An Instructional Analogy Between Unitizing and Fraction Division: Seventh-Graders' Conceptual Understandings of Division and Interpretations of Fractional Remainders

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ABSTRACT

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Anna Tomaszewski

Fraction division is not a simple concept. In fact, both students and teachers struggle to understand and explain this operation. This study examined whether using instructional analogies between unitizing and division were effective when teaching fraction division to seventh-grade students. The extent to which the analogies were explicitly supported varied by condition. The aim was to investigate whether condition differences existed in participants' understanding of fraction division, in their interpretations of fractional remainders, and in their understanding of unitizing. Fifty-one participants were randomly assigned to the explicit-links condition (n = 17), the implicit-links condition (n = 17), or the control condition (n = 17). Students in the explicit and implicit-links conditions were first presented with a unitizing problem in which six was regrouped using various units and then they were taught how to solve a fraction division equation. The instructor in the explicit-links condition made connections between both concepts by using gestures, spatial cues, relational language, and other cognitive supports. The link between both concepts was in no way emphasized for the participants in the implicit-links condition. Finally, participants in the control condition saw the fraction division instruction twice and did not receive unitizing instruction. No condition differences were found at posttest in participants' division explanations nor in their understandings of unitizing. When interpreting a fractional remainder, participants in the implicit-links condition referred to the original unit significantly more than participants in the other two conditions, but no differences were observed in their uses of the referent unit when interpreting the remainder. The present study is relevant to

teaching professionals as it provides new information about the ways children think of fraction division and the fractional remainder.

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Statement of the Problem

Mathematical knowledge cannot be exclusively characterized by memorized rules or step-by-step instructions. Recognizing and understanding the connections between different concepts in mathematics in a meaningful way is an important aspect of mathematical thinking (Hiebert & Lefevre, 1986). Fractions knowledge, for example, is linked to students' future success in algebra (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Conceptual understanding of fraction operations not only predicts students' later achievement in mathematics, but will also positively impact their future adult lives. In fact, the use of fractions is common in many careers, even in those that do not require formal mathematical training, and is essential in other everyday life situations, as well, such as following a recipe or even measuring the length of a wall (Lortie-Forgues, Tian, & Siegler, 2015).

Despite the importance of these concepts, both teachers and students seem to struggle with fraction arithmetic, especially fraction division. Operations with fractions are more complex than operations with whole numbers (Empson & Levi, 2011) and, in particular, division with fractions is considered one of the most challenging concepts taught in elementary school mathematics (Petit, Laird, & Marsden, 2010). Sharp and Welder (2014) explained that seventh-grade students, despite having had previous instruction on fractions and division in elementary school, still have difficulties with fraction division. For one, they struggle with correctly identifying remainders and labeling units. They also use inappropriate or inefficient techniques when dividing with fractions, such as modifying fraction division problems into ones with whole numbers or overly relying on repeated subtraction, a potentially correct but inefficient and time-consuming strategy. Sharp and Welder (2014) also observed that seventh-graders had more difficulties correctly using discrete representations (i.e., set of chocolates in a box or a row of children) than continuous representations of fractional quantities (i.e., area of a shape, distance).

Students must also be wary of transferring whole number concepts inappropriately to contexts involving operations with fractions (Graeber & Tirosh, 1990). For instance, a misconception that the result of a multiplication will always be larger than its initial number will be true for multiplication with whole numbers (e.g., 4 x 5 = 20), but will not always be true for multiplication with fractions (e.g., $4 \times \frac{1}{5} = \frac{4}{5}$). Similarly, the misconception that division always results in a smaller number is true for division with whole numbers (e.g., 20 ÷ 5 = 4), but not always for division with fractions (e.g., $20 \div \frac{1}{5} = 100$). Students are forced to re-evaluate their understandings of multiplication and division when operating with fractions, making the process of learning how to multiply or divide by a fraction more complex. Other reasons for difficulties with operations with fractions have also been identified. For instance, solving equations with fractional amounts involves learning new mathematical procedures and concepts that are not present in operations with whole numbers (Empson & Levi, 2011), including the introduction of a novel notational form ($\frac{a}{b}$) (Hasemann, 1981; Lortie-Forgues, Tian, & Siegler, 2015).

Fraction division also presents difficulties for elementary school teachers. In fact, teachers display similar misconceptions as their students (Van Steenbrugge, Lesage, Valcke, & Desoete, 2014) and consider fraction division as the most complex operation taught at the elementary school level (Newton, 2008). Because of the teachers' uncertainties about how to effectively teach this operation and their commonly observed conceptual difficulties, they often depend on procedures, such as "invert-and-multiply," often without addressing why the procedure works. This type of instruction results in students themselves acquiring procedural and superficial understandings of fraction division, which may explain their own misconceptions. In this context, where teachers have been observed to struggle with teaching a specific mathematical concept, it is important to determine what type of instruction would be effective and what type of supports students would need under various conditions. Research addressing these questions promises to prevent the development of misconceptions and foster students' conceptual understanding.

At a broad level, this study aims to investigate more deeply how the use of instructional analogies can support students' learning in mathematics and the degree to which explicit supports are necessary to facilitate conceptual learning. The aim is to design and test instruction that focuses on making conceptual connections through analogical reasoning. More specifically, the objective is to examine effective approaches to teaching fraction division conceptually, because such research can be translated into effective instructional methods in classrooms and in teacher preparation, leading to students' better understanding of the concepts involved in operations with fractions.

Literature Review

Division of fractions is the operation elementary school teachers are least certain about when teaching mathematics (Newton, 2008), especially because teachers display similar misconceptions about fraction division as elementary school students (Ball, 1990; Van Steenbrugge et al., 2014). Teachers are not always able to explain why the procedures work and rarely make connections between the procedures and their conceptual meanings while teaching, which suggests that students also struggle to learn the meaning behind the operation (Olanoff, Lo, & Tobias, 2014). Such instruction may not be most effective for students' understanding of fraction division, particularly in light of the research showing that teachers who emphasize connections between written mathematical fraction symbols and their conceptual meanings has been shown to be effective (Osana & Pitsolantis, 2013).

Research on making conceptual connections in mathematics instruction indicates that increases in students' conceptual knowledge can lead to observable improvements in procedure and strategy use (Hiebert & Wearne, 1996; Rittle-Johnson & Alibali, 1999). Students with strong conceptual understanding often create their own strategies, modify existing ones when solving a novel problem, and discuss the reasoning behind other children's invented procedures (Hiebert & Wearne, 1996). On the other hand, students with weak conceptual understanding can perform taught procedures correctly, but have difficulty explaining the concepts underlying the procedures used (Hiebert & Wearne, 1996).

It is important to determine what types of instruction will effectively facilitate students' conceptual understanding, because such knowledge can positively affect their flexibility in applying mathematical concepts to known or novel contexts. In order to determine whether instruction emphasizing connections would be effective, Rittle-Johnson and Koedinger (2009) compared the effects of a concepts-first teaching condition to that of an iterative condition on sixth-graders' understanding of place value and regrouping in decimal arithmetic. In the concepts-first condition, students received three conceptual lessons followed by three procedural lessons, while those in the iterative condition received conceptual lessons alternating with procedural lessons. Their findings suggested that students in the iterative condition learned more and made fewer mistakes on the problems than students in the concepts-first condition. The students who received the lessons iteratively also appropriately transferred acquired knowledge to novel arithmetic tasks. On the contrary, students in the concepts-first condition were more successful at applying the concepts to tasks that were familiar to them. Rittle-Johnson and Koedinger (2009) speculated that the students learned less in the concepts-first condition because they each saw each set of lessons as two different unrelated systems, instead of recognizing and reflecting on the conceptual similarities between the concepts and procedures (see also Resnick & Omanson, 1987).

Other studies also supported the notion that making connections is essential to acquiring mathematical knowledge. For example, Rittle-Johnson and Alibali (1999) demonstrated that teaching novel concepts of mathematical equivalence by comparing multiple problem-solving strategies presented simultaneously and pointing out the similarities between them positively impacted students' conceptual understanding and transfer accuracy. Similarly, Canobi (2009) found that seven- and eight-year-old students' accuracy on a series of arithmetic problems and their conceptual explanations of the commutativity property improved when they were presented with addition and subtraction problems that were sequentially organized to highlight the conceptual connections between them. These results suggest that instruction that encourages students to make conceptual connections could be more effective when teaching mathematics concepts than teaching concepts in isolation.

Making Connections in Mathematics Instruction through Analogies

Although there appear to be clear advantages to teaching concepts through comparisons, further research is necessary to determine how these connections should be supported by the teacher. In teaching mathematics, there are several representational systems that can be used to illustrate and explain concepts to students through comparisons, including equations, word problems, graphs, diagrams, manipulatives, and drawings. For example, an equation can be represented using standard notation (i.e., $1 + 2 = 3$), but the same relationships among the quantities can also be represented in a word problem (i.e., There is one boy in the playground. Two children join him. How many children are there in the playground now?) or even with manipulatives, which are concrete objects such as chips or blocks meant to reflect abstract concepts (i.e., one green chip and two red, totaling three chips).

Mathematics instruction can be designed to highlight a number of different types of comparisons, such as two different external representations of the same concept (i.e., instruction of place value concepts with two sets of manipulatives Base-ten blocks and Digi-Blocks; see Blondin, Tomaszewski, & Osana, 2017), worked-out problems sharing the same structure (Sidney & Alibali, 2017b) or different problem-solving strategies for the same problem (Richland & McDonough, 2010). Two representations that differ on the surface (i.e., color, shape, context) but correspond to the same underlying mathematical idea can be considered analogs to each other (Gentner, 1983; Gick & Holyoak, 1983). Therefore, analogies and analogical reasoning can be used as a theoretical framework for concept instruction in mathematics (English, 2004).

To illustrate how analogies can be used in a mathematics instruction, consider the example in Figure 1. As shown, *half-ness* can be represented with the standard notation $\frac{1}{2}$, but also with an area model, where half of a rectangle is shaded in. Both these representations, though different on the surface, correspond to the same concept of "*half*-ness." When these two structurally-similar representations of "*half*-ness" are compared, they become analogs to each other. The first analog, usually the more familiar one, is considered the *source* analog (Gentner & Colhoun, 2010). The second analog, usually the less familiar one or an entirely novel one, is considered the *target* analog (Gentner & Colhoun, 2010). In the case of one-half illustrated here, if a student were more familiar with the area model, this representation would be considered the source, while the standard notation (i.e., "1/2") would be the target.

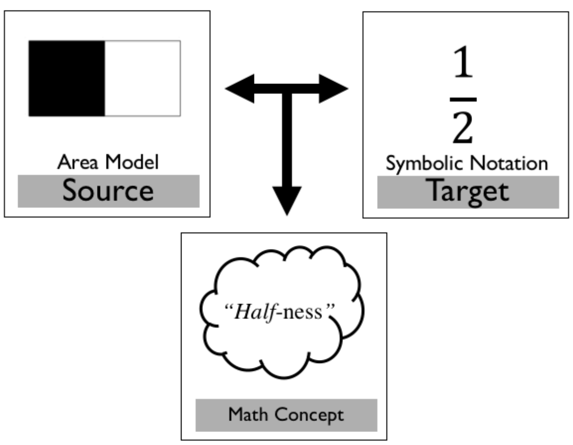


Figure 1. Instructional analogy of the mathematical concept of "*half*-ness."

According to developmental research, when children see the relational connections between analogical representations, the development of a deeper understanding of their shared conceptual structure is facilitated (Colhoun & Gentner, 2009; Richland & Simms, 2015). When students recognize the conceptual commonalities between a source and target analog, they engage in a process called *structure mapping* (Boulton-Lewis & Halfrod, 1992; Gentner, 1983;

Holyoak & Thagard, 1989). The learning that results from structure mapping can be identified or assessed through the transfer of the concept induced from both analogs to a novel context (English, 2004; Gick & Holyoak, 1983). Indeed, empirical evidence suggests that an integral part of good teaching involves encouraging students to make comparisons between representations (Gick & Holyoak, 1980; Osana & Pitsolantis, 2013).

Unitizing and Fraction Division

Using analogies in the mathematics classroom may be an effective way to encourage the development of conceptual understanding of fraction division. In my study, I propose instruction that will emphasize the conceptual connections between fraction division and the mathematical notion of unitizing. In the following section, I first provide a detailed description of both concepts and then explain why the two are analogous.

Defining unitizing. Lamon (2012) defined unitizing as a process of "constructing mental chunks in terms of which to think about a given quantity" (p. 104). In other words, unitizing can be understood as mentally assigning the value of a unit to an item or collection of items (Steffe & Olive, 2010). A unit can be understood as a quantity or measure representing one whole (Lee, 2017). Lee and Sztajn (2008) also explained a unit as "the item that can be repeated to quantify an object" (p. 21).

Units can be composed either of discrete sets of objects (e.g., a carton of eggs or a six-pack of juice) or continuous quantities (e.g., a chocolate bar, a pizza), depending on the representation chosen to represent the whole quantity (Lamon, 1996). For example, Figure 2 illustrates how to regroup the same quantity using a number of different units. Six muffins can be seen as one group of six muffins (here, the unit is six muffins), two groups of three muffins (here, the unit is three muffins), or even as six groups of one muffin (here, the unit is one muffin).

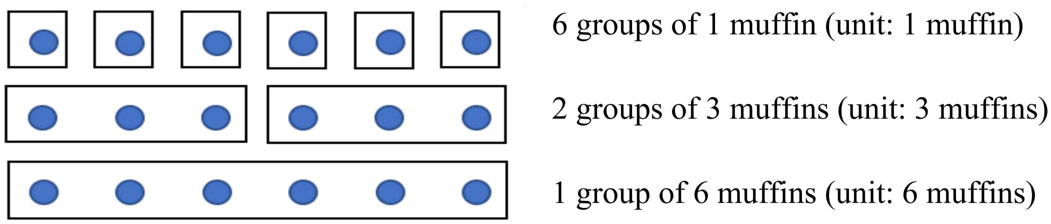


Figure 2. Example of unitizing based on different units.

When regrouping a quantity, as in the muffin example in Figure 2, we are *partitioning* it into equal-sized parts of a specific size, where the size of each part is the unit (Lamon, 2012; Olive, 1999; Steffe, 2002). If the partitioning occurs abstractly without physically partitioning the whole, the process is called *disembedding* (Norton & McCloskey, 2008). Lamon (1996) explained that children are naturally able to partition or decompose a quantity into smaller units, especially in contexts of sharing, but that recomposing a quantity by repeating, or *iterating*, a unit is more complex and develops over time.

Tobias, Roy, and Safi (2015) developed various tasks to teach preservice teachers about these components of unitizing. The tasks involved iterating units when solving whole-number problems in different bases, creating visual representations by iterating units and composite unit fractions, as well as composing and decomposing units to represent equivalent amounts using groupings of boxes, rolls, and pieces. Contextualized problems involving sharing items among sharers (e.g., five pizzas shared by four people) also encouraged discussions about the unit used when solving the problems. As evidenced by the Tobias et al. (2015) study, unitizing plays a central role in mathematics education, as it forms the conceptual basis for whole numbers (Lamon, 2005), fractions and rational numbers (Lamon, 2002; Thanheiser, 2009), place value (Cobb & Wheatley, 1988; Steffe, 2004), and operations (Tobias, 2013).

Defining fraction division. Division can be interpreted in two distinct ways: partitive or measurement division (Timmerman, 2014). The partitive model of division describes a situation in which the total amount and the number of groups is known, reflecting an equal sharing context (Empson & Levi, 2011). To solve, the total number of items is distributed equally into the number of groups. An example of a partitive division word problem for the whole-number equation 12 ÷ 2 = 6 is, “There are 12 chocolate bars. Two children split the bars equally between themselves. How many chocolate bars will they each get?” Here, the total number of bars is 12 and the number of groups is two. The solution, the unknown element in the equation, is the number of items in each group.

The reasoning needed to solve fraction division problems varies from the reasoning needed for whole-number division problems, because in the context of partitive division, the number of groups in the problem is no longer a whole number, but a fraction of a group (Empson & Levi, 2011). Empson and Levi (2011) noted that these types of problems, also called Partial Groups Problems, are more complex for children to solve, particularly when the divisor is a non-unit fraction (e.g., 2/3). In the problem, “Twenty-four cupcakes remain at the end of a birthday party. If these 24 cupcakes represent two-thirds of the cupcakes that were baked for the party, how many cupcakes were there at the beginning?” In this problem, the total number of cupcakes at the end of the party is known. We also know that these represent two-thirds of the initial number of cupcakes. The solution to the problem is the number of cupcakes in one full group.

The measurement model of division, on the other hand, describes a situation in which the total amount and the amount in each group is known, but the number of groups is unknown. A whole-number word problem reflecting the equation 12 ÷ 2 = 6, but through a measurement lens, would be, “There are 12 chocolate bars. If each child receives two chocolate bars, how many children can share the 12 chocolate bars?" Here, the total number of chocolate bars is 12 and the amount each child receives is two. The solution is the number of children sharing those 12 chocolates. This model of division could also be understood as "How many times does 2 fit into 12?" or "How many groups of two are there in 12?" The measurement division model provides a more accessible context to understand or visualize fraction division, especially when the divisor is a fractional amount (Ervin, 2017; Van de Walle et al., 2008). For instance, the word problem, "There are 24 cupcakes at the party. If each child receives only half a cupcake, how many children will be needed to eat all 24 cupcakes?" can also be envisioned as "How many groups of ½ are there in 24?"

Unitizing and measurement fraction division as analogs. The mathematical concepts of unitizing and measurement fraction division are analogous because they share the same conceptual structure of regrouping or reorganizing a certain quantity based on a new unit. Timmerman (2014) explained that the question, "What is the whole or unit?" is also central to students' interpretation of the quotient of a fraction division, and in particular, the fractional remainder (p. 117). This is especially the case of measurement division, where the dividend and quotient are interpreted using different units (Lee & Sztajn, 2008). Students find it challenging to interpret remainders of whole-number division situations, but the interpretation of fractional remainders appears to be even more challenging because of confusion over which unit should be used when interpreting the remainder (Lamberg & Wiest, 2015; Perlwitz, 2005).

Consider the following example as an illustration of unitizing in a fraction division problem. In any fraction division problem, the dividend and the divisor are interpreted using the original unit of one. For instance, in the equation $1 \div \frac{2}{3} = ?$, both the dividend (the amount that is divided) and the divisor (the amount by which the dividend is divided) are interpreted using the original unit (see Figure 3).

Original Unit Dividend (1 of 1 Original Unit) Divisor ($\frac{2}{3}$ of 1 Original Unit)

Figure 3. Dividend and divisor represented using the original unit.

However, to solve the equation, the quotient needs to be interpreted using a new unit, called the *referent*. When dividing by a fraction, “the divisor becomes the referent unit for the dividend” (Orrill, de Araujo, & Jacobson, 2010, p. 3). When dividing, one is technically asking “How many groups of the size of the referent unit enter into the dividend?” or in this specific context, “How many groups of $\frac{2}{3}$ enter into one?” As illustrated in Figure 4, the solution to $1 \div \frac{2}{3}$ is $1\frac{1}{2}$ because there are one and a half groups of $\frac{2}{3}$ (the referent unit) in the dividend. Using referent units flexibly, by keeping track of the unit, and recognizing when the unit shifts during problem solving, are important when reasoning with fractions (Jacobson & Izsák, 2015).

