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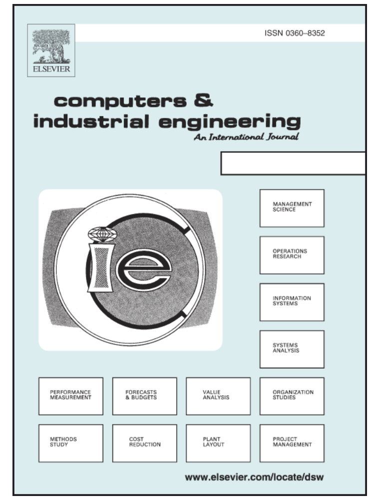

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Investigating Rank Reversal in Preference Relation Based on Additive Consistency: Causes and Possible Solutions

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Investigating Rank Reversal in Reciprocal Fuzzy Preference Relation Based on Additive Consistency: Causes and Solutions

Abstract:

Rank reversal is a common phenomenon in decision making. Rank reversal occurs when a new alternative is added to (or removed from) a set of alternatives, which causes change in the ranking order of the alternatives. This paper studies the possible causes of rank reversal in reciprocal preference relation based on additive consistency. Our investigation reveals that inconsistency of information is the main cause of this phenomena in preference relations followed by ranking score aggregation. We propose score aggregation methods to address the phenomenon of rank reversal. The proposed methods are illustrated using numerical examples. The results are better than other tested methods.

Keywords:

Preference relation; Multi criteria analysis; Rank Reversal; Additive consistency; AHP

1. Introduction

Multi-Criteria Decision-making (MCDM) is a field with many strengths, among which is its ability to assist decision-makers in solving difficult decisions involving conflicting criteria and to help them learn more about their preferences. However, some methods are known to exhibit a phenomenon called rank reversal. Rank reversal occurs when a new alternative is added to (or removed from) a set of alternatives, which causes a change in the ranking order of the alternatives (Barzilai & Golany, 1994). The literature on decision-making reveals that a number of methods suffer from this phenomenon. Some of them are Analytic Hierarchy Process (AHP) (Barzilai & Golany, 1994; Wang & Luo, 2009), Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) (Wang et al., 2007; Wang & Luo, 2009), ELimination and Choice Expressing Reality (ELECTRE), Preference Ranking Organisation Method for Enrichment Evaluations (PROMETHEE) (Frini et al., 2012; Mareschal et al., 2008), Data Envelopment analysis - Analytic hierarchy process (DEAHP), Borda-Kendall (BK) (Wang & Luo, 2009) and Weighted Sum Method (WSM)(Wang & Luo, 2009).

The rank reversal phenomenon has raised concerns against the use of affected methods, especially AHP. Rank reversal could be of two types: partial or total. Partial rank reversal happens to limited alternatives while other alternatives still have the same ordering. For example, suppose that the current ranking of three alternatives is $A_3 \succ A_1 \succ A_2$, such that alternative A_3 is preferred over alternative A_1 and A_2 respectively. However, when a new alternative A_4, which is not dominant, is introduced, the ranking could become $A_1 \succ A_3 \succ A_4 \succ A_2$. Notice that alternative A_3 now becomes second while alternative A_1 is first. This is called partial rank reversal. On the other hand, total rank reversal is the same as the partial rank reversal except that the whole ranking order is reversed. In this case, the best alternative becomes the worst and the worst becomes the best $A_2 \succ A_4 \succ A_1 \succ A_3$ (Dymova et al., 2013; Garcia-Cascales & Lamata, 2012).

Belton and Gear (1983) were the first to notice this phenomenon in AHP. Since then, the literature of MCDM has been in debate about the impact of this phenomenon, and the validity of the affected methods. Many researchers such as Dyer (1990), Schenkerman (1994), Perez (1995), and Leung and Cao (2001) criticized the exhibited methods, whereas researchers such as Saaty and Vargas (1984), Saaty (1987), Forman (1990), and Millet and Saaty (2000) argued for the legitimacy of this phenomenon.

