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Gaussian Approximations for Intra- and Intersystem Interference in RNSS

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***Abstract*—Spread spectrum multiple access (SSMA) is widely used by radionavigation satellite services (RNSS) to separate an increasing number of systems, satellites and signals within a common frequency band, but multiple access interference (MAI) needs to be carefully modeled and monitored for spectrum coordination or performance analysis of safety-critical services. Within the class of Gaussian approximations for MAI, we derive the very accurate conditional Gaussian approximation (CGA), being the first to consider a delay/Doppler channel, non-uniform symbol rates and arbitrary improper signaling.**

***Index Terms*—Global navigation satellite system (GNSS), spectral separation coefficient (SSC), intersystem interference.**

I. Introduction

RADIONAVIGATION satellite services (RNSS) provided by Global Positioning System (GPS) and other global or regional constellations rely on spread spectrum multiple access (SSMA) to separate multiple signals transmitted in a shared frequency band. However, as the signals are received asynchronously, multiple access interference (MAI) cannot be avoided at any point in time or space. Conservative RNSS performance analysis in the presence of MAI is of interest for spectrum coordination [1], [2], signal design, and for integrity assessment of safety-critical services such as space-based/ground-based augmentation systems (SBAS/GBAS).

Performance of SSMA has been studied extensively for time-limited [2]–[5] or band-limited [5]–[9] spreading waveforms. It is generally agreed that MAI at the output of a matched filter (MF) is accurately modeled by the conditional Gaussian approximation (CGA) [4], [6], when conditioned on the channel parameters. The considerably simpler standard Gaussian approximation (SGA) [3], [5] is accurate only in special cases. In particular, SGA and CGA are identical for band-limited IS-95-type systems [7], which employ aperiodic and second-order circular (*proper* [10]) spreading. With few exceptions (encrypted services), RNSS use periodically correlated spreading and binary (hence *improper*) symbol alphabets, so that the SGA must be considered mismatched here.

We revisit the CGA and extend it from the case of a single terrestrial service to multiple satellite services. The technical novelties compared with literature [2]–[9] are as follows:

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1) In addition to the variance, we derive the conjugate variance of MAI, which is relevant for improper alphabets;
2) A satellite-to-earth channel with considerable relative Doppler shifts is considered when deriving the CGA;
3) Arbitrary symbol rates among different services can be taken into account by including a Dirichlet kernel in the calculation of conditional second-order moments.

Numerical results show that the established spectral separation coefficient (SSC) [1], which is based on SGA, leads to overoptimistic results for modernized and legacy RNSS signals.

II. System Model

We consider K satellite signals received over an additive white Gaussian noise (AWGN) channel. Signals are processed independently, producing a delay estimate and a phase estimate per signal. These estimates, which are the crucial raw observables for satellite navigation, are obtained by widely linear combinations of an early, late and prompt MF output.

A. Received Signal

The receiver's pre-correlation baseband signal is the sum of K wide-sense cyclostationary (WSCS) signals and noise

$$y(t) = \sum_{k=1}^{K} \sqrt{C_k} \left(\sum_{n=-\infty}^{\infty} x_k^{(n)}(t; \boldsymbol{\theta}_k) \right) + \eta(t). \tag{1}$$

Each satellite signal is characterized by the respective received power C_k and synchronization parameter $\boldsymbol{\theta}_k = [\tau_k, \nu_k, \phi_k]^T$, including delay τ_k, Doppler shift ν_k and phase ϕ_k. Cyclostationarity is induced by transmission of random symbols $\{b_k^{(n)}\}$ with a rate of $1/T_k$. The contribution of the nth symbol is

$$x_k^{(n)}(t; \boldsymbol{\theta}_k) = b_k^{(n)} e^{\mathrm{j}2\pi\nu_k t} e^{\mathrm{j}\phi_k} q_k(t - \tau_k - nT_k). \tag{2}$$

The spreading waveforms $q_k(t)$ are normalized to the symbol duration as $\int_{-\infty}^{\infty} |q_k(t)|^2 \,\mathrm{d}t = T_k$, and they are band-limited to a common receiver bandwidth B in the sense that the Fourier transforms $Q_k(f) \triangleq \int_{-\infty}^{\infty} q_k(t) e^{-\mathrm{j}2\pi f t} \,\mathrm{d}t$ have compact support $[-B, B]$ for $k = 1, \ldots, K$. While $q_k(t)$ is deterministic, each sequence $\{b_k^{(n)}\}$ consists of independent and identically distributed symbols with $\mathrm{E}[b_k^{(n)}] \equiv \mu_k = 0$, $\mathrm{E}[|b_k^{(n)}|^2] \equiv \Sigma_{kk^*} = 1$ and $\mathrm{E}[(b_k^{(n)})^2] \equiv \Sigma_{kk} \in \mathbb{C}$, $|\Sigma_{kk}| \leq 1$. The sequences are independent for $k = 1, \ldots, K$. We assume $1/T_1 \ll B \ll f_c$, where f_c is the signals' common carrier frequency.