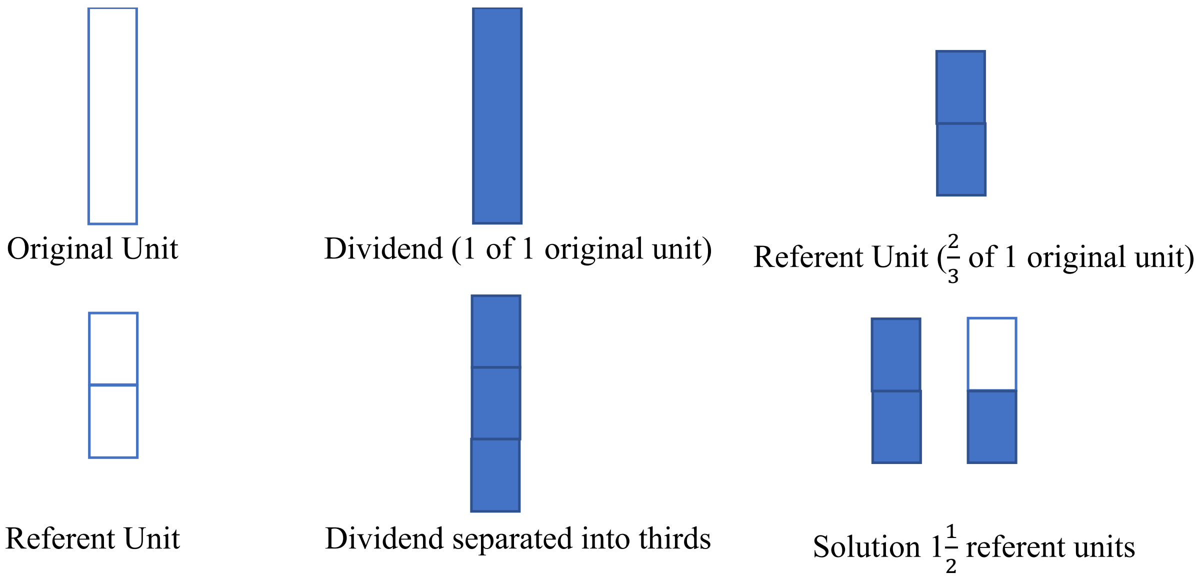


Figure 4. Solution to $1 \div \frac{2}{3} = ?$

In sum, unitizing is analogous to fraction division because an original set of objects (i.e., six) can be regrouped using various new “referent” units (as shown in Figure 2), just like a dividend is regrouped using the divisor as the referent unit (as shown in Figure 4). In this analogy, the notion of unitizing can be considered the *source* analog. Unitizing is more familiar to students because it is a concept that they are exposed to in other mathematical contexts, such as in place value (where 100 can be considered as one group of 100, 10 groups of 10, or 100 groups of one; Cobb & Wheatley, 1988), set representations or composite units (where a whole is represented using various items, such as a box of 16 cupcakes; Steffe, 2004), or even unit fractions (where a whole is divided into equal parts, such as one whole represented by five fifths; Lamon, 1996). Fraction division can be a *target* analog, because it is typically the concept that both students and teacher struggle with the most in elementary mathematics (Petit et al., 2010).

Supporting Analogical Reasoning through Instruction

A teaching framework based on analogical reasoning can potentially strengthen children’s conceptual understanding of mathematical concepts because of the emphasis on the common conceptual structures of both analogs. In the case where structure mapping focuses on the shared conceptual structure of two concepts, such as unitizing and fraction division, for example, the focus is no longer on the surface similarities of both analogs, but on the mathematical concepts embedded in both of them. Instruction with appropriately-supported analogs has been empirically demonstrated to strengthen students' conceptual understandings and their ability to make connections between analogs (Gick & Holyoak, 1983; Richland & Simms, 2015).

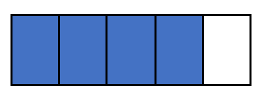
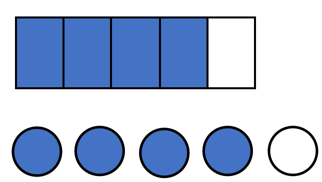
Children do not necessarily engage in effective connection-making on their own, however, because they are often distracted by surface features of analogs used during instruction (e.g., Gick & Holyoak, 1980). English (2004) explained that when students are presented with new mathematical word problems with the same structure but couched in different contexts (for example, one word problem about a market place and the second about a summer camp), they tend to be distracted by the context and have difficulty detecting the structural similarities without additional instructional support. Richland, Morrisson, and Holyoak (2006) also argued that young children are often distracted by surface features (i.e., color, shape, context) because of limitations in their executive functioning, specifically working memory and inhibitory control (Markman & Gentner, 1993). This distractibility, however, decreases with age as children's executive functioning skills develop (Gentner, 1988; Richland et al., 2006).

Although analogical reasoning has been shown to be developmental, students do not spontaneously make analogies within a school-based context. Even in adults, some analogical reasoning occurs spontaneously, while at other times, it needs specific intervention. Dunbar (2001) observed an analogical paradox where adults were able to reason spontaneously with analogies in everyday naturalistic situations, but required more supports to use analogies within a formal educational or experimental situation. Even if children reason analogically in their everyday lives while problem solving (English, 2004), making mathematical connections between different representations in the classroom context is often challenging and is not automatic (Hiebert, 1992; Osana, Przednowek, Cooperman, & Adrien, 2018; Resnick & Omason, 1987; Richland, Stigler, & Holyoak, 2012). Therefore, the teacher's role in explicitly guiding students' structure mappings between school-based analogs needs to be explored, especially because children's analogical reasoning skills predict their future mathematical reasoning (Buehl & Alexander, 2004).

Evidence for explicit instruction with analogies. Using highly-supported analogies in a classroom context is one way to support children's analogical reasoning in the mathematics classroom (Richland, Zur, & Holyoak, 2007). The use of cognitive supports has several benefits, including decreasing the demands on the children's working memories (Gentner, 1988; Gentner & Toupin, 1986; Rattermann & Gentner, 1998; Richland et al., 2006). Richland and her colleagues (2007) identified a number of explicit cognitive supports for instructional analogies by comparing eighth-grade mathematics instruction in the United States with instruction in Hong Kong and Japan. The authors randomly selected ten lessons per country from the Third International Mathematics and Science Study 1999 video database and qualitatively coded the teachers' use of supports of analogs and supports of structure mappings between analogs. The authors observed the teachers using a variety of supports, presented in Table 1, which included (a) presenting a familiar source, (b) presenting the source visually, (c) keeping the source and target visible simultaneously, (d) spatially aligning the analogs in a way that emphasizes their conceptual similarities, (e) using linking gestures between both analogs, and (f) using imagery.

Table 1

Definitions and Examples of Cognitive Supports based on Richland et al. (2007)

Cognitive Support	Definition	Examples
Familiarity of Source	Familiarity of the source can be defined as prior exposure to a specific analog (i.e., prior classroom experience or everyday life situations).	Use of fingers to count whole numbers Classroom materials students have already used (e.g. Base-Ten blocks) Mathematical concept already mastered by a student
Simultaneous visual presentation of source and target	Both the source and target analogs are presented visually and are kept visible throughout the entire process of structure mapping.	 $\frac{4}{5}$
Spatial alignment	The source and target analogs are spatially positioned in a manner that highlights their conceptual similarities.	

Linking gestures	Gestures that highlight the conceptual similarities between both analogs	Simultaneously or sequentially pointing to one analog and then the next. Capturing a concept of the source analog through a gesture and repeating the same gesture with the target.
Imagery	Visualizing the source: the analog can be a familiar object (e.g., scale in the context of equivalency) or a specific situation from the past (e.g., previously taught concepts).	A teacher addressing the class: "Remember when we were skip counting by twos last class, **visualize in your mind the number line we used**, now we are doing the same thing but instead of counting groups of two, we are counting groups of ten."

The effect on student learning of cognitive supports that highlight the explicit connections between two analogs has been supported through experimental and laboratory studies. For instance, both Sidney (2016) and Thompson and Opfer (2010) explained that using a familiar source facilitates students' connection-making, because the students are comparing a target to their prior knowledge in a specific domain. The other cognitive supports described by

Richland et al. (2007) have also been shown to facilitate structure mapping, including the simultaneous visual presentation of both analogs (Begolli & Richland, 2016; Christie & Gentner, 2010), spatial alignment that highlights the similarities between the source and target analogs (Matlen et al., 2014), gestures linking both analogs (Alibali et al., 2014; Alibali & Nathan, 2007), and the use of imagery (Richland et al., 2007).

Studies simulating classroom instruction have also established the effectiveness of Richland and colleagues' (2007) framework for mathematics teaching with analogies. In one study, Richland and McDonough (2010) performed two laboratory experiments using videotaped instruction to determine whether cognitive supports encouraged undergraduate students' transfer after instruction. The instructional focus of the first experiment was on permutation and combination problems, while the instructional focus of the second experiment was on problem-solving strategies in the context of proportionality. The mathematical concepts for both experiments were chosen because undergraduate students have been known to struggle with those concepts (e.g., Vanderstoep & Seifert, 1993).

Two conditions were part of the design: In one condition, the instruction incorporated highly-supported analogies and the second one used minimally-supported analogies. Highly-supported analogies included the visual presentation of the source, which was kept visible when compared to the target. The source and target were also spatially aligned and gestures were used to highlight the similarities between both analogs. Minimally-supported analogies incorporated the same source and target analogs, but no cognitive supports were used. The first experiment compared these two conditions in the context of comparing two problem types. The second experiment compared participants exposed to a highly-supported condition, a minimally-supported condition, and a no-analogy control group on a transfer task.

The results of the first experiment indicated that there were no differences between conditions on transfer to problems that were similar both on the surface and in structure to the problems presented during the intervention. However, participants in the highly-supported condition transferred better to structurally-similar problems that differed on the surface. The participants in the minimally-supported condition were more likely to apply the learned strategies to problems that were similar on the surface but not in structure, resulting in incorrect reasoning, possibly because of a reliance on surface features for comparison and transfer. The second experiment replicated the findings of the first experiment.

A similar study was conducted with fifth-grade students examining the degree to which cognitive supports should be used when teaching division involving both whole numbers and fractions (Richland & Hansen, 2013). All participants were instructed using the same division problems (more precisely, 24 ÷ 8 = ? as the source and $\frac{1}{3} \div \frac{1}{6} = ?$ as the target), but the level of support that was offered to the students varied. In the “high cuing” condition, the two problems were taught using all of the cognitive supports identified by Richland and colleagues (2007), except for imagery. This means that they supported the analogy using a familiar source (i.e., the whole-number problem presented first), kept the source visible when presenting the target, spatially aligned both equations on the board, and used gestures to link both analogs. In the minimal cuing condition, none of the above supports were used, and to ensure that the source was not familiar to the students, the order in which the two problems were presented in the high cuing condition was reversed. The results supported Richland and McDonough’s (2010) findings. Participants in the high-cuing condition were better at extending their learning from the analogy to transfer problems, but there was no difference between conditions on posttest problems that were highly similar to the problems presented during instruction, suggesting that using cognitive supports is possibly more effective for far transfer (complex analogies), but is less necessary for near transfer (simpler analogies).

Evidence for implicit instruction with analogies. Despite the Richland and McDonough (2010) and Richland and Hansen (2013) studies, conflicting evidence exists as to the minimum level of explicitness required to teach effectively with analogies. It is perhaps not necessary, or it could even be detrimental, to include all cognitive supports. Sidney and Alibali (2015) investigated whether students learned more about fraction division from surface-similar analogs (i.e., where the source would be a fraction addition or subtraction problem and the target a fraction division problem), or from structure-similar analogs (i.e., where the source would be a whole-number division problem and the target a fraction division problem). Using a two by two design, the authors also examined the difference between encouraging participants to make their own links between analogs and offering no such encouragements.

In the study, participants completed a worksheet either with fraction addition and subtraction problems (i.e., the worksheet contained the surface analog) or with whole-number division problems (i.e., the worksheet contained the structure analog). This activity was followed by instruction on fraction division. In the conditions without links, the initial worksheet, once completed, was collected by the researcher and removed from the participants' view for the remainder of the intervention. This reduced the possibility of making connections to the source analog, and participants were in no way encouraged to make any connections to the worksheet during the lesson. In the conditions with links, the participants kept the worksheet and the researcher asked them three times to refer to it. In this condition, the specific connections were not explicitly taught by the researcher, but the participants were encouraged to make connections if they could.

On measures of conceptual learning, all participants who received the structure analog worksheet performed better than those in the surface-analog condition, suggesting that a source analog that is more structurally similar to the target analog facilitates learning. Surprisingly, the greatest gains were achieved by participants in the structurally-similar analog condition who were not encouraged to make links. These findings stand in contrast to previous similar studies, in which participants were most successful after having been encouraged to make their own connections (Gentner et al., 2003; Osana, Adrien, & Duponsel, 2017; Rittle-Johnson & Star, 2007). Sidney and Alibali (2015) argued that in this context, encouraging the participants to make their own links, without providing any explicit supports, might have been a difficult and error-prone process.

Sidney and Alibali (2017b) further investigated the question of comparing whole-number division to measurement fraction division by creating three conditions: an implicit condition without any explicit links, an explicit condition in which explicit links were made by the teachers, and a control no-analogy condition. The study was meant to test the effects of teacher guidance with the use of cognitive supports (based on the framework of Richland et al., 2007) against an implicit condition in which students were not encouraged to make any connections between analogs.

In the implicit links condition, the instructor first presented two whole-number division problems, which were familiar to the students, followed by two fraction division problems. The whole-number division problems (source) were made visible, but were not kept visible when the fraction division problems were presented (target). Furthermore, parallel language was used for both analogs, but the instructor never explicitly explained the connections between the two analogs, nor did she invite the students to make such connections. In the explicit condition, both representations were kept visible, were spatially aligned to highlight their similarities, and the instructor used linking speech and gestures. In the control condition, no analogy was presented, but to maintain an equal amount of instruction, four fraction division problems were presented.

Surprisingly, even though the study's design (Sidney & Alibali, 2017b) closely resembled that of Richland and Hansen's (2013) study, the results diverged. In the Sidney and Alibali study, participants in the implicit-links conditions demonstrated higher conceptual learning than participants in both the explicit-links and no-analogy conditions. There were no significant differences between the conceptual performance of participants in explicit-links and no-analogy conditions. Also, participants in the explicit analogy condition had lower performance on procedural knowledge tasks than the participants in both the implicit- and no-analogy conditions. The finding that an implicit analogy supported higher conceptual and procedural gains than an explicitly-guided analogy brings into question the conditions under which cognitive supports are beneficial in instruction with analogies.

One explanation provided by Sidney and Alibali for their unexpected findings was that the lessons in the explicit condition were more complex because of the unfamiliar mathematical terms, or the visual representations contained irrelevant details. Another possible explanation comes from the theory of analogical priming (Leech, Mareschal, & Cooper, 2008), which would suggest that simply activating appropriate and related prior knowledge facilitates connections to a new concept. Thus, priming theory suggests that students can demonstrate analogical transfer without requiring the teacher to state explicitly the similarities between both analogs (Day & Goldstone, 2011; Green, Fugelsang, & Dunbar, 2006) and without additional cognitive supports (Sidney & Alibali, 2015; Sidney & Alibali, 2017b). Once students have activated a concept they have previously mastered, they are "primed," or biased, towards detecting the same concept, relation, or similarity in the target analog, without the need to map explicitly the similarities between both analogs (Day & Goldstone, 2012; Leech et al., 2008). Priming would suggest that the order in which knowledge is being activated might play an important role when teaching new concepts.

Present Study

I have described the importance of teaching fraction division conceptually, instead of solely relying on procedures, and proposed to investigate analogical reasoning as a framework for mathematics instruction. Teaching with an analogy, where the source analog is unitizing and the target is fraction division, provides a context in which the teacher has the opportunity to emphasize the connections between two analogous mathematical contexts. Supporting the structure mappings between both analogs might allow students to develop a stronger understanding of fraction division, remainders, and unitizing concepts, and the degree of support also has yet to be explored.

In my research, my objectives were to investigate whether analogies and analogical reasoning should be used as an instructional method in mathematics classrooms and to what degree teachers should support their students' structure mappings between concepts of unitizing and fraction division. In my study, participants were randomly assigned to three conditions. In the explicit-links condition, the connections between unitizing and fraction division were clearly stated by the researcher and supported using Richland et al.'s (2007) framework of cognitive supports. In the implicit-links condition, the connections between the analogs were left implicit. In the control group, no analogy was presented. The participants in the control group were only taught how to solve one fraction division problem, but to maintain the same amount of instruction as in the other two conditions, the participants in the no-analogy condition saw the same fraction division problem solved twice with no instruction on unitizing. Students' understanding of fraction division, their interpretation of the fractional remainder, and their understanding of unitizing was assessed before and after the intervention.

My specific research questions were as follows:

1. Are there condition differences in the students' understanding of fraction division, fractional remainders, and unitizing concepts after the intervention?
2. What is the nature of any changes from pretest to posttest in students' thinking about fraction division, fractional remainders, and unitizing concepts? If so, do these changes differ by condition?

Despite the few studies suggesting that implicit analogies are beneficial, the preponderance of research supports the use of explicit analogies in instruction (e.g., Christie & Gentner, 2010; Matlen et al., 2014; Richland et al., 2007; Richland & McDonough, 2010), which allowed me to make the following predictions for participants' understanding of fraction division, their interpretations of fractional remainders, and their understanding of unitizing. Because the emphasis of both the explicit- and implicit-links conditions was on participants' conceptual understanding of fraction division in relation to unitizing and because of the research that supports making connections explicit in mathematics instruction, I hypothesized that the performance in both these conditions would differ at posttest from the control condition on all measures. I also predicted that the participants in the explicit-links condition would demonstrate significantly higher scores following intervention than participants in the implicit-links condition because of the above-mentioned literature supporting the use of cognitive supports when teaching with analogies. The students in the control condition would not perform as well as those in the analogy conditions because the focus of instruction would be solely on solving a fraction division problem and not on the conceptual connections between unitizing and fraction division.

Method

Participants

Before data collection began, I provided the schools' principals with information about the study, and they introduced me to seven teachers who were interested in participating in the project. I met with the teachers to present the project in greater detail. Letters about the project were then sent home to the parents of the students in the participating classes. With the letter, there was a consent form for the parents to sign. Once the participants were identified, I asked them to sign an assent form before any data were collected. Only those students who gave their assent participated. The consent and assent forms are in Appendix A.

The study was conducted with seventh-grade students from two high schools in a Montreal metropolitan area school board ($N = 51$). The only inclusion criterion was for participants to have entered seventh-grade in September 2017. The participants in the sample were 37% female and 63% male. Their ethnicity data are presented in Table 2. The mean age of participants was 13 years and one month. The youngest participant was 12 years and four months old, while the oldest participant was 13 years and nine months old.

Table 2

Frequencies and Percentages of Participants' Ethnicities in Sample

Ethnicity	Frequency (Percentage of Sample)
White	27 (53%)
Black	8 (16%)
Black/White	2 (4%)
Asian	2 (4%)
Caribbean	1 (2%)
Black/Caribbean	1 (2%)
Black/White/Caribbean	1 (2%)
Black/Latin American	1 (2%)
White/Latin American	1 (2%)
Middle Eastern	1 (2%)
Aboriginal/Other	1 (2%)
Other	3 (6%)

Note. The total percentage adds up to 101% because of rounding. The categories are mutually exclusive, because new categories were created for participants who selected more than one ethnicity on the demographic questionnaire.

