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the Dynamics of Fluidic Membranes

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Abstract

We develop a finite element scheme to approximate the dynamics of two and three dimensional fluidic membranes in Navier–Stokes flow. Local inextensibility of the membrane is ensured by solving a tangential Navier–Stokes equation, taking surface viscosity effects of Boussinesq–Scriven type into account. In our approach the bulk and surface degrees of freedom are discretized independently, which leads to an unfitted finite element approximation of the underlying free boundary problem. Bending elastic forces resulting from an elastic membrane energy are discretized using an approximation introduced by Dziuk (2008). The obtained numerical scheme can be shown to be stable and to have good mesh properties. Finally, the evolution of membrane shapes is studied numerically in different flow situations in two and three space dimensions. The numerical results demonstrate the robustness of the method, and it is observed that the conservation properties are fulfilled to a high precision.

Key words. fluidic membranes, incompressible two-phase flow, parametric finite elements, virtual element method, Helfrich energy, Boussinesq–Scriven surface fluid, local surface area conservation

AMS subject classifications. 65M60, 65M12, 76M10, 76Z99, 92C05, 35Q35, 76D05

1   Introduction

The evolution of lipid bilayer membranes is driven by the bending energy, which involves the curvature of the membrane, and hydrodynamics. Lipid membranes typically form vesicles, i.e. bag-like structures containing fluid, which are surrounded by a possibly different fluid. The omnipresence of membranes in biological systems has led to a growing

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interest in vesicles over the past decades. Much of the work on vesicles was motivated by the fact that their shape at rest resembles the biconcave forms of red blood cells. It is the goal of this paper to introduce, and analyze, a finite element method of a model for the evolution of lipid membranes, which was introduced by Arroyo and DeSimone (2009).

The bending energy for a lipid membrane used in this paper is

$$E_\alpha(\Gamma) = \alpha\, E(\Gamma)\,, \quad \text{with} \quad E(\Gamma) = \tfrac{1}{2} \int_\Gamma \varkappa^2 \, \mathrm{d}\mathcal{H}^{d-1}\,,$$

where the bilayer is modelled as a closed hypersurface Γ in $\mathbb{R}^d$, $d = 2$ or 3. By $\varkappa$ we denote the mean curvature (the sum of the principal curvatures) of Γ, $\alpha \in \mathbb{R}_{>0}$ is the bending rigidity and $\mathrm{d}\mathcal{H}^{d-1}$ indicates integration with respect to the $(d-1)$-dimensional surface measure. In the simplest energetical model for vesicles one minimizes the energy $E_\alpha(\Gamma)$ under the constraints that the area of Γ is fixed and that Γ encloses a fixed volume. The latter is due to the fact that the osmotic conditions of the fluids surrounding the membrane lead to a fixed volume. Furthermore, the vesicle can be considered as locally incompressible, which leads to a fixed total surface area. For a deeper physical discussion of these conditions we refer to the overview article of Seifert (1997), where also other aspects of fluidic membranes and vesicles are thoroughly discussed.

In the fluid regions, Ω_- and Ω_+, inside and outside of the membrane, one requires the incompressible Navier–Stokes equations, i.e.

$$\rho\,(\vec{u}_t + (\vec{u}\,.\,\nabla)\,\vec{u}) - \nabla\,.\,\underline{\underline{\sigma}} = \rho\,\vec{f}, \qquad \nabla\,.\,\vec{u} = 0\,.$$

At typical temperatures the membrane itself is in a fluidic state, which leads to the fact that on the membrane the incompressible surface Navier–Stokes equations

$$\rho_\Gamma\,\partial_t^\bullet\,\vec{u} - \nabla_s\,.\,\underline{\underline{\sigma}}_\Gamma = [\underline{\underline{\sigma}}\,\vec{\nu}]_-^+ + \alpha\,f_\Gamma\,\vec{\nu}, \qquad \nabla_s\,.\,\vec{u} = 0$$

have to hold. Here $\underline{\underline{\sigma}}_\Gamma$ is the surface stress tensor, $\vec{\nu}$ is the unit normal on Γ and $[\underline{\underline{\sigma}}\,\vec{\nu}]_-^+$ describes stresses acting on the membrane via the normal stresses $\underline{\underline{\sigma}}\,\vec{\nu}$ from both sides of the membrane (see Section 2 for precise definitions). The operator $\nabla_s\,.$ is the surface divergence and $\nabla_s\,.\,\vec{u} = 0$ models the fact that the membrane is locally incompressible.

Interfacial fluid mechanics was first thoroughly discussed by Scriven (1960), generalizing earlier ideas of Boussinesq. In this context the surface stress tensor $\underline{\underline{\sigma}}_\Gamma$ was first introduced, and is hence called the Boussinesq–Scriven tensor. In addition, $\alpha\,f_\Gamma\,\vec{\nu}$ models forces acting on the membrane, which result from the curvature elasticity $E_\alpha(\Gamma)$. The forces act in a direction normal to the membrane and f_Γ is given as minus the first variation of $E(\Gamma)$, i.e.

$$f_\Gamma = -\Delta_s\,\varkappa - \varkappa\,|\nabla_s\,\vec{\nu}|^2 + \tfrac{1}{2}\,\varkappa^3\,, \tag{1.1}$$

where Δ_s is the surface Laplacian and ∇_s is the surface gradient.

In recent years, many papers have appeared which numerically approximate the L^2-gradient flow equation related to the Willmore energy $E(\Gamma)$, i.e.

$$\mathcal{V} = -\Delta_s\,\varkappa - \varkappa\,|\nabla_s\,\vec{\nu}|^2 + \tfrac{1}{2}\,\varkappa^3\,, \tag{1.2}$$

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where $\mathcal{V}$ is the normal velocity of the evolving membrane Γ. This geometric evolution equation is called Willmore flow, and we refer to Mayer and Simonett (2002); Clarenz *et al.* (2004); Dziuk (2008); Barrett *et al.* (2008b); Deckelnick and Dziuk (2009); Deckelnick and Schieweck (2010); Barrett *et al.* (2012); Franken *et al.* (2013) for different computational approaches to Willmore flow. Since the enclosed volume and the total surface area are preserved for lipid membranes, the volume and area preserving variant of (1.2), which is called Helfrich flow, is of particular interest. Helfrich flow has been considered numerically in e.g. Barrett *et al.* (2008b); Bonito *et al.* (2010). Other authors included additional physical effects in the geometrical model, such as lateral inhomogeneity and line tension effects, see Elliott and Stinner (2010); Mercker *et al.* (2013). Bonito *et al.* (2011) considered a fluid-membrane system in which forces resulting from the Willmore energy act on an interior flow. In their model surface area is maintained with the help of a global Lagrange multiplier.

As pointed out above, the membrane is locally incompressible and hence the condition $\nabla_s \cdot \vec{u} = 0$ should be enforced on the flow. This condition has been dealt with in numerical simulations by Salac and Miksis (2011, 2012); Laadhari *et al.* (2014) within a level set context, by Jamet and Misbah (2007); Aland *et al.* (2014) with the help of a phase field approach and by Hu *et al.* (2014) using an immersed boundary method. In these approaches the local incompressibility constraint on the membrane is enforced by a Lagrange multiplier leading to an inhomogeneous surface pressure. However, in the computations in the latter paper the constraint is relaxed by a spring-like elastic force. In addition, there exists work on the surface Stokes system without taking the bulk fluid flow into account. There the volume conservation is enforced by a global Lagrange multiplier. We refer to Rahimi and Arroyo (2012); Rodrigues *et al.* (2014) for numerical results using this modelling variant. The only numerical work taking simultaneously surface and bulk viscosity effects in the fluidic membrane evolution into account is by Arroyo *et al.* (2010). However, their results are restricted to the axisymmetric situation. In addition, their numerical method cannot be shown to be stable, as is the case for all of the above numerical methods.

We also refer to numerical work by McWhirter *et al.* (2009); Franke *et al.* (2011); Krüger *et al.* (2013); Shi *et al.* (2014) on the evolution of red blood cells, which study the influence of the elastic effects resulting from the cytoskeleton on the membrane evolution. Finally, we mention that analytical well-posedness issues for the model considered in this paper are currently being addressed by Köhne and Lengeler (2014); Lengeler (2014).

Building on earlier work by the present authors on two-phase flow and by Dziuk (2008) on Willmore flow, it is the main goal of this paper to introduce and analyze a numerical method for the full membrane evolution problem. Our numerical approach has the following features:

- The bulk and surface degrees of freedom are discretized with standard bulk and surface finite elements leading to an unfitted finite element method.

- The effects of the bulk fluid and of the fluidic membrane are taken into account

simultaneously. In particular, surface viscous effects are accounted for through the Boussinesq–Scriven law.

- Local volume and local membrane area conservation result naturally from the volume and surface incompressibility conditions. Local area conservation can be shown for a continuous-in-time semidiscrete variant of our proposed scheme. In addition, for a simple modification of our scheme, which can be interpreted as a virtual element method, see e.g. Beirão da Veiga *et al.* (2013), volume conservation properties can also be shown.

- Elastic forcing from the curvature energy $E(\Gamma)$ is taken into account, and this is discretized with the help of a weak formulation due to Dziuk (2008).

- Stability of a semidiscrete version can be shown. To our knowledge, this is the first stability result in the literature for a numerical approximation of the dynamics of fluidic membranes.

- The interface is advected with the help of the fluid velocity. In other fluid flow problems with a free boundary this typically leads to distortions of the parametric surface mesh, see the discussion in Bänsch (2001). However, in our case the local surface area conservation $\nabla_s \,.\, \vec{u} = 0$ guarantees that the surface mesh quality remains good during the evolution, see Remark 4.2 and the numerical simulations in Section 7. We also refer to Mikula *et al.* (2014), who designed a strategy for the tangential redistribution of mesh points by conserving the relative surface area during the evolution.

- Fully three dimensional simulations, without making any symmetry assumptions, have been performed, and to the knowledge of the authors this paper presents the first such numerical computations for the full fluidic membrane problem, i.e. taking the bulk viscosity, the surface viscosity and the local incompressibility of the bulk and surface fluid into account.

The outline of the paper is as follows. After introducing the governing equations in Section 2, we present a weak formulation in Section 3. This weak formulation is the basis for our semidiscrete and fully discrete finite element approximations, which are formulated and analyzed in Sections 4 and 5. After stating the solutions methods in Section 6, we present numerical simulations in Section 7.

2   Governing equations

In this section we state the equations governing the evolution of fluidic membranes, as introduced by Arroyo and DeSimone (2009). Let $\Omega \subset \mathbb{R}^d$ be a given domain, where $d = 2$ or $d = 3$. We seek a time dependent interface $(\Gamma(t))_{t\in[0,T]}$, $\Gamma(t) \subset \Omega$, which for all $t \in [0,T]$ separates Ω into a domain $\Omega_+(t)$, occupied by the outer phase, and a domain
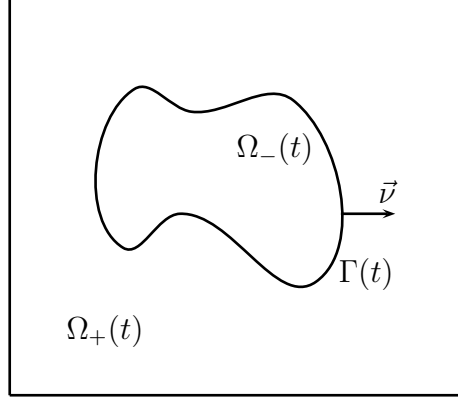
Figure 1: The domain Ω in the case $d = 2$.

$\Omega_-(t) := \Omega \setminus \overline{\Omega}_+(t)$, which is occupied by the inner phase, see Figure 1 for an illustration. For later use, we assume that $(\Gamma(t))_{t \in [0,T]}$ is a sufficiently smooth evolving hypersurface without boundary that is parameterized by $\vec{x}(\cdot, t) : \Upsilon \to \mathbb{R}^d$, where $\Upsilon \subset \mathbb{R}^d$ is a given reference manifold, i.e. $\Gamma(t) = \vec{x}(\Upsilon, t)$. Then

$$\vec{\mathcal{V}}(\vec{z}, t) := \vec{x}_t(\vec{q}, t) \qquad \forall \, \vec{z} = \vec{x}(\vec{q}, t) \in \Gamma(t) \tag{2.1}$$

defines the velocity of $\Gamma(t)$, and $\mathcal{V} := \vec{\mathcal{V}} \cdot \vec{\nu}$ is the normal velocity of the evolving hypersurface $\Gamma(t)$, where $\vec{\nu}(t)$ is the unit normal on $\Gamma(t)$ pointing into $\Omega_+(t)$. Moreover, we define the space-time surface $\mathcal{G}_T := \bigcup_{t \in [0,T]} \Gamma(t) \times \{t\}$.

Let $\rho(t) = \rho_+ \, \mathcal{X}_{\Omega_+(t)} + \rho_- \, \mathcal{X}_{\Omega_-(t)}$, with $\rho_\pm \in \mathbb{R}_{\geq 0}$, denote the fluid densities, where here and throughout $\mathcal{X}_\mathcal{A}$ defines the characteristic function for a set $\mathcal{A}$. Denoting by $\vec{u} : \Omega \times [0,T] \to \mathbb{R}^d$ the fluid velocity, by $\underline{\sigma} : \Omega \times [0,T] \to \mathbb{R}^{d \times d}$ the stress tensor, and by $\vec{f} : \Omega \times [0,T] \to \mathbb{R}^d$ a possible volume force, the incompressible Navier–Stokes equations in the two phases are given by

$$\rho \, (\vec{u}_t + (\vec{u} \cdot \nabla) \, \vec{u}) - \nabla \cdot \underline{\sigma} = \rho \, \vec{f} \qquad \text{in } \Omega_\pm(t) \,, \tag{2.2a}$$

$$\nabla \cdot \vec{u} = 0 \qquad \text{in } \Omega_\pm(t) \,, \tag{2.2b}$$

$$\vec{u} = \vec{g} \qquad \text{on } \partial_1 \Omega \,, \tag{2.2c}$$

$$\underline{\sigma} \, \vec{\mathrm{n}} = \vec{0} \qquad \text{on } \partial_2 \Omega \,, \tag{2.2d}$$

where $\partial \Omega = \partial_1 \Omega \cup \partial_2 \Omega$, with $\partial_1 \Omega \cap \partial_2 \Omega = \emptyset$, denotes the boundary of Ω with outer unit normal $\vec{\mathrm{n}}$. Hence (2.2c) prescribes a possibly inhomogeneous Dirichlet condition for the velocity on $\partial_1 \Omega$, which collapses to the standard no-slip condition when $\vec{g} = \vec{0}$, while (2.2d) prescribes a stress-free condition on $\partial_2 \Omega$. Throughout this paper we assume that $\mathcal{H}^{d-1}(\partial_1 \Omega) > 0$. We will also assume w.l.o.g. that $\vec{g}$ is extended so that $\vec{g} : \Omega \to \mathbb{R}^d$. In addition, the stress tensor in (2.2a) is defined by

$$\underline{\sigma} = \mu \, (\nabla \, \vec{u} + (\nabla \, \vec{u})^T) - p \, \underline{\mathrm{Id}} = 2 \, \mu \, \underline{D}(\vec{u}) - p \, \underline{\mathrm{Id}}, \,, \tag{2.3}$$

where $\underline{\mathrm{Id}} \in \mathbb{R}^{d \times d}$ denotes the identity matrix and $\underline{D}(\vec{u}) := \frac{1}{2}\left(\nabla\,\vec{u} + (\nabla\,\vec{u})^T\right)$ is the rate-of-deformation tensor, with $\nabla\,\vec{u} = \left(\partial_{x_j}\,u_i\right)_{i,j=1}^d$. Moreover, $p : \Omega \times [0,T] \to \mathbb{R}$ is the pressure and $\mu(t) = \mu_+\,\mathcal{X}_{\Omega_+(t)} + \mu_-\,\mathcal{X}_{\Omega_-(t)}$, with $\mu_\pm \in \mathbb{R}_{>0}$, denotes the dynamic viscosities in the two phases. On the free surface $\Gamma(t)$, the following conditions need to hold:

$$[\vec{u}]_-^+ = \vec{0} \qquad \text{on } \Gamma(t)\,, \tag{2.4a}$$

$$\rho_\Gamma\,\partial_t^\bullet\,\vec{u} - \nabla_s\,.\,\underline{\sigma}_\Gamma = [\underline{\sigma}\,\vec{\nu}]_-^+ + \alpha\,\vec{f}_\Gamma \qquad \text{on } \Gamma(t)\,, \tag{2.4b}$$

$$\nabla_s\,.\,\vec{u} = 0 \qquad \text{on } \Gamma(t)\,, \tag{2.4c}$$

$$\vec{\mathcal{V}}\,.\,\vec{\nu} = \vec{u}\,.\,\vec{\nu} \qquad \text{on } \Gamma(t)\,, \tag{2.4d}$$

where $\rho_\Gamma \in \mathbb{R}_{\geq 0}$ denotes the surface material density, $\alpha \in \mathbb{R}_{>0}$ is the bending rigidity and $\vec{f}_\Gamma := f_\Gamma\,\vec{\nu}$ is defined by (1.1). In addition, $\nabla_s\,.$ denotes the surface divergence on $\Gamma(t)$, and the surface stress tensor is given by

$$\underline{\sigma}_\Gamma = 2\,\mu_\Gamma\,\underline{D}_s(\vec{u}) - p_\Gamma\,\underline{\mathcal{P}}_\Gamma \quad \text{on } \Gamma(t)\,, \tag{2.5}$$

where $\mu_\Gamma \in \mathbb{R}_{>0}$ is the interfacial shear viscosity and p_Γ denotes the surface pressure, which acts as a Lagrange multiplier for the incompressibility condition (2.4c). Here

$$\underline{\mathcal{P}}_\Gamma = \underline{\mathrm{Id}} - \vec{\nu} \otimes \vec{\nu} \quad \text{on } \Gamma(t) \tag{2.6a}$$

and

$$\underline{D}_s(\vec{u}) = \tfrac{1}{2}\,\underline{\mathcal{P}}_\Gamma\left(\nabla_s\,\vec{u} + (\nabla_s\,\vec{u})^T\right)\underline{\mathcal{P}}_\Gamma \quad \text{on } \Gamma(t)\,, \tag{2.6b}$$

where $\nabla_s = \underline{\mathcal{P}}_\Gamma\,\nabla = (\partial_{s_1}, \ldots, \partial_{s_d})$ denotes the surface gradient on $\Gamma(t)$, and $\nabla_s\,\vec{u} = \left(\partial_{s_j}\,u_i\right)_{i,j=1}^d$. Moreover, as usual, $[\vec{u}]_-^+ := \vec{u}_+ - \vec{u}_-$ and $[\underline{\sigma}\,\vec{\nu}]_-^+ := \underline{\sigma}_+\,\vec{\nu} - \underline{\sigma}_-\,\vec{\nu}$ denote the jumps in velocity and normal stress across the interface $\Gamma(t)$. Here and throughout, we employ the shorthand notation $\vec{b}_\pm := \vec{b}\,|_{\Omega_\pm(t)}$ for a function $\vec{b} : \Omega \times [0,T] \to \mathbb{R}^d$; and similarly for scalar and matrix-valued functions. In addition,

$$\partial_t^\bullet\,\zeta = \zeta_t + \vec{u}\,.\,\nabla\,\zeta \qquad \forall\,\zeta \in H^1(\mathcal{G}_T) \tag{2.7}$$

denotes the material time derivative of ζ on $\Gamma(t)$. We compute $\partial_t^\bullet\,\zeta$ with the help of an extension of ζ to a neighborhood of $\mathcal{G}_T$. Here we stress that the derivative in (2.7) is well-defined, and depends only on the values of ζ on $\mathcal{G}_T$, even though ζ_t and $\nabla\,\zeta$ do not make sense separately for a function defined on $\mathcal{G}_T$; see e.g. Dziuk and Elliott (2013, p. 324). The system (2.2a–d), (2.3), (2.4a–d), (2.5) is closed with the initial conditions

$$\Gamma(0) = \Gamma_0\,, \quad \rho\,\vec{u}(\cdot,0) = \rho\,\vec{u}_0 \quad \text{in } \Omega\,, \quad \rho_\Gamma\,\vec{u}(\cdot,0) = \rho_\Gamma\,\vec{u}_0 \quad \text{on } \Gamma_0\,, \tag{2.8}$$

where $\Gamma_0 \subset \Omega$ and $\vec{u}_0 : \Omega \to \mathbb{R}^d$ are given initial data satisfying $\rho\,\nabla\,.\,\vec{u}_0 = 0$ in Ω, $\rho_\Gamma\,\nabla_s\,.\,\vec{u}_0 = 0$ on Γ_0 and $\rho_+\,\vec{u}_0 = \rho_+\,\vec{g}$ on $\partial_1\Omega$. Of course, in the case $\rho_- = \rho_+ = \rho_\Gamma = 0$ the initial data $\vec{u}_0$ is not needed. Similarly, in the case $\rho_- = \rho_+ = 0$ and $\rho_\Gamma > 0$ the initial data $\vec{u}_0$ is only needed on Γ_0. However, for ease of exposition, and in view of the unfitted nature of our numerical method, we will always assume that $\vec{u}_0$, if required, is given on all of Ω.

