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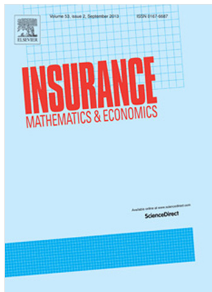

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Option pricing under regime-switching models: Novel approaches removing path-dependence*

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Abstract

A well-known approach for the pricing of options under regime-switching models is to use the regime-switching Esscher transform (also called regime-switching mean-correcting martingale measure) to obtain risk-neutrality. One way to handle regime unobservability consists in using regime probabilities that are filtered under this risk-neutral measure to compute risk-neutral expected payoffs. The current paper shows that this natural approach creates path-dependence issues within option price dynamics. Indeed, since the underlying asset price can be embedded in a Markov process under the physical measure even when regimes are unobservable, such path-dependence behavior of vanilla option prices is puzzling and may entail non-trivial theoretical features (e.g., time non-separable preferences) in a way that is difficult to characterize. This work develops novel and intuitive risk-neutral measures that can incorporate regime risk-aversion in a simple fashion and which do not lead to such path-dependence side effects. Numerical schemes either based on dynamic programming or Monte-Carlo simulations to compute option prices under the novel risk-neutral dynamics are presented.

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1 Introduction

Since their introduction in the economics literature by Hamilton (1989), regime-switching models have received extensive attention in the context of derivatives pricing. This can be explained by the ability of regime-switching models to reproduce stylized facts of financial log-returns such as fat tails, volatility clusters and momentum, see for instance Ang and Timmermann (2012). In particular, regime-switching models are used to price and hedge long-dated options; such models are sensible choices in such circumstances since the underlying asset of a long-dated option might go through multiple business cycles or varying financial conditions throughout the life of the option. Moreover, regime-switching dynamics allow recovering volatility smiles exhibited by empirical option prices, see Ishijima and Kihara (2005) and Yao et al (2006).

The pricing of long-dated equity options is extremely relevant in insurance; numerous long-term insurance contracts such as variable annuities and equity-linked insurance include embedded implicit options guaranteeing a minimum amount of benefits to be paid contingent on either the survival or the mortality of the policyholder. Consider for example a variable annuity including a Guaranteed Minimum Maturity Benefit (GMMB) rider. Under such a policy, the policyholder deposits an initial amount of savings into the policy account, which is then typically invested in a mutual fund. The insurer periodically collects fees from the policy account. In return, it guarantees that the account value will be worth at least a minimum guaranteed value at the maturity of the policy provided that the policyholder is alive at that date; in other words, the insurer promises to make up for any possible shortfall between the terminal account value at maturity and the guaranteed amount. For the insurer, such a promise consists in a short position on a put option over the policy account value with the strike being the guaranteed amount.

Implicit options embedded in long-term insurance contracts are illiquid, which entails that quantitative models are needed to value and hedge such options. Hardy (2003) pioneered the use of regime-switching models to value long-term options embedded in variable annuities. A non-exhaustive list of other papers which use this family of models to either price or hedge equity options attached to equity-linked insurance contracts or variable annuities is hereby provided:

Lin et al (2009), Jin et al (2011), Ng and Li (2011), Ngai and Sherris (2011), Wang and Yin (2012) Azimzadeh et al (2014), Fan et al (2015), Siu et al (2015), Ignatieva et al (2016), Wang et al (2017), Trottier et al (2018a), Trottier et al (2018b) and Ignatieva et al (2018).

The usual route to obtain a risk-neutral measure in the context of regime-switching models is to first assume the observability of regimes and then use the extended Girsanov principle (coinciding in this case with the regime-switching Esscher transform or the mean-correcting transform) which preserves the model specification and shifts the drift to the risk-free rate in all regimes. See for instance Elliott and Madan (1998) for the extended Girsanov principle, Bühlmann et al (1996), Gerber and Shiu (1996) and Goovaerts and Laeven (2008) for the Esscher transform, and Hardy (2001) and Buffington and Elliott (2002b) for their application to the regime switching context. Elliott et al (2005) provide a justification for using the latter transform by showing that it leads to the minimal entropy martingale measure. To handle regime latency, the typical approach found for instance in Liew and Siu (2010) is to compute the filtered risk-neutral distribution of the hidden regimes to obtain weights for derivatives prices associated with each regime which lead to a price in the context of regime unobservability. Note that failing to recognize that latent variables are unobserved can lead to systematic bias in option prices, see Bégin and Gauthier (2017).

The current paper shows that combining the usual Girsanov transform with the risk-neutral filter in the context of regime-switching models provides derivatives price dynamics exhibiting path-dependence even though the underlying asset price can be embedded in a Markov process under the physical measure.

The non-Markovian option price dynamics obtained through the usual pricing approach in the context of Markov-driven state variables is the main motivation for the current study. A legitimate perspective consists in accepting the presence of path-dependent derivatives prices even if the underlying asset price dynamics can be embedded within a Markov process, as the former are not incompatible with arbitrage pricing theory. However, we argue that the construction of martingale measures producing path-independent vanilla option prices in this context, which is the main objective of this paper, is relevant for multiple reasons.

A first motivation for the design of such risk-neutral measures is computational convenience. Indeed, modeling option prices from a dynamic perspective rather than a static one is very important since such dynamic models can be embedded into dynamic hedging performance assessment models, see for instance Trottier et al (2018a). Considering non-Markovian option prices can significantly hinder the computational tractability of such hedging schemes as it requires keeping track of additional state variables which increases the dimensionality of the underlying optimization problems. A second aspect motivating the construction of pricing measures generating path-independent derivatives prices is that the path-dependence feature is incompatible with equilibrium models using time-additive utility functions; this points towards non-trivial theoretical implications such as time non-separable preferences as in Garcia et al (2003). Although the construction of equilibrium models justifying the pricing measures from the current paper is not attempted, it is nevertheless relevant to identify pricing measures possessing properties that are not inconsistent with simple traditional equilibrium models. Finally, providing an interpretation for the relation between risk-neutral regime probabilities and observed market asset prices stemming from the regular pricing approach yielding path-dependence is not straightforward.

In the current paper, convenient alternative risk-neutral measures which remove the path-dependence feature are developed. Similarly to Christoffersen et al (2009), we do not attempt designing an equilibrium model recovering derivatives prices dynamics provided by the risk-neutral measures developed herein. Although very interesting, the latter work is scoped out from the current paper; we focus directly on the risk-neutral valuation relationship linking option prices to exogenously given regime-switching underlying asset price dynamics without characterizing the entire economic environment. A first approach considered is a modified version of the regime-switching Esscher transform that leads to the construction of a wide class of risk-neutral measures by engineering a dynamic transition matrix so as to yield option prices exhibiting the Markov property. Such risk-neutral measures are obtained by the successive alteration of transition probabilities and of the underlying asset drift. A second approach explores two different families of martingale measures whose Radon-Nikodym derivatives are measurable given the partially observable information. For the latter measures, option prices exhibit the Markov property, and furthermore the conditional distribution of the past (unobservable) regime trajectory given the asset full trajectory set is left unaltered. Under all of our introduced martingale measures, option prices can be calculated simply either through a dynamic program or a Monte-Carlo simulation.

Several other interesting papers from the regime-switching option pricing literature should be mentioned. Classical regime-switching dynamics were expanded by incorporating jumps, see Naik (1993), Elliott et al (2007) and Elliott and Siu (2013), feedback effects of asset prices on regime transition probabilities (Elliott et al, 2011), or GARCH feedback effects (Duan et al, 2002). Multiple types of derivatives were priced such as American options (Buffington and Elliott, 2002a), perpetual American options (Zhang and Guo, 2004), barrier options (Jobert and Rogers, 2006; Ranjbar and Seifi, 2015), and other exotic options such as Asian and lookback options (Boyle and Draviam, 2007). The incorporation to the market of an additional asset providing payoffs at regime switches which allows completing the market is investigated in Guo (2001) and Fuh et al (2012). The partial differential equations approach to price derivatives in regime-switching markets is presented in Mamon and Rodrigo (2005). Di Masi et al (1995) investigate mean-variance hedging in the presence of regimes. Various numerical schemes were developed to price options in the regime-switching context, such as trees (Bollen, 1998; Yuen and Yang, 2009), and the fast Fourier transform (Liu et al, 2006). Finally, alternative approaches to pricing such as equilibrium and stochastic games are considered in Garcia et al (2003) and Shen and Siu (2013).

The paper continues as follows. Section 2 introduces the regime-switching market. Section 3 illustrates the use of the mean-correcting martingale measure to price options under regime uncertainty. The non-Markov behavior of option prices under such a transform is discussed. Section 4 introduces a wide class of risk-neutral measures based on the successive alteration of transition probabilities and of the underlying asset drift. Section 5 explores two different families of martingale measures whose Radon-Nikodym derivatives are measurable given the asset trajectory. Section 6 concludes.

2 Regime-switching market

This section introduces the regime-switching market model. We adopt the shorthand notation $x_{1:n} \equiv (x_1, \ldots, x_n)$, and denote the conditional PDF of random variables X given Y by $f_{X|Y}$.

2.1 Regime-switching model

Consider a discrete time space $\mathcal{T} = \{0, \ldots, T\}$ and a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$. Define a regime process $h = \{h_t\}_{t=0}^{T-1}$ and an innovation process $z^{\mathbb{P}} = \{z_t^{\mathbb{P}}\}_{t=1}^{T}$ which are independent under $\mathbb{P}$. The process $z^{\mathbb{P}}$ is a strong standardized Gaussian white noise, i.e., $z^{\mathbb{P}}$ is a sequence of i.i.d. normal random variables with mean zero and unit variance. The process h is a hidden Markov chain. Possible values for regimes (the states of the Markov chain) are $h_t(\omega) \in \{1, \ldots, H\}$ for all $\omega \in \Omega$, where H is a positive integer. A risk-free asset is introduced and its price is given by $B_t = e^{rt}$ with r being the constant risk-free rate. A risky asset price process is defined by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t} \epsilon_j\right), \qquad t \in \mathcal{T}, \tag{2.1}$$

where the asset log-returns are given by

$$\epsilon_{t+1} = \mu_{h_t} + \sigma_{h_t} z_{t+1}^{\mathbb{P}}, \qquad t \in \{0, \ldots, T-1\}, \tag{2.2}$$

for some constants μ_i and σ_i, $i \in \{1, \ldots, H\}$. Although a Gaussian distribution is used for log-returns in each regime for simplicity, all the concepts from this paper could easily be generalized to non-Gaussian distributions without additional technical difficulty. Non-Gaussian innovations were considered in a regime-switching option pricing context for instance by Siu et al (2011).

The filtrations $\mathcal{G} = \{\mathcal{G}_t\}_{t=0}^{T}$, $\mathcal{H} = \{\mathcal{H}_t\}_{t=0}^{T}$ and $\mathcal{F} = \{\mathcal{F}_t\}_{t=0}^{T}$ are respectively defined as

$$\mathcal{G}_t = \sigma(S_0, \ldots, S_t), \qquad \mathcal{H}_t = \sigma(h_0, \ldots, h_t), \qquad \mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t. \tag{2.3}$$

$\mathcal{G}_t$ and $\mathcal{H}_t$ are sub-σ-algebras of $\mathcal{F}_t$. The filtration $\mathcal{G}$ is referred to as the partial information whereas $\mathcal{F}$ is called the full information. In practice, investors only have access to information $\mathcal{G}_t$ at time t as regimes are hidden variables. We assume that for all $j \in \{1, \dots, H\}$,

$$\mathbb{P}[h_{t+1} = j | \mathcal{G}_{t+1} \vee \mathcal{H}_t] = P_{h_t, j}, \tag{2.4}$$

where $P_{k,j}$ represents the probability of a transition $k \to j$ of the Markov chain h. This implies

$$\mathbb{P}[h_{t+1} = j | \mathcal{F}_t] = P_{h_t, j}.$$

This framework is known as a regime-switching (RS) model. The joint mixed PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under such model is (proof in Appendix A.1)

$$f^{\mathbb{P}}_{\epsilon_{1:T}, h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_0}(h_0) \prod_{t=2}^{T} P_{h_{t-2}, h_{t-1}} \prod_{t=1}^{T} \phi^{\mathbb{P}}_{h_{t-1}}(\epsilon_t), \tag{2.5}$$

where we have introduced the functions $\phi^{\mathbb{P}}_i$, $i \in \{1, \dots, H\}$, which are defined as

$$\phi^{\mathbb{P}}_i(x) = \frac{1}{\sigma_i} \varphi\left(\frac{x - \mu_i}{\sigma_i}\right), \qquad x \in \mathbb{R}, \tag{2.6}$$

with $\phi(z) \equiv \frac{e^{-z^2/2}}{\sqrt{2\pi}}$ denoting the standard normal PDF. Hence, $\phi^{\mathbb{P}}_i$ is the Gaussian density with mean μ_i and variance σ_i^2.

