Improved upper bounds and lower bounds on broadcast function

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Abstract

Improved upper bounds and lower bounds on broadcast function

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Given a graph $G = (V, E)$ and an *originator* vertex v, broadcasting is an information disseminating process of transmitting a message from vertex v to all vertices of graph G as quickly as possible. A graph G on n vertices is called *broadcast graph* if the broadcasting from any vertex in the graph can be accomplished in $\lceil \log n \rceil$ time. A broadcast graph with the minimum number of edges is called *minimum broadcast graph*. The number of edges in a minimum broadcast graph on n vertices is denoted by $B(n)$. A long sequence of papers present different techniques to construct broadcast graphs and to obtain upper bounds on $B(n)$. In this thesis, we study the compounding and the vertex addition broadcast graph constructions, which improve the upper bound on $B(n)$. We also present the first nontrivial general lower bound on $B(n)$.

Acknowledgments

I would like to express my deepest gratitude to my supervisor, Dr. Hovhannes A. Harutyunyan for his immeasurable support and guidance throughout the research.

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This thesis is dedicated to all pioneers.

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Chapter 1

Introduction

One-to-all information spreading is one of the major tasks on a modern interconnection network. This process, named *broadcasting*, originates from one node in the network, called *originator*, and finishes when every node in the network has the information. The broadcast time is one of the main measures of network performance.

Over the last four decades, a long sequence of research papers study broadcasting in networks under different models. These models differ at the number of originators, the number of receivers at each time unit, the distance of each call, the number of destinations, and other characteristics of the network. In this thesis, we focus on the classical model with the following assumptions:

- the network has only one originator;
- each call has only one informed node, the sender and one of its uninformed neighbors - the receiver;
- every call requires one time unit.

A network is modeled as a simple connected graph $G = (V, E)$, where the vertex set V represents the nodes in the network, and the edge set E represents the communication links.

Definition 1.1. The *broadcast scheme* is a sequence of parallel calls in a graph G originating from a vertex v. Each call, represented by a directed edge, defines the sender and the receiver. The broadcast scheme generates a *broadcast tree*, which is a directed spanning tree of the graph G rooted at the originator.

Definition 1.2. Let G be a graph on n vertices and v be the broadcast originator in G. $b(G, v)$ defines the minimum number of time units required to broadcast from the originator v in the graph G. The broadcast time of the graph G, $b(G) = \max\{b(G, v)|v \in V(G)\}$ defines the maximum number of time units required from any originator to broadcast in the graph G.

Note that $b(G) \geq \lceil \log n \rceil$, since the number of informed vertices can at most double during each time unit.

Definition 1.3. A graph G on n vertices is called *broadcast graph* if $b(G) = \lceil \log n \rceil$. A broadcast graph with the minimum number of edges is called *minimum broadcast graph* (mbg). This minimum number of edges is called broadcast function and denoted by $B(n)$.

From the application perspective mbgs represent the cheapest graphs (with minimum number of edges), where broadcasting can be accomplished in the minimum possible time.
In this big research area of broadcasting messages in a graph, there are two major topics:

(1) *minimum broadcast graph problem*, construct the minimum broadcast graph on n vertices with the given integer n, or determine the value of broadcast function $B(n)$;

(2) *broadcast time problem*, determine the broadcast time of a given graph, or find the optimal broadcast scheme of the graph.

1.1 Minimum broadcast graph problem

1.1.1 Minimum broadcast graphs

The study of minimum broadcast graphs and the broadcast function $B(n)$ has a long history. Farley, Hedetniemi, Mitchell and Proskurowski have introduced minimum broadcast graphs in [22]. In the same paper, they have defined the broadcast function, determined the values of $B(n)$, for $n \leq 15$ and $n = 2^k$, and proven that hypercubes are minimum broadcast graphs. Khachatrian and Haroutunian [50] and independently Dinneen, Fellows and Faber [17] have shown that Knödel graphs, defined in [52], are minimum broadcast graphs on $n = 2^k - 2$ vertices. Park and Chwa have proven that the recursive circulant graphs on 2^k vertices are minimum broadcast graphs [62]. The comparison of information dissemination properties of these three classes of minimum broadcast graphs can be found in [23]. Besides these three classes, there is no other known infinite construction of minimum broadcast graphs. The values of $B(n)$ have been also known for $n = 17$ [61], $n = 18, 19$ [9, 74], $n = 20, 21, 22$ [59], $n = 26$ [66, 76], $n = 27, 28, 29, 58, 61$ [66], $n = 30, 31$ [9], $n = 63$ [56], $n = 127$ [30] and $n = 1023, 4095$ [69].

1.1.2 Upper bounds on broadcast function

Since minimum broadcast graphs are difficult to construct, a long sequence of papers present different techniques to construct broadcast graphs in order to obtain upper bounds on $B(n)$. Furthermore, proving that a lower bound matches the upper bound is also extremely difficult, because most of the lower bound proofs are based only on vertex degree. However, minimum broadcast graphs except hypercubes and Knödel graphs on $2^k - 2$ vertices are not regular.

Upper bounds on $B(n)$ are given by constructions of sparse broadcast graphs. A. Farley has constructed broadcast graphs recursively by combining two or three smaller broadcast graphs and shows $B(n) \leq \frac{n}{2}\lceil \log n \rceil$ [21]. This construction has been generalized in [10] using up to seven small broadcast graphs. A tight asymptotic bound on $B(n) = \Theta(L(n) \cdot n)$ has been given in [27] by proving that $\frac{L(n)-1}{2}n \leq B(n) \leq (L(n)+2)n$, where $L(n)$ is the number of consecutive leading 1's in the binary representation of $n-1$. In [50], the compounding method has been introduced which uses vertex cover of graphs. This method constructs new broadcast graphs by forming the compound of several known broadcast graphs. In [7], the compounding method has been generalized to any n by using solid vertex cover. A compounding method using center vertices has been introduced in [73] and shown to be equivalent to the method of using solid vertex cover in [18]. The authors in [38] have continued on the line of compounding and introduced a method of also merging vertices. And more recently [3, 35], compounding binomial trees with hypercubes has improved the upper bound on $B(n)$ for many values of n.

Vertex addition is another approach to construct good broadcast graphs by adding several vertices to existing broadcast graphs [9]. [30] has continued on this line and added one vertex to Knödel graphs on $2^k - 2$ vertices. The added vertex is connected to every vertex in a dominating set of the Knödel graph. In [40], the same method has been applied to generalized Knödel graphs, in order to construct broadcast graphs on any odd number of vertices.

Adhoc constructions sometimes also provide good upper bounds. This method usually constructs broadcast graphs by adding edges to a binomial tree [27, 38].

Vertex deletion has been studied in [9, 38]. Several other constructions have been presented in [9, 26, 27, 38, 72–74].

1.1.3 Lower bounds on broadcast function

Lower bounds on $B(n)$ are also studied in the literature. In [26], Gargano and Vaccaro have shown $B(n) \geq \frac{n}{2}(\lfloor \log n \rfloor - \log(1+2^{\lceil \log n \rceil} - n))$, for any n. $B(n) \geq \frac{n}{2}(m-p-1)$ has been proved in [53], where m is the length of the binary representation $a_{m-1}a_{m-2}...a_1a_0$ of n and p is the index of the leftmost 0 bit. Harutyunyan and Liestman have studied k-broadcasting (every sender can inform at most k neighbors in each time unit) and given a lower bound on the broadcast function for k-broadcast graph in [39]. The latter bound has been the best known general lower bound for our model of broadcasting (which corresponds to the case $k = 1$ in [39]). Let $n = 2^m - 2^k + 1 - d$, $1 \leq k \leq m-2$ and $0 \leq d \leq 2^k - 1$. $B(n) \geq \frac{n}{2}(m-k)$.

Besides the general lower bounds, the lower bounds for special values have been studied. Labahn has shown $B(n) \geq \frac{m^2(2^m-1)}{2(m+1)}$ for $n = 2^m - 1$ in [56] by considering the broadcast tree rooted at a vertex with the minimum degree. Saclé has followed this method and gives tight lower bounds on $B(2^m-3)$, $B(2^m-4)$, $B(2^m-5)$ and $B(2^m-6)$ in [66]. Grigoryan and Harutyunyan have further shown that $B(2^m - 2^k + 1) \geq \frac{2^m-2^k+1}{2}(m - k + \frac{m(2k-1)-(k^2+k-1)}{m(m-1)-(k-1)})$. Better lower bounds on $n = 24, 25$ have been given in [5]. Note that $23 \leq n \leq 25$ are the only values of $n \leq 32$ for which $B(n)$ is not known.

1.2 Broadcast time problem

Since Slater, Cockayne, and Hedetniemi [70] have proven that determining the broadcast time for an arbitrary vertex in an arbitrary graph is NP-complete, many researchers spend a lot of efforts on approximation and heuristic algorithms to solve the problem.

An approximation finding the poise of a graph, which is an NP-hard problem, and an $O(\frac{\log^2 n}{\log\log n})$-approximation algorithm for broadcasting using the *poise* has been given in

[65]. The *poise* of a tree is the sum of its diameter and the maximum degree. An $O(\sqrt{n})$-additive approximation for graphs in general and an $O(\frac{\log n}{\log \log n})$-multiplicative approximation for graphs in the *open-path* model have been presented in [54]. In this model, every informed vertex can send the message to an uninformed vertex via a path of any length in each time unit. An $O(\log k)$-additive approximation for *multicasting* in a *heterogeneous* graph has been suggested in [4]. In *multicasting*, the process stops if every vertex in a given subset of all vertices are informed. A *heterogeneous* network connects devices with different operating systems and/or protocols. The NP-hardness of finding an efficient approximation for single-source broadcasting in a general graph and multi-source broadcasting in a ternary graph has been proven in [68]. An $O(\frac{\log k}{\log \log k})$-approximation for k-broadcasting has been given in [19]. And the approximation has been later improved of a ratio between $\Omega(\sqrt{\log n})$ and $O(\sqrt{k})$ in [20].

A heuristic *least-weight maximum matching* has been given in [67]. A *least-weight maximum matching* is a subset of the edges with all vertices are matched and the total weight of the edges are minimum. The genetic algorithm on broadcasting in random graphs has been studied in [47]. Two different heuristics for *gossiping* have been given in [6]. *Gossiping* is an all-to-all information spreading process in contrast to the one-to-all process of broadcasting. A heuristic of the optimal broadcast scheme on some simple topologies, for example rings and trees and almost optimal broadcast scheme on torus is presented in [43]. A simple search heuristic algorithm for multicasting on random networks has been given in [32]. The performance of the algorithm on three different random graphs has also been presented in the same paper. A new heuristic has been studied in [45]. This heuristic gives the optimal broadcast time for rings, trees, and grids if the originator is on the corner. Simulations have shown that this heuristic also gives good solutions for two different models of Internet and ATM networks.

1.3 Minimum broadcast graphs and the broadcast time problem on other models

Variant of broadcasting on different models have been also studied with respect to many application reasons.

The network reliability has been considered in the fault-tolerant broadcasting. The k fault-tolerant broadcast graph has been defined and the tradeoff among broadcast time, the number of link failures, and the number of edges of the graph has been studied in [58]. Minimum k fault-tolerant broadcast graphs for some values of n and k have been constructed in [1]. A survey of the fault-tolerant broadcasting and gossiping on networks has been given in [63].

A variant of broadcasting using universal lists aims reduce the local memory cost of each node in the network. In classical broadcasting, every node needs a large memory to store different transmission lists corresponding to different originators. In contrast with universal list, each vertex uses a little memory for a single ordered list of some of its neighbors and only informs its neighbors from its list in prescribed order. This model has been first studied for trees, rings, and 2D grids in [16]. Several different broadcst schemes with universal list for paths, k-ary trees, grids, complete graphs, and hypercubes have been designed in [51]. The *nonadaptive* broadcasting model with universal list has been studied in [41]. This model always uses the broadcast list. Thus, every nonoriginator sends the information back to the sender which it is received.

Considering a node in the real-world network may not know the topology of the whole network, knowledge bounded models have been studied. Radio broadcasting is one of the models, such that every node in the network only knows the topology in its knowledge range. A linear-time radio broadcast scheme on ad hoc multi-hop netwroks has been given in [11]. The trade-offs among the eccentricity of the originator, the broadcast time, and the number of transmissions of an ad hoc network has been studied in [14]. The effect of knowledge radius on broadcasting in geometric radio networks has been discussed in [15]. The same paper has also given a linear-time broadcast scheme when the knowledge radius is large. A survey of the studies related to radio broadcasting has been presented in [64]. In a more general case, the messy broadcasting problem, a random broadcast model has been studied if nodes know nothing about their neighbors' topology in [2]. The same paper has also given the solutions in trees. An algorithm solving the messy broadcasting problem with three different further assumptions on networks has been presented in [25]. The first algorithm for directed graphs and the bounds on the messy broadcast time of directed toris have been given in [12]. The trade-offs between knowledge radius of each vertex and the broadcast time have been studied in [13]. The order of informed neighbors of each vertex which gives the minimum messy broadcast time in a 2D torus has been discussed in [44]. The average-case messy broadcast time of stars, paths, cycles, d-ary trees, hypercubes, and complete graphs has been shown in [57]. The worst case of messy broadcasting based on three different models, which depends on how much a vertex knows about its senders and receivers has been analyzed in [31].

For more on broadcasting and gossiping in general see the following survey papers [24, 28, 42, 46, 48, 49, 55, 60, 69, 71, 75].

This thesis mainly focuses on general upper bounds and lower bounds on broadcast function. In Chapter 2, we review the existing constructions of broadcast graphs and general bounds. In Chapter 3, we follow the compounding method and the vertex addition method and further improve general upper bounds. In Chapter 4, we give a new general lower bound. Chapter 5 gives two minor results related to our research. Chapter 6 lists several topics for the future work. And Chapter 7 concludes the thesis.

Chapter 2

Previous results

This chapter reviews the major results on the topic of minimum broadcast graph. First, we introduce two types of minimum broadcast graphs. Then, we summarize the best known general upper and lower bounds on broadcast function $B(n)$.

2.1 Definitions and important graphs

Definition 2.1. A *hypercube* Q_k of order k, for any $k \in \mathbb{N}$ on 2^k vertices is recursively defined. When $k = 0$, the hypercube Q_0 is a single vertex. When $k \geq 1$, the hypercube Q_k can be constructed by having two copies of the hypercube Q_{k-1} (the vertices are labeled as $v_1, v_2, ..., v_{2^{k-1}}$) and connecting two vertices with the same label in the two copies.

It is easy to see that a hypercube Q_k is a k regular graph with $\frac{1}{2}k2^k = k2^{k-1}$ edges.

Definition 2.2. A *binomial tree* BT_k of order k has 2^k vertices for any $k \geq 0$. When $k = 0$, the binomial tree B_0 is a single vertex. When $k \geq 1$, the binomial tree B_k consists of two binomial trees BT_{k-1} having their roots r_1 and r_2 connected by an edge. Either of r_1 or r_2 is the root of the binomial tree B_k.

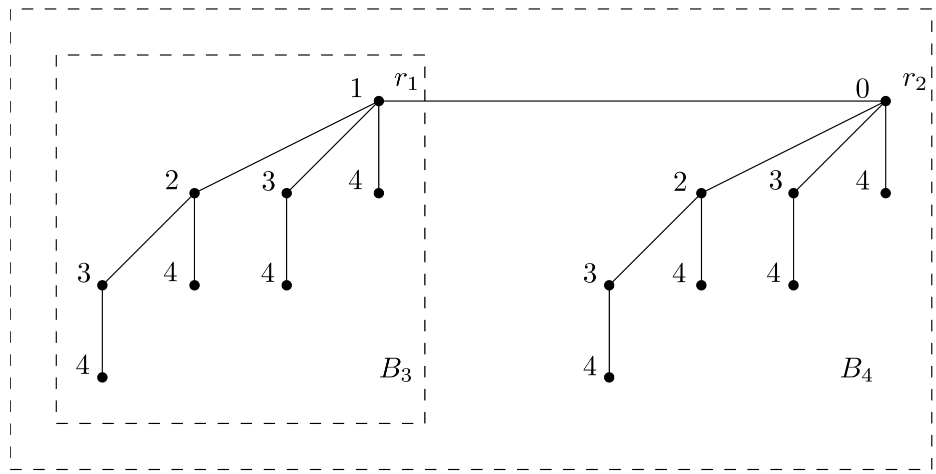


Figure 2.1: The binomial tree BT_4 is constructed by connecting the roots r_1 and r_2 of two binomial trees BT_3. The root of BT_4 can be either one of the roots r_1 or r_2. The numbers show the broadcast scheme from r_2 in BT_4.

Binomial trees are useful for constructing broadcast graphs, since the broadcast time of the root in a binomial tree BT_k is k which is the minimum possible time. It is easy to see that a binomial tree BT_k is a broadcast tree of any broadcast scheme from any vertex in a hypercube Q_k. Furthermore, any broadcast tree of a broadcast graph on n vertices is a subtree of $BT_{\lceil \log n \rceil}$. Figure 2.1 presents an example of a binomial tree BT_4, and a minimum time broadcast scheme from the root vertex r_2.

In 1975, Knödel defined a class of broadcast graphs on even number of vertices [52]. We follow the equivalent definition given in [38, 50].

Definition 2.3. A Knödel graph $KG_n = (V, E)$ is defined for even values of n, where the vertex set is $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$ and the edge set is $E = \{(v_x, v_y) | x + y \equiv 2^s - 1 \mod n, 1 \leq s \leq \lfloor \log n \rfloor\}$, where $0 \leq x, y \leq n - 1$.

By the definition above, if $(v_x, v_y) \in E$, we say that v_x and v_y are connected on dimension s. Furthermore, v_x is v_y's neighbor on dimension s or vice versa. The following broadcast scheme of a Knödel graph on n vertices is called *dimensional broadcast* scheme [8]. That is in the first $\lceil \log n \rceil - 1$ time units, every vertex with the message calls its neighbor on dimension t at time unit t, $1 \leq t \leq \lceil \log n \rceil - 1$. Then at the last time unit every vertex calls

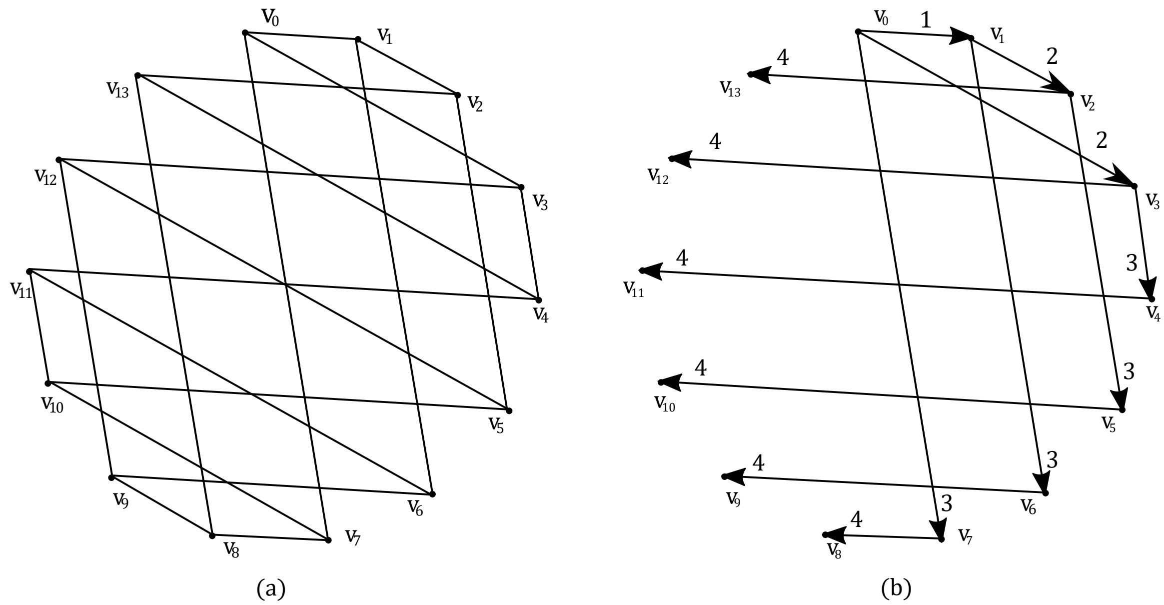


Figure 2.2: (a): an example of KG_{14}; (b): the broadcast scheme from v_0 in KG_{14}

its neighbor on dimension 1.

Let $n = 2^m - 2^k - 2a$, $0 \leq k \leq m-2$ and $0 \leq 2a \leq 2^{k-1}$. It is easy to see that KG_n has the following well known properties.

- KG_n is $(m-1)$-regular. Each vertex has $m - 1 = \lfloor \log n \rfloor$ dimensional neighbors. Each edge has dimension i for all $1 \leq i \leq m-1$
- KG_n is bipartite, v_i and v_j are adjacent only if i and j have different parities.
- KG_n has $\frac{n\lfloor \log n \rfloor}{2}$ edges.

Figure 2.2 shows one example of a Knödel graph on 14 vertices for $k = 4$ and the dimensional broadcast scheme from v_0 in KG_{14}.

Figure 2.3a and 2.3b show the bipartite representation and the corresponding dimensional broadcast scheme of KG_{14}. In particular, $1, 2, \cdots, m-1$ denotes the dimensional broadcast scheme, where at time unit i, all of the informed vertices call their i-th dimensional neighbors for all $i = 1, 2, \cdots, m-1$ and call their first dimensional neighbors at time unit m. Authors of [8] also show that any *cyclic shift* of dimensions, $s, s+1, \cdots, m-1, 1, 2,$

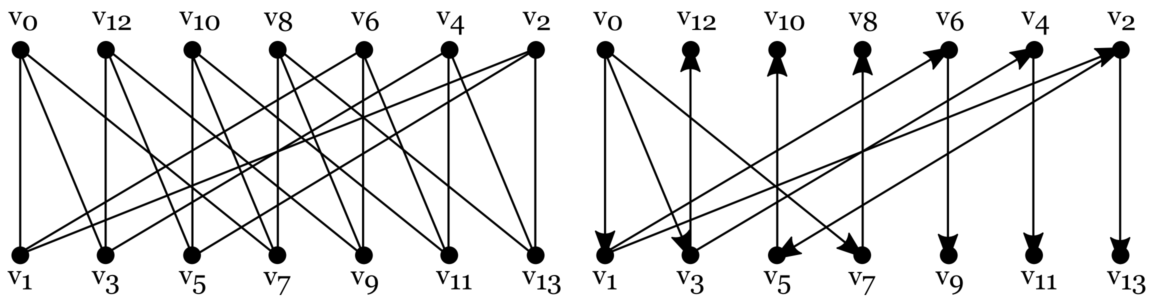


Figure 2.3: The Knödel graph KG_{14} and its broadcast scheme. An even vertex v_i locates its s'th dimensional neighbor at the 2^{s-1}'s position to the right of the odd vertex right under v_i (also v_i's first dimensional neighbor).

$\cdots$, $s-1$, s is also a valid broadcast scheme for the Knödel graph KG_n, and $1 \leq s \leq m-1$. The bipartite representation of a Knödel graph $KG(n)$ can be considered as a ring or an infinite tie if we take the modulus n of an arbitrary integer as the index of a vertex and repeating copies of $KG(n)$. For example, v_{n+2} is equivalence to v_2 and adjacent to v_1, v_3, v_7, $\cdots$.

Theorem 2.1. The Knödel graph KG_{2^m-2}, for $m \geq 3$ is a minimum broadcast graph.