It is not difficult to show that the conditions (2.2b) enforce volume preservation for the phases, while (2.4c) leads to the conservation of the total surface area $\mathcal{H}^{d-1}(\Gamma(t))$, see Section 3 below for the relevant proofs. As an immediate consequence we obtain that spheres remain spheres, and that spheres with a zero bulk velocity are stationary solutions.

Furthermore, we note that

$$\nabla_s \cdot \underline{\sigma}_\Gamma = 2\, \mu_\Gamma\, \nabla_s \cdot \underline{D}_s(\vec{u}) - \nabla_s \cdot [p_\Gamma\, \underline{\mathcal{P}}_\Gamma] = 2\, \mu_\Gamma\, \nabla_s \cdot \underline{D}_s(\vec{u}) - \nabla_s\, p_\Gamma - \varkappa\, p_\Gamma\, \vec{\nu}. \qquad (2.9)$$

Here $\varkappa$ denotes the mean curvature of $\Gamma(t)$, i.e. the sum of the principal curvatures $\varkappa_i$, $i = 1, \ldots, d-1$, of $\Gamma(t)$, where we have adopted the sign convention that $\varkappa$ is negative where $\Omega_-(t)$ is locally convex. In particular, it holds that

$$\Delta_s\, \vec{\mathrm{id}} = \varkappa\, \vec{\nu} =: \vec{\varkappa} \qquad \text{on } \Gamma(t), \qquad (2.10)$$

where $\Delta_s = \nabla_s \cdot \nabla_s$ is the Laplace–Beltrami operator on $\Gamma(t)$.

Assuming there are two solutions $\{(\Gamma(t), \vec{u}(\cdot, t), p^{(i)}(\cdot, t), p_\Gamma^{(i)}(\cdot, t))\}_{t\in[0,T]}$, $i = 1, 2$, to the problem (2.2a–d), (2.3), (2.4a–d), (2.5) and (2.8), then it follows from (2.3) and (2.9) that

$$\nabla\, \bar{p}_\pm = \vec{0} \qquad \text{in } \Omega_\pm(t), \qquad (2.11a)$$
$$\nabla_s\, \bar{p}_\Gamma + \varkappa\, \bar{p}_\Gamma\, \vec{\nu} = [\bar{p}\, \vec{\nu}]_-^+ \qquad \text{on } \Gamma(t), \qquad (2.11b)$$

where $\bar{p} = p^{(1)} - p^{(2)}$ and $\bar{p}_\Gamma = p_\Gamma^{(1)} - p_\Gamma^{(2)}$. Therefore $\bar{p}_\pm$ is constant on $\Omega_\pm(t)$. In addition, since $\nabla_s\, \bar{p}_\Gamma$ is tangential, we obtain that $\nabla_s\, \bar{p}_\Gamma = \vec{0}$, and hence $\bar{p}_\Gamma$ is a constant. Moreover, (2.11b) implies that $\varkappa\, \bar{p}_\Gamma = \bar{p}_+ - \bar{p}_-$. So if $\varkappa$ is not constant, which is the case if $\Gamma(t)$ is not a sphere, then $\bar{p}_\Gamma = 0$ and $\bar{p}_+ = \bar{p}_-$. Hence p_Γ in this case is unique, and p is unique in Ω up to an additive constant. If $\varkappa$ is constant, however, i.e. if $\Gamma(t)$ is a sphere, then p_Γ is only unique up to an additive constant, which is fixed by the two additive constants in the bulk phases. For more details see the discussion around (3.13) below.

Finally, we recall that the source term $\vec{f}_\Gamma = f_\Gamma\, \vec{\nu}$ in (2.4b), with f_Γ defined in (1.1), is the first variation of $E(\Gamma(t))$, i.e.

$$\tfrac{1}{2}\, \frac{\mathrm{d}}{\mathrm{d}t}\, \langle \varkappa, \varkappa \rangle_{\Gamma(t)} = -\, \langle f_\Gamma, \mathcal{V} \rangle_{\Gamma(t)} = -\, \left\langle \vec{f}_\Gamma, \vec{\mathcal{V}} \right\rangle_{\Gamma(t)},$$

where $\langle \cdot, \cdot \rangle_{\Gamma(t)}$ denotes the L^2–inner product on $\Gamma(t)$. It does not appear possible to derive a stable discretization of the system (2.2a–d), (2.3), (2.4a–d), (2.5) based on the formulation (1.1). Hence in this paper we will make use of the stable approximation of Willmore flow introduced in Dziuk (2008), which is based on a discretization of the curvature vector $\vec{\varkappa} = \varkappa\, \vec{\nu}$, and on the identity

$$\tfrac{1}{2}\, \frac{\mathrm{d}}{\mathrm{d}t}\, \langle \vec{\varkappa}, \vec{\varkappa} \rangle_{\Gamma(t)} = -\, \left\langle \nabla_s\, \vec{\varkappa}, \nabla_s\, \vec{\mathcal{V}} \right\rangle_{\Gamma(t)} - \left\langle \nabla_s \cdot \vec{\varkappa}, \nabla_s \cdot \vec{\mathcal{V}} \right\rangle_{\Gamma(t)} - \tfrac{1}{2}\, \left\langle |\vec{\varkappa}|^2\, \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec{\mathcal{V}} \right\rangle_{\Gamma(t)}$$
$$+\, 2\, \left\langle \nabla_s\, \vec{\varkappa}, \underline{D}_s(\vec{\mathcal{V}})\, \nabla_s\, \vec{\mathrm{id}} \right\rangle_{\Gamma(t)}. \qquad (2.12)$$

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3 Weak formulation

Before introducing our finite element approximation, we will state an appropriate weak formulation. With this in mind, we introduce the following function spaces for a given $\vec{b} \in [H^1(\Omega)]^d$:

$$\mathbb{U}(\vec{b}) := \{\vec{\varphi} \in [H^1(\Omega)]^d : \vec{\varphi} = \vec{b} \text{ on } \partial_1\Omega\}, \quad \mathbb{V}(\vec{b}) := L^2(0, T; \mathbb{U}(\vec{b})) \cap H^1(0, T; [L^2(\Omega)]^d),$$

$$\mathbb{V}_\Gamma(\vec{b}) := \{\vec{\varphi} \in \mathbb{V}(\vec{b}) : \vec{\varphi}|_{\mathcal{G}_T} \in [H^1(\mathcal{G}_T)]^d\}.$$

In addition, we let $\mathbb{P} := L^2(\Omega)$ and define

$$\widehat{\mathbb{P}} := \begin{cases} \{\eta \in \mathbb{P} : \int_\Omega \eta \, \mathrm{d}\mathcal{L}^d = 0\} & \text{if } \mathcal{H}^{d-1}(\partial_2\Omega) = 0, \\ \mathbb{P} & \text{if } \mathcal{H}^{d-1}(\partial_2\Omega) > 0. \end{cases}$$

Letting $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle_{\partial_2\Omega}$ denote the L^2–inner products on Ω and $\partial_2\Omega$, respectively, we recall from Barrett *et al.* (2014c) that it follows from (2.2a–d) and (2.3) that

$$(\rho[\vec{u}_t + (\vec{u} \,.\, \nabla)\,\vec{u}], \vec{\xi})$$
$$= \tfrac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\rho\,\vec{u}, \vec{\xi}) + (\rho\,\vec{u}_t, \vec{\xi}) - (\rho\,\vec{u}, \vec{\xi}_t) + (\rho, [(\vec{u} \,.\, \nabla)\,\vec{u}] \,.\, \vec{\xi} - [(\vec{u} \,.\, \nabla)\,\vec{\xi}] \,.\, \vec{u})\right]$$
$$+ \tfrac{1}{2}\,\rho_+ \left\langle \vec{u} \,.\, \vec{\mathrm{n}}, \vec{u} \,.\, \vec{\xi} \right\rangle_{\partial_2\Omega} \qquad \forall\,\vec{\xi} \in \mathbb{V}(\vec{0}) \qquad (3.1)$$

and

$$\int_{\Omega_+(t) \cup \Omega_-(t)} (\nabla \,.\, \underline{\sigma}) \,.\, \vec{\xi} \, \mathrm{d}\mathcal{L}^d = -2\,(\mu\,\underline{\underline{D}}(\vec{u}), \underline{\underline{D}}(\vec{\xi})) + (p, \nabla \,.\, \vec{\xi}) - \left\langle [\underline{\sigma}\,\vec{\nu}]_-^+, \vec{\xi} \right\rangle_{\Gamma(t)} \quad \forall\,\vec{\xi} \in \mathbb{U}(\vec{0}), \tag{3.2}$$

where we have also noted for symmetric matrices $\underline{\underline{A}} \in \mathbb{R}^{d \times d}$ that $\underline{\underline{A}} : \underline{\underline{B}} = \underline{\underline{A}} : \tfrac{1}{2}(\underline{\underline{B}} + \underline{\underline{B}}^T)$ for all $\underline{\underline{B}} \in \mathbb{R}^{d \times d}$. Only slip or free-slip conditions were considered in Barrett *et al.* (2014c), and so the boundary integral over $\partial_2\Omega$ did not appear there. But it is easily established that the more general (3.1) also holds, on noting Barrett *et al.* (2014c, (3.2)).

Similarly to (2.7) we define the following time derivative that follows the parameterization $\vec{x}(\cdot, t)$ of $\Gamma(t)$, rather than $\vec{u}$. In particular, we let

$$\partial_t^\circ \zeta = \zeta_t + \vec{\mathcal{V}} \,.\, \nabla\zeta \qquad \forall\,\zeta \in H^1(\mathcal{G}_T); \tag{3.3}$$

where we stress once again that this definition is well-defined, even though ζ_t and $\nabla\zeta$ do not make sense separately for a function $\zeta \in H^1(\mathcal{G}_T)$. On recalling (2.7) we obtain that $\partial_t^\circ = \partial_t^\bullet$ if $\vec{\mathcal{V}} = \vec{u}$ on $\Gamma(t)$. Moreover, for later use we note that

$$\langle \zeta, \nabla_s \,.\, \vec{\eta} \rangle_{\Gamma(t)} + \langle \nabla_s\,\zeta, \vec{\eta} \rangle_{\Gamma(t)} = -\langle \zeta\,\vec{\eta}, \vec{\varkappa} \rangle_{\Gamma(t)} \qquad \forall\,\zeta \in H^1(\Gamma(t)),\ \vec{\eta} \in [H^1(\Gamma(t))]^d \tag{3.4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \chi, \zeta \rangle_{\Gamma(t)} = \langle \partial_t^\circ \chi, \zeta \rangle_{\Gamma(t)} + \langle \chi, \partial_t^\circ \zeta \rangle_{\Gamma(t)} + \left\langle \chi \zeta, \nabla_s \, . \, \vec{\mathcal{V}} \right\rangle_{\Gamma(t)} \qquad \forall \, \chi, \zeta \in H^1(\mathcal{G}_T), \quad (3.5)$$

see Definition 2.11 and Lemma 5.2 in Dziuk and Elliott (2013), respectively.

The most natural weak formulation of the system (2.2a–d), (2.3), (2.4a–d), (2.5) uses the fluidic tangential velocity for the evolution of $\Gamma(t)$, and so (2.4d) is replaced by $\vec{\mathcal{V}} = \vec{u}$ on $\Gamma(t)$. It then follows from (2.4b), (2.9), and (3.4) that

$$\rho_\Gamma \left\langle \partial_t^\bullet \vec{u}, \vec{\xi} \right\rangle_{\Gamma(t)} + 2\,\mu_\Gamma \left\langle \underline{D}_s(\vec{u}), \underline{D}_s(\vec{\xi}) \right\rangle_{\Gamma(t)} - \left\langle p_\Gamma, \nabla_s \, . \, \vec{\xi} \right\rangle_{\Gamma(t)}$$
$$= \left\langle [\underline{\sigma} \, \vec{\nu}]_-^+, \vec{\xi} \right\rangle_{\Gamma(t)} + \alpha \left\langle \vec{f}_\Gamma, \vec{\xi} \right\rangle_{\Gamma(t)} \qquad \forall \, \vec{\xi} \in H^1(\mathcal{G}_T), \quad (3.6)$$

where we have noted for symmetric matrices $\underline{A} \in \mathbb{R}^{d \times d}$ that $\underline{\mathcal{P}}_\Gamma \, \underline{A} \, \underline{\mathcal{P}}_\Gamma : \underline{B} = \underline{\mathcal{P}}_\Gamma \, \underline{A} \, \underline{\mathcal{P}}_\Gamma : \frac{1}{2} \underline{\mathcal{P}}_\Gamma \, (\underline{B} + \underline{B}^T) \, \underline{\mathcal{P}}_\Gamma$ for all $\underline{B} \in \mathbb{R}^{d \times d}$. This weak formulation of the system (2.2a–d), (2.3), (2.4a–d), (2.5) is then given as follows. Find $\Gamma(t) = \vec{x}(\Upsilon, t)$ for $t \in [0, T]$ with $\vec{\mathcal{V}} \in [L^2(\mathcal{G}_T)]^d$, and functions $\vec{u} \in \mathbb{V}_\Gamma(\vec{g})$, $p \in L^2(0, T; \widehat{\mathbb{P}})$, $p_\Gamma \in L^2(\mathcal{G}_T)$, $\vec{\varkappa} \in [H^1(\mathcal{G}_T)]^d$ and $\vec{f}_\Gamma \in [L^2(\mathcal{G}_T)]^d$ such that for almost all $t \in (0, T)$ it holds that

$$\frac{1}{2} \left[\frac{\mathrm{d}}{\mathrm{d}t} (\rho \, \vec{u}, \vec{\xi}) + (\rho \, \vec{u}_t, \vec{\xi}) - (\rho \, \vec{u}, \vec{\xi}_t) + (\rho, [(\vec{u} \, . \, \nabla) \, \vec{u}] \, . \, \vec{\xi} - [(\vec{u} \, . \, \nabla) \, \vec{\xi}] \, . \, \vec{u}) + \rho_+ \left\langle \vec{u} \, . \, \vec{\mathrm{n}}, \vec{u} \, . \, \vec{\xi} \right\rangle_{\partial_2 \Omega} \right]$$
$$+ 2 \, (\mu \, \underline{D}(\vec{u}), \underline{D}(\vec{\xi})) - (p, \nabla \, . \, \vec{\xi}) + \rho_\Gamma \left\langle \partial_t^\circ \vec{u}, \vec{\xi} \right\rangle_{\Gamma(t)} + 2 \, \mu_\Gamma \left\langle \underline{D}_s(\vec{u}), \underline{D}_s(\vec{\xi}) \right\rangle_{\Gamma(t)}$$
$$- \left\langle p_\Gamma, \nabla_s \, . \, \vec{\xi} \right\rangle_{\Gamma(t)} = (\rho \, \vec{f}, \vec{\xi}) + \alpha \left\langle \vec{f}_\Gamma, \vec{\xi} \right\rangle_{\Gamma(t)} \qquad \forall \, \vec{\xi} \in \mathbb{V}_\Gamma(\vec{0}), \quad (3.7\mathrm{a})$$

$$(\nabla \, . \, \vec{u}, \varphi) = 0 \qquad \forall \, \varphi \in \widehat{\mathbb{P}}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.7\mathrm{b})$$

$$\langle \nabla_s \, . \, \vec{u}, \eta \rangle_{\Gamma(t)} = 0 \qquad \forall \, \eta \in L^2(\Gamma(t)), \qquad\qquad\qquad\qquad\qquad (3.7\mathrm{c})$$

$$\left\langle \vec{\mathcal{V}} - \vec{u}, \vec{\chi} \right\rangle_{\Gamma(t)} = 0 \qquad \forall \, \vec{\chi} \in [L^2(\Gamma(t))]^d, \qquad\qquad\qquad\qquad (3.7\mathrm{d})$$

$$\langle \vec{\varkappa}, \vec{\eta} \rangle_{\Gamma(t)} + \left\langle \nabla_s \, \vec{\mathrm{id}}, \nabla_s \, \vec{\eta} \right\rangle_{\Gamma(t)} = 0 \qquad \forall \, \vec{\eta} \in [H^1(\Gamma(t))]^d, \qquad\qquad (3.7\mathrm{e})$$

$$\left\langle \vec{f}_\Gamma, \vec{\chi} \right\rangle_{\Gamma(t)} = \langle \nabla_s \, \vec{\varkappa}, \nabla_s \, \vec{\chi} \rangle_{\Gamma(t)} + \langle \nabla_s \, . \, \vec{\varkappa}, \nabla_s \, . \, \vec{\chi} \rangle_{\Gamma(t)} + \frac{1}{2} \left\langle |\vec{\varkappa}|^2 \, \nabla_s \, \vec{\mathrm{id}}, \nabla_s \, \vec{\chi} \right\rangle_{\Gamma(t)}$$
$$- 2 \left\langle \nabla_s \, \vec{\varkappa}, \underline{D}_s(\vec{\chi}) \, \nabla_s \, \vec{\mathrm{id}} \right\rangle_{\Gamma(t)} \qquad \forall \, \vec{\chi} \in [H^1(\Gamma(t))]^d, \quad (3.7\mathrm{f})$$

as well as the initial conditions (2.8), where in (3.7d) we have recalled (2.1). Here (3.7a–e) can be derived analogously to the weak formulation presented in Barrett *et al.* (2014c,a), recall (3.1), (3.2), (3.6) and (2.10). In addition, (3.7f) is based on (2.12).