2.2 Regime mass function

Following François et al. (2014), we introduce $\eta^{\mathbb{P}} = \{\eta^{\mathbb{P}}_t\}_{t=0}^{T}$ where $\eta^{\mathbb{P}}_t = \left(\eta^{\mathbb{P}}_{t,1}, \dots, \eta^{\mathbb{P}}_{t,H}\right)$ is defined as the regime mass function process, or regime predictive density, with respect to the partial information:

$$\eta^{\mathbb{P}}_{t,j} \equiv \mathbb{P}[h_t = j | \mathcal{G}_t], \qquad j \in \{1, \dots, H\}. \tag{2.7}$$

The random vector $\eta^{\mathbb{P}}_t = \left(\eta^{\mathbb{P}}_{t,1}, \dots, \eta^{\mathbb{P}}_{t,H}\right)$ determines what are the probabilities at time t that the regime process is currently in each respective possible regime given the observable information.

Using Bayes' rule, the process $\eta^{\mathbb{P}}$ can be computed through the following recursion, see for instance, Elliott et al (1995) or François et al (2014):

$$\eta^{\mathbb{P}}_{t+1,i} = \frac{\sum_{j=1}^{H} P_{j,i}\,\phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}}{\sum_{\ell=1}^{H}\sum_{j=1}^{H} P_{j,\ell}\,\phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}} = \frac{\sum_{j=1}^{H} P_{j,i}\,\phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}}{\sum_{j=1}^{H}\phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}}, \qquad i \in \{1,\dots,H\}. \tag{2.8}$$

A direct consequence of this relation is the following proposition.

Proposition 2.1 (François et al 2014)**.** *The joint process* $\left\{\left(S_t, \eta^{\mathbb{P}}_t\right)\right\}_{t=0}^{T}$ *has the Markov property with respect to the filtration* $\mathcal{G}$ *under the physical measure* $\mathbb{P}$*.*

The conditional density of the stock price process under $\mathbb{P}$ is

$$f^{\mathbb{P}}_{S_{t+1}|S_{0:t}}(s|S_{0:t}) = \sum_{k=1}^{H} \eta^{\mathbb{P}}_{t,k} \frac{1}{s\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{\left[\log(s/S_t) - \mu_k\right]^2}{2\sigma_k^2}\right), \qquad s \geq 0, \tag{2.9}$$

which is a mixture of log-normal distributions with mixing weights $\eta^{\mathbb{P}}_t$.

3 The RS mean-correcting martingale measure

This section illustrates the traditional approach to option pricing based on a regime-switching version of the mean-correcting martingale measure as in Hardy (2001) and Elliott et al (2005). This procedure is shown to entail non-Markovian option price dynamics even though the underlying asset price process can be embedded in a Markov process under the physical measure.

3.1 Constructing the RS mean-correcting martingale measure

Consider a European-type contingent claim whose payoff at time T is $\Psi(S_T)$, for some Borel real function Ψ. For instance, a call option has a payoff $\Psi(S_T) = \max(S_T - K, 0)$ where $K \geq 0$ is the strike price. The problem considered in the current paper is to identify a suitable price process $\Pi = \{\Pi_t\}_{t=0}^{T}$ for the contingent claim, where Π_t represents the contingent claim price at time t. Since regimes are unobservable, only prices Π_t that are $\mathcal{G}_t$-measurable are considered as prices cannot depend on information that is unavailable to investors. In other words, option prices cannot directly be a function of regimes as this would entail regimes are observable by agents pricing the options. This approach is different from the one of Hardy (2001) where the option price depends on the currently prevailing regime. Considering option prices that are $\mathcal{G}_t$-measurable is consistent with the weak form of efficient markets as explained in Elliott and Madan (1998).

Define $\mathcal{Q}$ as the set of all probability measures $\mathbb{Q}$ that are equivalent to $\mathbb{P}$ and such that the discounted price process $\{e^{-rt}S_t\}_{t=0}^{T}$ is a $\mathcal{G}$-martingale under the measure $\mathbb{Q}$. Such probability measures are referred to as martingale measures. A well known result from option pricing theory (see, e.g., Harrison and Kreps, 1979, for a proof) is that the set of all pricing processes which do not generate arbitrage opportunities is characterized by

$$\left\{\Pi^{\mathbb{Q}} = \left\{e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Psi(S_T)|\mathcal{G}_t]\right\}_{t=0}^{T} : \mathbb{Q} \in \mathcal{Q}\right\}$$

up to some integrability conditions to ensure prices are finite. Because the market is incomplete under the regime-switching framework, an infinite number of martingale measures exist and solutions to the option pricing problem are thus not unique.

A common approach is to select a particular martingale measure under which the asset price dynamics remains in the same class of models. This approach is followed for instance by Hardy (2001) who considers a martingale measure under which the risky asset price returns are still a Gaussian regime-switching process with transition probabilities $P_{i,j}$, but where the drift in each regime μ_i is replaced by $r - \frac{1}{2}\sigma_i^2$. Such a martingale measure can be constructed using a regime-switching mean-correcting change of measure following the lines of Elliott et al (2005) who perform a similar exercise in a continuous-time framework. Note that the pricing method in Elliott et al (2005) is based on a version of the Esscher transform, which was called a regime-switching Esscher transform. Replacing μ_i by $r - \frac{1}{2}\sigma_i^2$ in (2.6), the joint mixed PDF of returns and regimes under such a risk-neutral measure $\mathbb{Q}$ is

$$f^{\mathbb{Q}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_0}(h_0)\prod_{t=2}^{T} P_{h_{t-2},h_{t-1}} \prod_{t=1}^{T} \phi^{\mathbb{Q}}_{h_{t-1}}(\epsilon_t), \tag{3.1}$$

where the functions $\phi_i^{\mathbb{Q}}$, $i \in \{1, \ldots, H\}$, are defined as

$$\phi_i^{\mathbb{Q}}(x) \equiv \frac{1}{\sigma_i}\phi\left(\frac{x - r + \frac{1}{2}\sigma_i^2}{\sigma_i}\right), \qquad x \in \mathbb{R}. \tag{3.2}$$

An assumption implicit to (3.1) is that the distribution of the initial regime h_0 is left untouched by the change of measure i.e. $f_{h_0}^{\mathbb{P}} = f_{h_0}^{\mathbb{Q}}$.

The following result (proven in the Online Appendix D.1) shows how to create a new probability measure under which the underlying asset price and regime dynamics matches the desired one.

Proposition 3.1. *Consider any joint mixed PDF for* $(\epsilon_{1:T}, h_{0:T-1})$ *denoted by* $f^{\mathbb{Z}}_{\epsilon_{1:T},h_{0:T-1}}$. *Then the measure defined by* $\mathbb{Z}[A] \equiv \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_A \frac{d\mathbb{Z}}{d\mathbb{P}}\right]$, *for all* $A \in \mathcal{F}_T$, *where*

$$\frac{d\mathbb{Z}}{d\mathbb{P}} \equiv \frac{f^{\mathbb{Z}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}, \tag{3.3}$$

is a probability measure. $\mathbb{Z}$ *is equivalent to* $\mathbb{P}$ *if and only if* $f^{\mathbb{Z}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})$ *is strictly positive almost surely. Furthermore, the joint mixed PDF of* $(\epsilon_{1:T}, h_{0:T-1})$ *under* $\mathbb{Z}$ *is* $f^{\mathbb{Z}}_{\epsilon_{1:T},h_{0:T-1}}$.

Note that Proposition 3.1 is analogous to Theorem 1.1 from Elliott and Madan (1998), but with regimes that are introduced in the market. By Proposition 3.1, we thus consider the measure $\mathbb{Q}$ generated by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{f^{\mathbb{Q}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}, \tag{3.4}$$

where $f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}$ and $f^{\mathbb{Q}}_{\epsilon_{1:T},h_{0:T-1}}$ are defined as before; see (2.5) and (3.1). Simplifying yields (see Online Appendix D.2)

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=1}^{T} \xi_t, \qquad \xi_t = e^{z_t^{\mathbb{P}}\lambda_t - \frac{1}{2}\lambda_t^2}, \tag{3.5}$$

where

$$\lambda_t \equiv -\frac{\mu_{h_{t-1}} - r + \frac{1}{2}\sigma^2_{h_{t-1}}}{\sigma_{h_{t-1}}}. \tag{3.6}$$

From (2.2), defining $z_t^{\mathbb{Q}} \equiv z_t^{\mathbb{P}} - \lambda_t$ yields

$$\epsilon_{t+1} = r - \frac{1}{2}\sigma_{h_t}^2 + \sigma_{h_t} z_{t+1}^{\mathbb{Q}}.$$

By Proposition 3.1, the joint PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under $\mathbb{Q}$ is $f^{\mathbb{Q}}_{\epsilon_{1:T}, h_{0:T-1}}$. Furthermore,

- $\{z_t^{\mathbb{Q}}\}_{t=1}^T$ are independent standard Gaussian random variables under $\mathbb{Q}$,
- $\{z_t^{\mathbb{Q}}\}_{t=1}^T$ and $\{h_t\}_{t=0}^{T-1}$ are independent processes under $\mathbb{Q}$
- $\mathbb{Q}[h_{t+1} = j | \mathcal{G}_{t+1} \vee \mathcal{H}_t] = \mathbb{Q}[h_{t+1} = j | \mathcal{F}_t] = P_{h_t, j}$.

3.2 Contingent claim pricing

The joint process $\big\{(S_t, h_t)\big\}_{t=0}^T$ possesses the Markov property under $\mathbb{Q}$ with respect to the filtration $\mathcal{F}$. The contingent claim price is thus given by

$$\begin{aligned}
\Pi_t^{\mathbb{Q}} &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}\Psi(S_T)\big|\mathcal{G}_t\big], \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}\Psi(S_T)\big|\mathcal{F}_t\big]\Big|\mathcal{G}_t\Big], \\
&= \mathbb{E}^{\mathbb{Q}}\big[g_t(S_t, h_t)\big|\mathcal{G}_t\big], \quad \text{by the Markov property of } \big\{(S_t, h_t)\big\}_{t=0}^T, \\
&= \sum_{k=1}^H \eta_{t,k}^{\mathbb{Q}}\, g_t(S_t, k), \qquad (3.7)
\end{aligned}$$

where $\eta_{t,j}^{\mathbb{Q}} \equiv \mathbb{Q}\left[h_t = j | \mathcal{G}_t\right]$, and with g_t, $t \in \{0, \ldots, T\}$, being real functions characterized by the following dynamic programming scheme starting with $g_T(s,k) = \Psi(s)$:

$$g_t(s,k) = \sum_{\ell=1}^H P_{k,\ell} \int_{-\infty}^{\infty} g_{t+1}\Big(se^{r-\sigma_k^2/2+\sigma_k z}, \ell\Big)\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz, \qquad t \in \{0, \ldots, T-1\}.$$

For European options, i.e., for $\Psi(s) = \max(s-K, 0)$, Hardy (2001) provides an explicit expression for g_t in the two regimes case.

The formula (3.7) illustrates the path-dependence feature generated by the RS mean-correcting transform. At time t, for an investor, $(S_t, \eta_t^{\mathbb{P}})$ completely characterizes the likelihood of every possible future scenarios under the physical measure $\mathbb{P}$ due to the Markov property of $(S, \eta^{\mathbb{P}})$

with respect to the partial information $\mathcal{G}$. Indeed, $f^{\mathbb{P}}_{S_{t+1:T}|\mathcal{G}_t} = f^{\mathbb{P}}_{S_{t+1:T}|S_t,\eta^{\mathbb{P}}_t}$. A purely forward-looking price setting mechanism would produce an option price at time t that is measurable with respect to $\sigma(S_t, \eta^{\mathbb{P}}_t)$. This is however not the case with the RS mean-correcting transform as the $\sigma(S_t, \eta^{\mathbb{Q}}_t)$-measurable derivative price $\Pi^{\mathbb{Q}}_t$ is a function of $\eta^{\mathbb{Q}}_t$ which is not $\sigma(S_t, \eta^{\mathbb{P}}_t)$-measurable in general since it depends on the whole path $S_0, \ldots, S_t$. The option price $\Pi^{\mathbb{Q}}_t$ therefore exhibits path-dependence (non-Markovian behavior) although the underlying asset payoff can be expressed as a function of the last observation of the $\mathcal{G}$-Markov process $(S, \eta^{\mathbb{P}})$ under $\mathbb{P}$. The Online Appendix B further illustrates the path-dependence feature in a simplified setting.