Design

The design of the study, including the three conditions and assessment tasks, are presented in Figure 5. In a three-group pretest-posttest experimental design, participants were assessed on their conceptual and procedural knowledge of fraction division, fractional remainders, and unitizing concepts before and after the intervention.

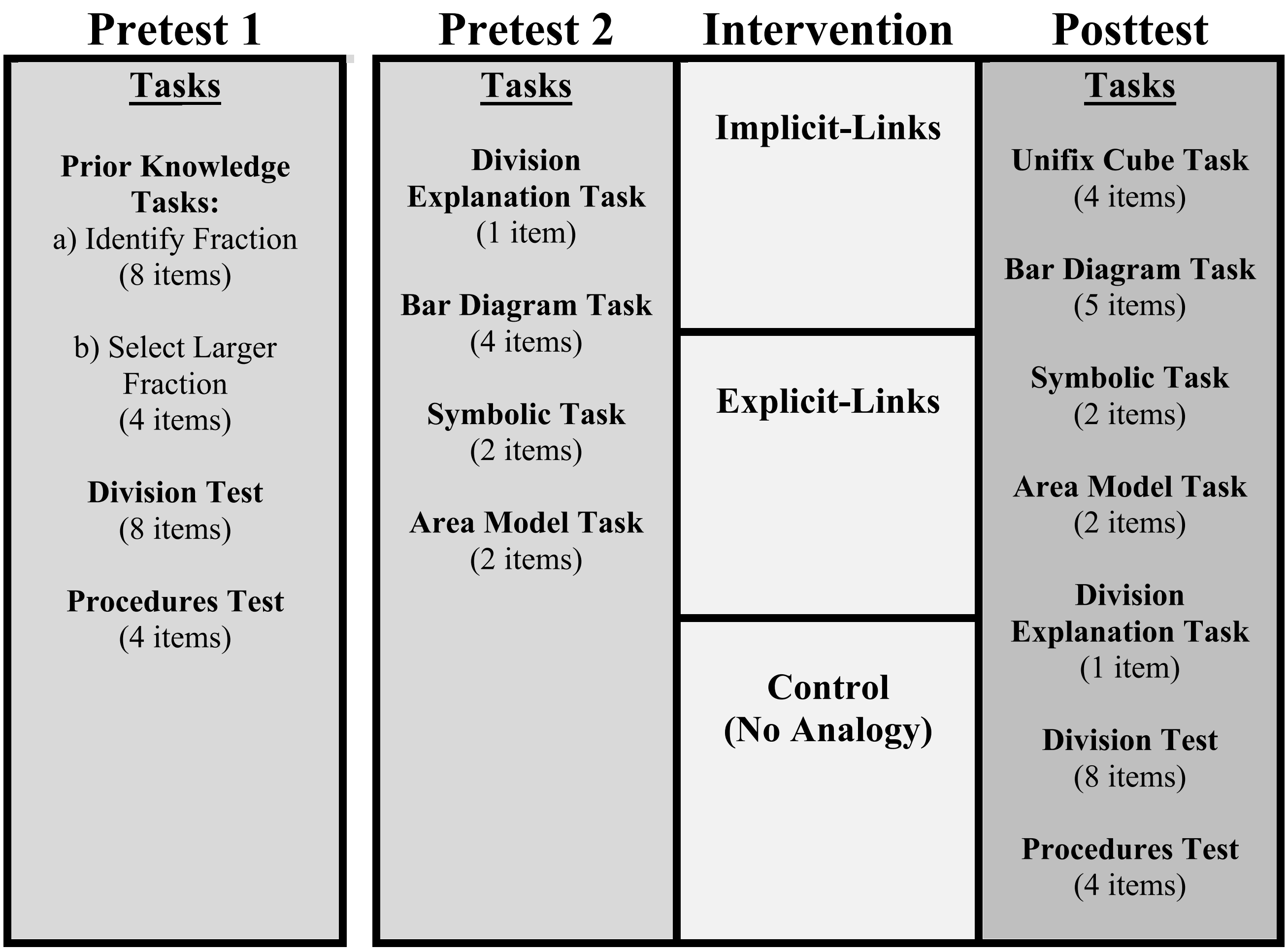


Figure 5. Study design including all assessment tasks and intervention conditions.

For the instructional intervention, participants were randomly assigned to one of the three conditions: (a) implicit-links, (b) explicit-links, and (c) control. In the implicit-links and explicit-links conditions, the participants received instruction on unitizing in the context of measurement fraction division. The conditions differed according to how explicit the links were between two analogs, which consisted of a unitizing problem and a measurement fraction division problem.

In the implicit-links condition, I delivered instruction on unitizing, followed by a demonstration of how to solve a measurement fraction division problem. The objective of this condition was to activate appropriate prior knowledge related to unitizing, but leave the connection between unitizing and division implicit (based on the analogical priming framework by Leech et al., 2008). In this condition, I did not explicitly identify the conceptual mappings between the two analogs (unitizing and measurement fraction division).

In the explicit-links condition, I delivered instruction on unitizing and fraction division identical to the one in the implicit-links condition, but here, I explicitly mapped the conceptual similarities between unitizing and fraction division for the students, applying the framework of cognitive supports for instructional analogies described by Richland et al. (2007). In this condition, I explicitly supported the conceptual mappings between both analogs, not only through verbal statements highlighting conceptual similarities, but also through the use of a number of cognitive supports that have been previously addressed in the literature. The cognitive supports I used from Richland et al. (2007)'s framework were familiar representations, simultaneous visual presentation of both analogs, spatial cues highlighting conceptual similarities, and linking gestures. Additionally, I referred to relational language to make the connections between both analogs more explicit (Vendetti, Matlen, Richland, & Bunge, 2015).

In the control group, I presented one fraction division problem twice to the participants and I did not deliver a lesson on unitizing; as such, there was no possible analogy between unitizing and measurement fraction division. Since no analogy was presented, no explicit supports were used either. If two different division problems had been presented to the participants, both problems could have been considered analogs to each other because they would have shared the same conceptual structure despite having different numbers in their equations. Repeating the same fraction division problem twice removed the possibility of participants making implicit links between the two cases, and ensured that participants in each condition received the same amount of instruction.

Instructional Intervention

Three different lessons were video-recorded, one for each condition (for the transcripts of each lesson, please refer to Appendix B). Unifix cubes were used as a visual support during the instruction in all three conditions. Unifix cubes are concrete objects that can snap together and be taken apart, and are often used in elementary mathematics classrooms to provide visual and concrete reifications of abstract concepts. Previous research has recommended using length models, such as number lines or bar diagrams, to represent fraction division and remainders because they more easily direct attention to units (Hackenberg, 2013; Steffe, 2002). Unifix cubes can be used to form towers to represent wholes (i.e., twelve towers of four cubes to represent the number 12). Such "tower-like" representations, lying flat on their sides, provide a perspective of length to each whole and can be physically broken apart when they need to be regrouped in both the unitizing and division contexts.

In the implicit-links conditions, the lesson began with the presentation of a unitizing activity. In the video, the instructor placed 24 cubes snapped together into six towers (see Panel A in Figure 6). Each tower, lying flat on a table, was made out of four connected Unifix cubes and the distance between each Unifix tower was about one inch. The instructor stated that these six towers represent the numerical quantity of six: "Here is six: one, two, three, four, five, six (point to each tower while counting)." The instructor then explained that each tower is composed of four fourths by saying: "Each whole has four fourths. One, two, three, four fourths (point to each cube while counting), so this is a whole made up of four fourths. The same goes for each of these (make a swooping pointing gesture to remaining five towers)."

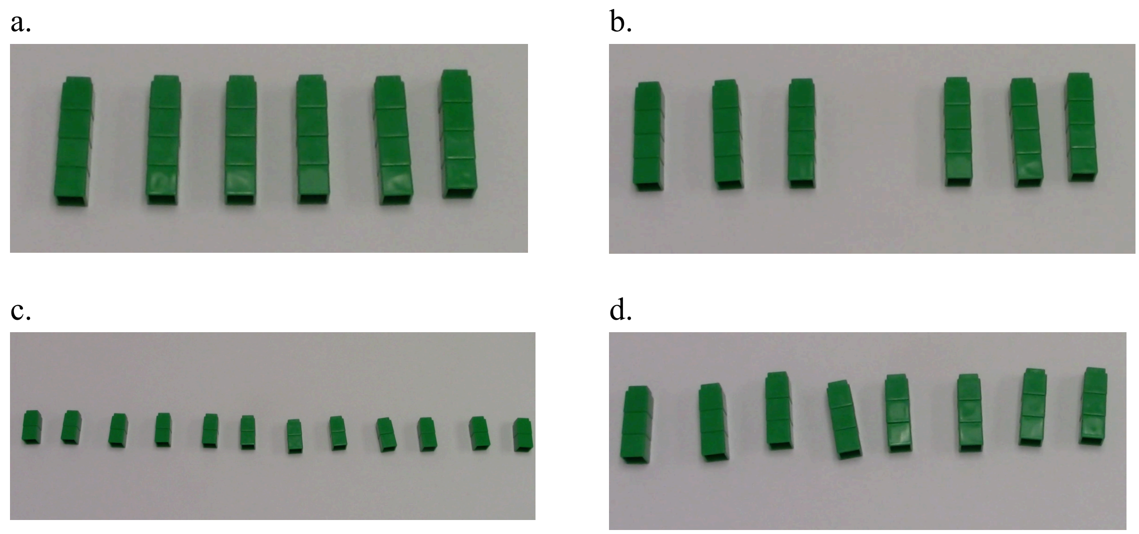


Figure 6. Screenshots of the configuration of the manipulatives used in the unitizing instruction in both the implicit-links and explicit-links conditions. The first photo shows how six can be represented as six groups of one (a). The second photo shows how six can be represented as two groups of three (b). The third photo shows how six can be represented as 12 groups of $\frac{1}{2}$ (c). The fourth photo shows how six can be represented as eight groups of $\frac{3}{4}$ (d).

The instructor then explained that the collection of six towers can be thought of in different ways: as one group of six, as two groups of three, as 12 groups of half, and as eight groups of three fourths (see Panels B, C, and D in Figure 6). The Unifix cubes were rearranged in each case to illustrate the above-mentioned groups. For example, when saying that six can be thought of as two groups of three, the six towers were physically rearranged into two separate groups of three towers with about a three-inch distance between both groups. Once these four different ways of regrouping six were presented, all the Unifix cubes were removed.

The second phase of the implicit-links intervention entailed demonstrating how to solve a measurement fraction division problem. The instructor in the video presented the equation ($4 \div \frac{3}{4} = ?$) on an index card and read it out loud: "Let's see what four divided by ¾ is." The instructor then brought out four towers to represent the dividend, each tower made up of four stacked Unifix cubes (see Panel A in Figure 7). The instructor explicitly stated that each tower represents one whole. The instructor also mentioned that each of these wholes is composed of four fourths and pointed to each fourth, counting them. The exact words used by the instructor were: “Each whole has four fourths. One, two, three, four fourths (points to each cube while counting), so this is a whole made up of four fourths. The same goes for these (make a swooping pointing gesture to remaining three towers).” The Unifix cubes used in this phase of the intervention were of a different color (red) than the ones used in the first phase (green).

a. b.

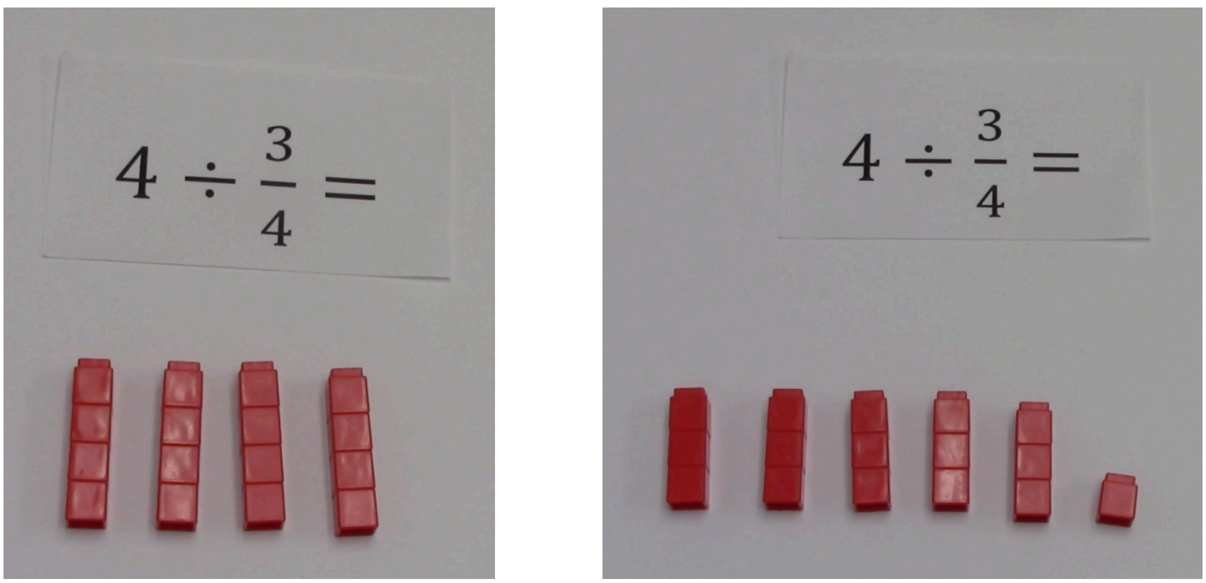


Figure 7. Screenshots of the materials used in the fraction division instruction in both the implicit-links and control conditions. The left panel shows four Unifix cube towers as representing the dividend prior to solving the problem $4 \div \frac{3}{4} = ?$ (a). The right panel shows the cubes representing the quotient of the problem: five groups of $\frac{3}{4}$ and $\frac{1}{3}$ of a group of $\frac{3}{4}$ (b).

The instructor then began solving the division problem by saying, “I want to find out how many groups of $\frac{3}{4}$ there are in four. Let’s see.” The instructor created groups of three cubes from the original set of cubes (“one, two, three… Here is one group of $\frac{3}{4}$”). The resulting rearrangement was five towers made up of three cubes with one cube remaining (see Panel B in Figure 7). To explain the fractional remainder and its interpretation as $\frac{1}{3}$ to the participants, the instructor said, “Here I have one piece out of the three I need to have a full group of $\frac{3}{4}$. I have $\frac{1}{3}$ of a group of $\frac{3}{4}$. This means that there are one group, two groups, three groups, four groups, five groups of $\frac{3}{4}$ and $\frac{1}{3}$ of a group of $\frac{3}{4}$ (gesturing to each group while counting). The answer to four divided by $\frac{3}{4}$ is $5\frac{1}{3}$.” The answer to the problem was only said out loud. It was not written down on the index card on which the equation was initially presented.

In the explicit-links condition, the same unitizing and fraction division problems were presented as in the implicit-links condition. The differences in instruction were in the cognitive supports used. The instructor first presented the unitizing problem, using the same script and materials as in the implicit-links condition. Once the unitizing activity was completed, the final set of green Unifix cubes (groups of $\frac{3}{4}$) was kept visible for the participants (see Figure 8). Then, the instructor presented the fraction division problem, using the same red cubes as in the implicit-links conditions.

a. b.

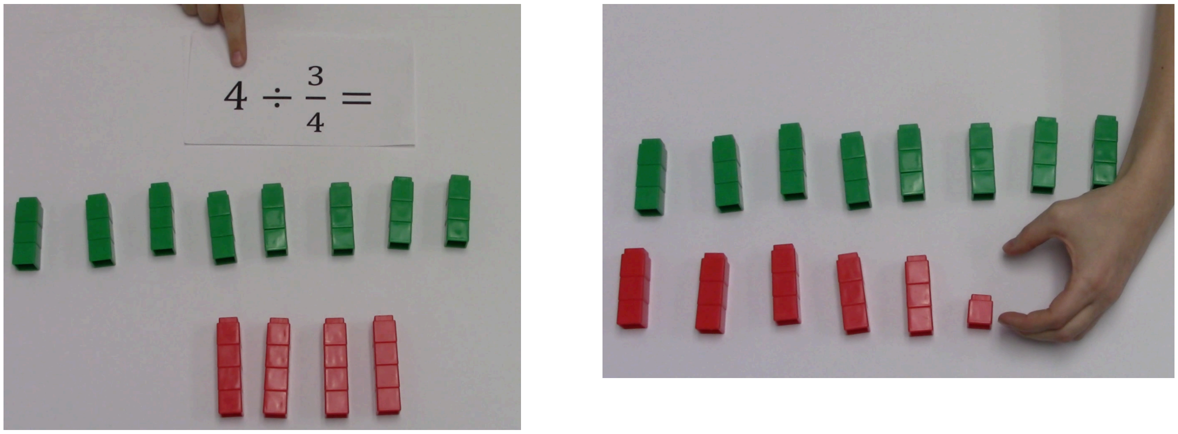


Figure 8. Screenshots of the configuration of materials used in the fraction division instruction in the explicit-links condition. The left panel shows how the dividend, represented by four red Unifix cube towers, is aligned with the last unitizing set (a). The right panel shows how the quotient of the problem, 5 groups of $\frac{3}{4}$ and $\frac{1}{3}$ of a group of $\frac{3}{4}$, is spatially aligned with the last unitizing set (b).

Throughout the part of the lesson on fraction division, the instructor referred back to the unitizing lesson using relational language. She began by saying, "Fraction division can be thought of in the same way as regrouping the number six based on different sized groups. Here is the six made out of groups of three fourths. Here are my four wholes, just like here I had the six wholes to start." The instructor made explicit connections like the ones above using relational language and made linking gestures between both representations. She provided visual supports by spatially aligning the elements of the fraction division problem to the unitizing set by arranging both the fraction division and unitizing Unifix cube sets horizontally one under the other and by keeping both sets visible throughout the explanation.

In the control condition, one fraction division problem was presented two times. The problem was identical to the one from both the implicit-links and explicit-links conditions (i.e., 4

$\div \frac{3}{4}$ = ?). Also, the instructor used the Unifix cubes in the same way as in the implicit-links condition and delivered an identical script. The instructor did not explain any unitizing concepts and did not provide the participants with any explicit supports.

Measures

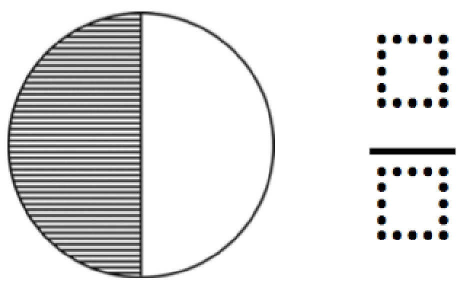
Participants' initial understanding of fraction division was assessed using a two-part pretest. Part 1 was designed to assess both conceptual and procedural knowledge of division. The first part of the pretest was a paper-and-pencil test composed of the prior knowledge tasks (Identify Fraction Task and Select Larger Fraction Task), the Division Test, and the Procedures Test. Part 2 was delivered in individual meetings with each participant, and included assessment tasks designed to assess the participants' conceptual understanding of fraction division, their interpretation of the fractional remainder, and their knowledge of unitizing. Posttest tasks were delivered in individual meetings following the intervention, and were designed to assess participants' thinking about division, remainders, and unitizing.