Proof. The dimensional broadcast scheme shows that Knödel graphs are broadcast graphs. Then, we prove that KG_{2^m-2} is minimum by contradiction. Assume there is a broadcast graph G on 2^m-2 vertices with strictly less than $\frac{1}{2}(m-1)(2^m-2)$ edges. By the pigeonhole principle, there must be one vertex $v \in G$ with degree at most $m-2$. Then considering the broadcasting from vertex v in the graph G, it takes at most $m-2$ time units to call v's neighbors. So in the graph G, at most 2^{m-2} vertices are informed at time unit $m-2$. Then v has no uninformed neighbors and it will be idle at time units $m-1$ and m. Thus, at time unit $m-1$ and m, at most $2^{m-2}-1$ and $2^{m-1}-2$ vertices are informed respectively. In total, at most 2^m-3 vertices in graph G are called within m time units, but G has 2^m-2 vertices. Therefore $B(2^m-2) \geq \frac{1}{2}(m-1)(2^m-2)$. This implies KG_{2^m-2} is a minimum broadcast graph. □

2.2 Upper bounds

2.2.1 Compounding method based on Knödel graphs and hypercubes

[38] has presented a broadcast graph construction by compounding an existing broadcast graph with a Knödel graph. Let $G = (V, E)$ be a broadcast graph on p vertices, KG_{2^m-2} be a Knödel graph on $2^m - 2$ vertices, and $\lceil \log p(2^m - 2) \rceil = \lceil \log p \rceil + m$. The construction of a new broadcast graph $G' = (V', E')$ on $p(2^m - 2)$ vertices is as follows:

- Create p copies of the Knödel graph KG_{2^m-2}, named $KG^1, KG^2, ..., KG^p$. Each KG^i has the vertex set $V^i = \{v_0^i, v_1^i, ..., v_{2^m-3}^i\}$, where $1 \leq i \leq p$.
- Define the vertex set $V' = \{v_j^i | v_j^i \in V^i \backslash \{v_0^i\}, 1 \leq i \leq p\} \cup \{v_0^i | i = 1 \text{ or } r + 1 \leq i \leq p, 2 \leq r \leq p - 1\}$.
- Define $E' = E_{local} \cup E_{product}$. E_{local} is the edge set of each KG^i and $E_{product} = \{(v_t^i, v_t^j) |\ t \text{ is odd}, (i, j) \in E\}$

The new vertex set contains all the vertices from p copies of Knödel graph KG_{2^m-2} and merging the vertices labeled v_0 in $r + 1$ copies of KG_{2^m-2}, where $2 \leq r \leq p - 1$. The construction selects all vertices with the same odd index from each copy of the Knödel graph KG_{2^m-2} and forms a copy of broadcast graph G. That is $2^{m-1} - 1$ copies of G in total. The number of vertices in the graph G' is $|V| = p(2^m - 2) - r$ and the number of edges is $|E| = (2^{m-1} - 1)(p(m - 1) + e)$, where e is the number of edges in G. Figure 2.4 is an example of the construction with KG_6, Q_2 and no merging vertices.

To show G' is a broadcast graph, we introduce the broadcast scheme. Recall that G is a broadcast graph. So broadcasting from any vertex $v \in G$ in the graph G takes $\lceil \log p \rceil$ time

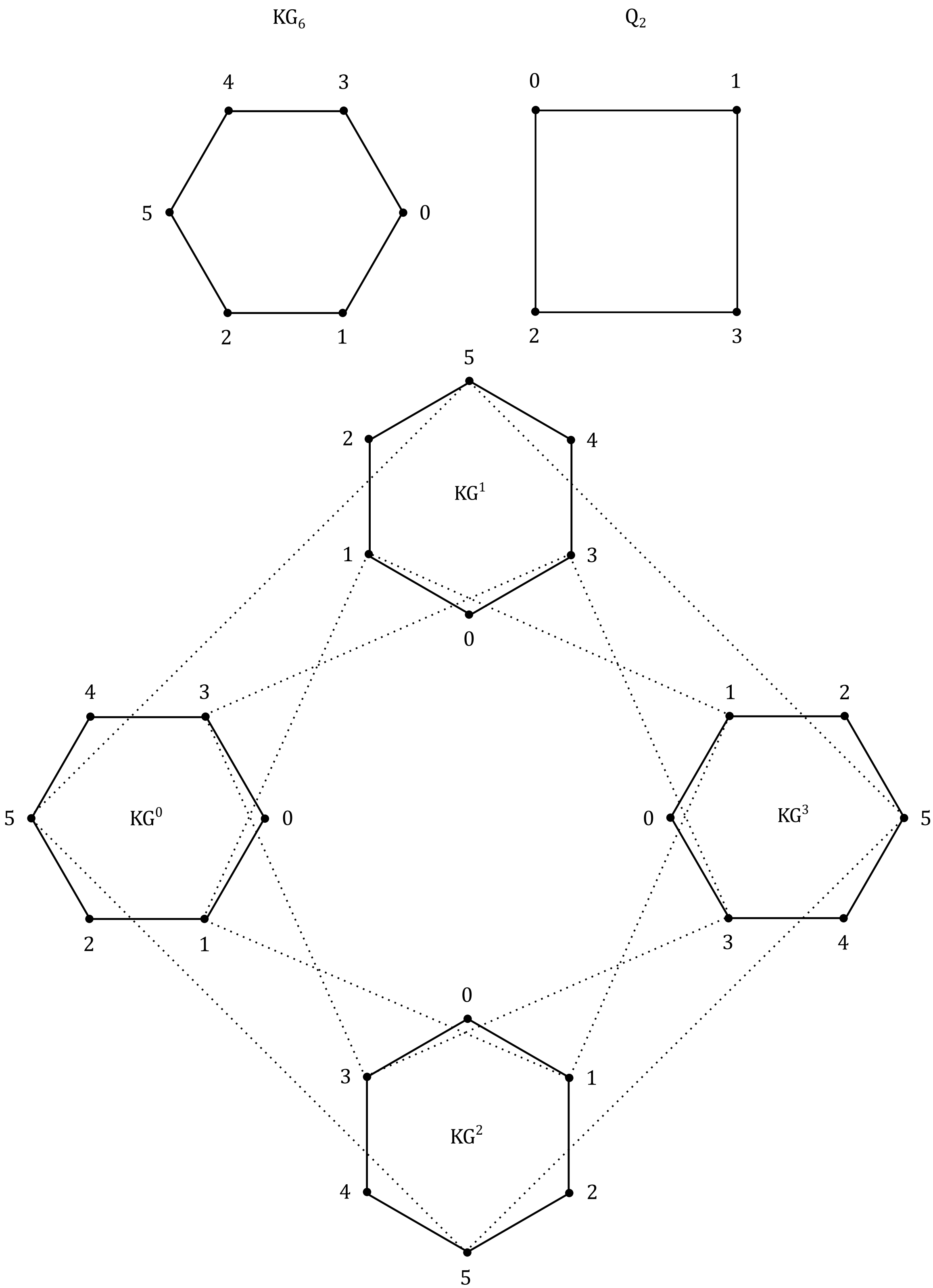


Figure 2.4: Construction of new broadcast graph with KG_6 and Q_2.

units.

- If the originator is vertex v_t^s, where $t \in [0, 2^m - 3]$ is odd and $0 \leq s \leq p-1$, then v is on a copy of G. The broadcasting from v_t^i in the copy of G takes $\lceil \log p \rceil$ time. Then each copy of KG_{2^m-2} has one vertex informed. Broadcasting from the vertex inside its copy of KG_{2^m-2} finishes in m time units, which means the total broadcast time in G' is equal to $\lceil \log p \rceil + m = \lceil \log(p(2^m - 2)) \rceil$.

- If the originator is a vertex v_t^s, where t is even, then it is not in a copy of G. It has to take $m-1$ time units to inform its neighbors inside its copy of KG_{2^m-2}. Let the neighbor b_i^s of originator v_t^s adjacent on dimension i is informed at time unit i, $1 \leq i \leq m-1$. Note that b_i^s is in the copy of G_i. Once a neighbor b_i^s is informed, it starts broadcasting on its copy of G_i immediately. Then at time unit $i + \lceil \log p \rceil$ every vertex in G_i is informed.

  Let v_t^j be the vertex in the copy of Knödel graph KG^j and b_i^j be the neighbors of v_t^j on dimension i, where $0 \leq i \leq m-1$ and $0 \leq j \leq p-1$. It is important to note that the vertex b_i^j and the vertex b_i^s are in the same copy of broadcast graph G_i. So the neighbor b_i^j of the vertex v_t^j on dimension i in the copy of KG^j is informed before time unit $i + \lceil \log p \rceil$ and can start broadcasting in KG^j at time unit $i + \lceil \log p \rceil + 1$. Consider the broadcasting from v_t^j in the copy of KG^j, b_i^j is informed at time unit i and finishes broadcasting all vertices on b_i^j's branch of the broadcast tree in the rest of $m-i$ time units. Thus, in the broadcast scheme originating from the vertex v_t^s in the graph KG_{2^m-2} compounded by the broadcast graph G, every vertex b_i^j is infromed at the right time unit and dimensionally broadcasts in its copy of KG^j in the rest of $k-i$ time units, except the vertex v_t^j. v_t^j can be informed by the vertex b_0^j at the last time unit, because b_0^j is idle. Therefore, the broadcasting finishes in $m + \lceil \log p \rceil$ time units.

This construction provides a method to create new broadcast graphs by compounding existing broadcast graphs. And more importantly, merging vertices enlarges the range of number of vertices, even for some odd numbers and prime numbers.

2.2.2 Compounding Binomial trees and hypercubes

[3] presents a new construction, compounding binomial trees and hypercubes. Their result is the tightest known general upper bound on broadcast function: for any $n = (2^{m-k} - 1)2^k - d$, where $m \geq 3$, $0 \leq k \leq m-2$, and $0 \leq d \leq 2^k - 1$, $B(n) \leq (m-k+1)n - (\frac{m}{2} + \frac{k}{2} + 1)2^{m-k} + k + 1$. We should notice that any $n \in [2^{m-1}+1, 2^m - 1]$ can be represented by $n = 2^m - 2^k - d$, where $0 \leq k \leq m-2$ and $0 \leq d \leq 2^k - 1$.

The new broadcast graph G consists of $2^{m-k} - 1$ binomial trees $BT_1, BT_2, ..., BT_{2^{m-k}-1}$ on 2^k vertices. Let vertex w be the leaf on the longest branch of binomial tree BT_1, which is at distance k from the root of BT_1. The root of each binomial tree $r_1, r_2, .., r^{2^{m-k}-1}$ and the vertex w (2^{m-k} vertices in total) form a hypercube Q of dimension $m-k$. So, in total there are $n = (2^{m-k} - 1)2^k$ vertices in graph G. Then we remove d leaves, where $0 \leq d \leq 2^k - 1$ from an arbitrary binomial tree B_j, $1 \leq j \leq 2^{m-k} - 1$ in the graph G to obtain general $n = (2^{m-k} - 1)2^k - d$ in the range $[2^{m-1} + 1, 2^m - 1]$.

The hypercube Q can be partitioned into $m - k + 1$ hypercubes $Q^{m-k-1}, Q^{m-k-2}, ...,$ Q^1,Q^0,Q^{01}, where Q^i for $1 \leq i \leq m-k-1$ is a hypercube of dimension i containing the roots from r_{2^i-1} to $r_{2^{i+1}-1}$. Q^0 and Q^{01} are both of dimension 0 and actually r_1 and m respectively.

The edges E of G are of three types. The set of edges in the hypercube Q is denoted by E_Q. The set of edges in binomial trees $BT_1, BT_2, ..., BT_{2^{m-k}-1}$ is denoted by E_T. And the set of edges connecting a tree vertex to some root vertices in the hypercube Q is denoted by $E_P = \{(u, r_l) | u \in BT_j, l = 2^i - 1, i \neq \lfloor \log j + 1 \rfloor, 0 \leq i \leq k-1, 0 \leq j \leq 2^{m-k} - 1\} \cup \{(u, r_l) | u \in BT_l\}$. In other words, the vertex u is a non-root vertex in the binomial tree BT_l. It is connected to the root r_l of its binomial tree BT_l and the root r_0, $r_1, r_3, ..., r_{2^i-1}$ except $r_{\lfloor \log j+1 \rfloor}$. In summary, every non-root vertex u has exactly one root neighbor in each of the hypercubes $Q^0, Q^1, ..., Q^{m-k-2}$.

The number of edges is counted separately. The hypercube of dimension $m-k-1$ has $|E_Q| = (m-k)2^{m-k-1}$ edges. $2^{m-k}-1$ binomial trees have $|E_T| = (2^{m-k}-1)(2^k-1)$ edges. If a non-root vertex u is not on the first level of its binomial tree, it is connected to $m-k$ roots. vertices in the hypercube. If the vertex u is on the first level, it is connected to $m-k-1$ roots. By simple calculations, $|E_P| = (m-k)(2^{m-k}-1)2^k - m2^{m-k} + k$. Then $d(m-k+1)$ edges are deleted after removing d leaves from a binomial tree. Thus, the total number of edges in the graph G is $|E| = (m-k+1)n - (\frac{m}{2} + \frac{k}{2} - 1)2^{m-k} + k + 1$ and the number of vertices is $n = 2^m - 2^k - d$.

Figure 2.5 gives an example of the construction above. The minimum time broadcast scheme for the constructed graph is as follows:

(1) If the originator is a vertex r_i in the hypercube, then it takes $m-k$ time units to finish informing all vertices in the hypercube. Then each vertex, a root of a binomial tree broadcasts to all vertices in its binomial tree in k time units. The broadcasting finishes in $m = \lceil \log n \rceil$ time units.

(2) If the originator is a non-root vertex u in a binomial tree, it takes $m-k$ time units to inform one vertex in each of the sub-hypercubes $Q_0, Q_1, ..., Q_{m-k-1}$. Once vertex r_i in hypercube Q^{m-k-i} is informed at time i, it broadcasts in hypercube Q^{m-k-i} in $m-k-i$ time units. Thus, the broadcasting in the whole hypercube (except vertex m) finishes in $m-k$ time units. Then the broadcasting in binomial trees informs all other non-root vertices in binomial trees. This broadcasting also takes m time units.

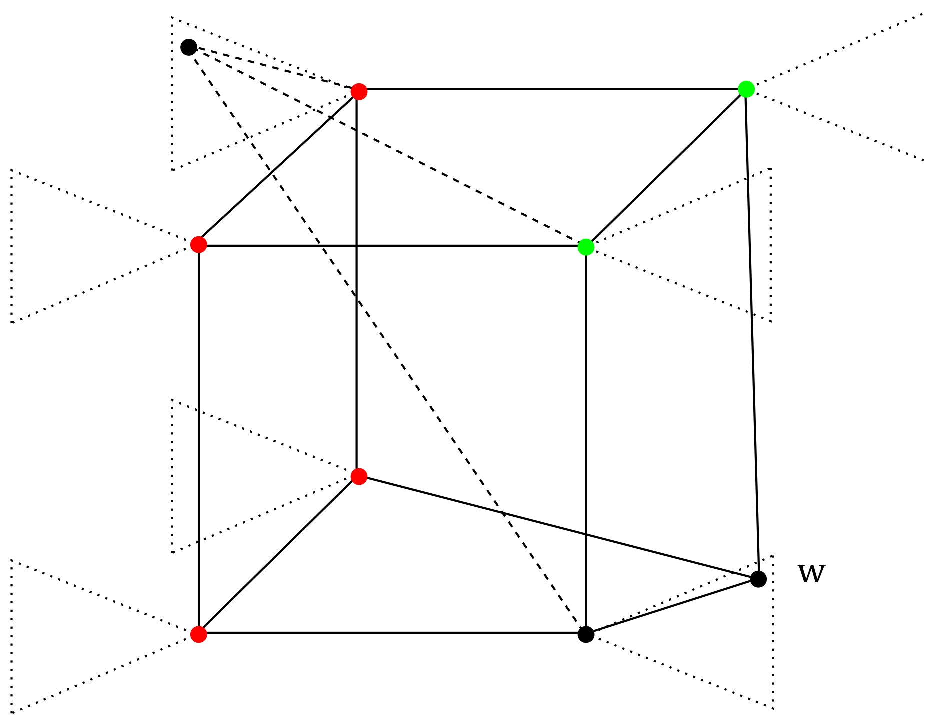


Figure 2.5: An example of compounding binomial trees and the hypercube Q_3. The hypercube is partitioned into Q^2, the hypercube on 4 vertices on the left side; Q^1, the hypercube on 2 vertices on the right upper corner; Q^0 and Q^{01}, two hypercubes on a single vertex. Each dashed triangle represents a binomial tree.

2.2.3 Vertex addition method using dominating set for KG_{2^m-2}

[30] introduces a vertex addition method for constructing broadcast graphs. This method adds one vertex connecting to every vertex in the dominating set of a Knödel graph.

Definition 2.4. The *dominating set* S of a graph $G = (V, E)$ is a subset of V such that any vertex in $V \backslash S$ is adjacent to at least one vertex in S. The *dominating number* of G, $\gamma(G)$ is the cardinality of a minimum dominating set S.

A dominating set of a Knödel graph KG_{2^m-2} is defined as follows.

Lemma 2.1. Let $KG_{2^m-2} = (V, E)$ be a Knödel graph on $2^m - 2$ vertices, with $V = \{v_0, v_1, ..., v_{2^m-3}\}$, where $m \geq 3$. Let S be the subset of V such that $S = \{v_{2p} | 0 \leq p \leq 2^{m-3} - 1\} \cup \{v_{2p+1} | 2^{m-2} \leq p \leq 2^{m-2} + 2^{m-3} - 1\}$. Then S is a dominating set of KG_{2^m-2} with dominating number $\gamma(KG_{2^m-2}) \leq 2^{m-2}$.

Proof. Let $S = S_1 \cup S_2$, $S_1 = \{v_{2p} | 0 \leq p \leq 2^{m-3} - 1\}$ and $S_2 = \{v_{2p+1} | 2^{m-2} \leq p \leq 2^{p-2} + 2^{p-3} - 1\}$. Intuitively, S_1 contains all even vertices in the first quarter of the indices and S_2 has all the odd vertices in the third quarter. Since a Knödel graph is bipartite and by definition of Knödel graphs, two vertices are adjacent only if they have different parities. Thus, every odd vertex in KG_{2^m-2} is adjacent to at least one vertex in S_1, while an even vertex is adjacent to at least one vertex in S_2. The proof has two cases:

(1) For v_j with odd j:

- If $0 \leq j \leq 2^{m-2} - 1$, $v_j \in \{v_1, v_3, ..., v_{2^{m-2}-1}\}$, each vertex can be denoted by $v_{2^{m-2}-1-t}$, where t is even and $0 \leq t \leq 2^{m-2} - 2$. Then by definition of S_1, $v_t \in S_1$. $v_{2^{m-2}-1-t}$ is adjacent to v_t on dimension $m-2$, since $2^{m-2}-1-t+t \equiv 2^{m-2} - 1 \mod 2^m - 2$.

- If $2^{m-2}+1 \leq j \leq 2^{m-1}-1$, similarly each vertex is denoted by $v_{2^{m-1}+1-t}$, t is even and $0 \leq t \leq 2^{m-2}-2$. Then $v_{2^{m-1}+1-t}$ and v_t are adjacent on dimension $m-1$.
- If $2^{m-1}+2^{m-2}+1 \leq j \leq 2^m-3$, each vertex is v_{2^m-3-t}, t is even and $0 \leq t \leq 2^{m-2}-4$. Then v_{2^m-3-t} is adjacent to v_{t+2} on dimension 1, since $2^m-3-t+t+2 = 2^m-1 \equiv 1 \mod 2^m-2$.

(2) For v_j with even j, there are also three similar cases. Vertices in S_2 are adjacent to vertices with indices between 2^{m-2} and $2^{m-1}-2$ on dimension 1, vertices with indices between 2^{m-1} and $2^{m-1}+2^{m-2}-4$ on dimension $m-2$ and vertices with indices between $2^{m-1}+2^{m-2}-4$ and 2^m-4 on dimension $m-1$.

□

The new broadcast graph G is a Knödel graph KG_{2^m-2} with one additional vertex w connecting to every vertex in the dominating set. The broadcast scheme of G is as follows:

(1) If the originator is an arbitrary vertex in the dominating set denoted by v, it dimensionally informs all vertices in the Knödel graph and is idle before the last time unit. Then it informs w in the last time unit.

(2) If the broadcasting originated at an arbitrary vertex not in the dominating set denoted by v, it must has a neighbor u on dimension i, $0 \geq i \geq k-1$ in the dominating set. v can inform all vertices in KG_{2^m-2} on dimension $i, i+1, ..., m-1, 0, ..., i-1$. Then u is idle at the last time unit. So, u can inform w at time unit m.

(3) If we broadcast from the additional vertex w, the broadcast scheme is similar to broadcasting from v_1. Consider the broadcast from v_1 in KG_{2^m-2}, the neighbor b_i of vertex v_1 on dimension i is informed at time unit $i+1$. It is easy to see that every neighbor b_i is in the dominating set except $v_{2^{m-1}-2}$. Thus, in the broadcasting from

vertex w in graph G, w plays the role of v_1 in the minimum time broadcasting in KG_{2^m-2}. In particular, following the dimensional broadcast scheme $1, 2, \cdots, m-1, 1$. Every neighbor b_i of vertex v_1 is informed at time unit i. At time unit $m-1$, w informs vertex $v_{2^{m-1}+1}$. Then at the last time unit, $v_{2^{m-1}+1}$ calls vertex $v_{2^{m-1}-2}$ and vertex v_0 calls vertex v_1. So, the broadcasting is finished after time unit m.

2.2.4 Dominating set of Knödel graphs

A dominating set of a Knödel graph is defined as follows in [30, 40]:

Theorem 2.2. [40] If $n = 2^{m-1}+2l, 1 \leq l \leq 2^{m-2}-1$, then $S = \{v_x | 2^{m-2} \leq x \leq 2^{m-1}-1\}$ is a dominating set of KG_n, and the domination number satisfies $\gamma(KG_n) \leq 2^{m-2}$.

Proof. (1) For v_x, $0 \leq x \leq 2^{m-2}-1$

$$2^{m-1} - 2^{m-2+1} \leq 2^{m-1} - x \leq 2^{m-1} - 1$$

$$2^{m-2} + 1 - 1 \leq 2^{m-1} - 1 - x \leq 2^{m-1} - 1$$

Then we define $v_z \in S$, $z = 2^{m-1} - 1 - x$. v_z and v_x are adjacent, since $z + x \equiv 2^{m-1} - 1 \mod (2^{m-1} + 2l)$

(2) For v_x, $2^{m-1} \leq x \leq 2^{m-1} + 2l - 1$, there exists $0 \leq y \leq 2l - 1$ such that $x = 2^{m-1} + 2l - 1 - y$. Then we construct $z = 2^{m-2} + y$ and $z \in S$. v_z and v_x are adjacent, since $x + z = 2^{m-1} + 2l + 2^{m-2} - 1 \equiv 2^{m-2} - 1 \mod 2^{m-1} + 2l$.

□

The dominating set constructed by Theorem 2.2 is not minimum. The minimum dominating set of Knödel graphs are unknown in general. But, [40] has given the minimum dominating set for some particular values of n and k.

Theorem 2.3. $\gamma(KG_n) = \frac{n}{m}$, where n is even, $m = \lceil \log n \rceil > 2$ is a prime, m divides n, and for any integer $d < m-1$ which is a divisor of $m-1$ satisfies $2^d \not\equiv 1 \mod m$. The dominating set $S = S_1 \cup S_2$, where $S_1 = \{v_{2mp} | 0 \le p < \frac{n}{2m}\}$ and $S_2 = \{v_{2mp-1} | 1 \le p < \frac{n}{2k}\} \cup \{n-1\}$

The proof of this theorem has three steps. First, we show that S is actually an independent set. Any two vertices in S are not adjacent. Then, any vertex $v_p \in V \backslash S$ is adjacent to at most one vertex in S. At the end, each v_p is adjacent to at least one vertex in S.

Proof. First, we prove that S is an independent set. Let v_x and v_y be two vertices in S. If v_x and v_y are adjacent, x and y cannot be both even or both odd. Otherwise it contradicts to the definition of KG_n. Therefore, we assume $x = 2m\alpha$ is even, $y = 2m\beta - 1$ is odd and v_x is adjacent to v_y without loss of generality. Then $x + y = 2m(\alpha + \beta) - 1 \equiv 2^j - 1 \mod n$, where $1 \le j \le m-1$, which is impossible since $2k(\alpha+\beta) \equiv 2^j \mod n$. But, k is a prime number, $\alpha + \beta > 0$ and $j > 0$.

If v_p of odd p is adjacent to v_x and v_y in S. x and y have to be even numbers. Assuming $x = 2m\alpha$, $y = 2m\beta$ and $\alpha > \beta$, then $x+p = 2m\alpha+p \equiv 2^i-1 \mod n$, $y+p = 2m\beta+p \equiv 2^j - 1 \mod n$ and $i > j$. If we subtract y from x, $x - y = 2m(\alpha - \beta) \equiv 2^i - 2^j \mod n$. By $v_{2m\alpha}, v_{2m\beta} \in S$, $0 \le \beta < \alpha < \frac{n}{2m}$. Thus, $\alpha - \beta < \frac{n}{2m}$ and $2m(\alpha - \beta) < n$. Then $2m(\alpha - \beta) = 2^i - 2^j$ and $\alpha - \beta = \frac{2^{j-1}(2^{i-j}-1)}{m}$. Since for any d, a divisor of $m-1$, $2^d \not\equiv 1 \mod m$, any $1 \le c \le m-2$, $2^c \not\equiv 1 \mod m$ by Lagrange's Theorem. Since $i \le m-1$ and $1 \le j$, $1 \le i-j \le m-2$, $2^{i-j} \not\equiv 1 \mod m$ and $2^{i-j} - 1 \not\equiv 0 \mod m$. Therefore, $\frac{2^{j-1}(2^{i-j}-1)}{m}$ is not an integer, which is a contradiction. If p is even we have the similar proof.