3.3 Stochastic discount factor representation

The path-dependence feature can be visualized through a *stochastic discount factor* (SDF) representation. As shown in the Online Appendix D.2, prices obey the following relationship:

$$\Pi^{\mathbb{Q}}_t = \mathbb{E}^{\mathbb{P}}\left[\Pi^{\mathbb{Q}}_{t+1} m^{\mathbb{Q}}_{t+1} \middle| \mathcal{G}_t\right], \qquad m^{\mathbb{Q}}_{t+1} = e^{-r} \frac{\sum_{i=1}^{H} \eta^{\mathbb{Q}}_{t,i}\, \phi^{\mathbb{Q}}_i(\epsilon_{t+1})}{\sum_{i=1}^{H} \eta^{\mathbb{P}}_{t,i}\, \phi^{\mathbb{P}}_i(\epsilon_{t+1})}. \tag{3.8}$$

Therefore, the SDF $m^{\mathbb{Q}}_t$ is not $\sigma(\epsilon_t, \eta^{\mathbb{P}}_{t-1})$-measurable. Pricing under $\mathbb{Q}$ in fact entails weighing prices at time $t+1$ based on the risk-neutral regime predictive probabilities $\eta^{\mathbb{Q}}_t$, thus in a path-dependent fashion. This could point toward complicated theoretical implications such as time non-separable preferences as in Garcia et al (2003). Note also that the SDF $m^{\mathbb{Q}}_t$ is not $\sigma(\epsilon_t, \eta^{\mathbb{Q}}_{t-1})$-measurable in general due to its dependence on $\eta^{\mathbb{P}}_t$.

4 A new family of RS mean-correcting martingale measures

This section shows how the concept of regime-switching mean-correcting change of measure can be adapted to yield a $\sigma(S_t, \eta^{\mathbb{P}}_t)$-measurable time-$t$ option price. The key takeaway is that the statistical properties of the regime process must be altered in suitable ways, i.e., so as to remove non-Markovian effects.

4.1 General construction of an alternative martingale measure

The joint mixed PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under any probability measure $\mathbb{M}$ can be expressed as

$$f^{\mathbb{M}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{M}}_{h_0}(h_0) f^{\mathbb{M}}_{\epsilon_1|h_0}(\epsilon_1|h_0) \times \prod_{t=2}^{T} f^{\mathbb{M}}_{h_{t-1}|h_{0:t-2},\epsilon_{1:t-1}}(h_{t-1}|h_{0:t-2}, \epsilon_{1:t-1}) f^{\mathbb{M}}_{\epsilon_t|h_{0:t-1},\epsilon_{1:t-1}}(\epsilon_t|h_{0:t-1}, \epsilon_{1:t-1}). \tag{4.1}$$

To obtain the martingale property, we apply a RS mean correction, i.e., we impose that conditionally on the current regime h_{t-1}, the distribution of the log-return ϵ_t is still Gaussian with a conditional variance $\sigma^2_{h_{t-1}}$ equal to the physical one and a conditional mean of $r - \frac{1}{2}\sigma^2_{h_{t-1}}$. Therefore,

$$f^{\mathbb{M}}_{\epsilon_t|h_{0:t-1},\epsilon_{1:t-1}} = \phi^{\mathbb{Q}}_{h_{t-1}}, \qquad t \geq 1, \tag{4.2}$$

where $\phi^{\mathbb{Q}}_i$, $i \in \{1, \ldots, H\}$, is defined as before; see (3.2).

Alterations on transition probabilities of the regime process are applied to remove non-Markovian effects on option prices. Consider a multivariate process $\psi = \{\psi_t\}_{t=1}^{T-1}$ where $\psi_t = \left[\psi_t^{(i,j)}\right]_{i,j=1}^{H}$ is a $\mathcal{G}_t$-measurable $H \times H$ random matrix for all $t \in \{0, \ldots, T-1\}$. Transition probabilities of the following form are assumed under $\mathbb{M}$:

$$f^{\mathbb{M}}_{h_{t-1}|h_{0:t-2},\epsilon_{1:t-1}}(h_{t-1}|h_{0:t-2}, \epsilon_{1:t-1}) = P_{h_{t-2},h_{t-1}} \psi_{t-1}^{(h_{t-2},h_{t-1})}, \qquad t \geq 2. \tag{4.3}$$

This imposes that for all $t \in \{1, \ldots, T-1\}$ and all $i, j \in \{1, \ldots, H\}$,

$$\psi_t^{(i,j)} > 0 \text{ almost surely}, \qquad \text{and} \qquad \sum_{j=1}^{H} P_{i,j}\psi_t^{(i,j)} = 1 \text{ almost surely}, \tag{4.4}$$

to ensure positiveness and normalization. Note also that the initial mass function of the first regime can be modified from $f^{\mathbb{P}}_{h_0}(h_0)$ to $f^{\mathbb{M}}_{h_0}(h_0)$ during the passage from $\mathbb{P}$ to $\mathbb{M}$. Coefficients $\psi_{t-1}^{(i,j)}$ alter transition probabilities and could therefore be used to represent aversion to regime transitions leading to adverse outcomes for trading agents in the market. However, the current work does not use coefficients $\psi_{t-1}^{(i,j)}$ for such purposes; they are used in a mechanical fashion to generate martingale measures which possess the desired property of yielding path-independent option prices.

By Proposition 3.1, such a measure $\mathbb{M}$ is constructed by the Radon-Nikodym derivative

$$\frac{d\mathbb{M}}{d\mathbb{P}} = \frac{f^{\mathbb{M}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})} = \frac{f^{\mathbb{M}}_{h_0}(h_0)}{f^{\mathbb{P}}_{h_0}(h_0)} \prod_{t=2}^{T} \psi_{t-1}^{(h_{t-2},h_{t-1})} \prod_{t=1}^{T} \xi_t, \tag{4.5}$$

where ξ_t is defined as in (3.5).

As shown in Appendix A.2, the risk-neutral mass function of regimes is given by

$$\eta^{\mathbb{M}}_{t+1,i} \equiv \mathbb{M}[h_{t+1} = i|\mathcal{G}_{t+1}] = \frac{\sum_{j=1}^{H} P_{j,i}\psi_{t+1}^{(j,i)}\phi_j^{\mathbb{Q}}(\epsilon_{t+1})\eta^{\mathbb{M}}_{t,j}}{\sum_{j=1}^{H}\phi_j^{\mathbb{Q}}(\epsilon_{t+1})\eta^{\mathbb{M}}_{t,j}}, \qquad t \in \{0, \dots T-1\}, \tag{4.6}$$

with $\eta^{\mathbb{M}}_{0,i} = f^{\mathbb{M}}_{h_0}(i)$.

Using (4.2) and (4.6), it is straightforward to show that

$$f^{\mathbb{M}}_{\epsilon_{t+1}|\epsilon_{1:t}}(\epsilon_{t+1}|\epsilon_{1:t}) = \sum_{i=1}^{H} \eta^{\mathbb{M}}_{t,i}\phi_i^{\mathbb{Q}}(\epsilon_{t+1}). \tag{4.7}$$

Hence, provided that $\eta^{\mathbb{M}}_t$ is $\sigma(\eta^{\mathbb{P}}_t)$-measurable for all $t \geq 0$, we have that the $\mathcal{G}_t$-conditional distribution of the log-return ϵ_{t+1} under $\mathbb{M}$ depends exclusively on $\eta^{\mathbb{P}}_t$. Furthermore, $\eta^{\mathbb{P}}_{t+1}$ is a function of $(\epsilon_{t+1}, \eta^{\mathbb{P}}_t)$, as shown by (2.8). Applying this reasoning recursively, it follows that the $\mathcal{G}_t$-conditional distribution of $\epsilon_{t+1:T}$ under $\mathbb{M}$ depends only on $\eta^{\mathbb{P}}_t$. This leads to the following result:

Proposition 4.1. *The joint process* $\left\{(S_t, \eta^{\mathbb{P}}_t)\right\}_{t=0}^{T}$ *has the Markov property with respect to the filtration* $\mathcal{G}$ *under the probability measure* $\mathbb{M}$ *if* $\eta^{\mathbb{M}}_t$ *is* $\sigma\left(\eta^{\mathbb{P}}_t\right)$*-measurable for all* $t \geq 0$.

Under the conditions stated in the above proposition, it follows that the option price

$$\Pi^{\mathbb{M}}_t = \mathbb{E}^{\mathbb{M}}\left[e^{-r(T-t)}\Psi(S_T)\middle|\mathcal{G}_t\right]$$

is $\sigma\big(S_t, \eta_t^{\mathbb{P}}\big)$-measurable by the Markov property. A simple way of designing a probability measure $\mathbb{M}$ satisfying such conditions is provided next.

4.2 A simple construction of an alternative martingale measure

In both Section 4.2 and Section 4.3 we assume that $P_{j,i} > 0$ for all $i, j \in \{1, \ldots, H\}$. Under this assumption, a special case is obtained by specifying the measure $\mathbb{M}$ through the conditions

$$f_{h_0}^{\mathbb{M}} = f_{h_0}^{\mathbb{P}}, \quad \text{and} \quad \psi_t^{(j,i)} = \frac{\eta_{t,i}^{\mathbb{P}}}{P_{j,i}} \quad \text{almost surely}, \quad i, j \in \{1, \ldots, H\}. \tag{4.8}$$

Substituting (4.8) in (4.6) yields

$$\eta_t^{\mathbb{M}} = \eta_t^{\mathbb{P}} \quad \text{almost surely}. \tag{4.9}$$

The condition from Proposition 4.1 requiring $\eta_t^{\mathbb{M}}$ to be $\sigma\big(\eta_t^{\mathbb{P}}\big)$-measurable for all $t \geq 0$ is thus trivially satisfied. As stated in Remark 4.1, it turns out that the martingale measure $\mathbb{M}$ obtained in this fashion has an interesting interpretation.

Remark 4.1. The martingale measure $\mathbb{M}$ obtained with (4.8) can be understood as a sequence of two consecutive changes of measures: one from the physical measure $\mathbb{P}$ to an equivalent measure $\tilde{\mathbb{P}}$ under which the statistical properties of returns are preserved, and another from $\tilde{\mathbb{P}}$ to $\mathbb{M}$ which induces the martingale property through a RS mean correction.

Indeed, assume $\tilde{\mathbb{P}}$ is a probability measure such that for all $t \in \{1, \ldots, T\}$ and all $j \in \{1, \ldots, H\}$,

$$\tilde{\mathbb{P}}[h_0 = j] = f_{h_0}^{\mathbb{P}}(j), \tag{4.10}$$

$$\tilde{\mathbb{P}}[h_t = j | \mathcal{G}_t \vee \mathcal{H}_{t-1}] = \eta_{t,j}^{\mathbb{P}}, \tag{4.11}$$

$$f_{\epsilon_t | h_{0:t-1}, \epsilon_{1:t-1}}^{\tilde{\mathbb{P}}} = f_{\epsilon_t | h_{0:t-1}, \epsilon_{1:t-1}}^{\mathbb{P}} = \phi_{h_{t-1}}^{\mathbb{P}}. \tag{4.12}$$

In other words, when passing from $\mathbb{P}$ to $\tilde{\mathbb{P}}$, only the transition probabilities are shifted, from

P_{h_{t-1},h_t} to $\eta^{\mathbb{P}}_{t,h_t}$. For such a measure $\tilde{\mathbb{P}}$, it can be shown (see Appendix A.3 for a proof) that

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t} = f^{\mathbb{P}}_{\epsilon_{t+1}|\mathcal{G}_t}. \tag{4.13}$$

This implies the joint distribution of log-returns is identical under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$, and thus the change of measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ preserves the statistical properties of the underlying asset S. Because regime-switching model adequacy and goodness-of-fit statistical tests are characterized by the distribution of the underlying process, there is no reason why $\mathbb{P}$ might be preferred to $\tilde{\mathbb{P}}$ when a regime-switching model is deemed appropriate for the price dynamics of some asset; both lead to the same joint distribution for the observed prices. Therefore, it cannot be distinguished whether a given price path is generated under $\mathbb{P}$ or under $\tilde{\mathbb{P}}$. Thus, $\tilde{\mathbb{P}}$ could even be viewed as the physical measure.