Prior Knowledge Tasks

Identify Fraction Task and Select Larger Fraction Task. The participants' prior knowledge was assessed at pretest with two tasks. First, they were asked to write fractions that corresponded to presented area models (for an example of this Identify Fraction Task, see Figure 9). This task assessed participants' conceptual understanding of part-whole interpretations of fractions. On a second task, participants selected the larger fraction between two symbolic fraction representations (e.g., $\frac{1}{4}$ or $\frac{1}{2}$) on four individual items (for an example of the Select Larger Fraction Task, see Figure 10). This task assessed participants' conceptual understanding of symbolic representations of fractions. Items on both tasks were assigned 1 point for a correct answer and 0 points for an incorrect answer, for a maximum of eight points on the Identify

Fraction Task and a maximum of four points on the Select Larger Fraction Task. The Identify Fraction score was the total number of points divided by the number of items on the task, resulting in percent scores. The Select Larger Fraction score was computed in the same way.

Question 1
For each picture below, write a fraction to show what part is shaded.



1.

Figure 9. Example of an item on the Identify Fraction Task.

Question 2
For each question below, circle the fraction that represents the larger amount.

1.

$$\frac{1}{2} \qquad \frac{3}{4}$$

Figure 10. Example of an item on the Select Larger Fraction Task.

Division Test

The participants' conceptual understanding of fraction division was assessed with the Division Test, an eight-item multiple-choice paper-and-pencil test. Two items on this test included whole-number division problems. One of these items had a solution with no remainder (20 ÷ 4 = ?) and one had a solution with a fractional remainder (57 ÷ 8 = ?). The remaining six items were composed of fraction division problems involving a whole number dividend and a fractional divisor, including both unit ($3 \div \frac{1}{6} = ?$; $10 \div \frac{1}{3} = ?$, $12 \div \frac{1}{5} = ?$) and non-unit fractions ($5 \div \frac{6}{10} = ?$; $20 \div \frac{4}{5} = ?$, $6 \div \frac{3}{10} = ?$). For each problem, the participants selected one response from four choices. The choices were in the form of word problems: one correct answer and three distractors. Participants were instructed to select the word problem that best represented the symbolic expression provided, but were not asked to solve the problems.

For each item, the correct response was a word problem that incorporated either partitive or measurement division. Half of the correct responses on the test had a partitive division context and the remaining half had a measurement division context. For whole number division items (for example, 20 ÷ 4 = ?), a word problem with a partitive division structure would be, “There were 20 cupcakes at the party. If four children share all the cupcakes, how many cupcakes will each child eat?” A word problem with a measurement division structure for the same equation would be, “There were 20 cupcakes at the party. If each child at the party ate exactly four cupcakes, and there were no more cupcakes left, how many children were at the party?” One of the three distractors was a word problem reflecting subtraction instead of division (e.g., There were 20 cupcakes at the party. Julianna ate four cupcakes. Now, how many cupcakes are left?). The second distractor involved inversing the quantities of the dividend and divisor (e.g., There are four cupcakes and 20 children. If shared equally, how much of a cupcake will each child get?). The final distractor involved multiplying by the value of the divisor instead of dividing (e.g., There are 20 children at the party. If each child eats four cupcakes, how many cupcakes were there altogether?).

Fraction division items had another set of distractors. This time, the participants chose between a correct answer and three incorrect word problems with structures that mirrored typical incorrect strategies used to model or solve fraction division operations (Sidney & Alibali, 2017a;

Siegler & Pyke, 2013). For example, for the fraction division item $3 \div \frac{1}{6} = ?$, the first distractor involved multiplying by the value of the divisor, instead of dividing (e.g., There are three children at the party. If each child ate $\frac{1}{6}$ of the cake, how much of the cake did they eat altogether?). The second distractor was based on multiplying by the reciprocal of the divisor (e.g., There are three children at the party. If each child eats six pieces of a cake, how many pieces of the cake did they eat altogether?). The final distractor required dividing by the reciprocal of the original divisor, thereby conflating the standard procedures for multiplication and division (e.g., There are three pieces of cake left. If six children share the remaining three pieces of cake, how much of a cake will each child eat?). Correct responses were assigned 1 point and incorrect responses 0 points, for a maximum of eight points on the Division Test. Each participant's score on the Division Test was calculated by summing the total number of points and dividing it on the task, resulting in percent scores.

Procedures Test

The Procedures Test assessed the participants' knowledge of division procedures. The Procedures Test included four division problems presented using standard symbolic notation. The items included one whole-number division ($27 \div 3 = ?$), one whole-number division with a fractional quotient ($50 \div 8 = ?$), and two fraction division problems, one with a whole number as dividend and a unit fractional divisor ($4 \div \frac{1}{3} = ?$), and the other with a whole number dividend and a non-unit fractional divisor ($5 \div \frac{2}{7} = ?$). The students could use any strategy to solve the problems and were asked to show their work. Correct answers were assigned 1 point and incorrect answers 0 points, for a maximum of 4 points on the Procedures Test. If a participant left an item blank, he or she would receive 0 points for that specific item. Each participant's score on the Procedures Test was calculated by summing the total number of points and dividing it by the number of items on the task, resulting in percent scores.

Division Explanation Task

The second part of the pretest took place in individual meetings with the participants, who were assessed on their understanding of concepts related to fraction division, fractional remainder, and units. On the Division Explanation Task, participants were asked to answer the question, "What does it mean to divide something by a fraction?" The coding rubric for participants' responses to this task is presented in Table 3.

Table 3

Definitions and Examples of Fraction Division Interpretations in the Division Explanation Task

Interpretation	Definition	Example
Fraction Division		
Measurement Interpretation	When dividing by a fraction, the participant is trying to find the number of times the fractional divisor can fit in the dividend.	"Let's say you have a dollar, you can divide it by quarters, like 1 over 4 and you can see how many quarters you need to make a dollar."
Partitive Interpretation	When dividing by a fraction, the participant is looking for the total amount of one group, while knowing the value of part of a group.	"I am looking for what the total amount of money I have, if a dollar is a quarter of my amount."
Whole Number Division		

Measurement Interpretation	The participant changes the fractional divisor into a whole number. In this new context, when dividing by a whole number, the participant is trying to find the number of times a whole number divisor can fit in the dividend.	“I want to find out how many people I can feed with 10 pizzas if each person eats 2 pizzas.”
Partitive Interpretation	The participant changes the fractional divisor into a whole number. In this new context, when dividing by a whole number, the participant is trying to find the size of a group, knowing the number of groups a total amount is being divided into.	“I want to know how much of a pizza will each person get if I share 5 pizzas with my three friends.”
General Division		
Measurement Interpretation	The participant provides a general measurement division explanation without mentioning whole numbers or fractions.	“I want to see how many equal sections of a certain amount will enter in my whole number”
Partitive Interpretation	The participant provides a general partitive division explanation	“I want to share an amount of food equally with my friends.”

	without mentioning whole numbers or fractions.	
Partitioning Approach		
Splitting or Grouping	The participant mentions the actions of splitting, cutting, sharing, breaking, separating, or making parts without providing any more detail. There is no mention of whole numbers or fractions. It is unclear whether the action of splitting results in equal groups. It is unclear whether the participant is looking for the number of groups (measurement division) or for the amount in each group (partitive division) following the partitioning action.	"To like split it into what it is being divided by…" "to cut it into different groups"
Other Codes		
Fraction seen as a Quotient	The fraction itself is defined as the quotient of the numerator divided by the denominator.	"It means to divide the numerator by the denominator to give you the answer."

Fraction seen as Part of a Whole	A fraction is described as a part of a whole.	"A fraction is a slice of a whole pizza."
Fraction seen as a Decimal or Percentage	A fraction is considered the same as their equivalent decimal number or percentage.	"If I have $\frac{4}{5}$, I'm getting a decimal of 0.8 or 80%."
Procedural Approaches	The participant describes a procedure when explaining what it means to divide by a fraction. It can be the standard division algorithm, an invented algorithm or even an incorrect one. The participant mentions the inverse relationship of division and multiplication within the use of a procedure.	"keep, switch, flip, multiply" "It means you have to inverse and multiply."
Multiplication	The participant explained multiplication instead of division with fractions.	"I am looking for half of a third." "I am looking for how much I will have if I have 3 groups of ¾."

Subtractive Understanding of Division	The participant sees division as a repeated subtraction of the value of the divisor. The participant can also describe a subtraction problem instead of a division.	“It’s when I remove a piece of a whole to give me the answer.” “I take the fraction and I take it away from the whole as many times as I need until I have nothing left.”
Finding the Simplest Form	The participant describes division as finding the simplest form of a fraction. For example, 3/6 would be ½ in its simplest form.	“I need to divide the numerator and the denominator by the same number to get a fraction that is easier to work with.”
Misconceptions about Fractions	The participant described a misconception about fractions, instead of explaining the meaning of fraction division.	“in a fraction, the denominator is always bigger and the numerator is always smaller”
Did Not Know	The participant did not know what dividing by a fraction meant.	“I don’t know… I really don’t.”
Vague or Unclear Explanation	The explanation was vague or unclear.	

As indicated in Table 3, participants' responses were coded using the following categories: (a) measurement division with fractions (i.e., "I want to see how many groups of the specific fraction will enter into the initial amount"), (b) partitive division with fractions (i.e., "I want to see how much is in one full group, if a fraction of a group and the total amount is known"), (c) measurement division with whole numbers (i.e., "I want to see how many groups of two there are in six"), (d) partitive division with whole numbers (i.e., "I want to distribute six muffins equally among my three friends"), (e) general measurement division (i.e., "I want to know how many groups of a certain amount go into a total amount"), and (f) general partitive division (i.e., "I want to share a certain amount into equal groups"). I also developed two categories of codes for responses that were conceptually less sophisticated. The first category was for general descriptions of partitioning, which were observed when participants referred to the action of splitting or grouping (i.e., "I am splitting an amount"). The second category was called Other and included a variety of responses, including definitions of fractions instead of definitions of division with fractions, procedural explanations, and also incorrect or unclear explanations of division.

Some of the codes in Table 3 were anticipated prior to data analysis, including interpreting division using the measurement approach, interpreting division using the partitive approach, and referring to procedures when describing division. All other codes were generated from the data while coding. To analyze the data, I reorganized the codes from Table 3 into higher-level categories based on two scenarios. In the first scenario, I looked at whether participants referred to division in quantitative or non-quantitative ways. The participants' responses were placed into the following categories: (a) division with fractions, (b) division with whole numbers, (c) division with non-specified quantities, and (d) other non-quantitative answers, regardless of whether participants referred to measurement or partitive division.

In the second scenario, I looked at the type of division interpretation the participants used when explaining division with fractions. The participants' responses were placed into the following categories, regardless of whether they referred to division in quantitative or non-quantitative ways: (a) measurement division, (b) partitive division, (c) measurement and partitive division, (d) general division, (e) splitting or grouping, and (f) other.

Bar Diagram Task

The participants were also assessed on their understanding of the fractional remainder using the Bar Diagram Task (inspired by Barnett-Clarke, Fisher, Marks, & Ross, 2010). At pretest, the task had four items. Two tasks involved dividing one by a non-unit fraction and the other two tasks involved dividing a whole number larger than one by a non-unit fraction.

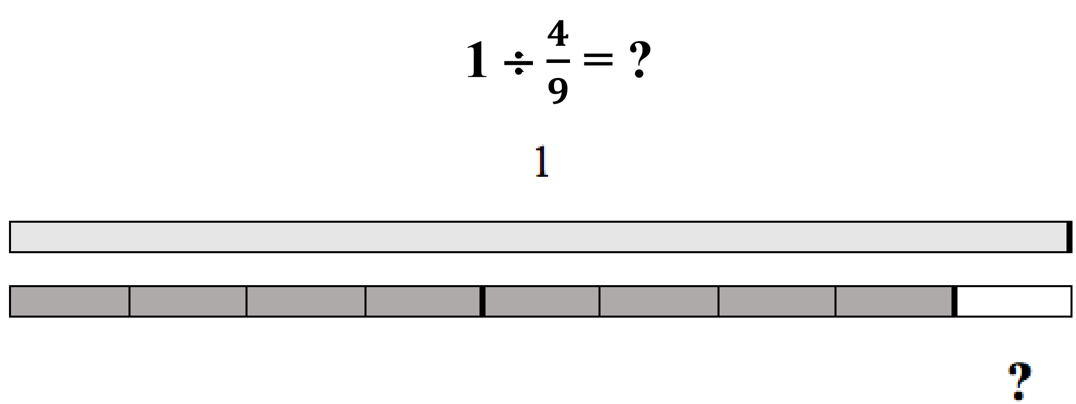


Figure 11. Example of a Bar Diagram Task.

On each item, the participant was presented with two bar diagrams, one representing the dividend and one representing the partitioned dividend in such a way that the divisor was illustrated (see Figure 11). For instance, for the equation ($1 \div \frac{4}{9} = ?$), the top bar represents "1," and the bottom bar is separated into three sections. The first two sections in grey on the bottom bar represent one full group of $\frac{4}{9}$ each, and the smaller section in white represents the remainder. The researcher read the question, "In this problem, Eric wanted to divide 1 by $\frac{4}{9}$. He drew the first bar to represent one. Then, he looked at how many groups of $\frac{4}{9}$ would go into one. He found two groups of $\frac{4}{9}$, but then he did not know what to do anymore. Help Eric figure out what the last piece on the bar represents." The researcher gestured to the whole, the groups created by the divisor, and then the white bar representing the remainder as she read through the problem.

Coding of participants' thinking about the fractional remainder at pretest and posttest were based on the type of unit the participant used to interpret the fractional remainder, as seen in Table 4 in Appendix C. Apart from the referent-unit and original-unit interpretations, which have been previously identified in research (e.g., Lambert & Wiest, 2015), the rest of the codes were developed based on the data provided by the participants. For instance, if for the equation $1 \div \frac{4}{9} = ?$, the participant identified the remainder as $\frac{1}{9}$, he or she would have incorrectly interpreted the remainder by using the original unit instead of the referent unit. If the participant identified the remainder as $\frac{1}{4}$, they would have correctly identified the remainder with the use of the referent unit. Some participants described a remainder using both the referent and original units (i.e., the remainder is $\frac{1}{9}$ and $\frac{1}{4}$ of my group of $\frac{4}{9}$). The participants could identify the remainder using an actual fractional amount as seen in the above examples (e.g., $\frac{1}{4}$ or $\frac{1}{9}$) or describe the remainder as part of a unit, without mentioning a specific fractional amount (e.g., I am missing a few more pieces to have another full group of $\frac{4}{9}$). Incorrect codes were identified in participants' answers as well, including, among others, interpreting the remainder using a new unit (e.g., the dividend is considered as the unit); not assigning a numerical value to the remainder, but referring to it as a “leftover” or “a missing piece”; demonstrating a whole number bias where the remainder was not a part of a group, but a whole group by itself; using a procedure to solve the equation; or concentrating on the superficial features of the remainder (e.g., the size or color of the bar).

Participants’ thinking about the remainder were assessed by collapsing categories seen in Table 4 in Appendix C into higher-level categories: All sub-codes related to the referent unit were collapsed into a general Referent category, all sub-codes related to the original unit were collapsed into an Original category, and finally all codes that did not refer to referent or original units were placed into the Other category. In some cases, the participants’ responses reflected thinking about both the referent and original units. I used one code, Referent or Original, for these responses according to the following criteria: (a) if the remainder based on the referent was expressed as a fractional quantity, regardless of whether the original unit was expressed as a fraction or simply identified more generally as part of an original unit, the response was placed in the Referent category, and (b) if the original unit was expressed as a fraction, but the referent unit thinking was expressed more generally as part of a referent group and not an explicit fraction, the response was placed in the Original category. For each participant, the Referent score was the number of times each participant interpreted the remainder using the Referent category, divided by the number of items. Original scores and Other scores were computed for each of the Original and Other categories in the same way.

Symbolic Task and Area Model Task

The final pretest tasks, the Symbolic and Area Model Tasks, assessed if participants could justify whether two quantities were equivalent by composing or decomposing the fractional amounts using different units. These two tasks were designed to assess participants’ understanding of unitizing, which entails either decomposing a fractional amount into unit fractions or composing a whole from unit fractions. These decomposing and composing actions are central to unitizing because when quantities are composed or decomposed, they are being regrouped into a certain amount by referring to a new unit (e.g., one group of $\frac{3}{3}$ can also be seen as three groups of $\frac{1}{3}$).

In the Symbolic Task, participants were asked to compare two fractional quantities presented using standard written notation. The fractional amounts in the first item were equivalent to each other ("Is $\frac{5}{3}$ the same as $1\frac{2}{3}$?") and not equivalent in the second item ("Is $\frac{5}{4}$ the same as $1\frac{3}{4}$?"). The Area Model Task had the same structure, but instead of providing equations to the participants, the two items presented the quantities using area models. In each item, one area model represented a certain number of unit fractions and the second area model represented a certain number of wholes and unit fractions (see a sample item in Figure 12). The participants were asked whether both area models represent the same amount or not: "Is this [point to first fractional representation in Panel a] the same as this [point to second fractional representation in Panel b]?"

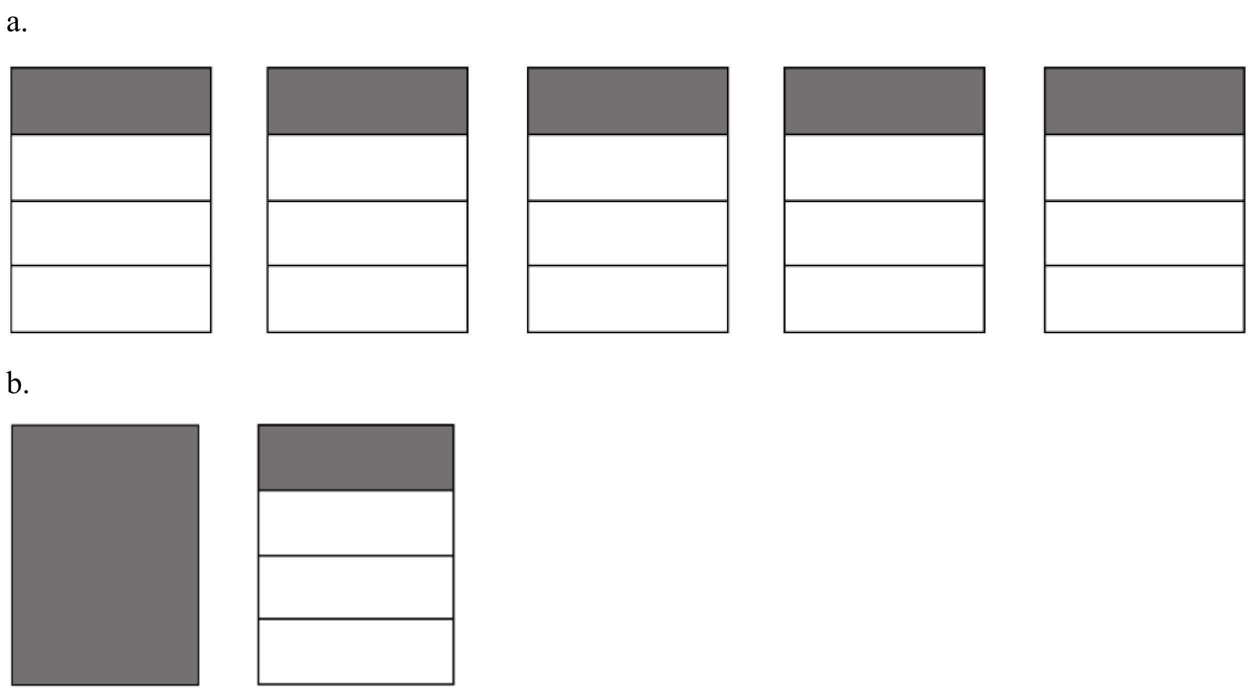


Figure 12. An example of an Area Model Task.