Each vertex in S has $m-1$ neighbors, because KG_n is $m-1$ regular. And there are $\frac{n}{2m} + \frac{n}{2m} - 1 + 1 = \frac{n}{m}$ vertices in S. The set S' of vertices adjacent to vertices in S has cardinality $\frac{n(m-1)}{m}$. By the previous two steps, $S' \cap S = \emptyset$. $S' \cup S$ has cardinality $\frac{n}{m} + \frac{n(m-1)}{m} = n$. Therefore, $S' \cup S = V$.To summarize, S is a dominating set of KG_n.

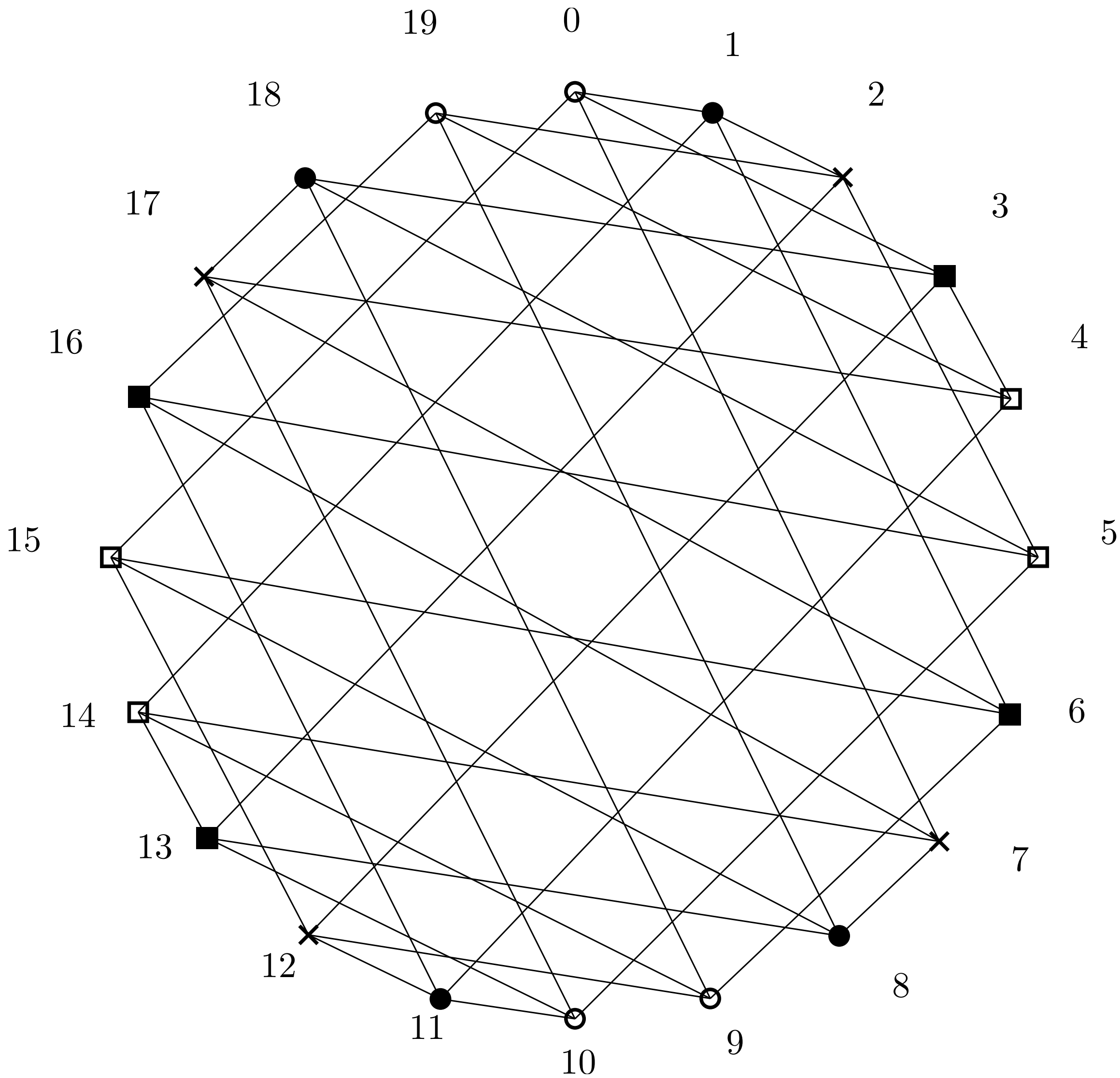


Figure 2.6: KG_{20} and dominating sets

Since S is also an independent set, then S is a minimum dominating set of KG_n. $\square$

Figure 2.6 shows one example of KG_{20} and the dominating sets. Since $n = 20$ is even and $\lfloor \log 20 \rfloor = 4$ is a divisor of 20, the dominating set $S = \{0\} \cup \{9, 10\} \cup \{19\}$. V is actually partitioned into 5 dominating sets: $\{0, 9, 10, 19\}$, $\{1, 8, 11, 18\}$, $\{2, 7, 12, 17\}$, $\{3, 6, 13, 16\}$ and $\{4, 5, 14, 15\}$.

Once the dominating set is defined, then a broadcast graph of $n + 1$ vertices can be constructed by adding one vertex adjacent to every vertex in the dominating set. The best general upper bound on broadcast function is $B(n) \leq \frac{n\lfloor \log n \rfloor}{2}$ by Knödel graphs for even n and $B(n) \leq \frac{(n-1)\lfloor \log n \rfloor}{2} + 2^{\lfloor \log n \rfloor - 2}$ for odd n. The above theorem improves the upper bound for odd, if $\lfloor \log n \rfloor > 2$ is a prime number and a divisor of n, then $B(n+1) \leq \frac{n\lfloor \log n \rfloor}{2} + \frac{n}{\lfloor \log n \rfloor} + \lceil \log n \rceil - 2$.

2.3 Lower bounds

First, we review the best existing general lower bound on $B(n)$ given in [39].

Theorem 2.4. Let $n = 2^m - 2^k - d$, where $1 \leq k \leq m - 2$ and $0 \leq d \leq 2^k - 1$.

$$B(n) \geq \frac{n}{2}(m - k)$$

Proof. Assume graph G is a broadcast graph on n vertices. Instead of directly estimating the number of edges in G, we show that the minimum degree of vertices in G is $m - k$ by contradiction. Then, the result will follow.

Assume there is a vertex u in graph G of degree $m - k - 1$. We consider the broadcasting originated from u. The maximum number of vertices are informed if every vertex is busy in the broadcasting. The broadcast tree rooted at u consists of complete binomial trees BT_{m-1}, BT_{m-2}, $\cdots$, B_{m-k-1} with their roots adjacent to u. The total number of vertices in this broadcast tree has the following number of vertices.

$$\begin{aligned}
&2^{m-1} + 2^{m-2} + \cdots + 2^{k+1} + 1 \\
=&2^m - 2^{k+1} + 1 \\
<&2^m - 2^k - d \\
=&n
\end{aligned}$$

Thus, we cannot inform all vertices in graph G. By contradiction, the minimum degree of

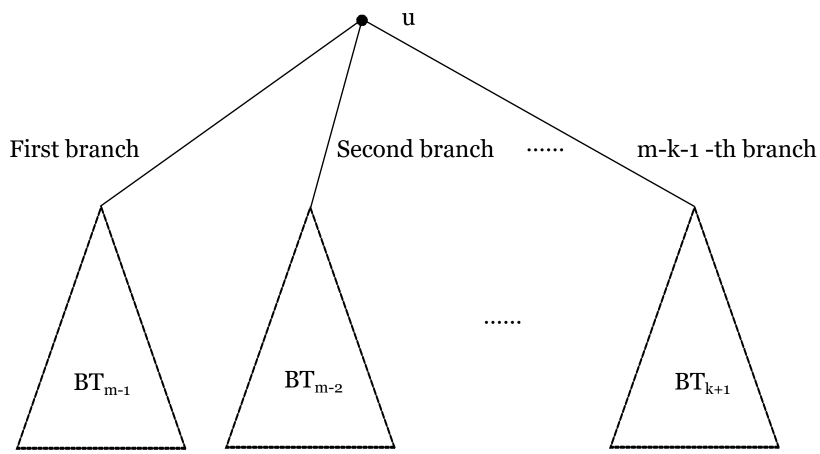


Figure 2.7: The optimal broadcast tree rooted at a vertex u of degree $m-k-1$

any vertex in G is at least $m-k$. Therefore, $B(n) \geq \frac{n}{2}(m-k)$. Figure 2.7 illustrates the broadcast tree if its root has degree $m-k-1$. □

Other than the general lower bound, [56] introduces an interesting lower bound on $B(n)$ for $n = 2^m - 1$. This lower bound is obtained by studying the degree of a certain vertex in the broadcast graph on $2^m - 1$ vertices.

Lemma 2.2 ([56]).

$$B(2^m - 1) \geq \frac{m^2(2^m - 1)}{2(m+1)}, \text{ for any } m \in \mathbb{N}$$

Proof. From the proof of Theorem 2.1, we know that there is no vertex of degree $m-2$ in a broadcast graph G on $2^m - 2$ vertices.

Then, consider the broadcasting from an originator u of degree $m-1$. u has to be idle at the last time unit m. So in order to inform $2^m - 2$ vertices (excluding the root u) in graph G, u has to call a vertex of degree at least m at the first time unit. Thus, every vertex of degree $m-1$ must have at least one neighbor of degree m. And a vertex of degree at least m can have at most m neighbors of degree $m-1$. Then graph G must have at least $\frac{2^m-1}{m+1}$ vertices of degree m or larger. Therefore, $B(2^m - 1) \geq \frac{m^2(2^m-1)}{2(m+1)}$, for any $m \geq 0$. □

By Fermat's little theorem, $m+1$ is a divisor of 2^m-1 only if $m+1$ is a prime number. If $m+1$ is not a divisor of 2^m-1, every vertex of degree $m-1$ cannot be adjacent to exactly one vertex of degree m. So, constructing such graphs holding Fermat's little theorem is intuitively easier than the other graphs. By following this idea, the minimum broadcast graph on 63 vertices is constructed when $m=6$ in [56] and $mbg(1023)$ and $mbg(4095)$ are constructed when $m=10$ and 12 respectively in [69].

2.4 A summary of the bounds on $B(n)$

Let $n = 2^m - 2^k - d$, $m \geq 3$, $0 \leq k \leq m-3$,and $0 \leq d \leq 2^k - 1$. After a simple comparison, we list the best known general upper bounds

$$UB(n) = \begin{cases} (m-k+1)n - (\frac{m}{2}+\frac{k}{2}+1)2^{m-k}+k+1, \\ \qquad \text{if } 2^{m-1}+1 \leq n \leq 2^m - 2^{\frac{1}{2}(m+3)} \text{ [3]}; \\ \frac{1}{2}(m-1)n, \\ \qquad \text{if } 2^m - 2^{\frac{1}{2}(m+3)} < n \leq 2^m \text{ for even } n \text{ [52]}; \\ \frac{1}{2}(m-1)n + 2^{m-2} - \frac{1}{2}(m-1), \\ \qquad 2^m - 2^{\frac{1}{2}(m+3)} < n \leq 2^m \text{ for odd } n \text{ [40]}. \end{cases}$$

and the best known general lower bound

$$LB(n) = \frac{n}{2}(m-k)$$

In the next chapter, we will improve both of the upper bound and the lower bound.

Chapter 3

New upper bounds

In this chapter, we continue the studies of compounding and vertex addition construction of broadcast graphs. Our new constructions improve the upper bounds on $B(n)$.

3.1 New compounding construction

3.1.1 Compounding Knödel graphs with binomial trees

In this section, we introduce a new broadcast graph construction similar to the compounding method in [3] for any $2^{m-1}+1 \leq n \leq 2^m - 1$, where $m \geq 5$, but using Knödel graphs as a base instead of a hypercube. The later comparison shows that this construction improve the upper bound on $B(n)$ for any $2^{m-1}+1 \leq n \leq 2^m - 2^{frac12(m+3)}$, where $n = 2^m - 2^k - d$, $m \geq 5$, $2 \leq k \leq m-2$, and $0 \leq d \leq 2^k - 1$. It is clear that any value of $n \in [2^{m-1}+1, 2^m - 2]$ can be represented as $n = 2^m - 2^k - d$, where $1 \leq k \leq m-2$ and $0 \leq d \leq 2^k - 1$. For convenience, we let $l = k-1$, $n = 2^m - 2^{l+1} - d$, $0 \leq l \leq m-3$, and $0 \leq d \leq 2^{l+1} - 1$ in the following constructions.

The new broadcast graph $L = (V, E)$ on $n = (2^{m-l} - 2)2^l$ vertices, where $m \geq 5$ and $0 \leq l \leq m-3$ is constructed from $2^{m-l} - 2$ copies of binomial tree of degree l, denoted by BT_0,BT_1,..., $BT_{2^{m-l}-3}$. The roots of the binomial trees denoted by r_i, form a Knödel graph $KG_{2^{m-l}-2}$ on $2^{m-l}-2$ vertices, $0 \leq i \leq 2^{m-l}-3$. Figure 3.1 presents the new construction for $m=6$ and $l=2$.

The next step of the construction is to delete d vertices from L, where $0 \leq d \leq 2^{l+1}-1$, in order to obtain any $2^{m-1}+1 \leq n \leq 2^m - 1$, the given number of vertices of the broadcast graph, where $m \geq 5$. This step can be done by deleting a leaf from any binomial tree repeatedly. Note that we do not delete the root of any binomial tree because it also belongs to $KG_{2^{m-l}-2}$. The number of deleted vertices is at most $2^{l+1}-1$.

Then the new construction connects the vertices of binomial trees BT_0, BT_1,..., $BT_{2^{m-l}-3}$ to $m-l-1$ vertices of $KG_{2^{m-l}-2}$.

Let r_i be the root of binomial tree B_i and r_h be the first dimensional neighbor of r_i in $KG_{2^{m-l}-2}$. By the definition of Knödel graph, $h \equiv 1-i \mod 2^{m-l}-2$. We connect each non-root vertex w in binomial tree BT_i to all the neighbors of r_h in $KG_{2^{m-l}-2}$. Let r_j denote these neighbors, $j+h \equiv j+1-i \equiv 2^s-1 \mod 2^{m-l}-2$ for all $s=1,2,...,m-l-1$. The edges of E of graph L are of three types: the edges in the Knödel graph $KG_{2^{m-l}-2}$ denoted by E_H, the edges in all binomial trees $BT_0, BT_1, ..., BT_{2^{m-l}-3}$ denoted by E_T, and the edges between vertex $w \in BT_i$ and some vertices in the Knödel graph denoted by E_P. Therefore, the set of edges of graph $L=(V,E)$ is defined as $E = E_H \cup E_T \cup E_P$, where $E_P = \{(w, r_j) | j+1-i \equiv 2^s - 1 \mod 2^{m-l}-2, 1 \leq s \leq m-l-1, w \in BT_i \setminus \{r_i\}, r_j \in KG_{2^{m-l}-2}\}$. Thus, the number of edges in L is $|E| = |E_H| + |E_T| + |E_P|$. The Knödel graph $KG_{2^{m-l}-2}$ has

$$|E_H| = \frac{(m-l-1)(2^{m-l}-2)}{2}$$

edges. All $2^{m-l}-2$ binomial trees $BT_0, BT_1, ...BT_{2^{m-l}-3}$ together have

$$|E_T| = (2^{m-l}-2)(2^l-1) - d$$

tree edges. To count the number of edges in E_P, each binomial tree has $2^l - l - 1$ vertices except the root and its l neighbors on the first level. In total, graph L has $(2^{m-l} - 2)(2^l - l - 1) - d$ such vertices remaining after removing d leaves. Each of these vertices needs $m - l - 1$ edges to connect to the vertices in the Knödel graph. And each of the vertices on the first level of any binomial tree (the l neighbors of the root within a binomial tree) needs $m-l-2$ additional edges connecting to the vertices of $KG_{2^{m-l}-2}$, since it is already adjacent to its root. Thus,

$$|E_P| = ((2^{m-l} - 2)(2^l - l - 1) - d)(m - l - 1) + (2^{m-l} - 2)l(m - l - 2)$$

The total number of edges of graph L is

$$|E| = (m - l)n - (m + l + 1)2^{m-l-1} + m + l + 1$$

In summary, graph L has $|V| = n$ vertices for any $n = 2^m - 2^{l+1} - d$, where $0 \leq l \leq m-3$ and $0 \leq d \leq 2^{l+1} - 1$, $2^{m-l} - 2$ vertices and edges of $KG_{2^{m-l}-2}$, and every vertex of any binomial tree BT_i, $0 \leq i \leq 2^{m-l} - 2$ is connected to $m - l - 1$ vertices of $KG_{2^{m-l}-2}$. Figure 3.1 demonstrates our construction of graph L for $l = 2$, $m = 6$, and $0 \leq d \leq 7$. We first construct a Knödel graph on $2^4 - 2$ vertices. The vertices of KG_{14} are labeled as $r_0, r_1, r_2, ..., r_{13}$. Each vertex of KG_{14} is attached a binomial tree on 4 vertices. Then, for example, we connect vertex $w \in BT_0$ to root vertices r_0, r_2 and r_6, which are the neighbors of r_1. In this particular example, if $d = 0$, $|E_H| = 21$, $|E_T| = 42$, $|E_P| = 98$, and $n = 56$.

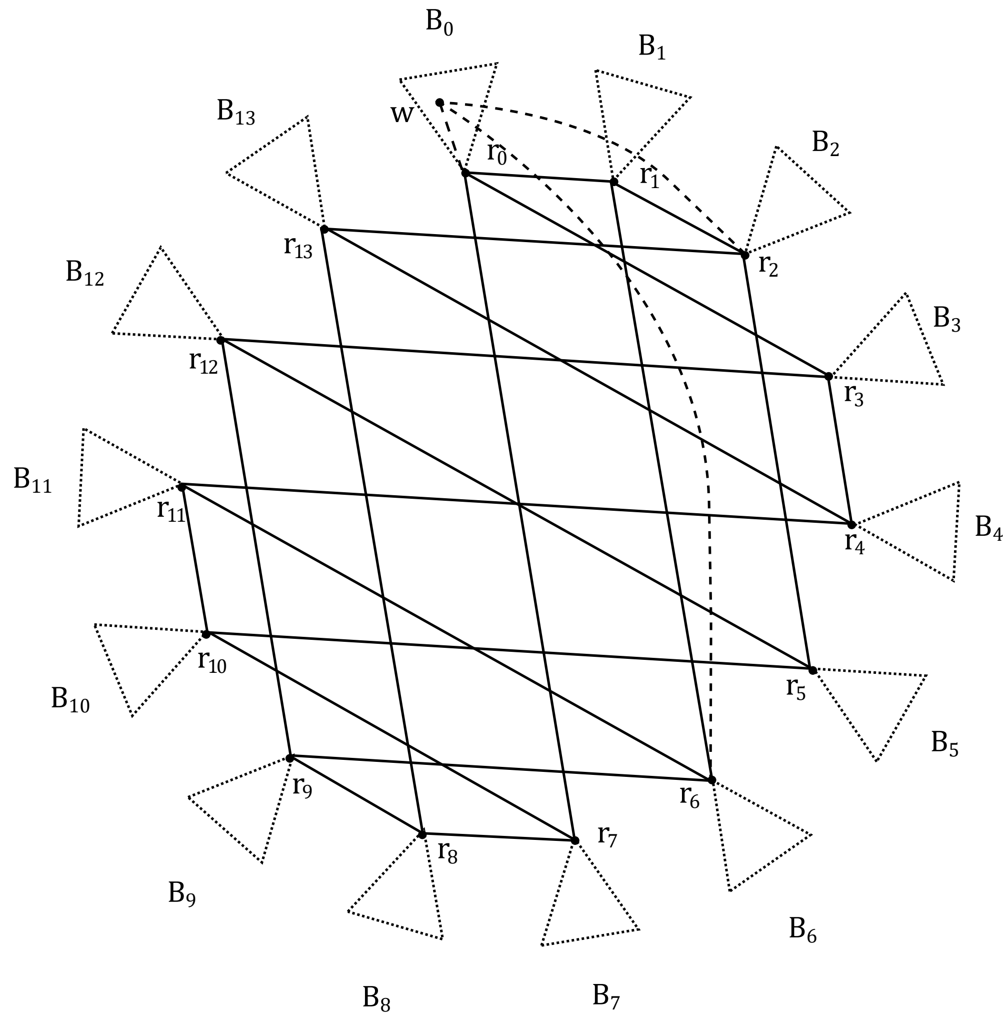


Figure 3.1: An example of L, when $m - l = 4$. Solid lines and vertices r_i form the Knödel graph KG_{14}. Each binomial tree of degree 2 is replaced by a dotted triangle. A tree vertex w of binomial tree BT_0 and the dashed edges show an example of the connections between a non-root vertex and the root vertices. w is connected to the neighbors of the first dimensional neighbor of the root vertex of tree BT_0.

Theorem 3.1. L is a broadcast graph and for any $n = 2^m - 2^{l+1} - d$, where $m \geq 5$, $1 \leq l \leq m - 3$, and $0 \leq d \leq 2^{l+1} - 1$

$$B(n) \leq (m - l)n - (m + l + 1)2^{m-l-1} + m + l + 1$$

Proof. It is clear that $n \in [2^{m-1} + 1, 2^m - 2]$ for any n above. Thus, $\lceil \log n \rceil = m$. To show that L is a broadcast graph, broadcast scheme for any originator is described below.

(1) If the originator is a root vertex r_i in $KG_{2^{m-l}-2}$, where $0 \leq i \leq 2^{m-l} - 3$, then the broadcast scheme of r_i consists of the broadcast scheme from originator r_i in $KG_{2^{m-l}-2}$ concatenated with the broadcast scheme in all binomial tree from their roots. r_i first completes broadcasting within the Knödel graph using dimensional broadcast scheme by time unit $m - l$. So, after time $m - l$ the roots of all binomial trees have the message. Then it takes l time units to broadcast in its binomial tree. Thus, the broadcasting in L completes in m time units.

(2) If the originator is a non-root vertex w in BT_i, $0 \leq i \leq 2^{m-l} - 3$; the broadcasting is more complicated. By our construction, w is adjacent to all the neighbors of r_h, which is the first dimensional neighbor of r_i - the root of binomial tree BT_i.
Consider the dimensional broadcast scheme of Knödel graphs from r_h in $KG_{2^{m-l}-2}$. r_h informs its neighbor on dimension t at time unit t for all $t = 1, 2, ..., m - l$. Since w is adjacent to all neighbors of r_h, w can play the role of r_h in the broadcast scheme from originator w in L. w informs the i-th dimensional neighbor of vertex r_h at time unit i, for all $i = 1, 2, ..., m-l-1$. Every informed vertex continues broadcasting as in the dimensional broadcast scheme from the originator r_h. As a result, every vertex in $KG_{2^{m-l}-2}$ except r_h can be informed by the same broadcast scheme from r_h in $KG_{2^{m-l}-2}$ at the same time, which is $m - l$. Then r_h can be informed by a call from r_i at time unit $m-l$. Note that since the degree of vertex r_i in $KG_{2^{m-l}-2}$ is $m-l-1$ and r_i is busy during the first $m - l - 1$ time units, then r_i is idle at time unit $m - l$, and so it can call vertex r_h. The first $m - l$ time units of the broadcast scheme from w in L is shown in Figure 3.2.

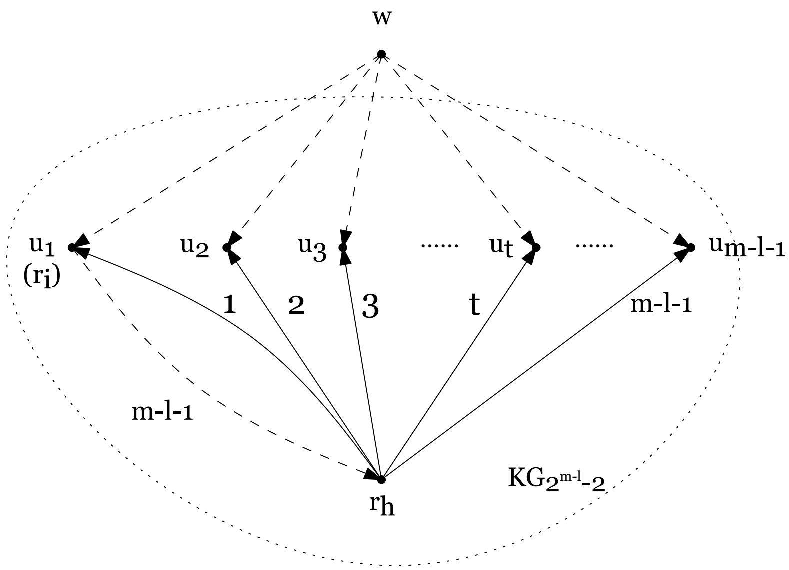


Figure 3.2: The broadcast scheme from w in L in the first $m-l$ time units. u_t, $1 \leq t \leq m-l-1$ is t dimensional neighbor of r_h. Solid arcs denote the calls of the broadcast scheme from originator r_h in $KG_{2^{m-l}-2}$. Dashed arcs denote the calls from originator w in L. All the other calls of the broadcast scheme from originator r_h in $KG_{2^{m-l}-2}$, and the broadcast scheme of originator w in graph L are the same. The numbers besides the arcs are the times of calls.