Next, let's see how the change of measure can be decomposed. As shown in Appendix A.4, the joint mixed PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under $\tilde{\mathbb{P}}$ is

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_0}(h_0) \prod_{t=2}^{T} \eta^{\mathbb{P}}_{t-1,h_{t-1}} \prod_{t=1}^{T} \phi^{\mathbb{P}}_{h_{t-1}}(\epsilon_t). \tag{4.14}$$

This implies the following representation of $\mathbb{M}$:

$$\frac{d\mathbb{M}}{d\mathbb{P}} \equiv \frac{d\mathbb{M}}{d\tilde{\mathbb{P}}}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}, \tag{4.15}$$

where

$$\frac{d\mathbb{M}}{d\tilde{\mathbb{P}}} = \frac{f^{\mathbb{M}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})} = \prod_{t=1}^{T} \xi_t, \tag{4.16}$$

with ξ_t defined as in (3.5), and

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})} = \prod_{t=2}^{T} \frac{\eta^{\mathbb{P}}_{t-1,h_{t-1}}}{P_{h_{t-2},h_{t-1}}}. \tag{4.17}$$

Therefore, $\mathbb{M}$ can be constructed by applying a regular Girsanov-type change of drift through

(4.16) to a measure $\tilde{\mathbb{P}}$ under which the risky asset has the same statistical properties as under the physical measure $\mathbb{P}$. This confirms the statement in Remark 4.1. In summary, the regime-switching mean-correcting change of measure can be used to yield Markovian option prices, but it must be applied on $\tilde{\mathbb{P}}$, rather than $\mathbb{P}$.

It is relevant to note that the pricing approach outlined in this section works because there are multiple joint distributions for $(\epsilon_{1:T}, h_{0:T-1})$ which allow recovering the same $\mathbb{P}$-distribution for $S_{1:T}$ characterized by (2.9). To use the terminology of Siu (2014), multiple original markets (i.e. specifications for the joint $\mathbb{P}$-dynamics of $\epsilon_{1:T}$ and $h_{0:T-1}$) can lead to the same filtered market dynamics, i.e. the $\mathbb{P}$-distribution of $\epsilon_{1:T}$. The approach followed in the current section consists in changing the measure to select a suitable original market dynamics without changing the filtered market dynamics to enable us to perform a change of drift which does not induce path dependence in option prices.

4.3 Incorporating regime uncertainty aversion

The condition (4.9) implies that regime unobservability risk is unpriced as the conditional distribution of the hidden regime h_t is left untouched by the passage from $\mathbb{P}$ to $\mathbb{M}$. The current section illustrates a generalization of the previous method which can incorporate regime unobservability risk aversion through a so-called *conversion function*. Such a function relates $\eta^{\mathbb{M}}$ to $\eta^{\mathbb{P}}$ by applying a distortion to the regime mass function process.

Definition 4.1. Consider functions $\zeta_k : [0,1]^H \to [0,1]$, $k \in \{1, \ldots, H\}$, having the property

$$\sum_{k=1}^{H} \zeta_k(\eta_1, \ldots, \eta_H) = 1, \quad \text{for all } (\eta_1, \ldots, \eta_H) \in [0,1]^H \text{ such that } \sum_{i=1}^{H} \eta_i = 1.$$

The function $\zeta = (\zeta_1, \ldots, \zeta_H)$ is referred to as a *conversion function.*

The $\psi_t^{(j,i)}$ from (4.3) characterizing the martingale measure $\mathbb{M}$ are determined to enforce the chosen conversion:

$$\eta^{\mathbb{M}}_{t,k} = \zeta_k\left(\eta^{\mathbb{P}}_t\right) \text{ almost surely for all } t \text{ and all } k. \tag{4.18}$$

By Proposition 4.1, path-dependence problems are purged when such a measure $\mathbb{M}$ is used as a martingale measure for pricing. From (4.4) and (4.6), the above condition involves using $\psi_t^{(i,j)}$ that are solutions of the following linear system of equations, for all $t \geq 1$:

$$\begin{aligned} \frac{\sum_{j=1}^{H} P_{j,i}\psi_t^{(j,i)}\phi_j^{\mathbb{Q}}(\epsilon_t)\zeta_j\left(\eta_{t-1}^{\mathbb{P}}\right)}{\sum_{j=1}^{H}\phi_j^{\mathbb{Q}}(\epsilon_t)\zeta_j\left(\eta_{t-1}^{\mathbb{P}}\right)} &= \zeta_i\left(\eta_t^{\mathbb{P}}\right), \qquad i \in \{1,\ldots,H\}, \\ \sum_{j=1}^{H} P_{i,j}\psi_t^{(i,j)} = 1, &\qquad i \in \{1,\ldots,H\}. \end{aligned} \tag{4.19}$$

The solutions are characterized in the proposition below whose proof is in Appendix A.5.

Proposition 4.2. *The system of equations* (4.19) *admits an infinite number of solutions. The trivial solution is*

$$\psi_t^{(j,i)} = \frac{\zeta_i\left(\eta_t^{\mathbb{P}}\right)}{P_{j,i}}, \qquad i,j \in \{1,\ldots,H\}. \tag{4.20}$$

A non-trivial solution to the system (4.19) is presented in the Online Appendix C.

Examples of conversion functions could include for instance:

- The identity conversion function:

$$\zeta_k(\eta_1,\ldots,\eta_H) = \eta_k, \tag{4.21}$$

- The softmax function: for some real constants a_i, b_i, with $i \in \{1,\ldots,H\}$,

$$\zeta_k(\eta_1,\ldots,\eta_H) = \frac{\exp(a_k + b_k\eta_k)}{\sum_{i=1}^{H}\exp(a_i + b_i\eta_i)}. \tag{4.22}$$

The identity conversion function case described in Section 4.2 would reflect risk-neutrality with respect to the risk associated with the unobservable current regime, whereas the softmax function could reflect risk-aversion to the current regime risk. Values for parameters (a_k, b_k) of the softmax function could be obtained through calibration using market option prices.

4.4 Price computation algorithms

Using martingale measures $\mathbb{M}$ described in the current section, options can be priced by means either of Monte-Carlo simulations or a dynamic programming approach. Both methods are outlined below.

4.4.1 Monte-Carlo simulations

A fairly simple recipe to simulate log-returns ϵ_t within a Monte-Carlo simulation under the measure $\mathbb{M}$ is given: at each $t = 0, \ldots, T-1$,

1. Calculate $\eta_t^{\mathbb{P}}$ from (2.8),
2. Calculate $\eta_{t,i}^{\mathbb{M}} = \zeta_i\big(\eta_t^{\mathbb{P}}\big)$, for $i \in \{1, \ldots, H\}$,
3. Draw ϵ_{t+1} from the Gaussian mixture (4.7).

4.4.2 Dynamic program

Dynamic programming can be used to price simple contingent claims. By Proposition 4.1, the option price is $\sigma\big(S_t, \eta_t^{\mathbb{P}}\big)$-measurable since (4.18). Hence

$$\Pi_t^{\mathbb{M}} = \mathbb{E}^{\mathbb{M}}\Big[e^{-r(T-t)}\Psi(S_T)\Big|\mathcal{G}_t\Big] = \pi_t^{\mathbb{M}}\big(S_t, \eta_t^{\mathbb{P}}\big),$$

for some real functions $\pi_0^{\mathbb{M}}, \ldots, \pi_T^{\mathbb{M}}$.

The functions $\pi_0^{\mathbb{M}}, \ldots, \pi_T^{\mathbb{M}}$ can be computed through a simple dynamic program provided by Proposition 4.3 which is proven in Appendix A.6.

Proposition 4.3. *For $i \in \{1, \ldots, H\}$ and $t \in \{0, \ldots, T-1\}$, define the functions*

$$\chi_{t+1,i}(\eta, \epsilon) \equiv \frac{\sum_{j=1}^{H} P_{j,i}\, \phi_j^{\mathbb{P}}(\epsilon)\eta_j}{\sum_{j=1}^{H} \phi_j^{\mathbb{P}}(\epsilon)\eta_j} \tag{4.23}$$

and

$$\chi_{t+1}\big(\eta_t^{\mathbb{P}}, \epsilon_{t+1}\big) = \Big(\chi_{t+1,1}\big(\eta_t^{\mathbb{P}}, \epsilon_{t+1}\big), \ldots, \chi_{t+1,H}\big(\eta_t^{\mathbb{P}}, \epsilon_{t+1}\big)\Big). \tag{4.24}$$

Then, for any $t \in \{0, \ldots, T-1\}$ and any possible value of S_t and $\eta_t^{\mathbb{P}}$:

$$\pi_t^{\mathbb{M}}\big(S_t, \eta_t^{\mathbb{P}}\big) = e^{-r}\sum_{k=1}^{H} \zeta_k\big(\eta_t^{\mathbb{P}}\big)\int_{-\infty}^{\infty} \pi_{t+1}^{\mathbb{M}}\Big(S_t e^{r-\sigma_k^2/2+\sigma_k z}, \chi_{t+1}\big(\eta_t^{\mathbb{P}}, r-\sigma_k^2/2+\sigma_k z\big)\Big)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz, \tag{4.25}$$

with $\pi_T^{\mathbb{M}}(S_T, \eta_T^{\mathbb{P}}) = \Psi(S_T)$ where Ψ is the payoff function.

Moreover, the dimension of the pricing functional can be reduced by one as stated below.

Remark 4.2. Because $\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} = 1$ almost surely (since they represent probabilities of a sample space partition), the function $\chi_{t+1,i}(\eta, \epsilon)$ only needs to be computed at points where $\eta_1 + \ldots + \eta_H = 1$. Because of this, we can drop $\eta_{t,H}^{\mathbb{P}}$ from the state variables since it is a known quantity when $\eta_{t,1}^{\mathbb{P}}, \ldots, \eta_{t,H-1}^{\mathbb{P}}$ are given. This reduces the dimension of the pricing functional by one since it is possible to write $\pi_t^{\mathbb{M}}(S_t, \eta_t^{\mathbb{P}}) = \bar{g}_t(S_t, \eta_{t,1}^{\mathbb{P}}, \ldots, \eta_{t,H-1}^{\mathbb{P}})$ for some function $\bar{g}_t$, $t \in \mathcal{T}$.

5 Martingale measures based on $\mathcal{G}_T$-measurable transforms

Martingale measures from the previous section possess the property that they alter the likelihood of past regimes given the full asset trajectory. Indeed, because the Radon-Nikodym derivative $\frac{d\mathbb{M}}{d\mathbb{P}}$ is not $\mathcal{G}_T$-measurable, there exists events $A \in \mathcal{F}_T$ such that

$$\mathbb{M}[A|\mathcal{G}_T] \neq \mathbb{P}[A|\mathcal{G}_T]. \tag{5.1}$$

For instance, the most probable regime trajectory could differ significantly under $\mathbb{M}$ (compared to under $\mathbb{P}$). Since a risk-neutral measure reflects risk-aversion and other considerations that affect equilibrium prices; as such it might be desirable not to alter the posterior regime distribution when there is no asset risk left, i.e., given $S_{0:T}$.

This section illustrates the construction of martingale measures which leave the $\mathcal{G}_T$-conditional distribution of past regimes unaffected by the change of measure. A first approach relies on the adaptation of the well-known Esscher transform to the latent regimes framework. A second approach, based on a regime-mixture approach, combines features of the Esscher transform and of martingale measures constructed in Section 4.

5.1 A conditional version of the Esscher transform

The Esscher transform is a popular concept in finance and insurance for the pricing of financial products, see for instance Gerber and Shiu (1994), Bühlmann et al (1996) and Bühlmann et al

(1998). As explained in Gerber and Shiu (1994), pricing derivatives through the Esscher transform is consistent with the presence of a representative agent optimizing his investments under a power utility function. It is therefore relevant to investigate whether it can be adapted to regime-switching models so as to provide a natural solution to path-dependence issues. The Esscher transform presented hereby is a particular case of the general pricing approach under heteroskedasticity of Christoffersen et al (2009). It can also be seen as a discrete-time version of the Esscher transform from Siu (2014) which is applied on the filtered market obtained through the application separation principle from filtering and optimal stochastic control theory.