In what follows we would like to derive an energy bound for a solution of (3.7a–f), where for ease of exposition we consider only the case $\vec{g} = \vec{0}$. All of the following considerations are formal, in the sense that we make the appropriate assumptions about the existence, boundedness and regularity of a solution to (3.7a–f). Firstly, it follows from (3.5), (3.7d)

and (3.7c) with $\eta = |\vec{u}|_{\Gamma(t)}|^2$ that

$$\tfrac{1}{2}\,\rho_\Gamma\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \vec{u}, \vec{u}\rangle_{\Gamma(t)} = \tfrac{1}{2}\,\rho_\Gamma\,\langle \partial_t^\circ\,|\vec{u}|^2, 1\rangle_{\Gamma(t)} + \tfrac{1}{2}\,\rho_\Gamma\,\left\langle \nabla_s\,.\,\vec{\mathcal{V}}, |\vec{u}|^2\right\rangle_{\Gamma(t)}$$
$$= \rho_\Gamma\,\langle \partial_t^\circ\,\vec{u}, \vec{u}\rangle_{\Gamma(t)} + \tfrac{1}{2}\,\rho_\Gamma\,\langle \nabla_s\,.\,\vec{u}, |\vec{u}|^2\rangle_{\Gamma(t)} = \rho_\Gamma\,\langle \partial_t^\circ\,\vec{u}, \vec{u}\rangle_{\Gamma(t)}. \qquad (3.8)$$

Now choosing $\vec{\xi} = \vec{u}$ in (3.7a), recall that $\vec{g} = \vec{0}$, $\varphi = p(\cdot, t)$ in (3.7b) and $\eta = p_\Gamma(\cdot, t)$ in (3.7c) yields, on combining with (3.8), that

$$\tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\left(\|\rho^{\frac{1}{2}}\,\vec{u}\|_0^2 + \rho_\Gamma\,\langle \vec{u}, \vec{u}\rangle_{\Gamma(t)}\right) + 2\,\|\mu^{\frac{1}{2}}\,\underline{D}(\vec{u})\|_0^2 + 2\,\mu_\Gamma\,\left\langle \underline{D}_s(\vec{u}), \underline{D}_s(\vec{u})\right\rangle_{\Gamma(t)}$$
$$+ \tfrac{1}{2}\,\rho_+\,\langle \vec{u}\,.\,\vec{\mathrm{n}}, |\vec{u}|^2\rangle_{\partial_2\Omega} = (\rho\,\vec{f}, \vec{u}) + \alpha\,\left\langle \vec{f}_\Gamma, \vec{u}\right\rangle_{\Gamma(t)}. \qquad (3.9)$$

Combining (3.9) with (2.12), on choosing $\vec{\chi} = \vec{f}_\Gamma$ in (3.7d) and $\vec{\chi} = \vec{\mathcal{V}}$ in (3.7f), yields that

$$\tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\left(\|\rho^{\frac{1}{2}}\,\vec{u}\|_0^2 + \rho_\Gamma\,\langle \vec{u}, \vec{u}\rangle_{\Gamma(t)} + \alpha\,\langle \vec{\varkappa}, \vec{\varkappa}\rangle_{\Gamma(t)}\right) + 2\,\|\mu^{\frac{1}{2}}\,\underline{D}(\vec{u})\|_0^2 + 2\,\mu_\Gamma\,\langle \underline{D}_s(\vec{u}), \underline{D}_s(\vec{u})\rangle_{\Gamma(t)}$$
$$+ \tfrac{1}{2}\,\rho_+\,\langle \vec{u}\,.\,\vec{\mathrm{n}}, |\vec{u}|^2\rangle_{\partial_2\Omega} = (\rho\,\vec{f}, \vec{u}). \qquad (3.10)$$

Moreover, we note that it immediately follows from (3.5) and (3.7c,d) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{H}^{d-1}(\Gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle 1, 1\rangle_{\Gamma(t)} = \left\langle 1, \nabla_s\,.\,\vec{\mathcal{V}}\right\rangle_{\Gamma(t)} = \langle 1, \nabla_s\,.\,\vec{u}\rangle_{\Gamma(t)} = 0. \qquad (3.11)$$

In addition, the volume of $\Omega_-(t)$ is preserved in time, i.e. the mass of each phase is conserved. To see this, choose $\vec{\chi} = \vec{\nu}$ in (3.7d) and $\varphi = (\mathcal{X}_{\Omega_-(t)} - \frac{\mathcal{L}^d(\Omega_-(t))}{\mathcal{L}^d(\Omega)})$ in (3.7b) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^d(\Omega_-(t)) = \left\langle \vec{\mathcal{V}}, \vec{\nu}\right\rangle_{\Gamma(t)} = \langle \vec{u}, \vec{\nu}\rangle_{\Gamma(t)} = \int_{\Omega_-(t)} \nabla\,.\,\vec{u}\,\mathrm{d}\mathcal{L}^d = 0. \qquad (3.12)$$

Recalling the argument on the uniqueness of the pressures p and p_Γ below (2.11a,b), we note the following LBB-type condition, provided that $\Gamma(t)$ is not a sphere:

$$\inf_{(\varphi,\eta)\in\widehat{\mathbb{P}}\times L^2(\Gamma(t))}\,\sup_{\vec{\xi}\in\mathbb{U}_\Gamma(\vec{0})} \frac{(\varphi, \nabla\,.\,\vec{\xi}) + \left\langle \eta, \nabla_s\,.\,\vec{\xi}\right\rangle_{\Gamma(t)}}{(\|\varphi\|_0 + \|\eta\|_{0,\Gamma(t)})\,(\|\vec{\xi}\|_1 + \|\underline{\mathcal{P}}_\Gamma\,\vec{\xi}|_{\Gamma(t)}\|_{1,\Gamma(t)})} \geq C > 0. \qquad (3.13)$$

Here we have defined the space $\mathbb{U}_\Gamma(\vec{0}) := \{\vec{\xi} \in \mathbb{U}(\vec{0}) : \underline{\mathcal{P}}_\Gamma\,\vec{\xi}|_{\Gamma(t)} \in H^1(\Gamma(t))\}$. In addition, $\|\eta\|_{0,\Gamma(t)}^2 := \langle \eta, \eta\rangle_{\Gamma(t)}$ and $\|\vec{\eta}\|_{1,\Gamma(t)}^2 := \langle \vec{\eta}, \vec{\eta}\rangle_{\Gamma(t)} + \langle \nabla_s\,\vec{\eta}, \nabla_s\,\vec{\eta}\rangle_{\Gamma(t)}$ for $\vec{\eta} \in H^1(\Gamma(t))$ This can be deduced from the pressure reconstruction result in Lengeler (2014) in the case $\partial_1\Omega = \partial\Omega$.

4 Semidiscrete finite element approximation

For simplicity we consider Ω to be a polyhedral domain. Then let $\mathcal{T}^h$ be a regular partitioning of Ω into disjoint open simplices o_j^h, $j = 1, \ldots, J_\Omega^h$. Associated with $\mathcal{T}^h$ are the finite element spaces

$$S_k^h := \{\chi \in C(\overline{\Omega}) : \chi|_o \in \mathcal{P}_k(o) \quad \forall\, o \in \mathcal{T}^h\} \subset H^1(\Omega), \qquad k \in \mathbb{N},$$

where $\mathcal{P}_k(o)$ denotes the space of polynomials of degree k on o. We also introduce S_0^h, the space of piecewise constant functions on $\mathcal{T}^h$. Let $\{\varphi_{k,j}^h\}_{j=1}^{K_k^h}$ be the standard basis functions for S_k^h, $k \geq 0$. We introduce $\vec{I}_k^h : [C(\overline{\Omega})]^d \to [S_k^h]^d$, $k \geq 1$, the standard interpolation operators, such that $(\vec{I}_k^h \vec{\eta})(\vec{p}_{k,j}^h) = \vec{\eta}(\vec{p}_{k,j}^h)$ for $j = 1, \ldots, K_k^h$; where $\{\vec{p}_{k,j}^h\}_{j=1}^{K_k^h}$ denotes the coordinates of the degrees of freedom of S_k^h, $k \geq 1$. In addition we define the standard projection operator $I_0^h : L^1(\Omega) \to S_0^h$, such that

$$(I_0^h \eta)|_o = \frac{1}{\mathcal{L}^d(o)} \int_o \eta \, \mathrm{d}\mathcal{L}^d \qquad \forall\, o \in \mathcal{T}^h\,.$$

Our approximation to the velocity and pressure on $\mathcal{T}^h$ will be finite element spaces $\mathbb{U}^h(\vec{g}) \subset \mathbb{U}(\vec{I}_k^h \vec{g})$, for some $k \geq 2$, and $\mathbb{P}^h(t) \subset \mathbb{P}$. For the former we assume from now on that $\vec{g} \in [C(\overline{\Omega})]^d$, while for the latter we assume that $S_1^h \subset \mathbb{P}^h(t)$. We require also the space $\widehat{\mathbb{P}}^h(t) := \mathbb{P}^h(t) \cap \widehat{\mathbb{P}}$. For the finite element spaces $(\mathbb{U}^h, \mathbb{P}^h)$ we may choose, for example, the lowest order Taylor–Hood element P2–P1, the P2–P0 element or the P2–(P1+P0) element on setting $\mathbb{U}^h = [S_2^h]^d \cap \mathbb{U}(\vec{I}_2^h \vec{g})$, and $\mathbb{P}^h = S_1^h$, S_0^h or $S_1^h + S_0^h$, respectively. We refer to Barrett *et al.* (2013, 2014c) for more details.

For the numerical approximation of the evolution of fluidic membranes it is desirable to maintain the surface area of the interface, recall (3.11), as well as the volume of the two phases, recall (3.12). In Barrett *et al.* (2013, 2014c) the present authors augmented the pressure space by the characteristic function of the inner phase in order to obtain discretizations that maintain the volume of the two phases. This enrichment of the pressure space is an example of an XFEM approach, and we refer to this particular approach as XFEM$_\Gamma$. Unfortunately, it does not appear possible to prove a discrete analogue of (3.11) for the XFEM$_\Gamma$ approach from Barrett *et al.* (2013, 2014c). Hence in this paper we will modify the XFEM$_\Gamma$ approach so that we obtain numerical approximations that satisfy discrete analogues of both (3.12) and (3.11). From a practical point of view, this approach is very close to the procedure in Barrett *et al.* (2013, 2014c). But the introduced modifications mean that the adjustments to the finite element approximations no longer have an interpretation within the XFEM framework. However, the new approach may be interpreted as an example of the recently proposed virtual element method, see below for further details.

The parametric finite element spaces in order to approximate e.g. $\vec{\varkappa}$ and p_Γ are defined as follows, see also Barrett *et al.* (2008a). Let $\Gamma^h(t) \subset \mathbb{R}^d$ be a $(d-1)$-dimensional *polyhedral surface*, i.e. a union of non-degenerate $(d-1)$-simplices with no hanging vertices (see Deckelnick *et al.* (2005, p. 164) for $d = 3$), approximating the closed surface $\Gamma(t)$. In

particular, let $\Gamma^h(t) = \bigcup_{j=1}^{J_\Gamma} \overline{\sigma_j^h(t)}$, where $\{\sigma_j^h(t)\}_{j=1}^{J_\Gamma}$ is a family of mutually disjoint open $(d-1)$-simplices with vertices $\{\vec{q}_k^h(t)\}_{k=1}^{K_\Gamma}$. Then let

$$\underline{V}(\Gamma^h(t)) := \{\vec{\chi} \in [C(\Gamma^h(t))]^d : \vec{\chi}|_{\sigma_j^h} \text{ is linear } \forall\, j = 1, \ldots, J_\Gamma\}$$
$$=: [W(\Gamma^h(t))]^d \subset [H^1(\Gamma^h(t))]^d,$$

where $W(\Gamma^h(t)) \subset H^1(\Gamma^h(t))$ is the space of scalar continuous piecewise linear functions on $\Gamma^h(t)$, with $\{\chi_k^h(\cdot,t)\}_{k=1}^{K_\Gamma}$ denoting the standard basis of $W(\Gamma^h(t))$, i.e.

$$\chi_k^h(\vec{q}_l^h(t), t) = \delta_{kl} \qquad \forall\, k, l \in \{1, \ldots, K_\Gamma\}, \ t \in [0, T]. \tag{4.1}$$

For later purposes, we also introduce $\pi^h(t) : C(\Gamma^h(t)) \to W(\Gamma^h(t))$, the standard interpolation operator at the nodes $\{\vec{q}_k^h(t)\}_{k=1}^{K_\Gamma}$, and similarly $\vec{\pi}^h(t) : [C(\Gamma^h(t))]^d \to \underline{V}(\Gamma^h(t))$. For scalar and vector functions η, ζ on $\Gamma^h(t)$ we introduce the L^2–inner product $\langle \cdot, \cdot \rangle_{\Gamma^h(t)}$ over the polyhedral surface $\Gamma^h(t)$ as follows

$$\langle \eta, \zeta \rangle_{\Gamma^h(t)} := \int_{\Gamma^h(t)} \eta \cdot \zeta \, \mathrm{d}\mathcal{H}^{d-1}.$$

If v, w are piecewise continuous, with possible jumps across the edges of $\{\sigma_j^h\}_{j=1}^{J_\Gamma}$, we introduce the mass lumped inner product $\langle \cdot, \cdot \rangle_{\Gamma^h(t)}^h$ as

$$\langle \eta, \zeta \rangle_{\Gamma^h(t)}^h := \tfrac{1}{d} \sum_{j=1}^{J_\Gamma} \mathcal{H}^{d-1}(\sigma_j^h) \sum_{k=1}^{d} (\eta \cdot \zeta)((\vec{q}_{j_k}^h)^-),$$

where $\{\vec{q}_{j_k}^h\}_{k=1}^d$ are the vertices of σ_j^h, and where we define $\eta((\vec{q}_{j_k}^h)^-) := \lim_{\sigma_j^h \ni \vec{p} \to \vec{q}_{j_k}^h} \eta(\vec{p})$.

Following Dziuk and Elliott (2013, (5.23)), we define the discrete material velocity for $\vec{z} \in \Gamma^h(t)$ by

$$\vec{\mathcal{V}}^h(\vec{z}, t) := \sum_{k=1}^{K_\Gamma} \left[\frac{\mathrm{d}}{\mathrm{d}t} \vec{q}_k^h(t) \right] \chi_k^h(\vec{z}, t). \tag{4.2}$$

Then, similarly to (3.3), we define

$$\partial_t^{\circ, h} \zeta = \zeta_t + \vec{\mathcal{V}}^h . \nabla \zeta \qquad \forall\, \zeta \in H^1(\mathcal{G}_T^h), \quad \text{where} \quad \mathcal{G}_T^h := \bigcup_{t \in [0,T]} \Gamma^h(t) \times \{t\}. \tag{4.3}$$

For later use, we also introduce the finite element spaces

$$W(\mathcal{G}_T^h) := \{\chi \in C(\mathcal{G}_T^h) : \chi(\cdot, t) \in W(\Gamma^h(t)) \quad \forall\, t \in [0, T]\},$$
$$W_T(\mathcal{G}_T^h) := \{\chi \in W(\mathcal{G}_T^h) : \partial_t^{\circ, h} \chi \in C(\mathcal{G}_T^h)\},$$

as well as

$$\mathbb{V}_{\Gamma^h}^h(\vec{g}) := \{\vec{\phi} \in H^1(0, T; \mathbb{U}^h(\vec{g})) : \vec{\chi} \in [W_T(\mathcal{G}_T)]^d, \text{ where } \vec{\chi}(\cdot, t) = \vec{\pi}^h\, [\vec{\phi}|_{\Gamma^h(t)}] \ \forall\, t \in [0, T]\}.$$

On differentiating (4.1) with respect to t, it immediately follows that

$$\partial_t^{\circ,h} \chi_k^h = 0 \quad \forall\, k \in \{1, \ldots, K_\Gamma\}, \tag{4.4}$$

see Dziuk and Elliott (2013, Lem. 5.5). It follows directly from (4.4) that

$$\partial_t^{\circ,h} \zeta(\cdot, t) = \sum_{k=1}^{K_\Gamma} \chi_k^h(\cdot, t) \frac{\mathrm{d}}{\mathrm{d}t} \zeta_k(t) \quad \text{on } \Gamma^h(t)$$

for $\zeta(\cdot, t) = \sum_{k=1}^{K_\Gamma} \zeta_k(t)\, \chi_k^h(\cdot, t) \in W(\Gamma^h(t))$, and hence $\partial_t^{\circ,h}\, \vec{\mathrm{id}} = \vec{\mathcal{V}}^h$ on $\Gamma^h(t)$.

We recall from Dziuk and Elliott (2013, Lem. 5.6) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\sigma_j^h(t)} \zeta\, \mathrm{d}\mathcal{H}^{d-1} = \int_{\sigma_j^h(t)} \partial_t^{\circ,h} \zeta + \zeta\, \nabla_s\,.\,\vec{\mathcal{V}}^h\, \mathrm{d}\mathcal{H}^{d-1} \quad \forall\, \zeta \in H^1(\sigma^h(t))\,, j \in \{1, \ldots, J_\Gamma\}, \tag{4.5}$$

which immediately implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \eta, \zeta \rangle_{\Gamma^h(t)} = \langle \partial_t^{\circ,h} \eta, \zeta \rangle_{\Gamma^h(t)} + \langle \eta, \partial_t^{\circ,h} \zeta \rangle_{\Gamma^h(t)} + \langle \eta\, \zeta, \nabla_s\,.\,\vec{\mathcal{V}}^h \rangle_{\Gamma^h(t)} \quad \forall\, \eta, \zeta \in W_T(\mathcal{G}_T^h). \tag{4.6}$$

Similarly, we recall from Barrett *et al.* (2014a, Lem. 2.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \eta, \zeta \rangle_{\Gamma^h(t)}^h = \langle \partial_t^{\circ,h} \eta, \zeta \rangle_{\Gamma^h(t)}^h + \langle \eta, \partial_t^{\circ,h} \zeta \rangle_{\Gamma^h(t)}^h + \langle \eta\, \zeta, \nabla_s\,.\,\vec{\mathcal{V}}^h \rangle_{\Gamma^h(t)}^h \quad \forall\, \eta, \zeta \in W_T(\mathcal{G}_T^h). \tag{4.7}$$

Similarly to (2.6a,b), we introduce

$$\underline{\mathcal{P}}_{\Gamma^h} = \underline{\mathrm{Id}} - \vec{\nu}^h \otimes \vec{\nu}^h \quad \text{on } \Gamma^h(t), \tag{4.8a}$$

and

$$\underline{D}_s^h(\vec{\eta}) = \tfrac{1}{2}\, \underline{\mathcal{P}}_{\Gamma^h} \left(\nabla_s\, \vec{\eta} + (\nabla_s\, \vec{\eta})^T \right) \underline{\mathcal{P}}_{\Gamma^h} \quad \text{on } \Gamma^h(t), \tag{4.8b}$$

where here $\nabla_s = \underline{\mathcal{P}}_{\Gamma^h} \nabla$ denotes the surface gradient on $\Gamma^h(t)$.

Given $\Gamma^h(t)$, we let $\Omega_+^h(t)$ denote the exterior of $\Gamma^h(t)$ and let $\Omega_-^h(t)$ denote the interior of $\Gamma^h(t)$, so that $\Gamma^h(t) = \partial\Omega_-^h(t) = \overline{\Omega_-^h(t)} \cap \overline{\Omega_+^h(t)}$. We then partition the elements of the bulk mesh $\mathcal{T}^h$ into interior, exterior and interfacial elements as follows. Let

$$\mathcal{T}_-^h(t) := \{o \in \mathcal{T}^h : o \subset \Omega_-^h(t)\},$$
$$\mathcal{T}_+^h(t) := \{o \in \mathcal{T}^h : o \subset \Omega_+^h(t)\},$$
$$\mathcal{T}_{\Gamma^h}^h(t) := \{o \in \mathcal{T}^h : o \cap \Gamma^h(t) \neq \emptyset\}.$$

Clearly $\mathcal{T}^h = \mathcal{T}_-^h(t) \cup \mathcal{T}_+^h(t) \cup \mathcal{T}_\Gamma^h(t)$ is a disjoint partition. In addition, we define the piecewise constant unit normal $\vec{\nu}^h(t)$ to $\Gamma^h(t)$ such that $\vec{\nu}^h(t)$ points into $\Omega_+^h(t)$. Moreover, we introduce the discrete density $\rho^h(t) \in S_0^h$ and the discrete viscosity $\mu^h(t) \in S_0^h$ as

$$\rho^h(t)|_o = \begin{cases} \rho_- & o \in \mathcal{T}_-^h(t), \\ \rho_+ & o \in \mathcal{T}_+^h(t), \\ \tfrac{1}{2}(\rho_- + \rho_+) & o \in \mathcal{T}_{\Gamma^h}^h(t), \end{cases} \quad \text{and} \quad \mu^h(t)|_o = \begin{cases} \mu_- & o \in \mathcal{T}_-^h(t), \\ \mu_+ & o \in \mathcal{T}_+^h(t), \\ \tfrac{1}{2}(\mu_- + \mu_+) & o \in \mathcal{T}_{\Gamma^h}^h(t). \end{cases}$$

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In what follows we will introduce a finite element approximation for the free boundary problem (2.2a–d), (2.3), (2.4a–d), (2.5). Here $\vec{U}^h(\cdot,t) \in \mathbb{U}^h(\vec{g})$ will be an approximation to $\vec{u}(\cdot,t)$, while $P^h(\cdot,t) \in \widehat{\mathbb{P}}^h(t)$ approximates $p(\cdot,t)$ and $P_\Gamma^h(\cdot,t) \in W(\Gamma^h(t))$ approximates $p_\Gamma(\cdot,t)$. When designing such a finite element approximation, a careful decision has to be made about the *discrete tangential velocity* of $\Gamma^h(t)$. The most natural choice is to select the velocity of the fluid, i.e. (3.7d) is appropriately discretized, and that is the approach we adopt in this paper. Overall, we then obtain the following semidiscrete continuous-in-time finite element approximation, which is the semidiscrete analogue of the weak formulation (3.7a–f).