For the Symbolic Task, which presented fractions using written notation, participants' answers were placed in three different categories: (a) characterizing unitizing by composing and/or decomposing wholes (e.g., "$\frac{5}{3}$ is the same as $1\frac{2}{3}$, because there are five thirds in total, three of which together compose a whole with two thirds remaining"), (b) using a correct procedure (e.g., "$\frac{5}{3}$ is the same as $1\frac{2}{3}$, because ((3x1) + 2) is 5, so 5 placed over 3 is $\frac{5}{3}$"), and (c) using an incorrect procedure (e.g., "$\frac{5}{3}$ is not the same as $1\frac{2}{3}$, because 3 x 2 is 6, but I also have the 1, so it is even more").

The Area Model Task was also coded with the same category of composing or decomposing wholes with a shift of unit was used to code participants' understanding of unitizing in the area model items. Two new codes were created to code this specific task. The Joining Strategy code was used for the case where a participant would use joining and separating strategies, but with no evidence of a shift in unit (e.g., "Here I have five colored pieces, and here

I also have five colored pieces, so they must be the same."). All other answers were coded as Other Strategy (e.g., referring to superficial features like color or shape, incorporating misconceptions about fractions in their responses, using different units to describe each fractional amount, or presenting an unclear explanation).

The Composing/Decomposing score for the Symbolic Task was the number of times the participant used composing or decomposing with a unit shift. These sums were divided by 2 to obtain percent scores. The same process was used to compute Correct Procedure and Incorrect Procedure scores. Scores for Composing/Decomposing, Joining Areas, and Other Strategy on the Area Model Task were computed in the same way.

Posttest

The posttest was composed of tasks that were isomorphic to some administered at pretest and one task not administered at pretest. The isomorphic tasks included the Division Explanation Task (one item), the Bar Diagram Task (five items), the Symbolic Task (two items), the Area Model Task (two items), the Division Test (eight items), and the Procedures Test (four items). On the Bar Diagram Task, there was one item that was not isomorphic to any item on the pretest. It consisted of dividing a mixed fraction by a non-unit fraction (i.e., $2\frac{1}{3} \div \frac{5}{6} = ?$). The participants had not been exposed to mixed fraction dividends, neither at pretest nor during the intervention. Each of these tasks was assessed using the same requirements as at pretest.

The new task administered at posttest was the Unifix Cube Task. Participants' interpretations of the fractional remainder were assessed at posttest by combining performance on the Unifix Cube Task and a version of the Bar Diagram Task that was isomorphic to the one administered at pretest.

Unifix Cube Task. The Unifix Cube Task consisted of four items and required the participants to interpret the remainder of four different equations ($4 \div \frac{3}{7} = ?$, $5 \div \frac{4}{5} = ?$, $7 \div \frac{2}{3} = ?$, and $6\frac{1}{2} \div \frac{5}{6} = ?$). These four items were presented by the researcher using the same script as for the Bar Diagram Task.

For each item, the researcher displayed the equation on a card and used a set of Unifix cubes to solve the problems. The Unifix cubes were organized into towers of stacked cubes ahead of time by the researcher to represent the quantities required in each problem. For example, the number of Unifix cube towers reflected the equation's dividend and the number of cubes in each tower was determined by the denominator of the equation's divisor. For the first equation ($4 \div \frac{3}{7} = ?$), the participants were shown four towers each made up of seven stacked Unifix cubes. To introduce this first item, the researcher said, "Amy wanted to do four divided by $\frac{3}{7}$" and placed the four Unifix cube towers in front of the participant. The researcher continued, "She first represented four using these blocks. Then, she looked at how many groups of $\frac{3}{7}$ would go into four (pointed to the dividend and divisor in the equation)". Here, the researcher separated the initial four towers composed of seven Unifix cubes each into nine smaller towers composed of three cubes each to represent the division. The researcher continued, "She found nine groups of $\frac{3}{7}$. One, two, three, four, five, six, seven, eight, nine (pointed to each newly-created tower of three cubes while counting). But then, Amy didn't know what to do with this last piece (pointed to the remaining Unifix cube). Can you tell me what this piece represents?" The same script was used to present the remaining three items.

The participants' answers were coded using the same rubric as for the Bar Diagram Task. Distinctions among the participants' interpretations of the fractional remainder were made based on the type of unit used. To evaluate participants' interpretations of the remainder at posttest, the data on the Unifix Cube Task and on the Bar Diagram Task were combined. The score for each category of interpretations (i.e., referent-unit interpretation, original-unit interpretation, and other interpretation) was calculated by summing the points on all nine items (four from the Unifix Cube Task and five from the Bar Diagram Task at posttest) and dividing by the number of items (i.e., nine).

Procedure

Part one of the pretest was a 28-minute paper-and-pencil test delivered to the participants in their mathematics classrooms. The paper-and-pencil test was composed of the two prior knowledge tasks (Identify Fraction and Select Larger Fraction Tasks), the Division Test, and the Procedures Test. Participants were given one minute and a half to answer the Identify Fraction Task and one minute to complete the Select Larger Fraction Task. For the eight items on the Division Test, the researcher instructed the participants to turn the page to the next item every three minutes. Finally, the participants were given one minute and a half to work on the four items on the Procedures Test, which were all presented on one page.

Part two of the pretest was delivered in an individual interview with each participant. The meetings were videotaped, but the camera was angled so that the participants' faces were not visible. At the beginning of the interview, the researcher asked a few demographic questions to obtain general information about the participants, namely date of birth, gender, and race/ethnicity. Then, the participant was presented with the Division Explanation Task, the Bar Diagram Task, the Symbolic Task, and the Area Model Task. Once the participant completed each item, the researcher asked the participant to explain his or her reasoning out loud, using specific prompts, described below.

After these three tasks, the researcher showed the student the instructional intervention video associated with his or her condition. The intervention was presented on an iPad, and the researcher did not provide feedback. Once the participant had watched the video, the iPad was removed. After the participants viewed the video, the researcher administered a 30-minute posttest beginning with the Unifix Cube Task. This was followed by a series of tasks that were isomorphic to six of the tasks given at pretest. These isomorphic tasks included the Bar Diagram Task, the Symbolic Task, the Area Model Task, the Division Explanation Task, the Division Test, and the Procedures Test.

There were no time limits placed on any of the tasks delivered individually to the participants. On all individually-delivered assessments, participants were prompted to explain their reasoning, except for the isomorphic Division and Procedures Tests. Prompts included questions such as: "How did you figure this out?" or "Can you explain to me how you got your answer?" If the participant appeared frustrated or tired, the researcher directed the participant to the next task. If the participant provided very clear explanations and did not appear to need additional prompts, the researcher said, "You explained yourself clearly. I understand how you figured that out," and continued to the next task.

Results

The main objective of my study was to investigate changes in children's thinking about fractions following an intervention on fraction division. My second objective was to determine the effects of making the links between unitizing concepts and measurement fraction division during instruction explicit. The students were randomly assigned to three instructional conditions: (a) the explicit-links condition, where the links between both analogs were directly stated and supported by the instructor, (b) the implicit-links condition, where children needed to make their own links between unitizing and fraction division concepts, and finally, (c) a control condition, in which children were only presented with a fraction division problem and no unitizing. In this results section, I discuss intervention effects on participants' conceptual understanding of fraction division, their interpretations of the fractional remainder, and their understanding of unitizing.

Fifty-one participants were randomly assigned to the three conditions, resulting in 17 participants per condition. Because of an administrative error, two participants had missing data on the Fraction Division Explanation Task at posttest, which meant that for this specific task, the implicit- and explicit-conditions had a total of 16 participants. In the Bar Diagram and Unifix Cube Tasks combined (also referred to as the Remainder Tasks), one participant's data from the explicit-links condition were excluded from the analysis because the participant used a procedure to describe the remainder, making it impossible to determine what conceptual interpretation he was using. This meant that for this specific task, the explicit-links condition had a total of 16 participants.

Condition Differences Prior to the Intervention

To determine equivalency of groups at pretest, I administered two prior knowledge measures before the intervention: the Identify Fraction Task and Select Larger Fraction Task. Mean scores and standard deviations on both prior knowledge measures by condition are presented in Table 5. To compare the performance in each condition on the Identify Fraction Task, I ran a one-way ANOVA, with condition as the between groups factor (explicit-links, implicit-links, control). I found no condition differences on the participants' performance on the task, $F(2, 48) = 1.18$, $p = .32$. These data should be interpreted with caution because of the apparent ceiling effect in each condition (see Table 5). Regarding the Select Larger Fraction Task, another one-way ANOVA, where the between groups factor was the condition (explicit-links, implicit-links, control), was run to determine whether there were condition differences. No differences were observed, $F(2, 48) = 1.71$, $p = .19$. In short, it appears that there is no significant difference between all three conditions on either of the two prior knowledge measures.

Table 5

Means and Standard Deviations of the Scores on the Identify Fraction and Select Larger Fraction Tasks by Condition

	Explicit-Links (n = 17)		Implicit-Links (n = 17)		Control (n = 17)	
Measures	*M*	*SD*	*M*	*SD*	*M*	*SD*
Identify Fraction	.94	.21	.99	.03	.91	.16
Select Larger Fraction	.81	.26	.90	.18	.74	.31

The intercorrelations between the two prior knowledge tasks and all outcome measures at posttest are presented in Table 6. The Select Larger Fraction Task was not correlated with any of the measures at posttest. Four individual significant correlations were observed between the outcomes and the Identify Fraction Task. Performance on the Identify Fraction Task and the Division Test in the implicit-links condition were negatively correlated. The Identify Fraction Task was negatively correlated with Other interpretations of the remainder on the Bar Diagram and Unifix Cube Tasks at posttest in both the explicit-links and control conditions. Also, the Identify Fraction Task was negatively correlated with the Other strategy on the Area Model Task in the control condition. All correlations between performance on the Identify Fraction Task and Other codes indicate that the better able students were in identifying a fraction prior to the intervention, the fewer instances of lower-level codes were observed at posttest. Both prior knowledge measures were dropped from further analyses because they did not consistently predict performance in all conditions on any of the outcome measures used in this study.

Table 6

Correlations between the Identify Fraction and Select Larger Fraction Tasks and Posttest Tasks by Condition

Measures		Explicit-Links	Implicit-Links	Control
Identify Fraction				
Division Test		.25	-.49*	-.11
Procedures Test		.28	-.12	.25
Remainder Tasks	Referent-Unit Interpretation	.33	-.33	.34
	Original-Unit Interpretation	.19	.31	.23
	Other Interpretation	-.56*	.13	-.53*
Symbolic Unitizing Task	Composing/ Decomposing	.13	.12	.21
	Correct Procedure	.20	-.24	.24
	Incorrect Procedure	-.29	.16	-.43
Area Model Unitizing Task	Composing/ Decomposing	.17	-.41	.22
	Joining Strategy	-.27	.21	.26
	Other Strategies	.13	.13	-.88**
Select Larger Fraction				
Division Test		.03	.27	-.20

Procedures Test		.14	.24	-.11
Remainder Tasks	Referent-Unit Interpretation	.07	-.17	.39
	Original-Unit Interpretation	.43	.09	-.36
	Other Interpretation	-.39	-.44	-.23
Symbolic Unitizing Task	Composing/ Decomposing	.26	.28	.28
	Correct Procedure	.08	.12	-.23
	Incorrect Procedure	-.28	-.37	-.03
Area Model Unitizing Task	Composing/ Decomposing	.22	.26	-.01
	Joining Strategy	-.32	-.33	.23
	Other Strategies	.14	.10	-.40

*$p < .05$. ** $p < .001$.

Intervention Effects on Conceptual Understanding of Fraction Division Concepts

In the following section, the effects of the intervention on participants' conceptual knowledge of fraction division are presented. More specifically, I ran analyses of covariance to test for differences between conditions following the intervention in participants' thinking about (a) fraction division, as assessed using the Division Test and on the Fraction Division Explanation Task, (b) fractional remainders, as assessed using both the Bar Diagram Task and Unifix Cube Task combined, and, (c) unitizing concepts, as assessed using the Symbolic Task and Area Model Task. Exact McNemar tests were also run to determine changes in each condition from pretest to posttest in the proportion of participants explaining division using fractions and changes in proportions of participants using the measurement interpretation to explain division. McNemar tests were also run to test for changes in the ways children thought about remainders by examining the proportions of participants in the Integrated Thinking and the Referent-Thinking profiles from pretest to posttest.

Conceptual Understanding of Fraction Division

Division Test. The division test was a multiple-choice written test in which participants were asked to select the word problem that best represented the structure of the division equation provided. Means and standard deviations of the Division Test scores at pretest and posttest as a function of condition are presented in Table 7.

Table 7

Mean Scores and Standard Deviations on the Division Test at Pretest and Posttest by Condition

	Explicit-Links (n = 17)		Implicit-Links (n = 17)		Control (n = 17)	
Measure	*M*	*SD*	*M*	*SD*	*M*	*SD*
Pretest	.46	.20	.39	.26	.33	.18
Posttest	.34	.25	.45	.29	.43	.24

Two separate analyses of covariance were run. The first was a one-way ANCOVA using mean scores on the Division Test at posttest as the dependent measure with condition as the between-groups factor (explicit-links, implicit-links, control) and the mean scores at pretest as the covariate. No significant effect of condition was found on the Division Test scores at posttest, $F(2, 47) = 1.39$, $p = .26$. The second analysis was a 3 (condition) x 2 (division type) ANCOVA with condition as the between groups factor and division type (measurement, partitive division) as the repeated measures factor, using pretest scores on the measurement division and partitive division items as covariates. Again, no effects of condition, $F(2, 46) = 1.31$, $p = .28$ or division type were found, $F(1, 46) = 0.97$, $p = .33$, nor was the condition by division type interaction significant, $F(2, 46) = 0.46$, $p = .63$. In short, no condition differences were observed in participants' conceptual understanding of fraction division at posttest, when controlling for their understanding at pretest.

Procedures Test. The Procedures Test was composed of two whole number division items and two fraction division items. Both types of division items had one item with a remainder and one without. Using this task, I tested for condition effects on students' procedural knowledge of division following instruction. The mean scores and standard deviations on the Procedures Test at pretest and posttest as a function of condition are presented in Table 8.

Table 8

Mean Scores and Standard Deviations on the Procedures Test at Pretest and Posttest by Condition

	Explicit-Links (*n* = 17)		Implicit-Links (*n* = 17)		Control (*n* = 17)	
Measure	*M*	*SD*	*M*	*SD*	*M*	*SD*
Pretest	.31	.25	.34	.18	.26	.30
Posttest	.32	.30	.38	.25	.37	.28

An ANCOVA was conducted to test for group differences at posttest on the Procedures Test, using the pretest scores as the covariate. No condition effects were found, $F(2, 47) = 0.51$, $p = .60$, which means that there were no differences between conditions at posttest on students' procedural knowledge of division when their pretest knowledge on the task was taken into account.

Division Explanation Task. On the Division Explanation Task, participants were asked to explain verbally what it means to divide something by a fraction. This task allowed me to assess changes in students' knowledge of fraction division from pretest to posttest. The participants' explanations of division were evaluated in two ways. I first looked at the types of quantitative or non-quantitative references made by the participants when explaining division and then I analyzed the type of interpretation participants referred to when discussing division (e.g., measurement or partitive division).

Changes in participants' quantitative and non-quantitative division interpretations. First, I assessed students' interpretations of division using specific references to quantities (i.e., division explained using fractions, division explained using whole numbers, or division explained using general unspecified quantities). The proportions of participants in each condition who made references to quantities in their explanations of division (fractions, whole numbers, general quantities, or non-quantitative responses) at pretest and posttest are presented in Table 9.

Table 9

Proportion of Participants in Each Condition at Pretest and Posttest Who Interpreted Division in Quantitative and Non-Quantitative Ways on the Fraction Division Explanation Task

Condition		Fractions	Whole Numbers	General Quantities	Non-Quantitative
Explicit-Links	Pretest	0.06	0.06	0.13	0.75
(n = 16)	Posttest	0.25	0.06	0	0.69
Implicit-Links	Pretest	0.06	0.06	0	0.88
(n = 16)	Posttest[a]	0.31	0	0.06	0.62
Control	Pretest[a]	0	0.12	0.12	0.77
(n = 17)	Posttest	0.12	0.06	0.24	0.58

[a] Total proportion in each identified row is 0.01 more or less than 1 because of rounding.

As seen in Table 9, increases in participants' references to fractions are observed across conditions. In the explicit-links condition, 25% of the participants referred to fractions in their explanations of what it means to divide at posttest compared to 6% at pretest, which amounts to an increase of 19 percentage points. In the implicit-links condition, 31% of participants referred to fractions at posttest, when only 6% did at pretest, amounting to an increase of 25 percentage points. Finally, an increase in participants' mentions of fraction division was also observed in the control condition: At posttest, 12% of participants mentioned fractions in their explanations compared to no mention at pretest. Another consistent change across conditions was a decrease in the number of participants who provided non-quantitative fraction division explanations from pretest to posttest. Also, it appears that whole numbers were used infrequently across conditions at both time points with little change from pretest and posttest. Changes in participants' thinking about fraction division can be summarized by an increase in the mention of fractions and a decrease in non-quantitative ways of explaining division.

Three exact McNemar tests were run, one per condition, to determine whether the proportions of participants who interpreted division using fractions changed from pretest to posttest. The proportion of participants in the analogy-based conditions who interpreted division using fractions did not significantly change from pretest to posttest (explicit-links condition, p = .25, and implicit-links condition, p = .13). There was also no significant change in the proportions of participants in the control condition who interpreted division using fractions from pretest to posttest (p = .50). In conclusion, the change in participants' use of fractions to explain division at posttest compared to pretest was not significant in any of the three conditions.

Changes in participants' references to division types in their division explanations.

Participants' responses to the Fraction Division Explanation Task were coded according to whether they referred to measurement or partitive interpretations in their explanations. I also placed responses in which neither the measurement nor the partitive interpretation was clearly identified into a General category, and responses in which participants only referred to elements of sharing or splitting into a Splitting/Grouping category. The proportions of participants whose responses reflected these categories when explaining the meaning of division at pretest and posttest are presented in Table 10.

Table 10

Proportion of Participants Who Explained Fraction Division by Referring to Division Types on the Fraction Division Explanation Task at Pretest and Posttest by Condition

Condition		Measurement	Partitive	Measurement and Partitive	General	Splitting/ Grouping	Other
Explicit-Links (*n* = 16)	Pretest[a]	0.13	0.13	0	0	0.19	0.56
	Posttest	0.25	0.06	0	0	0.19	0.50
Implicit-Links (*n* = 16)	Pretest	0.06	0	0	0.06	0.44	0.44
	Posttest[a]	0.31	0	0.06	0	0.06	0.56
Control (*n* = 17)	Pretest[a]	0.06	0.12	0	0.06	0.18	0.59
	Posttest[a]	0.18	0.24	0	0	0.06	0.53

[a] Total proportion in each identified row is 0.01 more or less than 1 because of rounding.

As seen in Table 10, the only change that occurred in participants' interpretations of division that is consistent across conditions is an increase in measurement division interpretations. In the implicit-links condition, 31% of the participants referred to measurement division when defining division at posttest compared to 6% at pretest, which amounts to an increase of 25 percentage points. There was an increase of 12 percentage points for both the explicit-links condition (13% at pretest and 25% at posttest) and the control condition (6% at pretest and 18% at posttest). There was only one participant in the implicit-links condition at posttest who interpreted division using both the measurement and partitive division interpretation. The largest observed change was the proportion of participants who referred to splitting or grouping in their explanations in the implicit-links condition at posttest (6%) when compared to at pretest (44%), which amounts to a decrease of 38 percentage points. In short, an increase in measurement division interpretations was observed across conditions following the intervention.

