of utilities.[2]

In this section, we will use the optimal strategy with claim, $H^*(X;\mathcal{H})$ and without a claim, $Z(\mathcal{H})$, defined by

$$\overline{U}(0;\mathcal{H}) = U(0;Z(\mathcal{H})) = U(X - p(X,0;\mathcal{H}); H^*(X;\mathcal{H})).$$

4.3.1 *General observations*

The following notation will be employed: the replicable claim's payoff is given by $\Delta \cdot S + q$, where q is the arbitrage-free price and Δ the replicating strategy, which is an element of the set of admissible continuous trading strategies $\mathcal{H} \subseteq L(S)$. We assume that q equals the continuous-trading utility indifference price.

Physical reality and eq. (4.6) limit the set of admissible strategies to a subset of practicable long-only strategies $\mathcal{K} \subseteq \mathcal{H} \cap \mathcal{P} \setminus \mathcal{S}$. Replacing $\mathcal{H}$ by $\mathcal{K}$ in eq. (4.1) will yield practicable optimal strategies and prices that substantially differ from their theoretical counterparts.

The following examples need a simple consequence of Jensen's inequality.

Lemma 4.1. *For any set $\mathcal{A} \subseteq \{H \in L(S) \mid \mathbb{E}[H \cdot S] \leq 0\}$ it holds $U(0;0) = \overline{U}(0;\mathcal{A})$.*

Proof. $U(0;0) \leq \overline{U}(0;\mathcal{A}) = \sup_{H\in\mathcal{A}} \mathbb{E}[u(H\cdot S)] \leq \sup_{H\in\mathcal{A}} u(\mathbb{E}[H\cdot S]) \leq u(0) = U(0;0)$. □

In our first example, we assume zero initial wealth, $W_1 := 0$, and a claim $X_1 := \Delta \cdot S + q$ bounded from above, $\Delta \geq 0$ and that the stock has zero excess return. Lemma 4.1 entails $Z(\mathcal{H}) = Z(\mathcal{K}) = 0$ and thus $H^*(X_1;\mathcal{H}) = -\Delta$, which

[2] For an early proof see Davis et al. (1993, Theorem 1), or for a more recent presentation Becherer (2003, eq. (3.8)), who calls this *elementary no-arbitrage consistency*.

is negative and by Theorem 4.1 every practicable approximation will have infinitely negative utility. Therefore, in practice it is favorable not to hedge at all. Furthermore, due to this impossibility to hedge, $p_1 := p(X_1, W_1; \mathcal{H}) = q$ is too high. If u is strictly concave, the agent pays too much and strictly decreases her utility when buying the option. We state this result which follows from eq. (4.6) and a strict version of Jensen's inequality without proof:

$$\overline{U}(X_1 - p_1; \mathcal{K}) = \sup_{H \in \mathcal{K}} \mathbb{E}[u((\Delta + H) \cdot S)] < u(0) = \overline{U}(0; \mathcal{K}). \tag{4.7}$$

However, the continuous-trading buy price is not always too high. Take, for example, the indifference price for closing the current risky position X_1. In this case, we have an initial portfolio $W_2 := X_1 - p_1$ and a payoff $X_2 := -X_1$. The continuous-trading price is of course $p_2 := p(X_2, W_2; \mathcal{H}) = -p_1$. For the realistic agent, this price is too low. By closing the position, she is able to eliminate the unhedgeable risk in her current portfolio. With eq. (4.7), we can see that buying X_2 for p_2 strictly increases the agent's utility:

$$\overline{U}(W_2 + X_2 - p_2; \mathcal{K}) = \overline{U}(0; \mathcal{K}) > \overline{U}(X_1 - p_1; \mathcal{K}) = \overline{U}(W_2; \mathcal{K}).$$

Another example, where buying a claim for the continuous indifference price improves the realistic agent's situation, i.e. where the continuous-trading price is again too low, is that of negative excess returns of the stock.

In this case, the optimal continuous strategy $Z(\mathcal{H})$ is short and the optimal practicable strategy is $Z(\mathcal{K}) = 0$ by Lemma 4.1. Now consider the claim with payoff $X_3 := Z(\mathcal{H}) \cdot S$. Of course, its arbitrage-free price is zero and thus $p(X_3, 0; \mathcal{H}) = 0$. However, buying this claim for 0 establishes the optimal short exposure which was not previously available to the practicable agent and thus strictly increases her utility. See Example 4.2 for concrete results in a log-normal model.

The assumption of continuous trading does not fail in situations in which the optimal continuous strategy on both sides of eq. (4.1) is long. One example is the situation where one put is sold from a portfolio whose optimal hedging position is long and only slightly reduced by the sale of one put. See Example 4.3 for concrete results in a log-normal model.

4.3.2 Concrete examples

For all of the following examples, we assume a risk-free rate of zero ($r = 0$), an agent with exponential utility ($u(x) = -e^{-\gamma x}$, with $\gamma = 0.01$), no initial portfolio ($W = 0$) and that the asset price follows a geometric Brownian motion with return μ:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

We use the usual $\mathcal{H} = \{H \in L(S) \mid H \cdot S \text{ bounded from below}\}$ and $\mathcal{K} = \mathcal{H} \cap \mathcal{P} \setminus \mathcal{S}$. This market is arbitrage-free with a unique martingale measure $\mathbb{Q}$ and the corresponding risk-neutral expectation $\mathbb{E}_{\mathbb{Q}}$.

Example 4.1. In the first example, we calculate the sell price of a put on GOOGL with payoff $X = (550 - S_T)^+$ at a current stock price of $S_0 = 600$, with $\mu = 0$ and $\sigma\sqrt{T} = 0.25$. Lemma 4.1 implies $Z(\mathcal{H}) = Z(\mathcal{K}) = 0$. The continuous-trading indifference sell price is given by the risk-neutral expectation of the payoff, $-p(-X, 0; \mathcal{H}) = \mathbb{E}_{\mathbb{Q}}[X] \approx 35.61$, and the optimal strategy $-\Delta$ is given by the Black and Scholes (1973) delta for a put, i.e. short. Hence as in eq. (4.7), the optimal practicable strategy is zero and the indifference sell price is:

$$-p(-X, 0; \mathcal{K}) = \frac{1}{\gamma} \ln \frac{\overline{U}(-X; \mathcal{K})}{\overline{U}(0; \mathcal{K})} = \frac{1}{\gamma} \ln \frac{U(-X; 0)}{U(0; 0)} = \frac{1}{\gamma} \ln \mathbb{E}\left[e^{\gamma X}\right] \approx 58.50.$$

Example 4.2. In the second example, we assume that every trading strategy $H \in \mathcal{H}$ is constant after a fixed time T, and take a look at the Hamilton-Jacobi-Bellman equation for $\overline{U}(0; \mathcal{H})$. Davis et al. (1993, eq. (4.30)) derive the optimal strategy:

$$Z(\mathcal{H})_t = \frac{\mu}{\gamma \sigma^2 S_t}. \tag{4.8}$$

This strategy replicates a claim with payoff $X := Z(\mathcal{H}) \cdot S = \frac{\mu}{\gamma}\left(\frac{T}{2} + \frac{1}{\sigma^2} \ln \frac{S_T}{S_0}\right)$ and optimal practicable utility $\overline{U}(X; \mathcal{K}) = U(X; 0)$, which can be seen from:

$$U(X; 0) \leq \overline{U}(X; \mathcal{K}) \leq \overline{U}(X; \mathcal{H}) = \overline{U}(0; \mathcal{H}) = U(0; Z(\mathcal{H})) = U(X; 0).$$

The continuous indifference price of this claim is $p(X, 0; \mathcal{H}) = \mathbb{E}_{\mathbb{Q}}[X] = 0$. However, in the case $\mu < 0$ and thus $Z(\mathcal{H}) < 0$ and by Lemma 4.1 $Z(\mathcal{K}) = 0$, its practicable indifference price is strictly positive:

$$p(X, 0; \mathcal{K}) = \frac{-1}{\gamma} \ln \frac{\overline{U}(X; \mathcal{K})}{\overline{U}(0; \mathcal{K})} = \frac{-1}{\gamma} \ln \frac{U(X; 0)}{U(0; 0)} = \frac{-1}{\gamma} \ln \mathbb{E}\left[e^{-\gamma X}\right] = \frac{\mu^2 T}{2\sigma^2 \gamma} > 0.$$

Example 4.3. This time we assume a positive excess return $\mu > 0$. According to eq. (4.8), the optimal strategy without any payoff is positive. The optimal continuous trading strategy after selling x put options with exercise price K is given by $H := H^*(-x(K - S_T)^+; \mathcal{H}) = Z(\mathcal{H}) - x\delta$, where δ is the Black and Scholes (1973) delta of the put. Simple analysis shows that $H \geq 0$ if and only if $x \leq \frac{\mu}{\gamma \sigma^2 K} =: x^*$. Consequently, only then can both $Z(\mathcal{H})$ and H be approximated by practicable strategies and the results obtained under the assumption of continuous trading carry over to the practicable case.

Assuming $\mu = 5\%$, $\sigma = 25\%$, and $K = 550$, we obtain $x^* \approx 0.15$, i.e. when selling more than that fraction of a put, the continuous trading price will differ from the practicable price.

We conclude with the observation that optimal continuous strategies and the corresponding indifference prices are not relevant for a realistic agent.

4.4 Implications for utility-based pricing

As in the previous section, we will look at dynamically replicable claims. Under the continuous trading assumption (with a suitable $\mathcal{H}$) the utility-based price of a replicable claim X is unique and given by the claim's arbitrage-free price (c.f. Kramkov and Hugonnier 2004):

$$B(X, W; \mathcal{H}) = \{\mathbb{E}_{\mathbb{Q}}[X]\}.$$

Under Assumption 4.1, this theoretical result is violated for a practical hedger, with strategies $\mathcal{K} \subseteq \mathcal{P} \cup \{0\}$, in various settings, two of which we will give here as examples. In the first setting, let us assume that the optimal hedge against a non-negative number of claims with payoff given by X is not to hedge at all, i.e.

$$\overline{U}(x + qX; \mathcal{K}) = U(x + qX; 0), \text{ if } q \geq 0. \tag{4.9}$$

Then, the following Theorem proves that utility-based prices for realistic agents with strictly concave u holding positive quantities of a claim are strictly lower than the claim's physical expectation value and thus for replicable claims strictly lower than the continuous trading result from above, whenever $\mathbb{E}[X] \leq \mathbb{E}_{\mathbb{Q}}[X]$.

Theorem 4.2. *If a claim X satisfies eq. (4.9), then for any real x and $q > 0$ it holds* $\sup B(X, x + qX; \mathcal{K}) \leq c$ *with*

$$c \equiv \frac{1}{q}\left(u^{-1}\left(\mathbb{E}\left[u(x + qX)\right]\right) - x\right) \leq \mathbb{E}[X].$$

If u is strictly concave, the last inequality becomes "$<$".

Proof. $c \leq \mathbb{E}[X]$ holds due to Jensen's inequality or the corresponding strict version for strictly concave u. Using eq. (4.9) and u's strict monotonicity, we can for any $b > c$ derive a violation of eq. (4.2), which proves that $(c, \infty] \not\subseteq B(X, x + qX; \mathcal{K})$:

$$\overline{U}(x+qX; \mathcal{K}) = \mathbb{E}\left[u(x + qX)\right] = u(x+qc) < u(x+qb) = \overline{U}(x+qX-q(X-b); \mathcal{K}).$$

□

Example 4.4 (Closing a short put position). In the exponential utility setting from Example 4.1 consider an agent holding one shorted put. The marginal price with continuous trading is $B(X, -X; \mathcal{H}) = \{\mathbb{E}_{\mathbb{Q}}[X]\} \approx \{35.61\}$. Yet, using the substitution $X \to -X$, Theorem 4.2 provides a lower bound for $B(X, -X; \mathcal{K})$ of $c \approx 58.50$. The exact value (given without proof) is significantly higher: $B(X, -X; \mathcal{K}) = \{\mathbb{E}[u'(-X)X]/\mathbb{E}[u'(-X)]\} \approx \{87.95\}$.

In the second setting, we assume negative excess returns of all market assets, a certain smoothness of u at the current wealth level and a given set of practicable strategies, $\mathcal{K} \subseteq \mathcal{P} \cup \{0\}$. The following theorem and example show that in this case the marginal price of any bounded (not necessarily replicable) claim is given by its physical expectation, which is again a large deviation from the continuous trading result.

Theorem 4.3. *If $\mathbb{E}[H \cdot S] \leq 0$ for all $H \in \mathcal{K}$, and u' exists at x, then $B(X, x; \mathcal{K}) = \{\mathbb{E}[X]\}$ for all bounded X.*

Proof. Due to $\mathbb{E}[H \cdot S] \leq 0$, Jensen's inequality and u's monotonicity, we have

$$\overline{U}(x + q(X - b); \mathcal{K}) \leq u(x + q(\mathbb{E}[X] - b)), \quad \text{for all } q \in \mathbb{R},$$

and $\overline{U}(x; \mathcal{K}) = u(x)$. This shows, that $b = \mathbb{E}[X]$ fulfills eq. (4.2). Now define $f(q) \equiv U(x + q(X - b); 0)$. Boundedness of X allows us to interchange limit and expectation:

$$\lim_{q \to 0} \frac{f(q) - f(0)}{q} = \mathbb{E}\left[\frac{d}{dq} u(x + q(X - b))\Big|_{q=0} \right] = u'(x)(\mathbb{E}[X] - b).$$

For every $b \neq \mathbb{E}[X]$ this limit is different from 0. Hence, there is some q, such that $\overline{U}(x + q(X - b); \mathcal{K}) \geq f(q) > f(0) = \overline{U}(x; \mathcal{K})$, proving that b violates eq. (4.2). □

Example 4.5 (Marginal put price under exponential utility). Due to Theorem 4.1, a realistic agent cannot short the stocks and thus cannot profit from falling stock prices. However, she can achieve short exposure through put options, and consequently purchasing them even for considerably more than the arbitrage-free price will improve her situation. In the setting of Example 4.4, but with $\mu T = -0.08$, the marginal price with continuous trading is again $B(X, x; \mathcal{H}) = \{\mathbb{E}_{\mathbb{Q}}[X]\} \approx \{35.61\}$, yet Theorem 4.3 gives $B(X, x; \mathcal{K}) = \{\mathbb{E}[X]\} \approx \{52.99\}$.

In analogy to the last section, we demonstrated utility-based prices obtained under the assumption of that continuous trading are too far off to be relevant for realistic agents.

4.5 Discussion

In this section, we address the question of whether there is an easy solution to the problem and give the answer: No.

The first attempt is to employ an economic argument to make the region in which the net position is never short—as in Example 4.3—large enough to cover the cases of interest. Still, outside this region it will always abruptly fail, even in dynamically complete markets. For a pricing and hedging theory this is not satisfactory.

Another attempt is to limit the unbounded losses of short positions through some stopping mechanism. If a stopping time τ exists such that the stopped process S^τ is bounded, then any practicable strategy in S^τ would have finite utility. How can the agent trade in the S^τ? Of course, it can be replicated using a continuous trading strategy and its arbitrage-free price at time t equals S^τ_t. However, under Assumption 4.1 such a strategy is not practicable, neither for the agent nor for the possible issuer of S^τ. The reason for this is that τ can be shorter than any fixed time which also lies at the heart of the problem discussed in this paper. While such products exist in practice (like guaranteed stops or barrier options with knock-out features), they do not resolve the issue. The only scenario in which the spread offered on S^τ enables the agent to approximate a continuous strategy in S, is one where Assumption 4.1 does not hold for the issuer of S^τ. Thus, nothing is gained and the problem not resolved.

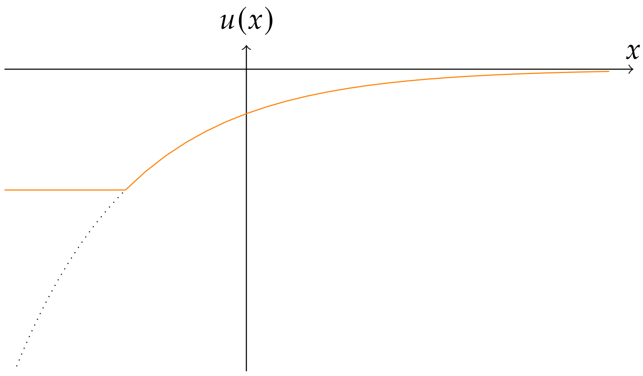

Figure 4.1: Capped exponential utility function.

If limiting the losses is not possible, one alternative could be arguing the fact that an upper bound on the losses always exists, e.g. through limited liability cooperations. Thus, the actor's utility cannot fall below a certain level which effectively corresponds to the situation depicted in Figure 4.1. Such a bound, however, breaks the concavity of the utility function and thus contradicts the risk aversion of the agent, as shown in the following example.

Assume a non-hedging agent is pricing a payoff f with a rare but severe possible loss event, $\mathbb{P}(f = 10.\overline{010}) = 99.9\%$ and $\mathbb{P}(f = -10\,000) = 0.1\%$. Although f has zero expectation, its indifference price under a utility function $u(x) = -e^{-(x^+)}$ is $p = 10.\overline{010}$.

One might ask why the existence of numerical algorithms for continuous trading results does not contradict our findings. After all, most numerical methods like PDE or tree methods somehow discretize the trading strategy (e.g. with mesh size δt) and thus describe a practicable hedging strategy, yet they yield finite results contrary to Theorem 4.1. The reason lies in the additional discretization of the state space (δS) which implies that cases 1(b) and 2(b) from Assumption 4.1 do not apply and the infinities of Theorem 4.1 do not occur. The finite continuum value is then obtained through a simultaneous limit $(\delta S, \delta t) \to 0$. Applying such methods to the practicable case requires keeping δt fixed in the limit $\delta S \to 0$. This limit diverges for short strategies and thus the numerical scheme will again produce the analytically correct, albeit problematic result of $-\infty$.

Last, we would like to address the reports of the seemingly good performance achieved by hedging strategies obtained under Assumption 4.1 (e.g. Mohamed 1994; Monoyios 2004a,b). In these studies, the distribution of the hedging error of different hedging strategies is compared on the basis of statistics such as mean, standard deviation and median. However, first optimizing the utility of the strategy and then looking at another performance measure defeats the purpose. Even if a utility optimizing strategy outperforms others, there will always be a better strategy: the optimal strategy under the measure of interest.

Besides this fundamental problem, the strategies considered in these studies are practicable approximations to the optimal continuous strategy. By Theorem 4.1, such a strategy has infinitely negative utility, if it contains a short position. Still, considering the implementation of such a strategy — e.g. as suggested by the studies above — implies that the agent simply does not have the postulated utility function.

We deem the transition to more suitable functions inevitable. More specifically, a utility function u should not exclude realistic behavior and ideally allow results obtained under the assumption of continuous trading to be applied practically. Obviously, to fulfill these requirements, u must not satisfy Assumption 4.1.

In addition to this necessary condition, we provide a sufficient condition in a simple setting. In accordance with intuition, u has to be defined on $\mathbb{R}$ and should not fall too fast as the wealth approaches negative infinity. Indeed, if S is a square integrable martingale, e.g. a geometric Brownian motion with no drift, it is sufficient for u to have a bounded left derivative. In this case, for any square integrable, replicable claim X there exists a sequence of simple processes $\{q_n\}_n$ that achieve the expected utility of a perfect hedge:

$$U(X; q_n) \xrightarrow[n\to\infty]{} u(\mathbb{E}_{\mathbb{Q}}[X]). \tag{4.10}$$

To see this, we resort to the definition of the stochastic integral, which ensures the existence of a sequence of simple processes $\{q_n\}_n$ such that $Y_n \equiv X - \mathbb{E}_{\mathbb{Q}}[X] + q_n{\cdot}S$ converges in mean square to zero. Now, if the upper bound of u's left derivative is given by C, then due to u's concavity and monotonicity, we get $|u(a) - u(b)| \leq C|a - b|$ and thus

$$|U(X; q_n) - u(\mathbb{E}_{\mathbb{Q}}[X])| \leq \mathbb{E}\left[|u(X + q_n{\cdot}S) - u(\mathbb{E}_{\mathbb{Q}}[X])|\right] \leq C\mathbb{E}[|Y_n|] \longrightarrow 0.$$

One utility function with a bounded derivative is $u(x) = kx - \sqrt{1 + (kx)^2}$ for any $k > 0$. The parameter k determines the absolute risk aversion at $x = 0$. This function exhibits a growing relative risk aversion for positive wealth, asymptotically approaches a constant relative risk aversion coefficient with value 2, and thus makes an economically viable candidate.