To emphasize the phenomenon of rank reversal, we point the reader to the *contraction consistency condition* mentioned by Pavlicic (2001) adopted from Amartya Sen that states:

Contraction consistency condition: *If alternative A is the best in the set of alternatives S such that $A \in S$, then it has to be the best in every subset $E \subset S$ where $A \in E$.*

This phenomenon could drive some decision-makers away from using methods known to have rank reversal, even if they are well-known. For instance, recently Anbaroglu et al. (2014) chose to use the Weighted Product Model (WPM) instead of relying on well-known and widely used models such as AHP and WSM just because it does not suffer from any kind of rank reversal issues. Furthermore, they commented on the problem of rank reversal as “a serious limitation” of the MCDM field, which could lead researchers to misunderstand the difference between examined alternatives. Therefore, a need for handling this phenomenon is necessary, at least for the experts who are not in favor of it. The literature on preference relations, especially multiplicative preference relations, links this phenomenon to the inconsistency of the data, the concept of pairwise comparison on which preference relations are based, the preference aggregation method, and the score aggregation method. To our knowledge, there is no complete study yet that investigates these three possible reasons for rank reversal in preference relations. There is one study, conducted by Leskinen and Kangas (2005), on the inconsistency of pairwise comparison based on a regression model. They concluded that inconsistency could lead to rank reversal. This phenomenon, however, does not occur when there is single criterion. But, in multiple criteria even if the data are consistent, the aggregation method (i.e. arithmetic mean) can result in rank reversals.

In this paper, our goal is to investigate how inconsistency and aggregation methods could lead to rank reversal in fuzzy preference relations.

The rest of the paper is organized as follows: in section 2 we present some preliminary knowledge on preference relations. In section 3, we present a review of rank reversal literature regarding possible causes and attempts to solve rank reversal. In section 4, we study the possible causes of rank reversal in preference relation, namely, inconsistency of preference relation, aggregation operators, and score aggregation method and their link to rank reversal. In section 5, we propose score aggregation methods that have better performance than the sum normalization methods in avoiding rank reversal. In section 6, we provide a numerical example. Finally in section 7, we present the conclusions.

2. *Preliminary Knowledge*

Definition 1 (Urena et al., 2015): A preference relation R is a binary relation defined on the set X that is characterized by a function $\mu_p: X \times X \rightarrow D$, where D is the domain of representation of preference degrees provided by the decision-maker.

Definition 2 (Xu, 2007): A fuzzy preference relation P on a finite set of alternatives X is represented by a matrix $P = \left(p_{ij}\right)_{n\times n} \subset X \times X$ with:

$p_{ij} \in [0,1], \;\; p_{ij} + p_{ji} = 1, \;\; p_{ii} = 0.5 \;\; \forall i, j = 1, \dots, n.$

$$P = \left(p_{ij}\right)_{nxn} = \begin{matrix} & A_1 & A_2 & \dots & A_n \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} 0.5 & p_{12} & \dots & p_{1n} \\ p_{21} & 0.5 & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & 0.5 \end{pmatrix}} \end{matrix}$$

when $p_{ij} > 0.5$ indicates that the expert prefers alternative x_i over alternative x_j; $p_{ij} < 0.5$ indicates that the expert prefers alternative x_j over alternative x_i; $p_{ij} = 0.5$ indicates that the expert is indifferent between x_i and x_j, thus, $p_{ii} = 0.5$.

Furthermore, the fuzzy preference relation $P = (p_{ij})_{n\times n}$ is additive consistent if and only if the following additive transitivity is satisfied (Meng & Chen, 2015; Urena et al., 2015; Herrera-Viedma et al., 2007a; Tanino, 1984):

$$F^1: \;\; p_{ik} = p_{ij} + p_{jk} - 0.5 \tag{1}$$

Definition 3 (Saaty, 1980): A multiplicative preference relation A on the set $X = \{x_1, x_2, \dots, x_n\}$

$$A = \left(a_{ij}\right)_{nxn} = \begin{matrix} & A_1 & A_2 & \dots & A_n \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{pmatrix}} \end{matrix}$$

of alternatives is defined as a reciprocal matrix $A = (a_{ij})_{n\times n} \subset X \times X$ with the following conditions:

$$a_{ij} > 0,\ \ a_{ij}a_{ji} = 1,\ \ a_{ii} = 1,\ \ \forall ij = 1, 2, \dots, n.$$

where a_{ij} is interpreted as the ratio of the preference intensity of the alternative x_i to x_j.

There are several numerical scales for the multiplicative preference relation; however, the most popular one is the 1-9 Saaty scale. $a_{ij} = 1$ means that alternative x_i and x_j are indifferent; $a_{ij} > 1$ implies that alternative x_i is preferred to x_j. As the ratio of intensity of (a_{ij}) increases, the stronger is the preference intensity of x_i over x_j. Thus, $a_{ij} = 9$ means that alternative x_i is absolutely preferred to x_j.