Three exact McNemar tests, one per condition, were run to determine whether the proportion of participants who explained division with the measurement interpretation changed from pretest to posttest. Proportions of participants in the analogy-based conditions who referred to measurement division on the Fraction Division Explanation Task did not change significantly from pretest to posttest (explicit-links conditions, $p = .63$, and implicit-links condition, $p = .13$), and neither did the proportions of participants in the control condition ($p = .63$). In short, no changes in measurement thinking about division were observed in any of the conditions following intervention.

Interpretations of the Fractional Remainder

Participants' interpretations of the fractional remainder were tested after the intervention using two tasks: the Bar Diagram Task and the Unifix Cube Task. Both tasks consisted of fraction division problems solved by a fictitious student using bar diagrams on the Bar Diagram Task and Unifix cubes on the Unifix Cube Task. The researcher asked the participants to interpret the remainder of each problem because it was left uninterpreted in the solution by the fictitious student.

Bar Diagram and Unifix Cube Tasks. Participants' interpretations of the fractional remainder were identified at pretest using the Bar Diagram Task and at posttest using their combined performance on the Unifix Cube Task and an isomorphic version of the Bar Diagram

Task. The students' remainder interpretations were classified into three general categories: referent-unit thinking, original-unit thinking, and other thinking. Three analyses of covariance, one per remainder interpretation category, were run to determine whether there were differences in thinking about the remainder at posttest, considering their pretest knowledge of the remainder as covariate. The mean scores and standard deviations of the Referent score, the Original score and the Other score on the Bar Diagram Task at pretest and the Bar Diagram and Unifix Cube Tasks combined at posttest are presented in Table 11.

Table 11

Mean and Standard Deviations of the Referent Score, Original Score and Other Score on the Bar Diagram and Unifix Cube Tasks at Pretest and Posttest by Condition

	Explicit-Links (*n* = 16)		Implicit-Links (*n* = 17)		Control (*n* = 17)	
Score	*M*	*SD*	*M*[a]	*SD*	*M*[a]	*SD*
Referent						
Pretest	.17	.20	.26	.29	.24	.27
Posttest	.60	.40	.49	.40	.56	.42
Original						
Pretest	.25	.30	.29	.28	.10	.20
Posttest	.15	.21	.43	.36	.10	.22
Other						
Pretest	.58	.31	.44	.31	.66	.36
Posttest	.25	.30	.07	.14	.33	.37

[a] Mean score totals for the implicit-links condition within pretest and within posttest are 0.01 less than 1 because of rounding. The mean score totals for the control condition within posttest is also 0.01 less than 1 because of rounding.

Referent-unit interpretations of the fractional remainder. A one-way analysis of covariance was run, where the dependent measure was the Referent score at posttest and the covariate was the corresponding score at pretest. No significant condition effect was found, $F(2, 46) = 0.51$, $p = .61$, which means that there were no differences among conditions at posttest in students' use of the referent unit to interpret the remainder when their pretest interpretations were taken into account.

Original-unit interpretations of the fractional remainder. A one-way ANCOVA was run to test for an effect of condition at posttest. The dependent measure was the Original score at posttest and the covariate was the corresponding score at pretest. Controlling for original-unit thinking at pretest, the ANCOVA revealed a significant effect of condition, $F(2, 46) = 5.82$ $p = .006$. Post hoc comparisons with Bonferroni corrections revealed a significant difference between the implicit-links condition and the control condition, $t(32) = 3.04$, $p = .01$, as well as between the implicit-links condition and the explicit-links condition, $t(31) = 2.85$, $p = .02$. There were no differences between the explicit-links condition and the control condition, $t(32) = 0.25$, $p = 0.99$. In short, these results revealed that participants in the implicit-links condition interpreted the remainder using the original unit more often than participants in the explicit-links and control conditions, who did not differ from each other.

Other interpretations of the fractional remainder. A one-way ANCOVA, where the dependent measure was the Other score at posttest and the covariate was their corresponding score at pretest, indicated no condition effects, $F(2, 46) = 2.18$, $p = .12$. This means that there were no differences between the conditions in participants' use of units other than the referent unit or original unit to interpret the remainder at posttest, when taking their pretest interpretations into account.

Changes in thinking about the fractional remainder. Having observed condition differences at posttest in participants' thinking about the original unit, I decided to explore whether there were qualitative differences in the ways participants interpreted remainders across items on the Bar Diagram Task at pretest and the Unifix Cubes and Bar Diagram Task (combined) at posttest. Did they always interpret the remainder using the same unit on all items, did they use the referent unit on one item and the original unit on another, or did they use the referent and original units simultaneously on one item when interpreting the remainder?

The participants were placed in profiles based on how they interpreted remainders across all items at pretest and again across all items at posttest. The Integrated Thinking profile was defined by responses where the remainder was identified using both the original and referent units on at least one item on a task. For example, a participant could say "This is ½ of the group of 2/5, so it is also 1/5." In this case, the participant used the referent unit to interpret the remaining piece by saying that it is half of 2/5, but also made a statement that the remainder is at the same time 1/5 of the original unit. The Separate Thinking profile was also based on both original and referent unit thinking, but only if they occurred on separate items on each task. For instance, the remainder could have been interpreted by a participant using the referent unit on the first item and the original unit on the second item. The Referent-Only Thinking profile described participants who interpreted the remainder using the referent unit on at least one item, without referring to the original unit in their interpretations on any other items. The Original-Only Thinking profile included participants who interpreted the remainder using the original unit on at least one item, without using the referent unit on any of the other items on the task. Finally, the

Other Thinking profile was characterized if no remainder on any of the items was interpreted using the referent or original units. The proportions of participants in each profile as a function of condition are presented in Table 12.

Table 12

Proportion of Participants in the Integrated Thinking, Separate Thinking, Referent-Only Thinking, Original-Only Thinking, and Other Thinking Profiles within each Condition at Pretest and Posttest

Condition		Integrated	Separate	Referent-Only	Original-Only	Other
Explicit-Links	Pretest	0.06	0.13	0.31	0.31	0.19
(*n* = 16)	Posttest	0.25	0.25	0.44	0.06	0
Implicit-Links	Pretest	0.29	0.12	0.18	0.29	0.12
(*n* = 16)	Posttest[a]	0.71	0.18	0.12	0	0
Control	Pretest	0	0.29	0.24	0	0.47
(*n* = 17)	Posttest[a]	0.24	0.12	0.53	0.12	0

[a] Total proportion in the row is 0.01 more than one because of rounding.

As seen in Table 12, an increase in participants' Integrated Thinking from pretest to posttest is observed in all three conditions. The largest change is observed in the implicit-links condition: At pretest, 29% of participants were in the Integrated Thinking profile and 71% at posttest, indicating an increase of 42 percentage points. In the explicit-links condition, 25% of the participants were placed into the Integrated Thinking profile at posttest compared to 6% at pretest, which amounts to an increase of 19 percentage points. In the control condition, there were no participants in this profile at pretest and at posttest 24% of participants demonstrated Integrated Thinking. There were also decreases in the number of participants in the Other Thinking profile from pretest to posttest in all conditions. In fact, across all condition, there were no participants in the Other Thinking profile at posttest, meaning that they demonstrated more conceptually-appropriate thinking after the intervention.

Six exact McNemar tests were run to determine whether there were significant changes in the proportions of participants in the Integrated Thinking and Referent-Only Thinking profiles between pretest and posttest. These two profiles were identified for further analysis because the Integrated Thinking profile demonstrated the highest flexibility of thinking about the remainder. Participants in this profile were able to think of the remainder using two units simultaneously, which very closely resembles the process of regrouping in unitizing. I also focused on the Referent-Only Thinking profile because using the referent unit to interpret a remainder does lead to the correct final answer when dividing, making it the most appropriate unit to use when performing a division operation.

The first three McNemar tests, one per condition, were run to determine whether there were changes in the proportion of participants in the Integrated Thinking profile following intervention. No significant change in the proportion of participants using Integrated Thinking from pretest to posttest was observed in the explicit-links condition ($p = .38$). In the implicit-links conditions, no differences were seen either ($p = 0.07$). There were also no significant differences in the proportion of participants reflecting Integrated Thinking at posttest in the control condition when compared to pretest proportions ($p = 0.13$).

A second set of McNemar tests was run to determine whether the proportions of participants in the Referent-Only Thinking profile changed from pretest to posttest in each condition. No significant changes in the proportion of participants using Referent-Only Thinking from pretest to posttest were observed in the implicit-links condition ($p = 0.99$) nor in the explicit-links condition ($p = 0.73$). No pretest to posttest differences were noted in the control group either ($p = 0.13$). In short, there were no significant changes in the proportions of participants in the Referent-Only Thinking profile following the intervention in any of the three conditions.

Understanding of Unitizing Concepts

In this section, participants' use of unitizing concepts when comparing two fractional quantities is discussed. Two tasks, each using different representations of fractions, were used to assess this: the Symbolic Task and the Area Model Task. Participants' strategies for comparing both fractional amounts are analysed at posttest, taking their strategies at pretest into account.

Symbolic Task. Participants' strategies for comparing fractions on the Symbolic Task were placed in one of three categories: (a) composing and decomposing wholes with unit shift, (b) using a correct procedure, and (c) using an incorrect procedure. The mean and standard deviations of the Composing/Decomposing score, Correct Procedure score, and Incorrect Procedure score on the Symbolic Task at pretest and posttest as a function of condition are presented in Table 13.

Table 13

Mean and Standard Deviations of the Composing/Decomposing Score, Correct Procedure Score, and Incorrect Procedure Score on the Symbolic Task at Pretest and Posttest by Condition

		Explicit-Links (*n* = 17)		Implicit-Links (*n* = 17)		Control (*n* = 17)	
Interpretations		*M*	*SD*	*M*	*SD*	*M*[a]	*SD*
Composing/Decomposing	Pretest	.18	.25	.06	.24	.26	.40
	Posttest	.21	.40	.18	.39	.26	.20
Correct Procedure	Pretest	.32	.39	.59	.48	.32	.39
	Posttest	.35	.46	.53	.51	.47	.45
Incorrect Procedure	Pretest	.50	.47	.35	.46	.41	.48
	Posttest	.44	.50	.29	.47	.26	.44

[a] Mean score totals for the control condition within pretest and within posttest are 0.01 less than 1 because of rounding.

Three one-way analyses of covariance were run to determine whether the participants' thinking about unitizing varied by condition after the intervention. The first ANCOVA that was run for this task had the Composing/Decomposing score at posttest as the dependent variable and the corresponding pretest score as covariate. No condition effects were found, $F(2,47) = 0.17$, $p = .84$. The second one-way ANCOVA had as dependent measure the Correct Procedure score at posttest, with the corresponding pretest score as covariate. Again, no condition effects were found, $F(2, 47) = 0.42$, $p = .66$. The third ANCOVA, where the dependent measure was the Incorrect Procedure score at posttest and the covariate was the corresponding score at pretest, revealed once more no condition effects, $F(2, 47) = 0.54$, $p = .58$. To conclude, no condition effects were found for any response category on the Symbolic Task, when taking into consideration corresponding performance at pretest.

Area Model Task. The final set of tests were run on the Area Model Task. The participants' answers on this task were coded based on whether they composed or decomposed wholes using a unit shift, joined areas without re-unitizing, or used other strategies. The means and standard deviations for the Compose/Decompose score, the Joining Areas score, and the Other Strategies score on the Area Model Task as a function of condition at pretest and posttest are presented in Table 14.

Table 14

Means and Standard Deviations of the Composing/Decomposing Score, Joining Areas Score, and Other Strategies Score on the Area Model Task at Pretest and Posttest by Condition

		Explicit-Links (n = 17)		Implicit-Links (n = 17)		Control (n = 17)	
Score		*M*	*SD*	*M*	*SD*	*M*[a]	*SD*
Composing/Decomposing	Pretest	.18	.30	.21	.40	.29	.40
	Posttest	.26	.40	.32	.43	.41	.44
Joining Areas	Pretest	.50	.43	.53	.48	.56	.43
	Posttest	.56	.43	.47	.45	.53	.45
Other Strategies	Pretest	.32	.47	.26	.44	.14	.29
	Posttest	.18	.35	.21	.40	.06	.24

[a] Mean score totals for the control condition within pretest are 0.01 less than 1 because of rounding.

Three one-way analyses of covariance were run to determine whether participants' strategies for solving the Area Model Task varied at posttest by condition. The first ANCOVA was run to test for condition effects in participants' use of composing or decomposing wholes with a unit shift at posttest. This ANCOVA had as dependent measure the Composing/Decomposing score at posttest with the corresponding pretest score as covariate. No condition effects were found, $F(2, 47) = 0.22$, $p = .81$. The second ANCOVA, testing for differences in condition in participants' use of joining areas without a unit shift, had as dependent measure the Joining Areas score at posttest and the corresponding pretest score as covariate. Again, no condition effects were observed, $F(2, 47) = 0.33$, $p = .72$. The final ANCOVA had as dependent measure the Other Strategies score at posttest with the corresponding pretest score as covariate. No condition effects were found, $F(2, 47) = 0.56$, $p = .58$. In short, no condition differences were found following intervention for composing/decomposing, joining areas, or other strategies when taking into account the pretest scores for each of these strategies. This suggests that there were no differences in unitizing understanding, demonstrated through the composing/decomposing with unit shift, following the intervention between the three conditions.

Discussion

The present study examined seventh-graders' understanding of fraction division concepts, following instruction using an analogy of unitizing and measurement division. The first objective of the study was to determine whether there would be condition differences at posttest in students' understanding of fraction division, fractional remainders, and unitizing concepts. The extent to which analogies should be explicitly supported by the teacher during instruction was investigated by looking at the effects of an analogy in which the links between both analogs were kept implicit and by looking at the effects of an analogy in which the links between analogs were made explicit by the teacher. The second objective was to determine the nature of any changes that occurred in students' thinking about these three fractional concepts (fraction division, fractional remainders, and unitizing) from pretest to posttest and whether there were condition differences in the nature of these changes. I hypothesized that there would be posttest differences between conditions on all fractional concepts when taking their corresponding pretest scores into consideration. I made the prediction that both analogy conditions would differ from the control condition on all measures at posttest, and that the students' performance at posttest in the explicit-links condition would be significantly higher than those in the implicit-links condition.

Following the pretest measures, participants were presented with a video-recorded intervention. They were randomly assigned to one of three conditions. In the explicit-links condition, participants received instruction on unitizing and fraction division, in which the similarities between both concepts were explicitly stated and emphasized by the teacher using cognitive supports, such as gestures, spatial cues, and relational language. The implicit-links condition had the same instruction on unitizing and fraction division, but the connections between both analogous concepts were left implicit. The control condition consisted of presenting the same fraction division two times without an analogy being present to ensure the same amount of instruction as the other two conditions. Following the intervention, I administered posttest tasks to measure participants' interpretations of remainders (the Unifix Cube Task and the Bar Diagram Task), their understanding of unitizing (the Symbolic Task and the Area Model Task), and their understanding of fraction division (Division Explanation Task, Division Test, and Procedures Test).

The analysis of students' prior knowledge, as indicated in the results section, did not reveal any condition differences. Because of the apparent ceiling effect, the Identify Fraction Task may not have been a suitable measure of students' conceptual understanding of part-whole interpretations of fractions. In addition, the Select Larger Fraction Task, intended to measure conceptual understanding of symbolic representations of fractions, was not correlated with any of the posttest measures. One might expect this prior knowledge measure to have been at least correlated to the Symbolic Task, on which students were asked to compare two fractional quantities represented in symbols. Because that was not the case, it brings into question whether the Select Larger Fraction Task itself was sensitive enough to measure what it was designed to measure.

Contrary to my predictions, there were no significant differences following the intervention between the conditions in students' conceptual understanding of division, as assessed by the Division Test and Fraction Division Explanation Task. There were, however, some interesting qualitative findings regarding the Fraction Division Explanation Task. Although condition differences were not significant, seventh-graders, regardless of condition, struggled to explain the meaning of fraction division at posttest, which aligns with the research identifying fraction division as the most complex concept taught at the elementary level (Newton, 2008).

On the same task (Fraction Division Explanation), less than a quarter of all participants across conditions referred to fractions when explaining fraction division at posttest. The remaining participants resorted to other quantities (e.g., whole numbers) or simply attempted to explain division without putting it in a quantitative context. Only about a quarter of the participants referred to measurement division in their interpretation and few participants referred to partitive division interpretations of fraction division. More than half of participants across conditions mentioned other interpretations of fraction division (i.e., referring only to procedures when explaining division, defining multiplication or subtraction instead of division, referring to misconceptions about fractions, or providing vague and unclear explanations).

My findings on students' conceptions of fraction division contribute to the literature, as few studies have revealed the variety of interpretations that children use to describe it. Sidney, Thompson, and Rivera (2018) investigated the effects of different representations used to solve fraction division problems. They compared students' performance using three different diagrams: number lines, rectangular areas, and circular areas. To identify the participants' conceptual understanding of fraction division, Sidney and colleagues (2018) investigated the strategies participants used when solving the division problems to determine whether they held a measurement or partitive interpretation of division. All other categories were coded into a "neither" category. The authors found that children who completed fraction division problems using number lines demonstrated a more conceptual approach to solving fraction division problems when compared to participants using rectangular and circular areas to solve the same problems. Their study investigated children's fraction division explanations using the measurement and partitive division approaches. My study contributes to the discussion on children's knowledge of division because, to my knowledge, no one has described the variety of different interpretations that students hold. I was able to reveal the nuances in students' thinking by asking them to explain in their own words what division with fractions means. Coding all types of responses (e.g., general division interpretations, mentions of partitioning quantities) went beyond identifying whether they used measurement or partitive interpretations in their explanations (Sidney & Alibali, 2017; Sidney et al., 2018).

Tirosh (2000) explained that the literature on children's mistakes when dividing fractions places them into three categories: (a) incorrectly following the procedure (e.g., inverting the dividend instead of the divisor), (b) misconceptions about division (e.g., the belief that the quotient has to be smaller than the dividend), and (c) misconceptions about fractions and operation properties (e.g., not recognizing that a fraction is one quantity or the belief that division is commutative). I observed each of these error categories in my participants' conceptual explanations as they attempted to define the meaning of dividing with fractions. The data demonstrated that students used procedural approaches to explain what it means to divide by a fraction and referred to misconceptions about the meaning of a fraction when talking about the division operation. The participants also confused division with fractions with other operations. For instance, in some cases, students described multiplication instead of division in their explanations (e.g., "I am looking for how much there will be if I have three groups of $\frac{2}{3}$") and as a subtraction (e.g., "Dividing is like taking away a part from the whole"). In other cases, my data revealed that some participants did not attempt to describe division at all, but instead focused on the meaning of a fraction, often providing correct quotient, part-whole, or decimal interpretations of fractions.