4.6 Conclusion

While utility indifference pricing and hedging as well as the so-called utility-based pricing are fruitful approaches in conquering the challenges of incomplete markets, we believe this work demonstrates that the use of utility functions that satisfy Assumption 4.1 contradicts even the most basic practical observations. This has demonstrable consequences for the practical applicability of results obtained under the assumptions of continuous trading. Even if mathematically elegant results rely on the simplicity of these utility functions, we plead for a replacement by more realistic ones.

While the last section should serve as a starting point in that matter, the broader question of general practical applicability of indifference and utility-based pricing will intimately depend on the combination of admissible strategies, the utility function, the price dynamics, the payoff and finally the kind of result to be applied and is left to further research.

We conclude with the managerial implication that the practical implementation of hedging strategies derived from continuous-time utility indifference pricing under Assumption 4.1 leads to far from optimal behavior.

5 THE PRICING EFFICIENCY OF EXCHANGE-TRADED COMMODITIES

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Abstract

Exchange-traded commodities (ETCs) open the commodity markets to both private and institutional investors. This paper is the first to examine the pricing efficiency and potential determinants of price deviations of this new class of derivatives based on daily data of 237 ETCs traded on the German market from 2006 to 2012. Given the unique size of the sample, we employ the premium/discount analysis, quadratic and linear pricing methods, as well as regression models. We find that the ETCs incur, on average, price deviations in their daily trading and are more likely to trade at a premium from their net asset values than at a discount. In addition, we examine the influence of certain factors such as management fees, commodity sectors, issuers, spread, assets under management, investment strategies, replication and collateralization methods on quadratic and linear price deviations.

5.1 Introduction

Exchange-traded commodities (ETCs) have evolved into a significant financial instrument within the commodity asset class. They were first introduced by Investor Resources Limited under its founder Graham Tuckwell in 2003 and listed on the Australian Securities Exchange. Marking a cornerstone for the development of commodity investing, they are designed to provide both institutional and private investors with exposure to a range of investment possibilities, from well-known commodities, such as gold, silver, or platinum, over exotic ones, such as lean hogs (e.g. Brooks 2008), to commodity futures or commodity indices. ETCs are open-ended passive derivative instruments that are listed on an exchange and traded like shares. This article is the first to explore the pricing efficiency of ETCs and to examine potential determinants of price deviations for the case of the German market which is the most important one for ETCs in Europe.

ETCs undoubtedly belong to the group of exchange-traded products (ETPs), which also include exchange-traded notes (ETNs) and exchange-traded funds (ETFs). ETCs, that sometimes are also called commodity ETFs, all share the following features: First, they are open-ended investments that are listed and continuously traded like shares on a stock exchange. Second, they are passive investments that track the performance of a given benchmark. Third, they use either a physical or synthetic replication method. Despite these common characteristics and the grouping as "exchange-traded" to increase the popularity of ETCs and ETNs in light of the success of ETFs, a clear distinction has to be made due to many structural and regulatory differences. ETCs are debt securities that enable investors to gain exposure to commodity markets without the requirement of physical delivery or futures trading. According to Lang (2009), they are undated and normally secured zero-coupon notes from a legal point of view. ETNs are also debt securities based on the performance of references outside the commodity sector, such as currencies or volatilities; however they are, unlike ETCs, generally non-collateralized and therefore bear the default risk of the issuer. By contrast, ETFs are collective investment funds, based on the performance of literally all held assets and are subject to strict regulatory requirements of the UCITS[1], which do not permit the replication of single commodities or less diversified indices.

For investors, ETCs are a means of gaining exposure to commodity returns. Therefore, the classification of ETCs with regard to investable resources may be based on the common classification of commodities. Even though commodities share unique investment characteristics separating them as a distinct asset class, there is a notable lack of homogeneity among the different types of commodities. In accordance with Engelke and Yuen (2008) and Fabozzi et al. (2008), commodities can traditionally be summarized as hard or soft commodities depending on

[1] The acronym UCITS stands for "Undertakings for Collective Investment in Transferable Securities" directive and is the regulatory framework for an investment vehicle that can be marketed across the European Union.

their degree of availability and perishability and further be categorized into five commodity sectors (agriculture, livestock, precious metals, industrial metals, and energy).

The first two sectors, agriculture and livestock, are among the soft commodities, which can be characterized as renewable, perishable, non-limited in quantitiy and typically grown products for consumption. In terms of ETCs, the agricultural sector comprises diverse sub segments such as corn, wheat, cotton, coffee, soybeans, soybean oil, cocoa, and sugar, whereas the livestock sector mainly bifurcates into live cattle and lean hogs.

The latter three sectors, precious metals, industrial metals and energy, can be considered as hard commodities which are non-renewable, non-perishable, limited in quantity, and typically extracted by a mining process or obtained from a non-agricultural source. Examples of investable precious metals are mainly gold, palladium, platin, rhodium, and silver. Moreover, industrial metals can be split into aluminium, lead, copper, nickel, zinc, and tin while the energy sector provides gasoline, fuel oil, crude oil, natural gas, and electricity as subsegments.

It is not only the limited attention of the academic literature to date, but also the positive outlook for the market of passive investment and the commodity sector in general (ibid.) which provide a motivation for our comprehensive analysis of ETCs. By identifying the reasons for the expected ongoing popularity of ETCs and other ETPs with regard to the investment horizon, Bienkowski (2007) mentions the easy access to the commodity markets previously reserved for sophisticated investors, the high liquidity, flexibility, transparency and appealing cost structure as major drivers.

A key consideration in the investigation of ETCs is the creation/redemption mechanism and the unique trading mechanism that ETCs have in common with ETFs and ETNs, which require a distinction between primary and secondary markets. In the primary market, ETC shares can be created and redeemed on an on-demand basis by the so-called authorized participants (APs). The issuer of an ETC is a special purpose vehicle (SPV) in the legal form of a limited liability company or a limited partnership created for the sole purpose of issuing ETCs and liable under the law of its incorporated country. APs are large financial institutions, brokers, or approved market makers that are contractually entitled to solely serve this role and directly operate with the SPV. For the creation of ETC units, the APs transfer securities or cash at the issuer's deposit in exchange for a block of a given number of ETC shares, often called "creation unit", which they split for a secondary market sale. The subscription price per unit is determined by the intrinsic value, the net asset value (NAV), which is calculated on a daily basis depending on the official price of the underlying asset. The process operates in reverse if ETC shares are redeemed. Due to this on-demand creation/redemption mechanism, ETCs can be defined as open-end investments.

The secondary market mainly takes place on stock exchanges[2] where the APs purchase and sell the ETC shares. Investors can trade and settle them at a

[2] Besides a stock market trading, over-the-counter (OTC) trading may also be possible.

price determined by the best bid and best ask within a defined spread during trading hours while market makers provide liquidity all day.

The existence of the creation/redemption process ensures that the price of the ETC shares is close to the NAV of the primary market (see Borsa Italiana 2009). Otherwise, Gastineau (2001) suggests the APs could exploit arbitrage opportunities. When ETC prices are lower than their respective NAVs, the APs will acquire the underlying securities and redeem ETC shares and vice versa.[3]

However, as in reality market imperfections exist, the factual pricing efficiency of theses derivative instruments is an important issue for investors and researchers. In our analysis we determine whether there are deviations between the prices of ETCs and their respective NAVs and consider 237 ETCs traded in Germany, which is the largest market for such ETCs in the euro zone.

Compared to previous studies on ETPs, this is one of the largest samples to analyze. Thus, we contribute to the literature not only by introducing a new asset class, but also by providing a data set of unique size and regional focus to further extend existing research of exchange-traded products. Despite its short market history of about six years and the occurrence of the financial crisis, we can identify a tremendous growth in the ETC assets under management (AUM) by more than a factor 10 from EUR 164 million up to EUR 23 096 million as well as in the number of products increasing by eight times from 31 to 276 products from November 2006 to June 2012.[4] With regard to the European ETC AUM by the end of 2011, we find a market share of nearly 70 % of the German market with AUM of EUR 19 951 million. In terms of turnover as of 2012, the German stock exchange "Deutsche Börse AG" is also the market leader with a turnover of EUR 7598 million for ETCs in the Euro area (Lan et al. 2013), followed by the "Börse Stuttgart AG".

Given these facts, we concentrate on the German market for the investigation of the ETCs from the perspective of a euro investor. We employ a number of various approaches: the premium/discount analysis (PD analysis), quadratic and linear pricing efficiency measures, and regression analysis. We first investigate the existence of price deviations based on daily figures of the ETCs under consideration and subsequently analyze potential influencing factors of these deviations. We find, on average, for all different price measures significant pricing deviations from theoretical fair values in the daily trading of ETCs. Aiming to detect influencing factors of the pricing mismatch, we run several multiple ordinary least squares (OLS) regressions which could explain the potential arbitrage opportunities of investors.

The remainder of the paper is organized as follows. We commence with a discussion of related literature, and then describe the data and methodology

[3]Thus, the APs operate as an important link between the secondary and primary market from which retail investors are usually excluded. The settlement by an independent clearing and settlement organization takes place on a normal $T + 2$ or $T + 3$ basis. In summary, a clear distinction between the primary and the secondary market including its market participants is crucial for a correct understanding of the whole ETC structure.

[4]The above given information are provided by the German stock exchange "Deutsche Börse AG".

we use in our empirical analysis. The next section introduces the variables and hypotheses developed as part of the regression analysis. Subsequently, we present and discuss our empirical results. Finally, a conclusion completes the paper.

5.2 Related literature

Since a number of studies is relevant for our analysis, we next discuss a selection of publications in the fields passive management, commodities in general, and ETPs.

Many authors are concerned with a general discussion about active and passive management approaches. Major studies by Jensen (1967), Lehmann and Modest (1987), Malkiel (1995), Gruber (1996), and Rompotis (2011a) are not able to find an outperformance of active investment solutions when compared to market indices, passively managed mutual funds or exchange-traded products. With a particular focus on commodities, Mankiewicz (2009) undertakes a comparative analysis between the active and passive management of commodity indices with regard to pension funds and discusses the suitability of passive financial instruments such as ETCs as alternative sources of return in a theoretical framework. Plante and Roberge (2007) describe the benefits of passive commodity investing relative to active approaches and find that theoretical sources of returns such as return on collateral and excess return of the GSCI index between 1970 and 2006 can be realized as actual returns.

Another fast-growing class of literature has shown substantial interest in commodities since the beginning of an increasing investor demand due to financial and sovereign crises and inflation fears. Fabozzi et al. (2008) as well as Anson et al. (2011) identify investment characteristics of commodities differentiating them from traditional asset classes like stock or bonds. Gorton and Rouwenhorst (2006) examine both a negative correlation between commodity futures and other asset classes like shares and bonds due to different behavioral patterns in the business cycle as well as a positive correlation with expected and unexpected inflation and changes in expected inflation. Several authors investigate the diversification benefits of commodities in a traditional portfolio consisting of stocks and bonds using different methods with different findings (e.g. Anson 1999; Belousova and Dorfleitner 2012; Bodie 1983; Stoll and Whaley 2010).

The literature has begun to cover the topic of exchange-traded products as financial innovations. Laying the foundations for further research approaches, Gastineau (2001) is the first to analyze ETPs in his study about the characteristics, mechanics and benefits of ETFs. Other follow-up studies provide an overview of ETFs (e.g. Deville 2008; Gastineau 2010) and ETNs (e.g. Wright et al. 2010) in great detail.

Since before the advent of ETPs, there has been another class of investment instruments that provide exposure to indices or other difficult-to-trade underlyings or with exotic features like principal protection or discounts. Structured products, like market-index certificates of deposits, discount certificates and reverse convertibles, which are issued by financial institutions, are derivative products, that are made up from more basic assets and derivatives. The literature concerned with the pricing of such products looks at deviations from the fair price, which is given by the capital required to set up a static hedge in exchange-traded derivatives. Chen and Kensinger (1990) were the first to note the severe mispricing, which could be traced back to the profit maximizing behavior of issuers that make rational use of their quasi monopoly (Grünbichler and Wohlwend 2005; Muck 2006; Wilkens et al. 2003). Another driver is hedging difficulty, which is for example higher for single stock underlyings compared to index underlyings, that have more liquid derivatives markets (Stoimenov and Wilkens 2005). Wallmeier and Diethelm (2009) also provided evidence for behavioral effects, like the irrational preference for overpriced products as long as they offer high coupons.

From the investor's perspective, structured products and ETCs serve a similar purpose, and pricing efficiency is an important topic in both classes. While the characteristics and drivers of the pricing anomalies are expected to be similar, the innovative, simpler and more transparent structures and mechanisms behind ETCs—devised in part to overcome the shortcomings of structured products—create the need for studies devoted to the peculiarities of ETCs.

However, we can only find incomprehensive studies that either focus on individual characteristics or are insufficient in terms of an in-depth review. Bienkowski (2010), for example, mainly presents a description of the development of commodity investments and addresses ETCs, especially oil ETCs and their various product strategies (long, short, forward, and leveraged positions) very briefly. In a further study with a sole focus on ETCs, Bienkowski (2007) depicts the backgrounds of the origins, the main advantages, and the general market development of ETCs based on assets under management and the number of existing products. In a similar introduction of ETCs, Brooks (2008) finds the predominance of precious metals ETCs in his global market analysis by sector and highlights the revolutionary role of ETCs in the opening of commodity markets to all investors. Despite the limited research on ETCs, many properties relating to ETPs, such as the creation/redemption process for the issuance and redemption of units (e.g. Gastineau 2001, 2010) are well explained in ETP literature. So far there is no study exploring the pricing efficiency of ETCs systematically.

The literature, then taking an empirical perspective on passive financial instruments, is often dedicated to various forms of price differences. Charupat and Miu (2011) distinguish between pricing efficiency and tracking errors in their study on leveraged ETFs. They describe pricing efficiency as the relationship between an ETF's prices and its respective net asset values while tracking errors refer to the ability of an ETF to replicate the underlying benchmark's return in the ETF's NAV return. In the view of ETFs, authors (e.g. Aber et al. 2009; Charupat and Miu 2011; Engle and Sarkar 2006; Jares and Lavin 2004; Kayali and Ozkan 2012; Lin and Chou 2006) analyze the relative price differences between the price and its net asset value in the so-called PD analysis.

Other publications by Kostovetsky (2003), Gallagher and Segara (2006), Rompotis (2008), Shin and Soydemir (2010), and Tzvetkova (2005) use quadratic and linear deviation measures based on the concepts developed by Roll (1992) and Rudolf et al. (1999) in their determination of deviations. Especially the few studies related to ETNs are important for our analysis. Wright et al. (2010) find, in their investigation of 65 globally traded ETNs in the period from 2008 to 2010, significant price deviations between the prices and their respective NAVs. By contrast, Diavatopoulos et al. (2011) suggest that the prices of 93 ETNs are significantly higher than their indicative prices due to a less liquid creation/ redemption process. Aroskar and Ogden (2012) employ five different measures to analyze both pricing efficiency and tracking errors in their sample of 25 ETNs divided into four categories in the period from 2008 to 2011; however, they find different results in their descriptive analysis of price deviations depending on the respective subcategory. Leung and Ward (2015) and Guo and Leung (2015) examine tracking errors of leveraged ETCs and demonstrate how a dynamic replication portfolio built from futures yields smaller tracking errors.

The literature about determinants of ETF tracking errors identifies several factors influencing the magnitude of errors. These are, among others, AUM, ETF trading volume (Buetow and Henderson 2012), management fee (Chu 2011), number of overlapping market hours as well as return differences between US and foreign markets (Johnson 2009), ETF age and standard deviations of returns (Rompotis 2011b). Physically replicating ETFs have smaller tracking errors than synthetically replicating ETFs (Fassas 2014). Schmidhammer et al. (2010) find that the tracking error of ETFs on the German stock index (DAX) is highly correlated with the price differences between DAX and DAX futures.

5.3 Data and methodology

5.3.1 Data

The data for our research covers 237 ETCs, with total assets under management exceeding EUR 21 billion as of June 2012, which are listed on the Frankfurt Stock Exchange on Xetra of Deutsche Börse AG or on regional stock exchanges, such as Stuttgart, and can be traded within trading hours of 9.00 a.m. to 5.30 p.m. on business days. The sample period of our daily data begins with the initial trading date of each ETC, the earliest with the start of the ETC trading in Germany on 03 November 2006, and ends on 25 July 2012. We constructed our dataset by comparing the 276 listed products of the German stock exchange to the product data available in Bloomberg, which only included prices and NAVs for 237 ETCs. The dataset analyzed differs from comparable studies of ETFs or ETNs in two ways. First, we focus on the German ETC market or, from a broader perspective, on the European market, which have not yet been investigated in academic literature. Second, by covering nearly all products available on the German market, the size of our dataset is significantly larger than that of comparable studies of other ETPs, which mostly include a range of between 5 and 100 investigation units (see Aber et al. 2009).

For the first part of our investigation, the ETC data consist of historical mid-prices as the average of bid and ask closing prices and of their net asset values (NAVs) as published by the issuers in euro currency for each ETC from its respective initiation date until 25 July 2012. The bid prices describe the highest prices a dealer will be prepared to pay whereas the ask prices are the lowest prices a dealer will be prepared to sell a security on a given day at. In accordance with Aroskar and Ogden (2012), we use mid-prices at closing of ETCs as they reflect more clearly the daily price movements of ETCs. Observations missing either bid price, ask price or NAV are removed from the respective ETC's data set.

The NAVs are computed by subtracting the liabilities from the portfolio value of the securities and dividing that figure by the number of outstanding shares. These are calculated once a day for each ETC, providing another argument for using mid-prices of ETCs. In the subsequent analysis, our computations are based on both prices and log returns[5] of mid-prices and NAVs.

For the second part of our empirical analysis, we extended our database by collecting additional information from stock exchanges, issuers' publications and Bloomberg. For each ETC, we gathered data on the following categories: Management fees, bid-ask-spreads, assets under management, age, issuers, commodity sectors, single versus broad-based ETCs, investment strategies, replication methods, and collateralization. Tables 5.1 and 5.2 provide a summary of our database for the categorical and the metric variables, respectively.

5.3.2 *Methodology*

In accordance with Tzvetkova (2005), due to the unique features of ETCs— and ETPs in general—their assessment as suitable investment vehicles to gain exposure to the underlying has two aspects. The first is usually measured by the tracking error (TE), which indicates how well the ETP's assets replicate the underlying benchmark (see e.g. Engle and Sarkar 2006; Frino and Gallagher 2002). The TE is predominantly determined by the way the ETC is set up and the execution skill of the management.

The second aspect is pricing efficiency. It measures how efficient the secondary market prices the ETC. Shares of an ETC can be bought (or created) on the primary market in exchange for its NAV per share and are basically— through the redemption mechanism or at termination or maturity—claims to the NAV per share. Consequently an ETC's *fair value*, and thus the comparison price for pricing efficiency is given by its NAV.

While both aspects are important, this study is concerned with the second aspect, which especially for ETCs deserves special attention. The reason for

[5] We use log returns instead of simple returns which are also widespread in the context of passive financial instruments in the academic literature and in practice, as the reliance on continuously compounded returns is more valid and suitable in the context of our further statistical computations.

Table 5.1: Frequency tables for categorical data. The data describe the sample of 237 ETCs traded on the German market.