Finally, $\eta(t)$ is complex baseband AWGN, which means that its real and imaginary part are independent AWGN processes each with two-sided power spectral density (PSD) $N_0/2$ [10].

B. Matched filter (MF) and coherent estimators

Without loss of generality, we consider $k = 1$ as the signal of interest. We focus on coherent estimators, which require knowledge of N symbols $b_1^{(1)}, \ldots, b_1^{(N)}$ and a reasonably accurate synchronization estimate $\hat{\boldsymbol{\theta}}_1 = [\hat{\tau}_1, \hat{\nu}_1, \hat{\phi}_1]^T$. If symbols and $\hat{\phi}_1$ are unavailable, noncoherent estimators with squaring loss can be used, for which our performance analysis may serve as a benchmark. The prompt MF output is defined as

$$\mathcal{P}_1 = \frac{1}{\sqrt{NT_1}} \sum_{n=1}^{N} \int_{-\infty}^{\infty} \left(x_1^{(n)}(t; \hat{\boldsymbol{\theta}}_1) \right)^* y(t)\, \mathrm{d}t. \tag{3}$$

The late/early outputs $\mathcal{L}_1$ and $\mathcal{E}_1$ are defined analogously, with a delay/advance of $\hat{\tau}_1$ in (3) by the correlator spacing $\epsilon > 0$.

Let the relative synchronization parameters be $\Delta\boldsymbol{\theta}_k \triangleq \boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_1$ for $k = 1, \ldots, K$. To refine the initial estimate $\hat{\boldsymbol{\theta}}_1$, the receiver produces high-resolution estimates of the unknown residuals $\Delta\phi_1$ and $\Delta\tau_1$. The coherent discriminator functions $\mathrm{Im}\{\mathcal{P}_1\}$ and $\mathrm{Re}\{\mathcal{E}_1 - \mathcal{L}_1\}$ are approximately linear in $\Delta\phi_1, \Delta\tau_1$, respectively, if $\hat{\boldsymbol{\theta}}_1 \approx \boldsymbol{\theta}_1$, and lead to the following well-known linear estimators realized by coherent tracking loops [11],

$$\Delta\hat{\phi}_1 = \frac{\mathrm{Im}\{\mathcal{P}_1\}}{\sqrt{NT_1 C_1}} \tag{4}$$

$$\Delta\hat{\tau}_1 = \frac{\mathrm{Re}\{\mathcal{E}_1 - \mathcal{L}_1\}}{S_1(\epsilon)\sqrt{NT_1 C_1}}. \tag{5}$$

Here, we used $S_1(\epsilon) = -\frac{1}{T_1}\int_{-B}^{B} |Q_1(f)|^2 4\pi f \sin(2\pi f \epsilon)\, \mathrm{d}f$. It is easily verified that (4) and (5) are unbiased for $\hat{\boldsymbol{\theta}}_1 \approx \boldsymbol{\theta}_1$.

III. Conditional Gaussian approximation (CGA)

For the CGA, we condition on $C \triangleq (C_1, \ldots, C_K)$ and $\Theta \triangleq (\hat{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_K)$, considering symbols and noise as random.

A. Variance of the carrier-phase estimator $\Delta\hat{\phi}_1$

Proposition 1. The conditional variance of the carrier-phase estimator (4) caused by MAI and AWGN is given by

$$\mathrm{Var}\big[\Delta\hat{\phi}_1\big] = \frac{N_0 + \sum_{k=2}^{K} C_k \left(\psi_k - \mathrm{Re}\{\tilde{\psi}_k\}\right)}{2NT_1 C_1}, \tag{6}$$