Now, every vertex r_j, $1 \leq j \leq 2^{m-l}-3$ in $KG_{2^{m-l}-2}$, which is also the root of BT_j, is informed after time $m-l$. Next, every root r_j broadcasts all vertices within its respective binomial tree in the remaining l time units. The broadcasting in L again takes m time units in total.

Therefore, L is a broadcast graph. And for any $n = 2^m - 2^{l+1} - d \in [2^{m-1}+1, 2^m-2]$, where $m \geq 5$, $0 \leq l \leq m-3$, and $0 \leq d \leq 2^{l+1}-1$

$$B(n) \leq (m-l)n - (m+l+1)2^{m-l-1} + m + l + 1$$

□

By substituting $l = k-1$,

Theorem 3.2.

$$B(n) \leq (m-k+1)n - (m+k)2^{m-k} + m + k,$$

where $n = 2^m - 2^k - d$, $m \geq 5$, $1 \leq k \leq m-2$, and $0 \leq d \leq 2^k - 1$

3.1.2 Combined compounding

The binomial-tree compounding method described above extends beyond using a Knödel graph on $2^l - 2$ vertices as a base graph. Finding a better base for compounding is possible. In this section, we combine the binomial-tree compounding method with the compounding method [38] and show that this combined compounding method further improves the general upper bound on $B(n)$.

The construction of a broadcast graph D on $n = (2^{m-l} - 2)2^q 2^{l-q} - d$ vertices, where $m \geq 5$, $1 \leq l \leq m-3$, $0 \leq q \leq l-1$, and $0 \leq d \leq 2^{l+1} - 1$, is as follows. One should notice that the representation of n follows the previous section if the equation of n is simplified.

First, we construct a broadcast graph C on $(2^{m-l} - 2)2^q$ vertices by hypercube compounding method in [38]. This construction creates 2^q copies of Knödel graph $KG_{2^{m-l}-2}$ denoted by $KG^1, KG^2, ..., KG^{2^q}$. Each vertex in graph C is denoted by r_i^j indicating the vertex r_i with index i from j'th copy of KG^j.

The edges of C are of two types: the edges E_H in the copies of Knödel graph $KG_{2^{m-l}-2}$ and the edges $E_C = \{(r_i^s, r_i^t) | i \text{ is odd and } (r^s, r^t) \in Q_q\}$, where Q_q is a hypercube of dimension q. The edges in E_C connect the vertices r_i^j from different copies of Knödel graph $KG_{2^{m-l}-2}$ with the same odd label i and form a hypercube Q^i of dimension q. Thus, graph C has two types of vertices. Vertex r_i^j is in a copy of hypercube Q_q when i is odd, and it is not in, otherwise. The construction, so far, is exactly the same as the hypercube compounding method in [38].

The next step of the combined compounding construction applies the binomial-tree compounding method to broadcast graph C as a base graph. The construction replaces each vertex r_i^j in C by a binomial tree B_i^j of degree $l-q$ on 2^{l-q} vertices with the root r_i^j. As in the binomial-tree compounding method, we remove d leaves from the binomial tree(s) to obtain a general value of n. Again, no root vertex is removed since it is a vertex in graph C.

The construction further adds two types of edges, E_{P1} and E_{P2}. Let w be a non-root vertex in graph D and vertex r_i^j be w's root. Each edge in E_{P1} connects w to all neighbors of r_i^j's first dimensional neighbor if i is odd; each edge in E_{P2} connects w to all neighbors of r_i^j, otherwise. Intuitively, every neighbor of each non-root vertex in the base graph C belongs to a distinct copy of the hypercube. Half of the non-root vertices are adjacent to their roots, and the others are adjacent to the neighbors of the first dimensional neighbor of their roots. We count the number of edges $|E_D|$ of graph D separately. The graph has 2^q copies of Knödel graph $KG_{2^{m-l}-2}$, $2^{m-l-1}-1$ copies of hypercube Q_q and $(2^{m-l}-2)2^q$ copies of binomial tree of degree $l-q$. Thus,

$$|E_H| = \frac{1}{2}(2^{m-l}-2)(m-l-1)2^q$$

$$|E_C| = \frac{1}{2}q2^q(2^{m-l-1}-1)$$

$$|E_T| = (2^{l-q}-1)(2^{m-l}-2)2^q$$

To count $|E_{P1}|$, every binomial tree has $2^{l-q}-(l-q)-1$ vertices adjacent to $m-l-1$ roots in the Knödel graph and $l-q$ vertices on the first level adjacent to $m-l-2$ roots (excluding its own root which is already counted in E_T). So,

$$|E_{P1}| = \frac{1}{2}(m-l-1)(2^{l-q}-(l-q)-1)(2^{m-l}-2)2^q + \frac{1}{2}(m-l-2)(l-q)(2^{m-l}-2)2^q$$

To count $|E_{P2}|$, every non-root vertex has $m-l-1$ neighbors connected by the edges in E_{P2}. Therefore,

$$|E_{P2}| = \frac{1}{2}(m-l-1)(2^{l-q}-1)(2^{m-l}-2)2^q$$

If d leaves are removed, $(m-l)d$ edges are also removed simultaneously because every leaf is associated to $m-l$ edges.

Thus, the total number of edges in graph D is

$$\begin{aligned}|E_D| &= |E_H| + |E_C| + |E_T| + |E_{P1}| + |E_{P2}| - (m-l)d \\ &= (m-l)n - (m-2q+1)(2^{m-l}-2)2^{q-1}\end{aligned}$$

$|E_D|$ is a function of m, l and q, but $n = (2^{m-l}-2)2^q 2^{l-q} - d = (2^{m-l}-2)2^l - d$, by our representation, is not a function of q. For a particular value of n, there are multiple ways to construct graph D with different values of integer q, $0 \leq q \leq l-1$. We analyze the monotonicity of the function $|E_D|$ to find when $|E_D|$ is the smallest.

$$\begin{cases} |E_D| \text{ increases when } \dfrac{m-1}{2} \leq q \leq l; \\ |E_D| \text{ decreases when } 0 \leq q \leq \dfrac{m-1}{2}. \end{cases}$$

$|E_D|$ valleys at $q = \frac{m-2}{2}$. However, $\frac{m-2}{2}$ is not an integer if m is odd. q has to be either $\frac{m-1}{2}$ or $\frac{m-3}{2}$. (The two possible values of q give the same value of $|E_D|$.) Furthermore, q is not necessarily larger than $\frac{m-1}{2}$ when $q = l-1 < \frac{m-1}{2}$. Therefore, $|E_D|$ is minimum when $q = min(\lfloor \frac{m-2}{2} \rfloor, l-1)$.

Figure 3.3 shows one example of the construction when $n = 96$. First, n is represented by $n = 2^7 - 2^5 = (2^3-2)2^4$, so $m - l = 3$ and $l = 4$. The value of q is decided by $min(\lfloor \frac{7-2}{2} \rfloor, 4-1) = 2$. Value n has the form $(2^3-2)2^2 2^2$. Next, the construction creates 4 copies of Knödel graph KG_6 denoted by KG^1, KG^2, KG^3, and KG^4. The odd vertices with the same label from different copies of KG_6, for example r_1^1, r_1^2, r_1^3 and r_1^4 are selected to form a hypercube Q_2 on 4 vertices. Then, every vertex in the current graph is attached a binomial tree B_2 on 4 vertices. Last, we connect vertex $v \in B_0^1$ and vertex $u \in B_1^1$, for example, to root r_5^1 and r_1^1 because r_5^1 and r_1^1 are the neighbors of r_0^1, which is v's root and r_1^1's first dimensional neighbor.

Theorem 3.3.

$$B(n) \leq (m-l)n - (2^{m-l}-2)(m-2q+1)2^{q-1},$$

where $n = 2^m - 2^{l+1} - d$, $m \geq 5$, $1 \leq l \leq m-3$, $0 \leq d \leq 2^{l+1}-1$, and $q = \min(\lfloor \frac{m-2}{2} \rfloor, l-1)$.

Proof. To show the theorem, we describe the broadcast scheme of graph D originating from any vertex. Each vertex can be a root or a non-root, and each binomial tree is in or not in a copy of the hypercube; therefore, the originator has four cases by combining the situations.

(1) If the originator is a root vertex r_i^j, and r_i^j is in a compounding hypercube, the broadcast scheme consists of three individual broadcast schemes. r_i^j informs all vertices inside its own copy of hypercube in the first q time units. Then, every copy of Knödel graph $KG_{2^{m-l}-2}$ has exactly one informed vertex. This vertex calls all vertices in its copy of $KG_{2^{m-l}-2}$ at time unit $m-l$. After this step, every copy of binomial tree has its root informed. The root informs all other vertices in time unit $l-q$. The broadcasting from vertex r_i^j in graph D finishes at time unit $q+m-l+l-q = m = \lceil \log n \rceil$. Figure 3.4 shows an example of the broadcast scheme originating from vertex r_1^1.

(2) If the originator is a root vertex r_i^j, and r_i^j is not in a compounding hypercube, the originator r_i^j calls all its neighbors $x_1^j, x_2^j, ..., x_{m-l-1}^j$ at time unit $m-l-1$. Once

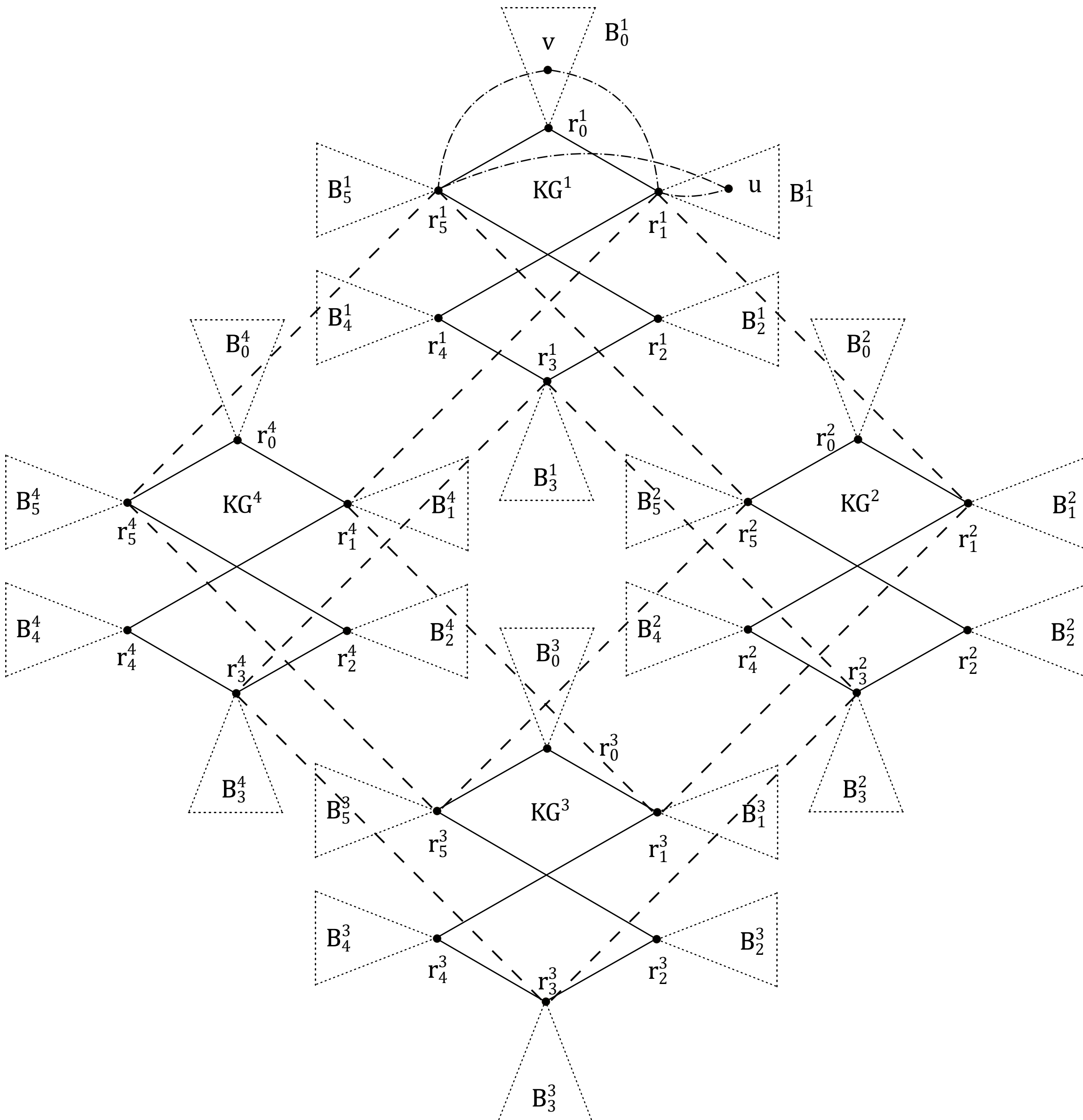


Figure 3.3: An example of the construction of graph D when $n = 96 = (2^3 - 2)2^2 2^2$. Solid lines are the edges in 4 copies of Knödel graph KG_6. Dashed lines are the edges in 3 copies of hypercube Q_2. Each dotted triangle represents a binomial tree B_2. The dotted dashed lines show the examples of the connections between the non-root vertices and the root vertices. v is connected to the neighbors of r_0^1, and u is connected to the neighbors of the first dimensional neighbor of its root r_1^1.

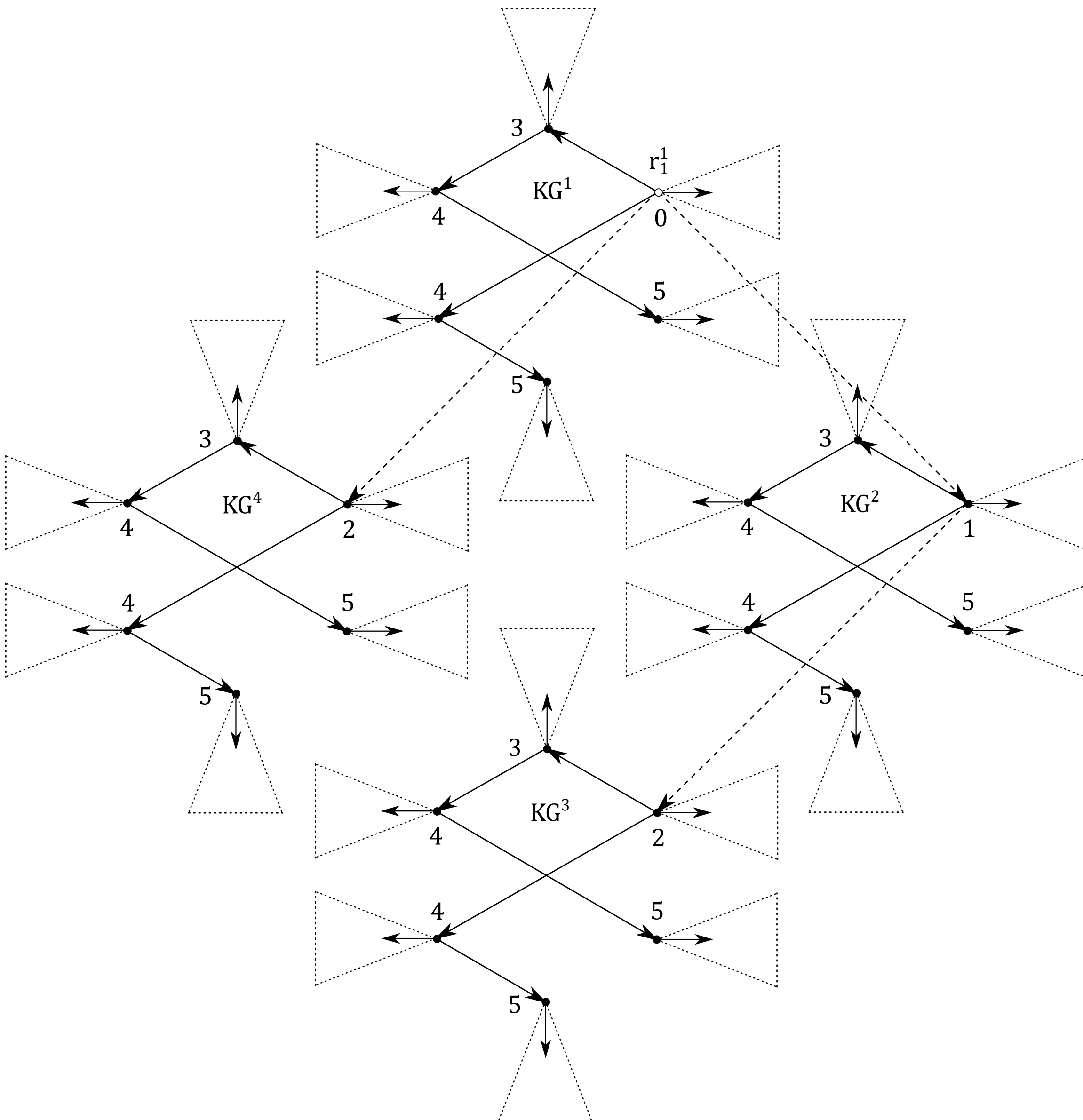


Figure 3.4: The broadcast scheme for the originator r_1^1 in the graph D on 96 vertices.

each x_p^j in the distinct copy of hypercube Q^p is informed at time unit p, it immediately starts informing all the vertices in Q^p and finishes at time unit $p + q$. During this step, every copy of Knödel graph KG^c has vertex x_p^c informed at time unit $p+q$, which is in the exactly same time unit as broadcasting from vertex r_i^c in graph D. (If root r_i^c is the originator in Case 1, it finishes broadcasting in its copy of hypercube at time unit q and informs its neighbor x_p^c in Knödel graph KG^c at time unit $p + q$.) Every vertex in KG^c except r_i^c is informed at the right time unit. Moreover, vertex r_i^c can be informed by vertex x_1^c at time unit $p+q$ since x_1^c is idle in the broadcasting from r_i^c in KG^c. Thus, every root vertex in the base graph C can be informed in time unit $m-l+q$. Then, each root vertex informs all vertices in its binomial tree in time unit $l - q$. All vertices in graph D are informed in time unit $m - l + q + l - q = m$. See Figure 3.5 for the example.

(3) If the originator w is a non-root vertex inthe binomial tree BT_i^j with the root r_i^j, and r_i^j is in a copy of the hypercube Q^i, w is adjacent to all the neighbors of r_f^j, which is r_i^j's first dimensional neighbor in the Knödel graph KG^j. Each neighbor of w (also a neighbor of r_f^j) is in a distinct copy of the hypercube. The originator w can play exactly the same role in broadcasting from vertex r_f^j in the graph D in Case 2. See Figure 3.5 for the similar example if the vertex r_0^1 is replaced by the vertex u.

(4) If the originator w is a non-root vertex, and w's root r_i^j is not in a copy of hypercube. By the definition of the graph D, w is adjacent to all the neighbors of the root vertex r_i^j. Each neighbor is again in a distinct copy of the hypercube. Similar to Case 3, the originator w can play the same role as vertex r_i^j in the broadcasting in the graph D. See Figure 3.5 for the similar example if r_0^1 is replaced by the vertex v.

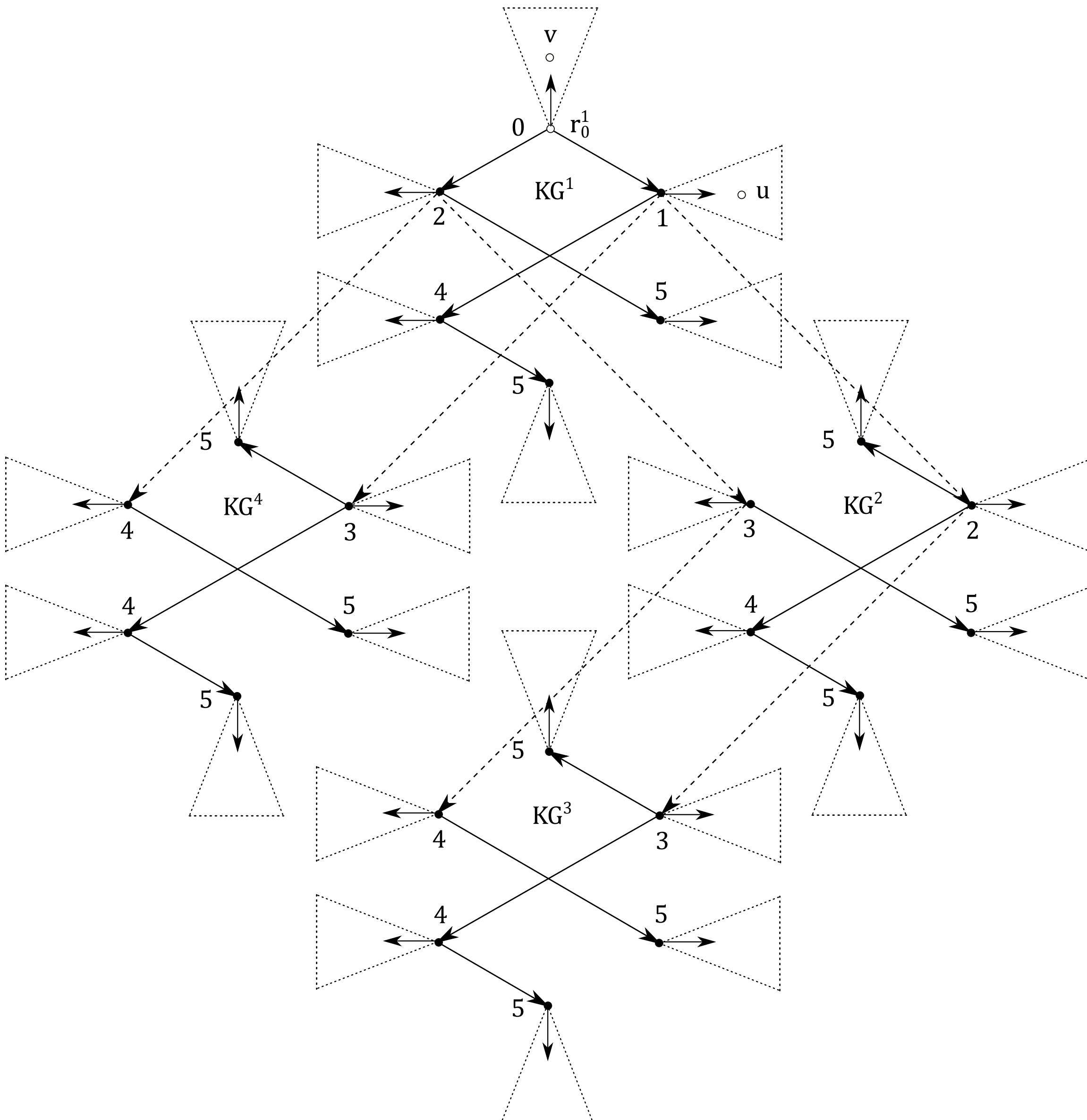


Figure 3.5: The broadcast scheme for the originator r_0^1 in the graph D on 96 vertices. We can also broadcast from the originator u or v by replacing r_0^1 using the method described in Figure 3.2. u is in a binomial tree attached to a copy of hypercube, while v is not.

Therefore, the graph D is a broadcast graph. For any $n = 2^m - 2^{l+1} - d$ in the interval $[2^{m-1} + 1, 2^m - 2]$,

$$B(n) \leq (m-l)n - (2^{m-l} - 2)(m - 2q + 1)2^{q-1},$$

where $m \geq 5$, $1 \leq l \leq m-3$, $0 \leq d \leq 2^{l+1} - 1$, and $q = \min(\lfloor \frac{m-2}{2} \rfloor, l-1)$. □

By substituting $l = k - 1$ in Theorem 3.3, we get

Theorem 3.4.

$$B(n) \leq (m-k+1)n - (2^{m-k+1} - 2)(m - 2q + 1)2^{q-1},$$

where $n = 2^m - 2^k - d$, $m \geq 5$, $2 \leq k \leq m-2$, $0 \leq d \leq 2^k - 1$, and $q = \min(\lfloor \frac{m-2}{2} \rfloor, k-2)$.