	Category	N
Issuer	Issuer 1	124
	Issuer 2	25
	Issuer 3	9
	Issuer 4	28
	Issuer 5	12
	Issuer 6	39
Sector	Precious metals	54
	Livestock	11
	Agriculture	52
	Industrial metals	38
	Energy	66
	Cross-sectional	16

	Category	N
Single vs. broad	Single-commodity	165
	Broad-commodity	72
Investment strat.	Long	128
	Short	42
	Leveraged Long	52
	Leveraged Short	15
Replication	Physical replication	19
	Synthetic replication	218
Collateralization	Coll. by securities	199
	Physical coll.	27
	No collateralization	9
	Third party coll.	2

Table 5.2: Descriptive statistics for metric variables of our sample of 237 ETCs traded on the German market. *Management fees* represents the average annual management fees as reported by the issuers, *Rel. Bid-ask spread* is the mean over each ETC's entire lifetime of the daily relative spreads between bid and ask closing prices, *Age* is measured as the difference between the initial trading date of the relevant ETC and the end of the investigation period. *AUM market share* is the mean over each ETC's quaterly share of AUM of all 276 listed ETCs.

	Min	Q1	Median	Mean	Q3	Max	SD
Management fees / %	0.20	0.49	0.49	0.66	0.98	0.98	0.23
Rel. bid-ask spread / %	0.19	0.69	1.25	1.54	2.19	8.90	1.21
Age / years	0.36	0.79	3.26	3.04	4.14	5.72	1.75
AUM market share / %	0.00	0.00	0.01	0.40	0.07	23.00	2.05

this is found in the way most ETCs are structured. For physically replicated ETCs—all of which are ETCs on precious metals or chopper—shares are created or redeemed in exchange for the physical underlying and thus ETCs do not engage in trading for tracking purposes (ETFS Metal Securities Ltd. 2016). As a consequence the NAV will always equal the underlying's spot price after fees, which is exactly the benchmark for these types of ETCs.

Then there are synthetically replicated ETCs that track their benchmark with the help of derivatives. For many of these ETCs (e.g. the largest synthetic ETC in our sample, the ETFS Agriculture, cf. ETFS Commodity Securities Limited 2016) the issuer enters into a swap agreement guaranteeing that on creation and redemption of ETC shares swap positions with predetermined conditions are automatically opened and closed. As in the case of physical replication, there is no actual tracking activity required. Therefore, the NAV equals the accumulated cash flows from the swap position, which is contractually specified to equal the benchmark.

Summing up, as the tracking error between the NAV and the underlying can be regarded as a minor issue for ETCs, we identify pricing efficiency as the primary concern for investors looking to participate in the commodity markets via ETCs.

Premium/discount analysis The objective of our study is to determine the daily pricing efficiency of ETCs before we identify potential factors influencing the pricing of ETCs in the German market. Therefore, we first apply specific quantification concepts that are able to measure potential differences between the price (yield) performance of ETCs and their respective benchmarks.

Consistent with past research on ETFs (e.g. Aber et al. 2009; Charupat and Miu 2011; Elton et al. 2002; Jares and Lavin 2004) and ETNs (e.g. Aroskar and Ogden 2012; Diavatopoulos et al. 2011), we measure the daily price deviations using PD analysis. In accordance with Aber et al. (2009), the relative price deviations are calculated for each ETC as follows:

$$\pi_t = \frac{P_t - \mathrm{NAV}_t}{\mathrm{NAV}_t}, \tag{5.1}$$

where π_t is the ETC's price deviation on day t, P_t is the midprice on day t, and NAV_t is the official net asset value on the same day t. When this deviation is positive (negative), the ETC is traded at a premium (discount). In case of $\pi_t = 0$, the pricing is perfect and, thus, the creation/redemption process does not allow arbitrage opportunities. The PD analysis serves well as a first indicator of the pricing deviation but due to its limited interpretation further methods must be implemented for a more thorough analysis.

Quadratic and linear pricing efficiency analysis The quadratic and linear pricing measures focus on return-based deviations as opposed to absolute deviations of the PD analysis (e.g. Roll 1992). We will analyze the pricing efficiency of the ETCs by means of different discrepancy measurement called the pricing efficiency (PE) methods. In general, the methods aim to reflect the extent to which a security's price deviates from its target value over a certain period of time (e.g. Frino and Gallagher 2002) and may be regarded as a form of quality measurement of a security. They are not only commonly used for a posteriori analysis of pricing and tracking errors, but also for tracking error minimization (see e.g. Gharakhani et al. 2014; Rudolf et al. 1999, in the context of index tracking). Therefore, we choose these measures over different methods, like regression approaches (e.g. Shin and Soydemir 2010).[6]

The quadratic pricing masures have been heavily discussed in the academic literature and been implemented in the expression of various statistical forms (e.g. Ammann and Tobler 2000; Roll 1992). Consistently with Ammann and Tobler (2000) we implement the pricing error volatility as the square root of the non-central second moment of the deviations in the framwork of quadratic pricing methods. As a first pricing measure, we define the pricing error volatility $\mathrm{PE_{VOL}}$ as:

$$\mathrm{PE_{VOL}} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_{P,t} - R_{B,t}\right)^2}, \tag{5.2}$$

where $R_{P,t}$ denotes the log return of the ETC's midprice in period t, $R_{B,t}$ the log return of the NAV as benchmark B in period t, and T the sample size.

This quadratic error definition is the most common quadratic pricing error in the academic literature due to its advantageous statistical properties. The $\mathrm{PE_{VOL}}$ reflects both random positive or negative deviations and a constant under- or outperformance of the underlying index. However, Rudolf et al. (1999) criticize the fact that eq. (5.2) is difficult to interpret from an investor's perspective and does not reflect investment objectives in an adequate way. Therefore, they suggest four linear error definitions as being more appropriate alternatives for the purpose of exemplifying an investor's risk attitude. The proposed alternative definitions are based on absolute deviations between the security's price and its target value instead of squared deviations. In addition, these pricing errors provide both consistency with expected utility maximization and explicit solutions. Considering all these benefits, we apply the following four linear pricing models in our empirical analysis. The $\mathrm{PE_{MAD}}$ captures the mean absolute deviations of the ETC's mid-price and its NAV by calculating the average of the absolute deviations between the mid-price returns and the NAV returns as follows:

$$\mathrm{PE_{MAD}} = \frac{1}{T}\sum_{t=1}^{T}\left|R_{P,t} - R_{B,t}\right|,$$

[6] In our framework, it would be necessary to aggregate α, β and standard errors into one deviation measure. Since there is no straight forward way to do this, we used the more proven concepts defined above.

where $R_{P,t}$ is the log return of the ETC's mid-price in period t, $R_{B,t}$ the log return of the NAV as benchmark B in period t, and T the sample size.

The PE_{MAX}, as our second linear pricing error, method focuses on the maximum deviation between the return differences and can be expressed as:

$$\text{PE}_{\text{MAX}} = \max_{t \in \{1,\dots,T\}} \left| R_{P,t} - R_{B,t} \right|.$$

The measure PE_{MAX} characterizes a worst-case-scenario in which greater deviations are to be expected compared to the two other PE concepts. Obviously, PE_{MAD} and the PE_{MAX} are symmetrical pricing error methods as neither distinguish between positive and negative deviations by only calculating absolute values. However, investors may also be interested in assessing the downside risk, i.e. the risk of mid-price returns being below the NAV returns. Consequently, we use two asymmetrical linear models as analogs to PE_{MAD} and PE_{MAX}, but with a restriction to negative deviations. For a proper notation, we define the set of all instants of time at which the return deviation is negative, i.e. $\mathcal{N} := \{t | R_{P,t} < R_{B,n}\}$ with cardinal number $N := |\mathcal{N}|$

The PE_{MADD} is the mean absolute downside deviation, i.e. deviations where the mid-price returns are less than the NAV returns, by the following formula:

$$\text{PE}_{\text{MADD}} = \frac{1}{N} \sum_{t \in \mathcal{N}} \left| R_{P,t} - R_{B,t} \right|.$$

Analogously, PE_{MAXD} is defined by:

$$\text{PE}_{\text{MAXD}} = \max_{t \in \mathcal{N}} \left| R_{P,t} - R_{B,t} \right|.$$

We will use the two abbreviations MeMs, short for mean-based measures (PE_{VOL}, PE_{MAD} and PE_{MADD}) and MaMs, short for maximum-based measures (PE_{MAX} and PE_{MAXD}).

5.4 Regression variables and hypotheses

Unlike other studies, we do not confine ourselves to measuring the different pricing error measures for all of the examined ETCs and listing the results in a table. In aiming to identify determining factors influencing the pricing efficiency, we go one step further. To this end, we regress all five pricing error measures of every one of the 237 ETCs—each calculated over the whole available time series—on several explanatory variables. These are operational ETC characteristics and general market factors comprising management fees, bid-ask-spreads, market share of assets under management, age, issuers, commodity sectors, single versus broad-based ETCs, investment strategies, replication methods, and collateralization forms. All of them could have an influence on pricing efficiency. Therefore, we formulate nine hypotheses (numerated H1 through H9), proposing expectable relationships of the explanatory variables on the pricing error measures. As many of the variables are categorial (with, say, c

categories), we use the standard approach of setting up $c - 1$ dummy variables for each category with one value being the reference category.

5.4.1 Costs

The cost analysis of ETCs plays a vital role in the investment decision of investors. ETCs as passive financial instruments are a relatively simple and inexpensive means of participation in the commodity markets compared to other products. On the contrary, a physical acquisition of commodities, if at all feasible, involves substantial costs due to storage, transportation and insurance and commodity futures are associated with considerable margin and rollover costs. Due to their passive investment structure, ETCs limit management costs for complex analysis tasks as well as transaction and distribution costs. Depending on the particular structure of the ETC, different cost components are to be distinguished and the individual investor may incur both direct and indirect costs.

One type of direct primary cost applying to all ETC structures is the management fee which covers management and administrative services of the issuer. These costs are often incorrectly stated as total expense ratios (TER), a term summarizing all cost components of ETFs. Since the management fee is to be specified explicitly in the issuers' publications and prospects and is applicable to all ETC products, it represents an appropriate basis for comparison. The management fee indicated as an annual percentage fee is deducted in equal parts from the ETC assets and varies according to different products and issuers. Besides these costs, physical ETCs may charge fees for storage and custody whereas derivative-based ETCs carry swap, collateral, index or licensing costs. However, these cost components are variable over time and, thus, often not clearly defined by the issuer. In addition to these primary costs, transaction fees[7] may be levied in the acquisition or selling process by brokers, custodian banks, or states. Some issuers charge varying ancillary fees for the creation and redemption of ETC units which are not applicable to investors in the secondary market and exchange markets. These fees increase if predefined threshold values of ETC units are not reached and are usually higher for the redemption than the creation of new shares.

Thus, the average annual management fees provide a proper basis for the cost analysis of ETCs as they are incurred for all products and are to be reported explicitly by the issuers. In our sample, management fees are rather low (with a median value below 5 %, cf. Table 5.2), with a tendency of lower values for plain vanilla ETCs. Therefore, we can view management fees as a proxy for the exoticness of the respective ETC as they are positively correlated with the effort in product management.

Moreover, in their analyses of ETF tracking errors, Chu (2011) and Rompotis (2011b) find a negative influence of management fees on pricing efficiency. The perspective taken above leads us to the same conclusion. Our first hypothesis

[7] Brokerage commissions, market fees, clearing and settlement costs as well as taxes and stamp duties are examples of direct or explicit trading costs (see D'Hondt and Giraud 2008).

(H1) conjectures a negative influence of management fees on the pricing efficiency because with more complex products arbitrage opportunities are harder to exploit, which is essentially a basic transaction costs argument.

5.4.2 *Spread*

Despite their difficult determination, indirect or implicit costs in the form of bid-ask spreads are also important in the cost analysis. We calculate the average relative bid-ask-spreads as the mean of the differences between the daily closing ask and bid prices divided by the closing mid prices.

Amihud and Mendelson (1991) interpret the bid-ask spread as a measurement of liquidity or, to be more precise, as the cost of immediate execution. The spread calculated between the ask (offer) and bid (sell) prices are to be as tight as possible and significantly influence the asset return. The bid-ask-spreads represent transaction costs imposed by market makers, which may negatively affect pricing. Delcoure and Zhong (2007) emphasize that higher bid-ask spreads could harm the effectiveness of the creation/redemption process by making arbitrage activities less attractive. In contrast, Buetow and Henderson (2012) use another variable as proxy for second market liquidity, namely ETF trading volumes, and also find a negative effect on the magnitude of tracking errors.

Therefore, we hypothesize a positive effect of the relative spread on the magnitude of pricing errors (H2).

5.4.3 *Market share of assets under management (AUM)*

Based on quarterly AUM data of all 276 listed ETCs, also extracted from Bloomberg, we calculate the quarterly market share of each ETC, by dividing the ETC's AUM by the sum of all ETCs' AUM. This number is then averaged over all quarters of the considered time period. We choose market share as dependent variable in the sense of the relative AUM, to exclude the influence of the rapid overall growth of the ETCs market in terms of AUM, from 0 in 2006 to EUR 23.1 billion in the second quarter of 2012.

A higher market share is expected to indicate a more mature, possibly more liquid, and above all more lucrative market for a potential arbitrageur. This effect can be explained by the fixed costs of implementation and monitoring, which are independent of the size of the market. This economies of scale argument is also used by Buetow and Henderson (ibid.) and Chu (2011), who suggest that ETF fund size has a positive effect on pricing efficiency.

Thus, our third hypothesis can be stated as follows: the relative AUM market share has positive effect on pricing efficiency (H3).

5.4.4 *Age*

The age, measured as the time in years between the initial trading date of the relevant ETC and the end of the investigation period, is used as a proxy for the age of a product and its market maturity.

To explain the overpricing of structured products in the German market, Wilkens et al. (2003) propose a life cycle hypothesis, conjecturing "that issuers orient their pricing toward the expected volume of purchases and sales.". It is indeed confirmed by numerous studies that overpricing is highest at initiation when the products are sold by the issuers and vanishes over the lifetime of the product, or even becomes negative, when the product is sold back to the issuer. ETFs are open ended, yet it is still expected that excess demand creates overpricing for younger products and we conjecture the variable to have a negative relationship with the mean based pricing errors (H4).

This reasoning cannot be applied to MaMs, which monotonically increase with the number of observations. Thus, we expect the opposite effect, in which age has a positive influence of the magnitude of the (maximal) pricing errors (H5).

5.4.5 Single-commodity versus broad-based ETCs

Through the usage of ETCs, investors gain exposure to one single commodity or to a basket of multiple commodities. Taking a position in single-commodity ETCs allows investors to invest in certain commodity markets without having to adhere to a certain level of diversification to meet regulatory requirements as is the case with ETFs. In addition, single-commodity ETCs are often used for short-term investment strategies and require a precise knowledge about the opportunities and risks associated with the respective commodity. By contrast, broad-based commodity ETCs offer the possibility of diversified investments in all commodity sectors, in combinations of commodity sectors, and in combinations of two or more or even all commodities of a sector through one single trade. These types of ETCs are more suitable for long term investment motives.

Single-commodity ETCs only replicate one underlying commodity whereas broad-based ETCs cover two or more commodities. Therefore, single-commodity ETCs may incur smaller replication costs compared to broad-commodity ETCs. In contrast, for structured products with equity underlyings, Stoimenov and Wilkens (2005) expect the opposite effect, as indices can have a more liquid derivatives market compared to single stocks. Even if the liquidity argument cannot be transferred to ETCs in general, there can be effects that make replication of broad underlyings less costly, namely lower volatility and lower average roll-yield effects due to diversification.

As it is possible for each of both effects to dominate the other, we conjecture an effect of the variable on the pricing efficiency (H6), but we do not have an ex-ante expectation on the direction.

5.4.6 Investment strategies

With regard to investment strategies, investors implement long, short, leveraged long, and leveraged short positions with or without currency hedging through the acquisition of ETCs. A long ETC, the simplest and most intuitive type of ETC, closely tracks the daily performance of its underlying. As with stocks, investors generate profits if the underlying's prices rise and, vice versa, losses if prices fall. Forward long ETCs, whose underlying are composed of longer-maturity forwards, are counted among the long-investment products as well. Short ETCs as counterparts to long ETCs are aimed to reflect the daily performance change of the respective underlying times minus one and, thus, behave inversely to their target benchmarks. Consequently, an investor profits from falling prices and loses in case of increasing prices.

However, the losses are limited to the amount invested, constituting a major difference to the classical short sale transactions with theoretically uncapped losses.

Leveraged long and leveraged short ETCs are relatively new types of ETC investments and are more suited to risk-taking investors for the purpose of speculating or hedging. A leverage of two to four leads to an above-average participation in value changes of the underlying on a daily basis.

For longer holding periods than one day the realized return does not necessarily correspond with the indicated leverage over the same period. As with short ETCs, the potential losses are limited to the investment total which illustrates a significant advantage of ETCs.

We propose that the investment strategies may also play a crucial role as determinants of the pricing error. Leung and Ward (2015) find that leveraged ETFs have significant tracking errors stemming from imperfect replication (they give an improved tracking performance by dynamic portfolios of futures). Moreover and Guo and Leung (2015) postulate the so called *volatility decay*, arising due to the convexity of the ideal leveraged underlying. Extending this argumentation, we expect the greater difficulty in performing leveraged and short strategies and the associated higher costs, to have a positive impact on the magnitude of the pricing errors (H7). In our regression models, the investment strategy *Long* serves as a reference category.

5.4.7 Replication methods

As passive financial instruments, ETCs use either a physical or synthetic approach to replicate the underlying benchmark accurately. Physical replication is achieved by buying the physical commodities or the securities of the relevant index. Physical ETCs often relate to spot prices of commodities or commodity baskets of precious metals, such as gold or silver, as they are relatively homogeneous, easy to standardize, and non-perishable. In comparison, the physical replication is less frequently applied in other commodity sectors as these investments are largely either unprofitable due to storage, transportation and insurance costs, or practically unimplementable. Ramaswamy (2011) emphasizes that the physical replication strategy can prove to be very costly especially in case of less liquid or broad-market underlyings with a daily change in their composition.

As a result, synthetic replication strategies are often employed to minimize costs and deviations from the underlying benchmark. In contrast to holding the underlying commodities directly, these derivative-based ETCs adopt both total return swaps and futures to gain exposure to their target commodities. The synthetic approach is usually effected by means of bilateral total return swap contracts in which generally the two parties exchange the total return of two designated financial instruments. At maturity, the ETC issuer transfers not only its assets in the form of cash or baskets of securities, which significantly deviate from the composition of the underlying benchmark to the swap counterparties but also the risk of deviations from the benchmark. The swap counterparties, which are often parent companies of the ETC issuer, in return transfer the respective total return of the ETC underlying for a given nominal exposure.[8] The daily offsetting of the swap transactions aims to mitigate the incurring exposure risk. Besides an issuer risk, the ETC investor bears a counterparty risk which describes the risk of insolvency of the swap counterparty.

An alternative synthetic replication method involves the use of futures contracts. Here, the ETC issuer acquires or sells futures contracts from a third party when units of ETCs are created or redeemed. This ETC structure can be found in the energy sector, in which the third parties are multinational oil companies with direct exposure to the relevant commodity and try to hedge their risk through the trading of futures.

The synthetic replication method is marketed by issuers as the superior replication method for tracking error minimization. Consequently, Fassas (2014) hypothesizes higher pricing efficiencies for synthetically replicated ETFs. However, he cannot confirm this statistically, possibly due to his small data set, which is less comprehensive than ours. Therefore, we expect a negative relation between the synthetic replication dummy and pricing efficiency (H8).

5.4.8 Collateralization

Due to their structure as debt notes, ETCs are subject to issuer credit risk. Issuers are special purpose vehicles (SPVs), corporations in the form of a limited liability company or a limited partnership, which are created for the sole purpose of issuing ETCs and are normally not rated by external rating agencies. Hence, ETCs are collateralized by physical holdings, securities pledging, or coverage by an independent third party to reduce the risk of an issuer's insolvency whereas only few ETCs dispense with collateral. For comparison with other exchange-traded products, ETFs are structured as funds whose assets invested are not part of the liquidation assets in the event of an issuer's bankruptcy. ETNs as debt notes are only backed by the credit-worthiness of their issuers which are mostly big financial institutions and hold more types of debt obligations.