The multiplicative preference relation $A = (a_{ij})_{n\times n}$ is called consistent if the following multiplicative transitivity is satisfied (Saaty, 1980):

$$a_{ij} = a_{ik}a_{kj},\ \ a_{ii} = 1,\ \ \forall i, j = 1, 2, \dots, n.$$

The AHP method, which uses multiplicative preference relations, decomposes complex problems into a hierarchy consisting of several levels, where the top level represents the goal and the lower levels consist of criteria, sub-criteria and alternatives respectively. The elements in each level are compared with each other through pair-wise comparison on a scale of 1-9 to find their relative importance. Then the weight for each element is computed using the eigenvector method. The same technique is used at the lower level with respect to a higher level element to find their relative importance (Saaty, 1980).

3. Literature Review

The purpose of this section is to explore the literature of MCDM to investigate possible causes of rank reversal phenomena. We will then cover the attempts of researchers to solve this issue. Thus, two main subsections will be explored: the literature of rank reversal causes and attempts to fix rank reversal.

3.1. The literature on rank reversal causes'

The literature on multiplicative preference relations, especially AHP, discusses three possible reasons behind rank reversal, see Table 1: inconsistency, pairwise comparison, and aggregation method. Dodd et al. (1995) claimed that Saaty's AHP misses a form of inconsistency within its model, which puts its results under doubt. Farkas et al. (2004) also blamed inconsistency in pairwise comparison for this issue. According to Paulson and Zahir (1995), judgmental uncertainty could also cause rank reversal. Chou (2012) referred the issue of rank reversal in AHP due to the aggregation method. However, researchers like Karapetrovic and Rosenbloom (1999) refused to link the problem to inconsistency. They argued that there is no direct relation between the consistencies or inconsistencies of pairwise comparison matrices and the occurrence of rank reversal. They stated that each could be considered as a separate problem. Ishizaka and Labib (2011) agreed with them and reported that rank reversal is independent of the consistency of the data and priority method. Moreover, they believed this phenomenon could happen in any additive model.

Other researchers like Schenkerman (1994) believed that the rank reversal in AHP is caused by normalization, and its scales seem arbitrary. He claimed that criteria weights are dependent on the measurements of the alternatives. Thus, any change in the number of alternatives and normalization imposes revising of the criteria weights. Correspondingly, Ishizaka and Labib (2011) claimed that the rank reversal phenomenon is related to the method rather than modeling procedure and it may not be resolved because aggregation of the standardized units is not simply interpretable, which has been even disputed by French school. Lai (1995) pointed out that rank reversal happens because of multiplying criteria weights by unrelated normalized scale of performance ratings. Dyer (1990) claimed that the problem is not just rank reversal but the AHP results are arbitrary. This is because the criteria weights may not be right due to the normalization procedure. Triantaphyllou (2001) agreed with Dyer that in AHP or any additive variants of it, ranking is arbitrary often which tends to generate rank reversal even if the data is perfectly consistent. According to Rosenbloom (1997), researchers tried to resolve this problem in AHP by proposing different normalization methods. Perez (1995) argued that the phenomenon of rank reversal is common in almost all of ordinal aggregation methods such as AHP. He claimed that rank reversal could be avoided if both criteria weights and performance ratings are generated from a common space of scales. On the other hand, Bana e Costa and Vansnick (2008) blamed the eigenvalue method. They stated that the priority vector violates a condition of order preservation, which makes use of AHP in decision-making very problematic.

Table 1: The causes' literature of rank reversal

Cause(s)	Author(s)	Claimed based on
Inconsistency	Dodd et al., (1995)	No proof
	Farkas et al., (2004)	No proof
Uncertainty	Paulson and Zahir (1995)	Simulation
Aggregation method	Chou (2012)	No proof
AHP method and score aggregation	Ishizaka and Labib (2011)	No proof
Normalization	Schenkerman (1994)	Examples
Multiplying criteria weights by	Lai (1995)	Based on comparison between MAUT and AHP*
unrelated normalized scale	Perez (1995)	Examples
AHP method (normalization)	Dyer (1990)	Examples
	Triantaphyllou (2001)	Examples
Normalization methods	Rosenbloom (1997)	No proof
Eigenvalue method	Bana e Costa and Vansnick (2008)	Examples