Given $\Gamma^h(0)$ and $\vec{U}^h(\cdot,0) \in \mathbb{U}^h(\vec{g})$, find $\Gamma^h(t)$ such that $\mathrm{id}\,|_{\Gamma^h(t)} \in \underline{V}(\Gamma^h(t))$ for $t \in [0,T]$, and functions $\vec{U}^h \in \mathbb{V}_{\Gamma^h}^h(\vec{g})$, $P^h \in \mathbb{P}_T^h := \{\varphi \in L^2(0,T;\widehat{\mathbb{P}}) : \varphi(t) \in \widehat{\mathbb{P}}^h(t)$ for a.e. $t \in (0,T)\}$, $P_\Gamma^h \in W(\mathcal{G}_T^h)$, $\vec{\kappa}^h \in [W(\mathcal{G}_T^h)]^d$ and $\vec{F}_\Gamma^h \in [W(\mathcal{G}_T^h)]^d$ such that for almost all $t \in (0,T)$ it holds that

$$\frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^h\,\vec{U}^h,\vec{\xi}\right) + \left(\rho^h\,\vec{U}_t^h,\vec{\xi}\right) - \left(\rho^h\,\vec{U}^h,\vec{\xi}_t\right) + \rho_+\left\langle\vec{U}^h.\,\vec{\mathrm{n}},\vec{U}^h.\,\vec{\xi}\right\rangle_{\partial_2\Omega}\right]$$
$$+ 2\left(\mu^h\,\underline{\underline{D}}(\vec{U}^h),\underline{\underline{D}}(\vec{\xi})\right) + \frac{1}{2}\left(\rho^h,[(\vec{U}^h.\,\nabla)\,\vec{U}^h].\,\vec{\xi} - [(\vec{U}^h.\,\nabla)\,\vec{\xi}].\,\vec{U}^h\right) - \left(P^h,\nabla.\,\vec{\xi}\right)$$
$$+ \rho_\Gamma\left\langle\partial_t^{\circ,h}\,\vec{\pi}^h\,\vec{U}^h,\vec{\xi}\right\rangle_{\Gamma^h(t)}^h + 2\,\mu_\Gamma\left\langle\underline{\underline{D}}_s^h(\vec{\pi}^h\,\vec{U}^h),\underline{\underline{D}}_s^h(\vec{\pi}^h\,\vec{\xi})\right\rangle_{\Gamma^h(t)}$$
$$- \left\langle P_\Gamma^h,\nabla_s.\,(\vec{\pi}^h\,\vec{\xi})\right\rangle_{\Gamma^h(t)} = \left(\rho^h\,\vec{f}^h,\vec{\xi}\right) + \alpha\left\langle\vec{F}_\Gamma^h,\vec{\xi}\right\rangle_{\Gamma^h(t)}^h \qquad \forall\,\vec{\xi} \in H^1(0,T;\mathbb{U}^h(\vec{0})),$$

$$\tag{4.9a}$$

$$\left(\nabla.\,\vec{U}^h,\varphi\right) = 0 \qquad \forall\,\varphi \in \widehat{\mathbb{P}}^h(t),\tag{4.9b}$$

$$\left\langle\nabla_s.\,(\vec{\pi}^h\,\vec{U}^h),\eta\right\rangle_{\Gamma^h(t)} = 0 \qquad \forall\,\eta \in W(\Gamma^h(t)),\tag{4.9c}$$

$$\left\langle\vec{\mathcal{V}}^h,\vec{\chi}\right\rangle_{\Gamma^h(t)}^h = \left\langle\vec{U}^h,\vec{\chi}\right\rangle_{\Gamma^h(t)}^h \qquad \forall\,\vec{\chi} \in \underline{V}(\Gamma^h(t)),\tag{4.9d}$$

$$\left\langle\vec{\kappa}^h,\vec{\eta}\right\rangle_{\Gamma^h(t)}^{(h)} + \left\langle\nabla_s\,\mathrm{id},\nabla_s\,\vec{\eta}\right\rangle_{\Gamma^h(t)} = 0 \qquad \forall\,\vec{\eta} \in \underline{V}(\Gamma^h(t)),\tag{4.9e}$$

$$\left\langle\vec{F}_\Gamma^h,\vec{\chi}\right\rangle_{\Gamma^h(t)}^h = \left\langle\nabla_s\,\vec{\kappa}^h,\nabla_s\,\vec{\chi}\right\rangle_{\Gamma^h(t)} + \left\langle\nabla_s.\,\vec{\kappa}^h,\nabla_s.\,\vec{\chi}\right\rangle_{\Gamma^h(t)} + \frac{1}{2}\left\langle|\vec{\kappa}^h|^2\,\nabla_s\,\mathrm{id},\nabla_s\,\vec{\chi}\right\rangle_{\Gamma^h(t)}^{(h)}$$
$$- 2\left\langle\nabla_s\,\vec{\kappa}^h,\underline{\underline{D}}_s^h(\vec{\chi})\,\nabla_s\,\mathrm{id}\right\rangle_{\Gamma^h(t)} \qquad \forall\,\vec{\chi} \in \underline{V}(\Gamma^h(t)),\tag{4.9f}$$

where we recall (4.2). Here we have defined $\vec{f}^h(\cdot,t) := \vec{I}_2^h\,\vec{f}(\cdot,t)$, where here and throughout we assume that $\vec{f} \in L^2(0,T;[C(\overline{\Omega})]^d)$. We observe that (4.9d) collapses to $\vec{\mathcal{V}}^h = \vec{\pi}^h\,\vec{U}^h\,|_{\Gamma^h(t)} \in \underline{V}(\Gamma^h(t))$, which on recalling (4.3) turns out to be crucial for the stability analysis for (4.9a–f). It is for this reason that we use mass lumping in (4.9d). The superscript $\cdot^{(h)}$ in (4.9e,f) means that we can consider the corresponding terms either with or without mass lumping. Here we note that the scheme (4.9d–f), with true integration used throughout, and with $\vec{U}^h$ in (4.9d) replaced by $\vec{F}^h$, is the stable approximation of Willmore flow from Dziuk (2008), see also Deckelnick and Dziuk (2009) for the case $d = 2$.

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In fact, for $d = 2$ we observe that

$$\left\langle \nabla_s \, \vec{\kappa}^h, \nabla_s \, \vec{\chi} \right\rangle_{\Gamma^h(t)} + \left\langle \nabla_s \, . \, \vec{\kappa}^h, \nabla_s \, . \, \vec{\chi} \right\rangle_{\Gamma^h(t)} - 2 \left\langle \nabla_s \, \vec{\kappa}^h, \underline{\underline{D}}^h_s(\vec{\chi}) \, \nabla_s \, \vec{\mathrm{id}} \right\rangle_{\Gamma^h(t)}$$

$$= \left\langle \nabla_s \, \vec{\kappa}^h, \nabla_s \, \vec{\chi} \right\rangle_{\Gamma^h(t)} - \left\langle \nabla_s \, . \, \vec{\kappa}^h, \nabla_s \, . \, \vec{\chi} \right\rangle_{\Gamma^h(t)} = \left\langle \vec{\kappa}^h_s \, . \, \vec{\nu}^h, \vec{\chi}_s \, . \, \vec{\nu}^h \right\rangle_{\Gamma^h(t)},$$

where $\cdot_s$ denotes differentiation with respect to arclength; compare also Barrett *et al.* (2012, (3.12a,b)).

In the following theorem we derive discrete analogues of (3.10) and (3.11) for the scheme (4.9a–f).

THEOREM. 4.1. *Let* $\{(\Gamma^h, \vec{U}^h, P^h, P^h_\Gamma, \vec{\kappa}^h, \vec{F}^h_\Gamma)(t)\}_{t \in [0,T]}$ *be a solution to* (4.9a–f). *Then, in the case* $\vec{g} = \vec{0}$, *it holds that*

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\| [\rho^h]^{\frac{1}{2}} \, \vec{U}^h \|_0^2 + \rho_\Gamma \left\langle \vec{U}^h, \vec{U}^h \right\rangle^h_{\Gamma^h(t)} + \alpha \left\langle \vec{\kappa}^h, \vec{\kappa}^h \right\rangle^{(h)}_{\Gamma^h(t)} \right) + 2 \, \| [\mu^h]^{\frac{1}{2}} \, \underline{\underline{D}}(\vec{U}^h) \|_0^2$$

$$+ 2 \, \mu_\Gamma \left\langle \underline{\underline{D}}^h_s(\vec{\pi}^h \, \vec{U}^h), \underline{\underline{D}}^h_s(\vec{\pi}^h \, \vec{U}^h) \right\rangle_{\Gamma^h(t)} + \tfrac{1}{2} \, \rho_+ \left\langle \vec{U}^h \, . \, \vec{\mathrm{n}}, |\vec{U}^h|^2 \right\rangle_{\partial_2 \Omega} = (\rho^h \, \vec{f}^h, \vec{U}^h) \, . \tag{4.10}$$

Moreover, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \chi^h_k, 1 \right\rangle_{\Gamma^h(t)} = 0 \quad \forall \, k \in \{1, \dots, K_\Gamma\} \tag{4.11}$$

and hence that

$$\frac{\mathrm{d}}{\mathrm{d}t} \, \mathcal{H}^{d-1}(\Gamma^h(t)) = 0 \, . \tag{4.12}$$

Proof. Choosing $\vec{\xi} = \vec{U}^h$ in (4.9a), recall that $\vec{g} = \vec{0}$, $\varphi = P^h$ in (4.9b) and $\eta = P^h_\Gamma$ in (4.9c) yields that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| [\rho^h]^{\frac{1}{2}} \, \vec{U}^h \|_0^2 + 2 \, \| [\mu^h]^{\frac{1}{2}} \, \underline{\underline{D}}(\vec{U}^h) \|_0^2 + \rho_\Gamma \left\langle \partial^{\circ,h}_t \, \vec{\pi}^h \, \vec{U}^h, \vec{U}^h \right\rangle^h_{\Gamma^h(t)} + \tfrac{1}{2} \, \rho_+ \left\langle \vec{U}^h \, . \, \vec{\mathrm{n}}, |\vec{U}^h|^2 \right\rangle_{\partial_2 \Omega}$$

$$+ 2 \, \mu_\Gamma \left\langle \underline{\underline{D}}^h_s(\vec{\pi}^h \, \vec{U}^h), \underline{\underline{D}}^h_s(\vec{\pi}^h \, \vec{U}^h) \right\rangle_{\Gamma^h(t)} = (\rho^h \, \vec{f}^h, \vec{U}^h) + \alpha \left\langle \vec{F}^h_\Gamma, \vec{U}^h \right\rangle^h_{\Gamma^h(t)} \, . \tag{4.13}$$

Moreover, similarly to (3.8), we note that (4.7), (4.9d) and (4.9c) with $\eta = \pi^h \, [|\vec{U}^h|_{\Gamma^h(t)}|^2]$ imply that

$$\frac{1}{2} \, \rho_\Gamma \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \vec{U}^h, \vec{U}^h \right\rangle^h_{\Gamma^h(t)} = \tfrac{1}{2} \, \rho_\Gamma \left\langle \partial^{\circ,h}_t \, \vec{\pi}^h \, [|\vec{U}^h|^2], 1 \right\rangle^h_{\Gamma^h(t)} + \tfrac{1}{2} \, \rho_\Gamma \left\langle \nabla_s \, . \, \vec{\mathcal{V}}^h, |\vec{U}^h|^2 \right\rangle^h_{\Gamma^h(t)}$$

$$= \rho_\Gamma \left\langle \partial^{\circ,h}_t \, \vec{\pi}^h \, \vec{U}^h, \vec{U}^h \right\rangle^h_{\Gamma^h(t)} + \tfrac{1}{2} \, \rho_\Gamma \left\langle \nabla_s \, . \, (\vec{\pi}^h \, \vec{U}^h), |\vec{U}^h|^2 \right\rangle^h_{\Gamma^h(t)}$$

$$= \rho_\Gamma \left\langle \partial^{\circ,h}_t \, \vec{\pi}^h \, \vec{U}^h, \vec{U}^h \right\rangle^h_{\Gamma^h(t)} \, . \tag{4.14}$$

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Similarly to (2.12), it is possible to show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \vec{\kappa}^h, \vec{\kappa}^h\right\rangle_{\Gamma^h(t)}^{(h)} = -\left\langle \nabla_s\,\vec{\kappa}^h, \nabla_s\,\vec{\mathcal{V}}^h\right\rangle_{\Gamma^h(t)} - \left\langle \nabla_s\,.\,\vec{\kappa}^h, \nabla_s\,.\,\vec{\mathcal{V}}^h\right\rangle_{\Gamma^h(t)}$$

$$-\frac{1}{2}\left\langle |\vec{\kappa}^h|^2\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\mathcal{V}}^h\right\rangle_{\Gamma^h(t)}^{(h)} + 2\left\langle \nabla_s\,\vec{\kappa}^h, \underline{D}_s^h(\vec{\mathcal{V}}^h)\,\nabla_s\,\vec{\mathrm{id}}\right\rangle_{\Gamma^h(t)},$$

see Dziuk (2008). Hence choosing $\vec{\chi} = \vec{F}_\Gamma^h$ in (4.9d) and $\vec{\chi} = \vec{\mathcal{V}}^h$ in (4.9f) yields that

$$\left\langle \vec{F}_\Gamma^h, \vec{U}^h\right\rangle_{\Gamma^h(t)}^h = \left\langle \vec{F}_\Gamma^h, \vec{\mathcal{V}}^h\right\rangle_{\Gamma^h(t)}^h = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \vec{\kappa}^h, \vec{\kappa}^h\right\rangle_{\Gamma^h(t)}^{(h)}. \tag{4.15}$$

The desired result (4.10) now directly follows from combining (4.13), (4.14) and (4.15).

Similarly to (3.11), it immediately follows from (4.6) and (4.4), on choosing $\eta = \chi_k^h$ in (4.9c), and on recalling from (4.9d) that $\vec{\mathcal{V}}^h = \vec{\pi}^h\,\vec{U}^h$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \chi_k^h, 1\right\rangle_{\Gamma^h(t)} = \left\langle \chi_k^h, \nabla_s\,.\,\vec{\mathcal{V}}^h\right\rangle_{\Gamma^h(t)} = 0, \tag{4.16}$$

which proves the desired result (4.11). Summing (4.11) for all $k = 1, \ldots, K_\Gamma$ then yields the desired result (4.12). $\square$

REMARK. 4.2. *We remark that (4.11) ensures that the area of the support of each basis function on $\Gamma^h(t)$ is conserved. In the case $d = 2$, and for J_Γ being odd, this is equivalent to each element σ_j^h maintaining its length. In particular, if $\Gamma^h(0)$ is equidistributed, then $\Gamma^h(t)$ will remain equidistributed throughout.*

The same result, for arbitrary $J_\Gamma \geq 2$ and for $d = 2$ and $d = 3$, can be obtained on replacing the space of continuous piecewise linear finite elements $W(\Gamma^h(t))$ for the surface pressure functions P_Γ^h, and for the test functions in (4.9c), with the space of discontinuous piecewise constant functions. Then (4.5), similarly to (4.16), immediately yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{d-1}(\sigma_j^h(t)) = 0 \qquad \forall\,j \in \{1, \ldots, J_\Gamma\}. \tag{4.17}$$

While this property may appear desirable at first, our numerical experience for the fully discrete variant of this modified (4.9a–f) indicates that in the case $d = 3$ the constraint (4.17) is too severe. Here we note that for typical triangulations of $\Gamma^h(t)$ it holds that $J_\Gamma \approx 2\,K_\Gamma$. It is for this reason that we prefer the scheme (4.9a–f) as stated.

We observe that it does not appear possible to prove a discrete analogue of (3.12) for the scheme (4.9a–f). The reason is that $\vec{\chi} = \vec{\nu}^h$ is not a valid test function in (4.9d). However, a procedure similarly to the XFEM$_\Gamma$ approach introduced by the authors in Barrett *et al.* (2013, 2014c) ensures that a modified variant of (4.9a–f) conserves the enclosed volumes. To this end, we introduce the vertex normal function $\vec{\omega}^h(\cdot, t) \in \underline{V}(\Gamma^h(t))$ with

$$\vec{\omega}^h(\vec{q}_k^h(t), t) := \frac{1}{\mathcal{H}^{d-1}(\Lambda_k^h(t))}\sum_{j \in \Theta_k^h}\mathcal{H}^{d-1}(\sigma_j^h(t))\,\vec{\nu}_j^h|_{\sigma_j^h(t)},$$

where for $k = 1, \ldots, K_\Gamma^h$ we define $\Theta_k^h := \{j : \vec{q}_k^h(t) \in \overline{\sigma_j^h(t)}\}$ and set $\Lambda_k^h(t) := \cup_{j \in \Theta_k^h} \overline{\sigma_j^h(t)}$. For later use we note that

$$\left\langle \vec{z}, w \, \vec{\nu}^h \right\rangle_{\Gamma^h(t)}^h = \left\langle \vec{z}, w \, \vec{\omega}^h \right\rangle_{\Gamma^h(t)}^h \qquad \forall \, \vec{z} \in \underline{V}(\Gamma^h(t)), \ w \in W(\Gamma^h(t)) \,. \tag{4.18}$$

We are now in a position to propose the following adaptation of (4.9a–f).

Given $\Gamma^h(0)$ and $\vec{U}^h(\cdot, 0) \in \mathbb{U}^h(\vec{g})$, find $\Gamma^h(t)$ such that $\vec{\mathrm{id}}|_{\Gamma^h(t)} \in \underline{V}(\Gamma^h(t))$ for $t \in [0, T]$, and functions $\vec{U}^h \in \mathbb{V}_{\Gamma^h}^h(\vec{g})$, $P^h \in \mathbb{P}_T^h$, $P_{\mathrm{sing}}^h \in L^2(0, T; \mathbb{R})$, $P_\Gamma^h \in W(\mathcal{G}_T^h)$, $\vec{\kappa}^h \in [W(\mathcal{G}_T^h)]^d$ and $\vec{F}_\Gamma^h \in [W(\mathcal{G}_T^h)]^d$ such that for almost all $t \in (0, T)$ it holds that

$$\frac{1}{2} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\rho^h \, \vec{U}^h, \vec{\xi} \right) + \left(\rho^h \, \vec{U}_t^h, \vec{\xi} \right) - (\rho^h \, \vec{U}^h, \vec{\xi}_t) + \rho_+ \left\langle \vec{U}^h . \vec{\mathrm{n}}, \vec{U}^h . \vec{\xi} \right\rangle_{\partial_2 \Omega} \right]$$

$$+ 2 \left(\mu^h \, \underline{D}(\vec{U}^h), \underline{D}(\vec{\xi}) \right) + \frac{1}{2} \left(\rho^h, [(\vec{U}^h . \nabla) \vec{U}^h] . \vec{\xi} - [(\vec{U}^h . \nabla) \vec{\xi}] . \vec{U}^h \right) - \left(P^h, \nabla . \vec{\xi} \right)$$

$$- P_{\mathrm{sing}}^h \left\langle \vec{\omega}^h, \vec{\xi} \right\rangle_{\Gamma^h(t)}^h + \rho_\Gamma \left\langle \partial_t^{\circ, h} \vec{\pi}^h \, \vec{U}^h, \vec{\xi} \right\rangle_{\Gamma^h(t)}^h + 2 \, \mu_\Gamma \left\langle \underline{D}_s^h(\vec{\pi}^h \, \vec{U}^h), \underline{D}_s^h(\vec{\pi}^h \, \vec{\xi}) \right\rangle_{\Gamma^h(t)}$$

$$- \left\langle P_\Gamma^h, \nabla_s . (\vec{\pi}^h \, \vec{\xi}) \right\rangle_{\Gamma^h(t)} = \left(\rho^h \, \vec{f}^h, \vec{\xi} \right) + \alpha \left\langle \vec{F}_\Gamma^h, \vec{\xi} \right\rangle_{\Gamma^h(t)}^h \qquad \forall \, \vec{\xi} \in H^1(0, T; \mathbb{U}^h(\vec{0})) \,, \tag{4.19a}$$

$$\left(\nabla . \vec{U}^h, \varphi \right) = 0 \qquad \forall \, \varphi \in \widehat{\mathbb{P}}^h(t) \qquad \text{and} \qquad \left\langle \vec{U}^h, \vec{\omega}^h \right\rangle_{\Gamma^h(t)}^h = 0 \tag{4.19b}$$

and (4.9c–f) hold. Of course, $\vec{\chi} = \vec{\omega}^h$ is a valid test function in (4.9d), and so combining with (4.19b) yields a discrete volume preservation property, as is shown in the following theorem.

THEOREM. 4.3. *Let* $\{(\Gamma^h, \vec{U}^h, P^h, P_{\mathrm{sing}}^h, P_\Gamma^h, \vec{\kappa}^h, \vec{F}_\Gamma^h)(t)\}_{t \in [0, T]}$ *be a solution to* (4.19a,b), (4.9c–f). *Then* (4.10) *holds if* $\vec{g} = \vec{0}$. *In addition,* (4.12) *and*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}^d(\Omega_-^h(t)) = 0 \tag{4.20}$$

hold.