where the (conjugate) variances due to MAI are

$$\begin{aligned}\psi_k = {} & \frac{1}{T_1 T_k} \sum_{m=-M_k}^{M_k} D_N^*\left(2\pi m \frac{T_1}{T_k}\right) \\ & \times e^{\mathrm{j}2\pi m \frac{\Delta\tau_k}{T_k}} \int_{-B}^{B} Q_1(f + \Delta\nu_k) Q_k^*(f) \\ & \times Q_1^*\left(f + \Delta\nu_k - \frac{m}{T_k}\right) Q_k\left(f - \frac{m}{T_k}\right) \mathrm{d}f,\end{aligned} \tag{7}$$

$$\begin{aligned}\tilde{\psi}_k = {} & \frac{\Sigma_{11}^* \Sigma_{kk}}{T_1 T_k} e^{\mathrm{j}4\pi\Delta\nu_k \hat{\tau}_1} e^{\mathrm{j}2\Delta\phi_k} \sum_{m=-M_k}^{M_k} D_N\left(2\pi m \frac{T_1}{T_k} + 4\pi\Delta\nu_k T_1\right) \\ & \times e^{-\mathrm{j}2\pi m \frac{\Delta\tau_k}{T_k}} \int_{-B}^{B} Q_1^*(f + \Delta\nu_k) Q_k(f) \\ & \times Q_1^*\left(\frac{m}{T_k} - f + \Delta\nu_k\right) Q_k\left(\frac{m}{T_k} - f\right) \mathrm{d}f.\end{aligned} \tag{8}$$

Here, we used $M_k = \lfloor 2BT_k \rfloor$ and the Dirichlet kernel

$$D_N(x) \triangleq \frac{1}{N}\sum_{n=1}^{N} e^{\mathrm{j}nx} = \begin{cases} 1 & \text{if } \frac{x}{2\pi} \in \mathbb{Z} \\ \frac{\sin(Nx/2)}{N\sin(x/2)} e^{\mathrm{j}x(N+1)/2} & \text{otherwise.} \end{cases} \tag{9}$$

Proof. We first recall from [10] that, for any complex random variable Y, $\mathrm{Var}[\mathrm{Im}\{Y\}] = \mathrm{Var}[Y]/2 - \mathrm{Re}\{\mathrm{Cov}[Y, Y^*]\}/2$ and apply this to the right-hand side of (4). Then we expand $\mathcal{P}_1$ into $K+1$ uncorrelated summands using (1), which represent the contributions of MAI, noise, and intersymbol interference (ISI). Variance and conjugate variance of MAI for $k \neq 1$ are given by $C_k\psi_k \triangleq \mathrm{Var}[\mathcal{P}_1]|_k$ and $C_k\tilde{\psi}_k \triangleq \mathrm{Cov}[\mathcal{P}_1, \mathcal{P}_1^*]|_k$, which are derived in the Appendix. The contribution of AWGN is well-known [5], [11, Sec. 5.2.9]. The ISI contributions $(C_1\psi_1, C_1\tilde{\psi}_1)$ are nonzero but negligible since $1/T_1 \ll B$ [5].

B. Variance of the time-delay estimator $\Delta\hat{\tau}_1$

Proposition 2. The conditional variance of the time-delay estimator (5) caused by MAI and AWGN is given by

$$\begin{aligned}\mathrm{Var}\big[\Delta\hat{\tau}_1\big] = {} & \Bigg(N_0\left(1 - \frac{1}{T_1}\int_{-B}^{B} |Q_1(f)|^2 \cos(4\pi f \epsilon)\, \mathrm{d}f\right) \\ & + \sum_{k=2}^{K} C_k \bigg(\frac{\varepsilon_k}{2} + \frac{\lambda_k}{2} - \mathrm{Re}\left\{\chi_k\right\} \\ & + \mathrm{Re}\left\{\frac{\tilde{\varepsilon}_k}{2} + \frac{\tilde{\lambda}_k}{2} - \tilde{\chi}_k\right\}\bigg)\Bigg) \Big/ \left(S_1^2(\epsilon) N T_1 C_1\right).\end{aligned} \tag{10}$$

