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Research Article

Smooth Kernel Estimation of a Circular Density Function: A Connection to Orthogonal Polynomials on the Unit Circle

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The circular kernel density estimator, with the wrapped Cauchy kernel, is derived from the empirical version of Carathéodory function that is used in the literature on orthogonal polynomials on the unit circle. An equivalence between the resulting circular kernel density estimator, to Fourier series density estimator, has also been established. This adds further weight to the considerable role of the wrapped Cauchy distribution in circular statistics.

1. Introduction

Consider an absolutely continuous (with respect to the Lebesgue measure) circular density $f(\theta)$, $\theta \in [-\pi, \pi)$; that is, $f(\theta)$ is 2π-periodic,

$$f(\theta) \geq 0 \quad \text{for } \theta \in \mathbb{R},$$
$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 1. \tag{1}$$

In the literature on modeling circular data, starting from the classical text of Mardia [1], the standard texts such as Fisher [2], Jammalamadaka and SenGupta [3], and Mardia and Jupp [4] cover parametric models along with many inference problems. More recently various alternatives to these classical parametric models, exhibiting asymmetry and multimodality, have been investigated with respect to their mathematical properties and goodness of fit to some real data; see Abe and Pewsey [5], Jones and Pewsey [6], Kato and Jones [7], Kato and Jones [8], Minh and Farnum [9], and Shimizu and Lida [10].

As elaborated in Nuñez-Antonio et al. [11], circular data possess characteristics such as high skewness or kurtosis and multimodality in many situations, for example, the data on directions of clinical vectorcardiogram (see Downs [12]), wind directions (see Fisher and Lee [13]), and animal orientation (see Oliveira et al. [14]). Such data are not well fitted by standard parametric models, and in such cases, semiparametric or nonparametric modeling may be considered more appropriate.

Fernändez-Durän [15] and Mooney et al. [16] considered modeling of circular data by semiparametric models based on mixture of circular normal and von Mises distributions whereas Hall et al. [17] and Bai et al. [18] considered nonparametric kernel-based density estimation for data on sphere and Fisher [19] and Taylor [20] considered kernel density estimation for circular data. Whereas Fisher [19] adapted the kernels used for linear data to the context of circular data, Taylor [20] used von Mises circular distribution replacing linear kernel used in the classical kernel density estimator that naturally maintains the periodicity in the resulting density estimator. More recently, Di Marzio et al. [21, 22] provided a theoretical basis for circular kernel density estimator by considering the general setting of nonparametric kernel density estimation on a d-dimensional torus; the special case of $d = 1$ provides circular kernel density estimator that is described below.

Given a random sample $(\theta_1, \ldots, \theta_N)$ from the density (1), the circular kernel density estimator is given by

$$\hat{f}(\theta; \kappa) = \frac{1}{N} \sum_{j=1}^{N} K_\kappa \left(\theta - \theta_j\right), \tag{2}$$

where $K_\kappa(\theta-\phi)$ is a circular kernel where κ is a concentration parameter and ϕ is the mean direction. Note that a circular kernel is usually chosen to be a circular density, unimodal and symmetric around its mean direction which is zero, and it is characterized by a concentration parameter κ which governs the amount of the smoothing (see, e.g., Di Marzio et al. [21, 22] for details). Classical examples of circular kernels include the von Mises density and the densities of wrapped normal and wrapped Cauchy distributions.

In this note we consider the circular kernel density estimator using the wrapped Cauchy kernel that is given by

$$\hat{f}_{\mathrm{WC}}(\theta;\rho)=\frac{1}{N}\sum_{j=1}^{N}K_{\mathrm{WC}}\left(\theta-\theta_j;0,\rho\right), \tag{3}$$

where

$$K_{\mathrm{WC}}(\theta;\mu,\rho)=\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)}, \qquad -\pi\le\theta<\pi, \tag{4}$$

and we show that it is derived from the empirical version of Carathéodory function, used in the literature on orthogonal polynomials on the unit circle. We also show that this approach leads to Fourier series density estimation; however no truncation of the series is required.

In Section 2, some basic results from the literature on orthogonal polynomials on the unit circle are presented first and then the strategy of estimating $f(\theta)$ is introduced. This in turn produces the nonparametric circular kernel density estimator given in (3). The details are in Section 3. The next section describes the Fourier series estimator of $f(\theta)$ that is shown to be equivalent to the circular kernel density estimator (2) in a limiting sense when wrapped Cauchy kernel is employed.