Proof. The proofs for (4.10) and (4.12) are analogous to the proofs in Theorem 4.1. In order to prove (4.20) we choose $\vec{\chi} = \vec{\omega}^h \in \underline{V}(\Gamma^h(t))$ in (4.9d) to yield that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}^d(\Omega_-^h(t)) = \left\langle \vec{\mathcal{V}}^h, \vec{\nu}^h \right\rangle_{\Gamma^h(t)} = \left\langle \vec{\mathcal{V}}^h, \vec{\nu}^h \right\rangle_{\Gamma^h(t)}^h = \left\langle \vec{\mathcal{V}}^h, \vec{\omega}^h \right\rangle_{\Gamma^h(t)}^h = \left\langle \vec{U}^h, \vec{\omega}^h \right\rangle_{\Gamma^h(t)}^h = 0 \,,$$

where we have used (4.18) and (4.19b). □

In order to interpret the adaptation in (4.19a,b) physically, we note the following. Of course, P^h and P_{sing}^h act as Lagrange multipliers for the conditions in (4.19b). Moreover, on noting that

$$\left(\nabla . \vec{\xi}, \mathcal{X}_{\Omega_-^h(t)} \right) = \int_{\Omega_-^h(t)} \nabla . \vec{\xi} \, \mathrm{d}\mathcal{L}^d = \left\langle \vec{\xi}, \vec{\nu}^h \right\rangle_{\Gamma^h(t)} \qquad \forall \, \vec{\xi} \in \mathbb{U}^h(\vec{0}) \,,$$

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we observe that replacing

$$\left\langle \vec{\omega}^h, \vec{\xi} \right\rangle_{\Gamma^h(t)}^h \quad \text{and} \quad \left\langle \vec{U}^h, \vec{\omega}^h \right\rangle_{\Gamma^h(t)}^h \tag{4.21}$$

in (4.19a,b) by

$$\left\langle \vec{\nu}^h, \vec{\xi} \right\rangle_{\Gamma^h(t)} \quad \text{and} \quad \left\langle \vec{U}^h, \vec{\nu}^h \right\rangle_{\Gamma^h(t)}$$

corresponds to augmenting $\mathbb{P}^h$ in (4.9a,b) with the single additional basis function $\mathcal{X}_{\Omega^h_-(t)}$. This is the XFEM$_\Gamma$ approach introduced by the authors in Barrett *et al.* (2013, 2014c), which for the schemes introduced there naturally leads to the conservation of the volume of the two phases. Such an XFEM interpretation is no longer possible for the modifications (4.21), as one cannot identify the corresponding additional basis function in the bulk. Therefore this can be viewed as an example of the recently proposed framework of virtual element methods, see e.g. Beirão da Veiga *et al.* (2013). In addition, on recalling (4.18) we have for a fixed time $t \in [0, T]$ that

$$\left| \left\langle \vec{\omega}^h, \vec{\xi} \right\rangle_{\Gamma^h(t)}^h - \left\langle \vec{\nu}^h, \vec{\xi} \right\rangle_{\Gamma^h(t)} \right| = \left| \left\langle \vec{\nu}^h, \vec{\pi}^h \vec{\xi} - \vec{\xi} \right\rangle_{\Gamma^h(t)} \right|$$

$$\leq \left\langle |\vec{\pi}^h \vec{\xi} - \vec{\xi}|, 1 \right\rangle_{\Gamma^h(t)} \to 0 \quad \text{as} \quad h_\Gamma(t) \to 0 \qquad \forall \, \vec{\xi} \in [C(\overline{\Omega})]^d, \tag{4.22}$$

if $\mathcal{H}^{d-1}(\Gamma^h(t))$ remains bounded as $h_\Gamma(t) \to 0$, where $h_\Gamma(t) := \max_{j=1,\ldots,J_\Gamma} \operatorname{diam}(\sigma_j^h(t))$. It follows from (4.22) that we can interpret $P^h(\cdot, t) + P^h_{\text{sing}}(t) \mathcal{X}_{\Omega^h_-(t)}$ as the natural approximation to the pressure $p(\cdot, t)$ arising from (4.19a,b), (4.9c–f).

5 Fully discrete finite element approximation

In this section we consider a fully discrete variant of the scheme (4.9a–f) from §4. Here we will choose the time discretization such that existence and uniqueness of the discrete solutions can be guaranteed, and such that we inherit as much of the structure of the stable schemes in Barrett *et al.* (2013, 2014c) as possible, see below for details.

We consider the partitioning $t_m = m\tau$, $m = 0, \ldots, M$, of $[0, T]$ into uniform time steps $\tau = T/M$. The time discrete spatial discretizations then directly follow from the finite element spaces introduced in §3, where in order to allow for adaptivity in space we consider bulk finite element spaces that change in time. For all $m \geq 0$, let $\mathcal{T}^m$ be a regular partitioning of Ω into disjoint open simplices o_j^m, $j = 1, \ldots, J_\Omega^m$. Associated with $\mathcal{T}^m$ are the finite element spaces S_k^m for $k \geq 0$. We introduce also $\vec{I}_k^m : [C(\overline{\Omega})]^d \to [S_k^m]^d$, $k \geq 1$, the standard interpolation operators, and the standard projection operator $I_0^m : L^1(\Omega) \to S_0^m$. Similarly, the parametric finite element spaces are given by

$$\underline{V}(\Gamma^m) := \{\vec{\chi} \in [C(\Gamma^m)]^d : \vec{\chi}|_{\sigma_j^m} \text{ is linear } \forall \, j = 1, \ldots, J_\Gamma\} =: [W(\Gamma^m)]^d \subset [H^1(\Gamma^m)]^d,$$

for $m = 0, \ldots, M-1$. Here $\Gamma^m = \bigcup_{j=1}^{J_\Gamma} \overline{\sigma_j^m}$, where $\{\sigma_j^m\}_{j=1}^{J_\Gamma}$ is a family of mutually disjoint open $(d-1)$-simplices with vertices $\{\vec{q}_k^m\}_{k=1}^{K_\Gamma}$. We denote the standard basis of $W(\Gamma^m)$

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by $\{\chi_k^m(\cdot, t)\}_{k=1}^{K_\Gamma}$. We also introduce $\pi^m : C(\Gamma^m) \to W(\Gamma^m)$, the standard interpolation operator at the nodes $\{\vec{q}_k^m\}_{k=1}^{K_\Gamma}$, and similarly $\vec{\pi}^m : [C(\Gamma^m)]^d \to \underline{V}(\Gamma^m)$. Throughout this paper, we will parameterize the new closed surface Γ^{m+1} over Γ^m, with the help of a parameterization $\vec{X}^{m+1} \in \underline{V}(\Gamma^m)$, i.e. $\Gamma^{m+1} = \vec{X}^{m+1}(\Gamma^m)$.

We also introduce the L^2–inner product $\langle \cdot, \cdot \rangle_{\Gamma^m}$ over the current polyhedral surface Γ^m, as well as the the mass lumped inner product $\langle \cdot, \cdot \rangle_{\Gamma^m}^h$. Similarly to (4.8a,b), we introduce

$$\underline{\mathcal{P}}_{\Gamma^m} = \underline{\mathrm{Id}} - \vec{\nu}^m \otimes \vec{\nu}^m \quad \text{on } \Gamma^m,$$

and

$$\underline{D}_s^m(\vec{\eta}) = \tfrac{1}{2} \underline{\mathcal{P}}_{\Gamma^m} \left(\nabla_s \vec{\eta} + (\nabla_s \vec{\eta})^T \right) \underline{\mathcal{P}}_{\Gamma^m} \quad \text{on } \Gamma^m,$$

where here $\nabla_s = \underline{\mathcal{P}}_{\Gamma^m} \nabla$ denotes the surface gradient on Γ^m.

Given Γ^m, we let Ω_+^m denote the exterior of Γ^m and let Ω_-^m denote the interior of Γ^m, so that $\Gamma^m = \partial \Omega_-^m = \overline{\Omega_-^m} \cap \overline{\Omega_+^m}$. We then partition the elements of the bulk mesh $\mathcal{T}^m$ into interior, exterior and interfacial elements as before, and we introduce $\rho^m, \mu^m \in S_0^m$, for $m \geq 0$, as

$$\rho^m|_{o^m} = \begin{cases} \rho_- & o^m \in \mathcal{T}_-^m, \\ \rho_+ & o^m \in \mathcal{T}_+^m, \\ \tfrac{1}{2}(\rho_- + \rho_+) & o^m \in \mathcal{T}_{\Gamma^m}^m, \end{cases} \quad \text{and} \quad \mu^m|_{o^m} = \begin{cases} \mu_- & o^m \in \mathcal{T}_-^m, \\ \mu_+ & o^m \in \mathcal{T}_+^m, \\ \tfrac{1}{2}(\mu_- + \mu_+) & o^m \in \mathcal{T}_{\Gamma^m}^m. \end{cases}$$

We introduce the following pushforward operator for the discrete interfaces Γ^m and Γ^{m-1}, for $m = 0, \ldots, M$. Here we set $\Gamma^{-1} := \Gamma^0$. Let $\vec{\Pi}_{m-1}^m : [C(\Gamma^{m-1})]^d \to \underline{V}(\Gamma^m)$ such that

$$(\vec{\Pi}_{m-1}^m \vec{z})(\vec{q}_k^m) = \vec{z}(\vec{q}_k^{m-1}), \qquad k = 1, \ldots, K_\Gamma, \qquad \forall \, \vec{z} \in [C(\Gamma^{m-1})]^d, \qquad (5.1)$$

for $m = 1, \ldots, M$, and set $\vec{\Pi}_{-1}^0 := \vec{\pi}^0$. Analogously to (5.1) we also introduce $\Pi_{m-1}^m : C(\Gamma^{m-1}) \to W(\Gamma^m)$. Similarly to (4.18), we note that

$$\langle \vec{z}, w \, \vec{\nu}^m \rangle_{\Gamma^m}^h = \langle \vec{z}, w \, \vec{\omega}^m \rangle_{\Gamma^m}^h \qquad \forall \, \vec{z} \in \underline{V}(\Gamma^m), \, w \in W(\Gamma^m),$$

where $\vec{\omega}^m := \sum_{k=1}^{K_\Gamma} \chi_k^m \, \vec{\omega}_k^m \in \underline{V}(\Gamma^m)$, and where for $k = 1, \ldots, K_\Gamma$ we let $\Theta_k^m := \{j : \vec{q}_k^m \in \overline{\sigma_j^m}\}$ and set $\Lambda_k^m := \cup_{j \in \Theta_k^m} \overline{\sigma_j^m}$ and $\vec{\omega}_k^m := \frac{1}{\mathcal{H}^{d-1}(\Lambda_k^m)} \sum_{j \in \Theta_k^m} \mathcal{H}^{d-1}(\sigma_j^m) \, \vec{\nu}_j^m$.

For the approximation to the velocity and pressure on $\mathcal{T}^m$ we use the finite element spaces $\mathbb{U}^m(\vec{g})$ and $\mathbb{P}^m$, which are the direct time discrete analogues of $\mathbb{U}^h(\vec{g})$ and $\mathbb{P}^h(t_m)$, as well as $\widehat{\mathbb{P}}^m \subset \widehat{\mathbb{P}}$. We also say that $(\mathbb{U}^m(\vec{0}), \mathbb{P}^m, W(\Gamma^m))$ satisfy the LBB$_\Gamma$ inf-sup condition if there exists a $C_0 \in \mathbb{R}_{>0}$, independent of h^m, such that

$$\inf_{(\varphi, \lambda, \eta) \in \widehat{\mathbb{P}}^m \times \mathbb{R} \times W(\Gamma^m)} \sup_{\vec{\xi} \in \mathbb{U}^m(\vec{0})} \frac{(\varphi, \nabla \cdot \vec{\xi}) + \lambda \left\langle \vec{\omega}^m, \vec{\xi} \right\rangle_{\Gamma^m}^h + \left\langle \eta, \nabla_s \cdot (\vec{\pi}^m \, \vec{\xi}|_{\Gamma^m}) \right\rangle_{\Gamma^m}}{(\|\varphi\|_0 + |\lambda| + \|\eta\|_{0,\Gamma^m})(\|\vec{\xi}\|_1 + \|\vec{\pi}^m \, \underline{\mathcal{P}}_{\Gamma^m} \, \vec{\xi}|_{\Gamma^m}\|_{1,\Gamma^m})} \geq C_0,$$
$$(5.2)$$

where $\|\eta\|_{0,\Gamma^m}^2 := \langle \eta, \eta \rangle_{\Gamma^m}$ and $\|\vec{\eta}\|_{1,\Gamma^m}^2 := \langle \vec{\eta}, \vec{\eta} \rangle_{\Gamma^m} + \langle \nabla_s \vec{\eta}, \nabla_s \vec{\eta} \rangle_{\Gamma^m}$ for $\vec{\eta} \in \underline{V}(\Gamma^m)$. Unfortunately, it does not appear possible to prove that (5.2) holds for e.g. $(\mathbb{U}^m(\vec{0}), \mathbb{P}^m) =$

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$([S_2^m]^d \cap \mathbb{U}(\vec{0}), S_1^m)$, because $\mathcal{T}^m$ and Γ^m are totally independent. Recall that also in the much simpler situation of the XFEM$_\Gamma$ approach from Barrett *et al.* (2013, 2014c), which corresponds to setting $\eta = 0$ in (5.2) and replacing $\left\langle \vec{\omega}^m, \vec{\xi} \right\rangle_{\Gamma^m}^h$ with $\left\langle \vec{\nu}^m, \vec{\xi} \right\rangle_{\Gamma^m}^h$, the authors were unable to show that an LBB condition holds.

Our proposed fully discrete equivalent of (4.19a,b), (4.9c–f) is then given as follows. Let Γ^0, an approximation to $\Gamma(0)$, as well as $\vec{\kappa}^0 \in \underline{V}(\Gamma^0)$ and $\vec{U}^0 \in \mathbb{U}^0(\vec{g})$ be given. For $m = 0, \ldots, M - 1$, find $\vec{U}^{m+1} \in \mathbb{U}^m(\vec{g})$, $P^{m+1} \in \widehat{\mathbb{P}}^m$, $P_{\text{sing}}^{m+1} \in \mathbb{R}$, $P_\Gamma^{m+1} \in W(\Gamma^m)$, $\vec{X}^{m+1} \in \underline{V}(\Gamma^m)$, $\vec{\kappa}^{m+1} \in \underline{V}(\Gamma^m)$ and $\vec{F}_\Gamma^{m+1} \in \underline{V}(\Gamma^m)$ such that

$$
\tfrac{1}{2} \left(\frac{\rho^m \vec{U}^{m+1} - (I_0^m \rho^{m-1}) \vec{I}_2^m \vec{U}^m}{\tau} + (I_0^m \rho^{m-1}) \frac{\vec{U}^{m+1} - \vec{I}_2^m \vec{U}^m}{\tau}, \vec{\xi} \right)
$$

$$
+ 2 \left(\mu^m \underline{\underline{D}}(\vec{U}^{m+1}), \underline{\underline{D}}(\vec{\xi}) \right) + \tfrac{1}{2} \left(\rho^m, [(\vec{I}_2^m \vec{U}^m \cdot \nabla) \vec{U}^{m+1}] \cdot \vec{\xi} - [(\vec{I}_2^m \vec{U}^m \cdot \nabla) \vec{\xi}] \cdot \vec{U}^{m+1} \right)
$$

$$
- \left(P^{m+1}, \nabla \cdot \vec{\xi} \right) - P_{\text{sing}}^{m+1} \left\langle \vec{\omega}^m, \vec{\xi} \right\rangle_{\Gamma^m}^h + \rho_\Gamma \left\langle \frac{\vec{U}^{m+1} - \vec{\Pi}_{m-1}^m (\vec{I}_2^m \vec{U}^m)|_{\Gamma^{m-1}}}{\tau}, \vec{\xi} \right\rangle_{\Gamma^m}^h
$$

$$
+ 2 \mu_\Gamma \left\langle \underline{\underline{D}}_s^m(\vec{\pi}^m \vec{U}^{m+1}), \underline{\underline{D}}_s^m(\vec{\pi}^m \vec{\xi}) \right\rangle_{\Gamma^m} - \left\langle P_\Gamma^{m+1}, \nabla_s \cdot (\vec{\pi}^m \vec{\xi}) \right\rangle_{\Gamma^m}
$$

$$
= \left(\rho^m \vec{f}^{m+1}, \vec{\xi} \right) + \alpha \left\langle \vec{F}_\Gamma^{m+1}, \vec{\xi} \right\rangle_{\Gamma^m}^h - \tfrac{1}{2} \rho_+ \left\langle \vec{U}^m \cdot \vec{\mathrm{n}}, \vec{U}^m \cdot \vec{\xi} \right\rangle_{\partial_2 \Omega} \qquad \forall \, \vec{\xi} \in \mathbb{U}^m(\vec{0}) ,
$$

$$
\tag{5.3a}
$$

$$
\left(\nabla \cdot \vec{U}^{m+1}, \varphi \right) = 0 \qquad \forall \, \varphi \in \widehat{\mathbb{P}}^m \qquad \text{and} \qquad \left\langle \vec{U}^{m+1}, \vec{\omega}^m \right\rangle_{\Gamma^m}^h = 0 , \tag{5.3b}
$$

$$
\left\langle \nabla_s \cdot (\vec{\pi}^m \vec{U}^{m+1}), \eta \right\rangle_{\Gamma^m} = 0 \qquad \forall \, \eta \in W(\Gamma^m) , \tag{5.3c}
$$

$$
\left\langle \frac{\vec{X}^{m+1} - \vec{\mathrm{id}}}{\tau}, \vec{\chi} \right\rangle_{\Gamma^m}^h = \left\langle \vec{U}^{m+1}, \vec{\chi} \right\rangle_{\Gamma^m}^h \qquad \forall \, \vec{\chi} \in \underline{V}(\Gamma^m) , \tag{5.3d}
$$

$$
\left\langle \vec{\kappa}^{m+1}, \vec{\eta} \right\rangle_{\Gamma^m}^{(h)} + \left\langle \nabla_s \vec{X}^{m+1}, \nabla_s \vec{\eta} \right\rangle_{\Gamma^m} = 0 \qquad \forall \, \vec{\eta} \in \underline{V}(\Gamma^m) , \tag{5.3e}
$$

$$
\left\langle \vec{F}_\Gamma^{m+1}, \vec{\chi} \right\rangle_{\Gamma^m}^h = \left\langle \nabla_s \vec{\kappa}^{m+1}, \nabla_s \vec{\chi} \right\rangle_{\Gamma^m} + \left\langle \nabla_s \cdot (\vec{\Pi}_{m-1}^m \vec{\kappa}^m), \nabla_s \cdot \vec{\chi} \right\rangle_{\Gamma^m}
$$

$$
+ \tfrac{1}{2} \left\langle |\vec{\Pi}_{m-1}^m \vec{\kappa}^m|^2 \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi} \right\rangle_{\Gamma^m}^{(h)} - 2 \left\langle \nabla_s (\vec{\Pi}_{m-1}^m \vec{\kappa}^m), \underline{\underline{D}}_s^m(\vec{\chi}) \nabla_s \vec{\mathrm{id}} \right\rangle_{\Gamma^m}
$$

$$
\forall \, \vec{\chi} \in \underline{V}(\Gamma^m) , \tag{5.3f}
$$

and set $\Gamma^{m+1} = \vec{X}^{m+1}(\Gamma^m)$. Here we have defined $\vec{f}^{m+1} := \vec{I}_2^m \vec{f}(\cdot, t_{m+1})$. We observe that (5.3a–f) is a linear scheme in that it leads to a linear system of equations for the unknowns $(\vec{U}^{m+1}, P^{m+1}, P_{\text{sing}}^{m+1}, P_\Gamma^{m+1}, \vec{X}^{m+1}, \vec{\kappa}^{m+1}, \vec{F}_\Gamma^{m+1})$ at each time level.

In the absence of the LBB$_\Gamma$ condition (5.2) we need to consider the reduced system

(5.3a,d–f), where $\mathbb{U}^m(\vec{0})$ in (5.3a) is replaced by $\mathbb{U}_0^m(\vec{0})$. Here we define

$$\mathbb{U}_0^m(\vec{b}) := \left\{ \vec{U} \in \mathbb{U}^m(\vec{b}) : (\nabla \cdot \vec{U}, \varphi) = 0 \ \forall \varphi \in \widehat{\mathbb{P}}^m, \left\langle \nabla_s \cdot (\vec{\pi}^m \vec{U}), \eta \right\rangle_{\Gamma^m} = 0 \ \forall \eta \in W(\Gamma^m) \right.$$
$$\left. \text{and} \ \left\langle \vec{U}, \vec{\omega}^m \right\rangle_{\Gamma^m}^h = 0 \right\},$$

for given data $\vec{b} \in [C(\overline{\Omega})]^d$.

In order to prove the existence of a unique solution to (5.3a–f) we make the following very mild well-posedness assumption.

($\mathcal{A}$) We assume for $m = 0, \dots, M-1$ that $\mathcal{H}^{d-1}(\sigma_j^m) > 0$ for all $j = 1, \dots, J_\Gamma$, and that $\Gamma^m \subset \Omega$.

THEOREM. 5.1. *Let the assumption* ($\mathcal{A}$) *hold. If the LBB$_\Gamma$ condition* (5.2) *holds, then there exists a unique solution* $(\vec{U}^{m+1}, P^{m+1}, P_{\text{sing}}^{m+1}, P_\Gamma^{m+1}, \vec{X}^{m+1}, \vec{\kappa}^{m+1}, \vec{F}_\Gamma^{m+1}) \in \mathbb{U}^m(\vec{g}) \times \widehat{\mathbb{P}}^m \times \mathbb{R} \times W(\Gamma^m) \times [\underline{V}(\Gamma^m)]^3$ *to* (5.3a–f). *In all other cases, on assuming that* $\mathbb{U}_0^m(\vec{g})$ *is nonempty, there exists a unique solution* $(\vec{U}^{m+1}, \vec{X}^{m+1}, \vec{\kappa}^{m+1}, \vec{F}_\Gamma^{m+1}) \in \mathbb{U}_0^m(\vec{g}) \times [\underline{V}(\Gamma^m)]^3$ *to the reduced system* (5.3a,d–f) *with* $\mathbb{U}^m(\vec{0})$ *replaced by* $\mathbb{U}_0^m(\vec{0})$.