One popular ETC structure comprises the collateralization through physical holdings, such as precious metals which are simple to be stored, standardized and associated as safe investments by investors. The posted collateral (e.g. gold, silver, or platinum) is equal to at least 100 % of the value of the issued ETC units calculated on each business day. They are stored in a certificated vault

[8] See Ramaswamy (2011) for further information.

of an eligible custodian and regularly audited in terms of available amounts or compliance with quality standards at the issuer's discretion. Furthermore, an independent security trustee receives a primary security interest and is allowed to take control of the vault in the case of a credit event. The investors themselves only have a limited right of recourse and may incur losses in the event of insolvency.

The collateralization by securities is based on the pledging of stocks, cash, money market funds, or fixed-income securities with excellent credit ratings. These are transferred to a pledge account of a custodian and safeguarded by an independent trustee. In addition, the collateral is subject to a daily mark-to-market evaluation ensuring that their target value reflects the value of the issued ETC units plus a security surcharge of up to 10 %. If the collateral value is less or certain collateral criteria are not met, the issuer will be requested to deposit additional funds. However, the pledging of securities is more risky for investors as the posted collateral may become worthless in extreme market conditions or not cover all liabilities due to changes in asset values.

A less common type of collateralization is the coverage by an independent third party with best credit ratings. The eligible collateral targeting at least 100 % of the issued ETCs fulfills the same requirements as before with the sole exception of bearing the credit risk of the third party.

The explanatory variables collateralization by securities, physical collateralization as well as collateralization by third parties are included as dummies to explicate the pricing of ETCs whereby the lack of collateralization is regarded as reference category. On the one hand, positive coefficients of the three explanatory variables could be expected due to higher costs related to the collateralization of securities. On the other hand, a lack of collateralization could also lead to negative return deviations in case of worse credit situations. As both assumptions are reasonable, we take both scenarios into account and test the effect in the following analyses (H9).

5.4.9 *Regression model*

In order to test the different hypotheses simultaneously, we estimate a multiple linear regression model using ordinary least squares (OLS) for each of the five different dependent variables PE_{VOL}, PE_{MAD}, PE_{MAX}, PE_{MADD}, and PE_{MAXD}.

Out data set comprises products from six different issuers, whose dummy variables are used as control variables. The issuer with the highest AUM value is used as reference category. Moreover, we also control for the commodity sector and set *cross-sectional* as reference category, i.e. those ETCs that track price changes across all commodity sectors.

The regression equation for each dependent variable $Y \in \{PE_{VOL}, PE_{MAD},$

$\mathrm{PE}_{\mathrm{MADD}}, \mathrm{PE}_{\mathrm{MAX}}, \mathrm{PE}_{\mathrm{MAXD}}\}$ takes the following form:

$$\begin{aligned} Y_j &= \beta_0 + \beta_1 \cdot \mathrm{Fee}_j + \beta_2 \cdot \mathrm{Spread}_j + \beta_3 \cdot \mathrm{AUM}_j + \beta_4 \cdot \mathrm{Age}_j \\ &+ \beta_5 \cdot \mathrm{Issuer}_j + \beta_6 \cdot \mathrm{Sector}_j + \beta_7 \cdot \text{Single-commodity}_j \\ &+ \beta_8 \cdot \mathrm{Strategy}_j + \beta_9 \cdot \mathrm{Replication}_j + \beta_{10} \cdot \mathrm{Collateralization}_j \\ &+ \epsilon_j, \end{aligned} \tag{5.3}$$

for ETC number $j \in \{1, \ldots, 237\}$.

To test the significance of the regressors we use t-statistics adjusted for heteroskedasticity by White (1980).

5.5 Empirical analysis

In this section, we first present the results of the PD analysis as well as the quadratic and linear pricing analysis, from which we deduce the pricing efficiency of ETCs in the German market. In particular, the analysis of the quadratic and linear pricing measures plays a substantial role for the subsequent investigation of potential factors influencing the price deviations of ETCs and is therefore considered in more detail.

5.5.1 *Premium/discount analysis*

Table 5.3 reports upon the summary statistics for the PD analysis using daily figures of the data sample of 237 ETCs during the investigation period from 04 November 2006 to 25 July 2012. For each ETC, we calculate the mean of the price deviations between the mid-prices and the NAVs according to eq. (5.1) as well as its standard deviations using the entire available corresponding times series. Then, we compute the fraction of days with premiums, i.e. positive deviations over the entire data history of each ETC.

The mean price deviation of the data sample is 0.09 % implying that the ETCs on average trade at a premium. The maximum positive price deviation is 4.69 % while the maximum negative price deviation is −0.41 %. On average, the standard deviation of all ETCs is 0.95 % which ranges from 0.15 % to 3.78 %, implying relatively large and greatly fluctuating price deviations. These results

Table 5.3: Results of PD analysis. The table presents brief summary statistics on the average price deviations, the standard deviations (SD) of the price deviation as well as the shares of premium and discount observations for all of the 237 investigated ETCs' times series. Min is the minimum, Max the maximum value.

	Min	Q1	Median	Mean	Q3	Max	SD
Average price deviation / %	−0.41	−0.03	0.02	0.09	0.10	4.69	0.36
SD of price deviation / %	0.15	0.59	0.78	0.95	1.14	3.78	0.59
Share of premium obs. / %	41.10	48.10	51.09	53.72	56.00	99.33	9.37

Table 5.4: Results of the quadratic and linear pricing methods. The table summarizes the descriptive statistics of the quadratic PE_{VOL} and linear pricing methods PE_{MAD}, PE_{MADD}, PE_{MAX}, and PE_{MAXD} based on daily data. The data period is from the inception of each ETC, the earliest being from 04 November 2006 to 25 July 2012. SD is the standard deviation, Min the minimum, and Max the maximum of all PE results. The dataset consists of 237 ETCs, for each of which the five PE measures are calculated.

	Min	Q1	Median	Mean	Q3	Max	SD
PE_{VOL} / %	0.21	0.75	1.03	1.29	1.50	5.54	0.86
PE_{MAD} / %	0.17	0.50	0.69	0.90	1.03	4.01	0.62
PE_{MADD} / %	0.16	0.50	0.70	0.91	1.02	4.74	0.65
PE_{MAX} / %	0.59	4.06	6.04	6.98	8.43	23.90	4.40
PE_{MAXD} / %	0.46	3.40	5.45	6.29	7.96	23.90	4.07

are only in part consistent with those of the previous literature on ETPs. Charupat and Miu (2011) suggest the existence of large price deviations and price volatilities based on higher results in their analysis of eight ETFs. However, Kayali and Ozkan (2012) determines, in his analysis, an average price deviation of −0.8 % whereas Elton et al. (2002) note a mean discount of −0.018 %. We find that half of the ETCs traded at a premium over their NAVs for at least 53.7 % of the time. This is consistent with the life cycle argumentation of Wilkens et al. (2003) (see also subsection 5.4.4).

5.5.2 *Quadratic and linear pricing efficiency measures*

Next, we calculate the quadratic and linear pricing error measures introduced above. Table 5.4 displays the summary statistics for the whole sample of 237 ETCs measured over the whole investigation period. It provides the mean, the standard deviation, the minimum, and the maximum pricing error size.

For PE_{VOL}, the mean of the pricing error of the sample is 1.29 % with a standard deviation of 0.86 %. Considering the range of the sample, we detect significant differences as the minimum within the sample is 0.21 % and the maximum is 5.54 %. The mean absolute deviation PE_{MAD} shows a lower mean pricing error of 0.90 % with a standard deviation of 0.62 %. In addition, the pricing deviations vary from a minimum of 0.17 % and a maximum of 4.01 %, indicating a tighter range within the sample. Considering PE_{MAX} as extreme value analysis, the sample average of all maximum deviations between the price and the NAV is 6.98 % and a standard deviation of 4.40 %, whereby the extreme values fluctuate between a minimum of 0.59 % and a maximum of 23.90 % in the sample.

A comparison of the preliminary results shows that the lowest values occur with PE_{MAD} followed by PE_{VOL} and PE_{MAX} as previously expected. The results for PE_{MADD}, depicting restriction to negative price deviations reveals a great similarity to the findings of PE_{MAD}. Only the maximum value of 4.74 % is slightly higher. From this, we can infer that a multitude of ETCs are likely to trade at negative pricing deviations from their NAVs. This view is consistent with Tzvetkova (2005) and Kayali and Ozkan (2012) who also report similar results in their analysis of ETFs. When looking at PE_{MAXD}, the daily pricing errors for the PE_{MAXD} are comparatively lower than those of the PE_{MAX}. Furthermore, all results are statistically different from zero at the 1 % level.

In summary, we state that the pricing of ETCs in the German market is far from being efficient according to the different PE measurement concepts.

Table 5.5 provides detailed results of the five pricing error measurement concepts in differentiation of various product characteristics, such as issuers, commodity sectors, single-commodity versus broad-commodity, investment strategies, replication methods and collateralization. We conduct an analysis of variance (ANOVA) by the F-test of Welch (1951) in order to test the differences between two or more means of the analyzed characteristics.

The results of PE_{VOL} are almost identical to PE_{MAD} and PE_{MADD} when scaled appropriately. To make the remaining differences between them visible, we include a version of the PE_{MAD} means scaled by $x = \text{Mean}(PE_{MAD})/\text{Mean}(PE_{VOL}) = 0.70$. Furthermore, due to the similarities among the measures, we focus on the differences between MeMs and MaMs—as introduced on Page 76.

In the *issuer* category we determine differences between the means of the six institutions issuing ETCs at the 0.1 % level for all measures. Furthermore, all measures give consistent results, with Issuers 1 and 6 showing the highest errors throughout. The MaMs are higher and show smaller relative differences between the issuers—as is expected.

When differentiating between *commodity sectors*, all measures show almost identical order. The lowest pricing error is found for cross-sectional ETCs, the highest being *energy* for MeMs and *agriculture* for MaMs. The differences in the group means for the different sectors are again highly significantly different at the 0.1 % level, with the exception of PE_{MAX} at 1 % and PE_{MAXD} at 5 %.

The differences between *issuers* and *sectors* underline the importance of including them as control variables in our regression model.

In view of *single-commodity* or *broad-commodity* ETCs, we note higher pricing errors for *single-commodity* ETCs than for *broad-commodity* ETCs under all five measures at the 0.1 % significance level. This result provides further insights on our hypothesis H6 regarding the direction of the influence of the variable and is consistent with the fact that *cross-sectional* ETCs—which are by definition *broad-commodity*—also show the lowest errors.

Considering *investment strategies*, lower price deviations occur for *short*, *long* and with a certain gap for *leveraged long* followed by *leveraged short*. These results are consistent over all five measures and are further supported by the applied ANOVA, which indicates a systematic difference between the group mean values of the various investment strategies at the 0.1 % level. This can be viewed as first supporting evidence in favor of H7, but only for the leveraged strategies.

Synthetic replication has lower pricing errors than *physical replication* for all pricing methods except for PE_{MAXD}. While the results are in line with our

Table 5.5: Dependence of the quadratic and linear pricing measures PE_{VOL}, PE_{MAD}, PE_{MADD}, PE_{MAX}, and PE_{MAXD} on various factors based on daily data for the German market. Additionally, there is a version of PE_{MAD}'s means scaled by $x = \text{Mean}(PE_{MAD}) / \text{Mean}(PE_{VOL}) = 0.70$ for better comparability. The data period is from the inception of each ETC, the earliest being from 04 November 2006, to 25 July 2012. The table lists the means within all subgroups of ETCs and the F-statistics resulting from an ANOVA comparing the means within each group of product characteristics. N represents the number of corresponding observations. ****, ***, **, * indicates statistical significance at the 0.1 %, 1 %, 5 %, 10 % level, respectively.

	PE_{VOL} / %		PE_{MAD} / %			PE_{MADD} / %		PE_{MAX} / %		PE_{MAXD} / %		
Determinant	Mean	F-statistic	Mean / x	Mean	F-statistic	Mean	F-statistic	Mean	F-statistic	Mean	F-statistic	N
Issuer 1	1.19		1.13	0.79		0.79		7.55		6.96		124
Issuer 2	1.00		0.99	0.69		0.69		6.23		5.58		25
Issuer 3	0.86		0.84	0.59		0.58		5.49		5.05		9
Issuer 4	1.01		1.03	0.72		0.72		4.70		4.11		28
Issuer 5	0.96		0.98	0.68		0.67		5.35		4.55		12
Issuer 6	2.21	8.06****	2.37	1.66	9.26****	1.69	9.14****	8.15	4.95****	7.04	5.78****	39
Precious metals	1.34		1.40	0.98		0.98		6.39		5.40		54
Livestock	0.88		0.80	0.56		0.56		6.04		5.72		11
Agriculture	1.28		1.21	0.84		0.84		8.18		7.68		52
Industrial metals	1.07		1.06	0.74		0.75		6.26		5.97		38
Energy	1.59		1.63	1.14		1.15		7.62		6.67		66
Cross-sectional	0.72	10.03****	0.68	0.48	13.02****	0.47	13.16****	4.84	3.46***	4.45	3.08**	16
Single-commodity	1.48		1.49	1.04		1.05		7.80		6.99		165
Broad-commodity	0.85	56.80****	0.84	0.59	53.69****	0.58	53.31****	5.11	31.87****	4.69	25.28****	72
Long	0.99		0.97	0.68		0.68		6.06		5.43		128
Short	1.00		0.98	0.69		0.69		5.57		4.99		42
Leveraged Long	1.87		1.87	1.30		1.26		9.48		8.36		52
Leveraged Short	2.67	20.18****	2.85	2.00	21.37****	2.20	20.09****	10.18	11.17****	10.18	11.04****	15
Physical replication	1.44		1.50	1.05		1.04		7.29		5.75		19
Synthetic replication	1.28	0.93	1.27	0.89	1.77	0.89	1.51	6.96	0.10	6.34	0.52	218
Coll. by securities	1.31		1.31	0.92		0.92		7.08		6.44		199
Physical coll.	1.26		1.29	0.90		0.90		6.44		5.29		27
No collateralization	0.86		0.84	0.59		0.58		5.49		5.05		9
Third party coll.	1.43	5.39**	1.25	0.88	8.72****	0.86	10.46****	11.93	2.23	10.75	1.79	2

hypothesis H8, the differences of the means are not statistically significant.

With regard to the type of *collateralization*, all measures induce a similar ranking, where *no collateralization* shows the smallest pricing errors. The ANOVA analysis indicates significant differences among the group means at the 0.1 % level. This gives support to our hypothesis H9, that cost savings due to a lack of collateralization improves pricing efficiency.

Overall, these results show that all five measures give qualitatively similar answers that are consistent with our hypotheses, except in the case of *single-commodity* ETCs.

5.5.3 Regression analysis

The main findings of our analysis can be found in Table 5.6, which presents the results of the OLS regression models based on eq. (5.3). It shows the determinants of the pricing efficiency of ETCs in the German market and is used in the following to assess the hypotheses of section 5.4. The dependent variables of the five different regression models are the quadratic and linear pricing measures of the dataset, while the independent variables are the different product characteristics described above.[9] The significance of the independent variables is tested by t-statistics adjusted for heteroskedasticity by White (1980).

The overall explanatory power is satisfactory, with adjusted R^2 in the range of 45 % to 61 %, and is the highest for PE_{MAD}, PE_{MADD}, closely followed by PE_{VOL} and then PE_{MAXD} and PE_{MAX}. This order is to be expected for the outlier-driven MaMs. As in the previous section all measures show comparable results and Table 5.6 again includes a scaled version of PE_{MAD} for better comparison with PE_{VOL}.

Corroborating the hypothesis H1, the management fee has a positive effect and is significant at the 1 % level on MeMs and only at the 10 % level for MaMs. In the case of PE_{VOL}, a coefficient of 0.91 implies that a 1 bps change in management fees will lead to an almost equal change in the pricing error volatility. This is in accordance with the findings from Chu (2011) and Rompotis (2011b).

As was expected by H2 and also seen by Delcoure and Zhong (2007), the bid-ask spread shows a 0.1 %-significant influence across all five dependent variables, with coefficients around 0.2 for MeMs and between 1.1 and 1.5 for MaMs.

The expected dependence on the AUM market share (H3) is also observed, in accordance with the results of Buetow and Henderson (2012) and Chu (2011). The coefficients are significant at the 1 % level, expect for for PE_{MAD} and PE_{MADD}, whose coefficients are closer to zero and do not follow the usual linear relationship with PE_{VOL}. For MaMs the coefficients are below −17, indicating a strong dampening effect of outliers for large ETCs. For example, the largest ETC's market share of 23 % explains the 4.3 p.p. increase in pricing efficiency measures by PE_{MAX} when compared to the smallest ETCs.

[9]The independent variable 'Issuer 3' was excluded from the model for it is identical to the variable 'no collateralization'.

Table 5.6: OLS regression results. The table displays the results of the OLS regression models, using the quadratic and linear pricing measures PE_{VOL}, PE_{MAD}, PE_{MADD}, PE_{MAX}, and PE_{MAXD} as dependent variables. It displays the estimated coefficients and the t-statistics for the various determinants. As in Table 5.5, for better comparability, there is another version of PE_{MAD} coefficients scaled by $x = \text{Mean}(PE_{MAD}) / \text{Mean}(PE_{VOL}) = 0.70$. The R^2, the adjusted R^2, the F-statistics, and the size of the dataset (N) are also provided. ****, ***, **, * indicates statistical significance at the 0.1 %, 1 %, 5 %, 10 % level, respectively.

	PE_{VOL} / %		**PE_{MAD} / %**			**PE_{MADD} / %**		**PE_{MAX} / %**		**PE_{MAXD} / %**	
Determinant	Coeff.	t-stat.	Coeff. / x	Coeff.	t-stat.	Coeff.	t-stat.	Coeff.	t-stat.	Coeff.	t-stat.
Constant	0.42	1.29	0.48	0.34	1.44	0.33	1.44	1.52	0.90	0.57	0.37
Management fees / %	0.91	2.82***	0.95	0.66	2.81***	0.67	2.97***	2.81	1.70*	2.77	1.96*
Rel. bid-ask spread / %	0.21	4.79****	0.22	0.16	4.92****	0.16	5.05****	1.49	5.27****	1.08	3.65****
AUM market share	−2.11	−2.78***	−1.89	−1.32	−2.43**	−1.31	−2.45**	−18.70	−3.36****	−17.01	−3.37****
Age / years	0.04	2.13**	0.03	0.02	1.32	0.02	1.37	0.87	6.17****	0.86	6.68****
Issuer 2	0.47	4.40****	0.50	0.35	4.67****	0.34	4.76****	3.09	3.54****	2.23	2.87***
Issuer 4	0.07	0.41	0.11	0.07	0.56	0.08	0.60	0.47	0.47	0.56	0.66
Issuer 5	0.19	1.52	0.21	0.15	1.77*	0.12	1.56	2.12	2.94***	1.32	1.78*
Issuer 6	1.09	4.79****	1.27	0.89	5.31****	0.83	5.67****	3.31	3.49****	1.80	2.47**
Precious metals	−0.07	−0.56	0.00	0.00	0.00	−0.02	−0.18	−0.77	−1.09	−0.93	−1.35
Livestock	−0.20	−1.34	−0.18	−0.13	−1.15	−0.12	−1.15	−1.71	−2.36**	−1.46	−1.95*
Agriculture	0.16	1.72*	0.17	0.12	1.92*	0.12	2.01**	0.28	0.46	0.52	0.85
Industrial metals	−0.28	−2.41**	−0.24	−0.17	−2.20**	−0.17	−2.23**	−1.86	−2.28**	−1.39	−1.73*
Energy	0.41	3.73****	0.50	0.35	4.55****	0.36	4.73****	0.82	1.24	0.41	0.62
Single-commodity	0.25	3.51****	0.17	0.12	2.44**	0.13	2.65***	2.26	4.44****	2.28	4.68****
Short	−0.18	−1.17	−0.18	−0.12	−1.11	−0.12	−1.16	−0.81	−1.10	−1.11	−1.68*
Leveraged Long	0.53	3.19***	0.51	0.36	3.01***	0.32	2.80***	2.95	3.37****	2.39	3.15***
Leveraged Short	0.90	2.58**	0.92	0.64	2.53**	0.90	3.23***	4.38	3.10***	5.82	4.53****
Synthetic replication	−0.55	−2.24**	−0.66	−0.46	−2.55**	−0.47	−2.64***	−1.11	−0.85	−0.33	−0.30
Coll. by securities	−0.39	−3.21***	−0.37	−0.26	−3.06***	−0.25	−3.04***	−3.91	−4.87****	−3.13	−3.69****
Physical coll.	0.11	0.55	0.11	0.08	0.55	0.08	0.60	−0.89	−0.74	−0.89	−0.80
Third party coll.	0.31	1.75*	0.14	0.10	0.93	0.08	0.77	5.69	2.97***	5.42	2.03**
R^2		0.61			0.64		0.64		0.51		0.50
Adjusted R^2		0.57			0.61		0.61		0.46		0.45
F-statistic		15.93****			18.22****		18.23****		10.72****		10.30****
N		237			237		237		237		237

The positive coefficient of the variable *age* does not confirm our expectation (H4) and thus also deviates from the life cycle hypothesis and results of Wilkens et al. (2003). The positive coefficient indicates that newer products have a better pricing efficiency than older ones. This may be attributed to the fact that newer products have to cope with great market competition and, thus, try to perfectly replicate the NAVs in their pricing. It should, however, be noted that it is not significantly different from zero for PE_{MAD} and PE_{MADD} and only at a 5 % level for PE_{VOL}.