Proof. For any pair (U, V) of complex random variables, $\mathrm{Var}[\mathrm{Re}\{U - V\}] = \mathrm{Var}[U]/2 + \mathrm{Var}[V]/2 - \mathrm{Re}\{\mathrm{Cov}[U, V]\} + \mathrm{Re}\{\mathrm{Cov}[U, U^*]/2 + \mathrm{Cov}[V, V^*]/2 - \mathrm{Cov}[U, V^*]\}$ is easy to show with [10]. We apply this to (5) and proceed as in the proof of Proposition 1; the MAI contributions $C_k\varepsilon_k \triangleq \mathrm{Var}[\mathcal{E}_1]|_k$, $C_k\tilde{\varepsilon}_k \triangleq \mathrm{Cov}[\mathcal{E}_1, \mathcal{E}_1^*]|_k$, $C_k\lambda_k \triangleq \mathrm{Var}[\mathcal{L}_1]|_k$, $C_k\tilde{\lambda}_k \triangleq \mathrm{Cov}[\mathcal{L}_1, \mathcal{L}_1^*]|_k$, $C_k\chi_k \triangleq \mathrm{Cov}[\mathcal{E}_1, \mathcal{L}_1]|_k$ and $C_k\tilde{\chi}_k \triangleq \mathrm{Cov}[\mathcal{E}_1, \mathcal{L}_1^*]|_k$ are derived in the Appendix for $k \neq 1$. The contribution of AWGN is well-known (cf., [11, Sec. 7.2.1]).

C. Signal-to-interference-plus-noise ratio (SINR)

The SINR of the prompt MF output is defined as $\mathrm{SINR} \triangleq |\mathrm{E}[\mathcal{P}_1]|^2/\mathrm{Var}[\mathcal{P}_1]$, where the denominator contains AWGN and MAI. It is often used as a single figure of merit, although it does not provide a full second-order characterization of $\mathcal{E}_1$, $\mathcal{L}_1$ and $\mathcal{P}_1$. Plain signal-to-noise ratio is $\mathrm{SNR} \triangleq |\mathrm{E}[\mathcal{P}_1]|^2/N_0$.

A useful quantity is the loss $\mathrm{SNR}/\mathrm{SINR} = 1 + \Psi$ with

$$\Psi = \sum_{k=2}^{K} \frac{C_k}{N_0}\psi_k \qquad \geq 0. \tag{11}$$

IV. Standard Gaussian approximation (SGA)

The SGA relies on the assumption of uniformly distributed relative time-delays and carrier-phases [7], [8]. To obtain the SGA for any of the performance measures (6), (10) or (11), all conditional moments are simply replaced by their expectations with respect to uniform $\Delta\tau_k \in [0, T_k)$ and $\Delta\phi_k \in [-\pi, \pi)$. For instance, $\mathrm{E}[\Psi]$ reduces to a weighted sum of the SSCs [1]

$$\mathrm{E}\left[\psi_k\right] = \int_{-B}^{B} \frac{\left|Q_1(f + \Delta\nu_k)\right|^2}{T_1} \frac{\left|Q_k(f)\right|^2}{T_k}\, \mathrm{d}f. \tag{12}$$

Note that CGA and SGA are identical for interference from an IS-95-type system, for which $BT_k \leq 0.5$ and $\Sigma_{kk} = 0$.

Table I
OVERVIEW OF RNSS SIGNALS L1 C/A [12] AND E1-B [13]

Signal	Σ_{kk}	$H_k(f)$	$A_k(f)$	T_k (ms)	C_k (dBW) [1]
E1-B	+1	$\text{CBOC}^{(1/11)}_{(6,1,+)}$	Memory	4	[−160.0, −157.0]
L1 C/A	−1	$\text{BPSK}_{(1)}$	Gold	20 × 1	[−158.5, −153.0]

(a)

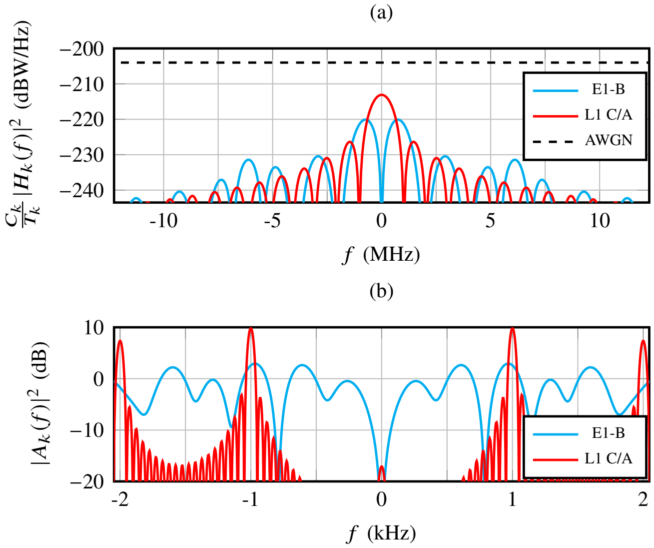

Fig. 1. (a) Pulse-only PSD for reference receiver bandwidth $B = 12.28$ MHz and maximum power C_k. (b) Code spectrum (exemplary). Addition of the respective components shown in (a) and (b) yields the PSD $\frac{C_k}{T_k}|Q_k(f)|^2$.