Proof. As the system (5.3a–f) is linear, existence follows from uniqueness. In order to establish the latter, we consider the homogeneous system. Find $(\vec{U}, P, P_{\text{sing}}, P_\Gamma, \vec{X}, \vec{\kappa}, \vec{F}_\Gamma) \in \mathbb{U}^m(\vec{0}) \times \widehat{\mathbb{P}}^m \times \mathbb{R} \times W(\Gamma^m) \times [\underline{V}(\Gamma^m)]^3$ such that

$$\frac{1}{2\tau} \left((\rho^m + I_0^m \rho^{m-1}) \vec{U}, \vec{\xi} \right) + 2 \left(\mu^m \underline{D}(\vec{U}), \underline{D}(\vec{\xi}) \right) - \left(P, \nabla \cdot \vec{\xi} \right) - P_{\text{sing}} \left\langle \vec{\omega}^m, \vec{\xi} \right\rangle_{\Gamma^m}^h$$
$$+ \frac{1}{2} \left(\rho^m, [(\vec{I}_2^m \vec{U}^m \cdot \nabla) \vec{U}] \cdot \vec{\xi} - [(\vec{I}_2^m \vec{U}^m \cdot \nabla) \vec{\xi}] \cdot \vec{U} \right)$$
$$+ \frac{1}{\tau} \rho_\Gamma \left\langle \vec{U}, \vec{\xi} \right\rangle_{\Gamma^m}^h + 2 \mu_\Gamma \left\langle \underline{D}_s^m(\vec{\pi}^m \vec{U}), \underline{D}_s^m(\vec{\pi}^m \vec{\xi}) \right\rangle_{\Gamma^m}$$
$$- \left\langle P_\Gamma, \nabla_s \cdot (\vec{\pi}^m \vec{\xi}) \right\rangle_{\Gamma^m} - \alpha \left\langle \vec{F}_\Gamma, \vec{\xi} \right\rangle_{\Gamma^m}^h = 0 \qquad \forall \vec{\xi} \in \mathbb{U}^m(\vec{0}), \tag{5.4a}$$

$$\left(\nabla \cdot \vec{U}, \varphi \right) = 0 \qquad \forall \varphi \in \widehat{\mathbb{P}}^m \qquad \text{and} \qquad \left\langle \vec{U}, \vec{\omega}^m \right\rangle_{\Gamma^m}^h = 0, \tag{5.4b}$$

$$\left\langle \nabla_s \cdot (\vec{\pi}^m \vec{U}), \eta \right\rangle_{\Gamma^m} = 0 \qquad \forall \eta \in W(\Gamma^m), \tag{5.4c}$$

$$\frac{1}{\tau} \left\langle \vec{X}, \vec{\chi} \right\rangle_{\Gamma^m}^h = \left\langle \vec{U}, \vec{\chi} \right\rangle_{\Gamma^m}^h \qquad \forall \vec{\chi} \in \underline{V}(\Gamma^m), \tag{5.4d}$$

$$\langle \vec{\kappa}, \vec{\eta} \rangle_{\Gamma^m}^{(h)} + \left\langle \nabla_s \vec{X}, \nabla_s \vec{\eta} \right\rangle_{\Gamma^m} = 0 \qquad \forall \vec{\eta} \in \underline{V}(\Gamma^m), \tag{5.4e}$$

$$\left\langle \vec{F}_\Gamma, \vec{\chi} \right\rangle_{\Gamma^m}^h - \langle \nabla_s \vec{\kappa}, \nabla_s \vec{\chi} \rangle_{\Gamma^m} = 0 \qquad \forall \vec{\chi} \in \underline{V}(\Gamma^m). \tag{5.4f}$$

Choosing $\vec{\xi} = \vec{U}$ in (5.4a), $\varphi = P$ in (5.4b), $\eta = P_\Gamma$ in (5.4c), $\vec{\chi} = \vec{F}_\Gamma$ in (5.4d), $\vec{\eta} = \vec{\kappa}$ in

(5.4e) and $\vec\chi = \vec{X}$ in (5.4f) yields that

$$\tfrac{1}{2}\left((\rho^m + I_0^m\,\rho^{m-1})\,\vec{U},\vec{U}\right) + 2\,\tau\left(\mu^m\,\underline{D}(\vec{U}),\underline{D}(\vec{U})\right) + \rho_\Gamma\left\langle\vec{U},\vec{U}\right\rangle_{\Gamma^m}^h$$

$$+\,2\,\tau\,\mu_\Gamma\left\langle\underline{D}_s^m(\vec{\pi}^m\,\vec{U}),\underline{D}_s^m(\vec{\pi}^m\,\vec{U})\right\rangle_{\Gamma^m} + \alpha\left\langle\vec\kappa,\vec\kappa\right\rangle_{\Gamma^m}^{(h)} = 0\,. \tag{5.5}$$

It immediately follows from (5.5), Korn's inequality and $\alpha > 0$, that $\vec{U} = \vec{0} \in \mathbb{U}^m(\vec{0})$ and $\vec\kappa = \vec{0}$. (For the application of Korn's inequality we recall that $\mathcal{H}^{d-1}(\partial_1\Omega) > 0$.) Hence (5.4d,f) yield that $\vec{X} = \vec{0}$ and $\vec{F}_\Gamma = \vec{0}$, respectively. Finally, if (5.2) holds then (5.4a) with $\vec{U} = \vec{0}$ and $\vec{F}_\Gamma = \vec{0}$ implies that $P = 0 \in \widehat{\mathbb{P}}^m$, $P_{\mathrm{sing}} = 0$ and $P_\Gamma = 0 \in W(\Gamma^m)$. This shows existence and uniqueness of $(\vec{U}^{m+1}, P^{m+1}, P_\Gamma^{m+1}, \vec{X}^{m+1}, \vec\kappa^{m+1}, \vec{F}_\Gamma^{m+1}) \in \mathbb{U}^m(\vec{g}) \times \widehat{\mathbb{P}}^m \times W(\Gamma^m) \times [\underline{V}(\Gamma^m)]^3$ to (5.3a–f). The proof for the reduced system is very similar. The homogeneous system to consider is (5.4a,d–f) with $\mathbb{U}^m(\vec{0})$ replaced by $\mathbb{U}_0^m(\vec{0})$. As before, we infer that (5.5) holds, which yields that $\vec{U} = \vec{0} \in \mathbb{U}_0^m(\vec{0})$, $\vec\kappa = \vec{0}$, and hence $\vec{X} = \vec{0}$ and $\vec{F}_\Gamma = \vec{0}$. $\square$

REMARK. 5.2. *The scheme (5.3a–f) clearly leads to a coupled system of linear equations. On replacing $\vec{F}_\Gamma^{m+1}$ in (5.3a) with $\vec{F}_\Gamma^m$ the system decouples into (5.3a–c) for $(\vec{U}^{m+1}, P^{m+1}, P_{\mathrm{sing}}^{m+1}, P_\Gamma^{m+1})$ and into (5.3d–f) for $(\vec{X}^{m+1}, \vec\kappa^{m+1}, \vec{F}_\Gamma^{m+1})$. Of course, the subsystem (5.3d–f) itself decouples into three equations for the three unknowns. While the decoupled system offers the advantage of being easier to solve, we found in practice that the coupled scheme (5.3a–f) preserved the surface area better than the decoupled scheme. An additional drawback of the decoupled scheme is that it is less stable and so in general needs smaller time steps than the coupled scheme (5.3a–f). The latter fact can partly be explained with the following observation.*

On replacing $\vec{F}_\Gamma^{m+1}$ with $\vec\kappa^{m+1}$, and in the case $\partial_1\Omega = \partial\Omega$, $\vec{g} = \vec{0}$ and $\rho_\Gamma = 0$, we obtain an unconditionally stable approximation for two-phase flow in the spirit of Barrett et al. (2014c), but with the additional side constraint (2.4c). In particular, for fixed bulk meshes in time one can show that

$$\tfrac{1}{2}(\rho^m\,\vec{U}^{m+1},\vec{U}^{m+1}) + \alpha\,\mathcal{H}^{d-1}(\Gamma^{m+1}) + \tfrac{1}{2}\left(\rho^{m-1}\,(\vec{U}^{m+1} - \vec{U}^m),\vec{U}^{m+1} - \vec{U}^m\right)$$

$$+\,2\,\tau\left(\mu^m\,\underline{D}(U^{m+1}),\underline{D}(U^{m+1})\right) + 2\,\tau\,\mu_\Gamma\left\langle\underline{D}_s^{m+1}(\vec{\pi}^{m+1}\,\vec{U}^{m+1}),\underline{D}_s^{m+1}(\vec{\pi}^{m+1}\,\vec{U}^{m+1})\right\rangle_{\Gamma^{m+1}}$$

$$\leq \tfrac{1}{2}(\rho^{m-1}\,\vec{U}^m,\vec{U}^m) + \alpha\,\mathcal{H}^{d-1}(\Gamma^m) + \tau\left(\rho^m\,\vec{f}^{m+1},\vec{U}^{m+1}\right),$$

see Barrett et al. (2014c, Theorem 4.1) and Barrett et al. (2014b, Theorem 4.2) for more details. It is for these reasons that we prefer the coupled scheme (5.3a–f).

6 Solution methods

As is standard practice for the solution of linear systems arising from discretizations of Stokes and Navier–Stokes equations, we avoid the complications of the constrained

pressure space $\widehat{\mathbb{P}}^m$ in practice by considering an overdetermined linear system with $\mathbb{P}^m$ instead. Introducing the obvious abuse of notation, the linear system (5.3a–c) for $\alpha = 0$, with $\widehat{\mathbb{P}}^m$ replaced by $\mathbb{P}^m$, can be formulated as: Find $(\vec{U}^{m+1}, P^{m+1}, P_{\text{sing}}^{m+1}, P_\Gamma^{m+1})$, such that

$$
\begin{pmatrix}
\vec{B}_\Omega & \vec{C}_\Omega & \vec{D}_\Omega & \vec{S}_{\Gamma,\Omega} \\
\vec{C}_\Omega^T & 0 & 0 & 0 \\
\vec{D}_\Omega^T & 0 & 0 & 0 \\
\vec{S}_{\Gamma,\Omega}^T & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\vec{U}^{m+1} \\
P^{m+1} \\
P_{\text{sing}}^{m+1} \\
P_\Gamma^{m+1}
\end{pmatrix}
=
\begin{pmatrix}
\vec{b} \\
0 \\
0 \\
0
\end{pmatrix} ,
\tag{6.1}
$$

where $(\vec{U}^{m+1}, P^{m+1}, P_{\text{sing}}^{m+1}, P_\Gamma^{m+1}) \in (\mathbb{R}^d)^{K_{\mathbb{U}}^m} \times \mathbb{R}^{K_{\mathbb{P}}^m} \times \mathbb{R} \times \mathbb{R}^{K_\Gamma}$ here denote the coefficients of these finite element functions with respect to the their standard bases. The definitions of the matrices and vectors in (6.1) directly follow from (5.3a–c), but we state them here for completeness in the case $\vec{g} = \vec{0}$. Let $i, j = 1, \ldots, K_{\mathbb{U}}^m$, $n, q = 1, \ldots, K_{\mathbb{P}}^m$ and $k, l = 1, \ldots, K_\Gamma$. Then

$$
\begin{aligned}
[\vec{B}_\Omega]_{ij} :={}& \left(\tfrac{\rho^m + I_0^m \rho^{m-1}}{2\tau} \phi_j^{\mathbb{U}^m}, \phi_i^{\mathbb{U}^m} \right) \underline{\text{Id}} + 2 \left(\left(\mu^m \underline{D}(\phi_j^{\mathbb{U}^m} \vec{e}_r), \underline{D}(\phi_i^{\mathbb{U}^m} \vec{e}_s) \right) \right)_{r,s=1}^d \\
& + \tfrac{1}{2} \left(\rho^m, [(\vec{I}_2^m \vec{U}^m \cdot \nabla) \phi_j^{\mathbb{U}^m}] \phi_i^{\mathbb{U}^m} - [(\vec{I}_2^m \vec{U}^m \cdot \nabla) \phi_i^{\mathbb{U}^m}] \phi_j^{\mathbb{U}^m} \right) \underline{\text{Id}}, \\
& + \tfrac{\rho_\Gamma}{\tau} \langle \varphi_j^{\mathbb{U}^m}, \varphi_i^{\mathbb{U}^m} \rangle_{\Gamma^m}^h \underline{\text{Id}} + 2\mu_\Gamma \left(\langle \underline{D}_s^m(\pi^m \phi_j^{\mathbb{U}^m} \vec{e}_r), \underline{D}_s^m(\pi^m \phi_i^{\mathbb{U}^m} \vec{e}_s) \rangle_{\Gamma^m} \right)_{r,s=1}^d \\
[\vec{C}_\Omega]_{iq} :={}& - \left(\phi_q^{\mathbb{P}^m}, (\nabla \cdot (\phi_i^{\mathbb{U}^m} \vec{e}_r)) \right)_{r=1}^d, \qquad [\vec{S}_{\Gamma,\Omega}]_{il} := - \left(\langle \chi_l^m, \nabla_s \cdot (\pi^m \phi_i^{\mathbb{U}^m} \vec{e}_r) \rangle_{\Gamma^m} \right)_{r=1}^d , \\
\vec{b}_i ={}& \left(\tfrac{I_0^m \rho^{m-1}}{\tau} \vec{I}_2^m \vec{U}^m + \rho^m \vec{f}^{m+1}, \phi_i^{\mathbb{U}^m} \right) + \tfrac{\rho_\Gamma}{\tau} \langle \vec{\Pi}_{m-1}^m \vec{U}^m |_{\Gamma^{m-1}}, \varphi_i^{\mathbb{U}^m} \rangle_{\Gamma^m}^h \\
& - \tfrac{1}{2} \rho_+ \langle (\vec{U}^m \cdot \vec{\text{n}}) \vec{U}^m, \phi_i^{\mathbb{U}^m} \rangle_{\partial_2 \Omega} ;
\end{aligned}
\tag{6.2}
$$

where $\{\vec{e}_r\}_{r=1}^d$ denotes the standard basis in $\mathbb{R}^d$, and where we have used the convention that the subscripts in the matrix notations refer to the test and trial domains, respectively. A single subscript is used where the two domains are the same. The entries of $\vec{D}_\Omega$, for $i = 1, \ldots, K_{\mathbb{U}}^m$, are given by $[\vec{D}_\Omega]_{i,1} := -\langle \phi_i^{\mathbb{U}^m}, \vec{\omega}^m \rangle_{\Gamma^m}^h$.

The only new term compared to previous works by the authors on two-phase flows, see Barrett *et al.* (2014c,b), is $\vec{S}_{\Gamma,\Omega}$. Here we note that

$$
\left(\langle \chi_l^m, \nabla_s \cdot (\pi^m \phi_i^{\mathbb{U}^m} \vec{e}_r) \rangle_{\sigma^m} \right)_{r=1}^d = \left(\langle \chi_l^m, \nabla_s (\pi^m \phi_i^{\mathbb{U}^m}) \cdot \vec{e}_r \rangle_{\sigma^m} \right)_{r=1}^d = \langle \chi_l^m, 1 \rangle_{\sigma^m} \nabla_s (\pi^m \phi_i^{\mathbb{U}^m}),
\tag{6.3}
$$

so that the following algorithm can be used to assemble $\vec{S}_{\Gamma,\Omega}$.

Algorithm 1: Calculate the matrix contributions for (6.3).

For all elements σ^m of Γ^m do

  Compute $\vec{G}_j = \nabla_s\,\chi^m_{Q_j}$, $j = 1,\ldots,d$ for the d vertices $\vec{Q}_1,\ldots,\vec{Q}_d$ of σ^m.

  For each vertex $\vec{Q}_j$ of σ^m, find the bulk element in which $\vec{Q}_j$ lies and denote the local S_2^m bulk basis functions on these elements with $\varphi_k^{local,j}$, $k = 1,\ldots,K$.

  For all $i = 1,\ldots,d$ do

    For all $j = 1,\ldots,d$ do

      For all $l = 1,\ldots,K$ do

        Add $\frac{1}{d}\,\mathcal{H}^{d-1}(\sigma^m)\,\varphi_l^{local,j}(\vec{Q}_j)\,\vec{G}_j$ to the contributions for

        $\left(\left\langle \chi^m_{global\_dof(i)}, \nabla_s\cdot(\pi^m\,\varphi^{\mathbb{U}^m}_{global\_dof(l)}\,\vec{e}_r)\right\rangle_{\Gamma^m}\right)_{r=1}^d$.

      end do

    end do

  end do

end do

In order to provide a matrix-vector formulation for the full system (5.3a–f), and in particular in view of (5.3f), we recall from Dziuk (2008, p. 64) that

$$2\left\langle \nabla_s\,\vec{\xi}, \underline{\underline{D}}^m_s(\vec{\chi})\,\nabla_s\,\vec{\mathrm{id}}\right\rangle_{\Gamma^m}$$

$$= \sum_{i,j=1}^d \left\langle (\nabla_s)_j\,(\vec{\xi})_i, (\nabla_s)_i\,(\vec{\chi})_j\right\rangle_{\Gamma^m} - \sum_{i,j=1}^d \left\langle (\vec{\nu}^m)_i\,(\vec{\nu}^m)_j\,\nabla_s\,(\vec{\xi})_j, \nabla_s\,(\vec{\chi})_i\right\rangle_{\Gamma^m}$$

$$+ \left\langle \nabla_s\,\vec{\xi}, \nabla_s\,\vec{\chi}\right\rangle_{\Gamma^m}$$

$$= \sum_{i,j=1}^d \left\langle (\nabla_s)_j\,(\vec{\xi})_i, (\nabla_s)_i\,(\vec{\chi})_j\right\rangle_{\Gamma^m} + \sum_{i,j=1}^d \left\langle (\delta_{ij} - (\vec{\nu}^m)_i\,(\vec{\nu}^m)_j)\,\nabla_s\,(\vec{\xi})_j, \nabla_s\,(\vec{\chi})_i\right\rangle_{\Gamma^m}.$$

Moreover, we observe that $\langle \nabla_s\cdot\vec{\xi}, \nabla_s\cdot\vec{\chi}\rangle_{\Gamma^m} = \sum_{i,j=1}^d \langle (\nabla_s)_j\,(\vec{\xi})_j, (\nabla_s)_i\,(\vec{\chi})_i\rangle_{\Gamma^m}$. Hence, in addition to (6.2), we introduce the following matrices, where $q = 1,\ldots,K^m_{\mathbb{U}}$, and $k,l = 1,\ldots,K_\Gamma$.

$$[\vec{\mathcal{B}}_\Gamma]_{kl} := \left(\langle[\nabla_s]_j\,\chi^m_l, [\nabla_s]_i\,\chi^m_k\rangle_{\Gamma^m}\right)_{i,j=1}^d, \qquad [\vec{\mathcal{R}}_\Gamma]_{kl} := \left\langle \nabla_s\,\chi^m_l\cdot\nabla_s\,\chi^m_k, \underline{\underline{\mathrm{Id}}} - \vec{\nu}^m\otimes\vec{\nu}^m\right\rangle_{\Gamma^m},$$

$$[\vec{A}_{\Gamma,\vec{\kappa}}]_{kl} := \tfrac{1}{2}\left\langle |\vec{\Pi}^m_{m-1}\,\vec{\kappa}^m|^2\,\nabla_s\,\chi^m_l, \nabla_s\,\chi^m_k\right\rangle_{\Gamma^m}^h \underline{\underline{\mathrm{Id}}}, \qquad [\vec{M}_{\Gamma,\Omega}]_{ql} := \left\langle \chi^m_l, \phi^{\mathbb{U}^m}_q\right\rangle_{\Gamma^m} \underline{\underline{\mathrm{Id}}},$$

$$[\vec{M}_\Gamma]_{kl} := \langle \chi^m_l, \chi^m_k\rangle^h_{\Gamma^m}\,\underline{\underline{\mathrm{Id}}}, \qquad [A_\Gamma]_{kl} := \langle \nabla_s\,\chi^m_l, \nabla_s\,\chi^m_k\rangle_{\Gamma^m}, \qquad [\vec{A}_\Gamma]_{kl} := [A_\Gamma]_{kl}\,\underline{\underline{\mathrm{Id}}}.$$

Here we have made use of the fact that

$$[\vec{\mathcal{B}}_\Gamma]_{kl} = \left(\langle\nabla_s\cdot(\chi^m_l\,\vec{e}_j), \nabla_s\cdot(\chi^m_k\,\vec{e}_i)\rangle_{\Gamma^m}\right)_{i,j=1}^d = \left(\langle(\nabla_s\,\chi^m_l)\cdot\vec{e}_j, (\nabla_s\,\chi^m_k)\cdot\vec{e}_i\rangle_{\Gamma^m}\right)_{i,j=1}^d$$

$$= \left(\langle[\nabla_s]_j\,\chi^m_l, [\nabla_s]_i\,\chi^m_k\rangle_{\Gamma^m}\right)_{i,j=1}^d.$$

Moreover, it clearly holds that $([\vec{\mathcal{B}}_\Gamma]_{kl})^T = [\vec{\mathcal{B}}_\Gamma]_{lk} =: [\vec{\mathcal{B}}^\star_\Gamma]_{kl}$.