*MAUT: Multiattribute Utility Theory

3.2. Attempts to fix rank reversal

The rank reversal phenomenon in AHP was initially observed by Belton and Gear in 1983 after they discovered that introducing a new similar alternative to the existing ones could reverse the ranking of the alternatives. They proposed a modified normalization method to overcome the rank reversal issue in the original AHP, which is later known as a Revised AHP. The revised method differs from the original AHP prioritization method where each criterion is divided by the max value with respect to it for all the alternatives. Later on, this method came to be known as the ideal model. Afterwards, Schenkerman (1994) claimed that in methods such as Referenced AHP, normalization to maximum entry (ideal model), normalization to minimum entry, and linking pins avoid rank reversal only when the criteria are quantitative. On the other hand, Saaty (1987) linked rank reversal with the existence of near or similar copies within the set of alternatives. To solve this issue, either the set of alternatives has to be revised or more criteria need to be considered. Saaty defines a near copy as an alternative that has close values within 10% for overall criteria. However, Dyer (1990) later criticized this suggestion.

Lootsma (1993), followed by Sheu (2004), claimed that using a geometric mean aggregation method in AHP helps to avoid rank reversal. Likewise, Ishizaka and Labib (2011) mentioned that using geometric mean in AHP prevents rank reversal since geometric mean in both row and column approaches produces the same results, unlike eigenvector methods. Barzilai and Golany (1994) stated that the rank reversal problem is related to the structure of AHP mainly through the additive aggregation rule. They argued that the multiplicative procedures such as the geometric mean and the weighted-geometric-mean aggregation rule are the solution. In fact, some studies have shown that multiplicative methods such as the weighted product model and the multiplicative AHP are immune against rank reversal (Wang & Triantaphyllou, 2008). Barzilai and Lootsma (1997) demonstrated that the multiplicative AHP method does not generate rank reversal by testing the method on Belton and Gear's (1983) example. Additionally, the multiplicative variants of the AHP tend to be more reliable and do not show any kind of rank reversal, which means they are perfect (Triantaphyllou, 2001). On the other hand, Buede and Maxwell (1995) pointed out that using geometric mean "will not eliminate rank reversal," contrary to removing normalization of the ratio scale, which guarantees immunity against rank reversal.

Farkas et al. (2004) developed an approach by determining the intervals for all possible occurrences of rank reversals. They demonstrated it for an example of a 3X3 matrix. Recently, Rodriguez et al. (2013) proposed a modification to the fuzzy AHP- TOPSIS method with a graphical approach for rank reversal detection and analysis. They claimed that this graphical approach increases the level of confidence in the results. However, they mentioned that the graphical approach is not suitable when large set of criteria is under consideration. Table 2 summarizes the attempts to avoid/solve rank reversal in AHP.

Table 2: Some attempts to solve AHP's rank reversal

Solution	Author(s)
Max normalization method	Belton and Gear (1983)
Exclude/remove near or similar copies of the alternatives	Saaty (1987)
Geometric mean aggregation method	Lootsma (1993); Sheu (2004); Ishizaka and Labib (2011); Barzilai and Golany (1994); Wang and Triantaphyllou (2008); Barzilai and Lootsma (1997)
Find the intervals of all rank reversals	Farkas et al. (2004)
Graphical approach	Rodriguez et al. (2013)

4. *Mathematical Investigation of Rank Reversal Causes in Preference Relations*

The literature reveals (summarized on Table 1) reveals that researchers link the phenomenon of rank reversal in preference relations to: inconsistency, uncertainty, aggregation method, pairwise comparison concept (AHP method), normalization, multiplying criteria weights by unrelated normalized scale and eigenvalue method. Observing these claims deeply, we notice that uncertainty of information results in inconsistent input. Normalization, multiplying criteria weights by unrelated normalized scale and eigenvalue method are all related to aggregation method. Researchers blame eigenvalue method for rank reversal because during normalization it produces different priority values in case of adding or deleting an alternative, which violates the order preservation concept. On the other hand, some researchers claim multiplying criteria weights by unrelated normalized scale (related to normalization procedure) is the cause since criteria weights usually are normalized solely based on their data domain which might differ from alternatives normalized scale. Hence, these causes could be reduced to inconsistency, aggregation method and pairwise comparison. The preference relations are built based on the later concept and it has been proven to be valid. In addition to that, Millet (1997) compared five different types of preference elicitation methods and concluded that preferences based on pairwise comparison are more accurate than the others. Therefore, in this paper we are going to investigate inconsistency and aggregation methods for fuzzy preference relations.