2. Some Preliminary Results on Orthogonal Polynomials on Unit Circle

Let $\mathbb{D}$ be the open unit disk, $\{z\mid |z|<1\}$, in the complex plane, and let μ be a continuous measure defined on the boundary $\partial\mathbb{D}$, that is, the circle $\mathbb{C}=\{z\mid |z|=1\}$. The point $z\in\mathbb{D}$ will be represented by $z=re^{i\theta}$ for $r\in[0,1)$, $\theta\in[0,2\pi)$ and $i=\sqrt{-1}$. The closure of $\mathbb{D}$ will be denoted by $\overline{\mathbb{D}}$.

Definition 1. A sequence of polynomials $\{\phi_n(z)\}$ defined on $\mathbb{C}$ are orthogonal with respect to a Borel measure μ, if they satisfy

$$\int_{\mathbb{C}}\phi_r(z)\overline{\phi_s(z)}d\mu(z)=\delta_{rs}, \tag{5}$$

where $\delta_{rs}>0$ for $r=s$ and it equals 0, otherwise.

The importance of these polynomials is in approximation of bounded functions $g(z)$ defined on the unit circle by the representation (see Cantero and Iserles [23])

$$g(z)=\sum_{n=1}^{\infty}\hat{g}_n\phi_n(z),\quad \text{where } \hat{g}_n=\frac{\langle g,\phi_n\rangle}{\|\phi_n\|_\mu^2}, \tag{6}$$

where

$$\langle g,h\rangle=\int_{|z|=1}g(z)\overline{h(z)}d\mu,\qquad \|g\|_\mu=\langle g,g\rangle^{1/2}. \tag{7}$$

Here $\overline{h(z)}$ refers to the complex conjugate of $h(z)$. The reader is referred to the excellent book on the subject of orthogonal polynomials on the unit circle by Simon [24]. We will use this result for approximating $F(z)$ defined below,

$$F(z)=\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)f(\theta)\,d\theta. \tag{8}$$

The above function is a "special function," known as Carathéodory function, that plays an important role in the study of orthogonal polynomials defined on the unit circle (see Simon [24], pp. 25). The orthogonal expansion of $F(z)$ with respect to the basis $\{1,z,z^2,\ldots\}$ is given by (see (4.15) and (4.16) of Simon [25])

$$F(z)=1+2\sum_{n=1}^{\infty}c_nz^n, \tag{9}$$

where

$$c_n=\int_{-\pi}^{\pi}e^{-in\theta}f(\theta)\,d\theta \tag{10}$$

is the jth trigonometric moment of the circular distribution with density f. The integrand in (8) involves the function

$$C(z,\omega)=\frac{\omega+z}{\omega-z} \tag{11}$$

that is known as the complex Poisson kernel in the theory of complex analysis. Its relation with the wrapped Cauchy kernel lies in the fact that

$$\operatorname{Re}C\left(re^{i\theta},e^{i\varphi}\right)=P_r(\theta,\varphi)=\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} \tag{12}$$

for $\theta,\varphi\in[-\pi,\pi)$ and $r\in[0,1)$, where "Re" denotes the real part. The above function is known as the real Poisson kernel and it is clearly related to the wrapped Cauchy kernel since

$$P_\rho(\theta,\varphi)=(2\pi)K_{\mathrm{WC}}(\theta-\varphi;0,\rho). \tag{13}$$

A standard result in complex analysis, known as the *Poisson representation* (see ([24], p. 27)), says that if g is analytic in a neighborhood of $\overline{\mathbb{D}}$, with $g(0)$ real, then for $z\in\mathbb{D}$,

$$g(z)=\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\operatorname{Re}\left(g\left(e^{i\theta}\right)\right)\frac{d\theta}{2\pi}. \tag{14}$$

This representation leads to the result (see (ii) in Section 5 of Simon [24]) that for Lebesgue a.e. θ

$$\lim_{r\uparrow1}F\left(re^{i\theta}\right)\equiv F\left(e^{i\theta}\right) \tag{15}$$

exists and that

$$f(\theta)=\frac{1}{2\pi}\operatorname{Re}\left(F\left(e^{i\theta}\right)\right)=\frac{1}{2\pi}\lim_{r\uparrow1}\operatorname{Re}F\left(re^{i\theta}\right). \tag{16}$$