The comparison of the upper bounds is given at the end of this chapter.

3.2 Improved vertex addition construction

3.2.1 New dimensional broadcast schemes for Knödel graph

The problem of finding all dimensional broadcast schemes in the Knödel graph is a very difficult problem [8]. In this section, we describe new dimensional broadcast schemes for Knödel graphs and use them to construct new broadcast graphs. Our first result generalizes the basic result of [8]. Since all results in this section are about Knödel graphs, which is defined only on even number of vertices, we further assume the the variable d is always even for the representation $n = 2^m - 2^k - d$.

Theorem 3.5. Let KG_n be a Knödel graph on n vertices, where $n = 2^m - 2^k - d$, $m \geq 5$ $2 \leq k \leq m-2$, and $0 \leq d \leq 2^k-2$. The dimensional broadcast scheme $1, 2, 3, ..., m-1, t$, where $1 \leq t \leq k$ is a valid broadcast scheme.

Proof. This dimensional broadcast scheme is same as the dimensional broadcast scheme given in [8], except the last dimension could be any $1 \leq t \leq k$ instead of only dimension 1. To prove that this broadcast scheme is valid, we only need to show that the uninformed vertices before the last time unit can be called by their neighbors on dimension t, for any $1 \leq t \leq k$ during the last time unit m.

Since a Knödel graph is regular and vertex transitive, then without loss of generality, we can assume that the originator is v_0. Since every vertex broadcasts on dimension s at time unit s for all $1 \leq s \leq m-1$, then the informed vertices after time s are v_0, v_1, $\cdots$, v_{2^s-1}. After $m-1$ time units, the informed vertices are $V_i = \{v_0, v_1, \cdots, v_{2^{m-1}-1}\}$ and thus, the uninformed vertices are $V_u = \{v_{2^{m-1}}, \cdots, v_{n-1}\}$.

To prove the validity of our broadcast scheme, we must show that every vertex $v_x \in V_u$ is adjacent to a vertex in V_i on dimension t, for any $1 \leq t \leq k$. Let $x = 2^{m-1} + c$, where $0 \leq c \leq 2^{m-1} - 2^k - d - 1$.

Assume v_x is adjacent to v_y on dimension t. Thus, we have

$$x + y \equiv 2^t - 1 \mod n$$

by the definition of Knödel graph,

$$x + y = n + 2^t - 1$$

since $x \geq 2^{m-1}$, and

$$2^{m-1} + c + y = n + 2^t - 1$$
$$y = 2^{m-1} - 2^k + 2^t - d - c - 1$$

From the above bounds on c, we get that $2^t \leq 2^{m-1} - 2^k + 2^t - d - c - 1 \leq 2^{m-1} - 1$. Thus, $0 < y \leq 2^{m-1} - 1$ and $v_y \in V_i$. Therefore, each vertex $v_x \in V_u$ has a neighbor $v_y \in V_i$ on dimension t, for any $1 \leq t \leq k$. Thus, every vertex broadcasts on dimension t at the last time unit. The broadcasting of KG_n is accomplished in m time units. □

We consider broadcasting on dimension t at the last time unit as $t-1$ dimension skipping from dimension 1. Once the cyclic shifts is applied similar to the one in [8], the dimension at the last time unit is also shifted. From the proof of Theorem 3.5, it actually follows that during time unit m, vertices $v_{2^t}, v_{2^t+1}, \cdots, v_{2^{m-1}+2^t-d-2^k-1}$ call vertices $v_{n-1}, v_{n-2}, \cdots, v_{2^{m-1}}$ respectively.
Our next result shows that the validity of a dimensional broadcast scheme with one left shift of the dimensions $1, 2, \cdots, m-1$ and then repeating dimension 1.

Theorem 3.6. Let KG_n be a Knödel graph on n vertices, where $n = 2^m - 2^k - d$, $m \geq 5$, $2 \leq k \leq m-2$ and $0 \leq d \leq 2^k - 2$. Dimensional broadcast scheme $m-1$, 1, $\cdots$, $m-2$, 1 is a valid broadcast scheme.

Proof. Without loss of generality, we again assume that the originator is v_0. At the first time unit, v_0 calls $v_{2^{m-1}-1}$ on dimension $m-1$. Then, during the time units $2, 3, \cdots, m-1$, v_0 informs the odd vertices of $I_1 = \{v_i | 1 \leq i \leq 2^{m-2} - 1, i \text{ is odd}\}$ and the even vertices of $I_1' = \{v_i | 0 \leq i \leq 2^{m-2} - 2, i \text{ is even}\}$, while $v_{2^{m-1}-1}$ informs the odd vertices of $I_2 = \{v_i | 2^{m-1} - 1 \leq i \leq 2^{m-1} + 2^{m-2} - 3, i \text{ is odd}\}$ and the even vertices in $I_2' = \{v_i | 2^{m-1} - 2^k - d + 2 \leq i \leq 2^{m-1} + 2^{m-2} - 2^k - d, i \text{ is even}\}$. Thus, after time unit $m-1$, the odd numbered uninformed vertices are

$$U_1 = \{v_i | 2^{m-2} + 1 \leq i \leq 2^{m-1} - 3, i \text{ is odd}\}$$
$$U_2 = \{v_i | 2^{m-1} + 2^{m-2} - 1 \leq i \leq 2^m - 2^k - d - 1, i \text{ is odd}\}$$

and the even numbered uninformed vertices are

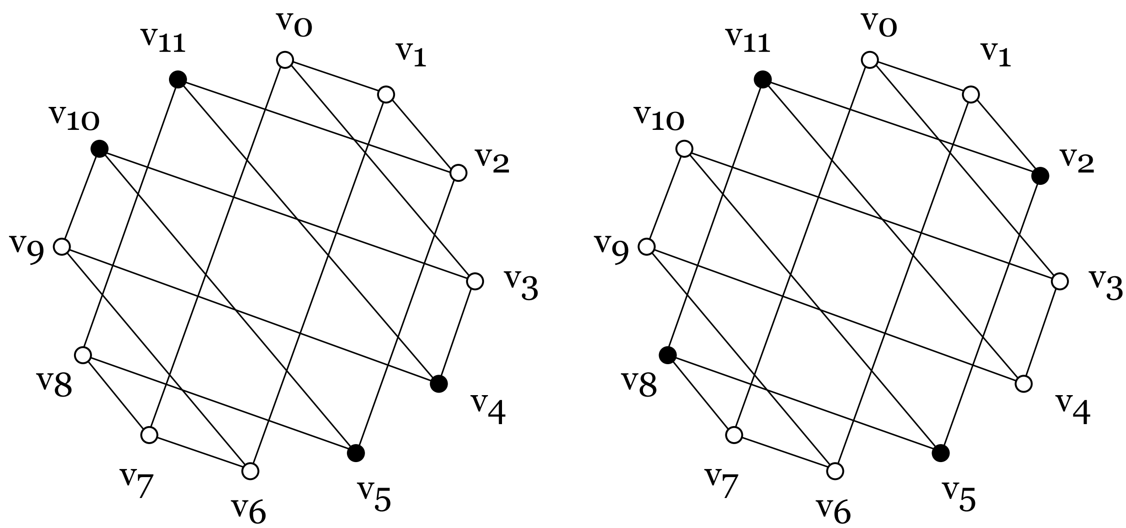
$$U'_1 = \{v_i | 2^{m-2} \leq i \leq 2^{m-1} - 2^k - d, i \text{ is even}\}$$
$$U'_2 = \{v_i | 2^{m-1} + 2^{m-2} - 2^k - d + 2 \leq i \leq 2^m - 2^k - d - 2, i \text{ is even}\}.$$

We can verify that the first dimensional neighbors of the vertices in U_1, U_2, U'_1, and U'_2 are all in I'_2, I'_1, I_2, and I_1 respectively. For example, the first dimensional neighbor of vertex $v_i \in U_1$ (where $i = 2^{m-2} + x$, x is odd, and $1 \leq x \leq 2^{m-2} - 3$) is vertex v_j, where $j = n+1-2^{m-2}-x = 2^m-2^k-d+1-2^{m-2}-x = 2^{m-1}-2^k-d+(2^{m-2}-x+1)$. $v_j \in I'_2$, since $4 \leq 2^{m-2} - x + 1 \leq 2^{m-2}$. Thus, at time unit m vertex $v_{2^{m-2}+x} \in U_1$,where x is odd, $1 \leq x \leq 2^{m-2} - 3$, receives the message from vertex $v_{2^{m-1}-2^k-d+(2^{m-2}-x+1)} \in I'_2$, for all $1 \leq x \leq 2^{m-2} - 3$.

We omit the proof of the other three pairs of subsets (U_2, I'_1), (U'_1, I'_2), and (U'_2, I_1) since all proofs are similar to the proof above.

Thus, broadcast on dimension 1 at the last time unit completes the broadcast of KG_n in m time units. □

Figure 3.6a shows KG_{12} and its dimensional broadcast scheme $3, 1, 2, 1$.



(a) (b)

Figure 3.6: Dimensional broadcast schemes for KG_{12} originating from v_0. The empty circles are the informed vertices before the last time unit. In (a), $I_1 = \{v_1, v_3\}$, $I_2 = \{v_7, v_9\}$, $I'_1 = \{v_0, v_2\}$, $I'_2 = \{v_6, v_8\}$, $U_1 = \{v_5\}$, $U_2 = \{v_{11}\}$, $U'_1 = \{v_4\}$, and $U'_2 = \{v_{10}\}$. So, the cyclic shift on dimension 3, 1, 2, and 1 gives a valid broadcast scheme. However, in (b), the cyclic shift on dimension 2, 3, 1, and 3 (skipping dimension 2) is not a valid broadcast scheme.

It turns out that the direct generalization of Theorem 3.5 with cyclic shifts similar to the result of [8] is not always true. Theorem 3.6 proves that one cyclic shift of dimensions $1, 2, \cdots, m-1$ and then repeating the second dimension (skipping dimension 1) also generates a valid dimensional broadcast scheme. However, our example in Figure 3.6b shows that two cyclic shifts of dimensions $1, 2, \cdots, m-1$ with repeating the second dimension (skipping dimension 1) does not always generate a valid dimensional broadcast scheme. In particular, the dimensional broadcast scheme $m-2, m-1, 1, \cdots, m-3, m-1$ is not valid for KG_{12} (when $m = 4$, $k = 2$, and $d = 0$). Again, without loss of generality the originator is v_0. In this particular case, the dimensional broadcast scheme $m-2, m-1, 1, \cdots, m-3, m-1$ is $2, 3, 1, 3$. Then, the informed vertices before the last time unit are v_0, v_1, v_3, v_4, v_6, v_7, v_9, and v_{10}. The uninformed vertices before the last time unit are v_2, v_5, v_8, and v_{11}. We can clearly see that broadcast on dimension 3 cannot complete broadcasting, neither does dimension 1. So, this cyclic shift is an invalid broadcast scheme, see Figure 3.6b for example.

3.2.2 Construction of broadcast graphs using newly obtained dimensional broadcast schemes

Since Knödel graph gives a good general upper bound on $B(n)$, but only for even values of n, vertex addition method from [30] adds one more vertex and some edges to Knödel graph to obtain new broadcast graphs for odd values of n. The construction uses dimensional broadcast schemes and their cyclic shifts.

Let vertex v_s be the neighbor of a particular originator v_i on dimension s, where $1 \leq s \leq m-1$ under valid broadcast scheme $s, s+1, \cdots m-1, 1, \cdots, s$. Then vertices v_s and v_i are idle at the last time unit. Then, the additional vertex added to the Knödel graph can be informed by vertex v_i if they are adjacent. Thus, v_i dominates all its s neighbors on different dimensions. Then, constructing a broadcast graph by adding one vertex to a Knödel graph is same as finding a dominating set. The same paper also introduces a dominating set of a Knödel graph of size 2^{m-2} and obtains an upper bound on $B(n)$ based on the dominating set.

$$B(n) \leq \frac{1}{2}n\lceil \log n \rceil + 2^{m-2}$$

This is the best known general upper bounds for odd n. However, if $m > 2$ is prime, m divides n, and for any integer $x < m-1$ which is a divisor of $m-1$, $2^x \not\equiv 1 \mod m$, KG_n has a dominating set of size $\frac{n}{m}$. Then [40] gives a better bound for these specific values of n. In particular,

$$\begin{aligned} B(n) &\leq \frac{1}{2}(n-1)\lfloor \log n \rfloor + \frac{n-1}{\lceil \log n \rceil} + \lfloor \log n \rfloor - 2 \\ &= \frac{1}{2}n\lfloor \log n \rfloor + \frac{n-1}{\lceil \log n \rceil} + \frac{1}{2}\lfloor \log n \rfloor - 2 \end{aligned}$$

In this subsection, we follow the track given by [30, 40] and construct a broadcast graph by adding one vertex to Knödel graph. The construction improves the general bound for almost all odd values of n to $B(n) \leq \frac{1}{2}n\lfloor\log n\rfloor + \lceil\frac{1}{7}n\rceil + \lfloor\log n\rfloor$.
Our method is similar to the one in [30, 40] but using 3-distance dominating sets. This also requires more careful consideration of the connections in Knödel graph.

Definition 3.1. Let $KG_n = (V, E)$ be a Knödel graph on n vertices, where $n = 2^m - 2^k - d$, $3 \leq k \leq m-2$, and $0 \leq d \leq 2^k - 2$, and let $e \equiv n \mod 14$. Define $U = \{v_0\} \cup \{v_{n-14a} | 1 \leq a \leq \frac{n}{14}\} \cup \{v_{14a+13} | 1 \leq a \leq \frac{n}{14}\} \cup X$, where $X = \{v_{e-7}\}$ if $e \geq 8$, and $X = \emptyset$ otherwise.

Theorem 3.7. U contains at least one idle vertex u at the last time unit under dimensional broadcast scheme $1, 2, 3, \cdots, m-1, 3$ from any originator in graph KG_n.

Proof. First, from Theorem 3.5, the dimensional broadcast scheme $1, 2, \cdots, m-1, 3$ is a valid dimensional broadcast scheme for KG_n. Here $t = 3 \leq k$ as stated in Theorem 3.5. So, we will show that under the dimensional broadcast scheme from any originator v_i, there is a vertex $u \in U$, such that v_i informs u and its third dimensional neighbor during the first 3 time units. Then, at the last time unit, when every vertex broadcasts on the third dimension, u and its third dimensional neighbor are both idle. We partition the vertices into 14 subsets and show the connections on dimensions 1, 2, and 3 between the sets.
For all $v_i \in V$, when i is even, $i = n - 14a + 2b$, where $0 \leq a \leq \frac{n}{14}$, $0 \leq b \leq 6$, and $14a - 2b \leq n - 2$. When i is odd, $i = 14a + 2b + 1$, where $0 \leq a \leq \frac{n}{14}$, $0 \leq b \leq 6$ and $14a + 2b + 1 \leq n - 1$. Then, we have 14 cases depending on the different parities of i and values of b.

(1) If i is even and $b = 0$, $i = n - 14a$ and $v_i \in U$ by definition. The broadcast originating from v_i makes vertex v_i idle at the last time unit. If i is odd and $b = 6$, the situation is the same.

(2) If i is odd and $b = 0$, 1, or 3, $i = 14a + 1$, $14a + 3$, or $14a + 7$. These three different vertices have a common neighbor $v_{n-14a} \in U$ on dimension 1, 2, and 3 respectively. Thus, if the originator is one of these vertices, v_{n-14a} and v_{14a+7} are informed in time unit 3. And $v_j \in U$ is idle at the last time unit.

If i is even and $b = 3$, 5, and 6, we have the same situation.

(3) If i is even and $b = 1$, $i = n - 14a - 2$. Vertex v_i is adjacent to vertex v_{14a+3}, which is the case we discussed above. So, v_i informs v_{14a+3} at the first time unit. v_{14a+3} informs $v_{n-14a} \in U$ at the second time unit. And v_{n-14a} informs v_{14a+7} at the third time unit. Thus, $v_{n-14a} \in U$ and its third dimensional neighbor v_{14a+7} are both informed after time unit 3, and v_{n-14a} is idle at the last time unit.

If i is odd and $b = 5$, we have the same case.

(4) If i is even and $b = 2$, $i = n - 14a - 4$. v_i is adjacent to vertex v_{14a+7} on dimension 2. So, we broadcast from v_i and inform v_{14a+7} at time unit 2. Then, v_{14a+7} informs v_{n-14a} at time unit 3. At the last time unit, $v_{n-14a} \in U$ is idle.

If i is odd and $b = 4$, the situation is the same.

(5) If i is odd and $b = 2$, $i = 14a + 5$. v_i is adjacent to $v_{n-14a-4}$ on dimension 1. So, v_i informs $v_{n-14a-4}$ at the first time unit. Then, if we just follow Case 4, vertex $v_{n-14a} \in U$ is idle at the last time unit. Again, we have the same case for even i and $b = 4$.

Therefore, for any originator v_i, there is always a vertex in U, which is idle at the last time unit. Figure 3.7 shows one example of set U. Note that the vertex set U is a 3-distance dominating set for KG_n.

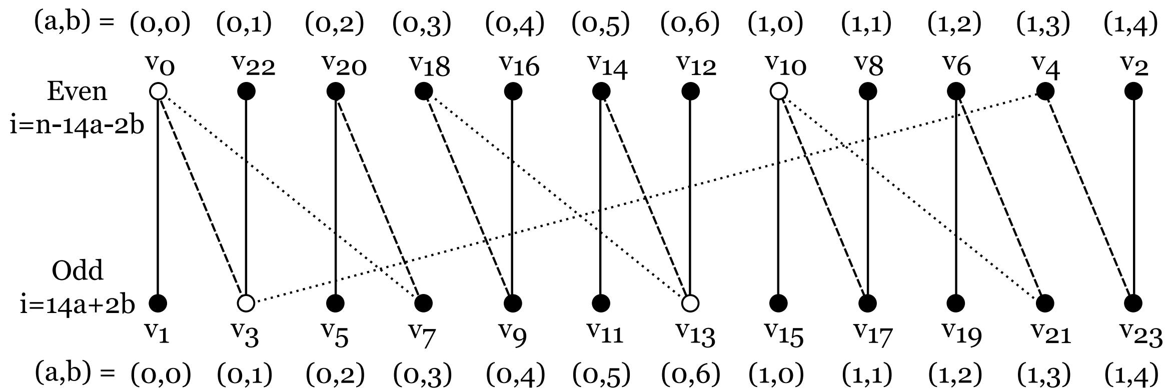


Figure 3.7: Knödel graph on $n = 24$ vertices with only a part of the edges involved in the dimensional broadcast scheme 1, 2, and 3. Solid lines are the edges on dimension 1. Dashed lines are the edges on dimension 2. And dotted lines are the edges on dimension 3. The circles are the vertices in vertex set U. We say that, for example, v_{10} covers v_6, v_8, v_{15}, v_{17}, v_{19}, and v_{21} on distance 3. So, vertex set U is a 3-distance dominating set of V. Note that v_3 is in U because $n \equiv e \mod 14$, $e = 10 \geq 8$, and $X = \{v_{10-7}\}$.

□

Next, we construct a new broadcast graph using the property of Knödel graph from Theorem 3.7 and a dimensional broadcast scheme from Theorem 3.5.

Let $KG_n = (V, E)$ be the Knödel graph on n vertices, where $n = 2^m - 2^k - d$, $3 \leq k \leq m-2$, and $0 \leq d \leq 2^k - 2$. Let $U = \{v_0\} \cup \{v_{n-14a} | 1 \leq a < \frac{n}{14}\} \cup \{v_{14a+13} | 1 \leq a < \frac{n}{14}\} \cup X$, where $X = \{v_{e-7}\}$ if $e \geq 8$, and $X = \emptyset$ otherwise. We add one vertex v and two types of edges E_1 and E_2 to KG_n, where $E_1 = \{(v, u) | u \in U\}$ and $E_2 = \{(v, v_i) | (v_1, v_i) \in E\}$. Then, the new graph is defined as $G = (V \cup \{v\}, E \cup E_1 \cup E_2)$.

By the definition of Knödel graph, $|E| = \frac{(m-1)n}{2}$. By the definition of vertex set U, $|E_1| = \lceil \frac{n}{7} \rceil$, if $n < 8 \mod 14$; or $\lceil \frac{n}{7} \rceil + 1$, otherwise. Since Knödel graph is $(m-1)$-regular, v_1 has $m-1$ neighbors. And one of the neighbors, v_0 is already adjacent to v, so $|E_2| = m - 2$. Thus, graph G has $\frac{1}{2}(m-1)n + \lceil \frac{n}{7} \rceil + m - 2 + x \leq \frac{1}{2}(m-1)n + \lceil \frac{n}{7} \rceil + m - 1$ edges, since $x \leq 1$.

Theorem 3.8. The graph G, constructed above, is a broadcast graph, and

$$B(n) \leq \frac{1}{2}(m-1)n + \lceil \frac{n-1}{7} \rceil + \frac{1}{2}(m-1)$$

where $n = 2^m - 2^k - d + 1$, $3 \leq k \leq m-2$, and $0 \leq d \leq 2^k - 2$.

The equation in Theorem 3.8 is slightly different from the equation above given for the number of edges in graph G. In Theorem 3.8, the number of vertices in graph G is equal to the number of vertices in Knödel graph KG_n plus one additional vertex. So, in the rest of the paper, we have KG_{n-1} instead of KG_n as a subgraph of G.

Proof. To prove the theorem, we show a broadcast scheme for any originator of graph G. Graph G has two types of vertices, the vertices in Knödel graph KG_{n-1} and the additional vertex. So, we have the following two cases.

(1) If the originator $v_i \in KG_{n-1}$, by Theorem 3.7, we know that there is a vertex $v_k \in U$ idle at the last time unit. Since every vertex in U is adjacent to the added vertex v, vertex v_k calls v at the last time unit.

(2) If the originator is the additional vertex v, v plays exactly the same role as vertex v_1, because v is adjacent to all v_1's neighbors. And at the last time unit, vertex v_0 informs v_1 and completes the broadcasting.

Thus, the broadcast time of graph G is the same as broadcast time of KG_{n-1}. Therefore G is a broadcast graph, and since $x \leq 1$ we obtain

$$B(n) \leq \frac{1}{2}(m-1)n + \lceil \frac{n-1}{7} \rceil + \frac{1}{2}(m-1)$$

□

We observe that the subset of vertices U defined above and used in Theorem 3.7 3.8 is a 3-distance dominating set for Knödel graph. The proof of this fact is simple.

For the veritces $\{v_1, v_{15}, \cdots\} \cup \{v_3, v_{17}, \cdots\} \cup \{v_7, v_{21}, \cdots\} \cup \{v_{n-6}, v_{n-20}, \cdots\} \cup \{v_{n-10}, v_{n-24}, \cdots\} \cup \{v_{n-12}, v_{n-26}, \cdots\}$, they have distance 1 to the vertices in U. The vertices in $\{v_9, v_{23}, \cdots\} \cup \{v_{11}, v_{25}, \cdots\} \cup \{v_{n-2}, v_{n-16}, \cdots\} \cup \{v_{n-4}, v_{n-18}, \cdots\}$ have distance 2 to the vertices in U. And the vertices in $\{v_7, v_{21}, \cdots\} \cup \{v_{n-6}, v_{n-20}, \cdots\}$ have distance 3 to the vertices in U. Thus, every vertex has at most distance 3 to a vertex in U.

3.2.3 A further improvement of the vertex addition method

In this section, we further improve the vertex addition method by following the same technique: making one particular vertex idle in the last time unit, but without using the dominating set. To simplify the notations, we let $n = 2^m - 2^k - d$ and $2a = 2^k + d$, where $1 \leq a \leq 2^{m-2} - 1$.

Observation 3.1. For any originator vertex $x \in W \subsetneq V(KG_n)$ there exist a broadcast scheme under which vertex $v_{n-2^{m-1}+2}$ is idle during the last time unit, where

$$\begin{aligned} W = &\{v_{n-2^{m-1}+2}\} && \text{(case 1)} \\ &\cup \{v_0, v_{n-2}, v_{n-4}, \cdots, v_{n-2a+4}\} \cup \{v_1, v_3, \cdots, v_{2a-3}\} && \text{(case 2)} \\ &\cup \{v_{2^{m-1}-1}, v_{2^{m-1}+1}, \cdots, v_{2^{m-1}+2a-5s}\} && \text{(case 3)} \end{aligned}$$

$n = 2^m - 2a$, and $1 \leq a \leq 2^{m-2} - 1$.