According to Chiclana et al. (2009), preference relations have three fundamental and hierarchical levels of rationality assumptions: 1) the first level requires indifference between any alternative x_i and itself, 2) the second level requires that if the decision-maker prefers x_i to x_j then they should not at the same time prefer x_j to x_i, and 3) the third level is related to transitivity among any three alternatives. There are a number of consistency properties in the literature. A few of these are: triangle condition, weak transitivity, max-min transitivity, max-max transitivity, restricted max-min transitivity, restricted max-max transitivity, multiplicative transitivity, and additive transitivity (Herrera-Viedma et al., 2004). Among these properties, additive and multiplicative transitivity are the most used and are equivalent to each other through a transformation function. The transitivity property is interpreted by the idea that the preference value of any two alternatives obtained directly by comparison should be equal to or greater than the preference value of an indirect alternative (intermediate alternative) that is between them (Herrera-Viedma et al., 2004). Furthermore, any property that enforces transitivity in the preferences is called a consistency property (Chiclana et al., 2009).

4.1. Additive consistency

From (1) two other formulations can be generated based on the characteristics of the reciprocal rule, $(p_{ij} + p_{ji} = 1)$, as follows:

- F^2: $p_{ik} = p_{jk} - p_{ji} + 0.5$ (2)
- F^3: $p_{ik} = p_{ij} - p_{kj} + 0.5$ (3)

Proposition 1: Let $P = (p_{ik})_{n\times n}$ be a fuzzy preference relation, then for every preference degree on P we can find its estimation based on the additive consistency through:

$$p_{ik} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right). \tag{4}$$

Proof: by taking the average of equations (1), (2) and (3) for p_{ik} for n alternatives, the following equation is generated:

$$p_{ik} = \frac{1}{3n}\left[\sum_{j=1}^{n}\left(p_{ij} + p_{jk} - 0.5\right) + \left(p_{jk} - p_{ji} + 0.5\right) + \left(p_{ij} - p_{kj} + 0.5\right)\right]$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{j=1}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \frac{1}{3n}\left(p_{ii} + p_{kk} + 4p_{ik} - 2p_{ki} + 1\right)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \frac{1}{3n}\left(0.5 + 0.5 + 4p_{ik} - 2(1 - p_{ik}) + 1\right)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \frac{1}{3n}\left(6p_{ik}\right)$$

$$\Rightarrow (3n)p_{ik} = \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \left(6p_{ik}\right)$$

$$\Rightarrow 3(n-2)p_{ik} = \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$\Rightarrow \boxed{p_{ik} = \frac{1}{3(n-2)}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)} \ \blacksquare$$

For a reciprocal fuzzy preference relation, (4) can be re-written as:

$$p_{ik} = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij} + p_{jk} - 0.5\right) \tag{5}$$

Definition 4: Let $P = (p_{ik})_{n \times n}$ be a given reciprocal fuzzy preference relation and $P^e = (p^e_{ik})_{n \times n}$ be the estimated fuzzy preference relation calculated by (5). Then the consistency degree of P is calculated by

$$CD(P, P^e) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} |p_{ik} - p^e_{ik}| \tag{6}$$

Thus, (6) is used to check the consistency degree of any reciprocal fuzzy preference relation. When $CD(P, P^e) = 1$ then P is perfectly consistent; keeping in mind that p_{ik} is a preference degree or preference intensity of alternative i over alternative k.

The additive consistency implies dependency between alternatives, which is clear from the transitivity property. Thus, any change in the examined set of the alternatives implies a possible change on the preference degrees. This is correct, especially if the provided information is not perfectly consistent. To illustrate this, let us assume that the provided information for a set of n alternatives is perfectly consistent. Then, if we remove an alternative (h) from the set, $P^n \rightarrow P^{n-1} = (p_{ik})_{n-1 \times n-1}$, or if we add an alternative (h) to the set, $P^n \rightarrow P^{n+1} = (p_{ik})_{n+1 \times n+1}$. Therefore, the remaining preference degrees from P^n after n is modified maintain their valuations only if (4) is satisfied. This can only happen if the original information and the new alternative are perfectly consistent.

Theorem 1: Based on additive consistency, a preference degree (p_{ik}^{n}) maintains its valuation after removing or adding an alternative h from n if

$$p_{ik}^{n-1} or\ p_{ik}^{n+1} = \frac{1}{3}(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh}) + \frac{1}{6} \quad \forall\ ik \in n \tag{7}$$

Otherwise the preference relation P^{n} or P^{n+1} is not perfectly consistent.