The conditional Esscher risk-neutral measure $\widehat{\mathbb{Q}}$ is defined by the Radon-Nikodym derivative

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = \prod_{t=1}^{T} \widehat{\xi}_t, \qquad \widehat{\xi}_t \equiv e^{-\theta_{t-1}} \left(\frac{S_t}{S_{t-1}} \right)^{\alpha_{t-1}}, \tag{5.2}$$

where $\{\theta_t\}_{t=0}^T$ and $\{\alpha_t\}_{t=0}^T$ are $\mathcal{G}$-adapted processes to be defined. As shown in Appendix A.7, the following condition, which is assumed to hold, ensures that $\widehat{\mathbb{Q}}$ is a probability measure:

$$\theta_t = \log \left(\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} \exp \left(\alpha_t \mu_k + \frac{1}{2} \alpha_t^2 \sigma_k^2 \right) \right). \tag{5.3}$$

Moreover, assuming this condition holds, as shown in Appendix A.8, the following condition is necessary and sufficient to ensure that $\widehat{\mathbb{Q}}$ is a risk-neutral measure:

$$\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} \exp \left(\alpha_t \mu_k + \frac{1}{2} \alpha_t^2 \sigma_k^2 \right) \left[1 - \exp \left(\mu_k + \alpha_t \sigma_k^2 + \frac{1}{2} \sigma_k^2 - r \right) \right] = 0. \tag{5.4}$$

A solution to this equation always exists since the left hand side tends to minus infinity as $\alpha_t \to \infty$ and to infinity as $\alpha_t \to -\infty$, on top of being a continuous function of α_t. Equation (5.4) can be solved numerically to determine α_t; the solution is a function of $\eta_t^{\mathbb{P}}$, and therefore (θ_t, α_t) is a function of $\eta_t^{\mathbb{P}}$.

Appendix A.9 shows that the distribution of returns under the measure $\widehat{\mathbb{Q}}$ is characterized by

$$\widehat{\mathbb{Q}}[\epsilon_{t+1} \leq x | \mathcal{G}_t] = \sum_{i=1}^{H} \hat{\eta}_{t,i}^{\mathbb{P}} \Phi\left(\frac{x - \mu_i - \alpha_t \sigma_i^2}{\sigma_i}\right), \qquad x \in \mathbb{R}, \tag{5.5}$$

where Φ is the standard Gaussian cumulative distribution function, and

$$\hat{\eta}_{t,i}^{\mathbb{P}} = \frac{\eta_{t,i}^{\mathbb{P}} \exp\left(\alpha_t \mu_i + \frac{1}{2}\alpha_t^2 \sigma_i^2\right)}{\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} \exp\left(\alpha_t \mu_k + \frac{1}{2}\alpha_t^2 \sigma_k^2\right)}. \tag{5.6}$$

The log-returns $\mathcal{G}_t$-conditional distribution under $\widehat{\mathbb{Q}}$ is therefore still a Gaussian mixture with modified mixing weights $\hat{\eta}_t^{\mathbb{P}}$ and means shifted from μ_i to $\mu_i - \alpha_t \sigma_i^2$ for each regime $i \in \{1, \ldots, H\}$. Note that the passage from $\eta_t^{\mathbb{P}}$ to $\hat{\eta}_t^{\mathbb{P}}$ is an instance of a conversion function since α_t is a function of $\eta_t^{\mathbb{P}}$ as shown by (5.4).

Equations (5.5)-(5.6) indicate the $\widehat{\mathbb{Q}}$ distribution of the log-return ϵ_{t+1} given $\mathcal{G}_t$ depends exclusively on $\eta_t^{\mathbb{P}}$ since α_t and $\hat{\eta}_t^{\mathbb{P}}$ are functions of $\eta_t^{\mathbb{P}}$. Furthermore, $\eta_{t+1}^{\mathbb{P}}$ is a function of $(\epsilon_{t+1}, \eta_t^{\mathbb{P}})$; see (2.8). Applying this reasoning recursively, it follows that the $\mathcal{G}_t$-conditional distribution of $\epsilon_{t+1:T}$ under $\widehat{\mathbb{Q}}$ depends only on $\eta_t^{\mathbb{P}}$. This leads to the following result:

Proposition 5.1. *The joint process* $\left\{(S_t, \eta_t^{\mathbb{P}})\right\}_{t=0}^{T}$ *has the Markov property with respect to the filtration* $\mathcal{G}$ *under the probability measure* $\widehat{\mathbb{Q}}$.

This result entails that the option price at time t is $\sigma\left(S_t, \eta_t^{\mathbb{P}}\right)$-measurable. Other theoretical properties satisfied by this measure are outlined in the remark below.

Remark 5.1. The risk-neutral measure $\widehat{\mathbb{Q}}$ displays the following properties:

- The option price $\Pi_t^{\widehat{\mathbb{Q}}} = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[e^{-r(T-t)}\Psi(S_T)\middle|\mathcal{G}_t\right]$ is $\sigma\left(S_t, \eta_t^{\mathbb{P}}\right)$-measurable.
- $\widehat{\xi}_t$ is $\mathcal{G}_t$-measurable for all $t \in \mathcal{T}$ and therefore $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \in \mathcal{G}_T$. Thus, the $\mathcal{G}_T$-conditional distribution of past risks is unaffected by the change of measure: $\widehat{\mathbb{Q}}[A|\mathcal{G}_T] = \mathbb{P}[A|\mathcal{G}_T]$, $\forall A \in \mathcal{F}_T$.
- If the martingale property is already satisfied under $\mathbb{P}$, i.e., $\phi_i^{\mathbb{Q}} = \phi_i^{\mathbb{P}}$ for all $i \in \{1, \ldots, H\}$, then there is no change of measure, i.e., $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = 1$ almost surely.[1]
- In the single-regime case ($H = 1$), $\widehat{\mathbb{Q}}$ reduces to the usual Esscher martingale measure $\mathbb{Q}$.

[1]This is because we then have $\alpha_t = \theta_t = 0$ almost surely for all t.

5.1.1 *Option pricing schemes*

A simple recipe is available to simulate log-returns under the measure $\widehat{\mathbb{Q}}$ within a Monte-Carlo simulation: at each $t = 0, \ldots, T-1$,

1. Calculate $\eta^{\mathbb{P}}_{t,i}$, $i \in \{1, \ldots, H\}$, from (2.8),
2. Solve numerically for α_t in (5.4),
3. Calculate $\hat{\eta}^{\mathbb{P}}_{t,i}$, $i \in \{1, \ldots, H\}$, from (5.6),
4. Draw ϵ_{t+1} from the Gaussian mixture (5.5).

Note that the second and third steps can be pre-calculated.

Simple contingent claims can also be priced by dynamic programming. Since the time-t option price is $\sigma\big(S_t, \eta^{\mathbb{P}}_t\big)$-measurable, it follows that for all $t \in \mathcal{T}$ there exists a function $\pi^{\widehat{\mathbb{Q}}}_t$ such that

$$\Pi^{\widehat{\mathbb{Q}}}_t \equiv \mathbb{E}^{\widehat{\mathbb{Q}}}\big[e^{-r(T-t)}\Psi(S_T)\,|\,\mathcal{G}_t\big] = \pi^{\widehat{\mathbb{Q}}}_t\big(S_t, \eta^{\mathbb{P}}_t\big).$$

The dynamic program that enables the recursive computation of the functions $\pi^{\widehat{\mathbb{Q}}}_t$ can be derived following the steps outlined in Section 4.4.2.

$$\pi^{\widehat{\mathbb{Q}}}_t\big(S_t, \eta^{\mathbb{P}}_t\big) = e^{-r} \sum_{k=1}^{H} \hat{\eta}^{\mathbb{P}}_{t,k} \int_{-\infty}^{\infty} \pi^{\widehat{\mathbb{Q}}}_{t+1}\Big(S_t e^{\mu_k - \alpha_t \sigma_k^2 + \sigma_k z}, \chi_{t+1}\big(\eta^{\mathbb{P}}_t, \mu_k - \alpha_t \sigma_k^2 + \sigma_k z\big)\Big) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz, \qquad (5.7)$$

with $\pi^{\widehat{\mathbb{Q}}}_T\big(S_T, \eta^{\mathbb{P}}_T\big) = \Psi(S_T)$ where Ψ is the payoff function, $\hat{\eta}^{\mathbb{P}}_t$ is defined as a function of $\eta^{\mathbb{P}}_t$ through (5.6), and χ_{t+1} is defined by (4.24).

5.2 A regime-mixture Esscher transform

We present now a new family of martingale measures based on a regime-mixture approach. A measure from this new family is denoted by $\bar{\mathbb{Q}}$. Similarly to the conditional Esscher transform $\widehat{\mathbb{Q}}$ from Section 5.1, the Radon-Nikodym derivative characterizing the new regime-mixture Esscher martingale measure $\bar{\mathbb{Q}}$ is $\mathcal{G}_T$-measurable. This implies the $\mathcal{G}_T$-conditional distribution of regimes $h_{0:T-1}$ is left untouched by the change of measure. Moreover, as for RS mean-correcting measures

$\mathbb{M}$, the risk-neutral one-period conditional distribution of asset log-returns is a mixture of Gaussian distribution whose mean is the risk-free rate minus the usual convexity correction. The regime-mixture approach therefore combines features of the two families of martingale measures previously considered, namely the new version of the RS mean-correcting measure $\mathbb{M}$ and the conditional Esscher transform $\widehat{\mathbb{Q}}$. We first explain how this measure can be derived.

The PDF of a trajectory under a probability measure $\bar{\mathbb{Q}}$ can be expressed as (see Appendix A.10)

$$f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T},h_{0:T-1}) = f^{\bar{\mathbb{Q}}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T)\prod_{t=1}^{T} f^{\bar{\mathbb{Q}}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}). \tag{5.8}$$

In comparison, the PDF under $\mathbb{P}$ is given by (see Appendix A.11)

$$f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T},h_{0:T-1}) = f^{\mathbb{P}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T)\prod_{t=1}^{T}\sum_{i=1}^{H}\eta^{\mathbb{P}}_{t-1,i}\,\phi^{\mathbb{P}}_i(\epsilon_t). \tag{5.9}$$

The regime-mixture Esscher martingale measure $\bar{\mathbb{Q}}$ is constructed by enforcing

$$f^{\bar{\mathbb{Q}}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T) = f^{\mathbb{P}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T), \tag{5.10}$$

$$f^{\bar{\mathbb{Q}}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}) = \sum_{i=1}^{H}\zeta_i(\eta^{\mathbb{P}}_{t-1})\phi^{\mathbb{Q}}_i(\epsilon_t), \qquad \forall t\in\{1,\ldots,T\}, \tag{5.11}$$

where ζ is the conversion function, and $\phi^{\mathbb{Q}}_i$, $i\in\{1,\ldots,H\}$, is defined as before; see (3.2). The property (5.10) states that the $\mathcal{G}_T$-conditional distribution of the regime trajectory is unaltered under $\bar{\mathbb{Q}}$. The property (5.11) states the $\mathcal{G}_{t-1}$-conditional distribution of the log-return ϵ_t under $\bar{\mathbb{Q}}$ is a Gaussian mixture with mixing weights given by the vector $\zeta(\eta^{\mathbb{P}}_{t-1})$, and means shifted from μ_i to $r-\frac{1}{2}\sigma_i^2$ for each regime $i\in\{1,\ldots,H\}$. The purpose of the latter condition is to ensure the martingale property is satisfied, and that regime risk is priced according to the chosen conversion function.

As shown in Appendix A.12 the Radon-Nikodym derivative is

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} = \prod_{t=1}^{T}\bar{\xi}_t, \qquad \bar{\xi}_t \equiv \frac{\sum_{i=1}^{H}\zeta_i(\eta^{\mathbb{P}}_{t-1})\phi^{\mathbb{Q}}_i(\epsilon_t)}{\sum_{i=1}^{H}\eta^{\mathbb{P}}_{t-1,i}\,\phi^{\mathbb{P}}_i(\epsilon_t)}. \tag{5.12}$$

Appendix A.13 shows that the distribution of returns under this measure is characterized by

$$\bar{\mathbb{Q}}[\epsilon_{t+1} \leq x | \mathcal{G}_t] = \sum_{i=1}^{H} \zeta_i(\eta_t^{\mathbb{P}}) \Phi\left(\frac{x - r + \frac{1}{2}\sigma_i^2}{\sigma_i}\right), \qquad x \in \mathbb{R}. \tag{5.13}$$

Hence, for any $s = 0, \ldots, T - t - 1$, the $\mathcal{G}_{t+s}$-conditional distribution of ϵ_{t+1+s} under $\bar{\mathbb{Q}}$ depends only on $\eta_{t+s}^{\mathbb{P}}$. Furthermore, by (2.8), $\eta_{t+s}^{\mathbb{P}}$ is a function of $\left(\epsilon_{t+s}, \eta_{t+s-1}^{\mathbb{P}}\right)$. The above reasoning, applied recursively, implies that the $\mathcal{G}_t$-conditional distribution of $\epsilon_{t+1:T}$ under $\bar{\mathbb{Q}}$ depends only on $\eta_t^{\mathbb{P}}$. The next proposition then follows.