Proof.

Case 1. If the originator is $v_{n-2^{m-1}+2}$, then consider the dimensional broadcast scheme $1, 2, \cdots, m-1, 1$. Vertex $v_{n-2^{m-1}+2}$ informs $v_{2^{m-1}-1}$ in the first time unit, or vice versa. Thus, in the last time unit, the two vertices are idle, and $v_{n-2^{m-1}+2}$ is one of them.

Case 2. If the originator is v_0 or v_1, and the dimensional broadcast scheme is $1, 2, \cdots, m-1, 1$, then $v_0, v_1, \cdots, v_{2^{m-1}-1}$ are informed, while $v_{2^{m-1}}, v_{2^{m-1}+1}, \cdots, v_{n-1}$ are uninformed before the last time unit. Then in the last time unit, every vertex broadcasts on dimension 1. The calls from odd vertices are

$$v_3 \rightarrow v_{n-2};$$

$$v_5 \rightarrow v_{n-4};$$

$$\vdots$$

$$v_{2^{m-1}-2a+1} \rightarrow v_{n-2^{m-1}+2a}.$$

And the calls from even vertices are

$$v_2 \rightarrow v_{n-1};$$

$$v_4 \rightarrow v_{n-3};$$

$$\vdots$$

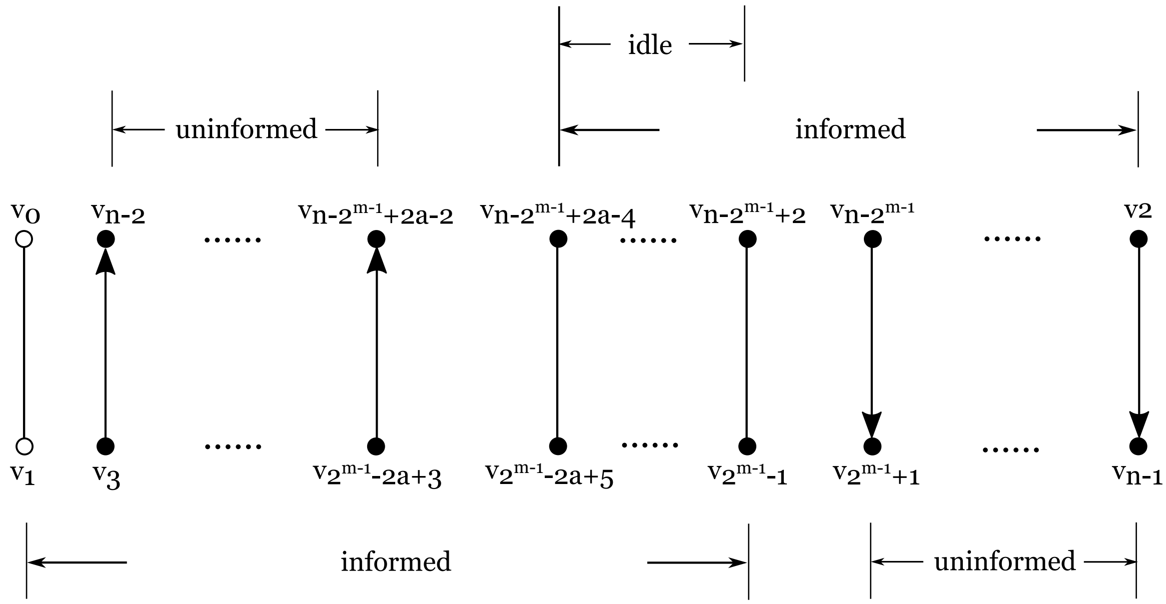
$$v_{n-2^{m-1}} \rightarrow v_{2^{m-1}+1}.$$

Thus, the odd vertices $v_{2^{m-1}-2a+3}, \cdots, v_{2^{m-1}-1}$ and the even vertices $v_{n-2^{m-1}+2}, \cdots, v_{n-2^{m-1}+2a-2}$ are idle in the last time unit, which includes vertex $v_{n-2^{m-1}+2}$.

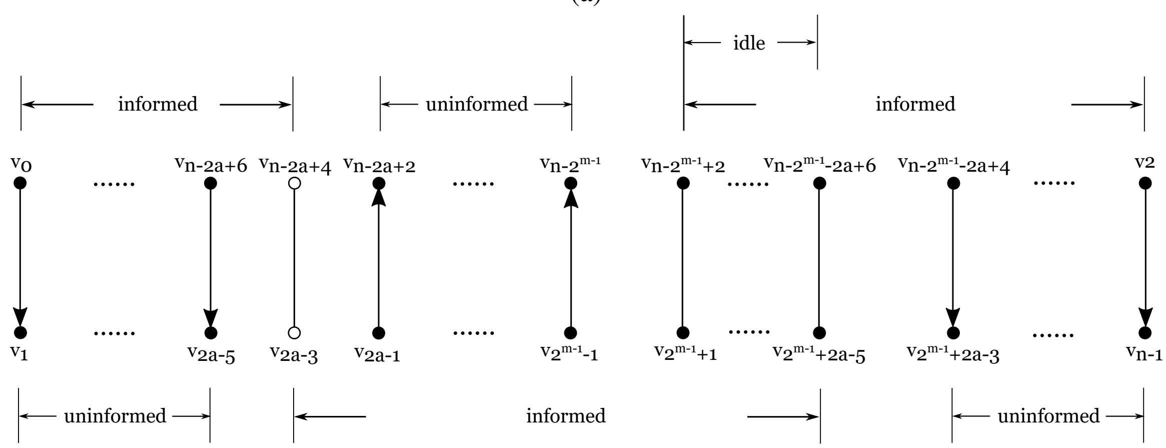
If we change the originator to be v_{n-2} or v_3 and keep the same broadcast scheme, the idle vertices become the odd vertices $v_{2^{m-1}-2a+5}, \cdots, v_{2^{m-1}+1}$ and the even vertices $v_{n-2^{m-1}}, \cdots, v_{n-2^{m-1}+2a-}$

Vertex $v_{n-2^{m-1+2}}$ is again idle in the last time unit.

Then, we can keep changing the originators v_{n-2a+4} and v_{2a-4} to keep $v_{n-2^{m-1+2}}$ always be idle. Figure 3.8 also shows this case by presenting two graphs.



(a)



(b)

Figure 3.8: 3.8a shows the observation when the originator is v_0 or v_1. In 3.8b, we right shift the originators to v_{n-2a+6} or v_{2a+5}. The originators are indicated by empty circles. All informed and uninformed vertices before the last time unit are indicated above or under the vertices. Arrows are the calls in the last time unit. And the idle vertices are also labeled. Vertex $v_{n-2^{m-1}+2}$ is always idle.

Case 3. Let the originator be v_0 or $v_{2^{m-1}-1}$ and the broadcast scheme be dimension $m-1, m-2, \cdots, 1, m-1$. Before the last time unit, the even vertices $v_0, v_{n-2}, \cdots v_{n-2^{m-1}+2}$ and the odd vertices $v_1, v_3, \cdots, v_{2^{m-1}-1}$ are informed. The even vertices $v_2, v_4, \cdots, v_{n-2^{m-1}}$ and the odd vertices $v_{2^{m-1}+1}, v_{2^{m-1}+3}, \cdots, v_{n-1}$ are uninformed. In the last time unit, when every vertex broadcasts on dimension $m-1$, the calls from even vertices are

$$v_{n-2} \rightarrow v_{2^{m-1}+1};$$
$$v_{n-4} \rightarrow v_{2^{m-1}+3};$$
$$\vdots$$
$$v_{n-2^{m-1}+2a} \rightarrow v_{n-1}.$$

And the calls from odd vertices are

$$v_{2^{m-1}-3} \rightarrow v_2;$$
$$v_{2^{m-1}-5} \rightarrow v_4;$$
$$\vdots$$
$$v_{2a-1} \rightarrow v_{n-2^{m-1}}.$$

Thus, the even vertices $v_{n-2^{m-1}+2}, \cdots, v_{n-2^{m-1}+2a-2}$ and the odd vertices $v_1, \cdots, v_{2a-3}$ are idle at the last time unit, which contains vertex $v_{n-2^{m-1}+2}$.

Similar to Case 2, we can also right shift the originators from v_0 to v_{n-2a+4} and from $v_{2^{m-1}-1}$ to $v_{2^{m-1}+2a-5}$ to keep vertex $v_{n-2^{m-1}+2}$ idle at the last time unit. Figure 3.9 and 3.10 show this case. □

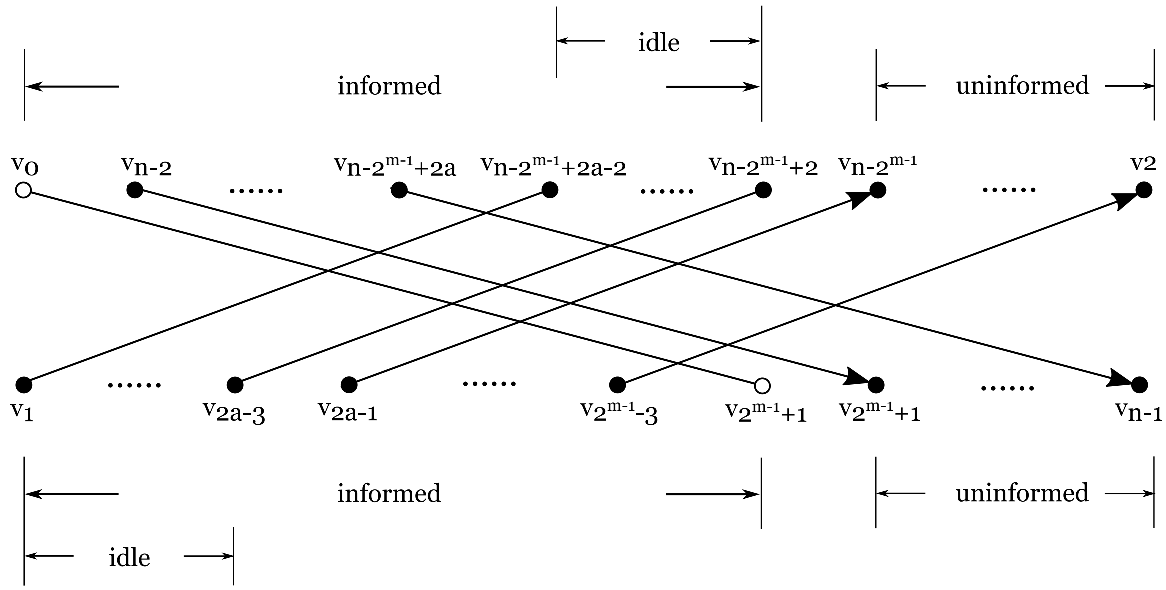


Figure 3.9: If the originator is v_0 or v_1, vertex $v_{n-2^{m-1}+2a-4}$ is idle in the last time unit.

By the same technique, we can prove that if the originator belongs to

$$\begin{aligned}&\{v_{2^{m-1}+2a-3}\}\\ \cup&\{v_{n-2a+2}, v_{n-2a}, \cdots, v_{n-6a+6}\}\\ \cup&\{v_{n-2^{m-1}}, v_{n-2^{m-1}-2}, \cdots, v_{2^{m-1}-a+4}\}\\ \cup&\{v_{2a-1}, v_{2a+1}, \cdots, v_{4a-5}\}\end{aligned}$$

vertex $v_{2^{m-1}+2a-3}$ is idle in the last time unit. And in general, if the indices are all modulo n, we have the following lemma.

Lemma 3.1. Let vertex v_i be an arbitrary vertex in KG_n, where $0 \le i \le n-1$. There exist a broadcast scheme from any originator of W such that v_i is idle during the last time

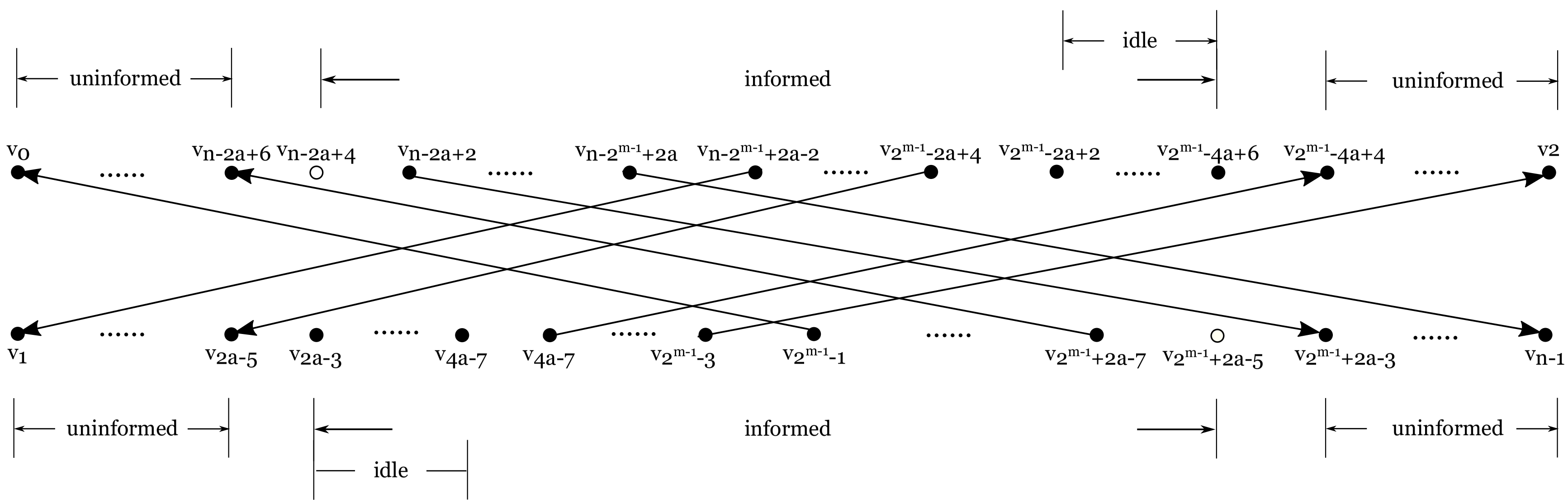


Figure 3.10: If the originator is v_{n-2a+6} or v_{2a+5}, vertex $v_{n-2^{m-1}+2a-4}$ is also always idle in the last time unit.

unit, where

$$
\begin{aligned}
W = &\{v_i\} \\
&\cup \left\{v_{2^{m-1}-2a+2+i}, v_{2^{m-1}-2a+4+i}, \cdots, v_{2^{m-1}-2+i}\right\} \\
&\cup \left\{v_{n+1-i}, v_{n+3-i}, \cdots, v_{n+2a-3-i}\right\} \\
&\cup \left\{v_{n-2^{m-1}+3-i}, v_{n-2^{m-1}+5-i}, \cdots, v_{n-2^{m-1}+2a-1-i}\right\}.
\end{aligned}
$$

We will borrow the word "dominating" from the dominating set used in the previous vertex addition method. We say that vertex x "broadcast dominates" the subset $W \subsetneq V$ if for any broadcast originator from W there exists a broadcast scheme under which vertex x is idle during the last time unit. For example, vertex v_i "broadcast dominates" the set of vertices the subset W from Lemma 3.1. Thus, to construct a broadcast graph by adding one vertex to a Knödel graph, we select the minimum number of vertices to "broadcast dominate" every vertex in the Knödel graph. Suppose the idle vertices $u_1, u_2, , \cdots u_l$ "broadcast dominate" sets $W_1, W_2, \cdots W_l$ respectively.

Figure 3.11 shows the way that the idle vertices are selected. Every idle vertex "broadcast dominates" two parts of the vertices separately. The part on the left has $2a-2$ vertices, $a-1$ vertices on each side, represented by a box. The other part on the right, a triangle, has only one vertex - the idle vertex on one side and $a-1$ vertices on the other side. The indices of the left most vertices in a box and the corresponding triangle differ by $2^{m-1}-2$. If the boxes and the triangles are selected as close as possible as they are in Figure 3.11, there is a gap containing $2a-4$ vertices between two pairs of triangles. In the figure, we select vertex $v_{2^{m-1}+4a-i-7}$ to "broadcast dominate" the vertices in the gap between the triangle W_2 and W_3. So, u_1, u_2, and $v_{2^{m-1}+4a-i-7}$"broadcast dominate" $4a-4$ vertices both on the left and the right sides. We define a new broadcast graph on odd number of vertices as follows.

Definition 3.2. Let H_n be a broadcast graph on $n = 2^m - 2a + 1$ vertices with Knödel graph KG_{n-1} as a subgraph, where $1 \leq a \leq 2^{m-2} - 1$. The vertex $v \notin KG_{n-1}$ is adjacent to every vertex in $U = U_o \cup U_e \cup U_g \cup N_0$, where $U_o = \{v_{n-2^{m-1}+2-x(4a-4)} | 0 \leq x \leq \lceil \frac{2^{m-1}-2a}{2a} \rceil\}$, $U_e = \{v_{2^{m-1}-1+x(4a-4)} | 0 \leq x \leq \lceil \frac{2^{m-1}-2a}{2a} \rceil\}$, $U_g = \{v_{6a-7+x(4a-4)} | 0 \leq x \leq \lceil \frac{2^{m-1}-2a}{2a} \rceil\}$, and N_0 consist of all neighbors of v_0.

Note that U_o consists of the odd idle vertices dominating the vertices in W_2, W_4, $\cdots$ in Figure 3.11, U_e consists of the even idle vertics dominating the vertices in W_1, W_3, $\cdots$, and U_g consists of the idle vertices dominating the vertices between the last box and the first triangle W_1.

Theorem 3.9. The graph H_n defined above is a broadcast graph and

$$\begin{aligned} B(n) &\leq \frac{1}{2}(m-1)n + \frac{3}{4}\lceil \frac{n+1}{2^k + 2d - 1} + 1 \rceil + m - 1 \\ &= \frac{1}{2}(m-1)n + \frac{3}{4}\lceil \frac{2^m}{2^m - (n+1)} \rceil + m - 1 \end{aligned}$$

where $n = 2^m - 2^k - 2d + 1$, $1 \leq k \leq m - 2$, and $0 \leq d \leq 2^{k-1} - 1$.

Proof. Since Lemma 3.1 ensures there is always one vertex idle, the broadcast scheme becomes trivial. If the originator is a vertex in the Knödel graph, then we know there is a vertex in U which is idle in the last time unit. Thus, the additional vertex v can be informed by the idle vertex in the last time unit. If the originator is the additional vertex v, v can act as vertex v_0, since it is adjacent to all neighbors of v_0.

Now we count the number of edges in H_n. The Knödel graph has $\frac{1}{2}(n-1)(m-1)$ edges. Each vertex in U has one more edge connecting to vertex v, that is $3(\lceil \frac{2^{m-1}-2a}{2a} \rceil + 1) + m - 1$ edges. Thus, graph H_n has $\frac{1}{2}(n-1)(m-1) + 3(\lceil \frac{2^{m-1}-2a}{2a} \rceil + 1) + m - 1$ edges in total. After substituting $2a = 2^k + 2d$ and $n = 2^m - 2^k - 2d + 1$ and some simple calculations, we have $B(n) \leq \frac{1}{2}(m-1)n + \frac{3}{4}\lceil \frac{n}{2^k+2d-1} + 1 \rceil = \frac{1}{2}(m-1)n + \frac{3}{4}\lceil \frac{2^m}{2^m-(n+1)} \rceil + m - 1$. □

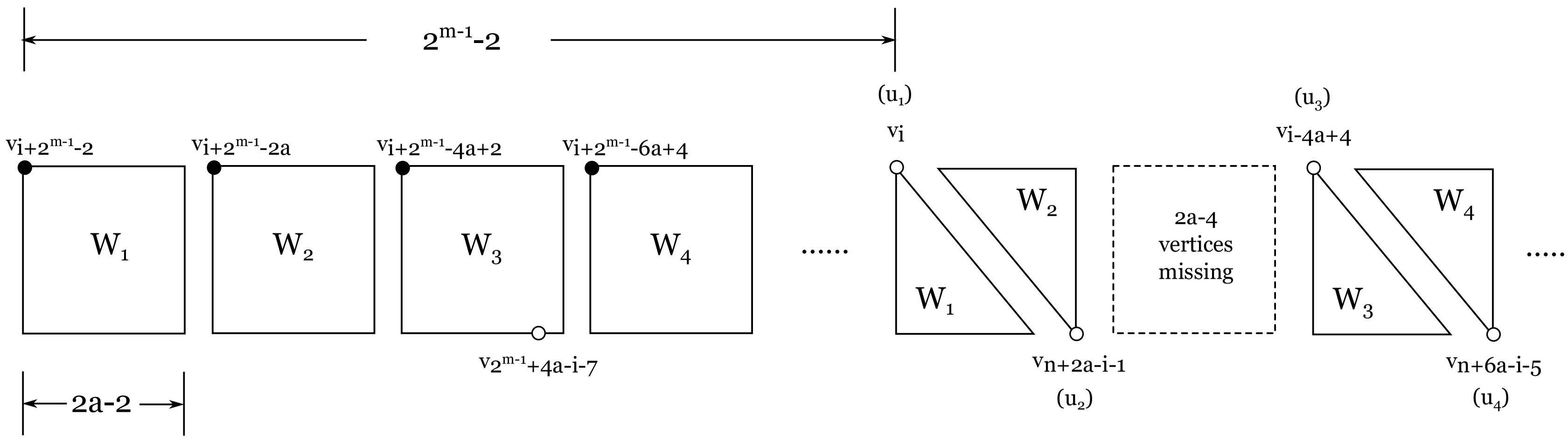


Figure 3.11: Selecting the idle vertices and the “broadcast dominated” sets. The idle vertices are empty cycles.

Note that the new upper bound on $B(n)$ above improves the existing general upper bounds on $B(n)$ that use the vertex addition method for $n = 2^m - 2^k - 2d$, $1 \leq k \leq m-2$, and $0 \leq d \leq 2^{k-1} - 1$. In particular, when $k = m-2$ and $d = 0$ then the new upper bound is approximately $\frac{1}{2}(m-1)n + \frac{n}{9}$. When $k = m/2$ and $d = 0$ then the new upper bound is $\frac{1}{2}(m-1)n + \frac{3}{4}\sqrt{n}$.

3.3 Comparing the new and the existing upper bounds

We denote the new upper bound in Theorem 3.2 by

$$NB_1 = (m-k+1)n - (m+k)2^{m-k} + m + k,$$

the one in Theorem 3.4 by

$$NB_2 = (m-k+1)n - (2^{m-k+1} - 2)(m - 2q + 1)2^{q-1},$$

the one in Theorem 3.8 by

$$NB_3 = \frac{1}{2}(m-1)n + \lceil \frac{n-1}{7} \rceil + \frac{1}{2}(m-1),$$

and the one in Theorem 3.9 by

$$NB_4 = \frac{1}{2}(m-1)n + \frac{3}{4}\lceil \frac{2^m}{2^m - (n+1)} \rceil + m - 1,$$

where $n = 2^m - 2^k - d$, $m \geq 5$, $2 \leq k \leq m-2$, $0 \leq d \leq 2^k - 1$, $q = \min(\lfloor \frac{m-2}{2} \rfloor, k-2)$, and NB_3 and NB_4 are only defined for odd n. Since both of NB_1 and NB_2 are given by the compounding method, NB_3 and NB_4 are given by the vertex addition method, we compare the two pairs of bounds separately.