Our hypothesis H5 conjecturing a positive coefficient for the variable *age* for MaMs can be confirmed at the 0.1 % significance level. The magnitude of the highest observed pricing error grows 0.86 p.p. per year on average. It should be noted, however, that this effect might be offset by growing AUM.

Hypothesis H6 can be confirmed. As in the previous section, *single-commodity* ETCs have higher pricing errors compared to *broad-commodity* ETCs—in analogy to the results of Stoimenov and Wilkens (2005). Except for PE_{MAD}, this effect is significant at or below the 1 % level.

The results of the regression analysis evidence the same effect as in Leung and Ward (2015), and thus confirm our hypothesis H7 concerning *leveraged investment strategies*. However, no significant effect can be found for *short*, which suggests that it is the leverage effect which causes price deviations.

Hypothesis H8 is supported only for MeMs and only at the 5 % level. The PE_{MADD} measure has a coefficient closer to zero, while being more significant (at the 1 % level). This indicates that when only negative deviations are considered, synthetic replication exhibits consistently higher pricing efficiency than physical replication. This result confirms the expected increase in explanatory power over the results of Fassas (2014), who did not find statistical evidence for the hypothesis, possibly due to a smaller sample size.

A negative coefficient for the variable *collateralization by securities* confirms hypothesis H9 at the 1 % level. The pricing efficiency benefits through reduced risk seems to dominate the higher costs associated with collateralization by liquid securities with highest credit quality.

The control variables show significant effects on both groups of measures, except for the *sector*, which has no significant effects on the MaMs. As one example take *Issuer 6*, whose ETCs have on average more than one additional percentage point of pricing error volatility compared to ETCs from *Issuer 1*. Further research is needed to identify drivers of these effects which are not explained by our hypotheses.

Compared to the reference category of *cross-sectional* ETCs, which were the clear leader in the previous section, the other sectors now show both positive and negative coefficients. The higher pricing efficiency visible in Table 5.5 now seems to be explained by the *broad-commodity* variable, as conjectured above.

Summarizing the differences between MeMs and MaMs, the results show the expected *age* dependence of the MaMs, some effects are missing in the *sector* and the *replication* coefficients are quite significant for the MeMs.

The regression results corroborate seven of our nine hypothesis and only partly our hypothesis on the effect of *investment strategy* (H6). They can not confirm our hypothesis on the effect of *age* (H4).

5.5.4 *Robustness test*

In order to test the robustness of the regression results, we split up the observation period into two sub-periods before and after the 1 October 2010, respectively. Then, for each of the two sub-periods, we perform regression analyses for the five pricing efficiency measures, based on the subset of prices and NAV values from the respective date range. ETCs that have no valid data reported within a sub-period are excluded from that sub-sample's regression.

The regression results for the first period are given in Table 5.7. It shows consistent results among the five measures and deviates from the full range regressions only moderately. To be more precise the following can be stated: *Management fee* shows no significant effect. The control variables *issuer*, and *sector* show differing significant effects, except for *energy*, which again has a significantly positive coefficient. The effect of *synthetic replication* is missing, instead there is a significantly positive coefficient for *physical collateralization*. Adjusted R^2 is slightly smaller, e.g. at 0.52 compared to 0.57 for $\mathrm{PE}_{\mathrm{VOL}}$. These differences can be explained by the fact that there are 64 ETCs missing that were present in the full regression, including all ETCs of *issuer 6* and all *leveraged short* ETCs.

The second sub-period (cf. Table 5.8), which includes 231 ETCs, gives results that are much closer to the full range regressions. They differ only in the following item: The positive coefficient for *age* is now significant and *AUM market share* is no longer significant, except for $\mathrm{PE}_{\mathrm{MAX}}$. The coefficient of *issuer 5* is now also significantly different from zero and the effects concerning the *investment strategy* are now less significant. For this sub-period, the adjusted R^2 is even higher, e.g. to 0.59 for $\mathrm{PE}_{\mathrm{VOL}}$.

All other effects as well as the signs and magnitudes of the coefficients are reproduced in the sub-periods. Consequently, this test confirms the robustness of our main findings.

5.6 Summary and conclusion

This study investigates ETCs, a very successful recent financial innovation, from an empirical perspective. ETCs are an important exchange-traded product enabling all kinds of investors to participate in commodity markets and have experienced considerable growth in both popularity and assets under management since their inception. However, as the field is still under-researched, this is the first examination of ETCs and their pricing efficiency in the euro marketplace with a special focus on the German market, which is the biggest ETC market in terms of product availability, assets under management, and turnover.

Our empirical examination of the pricing efficiency of 237 ETCs listed in the German market utilizes different measures. This study is not only unique concerning its focus on the German, or more generally the euro market, but also in the size of our dataset which is by far the largest in the ETP literature so far and allows a representative analysis since the start of the German ETC trading. We concentrate on the daily pricing deviations between the mid-prices of ask and bid prices and the theoretical fair values in form of the net asset values calculated in euro.

The measures employed include the premium/discount analysis based on prices and the pricing errors based on returns and result in the following findings: First, the ETCs incur, on average, pricing deviations in their daily trading. They are also more likely to trade at a premium from their theoretical fair prices. These outcomes are also supported by the five symmetrical pricing error measures, which provide deeper insight in the magnitude of deviation by alternate interpretation models.

Second, we use the pricing error measures as dependent variables in our regression analysis to find potential contributors to the pricing errors of ETCs. The results imply that a set of several variables influences pricing efficiency and thereby confirm seven out of nine of our hypotheses, namely those on management fee, spread, AUM market share, single- vs. broad-commodity, replication and collateralization. Mixed significance is found for investment strategy. No significant effects are found for our hypothesis on age.

Both the mean-based and the outlier driven maximum-based measures give comparable results. However, the observable and expected differences imply that the former is slightly better suited to assess the pricing efficiency of ETCs in the German market and slightly better explainable by economically relevant determinants.

ETCs as a passive, simple, and cost-effective financial innovation are likely to grow in investors' interests. For example, they are likely to play an important role in private pension plans due to their advantageous characteristics. However, the prosperous outlook of ETC investing is limited by the potential systemic risks arising from extensive passive investing and its influence on the commodity markets, the stock markets, and the whole economy.

Hence, the ETC pricing problem is of considerable importance and interest to private and institutional investors and may be extended in a global analysis. Another potential extension of our study is to derive concrete trading opportunities to exploit the existing pricing deviations.

Table 5.7: Results of the subsample regression in analogy to Table 5.6 based on data between 4 November 2006 and 30 September 2010. $x = \text{Mean}(\text{PE}_{\text{MAD}})/\text{Mean}(\text{PE}_{\text{VOL}}) = 0.69$.

	PE_{VOL} / %		PE_{MAD} / %			PE_{MADD} / %		PE_{MAX} / %		PE_{MAXD} / %	
Determinant	Coeff.	t-stat.	Coeff. / x	Coeff.	t-stat.	Coeff.	t-stat.	Coeff.	t-stat.	Coeff.	t-stat.
Constant	0.94	2.33**	1.03	0.71	2.54**	0.77	2.76***	−0.85	−0.32	0.36	0.15
Management fees / %	0.35	1.04	0.45	0.31	1.30	0.27	1.13	2.27	1.14	1.03	0.68
Rel. bid-ask spread / %	0.11	3.72****	0.11	0.08	4.16****	0.08	4.18****	0.96	3.47****	0.71	2.77***
AUM market share	−2.90	−3.43****	−2.81	−1.94	−3.25***	−1.86	−3.02***	−15.36	−1.78*	−17.64	−3.38****
Age / years	−0.12	−1.42	−0.15	−0.11	−2.02**	−0.12	−2.20**	0.67	0.83	0.54	0.66
Issuer 2	−0.16	−0.84	−0.13	−0.09	−0.68	−0.12	−0.91	1.18	0.74	0.51	0.31
Issuer 4	−1.09	−3.25***	−1.12	−0.77	−3.47****	−0.77	−3.39****	−3.69	−1.27	−3.41	−1.32
Issuer 5	−0.64	−2.35**	−0.65	−0.45	−2.63***	−0.51	−2.93***	−0.58	−0.25	−0.93	−0.40
Precious metals	−0.18	−1.71*	−0.19	−0.13	−2.01**	−0.13	−1.97*	−0.37	−0.44	−0.39	−0.47
Livestock	−0.10	−0.87	−0.08	−0.06	−0.68	−0.06	−0.65	−0.53	−0.64	−0.39	−0.47
Agriculture	0.29	3.60****	0.30	0.20	3.90****	0.21	4.04****	1.01	1.54	1.01	1.60
Industrial metals	0.11	1.13	0.16	0.11	1.77*	0.11	1.80*	0.03	0.04	0.30	0.33
Energy	0.53	5.63****	0.64	0.44	6.33****	0.44	6.57****	1.72	2.32**	1.45	2.00**
Single-commodity	0.29	4.32****	0.22	0.15	3.26***	0.16	3.53****	2.09	4.00****	2.12	4.16****
Short	−0.18	−0.96	−0.20	−0.14	−1.08	−0.16	−1.22	−1.26	−1.03	−1.26	−1.18
Leveraged Long	0.42	2.14**	0.40	0.28	2.07**	0.27	1.96*	2.45	1.79*	2.54	2.10**
Synthetic replication	−0.20	−1.12	−0.31	−0.22	−1.65	−0.21	−1.59	0.63	0.67	0.76	0.81
Coll. by securities	0.17	0.67	0.23	0.16	0.99	0.19	1.12	−1.86	−0.83	−1.54	−0.69
Physical coll.	1.00	2.70***	1.16	0.80	3.17***	0.79	3.05***	2.16	0.68	1.35	0.49
Third party coll.	0.10	0.42	−0.14	−0.10	−0.64	−0.16	−1.06	6.60	2.59**	5.46	1.82*
R^2		0.58			0.58		0.57		0.51		0.49
Adjusted R^2		0.52			0.53		0.51		0.45		0.42
F-statistic		10.91****			11.30****		10.57****		8.53****		7.68****
N		173			173		173		173		173

Table 5.8: Results of the subsample regression in analogy to Table 5.6 based on data between 1 October 2010 and 25 July 2012. $x = \text{Mean}(PE_{MAD}) / \text{Mean}(PE_{VOL}) = 0.71$.

	PE_{VOL} / %		**PE_{MAD} / %**			**PE_{MADD} / %**		**PE_{MAX} / %**		**PE_{MAXD} / %**	
Determinant	Coeff.	t-stat.	Coeff. / x	Coeff.	t-stat.	Coeff.	t-stat.	Coeff.	t-stat.	Coeff.	t-stat.
Constant	0.60	1.77*	0.67	0.48	1.96*	0.47	1.95*	2.09	1.21	1.15	0.79
Management fees / %	0.55	1.60	0.54	0.38	1.53	0.38	1.60	2.39	1.47	2.30	1.56
Rel. bid-ask spread / %	0.36	3.93****	0.37	0.26	4.22****	0.27	4.46****	1.54	3.91****	1.40	3.01***
AUM market share	−1.38	−1.72*	−1.46	−1.03	−1.79*	−0.99	−1.72*	−15.94	−3.39****	−10.19	−2.59**
Age / years	0.05	2.42**	0.05	0.03	2.67***	0.03	2.69***	0.46	4.31****	0.39	4.23****
Issuer 2	0.56	5.18****	0.54	0.38	5.16****	0.38	5.33****	3.22	4.02****	2.42	3.92****
Issuer 4	0.24	1.10	0.28	0.20	1.30	0.21	1.39	0.64	0.59	0.94	1.04
Issuer 5	0.43	3.00***	0.43	0.30	3.19***	0.29	3.22***	2.62	3.35****	2.11	2.48**
Issuer 6	1.32	5.32****	1.49	1.06	5.81****	1.01	6.24****	3.96	3.92****	2.57	3.35****
Precious metals	−0.11	−0.88	−0.04	−0.03	−0.34	−0.04	−0.44	−0.83	−1.28	−1.12	−1.88*
Livestock	−0.22	−1.85*	−0.23	−0.17	−2.11**	−0.15	−2.07**	−1.61	−2.33**	−1.44	−2.18**
Agriculture	0.14	1.42	0.13	0.09	1.48	0.09	1.47	1.07	1.82*	1.00	1.74*
Industrial metals	−0.41	−3.56****	−0.37	−0.26	−3.58****	−0.26	−3.57****	−2.08	−3.33***	−1.85	−3.16***
Energy	0.27	2.57**	0.33	0.23	3.50****	0.25	3.67****	0.72	1.17	0.16	0.28
Single-commodity	0.29	4.57****	0.24	0.17	4.17****	0.17	4.23****	1.58	3.88****	1.54	4.23****
Short	−0.20	−1.24	−0.23	−0.16	−1.39	−0.15	−1.38	−0.22	−0.29	−0.25	−0.39
Leveraged Long	0.42	2.17**	0.34	0.24	1.78*	0.20	1.55	2.75	3.07***	2.19	2.79***
Leveraged Short	0.86	2.40**	0.84	0.59	2.26**	0.85	2.99***	4.44	3.08***	5.92	4.59****
Synthetic replication	−0.59	−2.31**	−0.70	−0.49	−2.60***	−0.50	−2.69***	−1.96	−1.45	−1.03	−0.94
Coll. by securities	−0.56	−3.94****	−0.54	−0.38	−4.11****	−0.38	−4.14****	−3.00	−4.63****	−2.49	−3.66****
Physical coll.	−0.08	−0.38	−0.14	−0.10	−0.65	−0.10	−0.65	0.40	0.34	0.15	0.15
Third party coll.	−0.24	−1.59	−0.22	−0.16	−1.58	−0.17	−1.78*	1.05	1.23	0.56	0.66
R^2		0.63			0.66		0.66		0.52		0.54
Adjusted R^2		0.59			0.62		0.63		0.48		0.49
F-statistic		16.71****			19.11****		19.49****		10.95****		11.67****
N		231			231		231		231		231

6 ADDENDUM: PRICING AND HEDGING THE SANDBOX OPTION

The initial research questions of this dissertation were positioned around a new variant of option on the proceeds of the holder's trading activity, called the *sandbox option*. However, when it became clear that the literature lacks the appropriate tools to answer these questions, the focus of this dissertation shifted to developing such tools. After this detour, the following text now returns to the original questions and demonstrates how the results of chapters 2 and 3 can be used to answer them.

Remark 6.1. All quantities represent discounted values.

A sandbox option gives its holder the right to buy at expiration T the proceeds of their trading activity for the strike price M. During the live of the option, they are allowed to buy and sell from a predefined set of assets at market prices described by an N-dimensional process X. Trades are settled against a money market account with initial balance Y_0. The account accrues interest at the risk-free rate and borrowing is not allowed. $Y_t(\psi)$ denotes the portfolio value at time t for a given N-dimensional trading strategy ψ, representing the number of shares held of each asset at each time. The option's payoff is then given by $f = (Y_T - M)^+$.

At first, we solve the frictionless case, where $Y_t(\psi) \equiv \int_0^t \psi_s dX_s + Y_0$ is defined using the stochastic integral.

We work in the setting of subsection 2.4.3 and only extend the defining equation for the hedger's acceptance set $\mathcal{A}$ to also include conservative acceptance of the holder's decisions:

$$\left\{ f + H \;\middle|\; f \in \mathcal{A} \right\} = \left\{ f \middle| f \geq 0 \right\}^{\exists S \, \forall R},$$

with the newly introduced set of admissible trading strategies by the holder, R. With the help of Theorems 2.2 and 2.4 we can directly state the formula for the ask price (cf. Remark 3.2):

$$-P[\mathcal{A}](-f) = \sup_{\psi \in R} \sup_{Q \in \mathcal{M}} \mathbb{E}_Q\left[(Y_T(\psi) - M)^+ \right] \tag{6.1}$$

In a simple market, the following theorem provides an explicit solution to this optimization problem.

Theorem 6.1 (Complete market with constant volatilities)**.** *Assume, that X is a complete market, given by a vector of N geometric Brownian motions with covariance matrix Σ, defined by*

$$dX_t^i\, dX_t^j = X_t^i\, X_t^j\, \Sigma^{ij}\, dt, \quad \textit{for all } 1 \leq i, j \leq n.$$

Let X^k be the asset with the highest volatility and define $n := Y_0/X_0^k$. Then,

(1) the price in eq. (6.1) equals the price of n calls on X^k with moneyness M/n and expiration T.

(2) The pessimal (for the writer) trading strategy of the holder is to buy and hold n shares of X^k.

(3) The writer's super-replicating strategy is to hold the same position as the holder multiplied by the Black-Scholes delta of the call option from (1).

Proof. As the market is complete there is a unique sigma-martingale measure $\mathcal{M} = \{Q\}$, which removes the second supremum in eq. (6.1).

Define the vector of portfolio weights at time t using the component-wise product of two vectors: $q_t \equiv \psi_t {*} X_t/Y_t(\psi)$. The option's specifications restrict the set of allowed weights to

$$J \equiv \left\{q \in [0, \infty\rangle^N \;\middle|\; \textstyle\sum_i^N q^i \leq 1\right\}. \tag{6.2}$$

We will need the following trivial properties of dY_t:

$$dY_t = Y_t \sum_i^N q_t^i \frac{dX_t^i}{X_t^i}, \qquad \mathbb{E}_Q[dY_t] = 0, \text{ and} \qquad (dY_t)^2 = Y_t^2 q^T \Sigma q\, dt\,.$$

Results on stochastic control (e.g. Pham 2009) entail that there exists a value function v, with $v(0, Y_0) = \sup_{\psi \in R} \mathbb{E}_Q\left[(Y_T(\psi) - M)^+\right]$, that satisfies the following Hamilton-Jacobi-Bellman equation. For all $t \in [0, T\rangle$ and $y \in [0, \infty\rangle$,

$$\frac{\partial v}{\partial t} + y^2 \sup_{q \in J} \frac{1}{2} \frac{\partial^2 v}{\partial y^2} q^T \Sigma q = 0, \qquad\qquad v(T, y) = (y - M)^+.$$

To prove the first statement of the theorem, we show that

$$v(t, y) \equiv n \mathbb{E}_Q\left[\left(X_T^k - M/n\right)^+ \middle| X_t^k = y/n\right]$$

satisfies the HJB equation. The terminal condition is trivial. To prove the differential equation, we subtract from it the Feynman-Kac equation for v, yielding:

$$\sup_{q \in J} \frac{\partial^2 v}{\partial y^2} q^T \Sigma q = \frac{\partial^2 v}{\partial y^2} \Sigma^{kk}$$

The covariance matrix Σ is positive semi-definite and due to the convexity of the call price, it holds $0 \leq \frac{\partial^2 v}{\partial y^2}$. Therefore, the quadratic form on the l.h.s is convex (Rockafellar 1970, Theorem 4.5) and bounded on any bounded set. By Rockafellar (ibid., Corollary 32.3.4 and Theorem 19.1) the supremum is attained at one of J's extreme points. Consequently, it is given by the largest diagonal element of Σ, i.e. Σ^{kk}. This also proves the second statement of the theorem.