Proof: from(4), $p_{ik}^{n} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$, when we remove h from n we get:

$$p_{ik}^{n-1} = \frac{1}{3(n-1-2)} \sum_{\substack{j=1 \\ i\neq j\neq k \\ n\backslash\{h\}}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$= \frac{1}{3(n-3)} \sum_{\substack{j=1 \\ i\neq j\neq k \\ n\backslash\{h\}}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

For $j \neq i \neq k$ and $n\backslash\{h\} = n - 1$, then

$$p_{ik}^{n-1} = \frac{1}{3(n-3)} \sum_{\substack{j=1 \\ i\neq j\neq k \\ n\backslash\{h\}}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$= \frac{1}{3(n-3)} \left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \cdots \right.$$

$$\left. + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)} + 0.5\right)\right]$$

$$= \frac{1}{3(n-3)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\right]$$

$$+ \frac{n-3}{6(n-3)}$$

For $j \neq i \neq k$ and $h \in n$, then

$$p_{ik}^{n} = \frac{1}{3(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \cdots\right.$$

$$+ \left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)} + 0.5\right) + \cdots$$

$$\left. + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)} + 0.5\right)\right]$$

$$= \frac{1}{3(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right) + \cdots\right.$$

$$\left. + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)\right] + \frac{n-2}{6(n-2)}$$

Thus, the only way $p_{ik}^{n-1} = p_{ik}^{n}$ after removing h is if

$$\frac{1}{3(n-3)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\right] + \frac{1}{6}$$

$$= \frac{1}{3(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right)\right.$$

$$\left. + \cdots + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)\right] + \frac{1}{6}$$

$$\Rightarrow \frac{1}{3(n-3)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\right]$$

$$- \frac{1}{3(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots\right.$$

$$\left. + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)\right]$$

$$= \frac{1}{3(n-2)}\left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right)$$

Since,

$$\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)$$

$$= \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)$$

Then,

$$\frac{1}{3(n-3)(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots\right.$$

$$\left. + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\right]$$

$$= \frac{1}{3(n-2)}\left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right)$$

Multiply both sides by $3(n-3)(n-2)$, we get,

$$\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\right]$$

$$= (n-3)\left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right)$$

Thus,

$$\sum_{\substack{j=1\\ i\neq j\neq k\\ n\backslash h}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) = (n-3)(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh})$$

$$\frac{1}{3(n-3)} \sum_{\substack{j=1\\ i\neq j\neq k\\ n\backslash h}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) = \frac{1}{3(n-3)}(n-3)(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh})$$

$$\frac{1}{3(n-3)} \sum_{\substack{j=1\\ i\neq j\neq k\\ n\backslash h}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) = \frac{1}{3}(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh}) + \frac{1}{6}$$

$$p_{ik}^{n-1} = \frac{1}{3}(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh}) + \frac{1}{6}$$

Also, we get the same conclusion when h is added to n ■

This shows how removing or adding an alternative could affect the entire information, especially when they are inconsistent. Thus, introducing new information implies a change in the original information, particularly if the new information is not consistent. Usually decision-makers do not revise their assessments based on the new information. In general, the decision-makers compare two alternatives at a time; however, when we consider the consistency of the information, all the alternatives need to be considered. So the decision-makers do not revise their previous assessments on a pair of alternatives if another alternative is removed or a new one is added. Moreover in real life, most decision-makers are not consistent in their opinions. Thus, how should we handle acceptably inconsistent information in a way to avoid rank reversal? Saaty (1980) suggested that the acceptable inconsistency (consistency ratio) should be less than 10%. Later, Saaty (1994) suggested another 5% acceptable inconsistency level for 3x3 preference relation and 8% for 4x4 preference relation. Aguaron and Moreno-Jiménez (2003) followed Saaty's threshold suggestion and come up with 10% threshold for inconsistency measure for Geometric Mean Method. However, it seems there is no clear answer as to when a preference relation is considered to be inconsistent (Ishizaka and Labib, 2011).

4.2. Aggregation methods

Aggregation methods or operators are used to aggregate individual preference relations into a collective one. For example, in group decision-making, the individuals' preference relations are aggregated into a collective preference relation. There are many aggregation operators in the literature; however, the most common one is the weighted averaging operator. The weighted averaging operator is defined as follows (Zhang et al., 2016; Wu and Xu, 2012):

$$p_{ik}^{c} = \sum_{t=1}^{m} w_t \cdot p_{ik}^{t} \tag{8}$$

where w_t is the weight of decision-maker t such that $\sum_{t=1}^{m} w_t = 1$, p_{ik}^{t} is the given preference degree by decision-maker t, m is the number of decision-makers, and p_{ik}^{c} is the collective preference degree. The weighted averaging operator becomes an averaging operator when the decision-makers have equal weights.