Seventh-graders' interpretations of remainders were particularly interesting. Students produced a variety of interpretations for the remainder, mainly before the intervention. In fact, my findings revealed a larger variety of remainder interpretations than was found in previous research (e.g., Sharp & Adams, 2002; Zembat, 2017). Aside from the tendency to view the remainder as part of the referent or original unit, I observed that students invented new, but inappropriate, units to quantify the remainder. These new units either made sense in the context of the problem (e.g., dividend as unit) or they did not (e.g., the number of groups of the divisor as unit). The larger variety of interpretations is perhaps a result of the external knowledge representations used to solve the fractions problems; had I used word problems instead, for example, the students' responses may have been directed by the context of the problems, where fractional remainders are "grounded" in real-world contexts that can foster sense-making (Foster & Osana, 2018; Koedinger et al., 2008; Sharp & Welder, 2014; Zembat, 2017).

Contrary to my predictions, there were no condition differences following the intervention in participants' use of the referent unit when interpreting remainders, but differences were found in seventh-graders' use of the original unit when interpreting remainders. It makes sense for students to think of the remainder using the original unit, because it is that unit that is initially used to represent the dividend and the divisor. In the context of 1 divided by 2/3, for example, it is not inaccurate to say that the remainder is 1/3 of the original unit. One must make the unit shift to provide a correct answer to the division equation, however. An interpretation of the remainder using the original unit would not be mathematically incorrect, but it would not reflect a completed division operation. After the intervention, I observed the students in the implicit-links condition refer to the original unit more often than students in the other two conditions. Similar to the results of Sidney and Alibali (2017b), when students' prior knowledge of unitizing is activated, which may have occurred in the implicit-links condition because of the priming theory (Leech et al., 2008), students may have been prompted to make their own connections between unitizing and fraction division concepts. Sidney and Alibali found that participants in the implicit-links condition outperformed participants from the explicit-links condition on fraction division when whole number division was the source analog. The participants did not require the experimenter to make the links explicit between both concepts to develop significantly higher conceptual understanding than participants in the explicit-links condition.

I argue that it may have been beneficial for students in the implicit-links condition to construct their own mappings between unitizing and fraction division concepts. The instruction in the explicit-links condition that included clear verbal descriptions of the links between both concepts and provided a number of other cognitive supports may have restricted the students' thinking. In fact, the students in the control condition and the explicit-links condition, in which direct explanations and explicit supports were provided during instruction, either understood the unit shift (i.e., referred to the referent unit) or resorted to incorrect interpretations. They did not consider the original unit as a way of thinking about the remainder.

The effects of direct instruction on students' exploration was empirically tested by Bonawitz and colleagues (2011). In their study, the instructor showed children four to six years-old an unfamiliar toy that had four separate functions. The children were given the toy to play with and were assessed on how many of the four functions they would discover by interacting with the toy. In the Pedagogical condition, the experimenter showed the child only one of the functions and made no explicit statements about the three additional functions before giving the toy to the child. In the Interrupted condition, the experimenter also showed one of the functions of the toy to the participating child, but the experimenter pretended to stop her explanations suddenly to go take care of something she forgot to do, creating the illusion that she had not finished talking about the toy before giving it to the child. In the Naïve condition, the experimenter pretended to have just found the toy. The experimenter pretended to accidentally trigger the function (a squeaking sound), and after making a statement as if she was surprised by the function, she repeated the action before giving the toy to the child. In the Baseline condition, the experimenter did not introduce any of the functions of the toy before giving it to the child. The authors found children in the Pedagogical condition would spend less time exploring the toy, perform fewer actions on the toy, and ultimately discover fewer of the toy's functions than children in any of the other three conditions. In short, when the examiner explicitly demonstrated a function of the toy, children showed constrained exploration efforts and discovered fewer functions of the toy.

Although Bonawitz and colleagues (2011) did not directly test the effects of analogies, a similar process might have occurred in our study, where students from the explicit-links condition and the control condition, both of whom received direct instruction, were restricted from exploring other mathematically appropriate alternatives when interpreting the remainder. Bonawitz and colleagues (2011) explained this constrained exploration by theorizing that if a knowledgeable teacher shows that a function exists, the student assumes it exists. Similarly, if a knowledgeable teacher does not mention additional functions of the toy, students will assume that these additional functions do not exist. In the context of the present study, if a teacher makes a direct statement about interpreting the remainder using the referent unit, students may assume that no other interpretations are possible. Even if interpreting the remainder using the original unit is not ideal in the context of solving a division problem, it still reflects sophisticated, and not incorrect, thinking about the remainder relative to many of the other responses I observed.

I also found no differences in students' use of unitizing to compare fractions at posttest, regardless of the representation used in the task to identify each pair of fractions. I observed that students used different strategies when comparing fractions in symbolic notation than when the fractions were represented with area models, however, the most notable observation was that the majority of students used a procedure, either appropriately or inappropriately, when working with fraction symbols. In this symbolic context, many students were unable to explain conceptually why two fractional amounts were the same or different, even after being prompted to think of another way to explain how the fractions compared. In contrast, the students did not use an algorithm when comparing fractional quantities represented with area models. In this case, more than half of the students regrouped the areas (i.e., the shaded rectangles) in such a way to show that the two fractions either were or were not equivalent. When students engaged in this regrouping process, however, it was most often without implementing a unit shift. These results might suggest that the task itself was not sensitive enough to measuring unitizing concepts because students did not need to use a new unit to solve the problem correctly.

As mentioned in the literature review, unitizing is a concept that is central to dividing with fractions (Lamon, 2002; Thanheiser, 2009). However, few studies have measured children's understanding of unitizing in the context of division (e.g., Lamon, 1996). Most studies have concentrated on preservice teachers' or practicing teachers' knowledge of division with fractions and the role units play in dividing quantities (e.g., Lamberg & Weist, 2015; Lee, 2017; Zembat, 2017). Because little research on children's understanding of unitizing exists, the Symbolic and Area Model Tasks were inspired by preservice teachers' discussions in Tobias and colleagues (2015), who examined unit fractions and improper fractions. As mentioned above, I believe the task I created was not the most effective at measuring unitizing because students found ways that were not related to unitizing to correctly compare the two fractional amounts.

To create a more valid measure of unitizing knowledge, I would need to develop a task where the only mathematically correct way of solving it would involve a change of unit. One possibility would be to ask students to interpret a fractional amount using different units. For example, the researcher could ask students to measure or represent the quantity $3\,\frac{4}{5}$ using various units, such as 1, $\frac{1}{5}$, and $\frac{3}{5}$. To conclude the task, students might be asked to describe each unitizing action and explain whether the quantities are equal or not.

Limitations, Educational Implications, and Future Research

The present study is a contribution to the literature because it is the first to experimentally assess the effects of an intervention designed to augment students' conceptual knowledge of remainders in fraction division contexts. Nevertheless, there were limitations to the present study. First, the size of the sample was small. A larger sample would have yielded more power to the analyses, possibly revealing a greater number of significant results. The data were also collected closer to the end of the academic year (i.e., from mid-April to end of May), which made it difficult to control for the amount of instruction on fractions or rational numbers the students had received during the school year.

Despite having used a strong experimental design, there were still weaknesses regarding methodology that should be addressed in future studies. The observed ceiling effects on the Identify Fraction Task suggests that the measure was too easy for the participants, thereby reducing variability. Furthermore, the task itself might not have measured participants' conceptual understanding of part-whole representations of fractions, as I had originally intended. Participants could have counted the shaded versus not shaded areas of each area model without truly understanding the relationship between the two areas. The validity of the Select Larger Fraction Task also comes into question. The fractions being compared were simple (e.g., $\frac{1}{5}$ and $\frac{1}{10}$), which means that participants could have potentially used mental math to arrive at an answer (e.g., "$\frac{1}{5}$ when multiplied by $\frac{2}{2}$ will give me $\frac{2}{10}$, which is larger than $\frac{1}{10}$"), even within the one-minute time limit. If this were the case, the task might have measured participants' ability to use a mental math procedure to transform the pair of fractions into fractions with common denominators instead of their understanding of fractions represented symbolically. This could explain why the task was not correlated to any of the posttest measures. Siegler and Lortie-Forgues (2015) used a different task to assess the conceptual understanding of fraction magnitudes of preservice teachers, middle school students, and university students. The participants were asked to place 10 fractions between zero and one and another set of 10 fractions from zero to five on a number line to indicate the fractions' relative magnitudes. The fractions were chosen in such a way that the participants would not be successful on the task by simply looking at the magnitudes of the numerators or denominators. Such a task might have been more suitable for measuring participants' conceptual understanding of the symbolic representations of fractions.

In addition, the Division Test limited my ability to assess students' thinking because of its multiple-choice paper-and-pencil format. Because I chose to deliver the test in a whole group context, I was unable to determine what the students were thinking as they were choosing the answers: What did they truly understand when selecting the right answer? What aspects of a selected misconception were salient to them? A suggestion for future research would be to deliver the test in a one-on-one interview context. This would allow the researcher to ask each participant to explain why the word problem he or she selected reflected the given equation.

Another explanation for the nonsignificant results could have been that the instruction in the control condition was too similar to the instruction in both analogy conditions. The fraction division instruction in the control condition included similar vocabulary to the other two conditions (e.g., "here is another group of $\frac{3}{4}$" when performing the division of four divided by $\frac{3}{4}$). Even if there was no implicit or explicit connection to unitizing as a separate concept in the control group, the concept of unitizing inherent in division itself might have been too strongly emphasized for a condition in which no focus on unitizing was intended.

Interpreting students' thinking is an important skill for teaching professionals, as it informs them of where students might require scaffolding when they are learning and applying new mathematical concepts (Ball, Thames & Phelps, 2008; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). The findings of the present study are valuable to teaching professionals because the study provides teachers with a detailed and nuanced view of students' remainder interpretations. The findings inform teaching professionals that students can interpret remainders using interpretations other than ones based in the original or referent units, as previous research has indicated. Teaching professionals could ask their students to explain mathematical concepts as a way of measuring their understanding, and identifying students' misconceptions can help teachers develop appropriate instruction in response to their needs (Jacobs, Lamb, & Philipp, 2010).

The fact that students found it difficult to explain the meaning of dividing with fractions indicates that students need a stronger conceptual understanding of the operation itself, mirroring Ball's (1990) findings of elementary and high school teachers' conceptual understandings of division. Ball (1990) found that teachers struggled to select a correct word problem that best reflected a fraction division equation (a task similar to the Division Test). Furthermore, none of the elementary school teachers from the sample could represent the division $1\frac{3}{4}$ divided by $\frac{1}{2}$ with an appropriate visual representation. Ball (1990) found that teachers struggled with understanding division itself, even outside of the context of dividing by fractions. She explained that they mostly referred to the partitive interpretation, making it difficult to interpret division with fractions, which is more easily represented with the measurement interpretation (Ervin, 2017; Van de Walle et al., 2008).

The results may also suggest that length models, such as the stacked cubes and bar diagrams used in the present study, are themselves conducive to aligning the concepts of unitizing and measurement division, regardless the type of instruction provided (see also Sidney et al., 2018). Osana, Blondin, Alibali, and Donovan (2018), for example, found that not all instructional representations offer the same affordances for children's learning and transfer in mathematics (see also Belenky & Schalk, 2014). Additional research is needed to test the effects of different fraction representations on students' learning of fraction division and how different representations interact with instructional support.

Another educational contribution of the present study is the support for instruction that holds back providing direct explanations to students on the structural similarities of two analogs, such as unitizing and measurement fraction division. Students' interpretations of the remainder using the referent unit in the implicit-links condition did not differ from those of students in the other conditions, but their interpretations using the original unit did. This could mean that the students were more free to explore when connections between both analogs are more implicit than explicit. In the direct instruction of fraction division (i.e., control group) or in the explicit-links analogy condition, the distribution of students' answers was more bimodal than in the implicit-links condition, with large proportions of students who either used the referent unit to interpret the remainder on the one hand, or who interpreted the remainder using mathematically incorrect units or unclear statements on the other. Together, the results of the present study suggest that implicit analogies may be beneficial for students' learning in the mathematics classroom, but more research would be needed to support this claim.

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Appendix A

Consent and Assent Forms

Parental Consent Form

UNIVERSITÉ Concordia UNIVERSITY

INFORMATION AND CONSENT TO PARTICIPATE IN A RESEARCH STUDY

Study Title: Understanding Fraction Division in the Seventh-Grade

Researcher: Helena P. Osana
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Study Investigator: Anna Tomaszewski
Study Investigator's Contact Information: anna.tomaszewski@concordia.ca

Source of funding for the study: Concordia University

Your child is invited to participate in a research study about how students learn math. Your child has been asked to participate because he/she has entered grade 7 in September 2017. This form provides information about what participating would mean. Please read it carefully before deciding if you want your child to participate or not. If there is anything you do not understand, or if you want more information, please contact the researcher.

A. PURPOSE

The purpose of the research is to learn more about how children learn mathematics from different lessons, specifically related to fraction division.

B. PROCEDURES

If you decide that your child may participate in this research, he/she will first be asked to solve some math problems on his/her own during a 20-minute activity in class. All participating students will complete this 20-minute activity at the same time.

Your child will also meet individually on one occasion with a research assistant within one week of the 20-minute class activity. During this individual meeting, your child will solve a few math problems, then receive a brief mathematics lesson. After the lesson, your child will be asked to solve some additional problems on his/her own and to explain how he or she solved the problems. This meeting will be conducted in the school library and will last a maximum of one hour. **The research assistant will be present with your child at all times during the interview, and the school library staff will also be present in the room.**

The research assistant would also like to videotape the conversations they have with your child about math only. This will help us better understand how children think about math problems. **The focus of the recordings will be on your child's hands only. His or her face will never be captured on tape.** These video recordings will be used for research and training purposes only and will not be shared with anyone outside the research team.

C. RISKS AND BENEFITS

The direct benefit to your child is that he or she will practice solving math problems related to fraction division and will receive a one-on-one lesson with the research assistant about these math problems. The content is directly related to the mathematics curriculum your child is learning in school.

Video recordings and other data will be collected for the study. To minimize the risk to participants and to maximize confidentiality, we will ensure that throughout the interview only your child's hands are recorded and not their faces. Additionally, all recordings will be stored in Dr. Osana's locked laboratory on a password-protected device at all times.

During the individual interviews, the risks are minimal. It is possible that your child gets bored during the individual interviews or gets frustrated because the math questions may be different from the ones he or she experiences in class. If at any point and for whatever reason your child does not want to continue participating, we will stop all activity and bring your child back to the classroom.

D. CONFIDENTIALITY

We will gather the following information as part of this research:

- Your child's responses on a 28-min paper-and-pencil test.
- Videotaped interview (total 1 hour). Your child will solve some math problems, receive a math lesson, and solve some more math problems during the videotaped interview.
- Your child's age, gender, and ethnicity.

By allowing your child to participate, you agree to let the researchers have access to the collected information.

We will not allow anyone to access the information, except people directly involved in conducting the research (H. Osana, A. Tomaszewski, M. Alibali (Professor at the University of Wisconsin-Madison, USA, collaborator on this project), and [name of research assistants – TBD]), and except as described in this form. Only H. Osana and A. Tomaszewski will have access to your child's name. We will only use the information for the purposes of the research and training described in this form.

To verify that the research is being conducted properly, regulatory authorities might examine the information gathered. By participating, you agree to let these authorities have access to the information.

The information gathered will be kept strictly confidential. This means that the research team (i.e., H. Osana, A. Tomaszewski, and Research Assistants who will be hired to assist in the data collection) will know your child's real identity, but it will not be disclosed.

We will protect the information by keeping it on a password-protected computer in the research lab of H. Osana in the FG Building (FG 6.403) and in a locked filing cabinet in the same office. Only the research assistants on this project will have the password and the key to the filing cabinet. The office is kept locked at all times.

We intend to publish the results of the research, but it will not be possible to identify your child in the published results.

We will destroy the information five years after the last presentation or publication that is generated from this study.

In certain situations we might be legally required to disclose the information your child provides. This includes situations where discoveries, such as child abuse or an imminent threat of serious harm to specific individuals, are uncovered as a result of our interactions. If this kind of situation arises, we will disclose the information as required by law, despite what is written in this form.

E. CONDITIONS OF PARTICIPATION

You do not have to agree for your child to participate in this research. It is purely your decision. If you do agree for your child to participate, he/she can stop at any time. You can also ask that your child's information that was provided not be used, and your choice will be respected. If you decide that you do not want us to use your child's information, you must tell the researcher before June 15, 2018.

We will tell you if we learn of anything that could affect your decision to stay in the research.

There are no negative consequences for not participating, stopping in the middle, or asking us not to use your information.

We will not be able to offer you compensation if you are injured in this research. However, you are not waiving any legal right to compensation by signing this form.

F. PARENTS'/GUARDIANS' DECLARATION

I have read and understood this form. I have had the chance to ask questions and any questions have been answered. I agree for my child to participate in this research under the conditions described.

NAME (please print) ____

SIGNATURE ____

DATE ____

If you have questions about the scientific or scholarly aspects of this research, please contact Dr. Helena Osana at Concordia University, the Principal Investigator on the project. Her contact information is on page 1 of this form.

If you have concerns about ethical issues in this research, please contact the Manager, Research Ethics, Concordia University, 514.848.2424 ex. 7481 or oor.ethics@concordia.ca.

Optional Permission:

☐ I give permission for **non-identifiable** portions of the videos of my child to be shown to educational or scientific audiences (e.g., in a scientific presentation, class, or teacher training).

Child Assent Form

Research Study Title: Units and Fraction Division

Researchers

Dr. Helena Osana (helena.osana@concordia.ca)

Anna Tomaszewski (anna.tomaszewski@concordia.ca)

What is this study about?

This research study is being done to find out how lessons on fractions help people solve math problems. You are being asked if you want to be in this research study because you just started seventh grade.

What will I need to do if I am in this study?

If you choose to be in the study, this is what will happen. The researcher will give you some math problems to solve during a 20-minute period in the classroom. Everyone participating in the study will complete this first set of problems on their own but at the same time. You will then participate in an individual interview with me or one of my friends from the university. You will answer a few more math problems and then receive a short math lesson. After the lesson, you will be asked to solve some more math problems. The whole study will take about one hour.

Can I stop being in the study?

You may stop being in the study at any time, and there will be no penalty.

Will anything negative happen to me if I am in the study?

You may feel uncomfortable if you do not know how to solve some of the problems. We are not evaluating your performance. Instead, we are trying to understand how students think about fraction problems.

What good things might happen to me if I am in the study?

You may learn new ways to solve fraction problems. This will help you in your math class at school. You may feel good knowing that what we find out from this study may help other people some day.

Will anyone know I am in the study?

We would like to videotape you as you solve the problems, so that we can remember what you did. I will only videotape your hands and not your face. My two teachers and I will look at the videos to understand your thinking. No one else will see those videos, unless you agree that we can show parts of the videos in a class or in a scientific presentation. Your name will not be written anywhere on your answer sheet. When we are finished with this study we will write a report about what we learned. This report will not include your name or that you were in the study.

Who can I talk to about the study?

If you have any questions about the study or any concerns, you can talk to your parent or guardian or anyone on the research team. You can contact the research team at helena.osana@concordia.ca or anna.tomaszewski@concordia.ca.

What if I do not want to take part in this study?

You don't have to be in this study. It is up to you. You can decide whether or not you want to be in this study, and you can stop taking part if you want to. If you say okay now, but change your mind later, that's okay too. Just tell one of us.

Child Authorization:

Your parent or guardian has to give permission for you to be in this study if you decide you want to take part.

I have been told about the study and what it involves. I agree to be in this study. I have been told that I can stop at any time. I have asked and received answers to my questions. I may keep a copy of this form.

If you would like to be in the study, please fill out the lines below.