Before comparing NB_1 with NB_2 we first need to determine the value of q in NB_2. By the theorem, $q = min(\lfloor \frac{m-2}{2} \rfloor, k-2)$. So, calculations show that NB_2 is monotonically decreasing when $q \in [1, \lfloor \frac{m-2}{2} \rfloor]$ and is maximized for $q = 1$. Thus,

$$NB_2 \leq (m-k+1)n - (m-1)(2^{m-k+1} - 2)$$

and

$$NB_1 - NB_2 \geq (m-k-2)(2^{m-k} - 1)$$

Since $m - k \geq 2$, $NB_1 - NB_2 \geq 0$. Therefore, NB_2 is a better upper bound than NB_1.
Comparing NB_3 and NB_4 is simple. They have the first term in common. And the second term of NB_4 is clearly smaller than NB_3. So, NB_4 is the best upper bound given by the vertex addition method.
Next, we compare NB_2 and NB_4 for odd values of n. The first terms of the two bounds are the leading term. When $k \geq \frac{1}{2}(m+3)$, $\frac{1}{2}(m-1)n$ is larger than $(m-k+1)n$. Thus, NB_2 is a smaller bound when $n \leq 2^m - 2^{\frac{1}{2}(m+3)}$, and NB_4 is smaller otherwise.
Finally, we compare the new bounds with the old bound UB given in Section 2.4. In the

range $2^{m-1}+1 \leq n \leq 2^m - 2^{\frac{1}{2}(m+3)}$,

$$\begin{aligned}
& UB - NB_2 \\
=&(m-k+1)n - (\frac{m}{2}+\frac{k}{2}+1)2^{m-k}+k+1 \\
&\qquad - ((m-k+1)n - (2^{m-k+1}-2)(m-2q+1)2^{q-1}) \\
\geq&(m-k+1)n - (\frac{m}{2}+\frac{k}{2}+1)2^{m-k}+k+1 \\
&\qquad - ((m-k+1)n - (m-1)(2^{m-k+1}-2)) \\
=&(m-1)(2^{m-k+1}-2) - (\frac{m}{2}+\frac{k}{2}+1)2^{m-k}+k+1 \\
=&\frac{1}{2}(3m-k-6)2^{m-k} - 2m + k + 3 \\
>&0
\end{aligned}$$

Also, for the odd n in the range $2^m - 2^{\frac{1}{2}(m+3)} < n \leq 2^m$,

$$\begin{aligned}
& UB - NB_4 \\
=&\frac{1}{2}(m-1)n + 2^{m-2} - \frac{1}{2}(m-1) \\
&\qquad - (\frac{1}{2}(m-1)n + \frac{3}{4}\lceil\frac{2^m}{2^m-(n+1)}\rceil + m - 1) \\
=&2^{m-2} - \frac{1}{2}(m-1) - \frac{3}{4}\lceil\frac{2^m}{2^m-(n+1)}\rceil - m + 1 \\
<&0
\end{aligned}$$

if $n < 2^m - 3$. Thus, the best general upper bound is as follows.

$$B(n) \leq \begin{cases} (m-k+1)n - (2^{m-k+1}-2)(m-2q+1)2^{q-1}, \\ \qquad\qquad \text{if } 2^{m-1}+1 \leq n \leq 2^m - 2^{\frac{1}{2}(m+3)} \text{ (our new bound } UB_2\text{);} \\ \frac{1}{2}(m-1)n, \\ \qquad\qquad \text{if } 2^m - 2^{\frac{1}{2}(m+3)} < n \leq 2^m \text{ for even } n \text{ [52];} \\ \frac{1}{2}(m-1)n + \frac{3}{4}\lceil \frac{2^m}{2^m-(n+1)} \rceil + m - 1, \\ \qquad\qquad \text{if } 2^m - 2^{\frac{1}{2}(m+3)} < n \leq 2^m - 5 \text{ for odd } n \text{ (our new bound } UB_4\text{);} \\ \frac{1}{2}(m-1)n + 2^{m-2} - \frac{1}{2}(m-1), \\ \qquad\qquad \text{if } n = 2^m - 3 \text{ or } 2^m - 1 \text{ [40].} \end{cases}$$

Chapter 4

A new lower bound

This chapter proposes a new method to improve the general lower bound of the broadcast function $B(n)$ based on new observations of partitioning broadcast graphs.

4.1 Definitions and observations

Definition 4.1. A binomial tree BT_m on 2^m vertices of order m consists of

(1) a single vertex which is also the root, if $m = 1$;

(2) two copies of binomial trees BT_{m-1} having the two roots connected by an edge, if $m > 1$.

Definition 4.2. Let BT_m and BT_k be two binomial trees of order m and k respectively, and $m > k$. u is the root of BT_m. $BT_m \setminus BT_k$ is a tree obtained by removing a complete binomial tree BT_k from u in BT_m except the root u.

Figure 4.1 gives an example of a binomial tree and $BT_m \setminus BT_k$ for $m = 5$ and $k = 3$.

Definition 4.3. Let T be a broadcast tree of graph G originating from root u. Then $L_k(T)$, the first k broadcast level tree of T, consists of all the vertices of T which are informed in the first k time units following the broadcast scheme from originator vertex u in graph G.

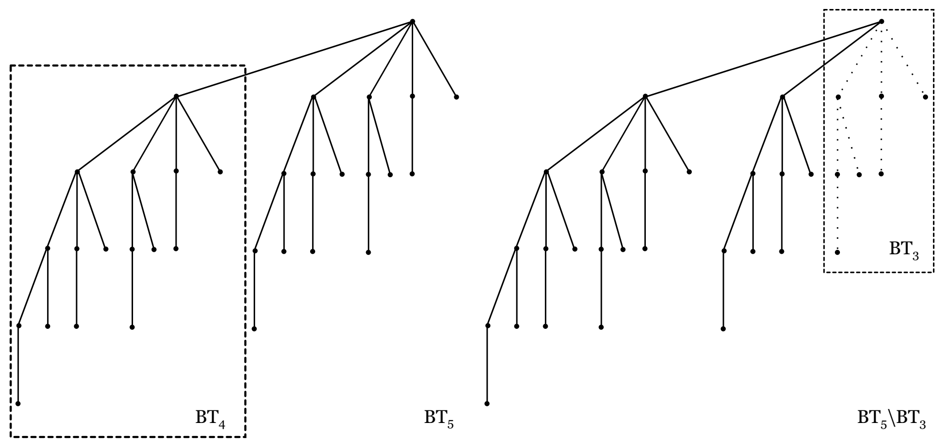


Figure 4.1: (a) is an example of a binomial tree BT_5. (b) the solid edges and the associated vertices give an example of $BT_5 \setminus BT_3$.

We know that any broadcast tree of a graph G on n vertices is a subtree of a binomial tree $BT_{\lceil \log n \rceil}$. So, the first k broadcast level tree $L_k(T)$ is a subtree of a binomial tree BT_k. Figure 4.2 gives one example of a broadcast tree BT_4 and its first 3 broadcast level tree. Let G be a minimum broadcast graph on $n = 2^m - 2^k + 1$ vertices, where $m \geq 3$ and $1 \leq k \leq m - 2$; u be a vertex of degree $m - k$ in G; T be the broadcast tree rooted at vertex u; and $L_k(T)$ be the first k broadcast level tree of T. If the neighbors of u are sorted in decreasing order of their degrees and the i-th neighbor corresponds to the i-th branch, we have the following observations.

Observation 4.1. $BT_m \setminus BT_k$ is a broadcast tree T of a broadcast graph G on $n = 2^m - 2^k + 1$ vertices rooted at a vertex u of degree $m - k$, where $m \geq 3$ and $1 \leq k \leq m - 2$.

Proof. Graph G has $2^m - 2^k + 1$ vertices, so the broadcasting must be completed in m time units. It is clear that during this m time broadcasting from originator u in $BT_m \setminus BT_k$ there are no idle vertices (informed vertices but not transferring the message). Thus, branches of the root u are complete binomial trees BT_{m-1}, BT_{m-2},$\cdots$, BT_k. There are in total $2^{m-1} + \cdots + 2^k + 1 = 2^m - 2^k + 1$ vertices, which is exactly the same as the number of

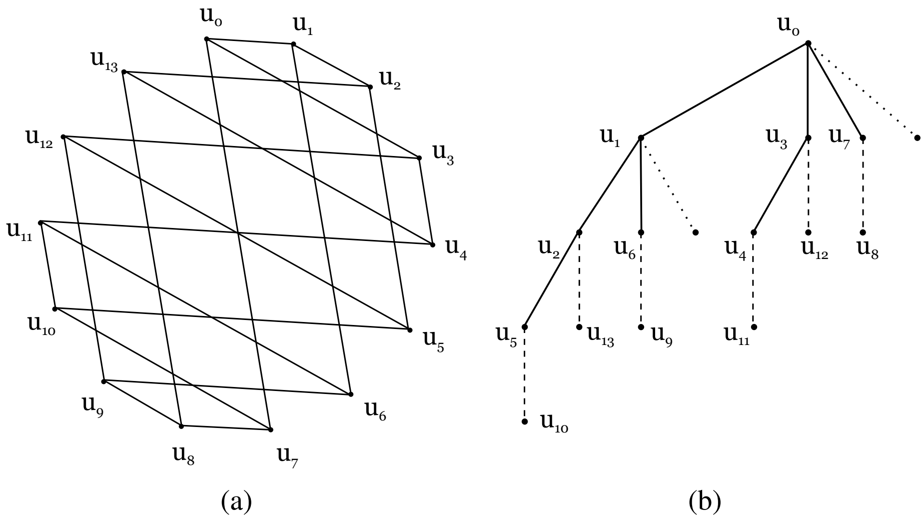


Figure 4.2: (a) is a broadcast graph G on 14 vertices. (b) is a binomial tree BT_4 on 16 vertices. 14 vertices with labels among them together with the solid and the dashed edges give a broadcast tree T of G. And the solid edges form a first 3 broadcast level tree $L_3(T)$.

vertices in G. Thus, the broadcast tree has to be $BT_m \setminus BT_k$.

□

Observation 4.2. Assume T is a broadcast tree of a broadcast graph G on $n = 2^m - 2^k + 1$ vertices rooted at vertex u of degree $m - k$, where $m \geq 3$. If $\frac{m}{2} \leq k \leq m - 2$, the i-th branch of u has 2^{k-i} vertices in $L_k(T)$, where $1 \leq i \leq m - k$.

Proof. If we ignore the first level (only one vertex: the root u), broadcast tree T becomes a forest of binomial trees BT_{m-1}, BT_{m-2}, $\cdots$, BT_k. So, the first k level broadcast tree $L_k(T)$ of T consists of the first $k-1$ level broadcast tree $L_{k-1}(BT_{m-1})$ of the first branch, $L_{k-2}(BT_{m-2})$ of the second branch, and $L_{k-i}(BT_{m-i})$ of the i-th branch in general. If $k \geq \frac{m}{2}$, then the last neighbor is informed at time unit $m - k \leq k$. Thus, $L_{2k-m}(BT_{m-k})$ is a binomial tree BT_{2k-m}. Then, each branch of $L_{k-i}(BT_{m-i})$ becomes a binomial tree BT_{k-i}. So, there are 2^{k-i} vertices on the i-th branch. □

Observation 4.3. Assume T is a broadcast tree of a broadcast graph G on $n = 2^m - 2^k + 1$ vertices rooted at vertex u of degree $m - k$, where $m \geq 3$. If w is an arbitrary vertex in

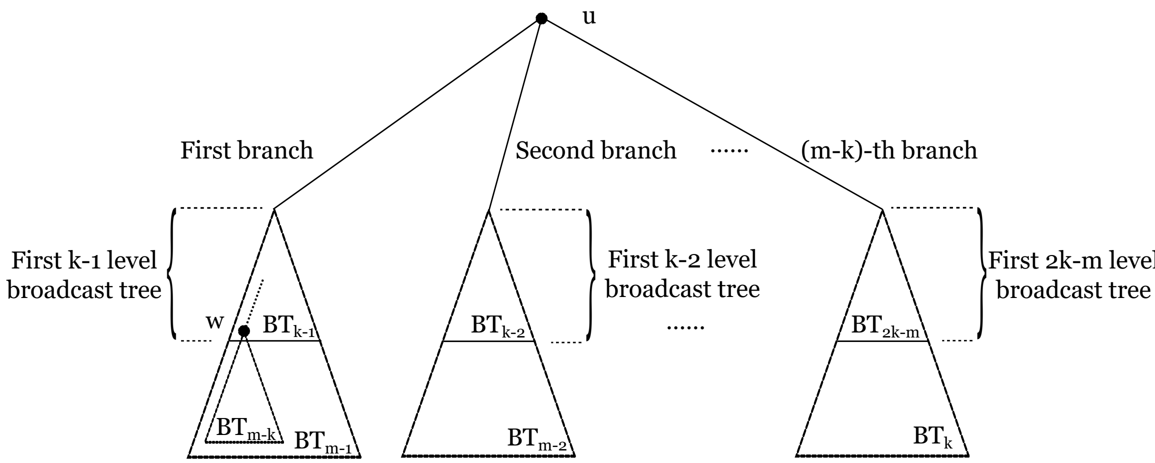


Figure 4.3: An example of a broadcast tree rooted at vertex u of degree $m-k$. The triangle at the i-th branch is a binomial tree BT_{m-i}, where $1 \leq i \leq m-k$. The upper part of each triangle is in the first $k-i$ broadcast level tree $L_{k-i}(BT_{m-i})$. And leaf w is an example of a vertex in $L_k(T)$. w has degree 1 in $L_k(T)$ and degree $m-k$ in $T - L_k(T)$. So, w has degree $m-k+1$ in broadcast tree T.

$L_k(T)$, then the w has degree strictly greater than $m-k$ in the broadcast tree T.

Proof. Observation 4.1 ensures that on i-th branch, BT_{m-i} is a complete binomial tree. So, $L_{k-i}(BT_{m-i})$ is indeed a complete binomial tree BT_{k-i} of order $k-i$, and it can be obtained by replacing every vertex in BT_{k-i} by a binomial tree BT_{m-k}. Thus, if a vertex w in $L_{k-i}(BT_{m-i})$ (which is BT_{m-k}) has degree a, then vertex w has degree $a+m-k$ in BT_{m-i} (also in broadcast tree T). Every leaf in any tree has the minimum degree 1. Therefore, any leaf in $L_k(T)$ gives the minimum degree $m-k+1 > m-k$ in broadcast tree T. Figure 4.3 shows an example of broadcast tree T when $k \geq \frac{m}{2}$. □

4.2 New lower bound

In this section, we first give a lower bound on $B(n)$ when $n = 2^m - 2^k + 1 - d$, where $\frac{m}{2} \leq k \leq m-2$ and $d = 0$. Then, we generalize the lower bound for any $0 \leq d \leq 2^k - 1$. That is we give a lower bound on $B(n)$ for all $2^{m-1} + 1 \leq n \leq 2^m - 2^{\frac{m}{2}+1} + 1$.

Theorem 4.1. Let $n = 2^m - 2^k + 1$, where $m \geq 3$ and $\lceil \frac{m}{2} \rceil \leq k \leq m-2$.

$$B(n) \geq \frac{n}{2}(m - k + \frac{1}{2} - \frac{1}{4m-4k+2}) + \frac{2^{k+1} - 2^{2k-m+1} - m + k}{2m-2k+1}$$

Proof. Observation 4.1 shows that the minimum degree of any vertex in G is $m-k$. So, we partition the vertices of G into V_{m-k}, the vertices of degree $m-k$; and V_{other}, other vertices. We also partition the edges into E_{m-k}, the edges connecting two vertices in V_{m-k}; E_{inter}, the edges connecting one vertex in V_{m-k} and one vertex in V_{other}; and E_{other}, the edges connecting two vertices in V_{other}. Let $|V_{m-k}|$, $|V_{other}|$, $|E_{m-k}|$, $|E_{inter}|$, and $|E_{other}|$ be the cardinality of each of the respective sets. It is easy to see $n = |V_{m-k}| + |V_{other}|$ and $e = |E_{m-k}| + |E_{inter}| + |E_{other}|$.

Case 1. If there is no vertex of degree $m-k$ in graph G, then the minimum degree is $m-k+1$, we have

$$e \geq \frac{n}{2}(m-k+1) \tag{1}$$

Case 2. If there is a vertex of degree $m-k$ in graph G, we consider the broadcast tree T originating from such a vertex u. In order to inform all vertices in graph G within m time units, every vertex except originator u cannot be idle during the minimum time broadcasting in G. So, the vertices informed by u (also the neighbors of u in broadcast tree T) must have degree m, $m-1$, ..., $k+1$. In other words the broadcast tree of originator u must be $BT_m \setminus BT_k$.

Since $k \geq \frac{m}{2}$, then the last neighbor of u has degree $k+1 > m-k$. Thus, there is no vertex of degree $m-k$ having a neighbor of degree $m-k$. Furthermore, if an edge is incident to a vertex of degree $m-k$, then it must be incident to a vertex of degree at least $m-k+1$.

$$|E_{m-k}| = 0$$

$$|E_{inter}| = (m-k)|V_{m-k}|$$

Again we consider the broadcast tree T and estimate $|E_{other}|$. By Observation 4.3, every vertex in the first k broadcast level tree $L_k(T)$ except the root u has degree greater than $m-k$. Thus, every edge except the ones on the first level in $L_k(T)$ has both of its end-points of degree greater than $m-k$. And by Observation 4.2, $L_k(T)$ becomes a forest of $L_{k-1}(BT_{m-1})$, $\cdots$, $L_{2k-m}(BT_{m-k})$ by ignoring the root and its incident edges. Then, $|E_{other}|$ can be estimated by counting the number of edges in the forest. Therefore,

$$|E_{other}| \geq 2^{k+1} - 2^{2k-m+1} - (m-k)$$

Combining $|E_{m-k}|$, $|E_{inter}|$, and $|E_{other}|$,

$$\begin{aligned}
e &= |E_{m-k}| + |E_{inter}| + |E_{other}| \\
e &\geq (m-k)|V_{m-k}| + 2^{k+1} - 2^{2k-m+1} - (m-k) \\
|V_{m-k}| &\leq \frac{e - 2^{k+1} - 2^{2k-m+1} - (m-k)}{m-k} \\
n - |V_{m-k}| &\geq n - \frac{e - 2^{k+1} + 2^{2k-m+1} + (m-k)}{m-k} \\
v_m + \cdots + v_{m-k+1} &\geq n - \frac{e - 2^{k+1} + 2^{2k-m+1} + (m-k)}{m-k} \qquad (2)
\end{aligned}$$

We have the following trivial inequalities.

$$
\begin{aligned}
2e &\geq (m-k)|V_{m-k}| + \cdots + mv_m \\
2e &\geq (m-k)n + v_{m-k+1} + 2v_{m-k+2} + \cdots + kv_m \\
2e &\geq (m-k)n + v_{m-k+1} + v_{m-k+2} + \cdots + v_m
\end{aligned} \tag{3}
$$

By substituting inequality (2) we get

$$
\begin{aligned}
2e &\geq (m-k)n + n - \frac{e - 2^{k+1} + 2^{2k-m+1} + (m-k)}{m-k} \\
e &\geq \frac{n}{2}(m-k+\frac{1}{2} - \frac{1}{4m-4k+2}) \\
&\quad + \frac{2^{k+1} + 2^{2k-m+1} + (m-k)}{2m-2k+1}
\end{aligned} \tag{4}
$$

Now we combine inequality (1) and inequality (4) given by the two different cases. Let RHS_1 and RHS_2 be the right hand side of the two inequalities respectively.

$$
\begin{aligned}
RHS_1 - RHS_2 &= \frac{m-k+1}{2(2m-2k+1)}n - \frac{2^{k+1} - 2^{2k-m+1} - m + k}{2m-2k+1} \\
&= \frac{1}{2(2m-2k+1)}((m-k+1)(2^m - 2^k + 1) \\
&\quad - (2^{k+2} - 2^{2k-m+2} - 2m + 2k)) \\
&\geq \frac{1}{2(2m-2k+1)}(3(2^{k+2} - 2^k + 1) \\
&\quad - (2^{k+2} - 2^{2k-m+2} - 2m + 2k)) \\
&= \frac{1}{2(2m-2k+1)}((2^{k+3} + 2^{k+1} + 3) \\
&\quad - (2^{k+2} - 2^{2k-m+2} - 2m + 2k)) \\
&> 0
\end{aligned}
$$

Thus, inequality (4) is the worst case and gives the lower bound, which completes the proof. ∎

Theorem 4.1 can be further generalized to other n.

Theorem 4.2. Let $n = 2^m - 2^k - d + 1$, where $m \geq 3$, $\lceil \frac{m}{2} \rceil \leq k \leq m-2$, and $0 \leq d \leq 2^k - 1$.

$$B(n) \geq \frac{n}{2}(m - k + \frac{1}{2} + \frac{\alpha - 1}{4m - 4k - 2\alpha + 2}) + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m - 2k + 1}$$

where

$$\alpha = \lfloor \frac{-W_{-1}(-2^{-d-2^{2k-m+1}+2k-m+1}\ln(2))}{\ln(2)} \rfloor - d - 2^{2k-m+1}$$

and $W_{-1}(x)$ is the lower branch of Lambert-W function.

Proof. Observation 4.1 is not true for general n; but when $k \geq \lceil \frac{m}{2} \rceil$, the minimum degree is always $m - k$. Assume a vertex r has degree $m - k - 1$ in a broadcast graph G on $n = 2^m - 2^k + 1 - d$ vertices, where $0 \leq d \leq 2^k - 1$. Minimum time broadcasting from originator r informs at most 2^{m-1} vertices on the first branch, 2^{m-2} vertices on the second branch, $\cdots$, and 2^{k+1} vertices on the last branch. Together with the originator r, there are $2^m - 2^{k+1} + 1$ vertices in total, which is $2^m - 2^{k+1} + 1 > 2^m - 2^{k+1} \geq 2^m - 2^k + 1 - d$. Thus, the minimum degree has to be $m-k$. Then, we have the two cases similar to theorem 4.1.

Case 1. If the minimum degree is greater than $m - k$, then

$$e \geq \frac{n}{2}(m - k + 1) \tag{5}$$

Case 2. If the minimum degree is $m - k$, we again have $|E_{m-k}|$, $|E_{inter}|$, and $|E_{other}|$ indicating the cardinalities of the different edge sets as in the proof of Theorem 4.1. However, the value of $|E_{m-k}|$, $|E_{inter}|$, and $|E_{other}|$ are different after removing d vertices.

Let u be a vertex of degree $m-k$. Assume α neighbors of u become degree $m-k$ after removing d vertices.

$$\begin{aligned}|E_{m-k}| + |E_{inter}| &\geq \frac{1}{2}\alpha|V_{m-k}| + (m-k-\alpha)|V_{m-k}| \\ &= \frac{1}{2}(2m-2k-\alpha)|V_{m-k}|\end{aligned}$$

$|E_{m-k}| + |E_{inter}|$ is minimized when α is maximized, which is the worst case for the lower bound. Consider the broadcast tree T rooted at vertex u. The neighbors s_1, s_2, $\cdots$, s_{m-k} of u have degree m, $m-1$, $\cdots$, $k+1$ respectively. And s_i is the root of a binomial tree BT_{m-i}. To maximize α, we remove vertices and make neighbors of u of degree $m-k$ from s_{m-k} to s_1, because the last neighbor s_{m-k} has the smallest degree. So, $2k-m+1$ neighbors of s_{m-k} are removed. $2^{2k-m+1}-1$ vertices are removed from the last branch. And the binomial tree BT_k attached to s_{m-k} becomes $BT_k \setminus BT_{2k-m+1}$.