Proposition 2: Let $P^t = (p_{ik}^t)_{n\times n}$ be a reciprocal fuzzy preference relation given by a decision-maker t. When all P^ts are perfectly consistent then the collective preference relation is also perfectly consistent.

Proof: from (8)

$$p_{ik}^c = \sum_{t=1}^{m} w_t \cdot p_{ik}^t$$

and from (5) $p_{ik}^t = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$ then

$$p_{ik}^c = \sum_{t=1}^{m} w_t \cdot p_{ik}^t = \sum_{t=1}^{m} w_t \cdot \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{t=1}^{m} w_t \cdot \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(\sum_{t=1}^{m} w_t \cdot p_{ij}^t + \sum_{t=1}^{m} w_t \cdot p_{jk}^t - \sum_{t=1}^{m} w_t \cdot 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^c + p_{jk}^c - 0.5\right) = p_{ik}^e \blacksquare$$

Similarly, when an arithmetic mean operator is used, the consistency is also maintained.

$$p_{ik}^c = \frac{1}{m} \sum_{t=1}^{m} p_{ik}^t \tag{8.1}$$

Proof: from (5) $p_{ik}^t = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$ then

$$p_{ik}^c = \frac{1}{m} \sum_{t=1}^{m} p_{ik}^t = \frac{1}{m} \sum_{t=1}^{m} \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$$

$$= \frac{1}{(n-2)} \cdot \frac{1}{m} \sum_{t=1}^{m} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(\frac{1}{m} \sum_{t=1}^{m} p_{ij}^t + \frac{1}{m} \sum_{t=1}^{m} p_{jk}^t - \frac{1}{m} \sum_{t=1}^{m} 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(p_{ij}^{c} + p_{jk}^{c} - 0.5\right) = p_{ik}^{e} \blacksquare$$

Proposition 3: The constructed collective preference relation by arithmetic mean operator gains the mean consistency degrees of the individuals' preference relations. Likewise, the constructed collective preference relation by the weighted averaging operator acquires the weighted averaging consistency degrees of the individuals' preference relations.

Proof: from (6)

$CD\left(P^{t}, P^{e(t)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| p_{ik}^{t} - p_{ik}^{e(t)} \right|$, then for $t = 1,2, \dots m$, we get:

$$CD\left(\sum_{t=1}^{m} P^{t}, \sum_{t=1}^{m} P^{e(t)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| \sum_{t=1}^{m} p_{ik}^{t} - \sum_{t=1}^{m} p_{ik}^{e(t)} \right|,$$

$$CD\left(\frac{1}{m} \sum_{t=1}^{m} P^{t}, \frac{1}{m} \sum_{t=1}^{m} P^{e(t)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| \frac{1}{m} \sum_{t=1}^{m} p_{ik}^{t} - \frac{1}{m} \sum_{t=1}^{m} p_{ik}^{e(t)} \right|,$$

$$\Rightarrow CD\left(\frac{1}{m} \sum_{t=1}^{m} P^{t}, \frac{1}{m} \sum_{t=1}^{m} P^{e(t)}\right) = CD\left(P^{c}, P^{e(c)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| p_{ik}^{c} - p_{ik}^{e(c)} \right|,$$

This is also true for the weighted averaging operator■

For an inconsistent preference relation, removal or addition of an alternative h could play a significant role in altering the ranking order of the alternatives if h is the outbalance element among the alternatives.

To illustrate this, first we define the following score aggregation method, which is called the sum normalization method:

$$S_i = \frac{\sum_{k=1}^{n} p_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}} = \frac{2}{n^2}\sum_{k=1}^{n} p_{ik} \tag{9}$$

where S_i is the score of alternative i and $\sum_{i=1}^{n} S_i = 1$. The higher the score of an alternative, the better it is.