Denoting the system matrix in (6.1) as $\begin{pmatrix} \vec{B}_\Omega & \vec{\mathcal{C}} \\ \vec{\mathcal{C}}^T & 0 \end{pmatrix}$, and letting $\widetilde{P}^{m+1} = (P^{m+1}, P^{m+1}_{\text{sing}},$ $P^{m+1}_\Gamma)^T$, then the linear system (5.3a–f), with numerical integration in (5.3e,f), can be written as

$$
\begin{pmatrix}
\vec{B}_\Omega & \vec{\mathcal{C}} & 0 & 0 & -\alpha\,\vec{M}_{\Gamma,\Omega} \\
\vec{\mathcal{C}}^T & 0 & 0 & 0 & 0 \\
(\vec{M}_{\Gamma,\Omega})^T & 0 & 0 & -\frac{1}{\tau}\vec{M}_\Gamma & 0 \\
0 & 0 & \vec{M}_\Gamma & \vec{A}_\Gamma & 0 \\
0 & 0 & -\vec{A}_\Gamma & 0 & \vec{M}_\Gamma
\end{pmatrix}
\begin{pmatrix}
\vec{U}^{m+1} \\
\widetilde{P}^{m+1} \\
\vec{\kappa}^{m+1} \\
\delta\vec{X}^{m+1} \\
\vec{F}^{m+1}_\Gamma
\end{pmatrix}
=
\begin{pmatrix}
\vec{b} \\
0 \\
0 \\
-\vec{A}_\Gamma\,\vec{X}^m \\
\vec{\mathcal{Z}}_\Gamma\,\vec{\kappa}^m + \vec{A}_{\Gamma,\vec{\kappa}}\,\vec{X}^m
\end{pmatrix}, \quad (6.4)
$$

where $\vec{\mathcal{Z}}_\Gamma := \vec{\mathcal{B}}_\Gamma - \vec{\mathcal{B}}^\star_\Gamma - \vec{\mathcal{R}}_\Gamma$. For the solution of (6.4) a Schur complement approach similarly to Barrett *et al.* (2014c) can be used. In particular, the Schur approach for eliminating $(\vec{\kappa}^{m+1}, \delta\vec{X}^{m+1}, \vec{F}^{m+1}_\Gamma)$ from (6.4) can be obtained as follows. Let

$$
\Theta_\Gamma := \begin{pmatrix}
0 & -\frac{1}{\tau}\vec{M}_\Gamma & 0 \\
\vec{M}_\Gamma & \vec{A}_\Gamma & 0 \\
-\vec{A}_\Gamma & 0 & \vec{M}_\Gamma
\end{pmatrix}.
$$

Then (6.4) can be reduced to

$$
\begin{pmatrix}
\vec{B}_\Omega + \alpha\,(0\ 0\ \vec{M}_{\Gamma,\Omega})\,\Theta_\Gamma^{-1}\begin{pmatrix}(\vec{M}_{\Gamma,\Omega})^T \\ 0 \\ 0\end{pmatrix} & \vec{\mathcal{C}} \\
\vec{\mathcal{C}}^T & 0
\end{pmatrix}
\begin{pmatrix}
\vec{U}^{m+1} \\
\widetilde{P}^{m+1}
\end{pmatrix}
$$
$$
= \begin{pmatrix}
\vec{b} + \alpha\,(0\ 0\ \vec{M}_{\Gamma,\Omega})\,\Theta_\Gamma^{-1}\begin{pmatrix} 0 \\ -\vec{A}_\Gamma\,\vec{X}^m \\ \vec{\mathcal{Z}}_\Gamma\,\vec{\kappa}^m + \vec{A}_{\Gamma,\vec{\kappa}}\,\vec{X}^m \end{pmatrix} \\
0
\end{pmatrix} \quad (6.5a)
$$

and

$$
\begin{pmatrix}
\vec{\kappa}^{m+1} \\
\delta\vec{X}^{m+1} \\
\vec{F}^{m+1}_\Gamma
\end{pmatrix}
= \Theta_\Gamma^{-1}\begin{pmatrix}
-(\vec{M}_{\Gamma,\Omega})^T\,\vec{U}^{m+1} \\
-\vec{A}_\Gamma\,\vec{X}^m \\
\vec{\mathcal{Z}}_\Gamma\,\vec{\kappa}^m + \vec{A}_{\Gamma,\vec{\kappa}}\,\vec{X}^m
\end{pmatrix}. \quad (6.5b)
$$

For the linear system (6.5a) well-known solution methods for finite element discretizations for the standard Navier–Stokes equations may be employed. We refer to Barrett *et al.* (2014c, §5), where we describe such solution methods in detail for a very similar situation.

7 Numerical results

For the bulk mesh adaptation we use the strategy from Barrett *et al.* (2014c), which results in a fine mesh size h_f around Γ^m and a coarse mesh size h_c further away from it. Here $h_f = \frac{2 \min\{H_1, H_2\}}{N_f}$ and $h_c = \frac{2 \min\{H_1, H_2\}}{N_c}$ are given by two integer numbers $N_f > N_c$,

where we assume from now on that the convex hull of Ω is given by $\times_{i=1}^{d}(-H_i, H_i)$. To summarize the discretization parameters we use the shorthand notation $n\,\text{adapt}_{k,l}$ from Barrett *et al.* (2014c). The subscripts refer to the fineness of the spatial discretizations, i.e. for the set $n\,\text{adapt}_{k,l}$ it holds that $N_f = 2^k$ and $N_c = 2^l$. For the case $d = 2$, in this paper, we have in addition that $K_\Gamma = J_\Gamma = 2^k + 1$, while for $d = 3$ it holds that $(K_\Gamma, J_\Gamma) = (1538, 3072)$ for $k = 5$. Finally, the uniform time step size for the set $n\,\text{adapt}_{k,l}$ is given by $\tau = 10^{-3}/n$, and if $n = 1$ we write $\text{adapt}_{k,l}$. We remark that we implemented the scheme (5.3a–f) with the help of the finite element toolbox ALBERTA, see Schmidt and Siebert (2005).

In all the numerical simulations we employ the scheme with numerical integration in (5.3e,f), i.e. we choose the superscript $\cdot^h$ in the two brackets $\cdot^{(h)}$. The initial data $\vec\kappa^0 \in \underline{V}(\Gamma^0)$ is always computed as the solution to

$$\left\langle \vec\kappa^0, \vec\eta \right\rangle_{\Gamma^0}^h + \left\langle \nabla_s \vec{\text{id}}, \nabla_s \vec\eta \right\rangle_{\Gamma^0} = 0 \qquad \forall\, \vec\eta \in \underline{V}(\Gamma^0)\,.$$

In addition, we employ the lowest order Taylor–Hood element P2–P1 in all computations and set $\vec{U}^0 = \vec{I}_2^0\,\vec{u}_0$. Unless stated otherwise we fix $\partial_1\Omega = \partial\Omega$, $\vec{g} = \vec{0}$ and $\vec{u}_0 = \vec{0}$. The volume force is always set to $\vec{f} = \vec{0}$. Moreover, we set all physical parameters to unity, i.e. $\rho_\pm = \mu_\pm = \rho_\Gamma = \mu_\Gamma = \alpha = 1$, unless stated otherwise.

At times we will discuss the discrete energy of the numerical solutions. On recalling Theorem 4.1 the discrete energy is defined by

$$\mathcal{E}^h(\Gamma^m, \vec\kappa^{m+1}) := \mathcal{E}_{kin}^h(\Gamma^m, \rho^m, \vec{U}^{m+1}) + \tfrac{1}{2}\alpha \left\langle \vec\kappa^{m+1}, \vec\kappa^{m+1} \right\rangle_{\Gamma^m}^h\,,$$

where

$$\mathcal{E}_{kin}^h(\Gamma^m, \rho^m, \vec{U}^{m+1}) := \tfrac{1}{2}\,\|[\rho^m]^{\frac{1}{2}}\,\vec{U}^{m+1}\|_0^2 + \tfrac{1}{2}\rho_\Gamma \left\langle \vec{U}^{m+1}, \vec{U}^{m+1} \right\rangle_{\Gamma^m}^h$$

represents the kinetic part of the discrete energy. For the simulation of vesicles the reduced volume is often mentioned as a characteristic number. In the case $d = 3$, and for the initial discrete interface Γ^0, this is defined as

$$v_r = \frac{3\,\mathcal{L}^3(\Omega_-^0)}{4\,\pi\,(\frac{\mathcal{H}^2(\Gamma^0)}{4\,\pi})^{\frac{3}{2}}} = \frac{6\,\pi^{\frac{1}{2}}\,\mathcal{L}^3(\Omega_-^0)}{(\mathcal{H}^2(\Gamma^0))^{\frac{3}{2}}}\,,$$

see e.g. Wintz *et al.* (1996). In a similar fashion, for the case $d = 2$ we define the reduced area as

$$a_r = \frac{\mathcal{L}^2(\Omega_-^0)}{\pi\,(\frac{\mathcal{H}^1(\Gamma^0)}{2\,\pi})^2} = \frac{4\,\pi\,\mathcal{L}^2(\Omega_-^0)}{(\mathcal{H}^1(\Gamma^0))^2}\,.$$

7.1 Numerical simulations in 2d

For all our two-dimensional simulations we choose the discretization parameters $2\,\text{adapt}_{9,4}$. In all the simulations presented here the areas of the two phases, as well as the length of

the interface, are well preserved, with the relative differences over time in each case being less than 0.2%. Moreover, the ratio

$$r_a := \max_{j \in \{1, \dots, J_\Gamma\}} \mathcal{H}^{d-1}(\sigma_j^m) / \min_{j \in \{1, \dots, J_\Gamma\}} \mathcal{H}^{d-1}(\sigma_j^m)$$

of the largest and smallest elements' lengths was always bounded by 1.005. Here we note that we always choose the initial polygon Γ^0 to be equidistributed.

We conducted the following shearing experiments on the domain $\Omega = (-2, 2)^2$ for an initial interface in the form of an ellipse, centred at the origin, with axis lengths 1 and 2.5, so that $a_r = 0.745$. In particular, we prescribe the inhomogeneous Dirichlet boundary condition $\vec{g}(\vec{z}) = (z_2, 0)^T$ on $\partial_1 \Omega = [-2, 2] \times \{\pm 2\}$. For the initial data $\vec{u}_0$ we choose the function $\vec{u}_0(\vec{z}) = \eta(z_2)\, \vec{e}_1$, where $\eta : [-2, 2] \to \mathbb{R}$ is a continuous piecewise linear function with $\eta(\pm 2) = \pm 2$ and $\eta(s) = 0$ if $|s| \le 1.5$. Hence $\vec{u}_0$ satisfies the required conditions $\nabla \cdot \vec{u}_0 = 0$ in Ω and $\nabla_s \cdot \vec{u}_0 = 0$ on $\Gamma(0)$, recall (2.8), and is such that $\vec{u}_0 = \vec{g}$ on $\partial_1 \Omega$. The remaining parameters are given by $\alpha = 0.05$, $\rho_\pm = \rho_\Gamma = 1$, $\mu_\Gamma = 0.05$ and either

$$\text{(a)} \quad \mu_+ = 1, \quad \mu_- = 1, \quad \text{or} \quad \text{(b)} \quad \mu_+ = 1, \quad \mu_- = 10. \tag{7.1}$$

The results can be seen in Figures 2 and 3. In the first case we observe that the evolution reaches a steady state in which the interfacial fluid rotates along the interface. This motion is often called tank treading, see e.g. Salac and Miksis (2012). The second example, on the other hand, leads to a rotation of the whole vesicle, and this is called tumbling.

The numerical simulation of a vesicle flowing through a constriction can be seen in Figure 4. This example shows that membranes can drastically deform in order to pass through a constriction. This resembles the remarkable properties of red blood cells, which show a similar behaviour when flowing through capillaries. Here we choose the initial shape of the interface to be an elongated tube of total dimension 0.2×1.5. This gives a reduced area of $a_r = 0.351$. As the computational domain we choose $\Omega = (-2, 2) \times (-1, 1) \setminus ((-1, 1) \times (-1, -0.5) \cup (-1, 1) \times (0.5, 1))$ with $\partial_2 \Omega = \{2\} \times (-1, 1)$ and $\partial_1 \Omega = \partial \Omega \setminus \partial_2 \Omega$. On the left boundary $\{-2\} \times [-1, 1]$ we prescribe the inhomogeneous boundary conditions $\vec{g}(\vec{z}) = (1 - z_2^2, 0)^T$ in order to model Poiseuille flow. In this computation we consider the quasi-static variant with $\rho_\pm = 0$. Moreover, we set $\alpha = 0.1$ and let $\rho_\Gamma = 0$ or $\rho_\Gamma = 15$. In the latter case the effect of inertia on the evolution is clearly visible.

An even more pronounced difference between $\rho_\Gamma = 0$ and $\rho_\Gamma > 0$ can be observed in our next simulation, where we start with an initial shape in the form of a smooth letter "C" with reduced area $a_r = 0.326$. As the computational domain we choose $\Omega = (-1, 1)^2$, and we let $\rho_\pm = \alpha = 1$. See Figures 5 and Figures 6 for the evolutions in the cases $\rho_\Gamma = 0$ and $\rho_\Gamma = 1$, respectively. In the latter case, the two arms of the vesicle swing up and down due to inertia, which is clearly visible in the plot of the kinetic energy as well.

7.2 Numerical simulations in 3d

As an example for a three-dimensional simulation, we consider the evolution for an initially flat plate of total dimension $4 \times 4 \times 1$, similarly to Barrett *et al.* (2008b, Fig. 15). As
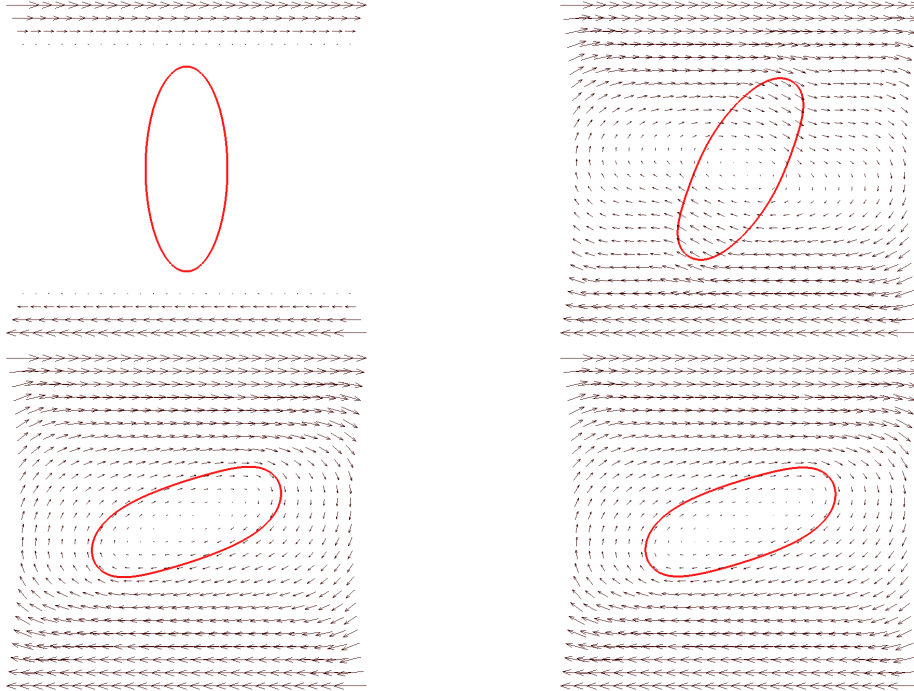
27

Figure 2: Shear flow with parameters as in (7.1a). The plots show the interface Γ^m, together with the discrete velocity $\vec{U}^m$ on $\overline{\Omega}$, at times $t = 0,\ 1,\ 3,\ 5$ (top left to bottom right).

the computational domain we choose $\Omega = (-2.5, 2.5)^3$, and we set $\rho_\Gamma = 0$. We note that the reduced volume for this shape is given by $v_r = 0.569$. As discretization parameters we choose adapt$_{5,2}$, so that the initial triangulation Γ^0 satisfies $(K_\Gamma, J_\Gamma) = (1538, 3072)$ and $r_a = 1.898$. The results for the scheme (5.3a–f) can be seen in Figure 7, where we note that the interface assumes the shape of a red blood cell. Plots of the discrete energies and of the ratio r_a are also shown in Figure 7. We note that the discrete energy is monotonically decreasing, while the ratio r_a always remains bounded below 2.1.

Acknowledgements

References

Aland, S., Egerer, S., Lowengrub, J., and Voigt, A. (2014). Diffuse interface models of locally inextensible vesicles in a viscous fluid. *J. Comput. Phys.*, **277**, 32–47.

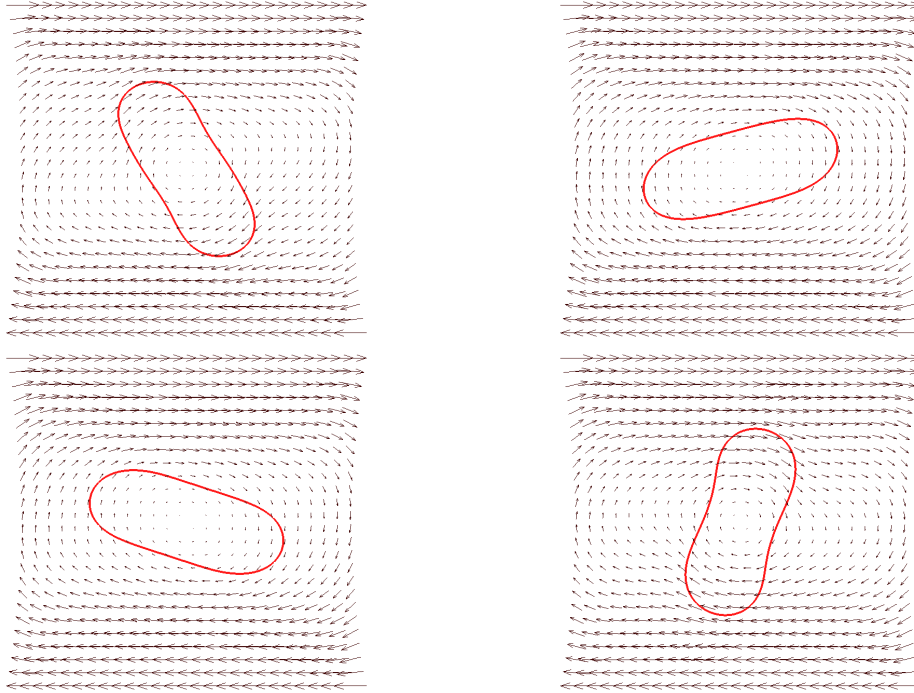Arroyo, M. and DeSimone, A. (2009). Relaxation dynamics of fluid membranes. *Phys. Rev. E*, **79**(3), 031915, 17.

Figure 3: Shear flow with parameters as in (7.1b). The plots show the interface Γ^m, together with the discrete velocity $\vec{U}^m$ on $\overline{\Omega}$, at times $t = 8$, 11, 14, 17 (top left to bottom right).

Arroyo, M., DeSimone, A., and Heltai, L. (2010). The role of membrane viscosity in the dynamics of fluid membranes. `http://arxiv.org/abs/1007.4934`.

Bänsch, E. (2001). Finite element discretization of the Navier–Stokes equations with a free capillary surface. *Numer. Math.*, **88**(2), 203–235.

Barrett, J. W., Garcke, H., and Nürnberg, R. (2008a). On the parametric finite element approximation of evolving hypersurfaces in $\mathbb{R}^3$. *J. Comput. Phys.*, **227**(9), 4281–4307.

Barrett, J. W., Garcke, H., and Nürnberg, R. (2008b). Parametric approximation of Willmore flow and related geometric evolution equations. *SIAM J. Sci. Comput.*, **31**(1), 225–253.

Barrett, J. W., Garcke, H., and Nürnberg, R. (2012). Parametric approximation of isotropic and anisotropic elastic flow for closed and open curves. *Numer. Math.*, **120**(3), 489–542.