Proposition 5.2. *The joint process* $\left\{(S_t, \eta_t^{\mathbb{P}})\right\}_{t=0}^{T}$ *has the Markov property with respect to the filtration* $\mathcal{G}$ *under the probability measure* $\bar{\mathbb{Q}}$.

This property entails that the option price $\Pi_t^{\bar{\mathbb{Q}}} = \mathbb{E}^{\bar{\mathbb{Q}}}\left[e^{-r(T-t)}\Psi(S_T)\middle|\mathcal{G}_t\right]$ is $\sigma\left(S_t, \eta_t^{\mathbb{P}}\right)$-measurable. Furthermore, the other properties stated in Remark 5.1 also hold for $\bar{\mathbb{Q}}$. Finally, since the underlying asset price joint distribution are identical under $\mathbb{M}$ and $\bar{\mathbb{Q}}$, the pricing algorithms are identical to those given in Section 4.4.

6 Conclusion

The current work shows that the usual approach to construct martingale measures in a regime-switching framework based on the correction of the drift for each respective regime (i.e., regime-switching mean correction) leads to path-dependence even for vanilla options. More precisely, even if the joint process $(S, \eta^{\mathbb{P}})$ comprising the underlying asset price and the regime mass function given observable information has the Markov property, vanilla derivatives prices at time t would not be a function strictly of the current value of the latter process, i.e., of $(S_t, \eta_t^{\mathbb{P}})$. The construction of multiple convenient martingale measures removing the path-dependence feature is illustrated in the current paper.

Our first approach is a modified version of the above concept of RS mean-correcting martingale measure; it also relies on RS mean correction to obtain the martingale property, but with the inclusion of transition probability transforms so as to recuperate the Markov property of option prices. This yields a very wide class of new martingale measures removing the path-dependence. This class includes an interesting special case which can be represented as the successive application of two changes of measures: a first one which allows retaining the exact same underlying asset statistical properties from the physical measure, and then a change of drift on each regime. Obtained generalizations allow for the pricing of regime uncertainty through conversion functions which distort the hidden regime distribution given the currently observed information.

A second approach developed is based on changes of measures whose Radon-Nikodym derivatives are $\sigma(S_0, \ldots, S_T)$-measurable, implying that they do not impact the conditional distribution of the regime hidden trajectory given the full asset trajectory. This approach embeds as a particular case the well-known Esscher transform.

Simple pricing procedures for contingent claims under the developed martingale measures based either on dynamic programming or Monte-Carlo simulations are also provided.

Potential further work includes determining if prices provided within the current study can be recovered through equilibrium schemes. The current paper relies on mathematical risk-neutralization arguments for the obtainment of derivatives prices without attempting to construct an underlying equilibrium model leading to the martingales measures that were designed herein (except for the conditional Esscher transform of Section 5.1 which we know is consistent with the presence of a representative agent maximizing his expected power utility function). Equilibrium schemes involving time separable preferences could be investigated to obtain path-independent option prices in the context of regime-switching models with latent regimes.

A Proofs

A.1 Proof of Eq. (2.5)

$$
\begin{aligned}
f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T},h_{0:T-1}) &= f^{\mathbb{P}}_{\epsilon_1,h_0}(\epsilon_1,h_0)\prod_{t=2}^{T} f^{\mathbb{P}}_{\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0) f^{\mathbb{P}}_{\epsilon_1|h_0}(\epsilon_1|h_0)\times\\
&\quad \prod_{t=2}^{T} f^{\mathbb{P}}_{\epsilon_t|\epsilon_{1:t-1},h_{0:t-1}}(\epsilon_t|\epsilon_{1:t-1},h_{0:t-1}) f^{\mathbb{P}}_{h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0)\prod_{t=2}^{T} P_{h_{t-2},h_{t-1}}\prod_{t=1}^{T}\frac{1}{\sigma_{h_{t-1}}}\varphi\left(\frac{\epsilon_t-\mu_{h_{t-1}}}{\sigma_{h_{t-1}}}\right),
\end{aligned}
$$

where the last equality follows from (2.2) and (2.4). Using definition (2.6) concludes the proof.

A.2 Proof of Eq. (4.6)

$$
\begin{aligned}
\eta^{\mathbb{M}}_{t+1,i} &= \mathbb{M}\left[h_{t+1}=i|\mathcal{G}_{t+1}\right],\\
&= \sum_{j=1}^{H}\mathbb{M}\left[h_{t+1}=i|\mathcal{G}_{t+1},h_t=j\right]\mathbb{M}\left[h_t=j|\mathcal{G}_{t+1}\right],\\
&= \sum_{j=1}^{H} P_{j,i}\psi^{(j,i)}_{t+1}\frac{f^{\mathbb{M}}_{h_t,\epsilon_{t+1}|\epsilon_{1:t}}(j,\epsilon_{t+1}|\epsilon_{1:t})}{f^{\mathbb{M}}_{\epsilon_{t+1}|\epsilon_{1:t}}(\epsilon_{t+1}|\epsilon_{1:t})}, \qquad \text{from (4.3)},\\
&= \sum_{j=1}^{H} P_{j,i}\psi^{(j,i)}_{t+1}\frac{f^{\mathbb{M}}_{h_t|\epsilon_{1:t}}(j|\epsilon_{1:t}) f^{\mathbb{M}}_{\epsilon_{t+1}|h_t,\epsilon_{1:t}}(\epsilon_{t+1}|j,\epsilon_{1:t})}{\sum_{k=1}^{H} f^{\mathbb{M}}_{h_t|\epsilon_{1:t}}(k|\epsilon_{1:t}) f^{\mathbb{M}}_{\epsilon_{t+1}|h_t,\epsilon_{1:t}}(\epsilon_{t+1}|k,\epsilon_{1:t})},\\
&= \sum_{j=1}^{H} P_{j,i}\psi^{(j,i)}_{t+1}\frac{\eta^{\mathbb{M}}_{t,j}\phi^{\mathbb{Q}}_j(\epsilon_{t+1})}{\sum_{k=1}^{H}\eta^{\mathbb{M}}_{t,k}\phi^{\mathbb{Q}}_k(\epsilon_{t+1})}, \qquad \text{from (4.2)}.
\end{aligned}
$$

A.3 Proof of Eq. (4.13)

$$
f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t}(x|\mathcal{G}_t) = \sum_{k=1}^{H} f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1},h_t|\mathcal{G}_t}(x,k|\mathcal{G}_t) = \sum_{k=1}^{H}\tilde{\mathbb{P}}[h_t=k|\mathcal{G}_t] f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|h_t,\mathcal{G}_t}(x|k,\mathcal{G}_t). \tag{A.1}
$$

Moreover,

$$\tilde{\mathbb{P}}[h_t = k|\mathcal{G}_t] = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\{h_t=k\}}|\mathcal{G}_t] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\{h_t=k\}}|\mathcal{G}_t, \mathcal{H}_{t-1}]\Big|\mathcal{G}_t\right] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\underbrace{\tilde{\mathbb{P}}[h_t = k|\mathcal{G}_t, \mathcal{H}_{t-1}]}_{=\eta^{\mathbb{P}}_{t,k} \text{ by } (4.11)}\Big|\mathcal{G}_t\right] = \eta^{\mathbb{P}}_{t,k}.$$

Similarly, it can be shown using (4.12) that

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|h_t,\mathcal{G}_t}(x|k,\mathcal{G}_t) = \phi^{\mathbb{P}}_k(x).$$

Using the above relations in (A.1) yields

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t}(x|\mathcal{G}_t) = \sum_{k=1}^{H} \eta^{\mathbb{P}}_{t,k}\phi^{\mathbb{P}}_k(x) = f^{\mathbb{P}}_{\epsilon_{t+1}|\mathcal{G}_t}(x|\mathcal{G}_t),$$

where the last equality is straightforward to prove. Hence, $f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t} = f^{\mathbb{P}}_{\epsilon_{t+1}|\mathcal{G}_t}$.

A.4 Proof of Eq. (4.14)

$$\begin{aligned}
f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T},h_{0:T-1}) &= f^{\tilde{\mathbb{P}}}_{\epsilon_1,h_0}(\epsilon_1,h_0)\prod_{t=2}^{T} f^{\tilde{\mathbb{P}}}_{\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0) f^{\tilde{\mathbb{P}}}_{\epsilon_1|h_0}(\epsilon_1|h_0)\times\\
&\quad \prod_{t=2}^{T} f^{\tilde{\mathbb{P}}}_{\epsilon_t|\epsilon_{1:t-1},h_{0:t-1}}(\epsilon_t|\epsilon_{1:t-1},h_{0:t-1}) f^{\tilde{\mathbb{P}}}_{h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0)\prod_{t=2}^{T}\eta^{\mathbb{P}}_{t-1,h_{t-1}}\prod_{t=1}^{T}\phi^{\mathbb{P}}_{h_{t-1}}(\epsilon_t), \qquad \text{from (4.11) and (4.12)}.
\end{aligned}$$

A.5 Proof of Proposition 4.2

The system (4.10) is equivalent to

$$\sum_{j=1}^{H}\tilde{\psi}^{(j,i)}_t\kappa_{t,j} = 0 \qquad \text{and} \qquad \sum_{j=1}^{H}\tilde{\psi}^{(i,j)}_t = 0, \qquad i\in\{1,\ldots,H\},$$

where we have defined

$$\tilde{\psi}_t^{(j,i)} \equiv P_{j,i}\psi_t^{(j,i)} - \zeta_i\big(\eta_t^{\mathbb{P}}\big), \qquad \kappa_{t,j} \equiv \phi_j^{\mathbb{Q}}(\epsilon_t)\zeta_j\big(\eta_{t-1}^{\mathbb{P}}\big)$$

Indeed, the trivial solution is, for all $i, j \in \{1, \ldots, H\}$,

$$\tilde{\psi}_t^{(j,i)} = 0 \qquad \Rightarrow \qquad \psi_t^{(j,i)} = \frac{\zeta_i\big(\eta_t^{\mathbb{P}}\big)}{P_{j,i}}. \tag{A.2}$$

The system has H^2 unknown values and $2H$ equations. If $H > 2$, the existence of a solution implies that an infinite number of solutions exist. Even if $H = 2$, we can show there exists an infinite number of solutions.

Indeed, the system can be written as follows for $H = 2$,

$$\underbrace{\begin{bmatrix} \kappa_{t,1} & 0 & \kappa_{t,2} & 0 \\ 0 & \kappa_{t,1} & 0 & \kappa_{t,2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{\equiv \mathcal{M}} \begin{bmatrix} \tilde{\psi}_t^{(1,1)} \\ \tilde{\psi}_t^{(1,2)} \\ \tilde{\psi}_t^{(2,1)} \\ \tilde{\psi}_t^{(2,2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\det \mathcal{M} = \kappa_{t,1}\kappa_{t,2} - \kappa_{t,1}\kappa_{t,2} = 0$, an infinity of solutions exist by the properties of homogeneous linear systems.