For the third statement it is enough to look at the random part of the differential of the price:

$$dv(t, Y_t(\psi)) + O(dt) = \frac{\partial v}{\partial y} dY_t = \frac{\partial}{\partial x} \mathbb{E}_Q \left[\left(X_T^k - M/n \right)^+ \middle| X_t^k = x \right] \psi_t \cdot dX_t$$

□

Real-world hedging always incurs transaction costs. As the hedger has to hold a scaled down version of the holder's position, it is obvious that frequent rebalancings by the holder will accumulate large sums of transaction costs for the hedger. Not so obvious are the answers the following questions:

What exactly is the pessimal trading strategy of the holder? How much larger will the ask price be? Do transaction costs render this contract useless by making it too expensive? Is there a qualitative difference between the cases $N = 1$ and $N > 1$? Can this problem be avoided by imposing similar transaction costs on the holder's trading?

Except for the derivation of eq. (6.1), the analysis of the frictionless case above utilized standard methods; the same methods that were used to derive prices for similar options (e.g. for the *passport option*, see Hyer et al. 1997; Ahn et al. 1999). However, they cannot be adjusted to answer the above questions.

The methods developed in chapter 3 fill this gap and make answering these questions straight forward. Working in the setting of section 3.4, we will derive a pricing formula for the optimally hedged sandbox option. In the following we setup only those quantities that are missing or different from section 3.4 and then state the result.

The holder's portfolio value is defined analogously to the result of the hedging activity in eq. (3.4). At time $t \in (s_{i-1}, s_i]$ it is given by:

$$Y_t(\psi) \equiv Y_0 + \sum_{j=1}^{i} \psi_{s_{j-1}} \cdot \left(X_{\min\{s_j, t\}} - X_{s_{j-1}} \right) \tag{6.3}$$

Furthermore, we assume there exists a pricing function $p_t^x : L_\infty \longrightarrow \mathbb{R}$ with $-\pi_t(-w) = p_t^{X_t}(w)$ for any w that is a function of present and future market prices. This assumption is satisfied e.g. if π is built from conditional expectations.

We require the hedging decisions to take values in some countable[1] set $W \subset \mathbb{R}^N$. The holder's positions have to stay within the contractually specified limits, i.e. $\psi_t{}^* X_t / Y_t(\psi) \in J$, with J from eq. (6.2).

[1] Note, that countability is no restriction of practical applications.

Theorem 6.2 (Realistic case)**.** *In the setting outlined above, the ask price of an optimally hedged sandbox option at time τ_i, for market price vector x, holder's portfolio value y and a hedger with current hedging position h, is given by $g_i(h,x,y)$, which is defined by the following recurrence relation:*

$$g_{n+1}(h,x,y) :=(y-M)^+ + c_{n+1}(h,0,x) \tag{6.4}$$

$$\begin{aligned} g_i(h,x,y) := \sup_{q\in J}\ \inf_{h'\in W} p^x_{\tau_i}\big(g_{i+1}(h',X_{\tau_{i+1}},y\big(1+q\cdot\delta_i X)\big)-h'\cdot\big(X_{\tau_{i+1}}-X_{\tau_i}\big)\big) \\ + c_i(h,h',x) \end{aligned} \tag{6.5}$$

Here, $(\delta_i X)^j := X^j_{\tau_{i+1}}/X^j_{\tau_i}-1$ is the vector of asset returns, $c_i(h,h',x)$ are the transaction costs for changing the position from h to h' at a current price of x at time τ_i and q is the holder's vector of portfolio weights.

Proof. Abbreviate the ask price at time s_{i-1} with $u_i \equiv \sup_{\psi\in S_{i-1}} -\eta_{\tau_i}(-f)\big[\psi\big]$ and show:

$$u_{n+1} = f + C_{n+1} \tag{6.6}$$

$$\begin{aligned} u_i &= \sup_{\psi\in S_{i-1}} -\sup_{\varphi\in R_i}\left(\pi_{\tau_i}\left(\inf_{\rho\in S_i}\eta_{\tau_{i+1}}(-f)\big[\rho\big]\big[\psi\big]\big[\varphi\big]+\varphi_{\tau_i}\cdot\big(X_{\tau_{i+1}}-X_{\tau_i}\big)\right)-C_i\big[\varphi\big]\right) \\ &= \sup_{\psi\in S_{i-1}}\ \inf_{\varphi\in R_i} -\pi_{\tau_i}\big(-u_{i+1}\big[\psi\big]\big[\varphi\big]+\varphi_{\tau_i}\cdot\big(X_{\tau_{i+1}}-X_{\tau_i}\big)\big)+C_i\big[\varphi\big] \end{aligned} \tag{6.7}$$

The first two equations is follow from Theorem 3.1 and eq. (3.13). The third follows from the inf/sup-duality and the definition of u_{i+1}. After applying Lemma 3.2 to both decisions in eq. (6.7), it can be easily checked by comparing the right hand sides of eqs. (6.6) and (6.7) with eqs. (6.4) and (6.5) that $g_i(h,X_{\tau_i},Y_{\tau_i}(\psi)) = u_i\big[\tau_{i-1}\mapsto h\big](\psi)$. □

If a pricing function p^x_t is provided — e.g. some monetary utility function or indifference price — then this theorem gives (at least numerically) the price, optimal writer and pessimal holder strategy and thus the answers to the above questions.

To answer the last question, an analogous derivation has to be performed with a modified version of eq. (6.3) that includes transaction costs.

If transaction costs are of the form of eq. (3.16) and p^x_t does not depend on x, then the price can be calculated using a simpler function $\widetilde{g}$, with $g_i(h,x,y) = \widetilde{g}_i(hx,y)$. This reduction in the complexity of the calculation is thinkable e.g. in cases where $\delta_i X$ is independent from X_{τ_i}, such as in models with constant volatility.

CONCLUSION

7

The results of the previous chapters have several theoretical and practical implications. The following summarizes the contributions of this dissertation.

Chapter 2 provides a characterization of the implicit assumptions underlying existing approaches to decisions embedded within derivative contracts. Known results can be replicated within our framework under the assumption of time-consistent conservative acceptance. Furthermore, the chapter demonstrates that complex combinations of a wide array of decisions can be handled in a consistent manner, independent from the market model and pricing function. Future research can leverage our framework to formally eliminate decisions from all kinds of pricing problems with minimal argumentative effort.

Our findings provide the ground for further research that explores the limits of conservative acceptance and its possible extensions — as motivated by Example 2.1.

Chapter 3 demonstrates how a unified approach to decisions makes hitherto unsolved problems accessible and that new results can be obtained by straightforward application of our methods. The model- and contract-independent hedging principle derived in this chapter can be used to solve future theoretical and practical hedging problems. The chapter also provides a first example of such an application with the following result: optimal hedging of American options is possible, constitutes a significant improvement over delta hedging and thus should be implemented in practice.

A second example is given in chapter 6. Here, the general hedging principle is applied to the problem of realistically pricing and hedging a new variant of option on the proceeds of the holder's trading activity, called the *sandbox option*.

To fully carry out the pricing procedures derived in chapters 2 and 3, a classical pricing function for derivatives without decisions is required. Finding utility-based candidates — analogously to the discussion in subsection 3.5.1 — that do not suffer from the shortcomings encountered in chapter 4 is a challenge left to future research.

Chapter 4 has two implications for the use of utility functions that fall "too fast" as the wealth approaches negative infinity. Firstly, indifference or utility-based prices and their hedging strategies derived for such utility functions lead to far from optimal behavior when approximated using discrete, i.e. practicable, strategies. Secondly, there exists no simple solution or workaround that would mitigate the problem. Its root lies in the fact that such utility functions forbid any asymptotic short exposure in the underlying asset and thus do not represent the behavior of realistic hedgers.

These findings should encourage future research to devise utility functions that carry over many of the mathematically elegant results from the indifference pricing literature without compromising practical applicability. Chapter 4 discusses a sufficient condition and an example to spur further explorations.

Chapter 5 provides evidence that prices of ETCs in the German market between 2002 and 2012 significantly deviate from the ETCs' fair prices. On average, ETCs traded at a premium over their NAVs. The results of the regression analysis corroborate the hypotheses about the influence of the following factors on the pricing efficiency of ETCs: management fee, relative bid-ask spread, AUM market share, single- vs. broad-commodity, type of replication, type of collateralization and leverage factor. No evidence is found for an effect of the ETC's age.

These findings are important to investors seeking ETCs to track certain individual or baskets of commodities. First, they show that investors cannot blindly rely on the advertised tracking properties of ETCs. Secondly, they can be used to select efficiently priced ETCs, thereby helping to achieve the desired investment strategy and diversification.

The model used in the regression analysis includes control variables for each ETC's issuer and sector. Several of these variables are found to have significant effects on pricing efficiency; effects which are not explained by the other variables. Identifying the mechanism behind these unexplained effects is left to future research.

APPENDICES TO CHAPTER 2

A

A.1 Proofs

Definition A.1 (Time t-acceptable premiums)**.** $\mathcal{K}\left[\mathcal{A}, f\right] = \left\{x \in L_t^- \;\middle|\; f - x \in \mathcal{A}\right\}$

A.1.1 Proof of Corollary 2.1

Proof. Equation (2.2) from Definition 2.4 follows directly from Axiom 2.1 and the definition of $\mathcal{A}$.

To prove t-compatibility assume $\mathbb{P}\big(\alpha_t\big(f_n\big)\big) = 1$ and therefore $\mathbb{P}\big(\alpha_t\big(f_n\big)\big|B_n\big) = 1$ for all $n \in \mathbb{N}$. Take any $x \in V_t$ and define $h \equiv x + \sum_n f_n \mathbb{1}_{B_n}$. Due to B_n's disjointness we have $h \overset{B_n}{=} f_n + x$ and thus by the "$\Longrightarrow$" direction of Axiom 2.1, $\mathbb{P}(\alpha_t(h)|B_n) = 1$ and furthermore by Corollary A.2.3 $\mathbb{P}(\alpha_t(h)) = 1$. Using the "$\Longleftarrow$" direction of Axiom 2.1 yields $\mathbb{P}\big(\alpha_t\big(\sum_n f_n \mathbb{1}_{B_n}\big)\big) = 1$. □

A.1.2 Proof of Theorem 2.1

This proof requires the following lemmata:

Lemma A.1. *Given $h \in \mathcal{X}^t$, $C \in \mathcal{F}_\infty$ and a non-empty set of payoffs $Y \subset \mathcal{X}_{\mathcal{T}}^t$, such that $h \overset{C}{<} \sup Y$, a sequence $(g_n) \subseteq Y$ exists as well as mutually disjoint events $\{B_n\} \in \mathcal{F}_t$, such that $B_n \subseteq \{g_n > h\} \cup \overline{C}$ and $\mathbb{P}(\bigcup_n B_n) = 1$.*

Proof. By Theorem A.1 and $Y \neq \emptyset$ a sequence $(g_n) \subseteq Y$ exists with pointwise supremum $g(\omega) \equiv \sup_n g_n(\omega)$ such that $g = \sup Y$. Define $D_n \equiv \{h < g_n\} \cup \overline{C}$ and $B_n \equiv D_n \setminus \bigcup_{m=1}^{n-1} B_m$. Disjointedness and $B_n \subseteq \{h < g_n\} \cup \overline{C}$ follow trivially. Next, show:

$$\bigcup_n B_n = \overline{C} \cup \bigcup_n \{h < g_n\} = \overline{C} \cup \{\exists n : h < g_n\} = \overline{C} \cup \{h < g\} = \overline{C} \cup (\{h < g\} \cap C) \tag{A.1}$$

The fourth from the least upper bound property of the supremum. By the hypothesis of the lemma we have $\mathbb{P}\big(\{h < g\} \cap C\big) = \mathbb{P}(C)$. Together with $\mathbb{P}$'s additivity for disjoint events, eq. (A.1) proves $\mathbb{P}(\bigcup_n B_n) = 1$ □

Lemma A.2. *$\infty \in \mathcal{A}$ and $-\infty \in \mathcal{K}\left[\mathcal{A}, f\right]$ for any proper t-acceptance set $\mathcal{A}$ and $f \in \mathcal{X}_{[t,\infty\rangle}$.*

Proof. Due to $\mathcal{A}$'s t-compatibility some $g \in \mathcal{A}$ exists. By eq. (2.2) from Definition 2.4 and $\infty \in V_t$ we can follow $g + \infty = \infty = f - (-\infty) \in \mathcal{A}$. i.e. $-\infty \in \mathcal{K}\left[\mathcal{A}, f\right]$. □

Lemma A.3. *For any proper t-acceptance set $\mathcal{A}$ and $f \in \mathcal{X}_{[t,\infty\rangle}$:*

$$0 < P[\mathcal{A}](f) \Longrightarrow f \in \mathcal{A}$$

Proof. By Lemma A.2 we can apply Lemma A.1 with $h = 0$, $Y = \mathcal{K}\left[\mathcal{A}, f\right]$ and $C = \Omega$ and get a sequence $(g_n) \in \mathcal{K}\left[\mathcal{A}, f\right]$ and a sequence (B_n) with $B_n \subseteq \{0 < g_n < \infty\}$ for any $n \in \mathbb{N}$. Define $x = \sum_n g_n \mathbb{1}_{B_n}$. By $f - g_n \in \mathcal{A}$, the other properties of B_n and t-compatibility we have $\sum_n (f - g_n)\mathbb{1}_{B_n} = f - x \in \mathcal{A}$. Furthermore, due to $x > 0$ and $x \in L_t$, we have $f = (f - x) + x$ and thus by eq. (2.2) from Definition 2.4: $f \in \mathcal{A}$. □

Lemma A.4. *For every cash invariant t-pricing function π and $B \in \mathcal{F}_t$, it holds:*

$$\pi\left(f + \infty \cdot \mathbb{1}_{\overline{B}}\right) = \pi\left(g + \infty \cdot \mathbb{1}_{\overline{B}}\right) \Longrightarrow \pi\left(f\right) \overset{B}{=} \pi\left(g\right)$$

Proof.

$$\begin{aligned}\pi\left(f\right) &\overset{B}{=} \pi\left(f\right) + \infty \cdot \mathbb{1}_{\overline{B}} = \pi\left(f + \infty \cdot \mathbb{1}_{\overline{B}}\right) \\ &= \pi\left(g + \infty \cdot \mathbb{1}_{\overline{B}}\right) = \pi\left(g\right) + \infty \cdot \mathbb{1}_{\overline{B}} \overset{B}{=} \pi\left(g\right)\end{aligned}$$

The second and fourth equation follow from cash invariance (Definition 2.8) and $\infty \cdot \mathbb{1}_{\overline{B}} \in L_t^+$. □

Proof of Theorem 2.1.1. First we show t-compatibility. $A[\pi]$ is not empty due to $\pi(\infty) \geq 0$, which follows from $0 \in \mathcal{X}_{[t,\infty\rangle}$, $\infty \in L_t^+$, cash invariance of π and the convention from Definition 2.1: $\pi(\infty) = \pi(0) + \infty = \infty \geq 0$.

For t-compatibility, take a sequence f_n with $\pi\left(f_n\right) \geq 0$, mutually disjoint $\{B_n\} \subseteq \mathcal{F}_t$ with $\mathbb{P}(\bigcup_n B_n) = 1$ and define $g \equiv \sum_n^\infty f_n \mathbb{1}_{B_n}$. We need to prove $\pi\left(g\right) \geq 0$. Using Corollary A.1.2 it remains to show $\pi\left(g\right) \overset{B_n}{\geq} 0$ for any n: The two functions $f_n + \infty \cdot \mathbb{1}_{\overline{B_n}}$ and $g + \infty \cdot \mathbb{1}_{\overline{B_n}}$ are identical and thus Lemma A.4 can be applied to prove $\pi\left(g\right) \overset{B_n}{=} \pi\left(f_n\right) \geq 0$.

Now prove the "⊆"-direction in eq. (2.2) from Definition 2.4. For any f with $0 \leq \pi\left(f\right)$, $g \in \mathcal{X}_{[t,\infty\rangle}$ and $x \in V_t$ with $g = f + x$ (pointwise) it holds:

$$0 \leq \pi\left(f\right) \leq \pi\left(f\right) + x = \pi\left(f + x\right) = \pi\left(g\right)$$

The first equality uses π's cash invariance (Definition 2.8).

For the "⊇"-direction: Assume for each $n \in \mathbb{N}$ it holds $f + \frac{1}{n} \in A[\pi]$, thus $0 \leq \pi\left(f + \frac{1}{n}\right)$ and by π's cash invariance $0 \leq \pi\left(f\right) + \frac{1}{n}$. By taking the limit $n \longrightarrow \infty$ yields the result: $f \in A[\pi]$. □

Proof of Theorem 2.1.2. Take any $f \in \mathcal{X}_{[t,\infty\rangle}$ and show:

$$\begin{aligned} P[A[\pi]](f) =& \sup\left\{x \in L_t^- \;\middle|\; f - x \in A[\pi]\right\} = \sup\left\{x \in L_t^- \;\middle|\; \pi_t(f-x) \geq 0\right\} \\ =& \sup\left\{x \in L_t^- \;\middle|\; \pi_t(f) \geq x\right\} = \pi_t(f) \end{aligned}$$

The first two equations are A's and P's definitions and the second uses cash invariance (Definition 2.8) of π and the convention from Definition 2.1. The last equation: The upper bound property of $\pi_t(f)$ is trivial and to prove the *least* upper bound property, assume $h \in \mathcal{X}_{[t,\infty\rangle}$ is another upper bound and define the sequence $g_n \equiv \min(n, \pi(f))$. By definition of a t-pricing function, we have $\pi(f) \in L_t^{\pm}$ and thus $g_n \in L_t^-$ and furthermore $\pi(f) \geq g_n$ for any $n \in \mathbb{N}$. As an upper bound h fulfills $h \geq g_n$, which in the limit $n \to \infty$ proves that $h \geq \pi(f)$ □

Proof of 2.1.3. It is a t-pricing function by Theorem A.1. It remains to show cash invariance.

Due to Lemma A.2 the following proof can ignore the case of empty sets. Take any $x \in L_t^+$ and define $g \equiv P[\mathcal{A}](f+x) = \sup \mathcal{K}\left[\mathcal{A}, f+x\right]$, $h \equiv P[\mathcal{A}](f) = \sup \mathcal{K}\left[\mathcal{A}, f\right]$, and the set $B \equiv \{x = \infty\}$.

1) $h + x \overset{B}{\leq} g$: By taking into account the convention from Definition 2.1, it is easy to see that for any $n \in \mathbb{N}$ it holds $f + x - (n - \infty \cdot \mathbb{1}_{\overline{B}}) = \infty$, which is element of $\mathcal{A}$ by Lemma A.2. Consequently, $n - \infty \cdot \mathbb{1}_{\overline{B}} \in \mathcal{K}\left[\mathcal{A}, f+x\right]$ and the upper bound property of the supremum g directly implies $n \overset{B}{\leq} g$ and in the limit $g \overset{B}{=} \infty \overset{B}{\geq} h + x$.

2) $h + x \overset{\overline{B}}{\leq} g$: Take any $y \in \mathcal{K}\left[\mathcal{A}, f\right]$ and define $z \equiv y + x \cdot \mathbb{1}_{\overline{B}} \in L_t^-$, such that $f + x - z = (f-y) \cdot \mathbb{1}_{\overline{B}} + \infty \cdot \mathbb{1}_B$. From $f - y \in \mathcal{A}$, $\infty \in \mathcal{A}$ (by Lemma A.2) and t-compatibility we derive $f + x - z \in \mathcal{A}$, i.e. $z \in \mathcal{K}\left[\mathcal{A}, f+x\right]$. The upper bound property of the supremum g and z's definition yield $g - x \overset{\overline{B}}{\geq} y$. Together with the previous result $g \overset{B}{=} \infty$ we get $g - x \geq y$ and thus $g - x$ is an upper bound of $\mathcal{K}\left[\mathcal{A}, f\right]$. By its definition, h has to be the *least* upper bound of this set, thus $h \leq g - x$, which proves the remaining $h + x \overset{\overline{B}}{\leq} g$.