V. Numerical Results

CGA and SGA are still conditioned on some or all of the parameters (C, Θ). For simplicity, we assume $\hat{\boldsymbol{\theta}}_1 = \boldsymbol{\theta}_1$ for Section V. The remaining dependency on constellation parameters can be removed numerically as follows:

- over a range of possible (C, Θ), determine the maximum (worst-case) of $\text{Var}[\Delta\hat{\phi}_1]$, $\text{Var}[\Delta\hat{\tau}_1]$ or Ψ,
- or invoke the joint distribution of (C, Θ) to obtain the distribution of $\text{Var}[\Delta\hat{\phi}_1]$, $\text{Var}[\Delta\hat{\tau}_1]$ or Ψ.

A. RNSS Signals

We consider GPS L1 C/A [12] and Galileo E1-B [13] at $f_c = 1575.42$ MHz as example for a legacy/modernized signal, respectively. Like almost all RNSS signals, both use a binary alphabet and direct-sequence SSMA, hence $|\Sigma_{kk}| = 1$ and $Q_k(f) = A_k(f)H_k(f)$, with a pulse $H_k(f)$ and a code $A_k(f)$. Details are given in Fig. 1 and Table I. Quite different from earlier works on frequency-domain CGA [5]–[8], we do not assume perfectly random code for $A_k(f)$, but consider the finite-length pseudorandom sequences from [12], [13].

We assume $N_0 = -204.0\,\frac{\text{dBW}}{\text{Hz}}$ for a low-noise receiver [1].

B. Constellation analysis

We consider 31 GPS satellites and 24 Galileo satellites to determine the cumulative density function (CDF) $F_\Psi(\Psi)$, using reference constellations [12], [14]. With L1 C/A and E1-B, there are $K = 55$ potentially active signals. For the power profile C, we assume the maximum C_k from Table I whenever the corresponding satellite appears with at least 5° elevation, and zero otherwise. Whenever $k = 1$ is active, we compute CGA and SGA for each constellation point (C, Θ) in time with a resolution of 6.5 Hz, and approximate $F_\Psi(\Psi)$ by their cumulative histograms. We consider ten sidereal days, as this is the least common multiple of the GPS and Galileo constellation periods. The receiver is located at 52° northern lattitude (Central Europe). The results in Fig. 2 reveal that the SGA tends to underestimate the tails of $F_\Psi(\Psi)$ by 4-8 dB. Moreover, L1 C/A SINR is dominated by L1 C/A (intrasystem) MAI rather than by AWGN 30% of the time. Meanwhile, E1-B SINR is barely affected by E1-B MAI, which is due to better spreading waveforms, fewer satellites but also lower power: if maximum E1-B received power is increased by 6 dB, for instance, intrasystem MAI will exceed AWGN 5% of the time.

[1]The given interval applies only to the reference receiver with the full bandwidth $B = 10.23$ MHz [12] or $B = 12.28$ MHz [13], respectively. For smaller B, parts of the PSD in Fig. 1 are unused, and C_k reduces accordingly.

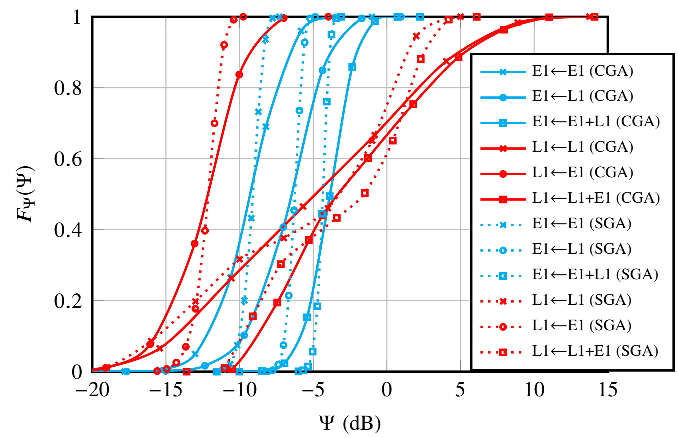

Fig. 2. CDF $F_\Psi(\Psi)$ of intrasystem (X←X), intersystem (X←Y), or combined (X←X+Y) MAI for $K = 55$ and $B = 2.046$ MHz.