A.6 Proof of Proposition 4.3

First,

$$
\begin{aligned}
\pi_t^{\mathbb{M}}(S_t, \eta_t^{\mathbb{P}}) &= \mathbb{E}^{\mathbb{M}}\left[e^{-r(T-t)}\Psi(S_T)\middle|\mathcal{G}_t\right], \\
&= \mathbb{E}^{\mathbb{M}}\left[e^{-r}\mathbb{E}^{\mathbb{M}}\left[e^{-r(T-(t+1))}\Psi(S_T)\middle|\mathcal{G}_{t+1}\right]\middle|\mathcal{G}_t\right], \\
&= e^{-r}\mathbb{E}^{\mathbb{M}}\left[\pi_{t+1}^{\mathbb{M}}(S_{t+1}, \eta_{t+1}^{\mathbb{P}})\middle|\mathcal{G}_t\right], \\
&= e^{-r}\sum_{k=1}^{H}\mathbb{M}[h_t = k|\mathcal{G}_t]\,\mathbb{E}^{\mathbb{M}}\left[\pi_{t+1}^{\mathbb{M}}(S_{t+1}, \eta_{t+1}^{\mathbb{P}})\middle|\mathcal{G}_t, h_t = k\right], \\
&= e^{-r}\sum_{k=1}^{H}\zeta_k(\eta_t^{\mathbb{P}})\mathbb{E}^{\mathbb{M}}\left[\pi_{t+1}^{\mathbb{M}}(S_{t+1}, \eta_{t+1}^{\mathbb{P}})\middle|S_t, \eta_t^{\mathbb{P}}, h_t = k\right], \qquad \text{by (4.18).}
\end{aligned}
\tag{A.3}
$$

Moreover, from (2.8), the definition (4.23) implies that

$$
\eta_{t+1,i}^{\mathbb{P}} = \chi_{t+1,i}(\eta_t^{\mathbb{P}}, \epsilon_{t+1}).
$$

and thus

$$
\eta_{t+1}^{\mathbb{P}} = \chi_{t+1}(\eta_t^{\mathbb{P}}, \epsilon_{t+1}). \tag{A.4}
$$

This means

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{M}}\left[\pi_{t+1}^{\mathbb{M}}(S_{t+1}, \eta_{t+1}^{\mathbb{P}})\middle|S_t, \eta_t^{\mathbb{P}}, h_t = k\right] \\
&= \mathbb{E}^{\mathbb{M}}\left[\pi_{t+1}^{\mathbb{M}}(S_t e^{\epsilon_{t+1}}, \chi_{t+1}(\eta_t^{\mathbb{P}}, \epsilon_{t+1}))\middle|S_t, \eta_t^{\mathbb{P}}, h_t = k\right], \\
&= \mathbb{E}^{\mathbb{M}}\left[\pi_{t+1}^{\mathbb{M}}\left(S_t e^{r-\sigma_k^2/2+\sigma_k z_{t+1}^{\mathbb{M}}}, \chi_{t+1}(\eta_t^{\mathbb{P}}, r-\sigma_k^2/2+\sigma_k z_{t+1}^{\mathbb{M}})\right)\middle|S_t, \eta_t^{\mathbb{P}}, h_t = k\right], \\
&= \int_{-\infty}^{\infty}\pi_{t+1}^{\mathbb{M}}\left(S_t e^{r-\sigma_k^2/2+\sigma_k z}, \chi_{t+1}(\eta_t^{\mathbb{P}}, r-\sigma_k^2/2+\sigma_k z)\right)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz.
\end{aligned}
\tag{A.5}
$$

Combining (A.3) and (A.5) yields the recursive formula (4.25) to obtain the option price $\Pi_t^{\mathbb{M}} = \pi_t^{\mathbb{M}}(S_t, \eta_t^{\mathbb{P}})$ from $\pi_{t+1}^{\mathbb{M}}$.

A.7 Proof of Eq. (5.3)

To ensure $\widehat{\mathbb{Q}}$ represents a change of probability measure, the following condition which guarantees that $\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right] = 1$ is assumed to hold for all $t \geq 0$:

$$\begin{aligned}
1 &= \mathbb{E}^{\mathbb{P}}\left[\widehat{\xi}_{t+1}\middle|\mathcal{G}_t\right], && \text{(A.6)}\\
&= e^{-\theta_t}\mathbb{E}^{\mathbb{P}}\left[\left(\frac{S_{t+1}}{S_t}\right)^{\alpha_t}\middle|\mathcal{G}_t\right],\\
&= e^{-\theta_t}\mathbb{E}^{\mathbb{P}}\left[\exp\left(\alpha_t\mu_{h_t} + \alpha_t\sigma_{h_t} z^{\mathbb{P}}_{t+1}\right)\middle|\mathcal{G}_t\right],\\
&= e^{-\theta_t}\sum_{k=1}^{H}\eta^{\mathbb{P}}_{t,k}\exp\left(\alpha_t\mu_k + \frac{1}{2}\alpha_t^2\sigma_k^2\right),\\
\Rightarrow \theta_t &= \log\left(\sum_{k=1}^{H}\eta^{\mathbb{P}}_{t,k}\exp\left(\alpha_t\mu_k + \frac{1}{2}\alpha_t^2\sigma_k^2\right)\right). && \text{(A.7)}
\end{aligned}$$

Next, let's prove that $\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right] = 1$. The following property will be useful:

$$\widehat{\xi}_s \text{ is } \mathcal{G}_t\text{-measurable}, \qquad \forall s \leq t. \tag{A.8}$$

It thus follows that for all $t \geq 1$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T}\widehat{\xi}_s\middle|\mathcal{G}_{t-1}\right] &= \mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T-1}\widehat{\xi}_s\underbrace{\mathbb{E}^{\mathbb{P}}\left[\widehat{\xi}_T\middle|\mathcal{G}_{T-1}\right]}_{=1,\text{ by (A.6)}}\middle|\mathcal{G}_{t-1}\right], \qquad \text{by (A.8)},\\
&\vdots \quad \text{(applying recursively)}\\
&= 1. && \text{(A.9)}
\end{aligned}$$

In particular, for $t = 1$ the above statement is equivalent to $\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right] = 1$.

A.8 Proof of Eq. (5.4)

To ensure $\widehat{\mathbb{Q}}$ is a martingale measure, the following risk-neutral condition must hold:

$$
\begin{aligned}
e^r &= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\frac{S_{t+1}}{S_t}\middle|\mathcal{G}_t\right], \\
&= \frac{\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\middle|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\middle|\mathcal{G}_t\right]}, \\
&= \frac{\prod_{n=1}^{t}\widehat{\xi}_n\,\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\prod_{n=t+1}^{T}\widehat{\xi}_n\middle|\mathcal{G}_t\right]}{\prod_{n=1}^{t}\widehat{\xi}_n\,\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+1}^{T}\widehat{\xi}_n\middle|\mathcal{G}_t\right]}_{=1,\text{ by (A.9)}}}, \qquad \text{by (A.8)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\prod_{n=t+1}^{T}\widehat{\xi}_n\middle|\mathcal{G}_{t+1}\right]\middle|\mathcal{G}_t\right], \\
&= \mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\widehat{\xi}_{t+1}\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+2}^{T}\widehat{\xi}_n\middle|\mathcal{G}_{t+1}\right]}_{=1,\text{ by (A.9)}}\middle|\mathcal{G}_t\right], \qquad \text{by (A.8)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[e^{-\theta_t}\left(\frac{S_{t+1}}{S_t}\right)^{\alpha_t+1}\middle|\mathcal{G}_t\right], \\
&= e^{-\theta_t}\mathbb{E}^{\mathbb{P}}\left[\exp\left((\alpha_t+1)\mu_{h_t}+(\alpha_t+1)\sigma_{h_t}z_{t+1}^{\mathbb{P}}\right)\middle|\mathcal{G}_t\right], \\
&= e^{-\theta_t}\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left((\alpha_t+1)\mu_k+\frac{1}{2}(\alpha_t+1)^2\sigma_k^2\right).
\end{aligned} \tag{A.10}
$$

Combining (5.3) and (A.10) yields

$$
\begin{aligned}
&\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left(\alpha_t\mu_k+\frac{1}{2}\alpha_t^2\sigma_k^2\right)=\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left((\alpha_t+1)\mu_k+\frac{1}{2}(\alpha_t+1)^2\sigma_k^2-r\right), \\
\Rightarrow\ &\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left(\alpha_t\mu_k+\frac{1}{2}\alpha_t^2\sigma_k^2\right)\left[1-\exp\left(\mu_k+\alpha_t\sigma_k^2+\frac{1}{2}\sigma_k^2-r\right)\right]=0.
\end{aligned}
$$

A.9 Proof of Eq. (5.5)

$$
\begin{aligned}
\widehat{\mathbb{Q}}\left[\epsilon_{t+1} \leq x | \mathcal{G}_t\right] &= \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}} \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \Big| \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \Big| \mathcal{G}_t\right]}, \\
&= \frac{\prod_{n=1}^{t} \widehat{\xi}_n \, \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}} \prod_{n=t+1}^{T} \widehat{\xi}_n \Big| \mathcal{G}_t\right]}{\prod_{n=1}^{t} \widehat{\xi}_n \, \underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+1}^{T} \widehat{\xi}_n \Big| \mathcal{G}_t\right]}_{=1, \text{ by } (A.9)}}, \qquad \text{by } (A.8), \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}} \widehat{\xi}_{t+1} \underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+2}^{T} \widehat{\xi}_n \Big| \mathcal{G}_{t+1}\right]}_{=1, \text{ by } (A.9)} \Big| \mathcal{G}_t\right], \qquad \text{by } (A.8), \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}} e^{-\theta_t + \alpha_t \epsilon_{t+1}} \Big| \mathcal{G}_t\right], \\
&= e^{-\theta_t} \sum_{i=1}^{H} \eta_{t,i}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\mu_i + \sigma_i z_{t+1}^{\mathbb{P}} \leq x\}} e^{\alpha_t \mu_i + \alpha_t \sigma_i z_{t+1}^{\mathbb{P}}} \Big| \mathcal{G}_t, h_t = i\right].
\end{aligned} \tag{A.11}
$$

Furthermore,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\mu_i + \sigma_i z_{t+1}^{\mathbb{P}} \leq x\}} e^{\alpha_t \mu_i + \alpha_t \sigma_i z_{t+1}^{\mathbb{P}}} \Big| \mathcal{G}_t, h_t = i\right] &= \int_{-\infty}^{(x-\mu_i)/\sigma_i} e^{\alpha_t \mu_i + \alpha_t \sigma_i z} \phi(z) dz, \\
&= \int_{-\infty}^{(x-\mu_i)/\sigma_i} e^{\alpha_t \mu_i + \alpha_t \sigma_i z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \\
&= \int_{-\infty}^{(x-\mu_i)/\sigma_i} e^{\alpha_t \mu_i + \alpha_t^2 \sigma_i^2/2} \frac{1}{\sqrt{2\pi}} e^{-(z-\alpha_t\sigma_i)^2/2} dz, \\
&= e^{\alpha_t \mu_i + \alpha_t^2 \sigma_i^2/2} \Phi\left(\frac{x-\mu_i}{\sigma_i} - \alpha_t \sigma_i\right).
\end{aligned} \tag{A.12}
$$

Plugging (A.7) and (A.12) in (A.11), we obtain

$$
\begin{aligned}
\widehat{\mathbb{Q}}\left[\epsilon_{t+1} \leq x | \mathcal{G}_t\right] &= e^{-\theta_t} \sum_{i=1}^{H} \eta_{t,i}^{\mathbb{P}} e^{\alpha_t \mu_i + \alpha_t^2 \sigma_i^2/2} \Phi\left(\frac{x-\mu_i-\alpha_t\sigma_i^2}{\sigma_i}\right), \\
&= \sum_{i=1}^{H} \frac{\eta_{t,i}^{\mathbb{P}} e^{\alpha_t \mu_i + \alpha_t^2 \sigma_i^2/2}}{\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} e^{\alpha_t \mu_k + \alpha_t^2 \sigma_k^2/2}} \Phi\left(\frac{x-\mu_i-\alpha_t\sigma_i^2}{\sigma_i}\right), \\
&= \sum_{i=1}^{H} \hat{\eta}_{t,i}^{\mathbb{P}} \Phi\left(\frac{x-\mu_i-\alpha_t\sigma_i^2}{\sigma_i}\right).
\end{aligned}
$$

A.10 Proof of Eq. (5.8)

The PDF of a trajectory $(\epsilon_{1:T}, h_{0:T-1})$ under a generic probability measure $\bar{\mathbb{Q}}$ can be expressed as

$$f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T}}(\epsilon_{1:T}) f^{\bar{\mathbb{Q}}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T), \tag{A.13}$$

since $\mathcal{G}_T \equiv \sigma(\epsilon_{1:T})$. Moreover,

$$\begin{aligned} f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T}}(\epsilon_{1:T}) &= f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T-1}}(\epsilon_{1:T-1}) f^{\bar{\mathbb{Q}}}_{\epsilon_T|\mathcal{G}_{T-1}}(\epsilon_T|\mathcal{G}_{T-1}), \\ &\vdots \quad \text{(applying recursively)} \\ &= \prod_{t=1}^{T} f^{\bar{\mathbb{Q}}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}). \end{aligned} \tag{A.14}$$

Combining (A.13) and (A.14) yields (5.8).

A.11 Proof of Eq. (5.9)

The expression (5.8) also holds for $\mathbb{P}$, i.e.,

$$f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T) \prod_{t=1}^{T} f^{\mathbb{P}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}). \tag{A.15}$$

Plugging the following concludes the proof:

$$f^{\mathbb{P}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}) = \sum_{i=1}^{H} \underbrace{\mathbb{P}[h_{t-1} = i|\mathcal{G}_{t-1}]}_{\equiv \eta^{\mathbb{P}}_{t-1,i}} \underbrace{f^{\mathbb{P}}_{\epsilon_t|\mathcal{G}_{t-1},h_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}, i)}_{=\phi^{\mathbb{P}}_i(\epsilon_t)}. \tag{A.16}$$

A.12 Proof of Eq. (5.12)

The Radon-Nikodym derivative is (from Proposition 3.1)

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} \equiv \frac{f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}. \tag{A.17}$$

Plugging Equation (5.8), (5.9), (5.10) and (5.11) yields (5.12).