3) $g \leq h + x$: For any $y \in \mathcal{K}\left[\mathcal{A}, f+x\right]$ it holds $f - (y - x) = f + x - y$ (taking into account the convention from Definition 2.1).Thus due to $f + x - y \in \mathcal{A}$ we have $y - x \in \mathcal{K}\left[\mathcal{A}, f\right]$. The upper bound property of the supremum h entails $y - x \leq h$. Or in other words, $h + x$ is an upper bound of the set $\mathcal{K}\left[\mathcal{A}, f+x\right]$. By its definition, g has to be the *least* upper bound of the assertion follows. □

Proof of "⊇" in Theorem 2.1.4. Take any $f \in \mathcal{A}$. It directly follows that $0 \in \mathcal{K}\left[\mathcal{A}, f\right]$, and by the upper bound property of the supremum we have $P[\mathcal{A}](f) \geq 0$. The assertion follows from Definition 2.9 of $A[P[\mathcal{A}]]$. □

Proof of "⊆" in Theorem 2.1.4. Take any $f \in A[P[\mathcal{A}]]$. By Definition 2.9 this implies $0 \leq P[\mathcal{A}](f)$. For any $x \in V_t$, $P[\mathcal{A}]$'s cash invariance ensures $0 < x \leq$

$x + P[\mathcal{A}](f) = P[\mathcal{A}](f + x)$. Now, Lemma A.3 implies $f + x \in \mathcal{A}$. Using eq. (2.2) from Definition 2.4 we arrive at $f \in \mathcal{A}$. □

A.1.3 *Proof of Corollary 2.2*

Proof. Define $\pi \equiv P[\mathcal{A}]$. We have to show that $\sup \mathcal{K}[\mathcal{A}, f] \in \mathcal{K}[\mathcal{A}, f]$, i.e. $\pi(f) \in L_t^-$ and $f - \pi(f) \in \mathcal{A}$. The first statements follows from Theorem A.1 and the hypotheses $\pi(f) < \infty$. Now, prove the second statement using

$$\pi(f - \pi(f)) = \pi(f) - \pi(f) \geq 0 \Longrightarrow f - \pi(f) \in A[P[\mathcal{A}]] = \mathcal{A},$$

which follows from cash invariance, Definition 2.9 and Theorem 2.1.4. □

A.1.4 *Proof of Corollary 2.3*

Proof. Assume $B \in \mathcal{F}_t$, $f, g \in \mathcal{X}$ and $f \overset{B}{=} g$ and define $f' = f + \infty \cdot \mathbb{1}_{\overline{B}}$ and $g' = g + \infty \cdot \mathbb{1}_{\overline{B}}$. By Lemma A.4 it remains to show $\pi(f') = \pi(g')$, which by cash invariance and Theorem 2.1.2 follows from

$$f' - x \in A[\pi] \Leftrightarrow g' - x \in A[\pi].$$

We prove the "⇒"-direction of this equation, the other follows analogously. Assume $f' - x \in A[\pi]$. It holds for every $y \in V_t$ and $h = (g' - x) + y$, that $h = (f' - x) + y$ and thus by Theorem 2.1.1 and eq. (2.2) from Definition 2.4, $h \in A[\pi]$, which by the other direction of eq. (2.2) from Definition 2.4 proves $g' - x \in A[\pi]$. □

A.1.5 *Properties of* inf *and* sup

Lemma A.5. *The following two mappings inherit π's cash invariance:*

$$f \mapsto \sup_{\varphi \in S} \pi(f[\varphi]) \text{ (only if } S \neq \emptyset\text{) and} \quad f \mapsto \inf_{\varphi \in S} \pi(f[\varphi])$$

Proof of the sup*-version.* By Definition 2.8, we need to show for any $x \in L_t^+$ and $f \in \mathcal{X}_{[t,\infty\rangle}$:

$$\sup_{\varphi \in S} \pi((f + x)[\varphi]) = \sup_{\varphi \in S} (\pi(f[\varphi]) + x) = x + \sup_{\varphi \in S} \pi(f[\varphi])$$

The first equation follows from the facts that fixation commutes with addition (trivially by Definition 2.3), $x[\varphi] = x$ (by Remark 2.4), and finally from cash invariance of π.

For the second equation we need $S \neq \emptyset$ and Theorem A.1, together ensuring the existence of a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq S$, such that $g(\omega) \equiv \sup_{\varphi\in S}(\pi(f[\varphi])(\omega) + x(\omega))$ and $g = \sup_{n\in\mathbb{N}}(\pi(f[\varphi_n]) + x)$. It remains to show for all $\omega \in \Omega$:

$$g(\omega) = x(\omega) + \sup_{n\in\mathbb{N}}(\pi(f[\varphi_n])(\omega)) \tag{A.2}$$

For all ω with $x(\omega) = \infty$, this follows directly from the convention introduced in Definition 2.1. For all other ω's, we have $x(\omega) - x(\omega) = 0$ and thus eq. (A.2) follows from translational invariance of the supremum.

$\square$

Proof of the inf*-version.* This prove is completely analogous with the only difference being that the S can be empty. However, in this case the assertion follows directly from the convention $\inf \varnothing = \infty$ and the one introduced in Definition 2.1. $\square$

Lemma A.6. *For any $f \in \mathcal{X}_{[t,\infty\rangle}$, $h, g \in L_t^{\pm}$, local (Corollary 2.3) t-pricing function π, t-compatible S and $x \in V_t$ it holds:*

If $h < \infty$ and $h \leq \sup_{\varphi \in S} \pi\big(f\big[\varphi\big]\big)$, then a $\psi \in S$ exists with $h \leq \pi\big(f\big[\psi\big]\big) + x$.

If $-\infty < g$ and $\inf_{\varphi \in S} \pi\big(f\big[\varphi\big]\big) \leq g$, then a $\psi \in S$ exists with $\pi\big(f\big[\psi\big]\big) \leq g + x$.

Proof of the inf*-version.* As locality is invariant under a $\pi \mapsto -\pi$ substitution an inf-version of Lemma A.6 can be derived from it using the inf/sup-duality, $-\sup A = \inf(-A)$. This derivation also relies on $x \in L_t^+$. $\square$

Proof of the sup*-version.* Using cash invariance (Lemma A.5) and $h < \infty$ we have $h \overset{C}{<} \sup_{\varphi \in S} \pi\big((f + x)\big[\varphi\big]\big)$ with $C \equiv \{-\infty < h\}$. Let $\{B_n\}$ be as in Lemma A.1 with $g_n = \pi\big(f[a_n] + x\big)$ for some sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq S$.

Define $b = \sum_n^\infty a_n \mathbb{1}_{B_n}$. By the hypothesis of our lemma that S is t-compatible (Definition 2.5) and B_n's properties we have $b \in S$. By $\sum_n^\infty a_n \mathbb{1}_{B_n} \overset{B_n}{=} a_n$ and Definitions 2.2 and 2.3 we have $f[b] \overset{B_n}{=} f[a_n]$ and locality yields $g_n \overset{B_n}{=} \pi\big(f\big[b\big] + x\big)$. From $B_n \subseteq \{g_n > h\} \cup \{h = -\infty\}$ and cash invariance we obtain $h \overset{B_n}{\leq} \pi\big(f\big[b\big]\big) + x$ and the lemma now follows from $\mathbb{P}(\bigcup_n B_n) = 1$ and Corollary A.1.2. $\square$

A.1.6 Proof of Theorem 2.2

Proof. Define $\pi \equiv P[\mathcal{A}]$ and prove for any $f \in \mathcal{X}_{[t,\infty\rangle}$:

$$0 \leq \inf_{\varphi \in S} \pi\big(f\big[\varphi\big]\big) \iff \big(\forall a \in S : 0 \leq \pi\big(f\big[a\big]\big)\big) \iff \big(\forall a \in S : f\big[a\big] \in \mathcal{A}\big) \tag{A.3}$$

The first equality follows from the infimum's defining greatest lower bound property and the second from Theorem 2.1.4 and $\mathcal{A}$'s properness.

Furthermore, by Theorem 2.1.3 we follow that π is cash invariant and thus by Lemma A.5 (appendix A.1.5) the inf-pricing function is also cash invariant. Equation (A.3) shows that its dual acceptance set, which is automatically proper by Theorem 2.1.1, equals $\mathcal{A}^{\forall S}$.

The first sentence of the second paragraph follows directly from the lower bound property of the supremum and the last senctence from Lemma A.6 (with $g \to \inf_{\varphi \in S} \pi\big(f\big[\varphi\big]\big)$). $\square$

A.1.7 *Proof of Theorem 2.3*

Proof of Theorem 2.3. The proof follows in analogy to the proof of Theorem 2.2 (appendix A.1.6). We are required to show for any $f \in \mathcal{X}_{[t,\infty\rangle}$:

$$0 \leq \sup_{\varphi \in S} \pi\big(f\big[\varphi\big]\big) \Longleftrightarrow \big(\forall x \in V_t : \exists a \in S : f\big[a\big] + x \in \mathcal{A}\big)$$

To prove "$\Longleftarrow$", note that for any $n \in \mathbb{N}$ an $a_n \in S$ with $f\big[a_n\big] + \frac{1}{n} \in \mathcal{A}$ exists. Theorem 2.1.4, π's cash invariance (by Theorem 2.1.3) and the upper bound property of the supremum imply $0 \leq \pi\big(f\big[a_n\big] + \frac{1}{n}\big) \leq \frac{1}{n} + \sup_{\varphi \in S} \pi\big(f\big[\varphi\big]\big)$. The assertion then follows after taking the limit $n \longrightarrow \infty$.

For the other direction, due to cash invariance and locality (Corollary 2.3) we can apply Lemma A.6 with $h = 0$. It shows that for any $x \in V_t$ an $a \in S$ such that $0 \leq \pi\big(f\big[a\big]\big) + x$ exists and thus by cash invariance and Theorem 2.1.4 we have $f\big[a\big] + x \in \mathcal{A}$.

If the supremum is finite, the upper bound property of the supremum and Lemma A.6 (with $h \rightarrow \sup_{\varphi \in S} \pi\big(f\big[\varphi\big]\big)$) ensure, that for any finite $x \in V_t$, there exists some $\psi \in S$ with:

$$\sup_{\varphi \in S} \pi\big(f\big[\varphi\big]\big) - x \leq \pi\big(f\big[\psi\big]\big) \leq \sup_{\varphi \in S} \pi\big(f\big[\varphi\big]\big)$$

This proves the last statement of the theorem. □

A.1.8 *Proof of Theorem 2.4*

This proof uses the abbreviation $\varphi \cdot X = \int_0^\infty \varphi_t \cdot \mathrm{d}X_t$.

Proof. Take any $f \in L_\infty$ bounded from above by some $z \in \mathbb{R}$, define $\mathcal{C} \equiv \mathcal{K}\big[\mathcal{A}, f\big]$ (Definition A.1) and show:

$$\begin{aligned} \mathcal{C} =& \Big\{x \in L_0^- \;\Big|\; f - x + H \in \big\{g \big| g \geq 0\big\}^{\exists S}\Big\} \\ =& \big\{x \in L_0^- \;\big|\; \forall y \in V_0, \exists \varphi \in S : f - x + \varphi \cdot X + y \geq 0\big\} \end{aligned} \tag{A.4}$$

The first equation follows from the definition of $\mathcal{A}$. The second uses Definition 2.11 and H's definition.

Now, define $\mathcal{D} = \big\{x \in L_0^- \;\big|\; \exists \varphi \in S : f + \varphi \cdot X \geq x\big\}$ and show

$$\sup \mathcal{D} = \sup \mathcal{C} \tag{A.5}$$

The "$\leq$" inequality follows directly from the trivial $\mathcal{D} \subseteq \mathcal{C}$. To prove the reverse equality by $\sup \mathcal{C}$'s *least* upper bound property, it remains shown that $\sup \mathcal{D}$ is an upper bound of $\mathcal{C}$: For any $x \in \mathcal{C}$ and $n \in \mathbb{N}$ it follows from eq. (A.4) that there exists a φ in S, such that $f + \varphi \cdot X \geq x - \frac{1}{n}$. Consequently, $x - \frac{1}{n} \in \mathcal{D}$ and thus $x - \frac{1}{n} \leq \sup \mathcal{D}$, which also holds in the limit $n \longrightarrow \infty$.

We can complete the proof:

$$\pi(f) = \sup C = \sup \mathcal{D} = \inf_{Q \in \mathcal{M}} \mathbb{E}_Q[f]$$

The first equations uses Definition 2.6 of the pricing function, the second uses eq. (A.5), and the thrid Theorem 5.12 of Delbaen and Schachermayer (1998, p. 246) with the substitution $g \to -f$ and the infimum-supremum duality. □

A.1.9 *Proof of Theorem 2.5*

This proof requires the following lemma

Lemma A.7. *Given a proper $\mathcal{A}^0$ with its dual pricing function π^0, it holds for all $f \in \mathcal{X}_\emptyset$ and t-acceptance sets $\mathcal{A}$ with $\mathcal{A} \cap \mathcal{X}_\emptyset = \mathcal{A}^0$*

$$P[\mathcal{A}](f) = \pi^0(f).$$

Proof. By Definition 2.6 and $\mathcal{A} \cap \mathcal{X}_\emptyset = \mathcal{A}^0$ it remains to show $\mathcal{K}[\mathcal{A}, f] = \mathcal{K}[\mathcal{A} \cap \mathcal{X}_\emptyset, f]$ for any $f \in \mathcal{X}_\emptyset$, which follows from

$$f - x \in \mathcal{X}_\emptyset \text{ for all } x \in L_t^-, f \in \mathcal{X}_\emptyset.$$

□

Proof of Theorem 2.5. By Theorems 2.2 and 2.3 it holds for any i with $\tau_i \in \mathbf{T}_c$ and $f \in \mathcal{X}_{[t,\infty\rangle}$:

$$P[\mathcal{D}_i](f) = \inf_{a_i \in S_i} \sup_{a_{i+1} \in S_{i+1}} P[\mathcal{D}_{i+2}](f[a_i][a_{i+1}])$$

We can use this relation recursively until we reach $P[\mathcal{D}_{n+1}](g) = P[\mathcal{A}^0](g)$, with $g \equiv f[a_1]\dots[a_n]$. By Definition 2.3 it is easy to see that $g \in \mathcal{X}_\emptyset$ and thus using $\mathcal{A}^0 \subseteq \mathcal{X}_\emptyset$ and Lemma A.7 we have $P[\mathcal{A}^0](g) = \pi^0(g)$. □

A.1.10 *Proof of Corollary 2.4*

Proof of the "if". Define $g = \pi_s(f) - \pi_s(0)$ and $B \equiv \{\pi_s(f) > -\infty\}$ ($B = \{g > -\infty\}$ due to $|\pi_\cdot(0)| < \infty$). Assuming eq. (2.5) we can prove Definition 2.13 by showing

$$\pi_s(g) = \pi_s(f).$$

1) We have

$$\pi_s(g) \overset{B}{=} \pi_s(0 + g \cdot \mathbb{1}_B) \overset{B}{=} \pi_s(0) + \pi_s(f) - \pi_s(0) = \pi_s(f)$$

The first equation uses locality (by Corollary 2.3), the second cash invariance and $g \cdot \mathbb{1}_B \in L_t^+$ and the third $|\pi_\cdot(0)| < \infty$.

2) $\pi_s(g) \overset{\overline{B}}{\geq} \pi_s(f)$ is trivial.
3) Note that $g \leq \infty \cdot \mathbb{1}_B - n$ for all $n \in \mathbb{N}$. Using monotonicity and cash invariance we have $\pi_s(g) \leq \infty \cdot \mathbb{1}_B - n + \pi_s(0)$ for any n and therefore due to $|\pi_\cdot(0)| < \infty$: $\pi_s(g) \overset{\overline{B}}{\leq} -\infty \overset{\overline{B}}{=} \pi_s(f)$ □

Proof of the "only if". Assume $\pi_\cdot$ satisfies Definition 2.13 and show

$$\pi_s(f) \geq \pi_s(g) \Longrightarrow \pi_t(\pi_s(f)) \geq \pi_t(\pi_s(g)) \Longrightarrow \pi_t(f) \geq \pi_t(g)$$

The first implication follows from monotonicity of π_t and the second from time consistency. □

A.1.11 Proof of Lemma 2.1

Proof. The first statement follows directly from the lemma below. To prove eq. (2.4), take any $\varphi \in \Phi$ and show:

$$\pi(f+g)(\varphi) = \pi\Big((f+g)\big[\varphi|_{\langle -\infty,t\rangle}\big]\Big) = (\pi(f)+g)(\varphi)$$

The second equality follows from cash invariance of π, $g \in \mathcal{X}_{\langle -\infty,t\rangle}$ and $g(\varphi) \in L_t^+$. □

Lemma A.8. *For any $f \in \mathcal{X}_\mathbf{T}$ and cash invariant t-pricing function π, the mapping $\varphi \mapsto \pi\Big(f\big[\varphi|_{\langle -\infty,t\rangle}\big]\Big)$ is an element of $\mathcal{X}^t_{\mathbf{T}\cap\langle -\infty,t\rangle}$.*

Proof. That $\pi\Big(f\big[\varphi|_{\langle -\infty,t\rangle}\big]\Big)$ is well-defined an $\mathcal{F}_t$-measurable, follows from the definition of a t-pricing function in subsection 2.3.2.

To prove the remaining properties in Definition 2.2 take $B \in \mathcal{F}_t$ and $\varphi, \psi \in \Phi$ with $\psi_s \stackrel{B}{=} \varphi_s$ for all $s \in \mathbf{T} \cap \langle -\infty, t\rangle \cap \mathbf{T}_d$. The required equation

$$\pi\Big(f\big[\varphi|_{\langle -\infty,t\rangle}\big]\Big) \stackrel{B}{=} \pi\Big(f\big[\psi|_{\langle -\infty,t\rangle}\big]\Big)$$

follows directly from locality (Corollary 2.3) and

$$f\big[\varphi|_{\langle -\infty,t\rangle}\big](\rho) \stackrel{B}{=} f\big[\psi|_{\langle -\infty,t\rangle}\big](\rho), \text{ for any } \rho \in \Phi,$$

which in turn follows from $f \in \mathcal{X}_\mathbf{T}$ and Definitions 2.2 and 2.3. □

A.1.12 Proof of Theorem 2.6

Proof of time consistency of $P[\mathcal{A}_\cdot]$. Define $\pi_\cdot \equiv P[\mathcal{A}_\cdot]$. By Lemma A.8 we have $\pi_s(f) \in \mathcal{X}_{[t,s\rangle} \subseteq \mathcal{X}_{[t,\infty\rangle}$ and thus by Definition 2.6 it remains to show $\mathcal{K}\big[\mathcal{A}_t, f\big] = \mathcal{K}\big[\mathcal{A}_t, \pi_s(f) - \pi_s(0)\big]$. Take any $x \in L_t^-$ and prove:

$$f - x \in \mathcal{A}_t \Longleftrightarrow \pi_s(f-x) - \pi_s(0) \in \mathcal{A}_t \Longleftrightarrow \big(\pi_s(f) - \pi_s(0)\big) - x \in \mathcal{A}_t$$

The two equivalences follow from eq. (2.6) and cash invariance (Theorem 2.1.3 and Definition 2.8). □

Proof of time consistency of $A[\pi_\cdot]$. Define $\mathcal{A}_\cdot \equiv A[\pi_\cdot]$ and show:

$$f \in \mathcal{A}_t \Longleftrightarrow \pi_t(f) \geq 0 \Longleftrightarrow \pi_t\big(\pi_s(f) - \pi_s(0)\big) \geq 0 \Longleftrightarrow \pi_s(f) - \pi_s(0) \in \mathcal{A}_t$$

The first and third equivalences follow from Definition 2.9. The second from π's time consistency. □

A.1.13 Proof of Theorem 2.7

This proof requires the following lemma

Lemma A.9. *Take a proper $\mathcal{A}^0_\cdot$ with dual $\pi^0_\cdot$ and some $\mathcal{A}_\cdot$ with dual $\pi_\cdot$ that satisfies $\mathcal{A}_\cdot \cap \mathcal{X}_\varnothing = \mathcal{A}^0_\cdot$. Taking into account Lemma 2.1, it holds $\pi_t(f) = \pi^0_t(f)$ for any payoff with no decisions at or after time t.*

Proof. This follows directly from Lemmas 2.1 and A.7 and $f\left[\varphi|_{\langle-\infty,t\rangle}\right] \in \mathcal{X}_\varnothing$ for all $\varphi \in \Phi$. □

Proof of Theorem 2.7. Define $\pi_\cdot \equiv P[\mathcal{A}_\cdot]$. Show for all $j \in \{i, \dots, n\}$ with $\tau_j \in \mathbf{T}_a$, $\tau_{j-1} < s \le \tau_j$ and $g \in \mathcal{X}$ or $s = \tau_{j-1}$ and $g \in \mathcal{X}_{\mathcal{T}\setminus\{s\}}$:

$$\pi_s(g) = \pi_s\Big(\pi_{\tau_j}(g)\Big) = \pi^0_s\Big(\pi_{\tau_j}(g)\Big) = \pi^0_s\left(\inf_{a_j \in S_j} \pi_{\tau_j}(g[a_j])\right)$$

The first equation follows from $\mathcal{A}_\cdot$'s time consistency, $\pi^0_\cdot$'s normalization and Theorem 2.6. The second follows from Lemma A.9 with $\pi_{\tau_j}(g) \in \mathcal{X}_{\langle-\infty,s\rangle}$ (by Lemma A.8). And the third from Remark 2.5, eq. (2.7) and Theorem 2.3.