C. Worst-case analysis

A worst-case analysis is particularly useful to study MAI for a single interferer ($K = 2$) as a function of the receiver configuration (N, B, ϵ), interferer-to-signal ratio C_2/C_1, or signal-to-noise-density ratio C_1/N_0, without the need to simulate full constellations. For the powers C, we use the minimum C_k from Table I for $k = 1$ and the maximum C_k for $k = 2$. The range of possible synchronization parameters Θ can simply be described by $0 \le \tau_k \le T_k$, $|\phi_k| \le \pi$ and $|\nu_k| \le 4$ kHz, $k = 1, 2$. Results in Figs. 3 and 4 show that MAI may exceed the impact of thermal noise even for $K = 2$. While longer coherent integration times NT_1 can only improve the overall performance, small correlator spacings ϵ can effectively suppress MAI for time-delay estimation.

VI. Conclusion

Other than IS-95-type systems, most RNSS use a recurring spreading waveform for transmission of each binary symbol, so that the signal statistics are neither proper nor aperiodic. Therefore, the SGA and currently used spectrum coordination standards [1] lead to an inaccurate assessment of MAI for

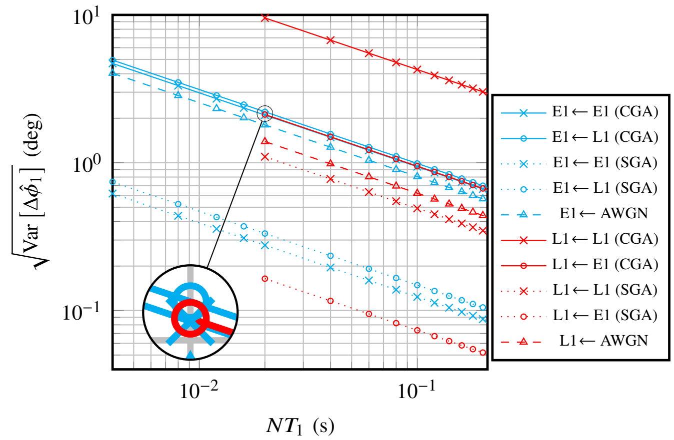

Fig. 3. Carrier-phase standard deviation vs. coherent integration time for $B = 2.046$ MHz. Contributions of AWGN and worst-case intrasystem (X←X) or intersystem (X←Y) MAI for $K = 2$.

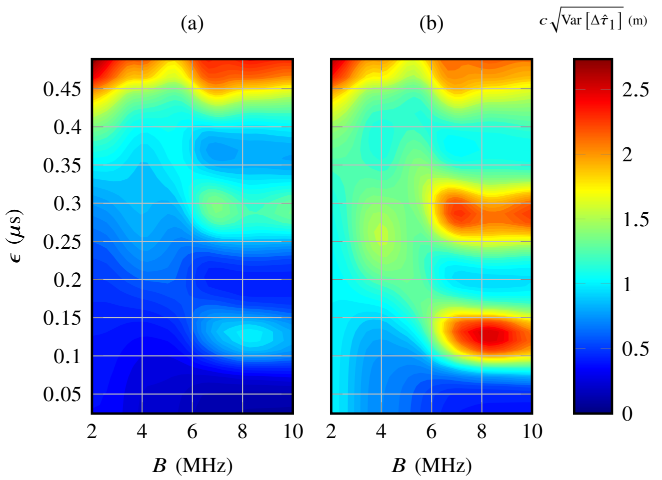

Fig. 4. Time-delay standard deviation vs. bandwidth and correlator spacing with E1-B for $NT_1 = 80$ ms. (a) CGA for worst-case intrasystem MAI ($K = 2, C_2/C_1 = 3$ dB). (b) Contribution of AWGN ($C_1/N_0 \approx 44$ dB-Hz). Speed of light is given by $c = 2.998 \times 10^8$ m/s.

legacy and even modernized RNSS signals. We derived accurate, receiver-specific expressions for satellite navigation performance in the presence of improper and WSCS interference based on the CGA. As RNSS are beginning to transform from being noise-limited to interference-limited in some frequency bands, the proposed analytical and numerical methodology will be important for receiver and operator parties.