In general, to make s_i of degree $m-k$, $2^{k-i+1}-1$ vertices are removed from the i-th branch. Thus, if α neighbors of u are of degree $m-k$, we need to remove $2^{2k-m+1}-1+2^{2k-m+2}-1+\cdots+2^{2k-m+\alpha}-1 = 2^{2k-m+\alpha+1}-2^{2k-m+1}-\alpha$ vertices from broadcast tree T. Since the number of removed vertices cannot exceed d, we have the following inequality:

$$d \geq 2^{2k-m+\alpha+1} - 2^{2k-m+1} - \alpha \tag{6}$$

$$2^{2k-m+1}2^{\alpha} \leq d + \alpha + 2^{2k-m+1}$$

$$2^{\alpha} \leq 2^{2k-m+1}\alpha + 2^{-(2k-m+1)}d + 1$$

Let $\alpha = -x - d - 2^{2k-m+1}$

$$
\begin{aligned}
2^{-x-d-2^{2k-m+1}} &\leq -x2^{-(2k-m+1)} \\
-2^{-2^{2k-m+1}-d+2k-m+1} &\geq x2^{x} \\
-2^{-2^{2k-m+1}-d+2k-m+1}\ln(2) &\geq x\ln(2)e^{x\ln(2)} \qquad (7)
\end{aligned}
$$

The right hand side of inequality (7) has the form $z \cdot e^{z}$. It can be solved by Lambert-W function $W(z \cdot e^{z}) = z$. However, $W(z)$ is a multivalued relation. $W(z)$ increases when $z \geq -\frac{1}{e}$ and $W(z) \geq -1$; while it decreases when $-\frac{1}{e} \leq z < 0$ and $W(z) \leq -1$. Let $W_0(z)$ and $W_{-1}(z)$ define the two single-valued function for the two different branches of $W(z)$ respectively. We need to estimate the value of $x\ln(2)$ to decide which single-valued function is used. We know that $\alpha \geq 0$, $0 \leq d \leq 2^{k} - 1$, and $\frac{m}{2} \leq k \leq m - 2$.

$$
\begin{aligned}
-x - d - 2^{2k-m+1} &\geq 0 \\
-x - 2^{2k-m+1} &\geq 0 \\
-x &\geq 2 \\
x\ln(2) &< -1
\end{aligned}
$$

Thus, $W_{-1}(z)$ is used.

$$
W_{-1}(-2^{-2^{2k-m+1}-d+2k-m+1}\ln(2)) \leq x\ln(2)
$$

Solve α by substitution.

$$
\alpha \leq -\frac{W_{-1}(-2^{-2^{2k-m+1}-d+2k-m+1}\ln(2))}{\ln(2)} - d - 2^{2k-m+1}
$$

Since α is an integer,

$$\alpha = \lfloor -\frac{W_{-1}(-2^{-2^{2k-m+1}-d+2k-m+1}\ln(2))}{\ln(2)} \rfloor - d - 2^{2k-m+1}$$

$|E_{other}|$ is analyzed as in the proof of Theorem 4.1 by counting the number of vertices in the first k broadcast level tree $L_k(T)$. If all the removed d vertices are in $L_k(T)$, then we have a trivial bound as follows.

$$|E_{other}| \geq 2^{k+1} - 2^{2k-m+1} - (m-k) - d$$

Therefore, we have the following inequality

$$e \geq \frac{1}{2}(2m - 2k - \alpha)|V_{m-k}| + 2^{k+1} - 2^{2k-m+1} - (m-k) - d$$

After reformatting,

$$v_m + \cdots + v_{m-k+1} \geq n - \frac{2e - 2^{k+1} + 2^{2k-m+1} + (m-k) + d}{2m - 2k - \alpha}$$

Then, by substituting the inequality to $2e \geq (m-k)|V_{m-k}| + \cdots + mv_m$ and by the similar technique given in the proof of Theorem 4.1,

$$e \geq \frac{n}{2}(m-k+1)\frac{2m-2k-\alpha}{2m-2k-\alpha+1} + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m-2k+1} \tag{8}$$

Again by the similar comparison, we can see that this bound is worse than bound 5 given in the first case. Thus, inequality (8) is the general lower bound on broadcast function, which completes the proof. □

4.3 Comparing the new and the existing lower bounds

Before comparing the old and the new lower bound, we first estimate the value of α. Let $n = 2^m - 2^k - d + 1$, $m \geq 3$, $\lceil \frac{m}{2} \rceil \leq k \leq m-2$, and $0 \leq d \leq 2^k - 1$. Recall Inequality 6.

$$\begin{aligned} d &\geq 2^{2k-m+\alpha+1} - 2^{2k-m+1} - \alpha \\ d &> 2^{2k-m+\alpha+1} \\ \log d &> 2k - m + \alpha + 1 \\ \alpha &< \log d - 2k + m - 1 \end{aligned} \tag{9}$$

Since $d \leq 2^k - 1 < 2^k$,

$$\alpha < m - k - 1$$

Let the new lower bound in Theorem 4.2 be NB_5. We compare it with the lower bound given in Section 2.4.

$$\begin{aligned} &NB_5 - LB \\ =&\frac{n}{2}(m-k+1)\frac{2m-2k-\alpha}{2m-2k-\alpha+1} + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m-2k+1} - \frac{n}{2}(m-k) \\ =&\frac{n}{2}\frac{m-k-\alpha}{2m-2k-\alpha+1} + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m-2k+1} \\ =&\frac{n}{2}(1 - \frac{m-k}{2m-2k-\alpha+1}) + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m-2k+1} \end{aligned}$$

By substituting Inequality 9,

$$\begin{aligned}
&NB_5 - LB \\
>&\frac{n}{2}(1 - \frac{m-k}{m-k+2}) + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m - 2k + 1} \\
>&\frac{n}{m-k+2} + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m - 2k + 1}
\end{aligned}$$

which is definitely positive. Therefore, the new lower bound is larger than the old one. Since the new lower bound is only valid when $k \geq \frac{m}{2}$, the best general lower bounds are as follows.

$$B(n) \geq \begin{cases} \frac{n}{2}(m-k+1)\frac{2m-2k-\alpha}{2m-2k-\alpha+1} + \frac{2^{k+1}-2^{2k-m+1}-m+k-d}{2m-2k+1}, \\ \qquad \text{if } 2^{m-1}+1 \leq n \leq 2^m - 2^{\frac{m}{2}} \text{ (the new bound);} \\ \frac{n}{2}(m-k), \\ \qquad \text{if } 2^m - 2^{\frac{m}{2}} < n \leq 2^m \text{ [39].} \end{cases}$$

Chapter 5

Minor results

This chapter lists two minor results related to the topic of minimum broadcast graphs.

5.1 A possibly existing minimum broadcast graph on $2^m - 2$ vertices

In [56], the author gives the lower bound on $B(2^m - 1)$ and also constructs a broadcast graph on 63 vertices with the same number of edges as the lower bound. The author of [69] constructs the minimum broadcast graph on 1023 and 4095 vertices by following the same idea and also conjectures that the star-cycle graph is the minimum broadcast graph on $2^m - 1$ vertices, where $m + 1$ is a prime number. The star-cycle graph is defined as follows.

Definition 5.1. The vertices of a star-cycle graph are of two types: $\frac{m(2^m-1)}{m+1}$ cycle vertices of degree $m - 1$, denoted by $c_0, c_1, \cdots, c_{\frac{m(2^m-1)}{m+1}-1}$; and $\frac{2^m-1}{m-1}$ star vertices of degree m, denoted by $s_0, s_1, \cdots, s_{\frac{2^m-1}{m+1}-1}$.
The edges are also of two types: the edges between two cycle vertices (c_i, c_j), where $|i - j| \equiv 2^{2i} \mod \frac{m(2^m-1)}{m+1}$ and $0 \le i \le \frac{m-4}{2}$; and the edges between a cycle vertex and a star vertex (c_i, s_j), where $i \equiv j \mod \frac{2^m-1}{m+1}$.

It is currently unknown that if a star-cycle graph is a minimum broadcast graph for any m, because there is no broadcast scheme for the graph in general. However, if it is a minimum broadcast graph, then one can construct a new minimum broadcast graph on $2^{m+1} - 2$ vertices, distinct to Knödel graphs.

Let S be a star-cycle graph on $2^m - 1$ vertices as defined above and S' be a copy of S. Then we construct a new graph G on $2^{m+1} - 2$ vertices by connecting all cycle vertices with the same label in S and S'.

To show graph G is a broadcast graph, we give the broadcast scheme. If the originator v is a cycle vertex in S (or S'), it informs its copy v' in S' (or S) in the first time unit. Then, v and v' use the broadcast scheme of the star-cycle graph to finish the broadcasting. If the originator is a star vertex s_a, it informs the cycle vertex $c_{a\times i}$ at time unit i. Then the cycle neighbors of s_a informs their copies $c'_{a\times i}$ in S' at time unit $i+1$. Next, every vertex follows the broadcast scheme of the star-cycle graph to inform all vertices except s'_a, the copy of the originator in S'. s'_a can be informed by c'_a at the last time unit, because the first informed cycle vertex must be idle at the last time unit.

Next, we show that graph G is not isomorphic to the Knödel graph on $2^{m+1} - 2$ vertices. By the definition of G, there is a path $c_0, c_1, \cdots, c_{\frac{2^m-1}{m+1}}$ of length $\frac{2^m-1}{m+1} + 1$. And since c_0 and $c_{\frac{2^m-1}{m+1}}$ are both adjacent to s_0, there is a cycle of length $\frac{2^m-1}{m+1} + 2$ in graph G. We know that $2^m - 1$ and $m + 1$ are both odd. So, $\frac{2^m-1}{m+1}$ is also odd and so is $\frac{2^m-1}{m+1} + 2$. Thus, graph G has an odd cycle. But, a Knödel graph is bipartite. The two graphs are not isomorphic.

This fact also shows us there may be another way to construct a minim broadcast graph on $2^m - 1$ vertices, which matches the lower bound in [56]. We can try to construct a new minimum broadcast graph on $2^{m+1} - 2$ vertices first, and decompose the graph to obtain a minimum broadcast graph on $2^m - 1$ vertices second.

5.2 Minimum regular broadcast graphs

With the respect to the restrictions of the classic model given in the introduction, we further add the regular property to broadcast graphs in addition. Similar to minimum broadcast graphs, we define the regular broadcast function $B_r(n)$ as the number of edges in the minimum regular broadcast graph on n vertices. Then, all the works done on the classic model can also be extended to the regular model, such as the exact value, upper bound, lower bound, and the monotonicity of $B_r(n)$.

Here is a list of known facts about $B_r(n)$.

(1) If a minimum broadcast graph is regular, then this graph is a minimum regular broadcast graph. And the converse is not necessarily true.

(2) $B_r(2^m) = \frac{1}{2}m2^m$, which is given by hypercubes;

(3) $B_r(2^m - 2) = \frac{1}{2}(m-1)(2^m - 2)$, which is given by Knödel graphs;

(4) $B_r(2^m - 2^k) \geq \frac{1}{2}(m - \frac{k}{2} - \frac{1}{2})(2^m - 2^k)$, where $k \geq 3$ is odd.

The upper bound on $B_r(2^m - 2^k)$ is given by the modified compounding construction in [38]. The original construction put the odd vertices in each copy of Knödel graph into a hypercube of dimension $k-1$. In the modified construction, the odd vertices with the same label form a hypercube of dimension x, while the even vertices with the same label form a hypercube of dimension $k-1-x$. So, any originator broadcasts in its own hypercube first and acts as an even vertex in the broadcast scheme in [38] to finish broadcasting. Then, all odd vertices have degree $m-k+x$, and even vertices have degree $m-k+k-1-x = m-x-1$. If $m-k+x = m-x-1$ or $x = \frac{k-1}{2}$, the constructed graph is regular. Figure 5.1 gives an example of the construction when $m=5$ and $k=3$, using Knödel graph KG_6 and hypercube HQ_2.

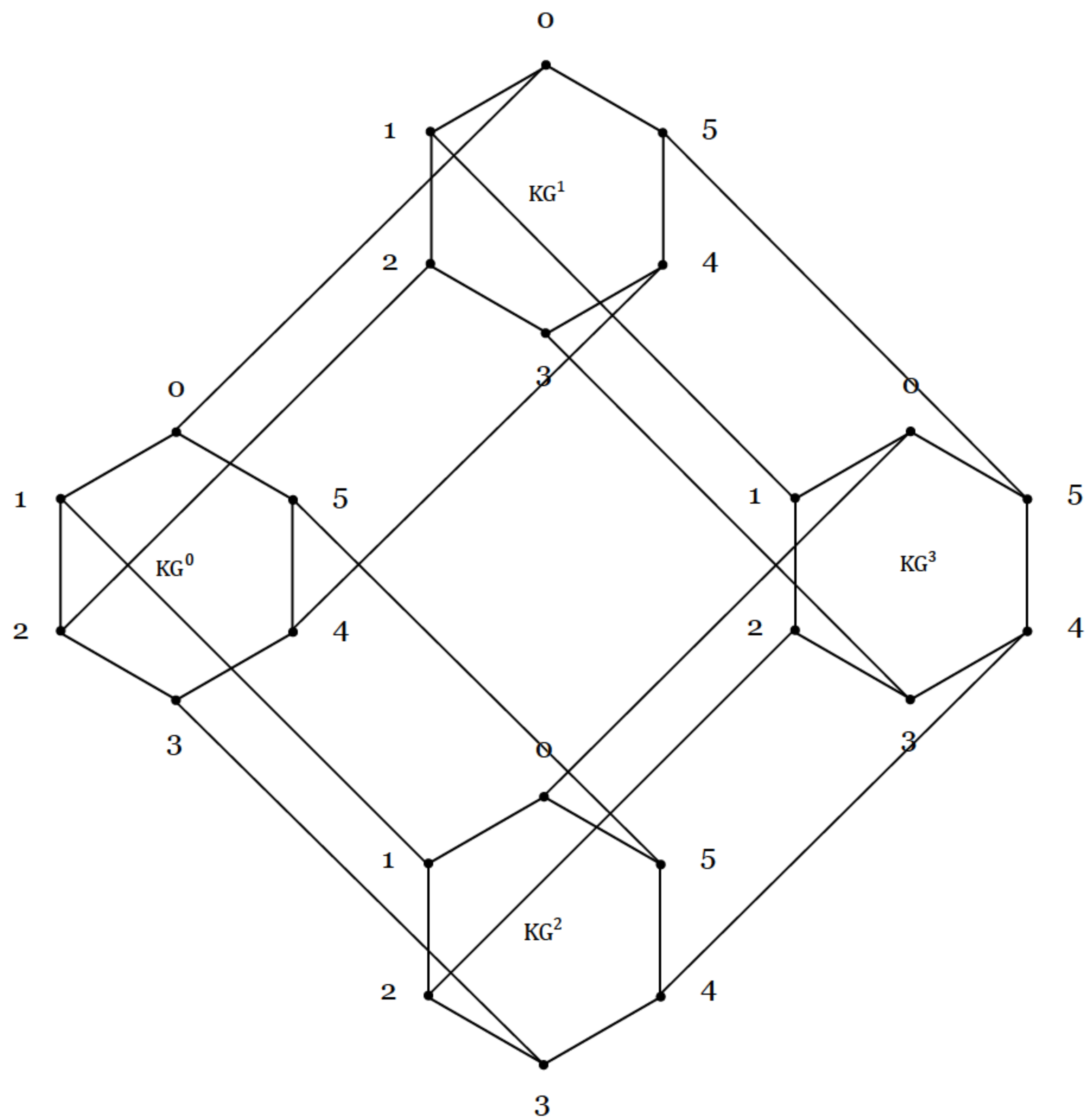


Figure 5.1: An example of the modified compounding construction

Other than the problems similar to the classic model, there is one completely different question. What is the maximum n such that there is an $m - a$ regular broadcast graph, where $2^{m-1} + 1 \leq n \leq 2^m$ and $0 \leq a \leq m - 3$? Here is an upper bound obtained by counting the maximum number of vertices in the possible broadcast tree.

Assume the broadcast graph is $m - a$ regular. Let n_x be the number of informed vertices up to time unit x and c_x be the number of vertices such that all of its neighbors are informed up to time unit x. Since the graph is $m-a$ regular, the originator is busy in the first $m-a$ time units. Then, $n_{m-a} = 2^{m-a}$ and $c_0 = c_1 = \cdots = c_{m-a-1} = 0$. The originator and its first informed neighbor have informed all their neighbors in time unit $m-a$. So, $c_{m-a} = 2$. In the next time unit $m-a+1$, these two vertices are idle, which implies $n_{m-a+1} = 2n_{m-a} - c_{m-a}$ and $c_{m-a+1} = 2c_{m-a}$. Thus, we have the recurrence relation.

$$\begin{cases} n_{m-a+i} = 2n_{m-a+i-1} - c_{m-a+i-1} & \text{if } i > 0 \\ n_{m-a} = 2^{m-a} & \text{if } i = 0 \\ c_{m-a+i} = 2c_{m-a+i-1} & \text{if } i > 0 \\ c_{m-a} = 2 & \text{if } i = 0 \end{cases}$$

After solving the recurrence relation, $n_{m-a+i} = 2^{m-a+i} - i2^i$. We also need to make sure that $2^{m-a+i-1} + 1 \leq n^{m-a+i}$ otherwise the broadcast time will exceed $\lceil \log n \rceil$. Therefore, $i2^i \leq 2^{m-a+i-1} - 1$. (Lamber W function is again needed to solve this inequality.)

Thus, $m-a$ regular broadcast graphs are only possible when $2^{m-1}+1 \leq n \leq 2^{m-a+i} - i2^i$, where $i2^i \leq 2^{m-a+i-1} - 1$.

If $a = 3$ and $i = 3$, the upper bound of $m-3$ regular broadcast graph is $n = 2^m - 24$. There is no $m-3$ regular broadcast graph when $n = 2^m - 8$. Furthermore, the construction above gives us an $m-2$ regular broadcast graph on $2^m - 8$ vertices. Thus, this graph is a minimum regular broadcast graph and

Theorem 5.1. $B_r(2^m - 8) = \frac{1}{2}(m-2)(2^m - 8)$.

Chapter 6

Future work

This chapter gives three topics that we can further study in the future.

6.1 Generalizing the compounding method

Section 3.1.2 introduces a new method of compounding binomial trees and Knödel graphs. In the future, the compounding method can be further improved.

Let $G = (V, E)$ be an arbitrary broadcast graph on n vertices, $v \in G$ be a random vertex with degree $k < \lceil \log n \rceil$ and $b_1, b_2, ..., b_k \in G$ be the neighbors of v. Then we can construct a new broadcast graph $G' = (V \cup \{w\}, E \cup E_w)$, where w is an additional vertex to graph G and $E_w = \{(w, b_i) | 1 \leq i \leq k\}$ (or one more edge (w, v) if every b_i is busy in the broadcasting from v. So, w has to inform v at the last time unit). Since vertex v and the additional vertex w share the common neighbors, w can play the same role in the broadcasting from w in graph $G \cup w$. Thus, every vertex in G except v can be informed in $\lceil \log n \rceil$ time units. To inform v, there are two cases. If there is a neighbor b_j, $1 \leq j \leq k$ which is idle after a time unit $0 \leq t < \lceil \log n \rceil$, vertex v can be informed by b_j. If there is no idle neighbor in the broadcasting, v has to be informed by w. The construction needs one edge connecting w and v. Also $deg(w) \leq \lceil \log n \rceil - 1$ is also important, where $deg(w)$

is the degree of vertex w.

The next step of the construction is replacing every vertex in graph G by a binomial tree B_k on 2^k vertices and connect every non-root vertex to v's neighbors (and v, if the case above happens). Every non-root vertex can be considered as vertex w described above. Thus, broadcasting from any non-root vertex in the graph G compounded by binomial tree B_k takes $k + \lceil \log n \rceil$ time units.

To minimize the edges used in the construction, we need to carefully select v with the minimum degree. Therefore, the first topic will be investigated in the future is finding the broadcast scheme from a root vertex in the graph described above and finding the best broadcast graph G and vertex v for the generalized compounding method.

6.2 More vertex addition methods

In the thesis, the vertex addition method is greatly improved by using different broadcast schemes of Knödel graph. However, we currently do not know if it is the best result we can achieve by this method. We believe that the degree of the additional vertex is no smaller than $\Theta(\frac{n}{2^m - d})$ and the vertex addition method result is $B(n) \leq \frac{n}{2}(m-1) + c\frac{n}{2^m - d}$, where $c \geq 1$ is a constant. In other words, our new result is approaching to the optimal solution.

The reason is that all dimensional broadcast schemes except $1, 2, \cdots, m-1, 1$ and $m-1, 1, 2, \cdots, m-1$ create complete different broadcast trees and make different vertices idle in the last time unit for different Knödel graphs. Figure 6.1 gives an example of one broadcast scheme acting differently in different Knödel graphs. In KG_{10}, v_4 and v_7 are idle in the third time unit, but no vertex is idle in the last time unit. In contrast, there is no vertex idle in the third time unit, v_6 and v_9 are idle in the last time unit in KG_{12}.

This fact implies that the vertex addition method can only use two broadcast schemes: $1, 2, \cdots, m-1, 1$ and $m-1, 1, 2, \cdots, m-1$ in general. It also provides a strong evidence

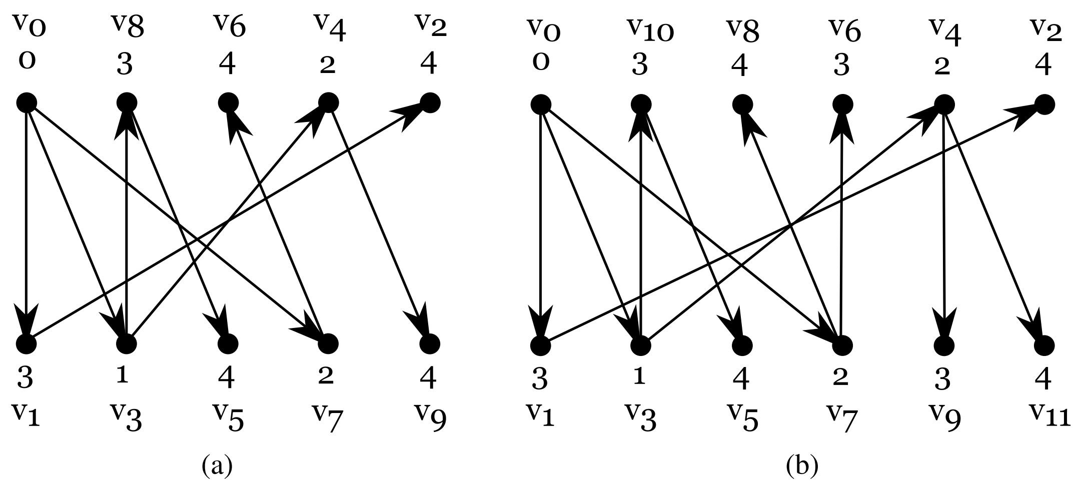


Figure 6.1: The examples of dimensional broadcast scheme 2,3,1,2 in Knödel graph KG_{10} and KG_{12}. The numbers beside the vertices indicate the informed time unit.

to support our estimation on the degree of the additional vertex.

If this topic is fully studied, the research on the vertex addition method based on Knödel graph is complete.

6.3 More lower bounds

There are two more works related to lower bounds that can be further explored in the future.

The first work is generalizing the lower bound to any $2^{m-1} + 1 \leq n \leq 2^m$. We assume $\frac{m}{2} \leq k \leq m - 2$ in the previous estimation. However when $2^m - 2^{\frac{m}{2}} \leq n \leq 2^m$, the value of k is smaller than $\frac{m}{2}$. Then, our lower bound is invalid. So, generalizing the lower bound to $k < \frac{m}{2}$ can be done by a similar technique in the future.

The second work is combining our result with the result given in [29]. In the paper, the authors estimate the number connections between vertices of different degrees and give the following inequalities. Let $n = 2^m - 2^k + 1$ and v_i be the number of vertices of degree i.

$$\sum_{i \geq m} (i-1)v_i \geq v_{m-k},$$

$$\sum_{i \geq m-1} (i-1)v_i \geq 2v_{m-k},$$

$$\sum_{i \geq m-2} (i-1)v_i \geq 3v_{m-k},$$

$$\cdots$$

$$\sum_{i \geq m-k+1} (i-1)v_i \geq kv_{m-k},$$

In our estimation, Inequality 3

$$2e \geq (m-k)n + v_{m-k+1} + v_{m-k+2} + \cdots + v_m$$

is independent to all inequalities above. Thus, there is a way to combine our new lower bound with the inequalities and may further improve the result.

Chapter 7

Conclusion

This thesis discusses the general upper bound and the general lower bound on broadcast function.

The upper bound is given by broadcast graph construction. When $2^m - 1 + 1 \leq n \leq 2^m - 2^{\frac{1}{2}(m+3)}$, the compounding construction gives the best upper bound. When n is even and $2^m - 2^{\frac{1}{2}(m+3)} < n \leq 2^m$, Knödel graphs give the best upper bound; while if n is odd, the vertex addition method based on Knödel graph obtains the lowest upper bound.

Our research improves both the compounding method and the vertex addition method. The improvement of the compounding method has two phases. First, we use the Knödel graph on $2^m - 2$ vertices instead of hypercubes in [3]. The Knödel graph on $2^m - 2$ vertices has the advantage comparing with hypercubes because every vertex in the Knödel graph has one fewer degree, and two vertices are always idle in the last time unit of broadcasting. In the second phase, we discover that every non-root vertex in binomial trees can act as any vertex in the base graph. So, a base graph with the smaller minimum degree possibly improves the construction. Thus, we use the compounded graph in [38] as the base and further reduce the upper bound.

The improvement on the vertex addition method also has two steps. In the first step, we use the newly discovered broadcast schemes to improve the broadcast graph construction when $n = 2^m - 2^k - d$ and $2^k + d$ is large. The new broadcast schemes make more vertices idle in the last time unit of broadcasting, which allows us using 3-distance dominating set instead of 1-distance dominating set. And in the second step, the dimensional broadcast schemes on $1, 2, \cdots, m-1, 1$ and $m-1, 1, 2, \cdots, m-1$ always make a block of vertices idle in the last time unit. Therefore, if one vertex v in the block of vertices is adjacent to the additional vertex, vertex v "dominates" a block of vertices of size $\Theta(\frac{n}{2^k - d})$, which greatly improves the upper bound.

For the general lower bound, we introduce the first non-trivial general lower bound on $B(n)$ by partitioning the vertices and the edges. However, this lower bound has two limitations: only valid when $n \leq 2^m - 2^{\frac{m}{2}}$ and not as good as the bound when $n = 2^m - 2^k + 1$ in [29]. To overcome the first limitation, we can generalize our lower bound to any n, but it needs more analysis based on different assumptions. For the second limitation, we can combine our bound with some results in [29] and further improve the lower bound.

We present some of the results of the thesis in [33–37].

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