For j with $\tau_j \in \mathbf{T}_c$, Theorem 2.2 would be needed instead of Theorem 2.3 and inf instead of sup.

To prove the theorem we start with $s \to t$ and $g \to f$. Then we can apply this equation recursively with $j \to j+1$, $s \to \tau_j$ due to $g[a_j] \in \mathcal{X}_{\mathcal{T}\setminus\{\tau_j\}}$. □

A.2 Mathematical theorems

Theorem A.1 (Existence of the essential supremum)**.** *Suppose that the measure space $(\Omega, \mathcal{F}, \mu)$ is σ-finite. Then the essential supremum of a collection $\mathcal{S}$ of measurable functions into the set $\overline{\mathbb{R}}$ exists. Furthermore, if $\mathcal{S}$ is nonempty then a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ exists such that its pointwise supremum equals (almost surely) the essential supremum of $\mathcal{S}$.*

Proof. See Chapter V.18 in Doob (1994). □

Corollary A.1 (Properties of conditionally almost sure)**.** *Let $\mathbf{D}$ be a countable collection of sets with positive probability and $A, B \in \mathbf{D}$.*

(1) $\mathbb{P}(B \setminus C) = 0 \Longleftrightarrow \mathbb{P}(C|B) = 1$

(2) $(\forall A \in \mathbf{D} : \mathbb{P}(C|A) = 1) \Longrightarrow \mathbb{P}(C|\bigcup \mathbf{D}) = 1$

(3) $\mathbb{P}(B|A) = \mathbb{P}(C|B) = 1 \Longrightarrow \mathbb{P}(C|A) = 1$

Proof of Corollary A.1.1. The statement follows from:

$$\mathbb{P}(B) = \mathbb{P}((B \setminus C) \cup (B \cap C)) = \mathbb{P}(B \setminus C) + \mathbb{P}(C \cap B) = \mathbb{P}(B \setminus C) + \mathbb{P}(C|B)\mathbb{P}(B)$$

Using Corollaries A.2.1 and A.2.2, additivity of $\mathbb{P}$ for disjoint sets and the definition of conditional probability. □

Proof of Corollary A.1.2. By Corollary A.1.1 it follows from the antecedent for any $A \in \mathbf{D}$: $\mathbb{P}(A \setminus C) = 0$. By using Corollary A.2.3 with subadditivity of $\mathbb{P}$ we can follow $\mathbb{P}((\bigcup \mathbf{D}) \setminus C) = 0$, which again with Corollary A.1.1 yields the result. □

Proof of Corollary A.1.3. Using Corollary A.1.1 we can follow from the assumption: $\mathbb{P}(A \setminus B) = \mathbb{P}(B \setminus C) = 0$. By using Corollary A.2.4 with monotonicity and subadditivity of $\mathbb{P}$, it follows $\mathbb{P}(A \setminus C) = 0$, which again with Corollary A.1.1 yields the result. □

Corollary A.2 (Properties of sets). *For all sets A, B, C and collections of sets $\mathbf{D}$, it holds: (1) $(A \setminus B) \cap (A \cap B) = \emptyset$, (2) $(A \setminus B) \cup (A \cap B) = A$, (3) $\bigcup_{E \in \mathbf{D}} (E \setminus A) = (\bigcup \mathbf{D}) \setminus A$, and (4) $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$.*

The proofs are left to the reader. □

Corollary A.3 (Convergence in probability preserves equality almost surely). *For any set $B \in \mathcal{F}_\infty$ and two sequences of random variables $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ that converge in probability to f and g, respectively, it holds $f \overset{B}{=} g$, if $B \subseteq \{f_n = g_n\}$.*

Proof. Simple set manipulations yield:

$$B \setminus \{f = g\} = B \cap \{\exists m \in \mathbb{N} : |f - g| > \tfrac{1}{m}\} = \bigcup_{m \in \mathbb{N}} B \cap \{|f - g| > \tfrac{1}{m}\}$$

Consequently, by Corollary A.1.1 and monotonicity of $\mathbb{P}$, $f \overset{B}{=} g$ is equivalent to

$$\mathbb{P}\big(B \cap \{|f - g| > \tfrac{1}{m}\}\big) = 0 \text{ for all } m \in \mathbb{N} \tag{A.6}$$

We prove eq. (A.6) in two steps. First show for any $m, n \in \mathbb{N}$:

$$B \cap \{|f - g| > \tfrac{1}{m}\} \subseteq \{|f - f_n| + |g - g_n| > \tfrac{1}{m}\} \subseteq \{|f - f_n| > \tfrac{1}{2m}\} \cup \{|g - g_n| > \tfrac{1}{2m}\}$$

The first "$\subseteq$" uses $B \subseteq \{f - g = (f - f_n) - (g - g_n)\}$, together with the triangle inequality. By definition of convergence in probability we can then prove:

$$\mathbb{P}\big(B \cap \{|f - g| > \tfrac{1}{m}\}\big) \leq \mathbb{P}\big(\{|f - f_n| > \tfrac{1}{2m}\}\big) + \mathbb{P}\big(\{|g - g_n| > \tfrac{1}{2m}\}\big) \to 0$$

□

Corollary A.4 (Integration preserves equality almost surely). *For any set $B \in \mathcal{F}_\infty$, integrator X and integrands a_1, b_2, it holds $\int a_1 \, \mathrm{d}X \overset{B}{=} \int a_2 \, \mathrm{d}X$, if $B \subseteq \{a_1 = a_2\}$.*

Proof. Let $(a_i^n)_{n<\infty}$ be a sequence of simple integrands approximating a_i. We can choose them such that $a_1^n = a_2^n$ on B for all n. Define $A_i \equiv \int_t^T a_i \mathrm{d}X \equiv \operatorname{plim}_{n\to\infty} A_i^n$ (limit in probability) with $A_i^n \equiv \int_t^T a_i^n \mathrm{d}X$. These integrals of simple integrands are defined path-wise and thus $B \subseteq \{A_1^n = A_2^n\}$ for each n. By Corollary A.3 it holds $P(A_1 = A_2 | B) = 1$. □

APPENDICES TO CHAPTER 3

B

B.1 Pointwise defined payoffs

Corollary B.1. *Given a function $g : \Omega \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ and a sequence of times $(\tau_i)_{i\in\mathbb{N}}$, define f:*

$$f(\varphi)(\omega) = g\big(\omega, \big(\varphi_{\tau_i}(\omega)\big)_{i\in\mathbb{N}}\big), \text{ for all } \varphi \in \Phi \text{ and } \omega \in \Omega \tag{B.1}$$

If $f(\varphi)$ is $\mathcal{F}_\infty$-measurable for all $\varphi \in \Phi$, then $f \in \mathcal{X}_{\{\tau_i\}_{i\in\mathbb{N}}}$.

Proof. As in Definition 3.1, we assume some set $B \in \mathcal{F}_\infty$ and two decision procedures with $\psi_{\tau_i} \overset{B}{=} \varphi_{\tau_i}$ for all i. As a direct consequence of eq. (B.1) we have $\bigcap_{i\in\mathbb{N}}\{\varphi_{\tau_i} = \psi_{\tau_i}\} \subseteq \{f(\varphi) = f(\psi)\}$ and applying basic set operations yields:

$$B \setminus \{f(\varphi) = f(\psi)\} \subseteq B \setminus \bigcap_{i\in\mathbb{N}}\{\varphi_{\tau_i} = \psi_{\tau_i}\} = \bigcup_{i\in\mathbb{N}}\big(B \setminus \{\varphi_{\tau_i} = \psi_{\tau_i}\}\big)$$

From monotonicity and sub-additivity of the probability measure we follow:

$$\mathbb{P}\big(B \setminus \{f(\varphi) = f(\psi)\}\big) \le \sum_{i\in\mathbb{N}} \mathbb{P}\big(B \setminus \{\varphi_{\tau_i} = \psi_{\tau_i}\}\big)$$

Due to $\mathbb{P}\big(\{\varphi_{\tau_i} = \psi_{\tau_i}\}\big|B\big) = 1$ for all i (by assumption) and Corollary B.3.1 both sides of this inequality are zero and thus by the same Corollary $\mathbb{P}\big(\{f(\varphi) = f(\psi)\}\big|B\big) = 1$.

□

B.2 Proofs

B.2.1 *Proof of Lemma 3.1*

Proof. Take any $\varphi \in \Phi$ and show:

$$\begin{aligned} P[\mathcal{B}_t(\varphi)](f)(\varphi) &= \sup\Big\{x \in L_t^- \;\Big|\; f\big[\varphi|_{\langle-\infty,t\rangle}\big] - x \in \mathcal{B}_t(\varphi)\Big\} \\ &= \sup\Big\{x \in L_t^- \;\Big|\; \big(f + H_i\big)\big[\varphi|_{\langle-\infty,t\rangle}\big] - x \in \mathcal{A}_t\Big\} = \pi_t\big(f + H_i\big)(\varphi) \end{aligned}$$

The first and third equation use the definition of $P[\mathcal{A}]$ (Definition 3.7). The second uses the definition of $\mathcal{B}_t$, cf. eq. (3.5). □

B.2.2 *Proof of Theorem 3.1*

Proof. To prove eq. (3.7), we start with Lemma 3.1 and apply cash invariance (Definition 3.5, applicable due to $H_{\tau_{n+1}} = -C_{n+1} \in \mathcal{X}^{\tau_{n+1}}_{\{\tau_n\}}$).

To prove eq. (3.8), we define $\Delta_a^b \equiv H_a - H_b$. By H's definition in eq. (3.4) and Axiom 3.1 we can infer for any $\tau_i < t \leq \tau_{i+1}$ and $\varphi \in \Phi$:

$$\begin{aligned}&\Delta^t_{\tau_i}(\varphi) = \varphi_{\tau_i} \cdot \left(X_t - X_{\tau_i}\right) - C_i(\varphi) \in L_t \\ &\text{and thus } \Delta^t_{\tau_i} \in \mathcal{X}_{\{\tau_{i-1},\tau_i\}}\end{aligned} \tag{B.2}$$

$\mathcal{A}_.$'s time consistency (and $\pi_.$'s normalization) with Theorem B.3 implies $\pi_.$'s time consistency. Now we can prove eq. (3.8) by eliminating the two decisions at τ_i and s_i:

$$\begin{aligned}
\eta_{\tau_i}(f) &= \pi_{\tau_i}\left(f + H_{\tau_{i+1}} + \Delta^{\tau_{i+1}}_{\tau_i}\right) \\
&= \pi_{\tau_i}\left(\pi_{\tau_{i+1}}\left(f + H_{\tau_{i+1}}\right) + \Delta^{\tau_{i+1}}_{\tau_i}\right) \\
&= \sup_{\varphi \in R_i} \pi_{\tau_i}\left(\left(p + \Delta^{\tau_{i+1}}_{\tau_i}\right)[\varphi]\right) \\
&= \sup_{\varphi \in R_i} \pi_{\tau_i}\left(\inf_{\psi \in S_i} \pi_{s_i}\left(\left(p + \Delta^{\tau_{i+1}}_{\tau_i}\right)[\varphi][\psi]\right)\right) \\
&= \sup_{\varphi \in R_i} \pi_{\tau_i}\left(\inf_{\psi \in S_i} \pi_{s_i}\left(p[\varphi][\psi] + \varphi_{\tau_i} \cdot \left(X_{\tau_{i+1}} - X_{\tau_i}\right)\right)\right) - C_i[\varphi]
\end{aligned}$$

The first equation follows from Lemma 3.1 and Δ's definition. The second uses time consistency (Definition 3.8) to introduce $\pi_{\tau_{i+1}}$ and eq. (B.2) with cash invariance to pull out Δ. The third applies Axiom 3.2 and Theorem B.2 to introduce the supremum. Furthermore it applies Lemma 3.1 in reverse and abbreviates $p \equiv \eta_{\tau_{i+1}}(f)$. The fourth uses again time consistency and applies Axiom 3.2 and Theorem B.1 to introduce $\inf \pi_{s_i}$. The fifth expands Δ and pulls out $C_i[\varphi]$ using cash invariance. □

B.2.3 *Proof of Lemma 3.2*

Proof "≥". Follows directly from monotonicity of the supremum[1] and the fact that $t \mapsto a \in S$ for any $a \in D_t$ by Definition of S. □

Proof "≤". Take any $\varphi \in S$. We have either $D_t = \{a_n\}_{n\in\mathbb{N}}$ or $\Omega = \{\omega_n\}_{n\in\mathbb{N}}$. In the latter case define $a_n \equiv \varphi_t(\omega_n)$ ($\in D_t$ by Definition of $\Phi_\mathbf{T}$ in eq. (3.1)). In both cases define $B_n \equiv \{\varphi_t = a_n\}$ for any n. It trivially holds $\bigcup_n B_n = \Omega$. By definition of S, φ is adapted and thus $B_n \in \mathcal{F}_t$. By Definition 3.1 of $\mathcal{X}$, we have $f[\varphi] \overset{B_n}{=} f[t \mapsto a_n]$ and thus from locality (by Corollary B.2) we can follow:

$$\pi(f[\varphi]) \overset{B_n}{=} \pi(f[t \mapsto a_n]) \leq \sup_{a \in D_t} \pi(f[t \mapsto a])$$

[1] $\sup A \leq \sup B$, if $A \subseteq B$

$\bigcup_n B_n = \Omega$ together with Corollary B.3.2 yields

$$\pi\big(f[\varphi]\big) \le \sup_{a \in D_t} \pi\big(f[t \mapsto a]\big).$$

The assertion now follows from the least upper bound property of the supremum.

□

B.2.4 *Proof of Theorem 3.2*

Proof. Equation (3.14) follows from Theorem 3.1 and eq. (3.7) due to $f_{n+1} = 0$ and π's normalization.

Now prove eq. (3.15). With p's definition and Theorem 3.1 and eq. (3.8) we get:

$$\begin{aligned} p_i &= -\sup_{\varphi \in R_i} \pi_{\tau_i}\left(\inf_{\psi \in S_i} \pi_{s_i}\big(\eta_{\tau_{i+1}}(-f_i)[\varphi][\psi] + \varphi_{\tau_i} \cdot (X_{\tau_{i+1}} - X_{\tau_i})\big)\right) - C_i[\varphi] \\ &= \inf_{\varphi \in R_i} -\pi^0_{\tau_i}\left(\inf_{\psi \in S_i} \eta_{\tau_{i+1}}(-f_i)[\varphi][\psi] + \varphi_{\tau_i} \cdot (X_{\tau_{i+1}} - X_{\tau_i})\right) + C_i[\varphi] \end{aligned}$$

The last equation uses the inf/sup-duality and eliminates π_{s_i} using eq. (3.13) and π's cash invariance and normalization. Furthermore, as the argument of the outer pricing function does not depend on any decision, we can replace the outer pricing function by its version without decisions (according to Lemma B.1). It remains to show:

$$\begin{aligned} \inf_{\psi \in S_i} \eta_{\tau_{i+1}}(-f_i)[\psi] &= \min\{\eta_{\tau_{i+1}}(-g_{s_i}), \eta_{\tau_{i+1}}(-f_{i+1})\} \\ &= -\max\{g_{\tau_{i+1}} - \eta_{\tau_{i+1}}(0), p_{i+1}\} \end{aligned}$$

The first equation uses eq. (3.11) after an application of Lemma 3.2, possible due to the explicit definition of S in eq. (3.12) and the fact that D_{s_i} from eq. (3.9) is finite. The second equation employs eq. (3.13), $\eta_\cdot$'s cash invariance, p's definition and the min/max-duality. □

B.3 Results from Gerer and Dorfleitner (2016b, referenced hereafter as GD16)

Lemma B.1 (Lemma A.9 in GD16)**.** *Take a $\mathcal{A}^0_\cdot$ with dual $\pi^0_\cdot$ and some $\mathcal{A}_\cdot$ with dual $\pi_\cdot$ that satisfies $\mathcal{A}_\cdot \cap X_\varnothing = \mathcal{A}^0_\cdot$. Taking into account Remark 3.3, it holds $\pi_t(f) = \pi^0_t(f)$ for any payoff with no decisions at or after time t.*

Definition B.1 (Locality, Corollary 3.3 in GD16)**.** *A t-pricing function π is called* local*, if*

$$\pi(f) \overset{B}{=} \pi(g), \text{ for all } B \in \mathcal{F}_t \text{ and } f \overset{B}{=} g$$

Corollary B.2 (Corollary 3.3 in GD16)**.** *Cash invariance implies locality.*

Theorem B.1 (Theorem 4.1 in GD16)**.** *If $\mathcal{A}$ is a proper t-acceptance set with price π, then $\mathcal{A}^{\forall S}$ also is a proper t-acceptance set and its pricing function is given by*

$$P[\mathcal{A}^{\forall S}](f) = \inf_{\varphi \in S} \pi\left(f\left[\varphi\right]\right) \text{ for all } f \in \mathcal{X}_{[t,\infty\rangle}$$

The agent's price for any actual decisions procedure followed by the counterparty is always higher than this price. The agent can keep the difference in the form of his or her own profit. However, the counterparty can make this profit arbitrarily small (if the infimum is finite).

Theorem B.2 (Theorem 4.2 in GD16)**.** *If S is t-compatible and $\mathcal{A}$ is a proper t-acceptance set with price π, then $\mathcal{A}^{\exists S}$ is also a proper t-acceptance set and its pricing function is given by:*

$$P[\mathcal{A}^{\exists S}](f) = \sup_{\varphi \in S} \pi\left(f\left[\varphi\right]\right) \text{ for all } f \in \mathcal{X}_{[t,\infty\rangle}$$

While this price can be higher than the price for an actually realized decision procedure by the agent, he or she can make this loss arbitrarily small (if the supremum is finite).

Theorem B.3 (Theorem 5.1 in GD16)**.** *An acceptance family is time consistent (cf. Definition 3.8) if and only if its dual pricing family is time consistent.*

Corollary B.3 (Properties of conditionally almost sure, Corollary B.1 in GD16)**.** *Let $\mathbf{D}$ be a countable collection of sets with positive probability and $A, B \in \mathbf{D}$.*

(1) $\mathbb{P}(B \setminus C) = 0 \Longleftrightarrow \mathbb{P}(C|B) = 1$

(2) $(\forall A \in \mathbf{D} : \mathbb{P}(C|A) = 1) \Longrightarrow \mathbb{P}(C|\bigcup \mathbf{D}) = 1$

(3) $\mathbb{P}(B|A) = \mathbb{P}(C|B) = 1 \Longrightarrow \mathbb{P}(C|A) = 1$

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