Barrett, J. W., Garcke, H., and Nürnberg, R. (2013). Eliminating spurious velocities with a stable approximation of viscous incompressible two-phase Stokes flow. *Comput. Methods Appl. Mech. Engrg.*, **267**, 511–530.

Barrett, J. W., Garcke, H., and Nürnberg, R. (2014a). On the stable numerical approximation of two-phase flow with insoluble surfactant. *M2AN Math. Model. Numer. Anal.* (to appear, DOI: 10.1051/m2an/2014039).
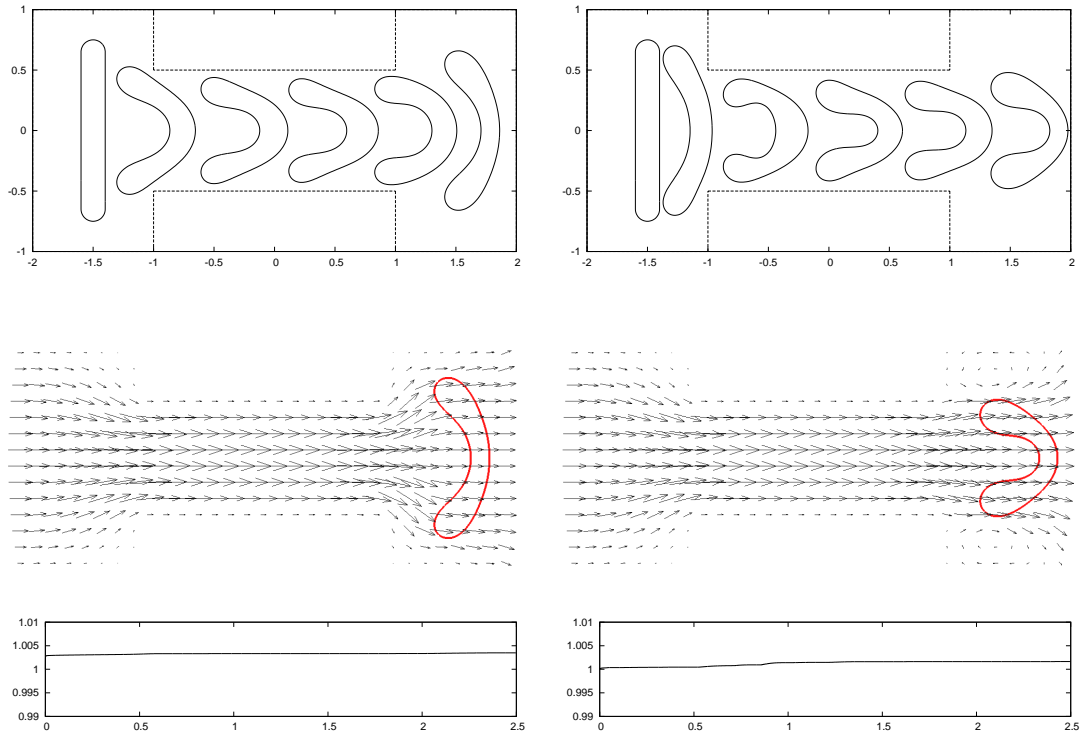
Figure 4: Flow through a constriction for the scheme (5.3a–f). Left for $\rho_\Gamma = 0$ and right for $\rho_\Gamma = 15$, where the plots show the interface Γ^m at times $t = 0$, 0.5, 1, 1.5, 2, 2.5. The middle row shows a visualization of $\vec{U}^m$ at time $t = 2.5$, and below are plots of the ratio r_a.

Barrett, J. W., Garcke, H., and Nürnberg, R. (2014b). Stable numerical approximation of two-phase flow with a Boussinesq–Scriven surface fluid. `http://arxiv.org/abs/1404.5519`.

Barrett, J. W., Garcke, H., and Nürnberg, R. (2014c). A stable parametric finite element discretization of two-phase Navier–Stokes flow. *J. Sci. Comp.* (to appear, DOI: 10.1007/s10915-014-9885-2).

Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L. D., and Russo, A. (2013). Basic principles of virtual element methods. *Math. Models Methods Appl. Sci.*, **23**(1), 199–214.

Bonito, A., Nochetto, R. H., and Pauletti, M. S. (2010). Parametric FEM for geometric biomembranes. *J. Comput. Phys.*, **229**(9), 3171–3188.

Bonito, A., Nochetto, R. H., and Pauletti, M. S. (2011). Dynamics of biomembranes: effect of the bulk fluid. *Math. Model. Nat. Phenom.*, **6**(5), 25–43.
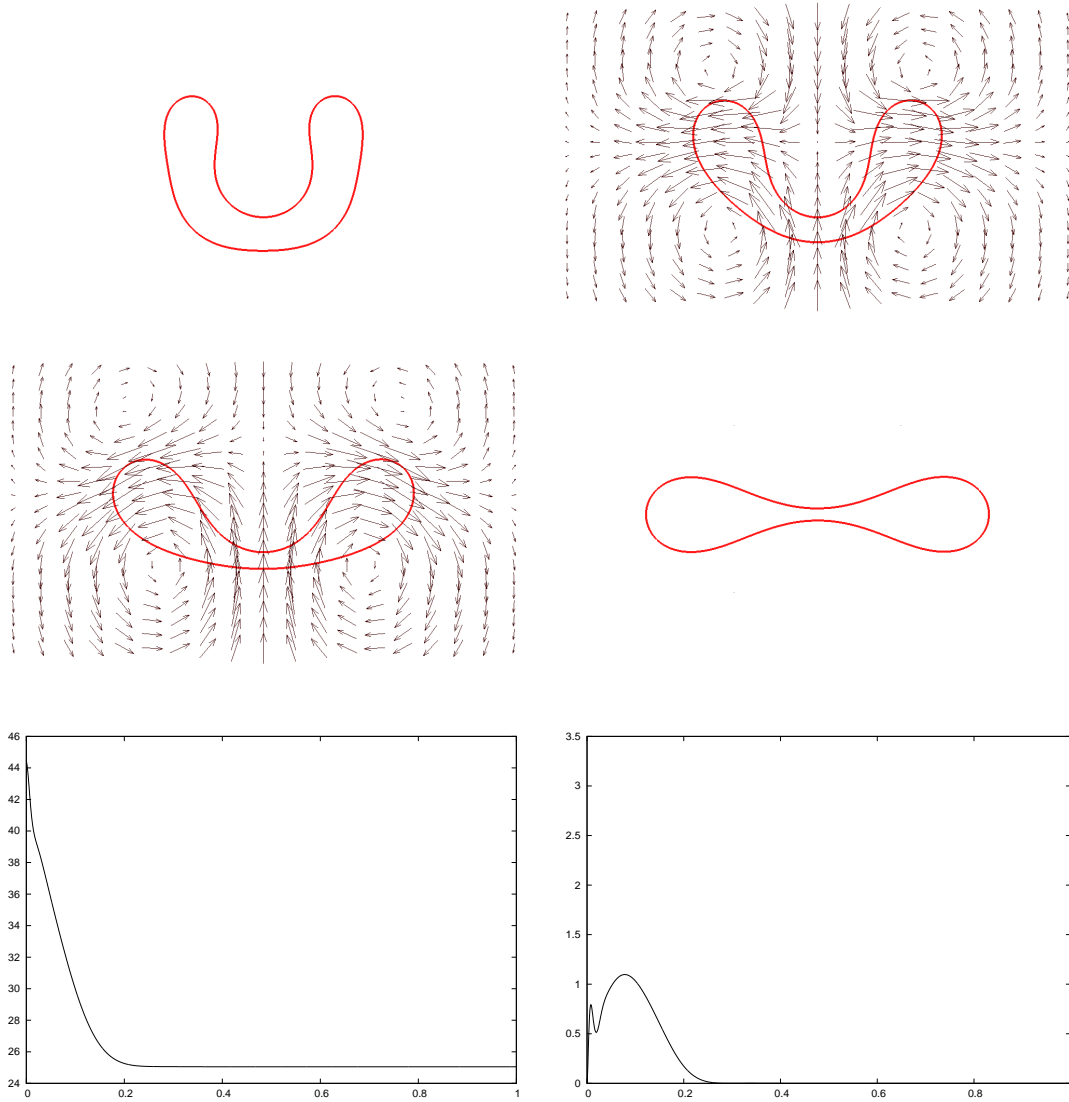
Figure 5: Flow for a smooth letter "C" for $\rho_\Gamma = 0$. The plots show the interface Γ^m, together with the discrete velocity $\vec{U}^m$ on $[-1, 1] \times [-0.5, 0.5]$, at times $t = 0,\ 0.05,\ 0.1,\ 1$. Below are plots of the discrete energy and the discrete kinetic energy.
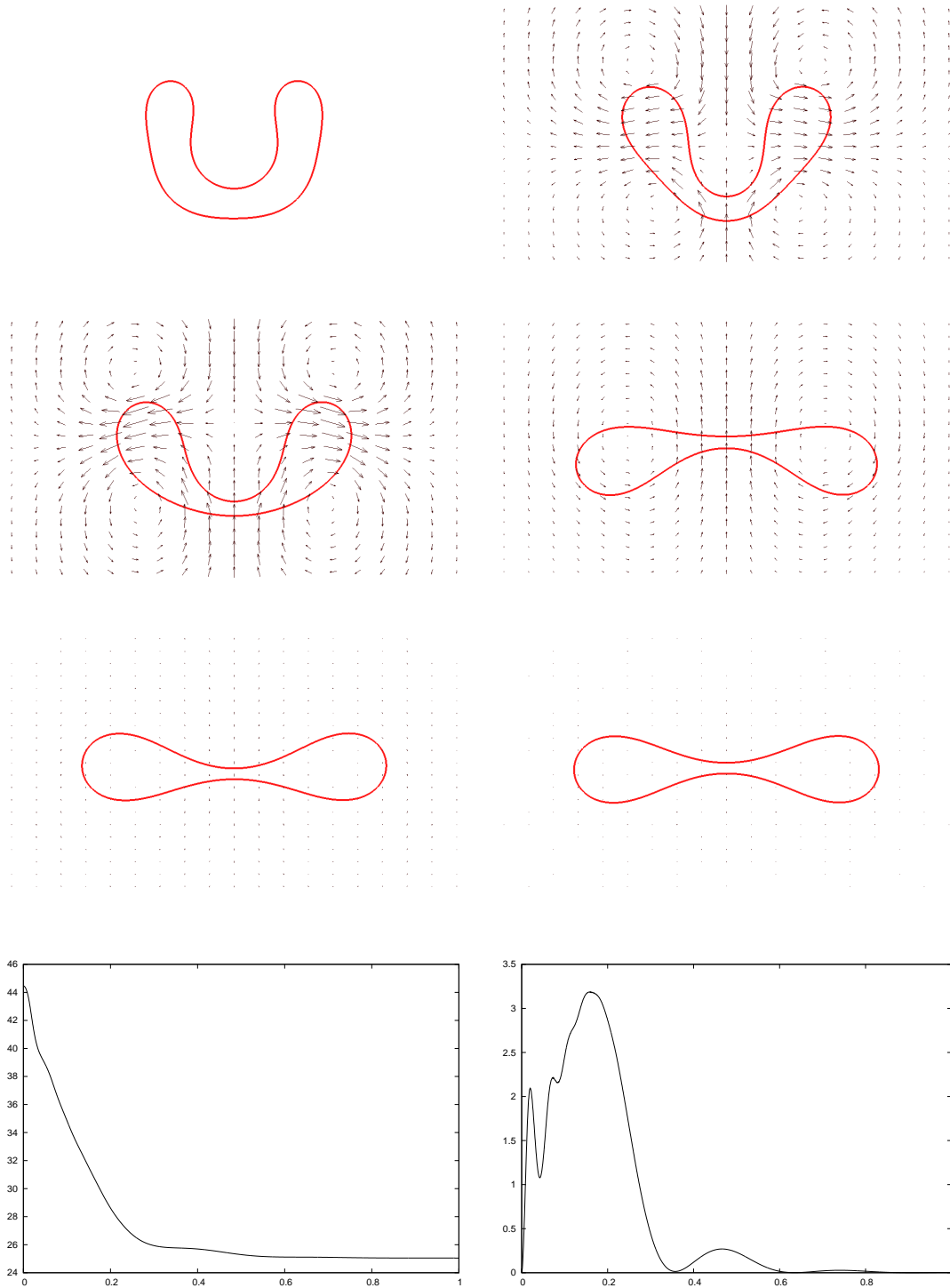
Figure 6: Flow for a smooth letter "C" for $\rho_\Gamma = 1$. The plots show the interface Γ^m, together with the discrete velocity $\vec{U}^m$ on $[-1,1] \times [-0.5, 0.5]$, at times $t = 0, 0.05, 0.1, 0.3, 0.6, 1$. Below are plots of the discrete energy and the discrete kinetic energy.
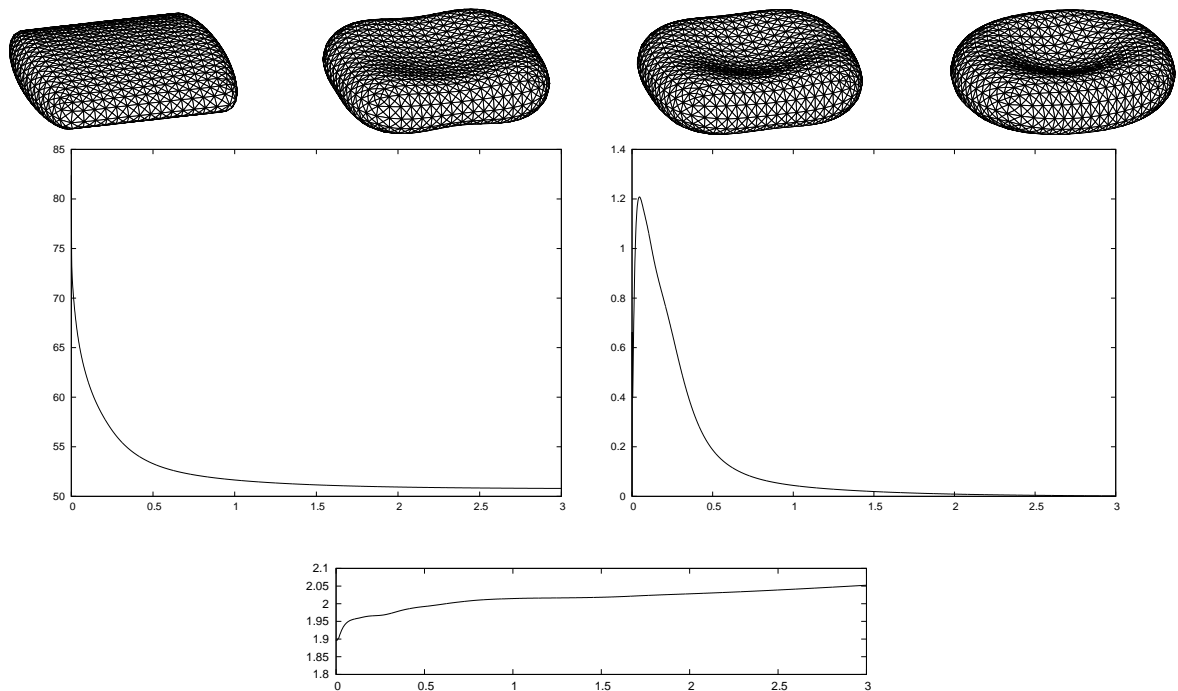
Figure 7: Flow for a flat plate of dimensions $4 \times 4 \times 1$ for the scheme (5.3a–f). The triangulations of Γ^m at times $t = 0$, 0.5, 1, 3. The middle row shows plots of the discrete energy and the discrete kinetic energy, while a plot of the ratio r_a can be found at the bottom.

Clarenz, U., Diewald, U., Dziuk, G., Rumpf, M., and Rusu, R. (2004). A finite element method for surface restoration with smooth boundary conditions. *Comput. Aided Geom. Design*, **21**(5), 427–445.

Deckelnick, K. and Dziuk, G. (2009). Error analysis for the elastic flow of parametrized curves. *Math. Comp.*, **78**(266), 645–671.

Deckelnick, K. and Schieweck, F. (2010). Error analysis for the approximation of axisymmetric Willmore flow by C^1-finite elements. *Interfaces Free Bound.*, **12**(4), 551–574.

Deckelnick, K., Dziuk, G., and Elliott, C. M. (2005). Computation of geometric partial differential equations and mean curvature flow. *Acta Numer.*, **14**, 139–232.

Dziuk, G. (2008). Computational parametric Willmore flow. *Numer. Math.*, **111**(1), 55–80.

Dziuk, G. and Elliott, C. M. (2013). Finite element methods for surface PDEs. *Acta Numer.*, **22**, 289–396.

Elliott, C. M. and Stinner, B. (2010). Modeling and computation of two phase geometric biomembranes using surface finite elements. *J. Comput. Phys.*, **229**(18), 6585–6612.

Franke, T., Hoppe, R. H. W., Linsenmann, C., Schmid, L., Willbold, C., and Wixforth, A. (2011). Numerical simulation of the motion of red blood cells and vesicles in microfluidic flows. *Comput. Vis. Sci.*, **14**(4), 167–180.

Franken, M., Rumpf, M., and Wirth, B. (2013). A phase field based PDE constrained optimization approach to time discrete Willmore flow. *Int. J. Numer. Anal. Model.*, **10**(1), 116–138.

Hu, W.-F., Kim, Y., and Lai, M.-C. (2014). An immersed boundary method for simulating the dynamics of three-dimensional axisymmetric vesicles in Navier–Stokes flows. *J. Comput. Phys.*, **257**, 670–686.

Jamet, D. and Misbah, C. (2007). Towards a thermodynamically consistent picture of the phase-field model of vesicles: Local membrane incompressibility. *Phys. Rev. E*, **76**, 051907.

Köhne, M. and Lengeler, D. (2014). Analysis of a single layer membrane model. (in preparation).

Krüger, T., Gross, M., Raabe, D., and Varnik, F. (2013). Crossover from tumbling to tank-treading-like motion in dense simulated suspensions of red blood cells. *Soft Matter*, **9**(37), 9008–9015.

Laadhari, A., Saramito, P., and Misbah, C. (2014). Computing the dynamics of biomembranes by combining conservative level set and adaptive finite element methods. *J. Comput. Phys.*, **263**, 328–352.

Lengeler, D. (2014). On a Stokes-type system arising in fluid vesicle dynamics. (in preparation).

Mayer, U. F. and Simonett, G. (2002). A numerical scheme for axisymmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow. *Interfaces Free Bound.*, **4**(1), 89–109.

McWhirter, J. L., Noguchi, H., and Gompper, G. (2009). Flow-induced clustering and alignment of vesicles and red blood cells in microcapillaries. *Proc. Natl. Acad. Sci. USA*, **106**(15), 6039–6043.

Mercker, M., Marciniak-Czochra, A., Richter, T., and Hartmann, D. (2013). Modeling and computing of deformation dynamics of inhomogeneous biological surfaces. *SIAM J. Appl. Math.*, **73**(5), 1768–1792.

Mikula, K., Remešíková, M., Sarkoci, P., and Ševčovič, D. (2014). Manifold evolution with tangential redistribution of points. *SIAM J. Sci. Comput.*, **36**(4), A1384–A1414.

Rahimi, M. and Arroyo, M. (2012). Shape dynamics, lipid hydrodynamics, and the complex viscoelasticity of bilayer membranes. *Phys. Rev. E*, **86**, 011932.

Rodrigues, D. S., Ausas, R. F., Mut, F., and Buscaglia, G. C. (2014). A semi-implicit finite element method for viscous lipid membranes. `http://arxiv.org/abs/1409.3581`.

Salac, D. and Miksis, M. (2011). A level set projection model of lipid vesicles in general flows. *J. Comput. Phys.*, **230**(22), 8192–8215.

Salac, D. and Miksis, M. J. (2012). Reynolds number effects on lipid vesicles. *J. Fluid Mech.*, **711**, 122–146.

Schmidt, A. and Siebert, K. G. (2005). *Design of Adaptive Finite Element Software: The Finite Element Toolbox ALBERTA*, volume 42 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin.

Scriven, L. E. (1960). Dynamics of a fluid interface: Equation of motion for Newtonian surface fluids. *Chem. Eng. Sci.*, **12**(2), 98–108.

Seifert, U. (1997). Configurations of fluid membranes and vesicles. *Adv. Phys.*, **46**, 13–137.

Shi, L., Pan, T.-W., and Glowinski, R. (2014). Three-dimensional numerical simulation of red blood cell motion in Poiseuille flows. *Internat. J. Numer. Methods Fluids*, **76**(7), 397–415.

Wintz, W., Döbereiner, H.-G., and Seifert, U. (1996). Starfish vesicles. *Europhys. Lett.*, **33**, 403–408.