3. Smooth Circular Density Estimator Derived from an Estimator of $F(z)$

Recognizing the expression for $F(z)$ as the expectation of $((e^{i\theta}+z)/(e^{i\theta}-z))$, its empirical version is given by

$$F_N\left(re^{i\theta}\right)=\frac{1}{N}\sum_{j=1}^{N}\left(\frac{e^{i\theta_j}+re^{i\theta}}{e^{i\theta_j}-re^{i\theta}}\right). \tag{17}$$

Thus we may define an estimator of $f(\theta)$ motivated by $F_N(z)$, the identities (15) and (16), as

$$\hat{f}_r(\theta)=\frac{1}{2\pi}\operatorname{Re}F_N\left(re^{i\theta}\right). \tag{18}$$

Now we show that the above estimator is of the same form as the circular density estimator given in (3). Recognize that

$$F_N\left(re^{i\theta}\right)=\frac{1}{N}\sum_{j=1}^{N}C\left(re^{i\theta},\omega_j\right), \tag{19}$$

where $\omega_j=e^{i\theta_j}$; then using (12), we have

$$\operatorname{Re}F_N\left(re^{i\theta}\right)=\frac{1}{N}\sum_{j=1}^{N}P_r\left(\theta,\theta_j\right), \tag{20}$$

and therefore

$$\begin{aligned}\hat{f}_r(\theta)&=\frac{1}{(2\pi)N}\sum_{j=1}^{N}P_r\left(\theta,\theta_j\right)\\&=\frac{1}{N}\sum_{j=1}^{N}K_{\mathrm{WC}}\left(\theta-\theta_j;0,r\right),\end{aligned} \tag{21}$$

which is of the same form as in (3).

4. A Connection of the Circular Density Estimator to Fourier Series Estimator

The orthogonal series for $F(z)$ given in (9) may be directly used to define an estimator of $f(\theta)$. This method also provides the same estimator as derived in the previous section that is demonstrated as follows. Estimating the coefficients c_n, $n=1,2,\ldots$, by

$$\hat{c}_n=\frac{1}{N}\sum_{j=1}^{N}e^{-in\theta_j}, \tag{22}$$

an estimator of $F(z)$ is given by

$$\hat{F}(z)=1+\sum_{n=1}^{\infty}\hat{c}_n z^n. \tag{23}$$

Substituting the expression for $\hat{c}_n$ from (22), we can write

$$\begin{aligned}\hat{F}(z)&=1+\frac{2}{N}\sum_{j=1}^{N}\left\{\sum_{n=1}^{\infty}e^{-in\theta_j}z^n\right\}\\&=1+\frac{2}{N}\sum_{j=1}^{N}\left\{\sum_{n=1}^{\infty}\left(\overline{\omega}_j z\right)^n\right\};\\ \omega_j=e^{in\theta_j}&=1+\frac{2}{N}\sum_{j=1}^{N}\left(\frac{\overline{\omega}_j z}{1-\overline{\omega}_j z}\right)\\&=\frac{2}{N}\sum_{j=1}^{N}\left(\frac{1}{2}+\frac{\overline{\omega}_j z}{1-\overline{\omega}_j z}\right)\\&=\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1+\overline{\omega}_j z}{1-\overline{\omega}_j z}\right)\\&=\frac{1}{N}\sum_{j=1}^{N}C\left(z,\omega_j\right),\end{aligned} \tag{24}$$

which is the same as $F_N(z)$ given in (19).

Equation (9) may be used to derive the Fourier series estimator. The reader may be referred to Efromovich [26] for the details about Fourier series density estimator. Truncating the infinite sum in (9) at some large index n^*, we have an estimator of $f(\theta)$

$$\hat{f}_S(\theta)=\frac{1}{2\pi}+\frac{1}{\pi N}\sum_{j=1}^{N}\sum_{n=1}^{n^*}\cos n\left(\theta-\theta_j\right). \tag{25}$$

Here n^* is chosen according to some criteria, for example, to minimize the integrated squared error. The reader may be referred to Efromovich [26]. Thus we have two contrasting situations; in one we have to choose n^* and in the other case we have to choose a suitable concentration parameter (ρ). Numerically as well as technically, the second choice, that is, the circular kernel density estimator, may be considered more suitable.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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