Appendix

Conditional second-order moments of MAI

For a generic $\alpha \in \{-\epsilon, 0, \epsilon\}$ and $\hat{\boldsymbol{\theta}}_1(\alpha) \triangleq [\hat{\tau}_1 + \alpha, \hat{\nu}_1, \hat{\phi}_1]^T$, we consider the random contribution from the mth symbol of the kth signal to the nth MF output for unit power $C_k = 1$

$$\begin{aligned} X_{1,k}^{(n,m)}(\alpha) &\triangleq \frac{1}{\sqrt{T_1}} \int_{-\infty}^{\infty} \left(x_1^{(n)}(t; \hat{\boldsymbol{\theta}}_1(\alpha))\right)^* x_k^{(m)}(t; \boldsymbol{\theta}_k)\,\mathrm{d}t \\ &= \frac{\left(b_1^{(n)}\right)^* b_k^{(m)} e^{\mathrm{j}\Delta\phi_k}}{\sqrt{T_1}} \int_{-\infty}^{\infty} Q_1^*(f - \hat{\nu}_1) Q_k(f - \nu_k) \\ &\quad \times e^{\mathrm{j}2\pi\left((nT_1+\hat{\tau}_1+\alpha)(f-\hat{\nu}_1)-(mT_k+\tau_k)(f-\nu_k)\right)}\,\mathrm{d}f, \quad (13) \end{aligned}$$

where the equation follows with Plancherel's theorem [15]. With Poisson's summation formula [15], we have for $k \neq 1$

$$\begin{aligned} &\mathrm{E}\left[\left(\sum_{m_1=-\infty}^{\infty} X_{1,k}^{(n,m_1)}(\alpha_1)\right)^{(*)} \sum_{m_2=-\infty}^{\infty} X_{1,k}^{(n,m_2)}(\alpha_2)\right] \\ &= \sum_{m=-\infty}^{\infty} e^{\mathrm{j}\left(\Delta\phi_k \overset{+}{(-)} \Delta\phi_k\right)} e^{\mathrm{j}2\pi(nT_1+\hat{\tau}_1)\left(\Delta\nu_k \overset{+}{(-)} \Delta\nu_k \overset{+}{(-)} \frac{m}{T_k}\right)} e^{(\pm)\mathrm{j}2\pi\frac{m}{T_k}\tau_k} \\ &\quad \times \int_{-\infty}^{\infty} e^{\mathrm{j}2\pi\left(\alpha_2\left((\pm)f_1 + \Delta\nu_k \overset{+}{(-)} \nu_k \overset{+}{(-)} \frac{m}{T_k}\right) \overset{+}{(-)} \alpha_1(f_1-\hat{\nu}_1)\right)} \\ &\quad\quad \times \left(Q_1^*(f_1 - \hat{\nu}_1) Q_k(f_1 - \nu_k)\right)^{(*)} Q_k\left((\pm)\, f_1 \overset{+}{(-)} \nu_k \overset{+}{(-)} \frac{m}{T_k}\right) \\ &\quad\quad \times Q_1^*\left((\pm)\, f_1 + \Delta\nu_k \overset{+}{(-)} \nu_k \overset{+}{(-)} \frac{m}{T_k}\right) \mathrm{d}f_1\, \frac{\Sigma_{11^{(*)}}^* \Sigma_{kk^{(*)}}}{T_1 T_k}. \quad (14) \end{aligned}$$

We summate this result over $n = 1, \ldots, N$ as any two random variables of the form (13) are uncorrelated for unequal n. Finally, dividing by N and substituting $f \leftarrow f_1 - \nu_k$ yields

- the prompt (conjugate) variance $\psi_k, \tilde{\psi}_k$ for $\alpha_1 = \alpha_2 = 0$,
- the early (conjugate) variance $\varepsilon_k, \tilde{\varepsilon}_k$ for $\alpha_1 = \alpha_2 = -\epsilon$,
- the late (conjugate) variance $\lambda_k, \tilde{\lambda}_k$ for $\alpha_1 = \alpha_2 = \epsilon$,
- the early/late (conjugate) covariance $\chi_k, \tilde{\chi}_k$ for $\alpha_1 = -\alpha_2 = \epsilon$.

Conjugate variances/covariances are obtained if the operators in brackets $^{(*)}$, $\overset{+}{(-)}$, $(\pm)$ are ignored, and variances/covariances otherwise.

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