A.13 Proof of Eq. (5.13)

The following property will be useful:

$$\bar{\xi}_s \text{ is } \mathcal{G}_t\text{-measurable}, \qquad \forall s \leq t. \tag{A.18}$$

Also, note that for all $t \geq 1$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[\bar{\xi}_t|\mathcal{G}_{t-1}] &= \int_{-\infty}^{\infty} \left\{ \frac{\sum_{i=1}^{H} \zeta_i(\eta_{t-1}^{\mathbb{P}})\phi_i^{\mathbb{Q}}(x)}{\sum_{i=1}^{H} \eta_{t-1,i}^{\mathbb{P}}\, \phi_i^{\mathbb{P}}(x)} f_{\epsilon_t|\mathcal{G}_{t-1}}(x|\mathcal{G}_{t-1}) \right\} dx, \\
&= \int_{-\infty}^{\infty} \left\{ \frac{\sum_{i=1}^{H} \zeta_i(\eta_{t-1}^{\mathbb{P}})\phi_i^{\mathbb{Q}}(x)}{\sum_{i=1}^{H} \eta_{t-1,i}^{\mathbb{P}}\, \phi_i^{\mathbb{P}}(x)} \sum_{i=1}^{H} \eta_{t-1,i}^{\mathbb{P}}\, \phi_i^{\mathbb{P}}(x) \right\} dx, \\
&= \sum_{i=1}^{H} \zeta_i(\eta_{t-1}^{\mathbb{P}}) \underbrace{\left[\int_{-\infty}^{\infty} \phi_i^{\mathbb{Q}}(x)dx \right]}_{=1}, \\
&= 1.
\end{aligned} \tag{A.19}$$

Furthermore, for all $t \geq 1$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T} \bar{\xi}_s \middle| \mathcal{G}_{t-1} \right] &= \mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T-1} \bar{\xi}_s \underbrace{\mathbb{E}^{\mathbb{P}}\left[\bar{\xi}_T \middle| \mathcal{G}_{T-1}\right]}_{=1, \text{ by (A.19)}} \middle| \mathcal{G}_{t-1} \right], \qquad \text{by (A.18)}, \\
&\vdots \quad \text{(applying recursively)} \\
&= 1.
\end{aligned} \tag{A.20}$$

We are now ready to carry out the main proof:

$$
\begin{aligned}
\bar{\mathbb{Q}}[\epsilon_{t+1} \le x|\mathcal{G}_t] &= \mathbb{E}^{\bar{\mathbb{Q}}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\middle|\mathcal{G}_t\right], \\
&= \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}\middle|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}\middle|\mathcal{G}_t\right]}, \\
&= \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\prod_{s=1}^{T}\bar{\xi}_s\middle|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\prod_{s=1}^{T}\bar{\xi}_s\middle|\mathcal{G}_t\right]}, \\
&= \frac{\prod_{s=1}^{t}\bar{\xi}_s\,\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\prod_{s=t+1}^{T}\bar{\xi}_s\middle|\mathcal{G}_t\right]}{\prod_{s=1}^{t}\bar{\xi}_s\,\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t+1}^{T}\bar{\xi}_s\middle|\mathcal{G}_t\right]}_{=1,\text{ by (A.20)}}}, \qquad \text{by (A.18)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\bar{\xi}_{t+1}\,\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t+2}^{T}\bar{\xi}_s\middle|\mathcal{G}_{t+1}\right]}_{=1,\text{ by (A.20)}}\middle|\mathcal{G}_t\right], \qquad \text{by (A.18)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\bar{\xi}_{t+1}\middle|\mathcal{G}_t\right], \\
&= \int_{-\infty}^{x}\left\{\frac{\sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\phi_i^{\mathbb{Q}}(y)}{\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}\,\phi_i^{\mathbb{P}}(y)}f^{\mathbb{P}}_{\epsilon_{t+1}|\mathcal{G}_t}(y|\mathcal{G}_t)\right\}dy, \\
&= \int_{-\infty}^{x}\left\{\frac{\sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\phi_i^{\mathbb{Q}}(y)}{\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}\,\phi_i^{\mathbb{P}}(y)}\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}\,\phi_i^{\mathbb{P}}(y)\right\}dy, \\
&= \sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\left[\int_{-\infty}^{x}\phi_i^{\mathbb{Q}}(y)dy\right], \\
&= \sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\Phi\left(\frac{x-r+\frac{1}{2}\sigma_i^2}{\sigma_i}\right).
\end{aligned}
$$

References

Ang A, Timmermann A (2012) Regime changes and financial markets. Annual Review of Financial Economics 4(1):313–337

Azimzadeh P, Forsyth PA, Vetzal KR (2014) Hedging costs for variable annuities under regime-switching. In: Hidden Markov models in finance, Springer, pp 133–166

Bégin JF, Gauthier G (2017) Price bias and common practice in option pricing, working paper. Simon Fraser University

Bollen NP (1998) Valuing options in regime-switching models. The Journal of Derivatives 6(1):38–49

Boyle P, Draviam T (2007) Pricing exotic options under regime switching. Insurance: Mathematics and Economics 40(2):267–282

Buffington J, Elliott RJ (2002a) American options with regime switching. International Journal of Theoretical and Applied Finance 5(05):497–514

Buffington J, Elliott RJ (2002b) Regime switching and European options. In: Stochastic Theory and Control, Springer, pp 73–82

Bühlmann H, Delbaen F, Embrechts P, Shiryaev AN (1996) No-arbitrage, change of measure and conditional Esscher transforms. CWI Quarterly 9(4):291–317

Bühlmann H, Delbaen F, Embrechts P, Shiryaev AN (1998) On Esscher transforms in discrete finance models. ASTIN Bulletin: The Journal of the IAA 28(2):171–186

Christoffersen P, Elkamhi R, Feunou B, Jacobs K (2009) Option valuation with conditional heteroskedasticity and nonnormality. The Review of Financial Studies 23(5):2139–2183

Di Masi GB, Kabanov YM, Runggaldier WJ (1995) Mean-variance hedging of options on stocks with Markov volatilities. Theory of Probability & Its Applications 39(1):172–182

Duan JC, Popova I, Ritchken P, et al (2002) Option pricing under regime switching. Quantitative

Finance 2(116-132):209

Elliott RJ, Madan DB (1998) A discrete time equivalent martingale measure. Mathematical Finance 8(2):127–152

Elliott RJ, Siu TK (2013) Option pricing and filtering with hidden Markov-modulated pure-jump processes. Applied Mathematical Finance 20(1):1–25

Elliott RJ, Aggoun L, Moore JB (1995) Hidden Markov models: Estimation and control, vol 29. Springer Science & Business Media

Elliott RJ, Chan L, Siu TK (2005) Option pricing and Esscher transform under regime switching. Annals of Finance 1(4):423–432

Elliott RJ, Siu TK, Chan L, Lau JW (2007) Pricing options under a generalized Markov-modulated jump-diffusion model. Stochastic Analysis and Applications 25(4):821–843

Elliott RJ, Siu TK, Badescu A (2011) On pricing and hedging options in regime-switching models with feedback effect. Journal of Economic Dynamics and Control 35(5):694–713

Fan K, Shen Y, Siu TK, Wang R (2015) Pricing annuity guarantees under a double regime-switching model. Insurance: Mathematics and Economics 62:62–78

François P, Gauthier G, Godin F (2014) Optimal hedging when the underlying asset follows a regime-switching Markov process. European Journal of Operational Research 237(1):312–322

Fuh CD, Ho KWR, Inchi H, Wang RH (2012) Option pricing with Markov switching. Journal of Data Science 10(3):483–509

Garcia R, Luger R, Renault E (2003) Empirical assessment of an intertemporal option pricing model with latent variables. Journal of Econometrics 116(1):49–83

Gerber HU, Shiu ES (1994) Option pricing by esscher transforms (with discussion). Transactions of the Society of Actuaries 46:99–191

Gerber HU, Shiu ES (1996) Actuarial bridges to dynamic hedging and option pricing. Insurance: Mathematics and Economics 18(3):183–218

Goovaerts MJ, Laeven RJ (2008) Actuarial risk measures for financial derivative pricing. Insurance: Mathematics and Economics 42(2):540–547

Guo X (2001) Information and option pricings. Quantitative Finance 1(1):38–44

Hamilton JD (1989) A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica 57(2):357–384

Hardy MR (2001) A regime-switching model of long-term stock returns. North American Actuarial Journal 5(2):41–53

Hardy MR (2003) Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance. John Wiley & Sons, Ltd.

Harrison JM, Kreps DM (1979) Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20(3):381–408

Ignatieva K, Song A, Ziveyi J (2016) Pricing and hedging of guaranteed minimum benefits under regime-switching and stochastic mortality. Insurance: Mathematics and Economics 70:286–300

Ignatieva K, Song A, Ziveyi J (2018) Fourier space time-stepping algorithm for valuing guaranteed minimum withdrawal benefits in variable annuities under regime-switching and stochastic mortality. ASTIN Bulletin: The Journal of the IAA 48(1):139–169

Ishijima H, Kihara T (2005) Option pricing with hidden Markov models, Institute of Finance, Waseda University

Jin Z, Wang Y, Yin G (2011) Numerical solutions of quantile hedging for guaranteed minimum death benefits under a regime-switching jump-diffusion formulation. Journal of Computational and Applied Mathematics 235(8):2842–2860

Jobert A, Rogers LC (2006) Option pricing with Markov-modulated dynamics. SIAM Journal on Control and Optimization 44(6):2063–2078

Lamberton D, Lapeyre B (2007) Introduction to Stochastic Calculus Applied to Finance. CRC Press

Liew CC, Siu TK (2010) A hidden Markov regime-switching model for option valuation. Insurance: Mathematics and Economics 47(3):374–384

Lin XS, Tan KS, Yang H (2009) Pricing annuity guarantees under a regime-switching model. North American Actuarial Journal 13(3):316–332

Liu R, Zhang Q, Yin G (2006) Option pricing in a regime-switching model using the fast Fourier transform. International Journal of Stochastic Analysis 2006

Mamon RS, Rodrigo MR (2005) Explicit solutions to European options in a regime-switching economy. Operations Research Letters 33(6):581–586

Naik V (1993) Option valuation and hedging strategies with jumps in the volatility of asset returns. The Journal of Finance 48(5):1969–1984

Ng ACY, Li JSH (2011) Valuing variable annuity guarantees with the multivariate Esscher transform. Insurance: Mathematics and Economics 49(3):393–400

Ngai A, Sherris M (2011) Longevity risk management for life and variable annuities: The effectiveness of static hedging using longevity bonds and derivatives. Insurance: Mathematics and Economics 49(1):100–114

Ranjbar HR, Seifi A (2015) A path-independent method for barrier option pricing in hidden Markov models. Physica A: Statistical Mechanics and its Applications 440:1–8

Shen Y, Siu TK (2013) Stochastic differential game, Esscher transform and general equilibrium under a Markovian regime-switching Lévy model. Insurance: Mathematics and Economics 53(3):757–768

Siu CC, Yam SCP, Yang H (2015) Valuing equity-linked death benefits in a regime-switching framework. ASTIN Bulletin: The Journal of the IAA 45(2):355–395

Siu TK (2014) A hidden Markov-modulated jump diffusion model for European option pricing. In: Hidden Markov Models in Finance, Springer, pp 185–209

Siu TK, Fung ES, Ng MK (2011) Option valuation with a discrete-time double Markovian regime-switching model. Applied Mathematical Finance 18(6):473–490

Trottier DA, Godin F, Hamel E (2018a) Local hedging of variable annuities in the presence of basis risk. ASTIN Bulletin: The Journal of the IAA 48(2):611–646

Trottier DA, Godin F, Hamel E (2018b) On fund mapping regressions applied to segregated funds hedging under regime-switching dynamics. Risks 6(3):78

Wang CW, Yang SS, Huang JW (2017) Analytic option pricing and risk measures under a regime-switching generalized hyperbolic model with an application to equity-linked insurance. Quantitative Finance 17(10):1567–1581

Wang Y, Yin G (2012) Quantile hedging for guaranteed minimum death benefits with regime switching. Stochastic Analysis and Applications 30(5):799–826

Yao DD, Zhang Q, Zhou XY (2006) A regime-switching model for European options. Stochastic Processes, Optimization, and Control Theory: Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems pp 281–300

Yuen FL, Yang H (2009) Option pricing in a jump-diffusion model with regime switching. ASTIN Bulletin: The Journal of the IAA 39(2):515–539

Zhang Q, Guo X (2004) Closed-form solutions for perpetual American put options with regime switching. SIAM Journal on Applied Mathematics 64(6):2034–2049