Child's Printed Name: ____

Child's Signature: ____

Date: ___________________________

Optional Permission:

___I give permission for portions of the **non-identifiable** videos to be shown to educational or scientific audiences (e.g., in a scientific presentation, class, or teacher training).

Appendix B

Transcript of Intervention Lessons

Implicit Condition Lesson Transcript

PART ONE **Unitizing** (Note: The unitizing problem will be presented using green Unifix cubes. For this problem, the unit will be composed of 4 Unifix cubes stacked together to form a tower).
1. Display 6 towers made out of 4 unifix cubes each. Make sure there is about an inch distance between each tower. 2. **Here is 6.** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **1,2,3,4,5,6.** *(point to each tower while counting).* 3. **Each whole** *(point quickly to each of the 6 Unifix towers one at a time from left to right)* **has 4 fourths.** 4. **1, 2, 3, 4 fourths** *(point to each fourth while counting starting from the top of the tower and going down)*. **So, this is a whole** *(point to the first tower)* **made up of 4 fourths.** 5. **The same goes for all of those** *(make a swooping pointing gesture to remaining 5 towers).* 6. **I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as one group of 6.** 7. **1** *(make a circular gesture around all 6 towers)* 8. **I can also think of this 6 in a different way. I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as 2 groups of 3.** 9. Rearrange the 6 towers into 2 groups of 3 towers by leaving a two-inch distance between both groups. 10. **1** *(make a circular gesture around first group of 3 towers)*, **2** *(make a circular gesture around second group of 3 towers).* 11. **I can also think of this 6 in a different way. I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as 12 groups of ½.** 12. Rearrange the 6 towers into 12 groups of ½ towers by leaving a two-inch distance between each group. 13. **1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.** *(make a circular gesture around each ½ tower while counting).* 14. **I can also think of this 6 in a different way. I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as 8 groups of ¾.** 15. Rearrange the 6 towers into 8 groups of 3/4 towers by leaving a two-inch distance between each group. 16. **1, 2, 3, 4, 5, 6, 7, 8.** *(make a circular gesture around each ¾ tower while counting).* 17. Remove all Unifix cubes related to unitizing from participant's view.

PART TWO
Fraction Division

$$4 \div \frac{3}{4} = ?$$

(Note: The fraction division will be presented using red Unifix cubes. For this problem, the unit will be composed of 4 Unifix cubes stacked together to form a tower).

1. Place index card with equation in front of child.
2. **Let's look at this problem***(point to the equation on index card)* **In this problem, I want to divide 4 by ¾.** *(point to the parts of the equation 4, ¾).* **Here are my 4 wholes** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 4 towers).* **1, 2, 3, 4** *(point to each tower while counting)*
3. **In this problem** *(point to the equation on index card)***, each whole** *(point to each of the 4 Unifix towers one at a time)* **has 4 fourths.**
4. **1, 2, 3, 4 fourths** *(point to each fourth while counting starting from the top of the tower and going down).* **So, this is a whole** *(point to the first tower)* **made up of 4 fourths.**
5. **The same goes for these** *(make a swooping pointing gesture to remaining 3 towers).*
6. **I want to find out how many groups of ¾** *(point to ¾ in equation on index card)* **there are in 4** *(point to the 4 Unifix towers).* **Let's see.**
7. **1, 2, 3…** *(point to each "fourth" while counting; once counted remove 3 fourths or 3 stacked Unifix cubes)* **Here is one group of ¾.** *(Place the newly created tower of 3 Unifix cubes to the left of the set of the original towers)*
8. **1, 2, 3…** *(point to each "fourth" while counting; once counted remove 3 fourths or 3 stacked Unifix cubes)* **Here is another group of ¾** *(Place the newly created tower of 3 Unifix cubes to the left of the first ¾ tower).*
9. <u>Repeat instructions in point 9 more three times.</u>
10. **Here I have 1 piece (point to the remaining Unifix cube) out of the 3** *(make a pointing gesture with index figure and thumb that encompasses the size of the 3 pieces – pointing to the cube with the thumb and with the index pointing to the top "empty" space where the third cube would be)* **I need to have a full group of ¾. I have 1/3** *(point with thumb to the bottom of cube and index the top of the cube)* **of a group of ¾** *(point with thumb to the bottom of cube and index to the empty space where the third cube would be).*
11. **This means that there are 1 group, 2 groups, 3 groups, 4 groups, 5 groups of ¾** *(point to each ¾ tower while counting)* **and 1/3 of a group of ¾** *(point to the remainder).*
12. **The answer to 4** *(point to the 4 in the equation on the index card)* **divided by ¾** *(point to ¾ in the equation on the index card)* **is 5 1/3** *(trace along the bottom of the set of towers stopping after the 5 towers when saying "5" and then at the remainder when saying "1/3").*
13. Remove all Unifix cubes.

Explicit Condition Lesson Transcript

PART ONE

Unitizing

(Note: The unitizing problem will be presented using green Unifix cubes. For this problem, the unit will be composed of 4 Unifix cubes stacked together to form a tower).

1. Display 6 towers made out of 4 unifix cubes each. Make sure there is about an inch distance between each tower.
2. **Here is 6.** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **1,2,3,4,5,6.** *(point to each tower while counting).*
3. **Each whole** *(point quickly to each of the 6 Unifix towers one at a time from left to right)* **has 4 fourths.**
4. **1, 2, 3, 4 fourths** *(point to each fourth while counting starting from the top of the tower and going down)***. So, this is a whole** *(point to the first tower)* **made up of 4 fourths.**
5. **The same goes for all of those** *(make a swooping pointing gesture to remaining 5 towers).*
6. **I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as one group of 6.**
7. **1** *(make a circular gesture around all 6 towers)*
8. **I can also think of this 6 in a different way. I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as 2 groups of 3.**
9. Rearrange the 6 towers into 2 groups of 3 towers by leaving a two-inch distance between both groups.
10. **1** *(make a circular gesture around first group of 3 towers)***, 2** *(make a circular gesture around second group of 3 towers).*
11. **I can also think of this 6 in a different way. I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as 12 groups of ½.**
12. Rearrange the 6 towers into 12 groups of ½ towers by leaving a two-inch distance between each group.
13. **1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.** *(make a circular gesture around each ½ tower while counting).*
14. **I can also think of this 6 in a different way. I can think of this 6** *(swooping pointing gesture with index finger from the left to the right along the bottom of the 6 towers)* **as 8 groups of ¾.**
15. Rearrange the 6 towers into 8 groups of 3/4 towers by leaving a two-inch distance between each group.
16. **1, 2, 3, 4, 5, 6, 7, 8.** *(make a circular gesture around each ¾ tower while counting).*
17. Keep the last set of Unifix cubes related to unitizing in participant's view.

PART TWO
Fraction Division

$$4 \div \frac{3}{4} = ?$$

(Note: The fraction division will be presented using red Unifix cubes. For this problem, the unit will be composed of 4 Unifix cubes stacked together to form a tower).

1. Place index card with equation in front of child.
2. **Fraction division** *(point to the equation on index card)* **can be thought of in the same way as regrouping the number 6** *(point to the unitizing set)* **based on different sized groups. Here** *(point to the unitizing set)* **is the 6 made out of groups of ¾s.**
3. **Let's look at this problem***(point to the equation on index card)* **In this problem, I want to divide 4 by ¾.** *(point to the parts of the equation 4, ¾).* **Here are my 4 wholes** *(point to the 4 wholes),* **just like here I had the 6 wholes to start** *(point to the unitizing set).*
4. **In this problem** *(point to the equation on index card)***, each whole** *(point to each of the 4 Unifix towers one at a time)* **has 4 fourths.**
5. **1, 2, 3, 4 fourths** *(point to each fourth while counting starting from the top of the tower and going down).* **So, this is a whole** *(point to the first tower)* **made up of 4 fourths.**
6. **The same goes for these** *(make a swooping pointing gesture to remaining 3 towers).*
7. **I want to find out how many groups of ¾** *(point to ¾ in equation on index card)* **there are in 4** *(point to the 4 Unifix towers),* **just like when I wanted to know how 6** *(point to unitizing set)* **could be thought of in groups of ¾. Let's see.**
8. **1, 2, 3…** *(point to each "fourth" while counting; once counted remove 3 fourths or 3 stacked Unifix cubes)* **Here is one group of ¾.** *(Spatially align the created ¾ tower with the first ¾ tower in the unitizing set – the fraction division set (red Unifix cubes) should be placed below the unitizing set (green Unifix cubes)).*
9. **1, 2, 3…** *(point to each "fourth" while counting; once counted remove 3 fourths or 3 stacked Unifix cubes)* **Here is another group of ¾** *(Spatially align the second created ¾ tower with the second ¾ tower in the unitizing set – the fraction division set (red Unifix cubes) should be placed below the unitizing set (green Unifix cubes)).*
10. <u>Repeat instructions in point 9 more three times.</u>
11. **Here I have 1 piece (point to the remaining Unifix cube) out of the 3** *(make a pointing gesture with index figure and thumb that encompasses the size of the 3 pieces – pointing to the cube with the thumb and with the index pointing to the top "empty" space where the third cube would be)* **I need to have a full group of ¾. I have 1/3** *(point with thumb to the bottom of cube and index the top of the cube)* **of a group of ¾** *(point with thumb to the bottom of cube and index to the empty space where the third cube would be).*
12. **This means that there are 1 group, 2 groups, 3 groups, 4 groups, 5 groups of ¾** *(point to each ¾ tower while counting)* **and 1/3 of a group of ¾** *(point to the remainder).*
13. **Just like here when I had 1 group, 2 groups, 3 groups, 4 groups, 5 groups, 6 groups, 7 groups, 8 groups of ¾.** *(point to the towers in the unitizing set while counting).*
14. **The answer to 4** *(point to the 4 in the equation on the index card)* **divided by ¾** *(point to ¾ in the equation on the index card)* **is 5 1/3** *(trace along the bottom of the set of*

towers stopping after the 5 towers when saying "5" and then at the remainder when saying "1/3").

15. Remove all Unifix cubes.

Control Condition Lesson Transcript

PART ONE

$$4 \div \frac{3}{4} = ?$$

(Note: The fraction division will be presented using red Unifix cubes. For this problem, the unit will be composed of 4 Unifix cubes stacked together to form a tower).

1. Place index card with equation in front of child.
2. **Let's look at this problem***(point to the equation on index card)* **In this problem, I want to divide 4 by ¾.** *(point to the parts of the equation 4, ¾).* **Here are my 4 wholes** *(point to the 4 wholes).*
3. **In this problem** *(point to the equation on index card)*, **each whole** *(point to each of the 4 Unifix towers one at a time)* **has 4 fourths.**
4. **1, 2, 3, 4 fourths** *(point to each fourth while counting starting from the top of the tower and going down).* **So, this is a whole** *(point to the first tower)* **made up of 4 fourths.**
5. **The same goes for these** *(make a swooping pointing gesture to remaining 3 towers).*
6. **I want to find out how many groups of ¾** *(point to ¾ in equation on index card)* **there are in 4** *(point to the 4 Unifix towers).* **Let's see.**
7. **1, 2, 3…** *(point to each "fourth" while counting; once counted remove 3 fourths or 3 stacked Unifix cubes)* **Here is one group of ¾.** *(Place the newly created tower of 3 Unifix cubes to the left of the set of the original towers)*
8. **1, 2, 3…** *(point to each "fourth" while counting; once counted remove 3 fourths or 3 stacked Unifix cubes)* **Here is another group of ¾** *(Place the newly created tower of 3 Unifix cubes to the left of the first ¾ tower).*
9. <u>Repeat instructions in point 9 more three times.</u>
10. **Here I have 1 piece (point to the remaining Unifix cube) out of the 3** *(make a pointing gesture with index figure and thumb that encompasses the size of the 3 pieces – pointing to the cube with the thumb and with the index pointing to the top "empty" space where the third cube would be)* **I need to have a full group of ¾. I have 1/3** *(point with thumb to the bottom of cube and index the top of the cube)* **of a group of ¾** *(point with thumb to the bottom of cube and index to the empty space where the third cube would be).*
11. **This means that there are 1 group, 2 groups, 3 groups, 4 groups, 5 groups of ¾** *(point to each ¾ tower while counting)* **and 1/3 of a group of ¾** *(point to the remainder).*
12. **The answer to 4** *(point to the 4 in the equation on the index card)* **divided by ¾** *(point to ¾ in the equation on the index card)* **is 5 1/3** *(trace along the bottom of the set of towers stopping after the 5 towers when saying "5" and then at the remainder when saying "1/3").*
13. Remove all Unifix cubes.

PART TWO: REPEAT EXACT SAME INSTRUCTION A SECOND TIME

Appendix C

Table 4

Remainder Interpretations Observed in the Bar Diagram Task and Unifix Cube Task

Interpretation of Remainder	Definition	Examples
CATEGORY 1: Remainder Interpretations		
REFERENT		
Remainder interpreted using the referent unit	The remainder is clearly interpreted using the divisor as the unit (e.g., in the equation $1 \div \frac{2}{5}$, $\frac{2}{5}$ would be the referent unit), reflecting the form of a fractional amount.	In the equation, 1 divided by $\frac{2}{5}$, the remainder would be interpreted as: • $\frac{1}{2}$ • 1 out of the 2 blocks needed to make a whole.
Remainder identified as a partial group of the divisor	The remainder is interpreted as part of a group of the divisor. The remainder is not assigned a specific numerical value, but there is an implicit use of the referent unit.	In the equation, 1 divided by $\frac{2}{5}$, the remainder would be interpreted as "a piece of the groups of $\frac{2}{5}$" or an "incomplete group of $\frac{2}{5}$."
ORIGINAL		

Remainder interpreted using the original unit	The remainder is interpreted using the unit of 1 (e.g., in the equation 1 ÷ $\frac{2}{5}$, 1 or $\frac{5}{5}$ would be considered the original unit).	In the equation, 1 divided by $\frac{2}{5}$, the remainder would be interpreted as $\frac{1}{5}$, 1 over 5, or 1 out of the 5 blocks I need to have a full original group.
Remainder identified as partial group of original whole	The remainder is interpreted as part of the original unit. The remainder is not assigned a specific numerical value, but there is an implicit use of the original unit.	In the equation, 1 divided by $\frac{2}{5}$, the remainder would be interpreted as "a piece of the groups of $\frac{5}{5}$". "I need four more pieces to have a whole (where there is 1 remainder and 5 make a whole)."
REFERENT AND ORIGINAL		
Remainder interpreted using both the referent and original units	The remainder is explicitly interpreted using both the original and referent units (e.g., in the equation 1 ÷ $\frac{2}{5}$, both 1 and $\frac{2}{5}$ would be used to interpret the remainder).	In the equation, 1 divided by $\frac{2}{5}$, the remainder would be interpreted as both $\frac{1}{5}$ and $\frac{1}{2}$. "This is $\frac{1}{2}$ of $\frac{2}{5}$, so it is $\frac{1}{5}$."
NEW UNIT INTERPRETATIONS		

Remainder interpreted using a new unit	The remainder is interpreted using the dividend as the unit (e.g. in the equation $3 \div \frac{2}{5}$, the dividend 3 would be considered as the unit).	In the equation, 3 divided by $\frac{2}{5}$, the 3 would be considered as the new unit of interpretation. This new unit of 1 would be composed of 3 original wholes, making the unit equivalent to 15 original fifths. The remainder would be interpreted as 1/15 of the new unit.
Remainder identified as part of a new unit	The remainder is interpreted using the dividend as the unit (e.g., in the equation $3 \div \frac{2}{5}$, the dividend 3 would be considered as the unit); however, no numerical value is assigned to the remainder.	"This (points to the remainder) is part of this (points to the dividend of the equation or the bars representing the dividend)." "This is not a complete group. I need 14 more to have a full group of 15 pieces."
OTHER PART OF A GROUP INTERPRETATIONS		
Remainder identified as part of an non-identified group	The participant interprets the remainder as part of a group, but does not explain which group it is	"This is just a part of the next group" "I need some more pieces, this one is not full."

	part of or what size the full group (unit) should have.	
Remainder identified as part of a new incorrect unit	The participant interprets the remainder using an inappropriate unit and referring without any numerical value.	"This remainder is part of this (points to the whole colored section of the bar)."
Remainder interpreted using a new non-logical unit	The participant interprets the remainder using an inappropriate unit and referring to the remainder with a fractional amount.	The unit used to interpret the remainder is the number of referent whole groups (e.g., in 3 divided by $\frac{2}{5}$, 7 groups of 2/5 would be the unit of interpretation).
Leftover	No numerical value is assigned to the remainder. It is simply treated as the leftover piece from the division.	"the remainder" "the mystery piece" "the leftover" "the extra unnecessary piece"
CATEGORY 2: OTHER INTERPRETATIONS		
Whole-Number Interpretation of the Remainder	The participant interprets the remainder as a whole number instead of a fractional amount.	The participant counts all sections on the second bar and concludes the last piece represents the last piece in the sequence (1, 2, 3, 4, 5, 6, 7, 8, 9. Oh it's 9).

Remainder identified through correct procedure	The participant solves the problem in a correct procedural way. (May be used as a support of a previous remainder explanation)	In the equation, 1 divided by $\frac{2}{5}$, the participant would simply solve the equation using the algorithm and say the answer is $2\frac{1}{2}$.
Remainder identified through incorrect procedure	The participant solves the problem in an incorrect procedural way. (May be used as a support of a previous remainder explanation)	In the equation, 1 divided by $\frac{2}{5}$, the participant would describe irrelevant or incorrect calculations when solving the problem (e.g., "1 minus 2/5 instead of 1 divided by 2/5").
Non-Remainder Identified	The participant ignored the remainder or identified something that is not the remainder as the remainder.	The participant says the remainder is 6/7 because there are 6 colored parts in total in a bar made of 7 parts.
Identified the missing part of a full referent whole, instead of the remainder	The participant identified the missing part which with the remainder would form a referent whole.	In a problem where the divisor is 4/5 and the correct remainder is 1/4, the participant responds, "This is 3/4, because I am missing 3 blocks to have a full group".

Identified the missing part of a full original whole, instead of the remainder	The participant identified the missing part which with the remainder would form an original whole, instead of the remainder.	In a problem where the divisor is 4/5 and the correct remainder is 1/4, the participant responds: “This is 3/5, because I am missing 3 blocks to have a full group”.
Divisor	The participant identifies the remainder as having the value of the divisor.	“This (points to remainder) is 3/7 (points to the 3/7 divisor in equation).”
Remainder identified as part of the final answer	The remainder is identified using the complete answer to the division.	In the equation, 1 divided by $\frac{2}{5}$, the remainder would be identified as $2\frac{1}{2}$.
Superficial features of the equation and/or of the bar diagram identified	The participant describes something he/she is seeing without making comments related to parts or groups. The surface features can be perceptual (e.g., colors, shape) or related to the	“This last piece is this whole over here (points to the dividend)” “This last piece is this invisible 1 that you put under your whole when doing calculations.” “This is the piece that was not colored in.”

	equation (e.g., pointing to dividend, division sign, denominator or numerator without conceptual connections).	"It's a rectangle."
Superficial features of the equation and/or of the bar diagram used as a support to explain another remainder interpretation	Perceptual features of the bar diagram and/or of the equation are used to support their non-superficial remainder explanations.	"There are 4 colored blocks and one non-colored, so I know that this remainder is ¼."
Did Not Know/Guess	The participant did not know what the remainder represented or incorrectly guessed what the piece represented.	"I don't know what this piece represents." "I guess it's maybe 1/5, but I don't know."
Uninterpretable	The participant's train of thought is uninterpretable.	