Theorem 2: Let the sum normalization method, equation(9), be the way to generate the ranking scores for the alternatives, then the following are true if alternative h is removed:

For $S_i^n > S_{i\prime}^n$ and $\sum_{k=1}^{n-1} p_{ik} \neq \sum_{k=1}^{n-1} p_{i\prime k}$ then,

$S_i^{n-1} > S_{i'}^{n-1}$ if and only if

$$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih} \tag{9.1}$$

For $S_i^n > S_{i\prime}^n$ and $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i\prime k}$ then

$$S_i^{n-1} = S_{i\prime}^{n-1} \tag{9.2}$$

$$p_{i'h} < p_{ih} \tag{9.3}$$

Proof: for (9.1),

$\sum_{k=1}^{n} p_{ik} = \sum_{k=1}^{n-1} p_{ik} + p_{ih}\ \forall i \in n$, substitute this into(9),

$S_i = \frac{2}{n^2}\sum_{k=1}^{n} p_{ik} = \frac{2}{n^2}[\sum_{k=1}^{n-1} p_{ik} + p_{ih}]$,

For $S_i^n > S_{i\prime}^n$ we get

$$\frac{2}{n^2}\left[\sum_{k=1}^{n-1} p_{ik} + p_{ih}\right] > \frac{2}{n^2}\left[\sum_{k=1}^{n-1} p_{i\prime k} + p_{i\prime h}\right] \Longrightarrow \sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih}$$

Since $S_i^{n-1} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{ik}$ and $S_{i\prime}^{n-1} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{i\prime k}$, then $S_i^{n-1} > S_{i'}^{n-1}$ ■

However, when $S_i^n > S_{i\prime}^n$ but $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i\prime k}$ then

$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih} \Rightarrow p_{i'h} - p_{ih} < 0$ since $p_{i'h} < p_{ih} \Longrightarrow S_i^n > S_{i\prime}^n$ and since $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i'k}$ then $S_i^{n-1} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{ik} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{i\prime k} = S_{i\prime}^{n-1}$, this completed the proof ■

This is also true when an alternative h is added. Therefore, rank reversal could occur when (9.1) and (9.3) are not satisfied.

Example 1: Suppose a decision-maker provides his assessments for one criterion on four alternatives using following reciprocal fuzzy preference relation:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.55 & 0.62 & 0.65 \\ 0.45 & 0.5 & 0.7 & 0.75 \\ 0.38 & 0.3 & 0.5 & 0.85 \\ 0.35 & 0.25 & 0.15 & 0.5 \end{pmatrix} \end{array}$$

Based on (6), the consistency degree of this preference relation is 82%. By using (9) the following ranking scores are generated:

$$A_2(0.3) > A_1(0.29) > A_3(0.254) > A_4(0.156)$$

However, when alternative A_4 is removed, the consistency degree increases to 87% with the following ranking scores:

$$A_1(0.371) > A_2(0.367) > A_3(0.262)$$

Notice that $A_1$and A_2 have been reversed. This is because A_4 was the outbalance element that differentiating between $A_1$and A_2. In fact, this happens because (9.1) is violated:

$\sum_{k=1}^{n-1} p_{1k} = 0.5 + 0.55 + 0.62 = 1.67, \sum_{k=1}^{n-1} p_{2k} = 0.45 + 0.5 + 0.7 = 1.65,$ $p_{14} = 0.65$ and $p_{24} = 0.75,$

$S_2^4(0.3) > S_1^4(0.29)$ Thus $S_2^3 > S_1^3$ only if

$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih}$

But $\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} \ngtr p_{i'h} - p_{ih} \Longrightarrow -0.02 < 0.1.$

Theorem 3: For any perfectly consistent reciprocal preference relation, (9.1), (9.2), and (9.3) are satisfied by the additive consistency.

Proof: from (7)

$p_{ik}^{n-1} = \frac{1}{3}(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh}) + \frac{1}{6}$ Then

$\sum_{k=1}^{n-1} p_{ik} = \frac{1}{3}(2p_{ih} + 2p_{h1} - p_{hi} - p_{1h}) + \frac{1}{6} + \frac{1}{3}(2p_{ih} + 2p_{h2} - p_{hi} - p_{2h}) + \frac{1}{6} + \cdots + \frac{1}{3}\left(2p_{ih} + 2p_{h(n-1)} - p_{hi} - p_{(n-1)h}\right) + \frac{1}{6}$, and

$\sum_{k=1}^{n-1} p_{i'k} = \frac{1}{3}(2p_{i'h} + 2p_{h1} - p_{hi'} - p_{1h}) + \frac{1}{6} + \frac{1}{3}(2p_{i'h} + 2p_{h2} - p_{hi'} - p_{2h}) + \frac{1}{6} + \cdots + \frac{1}{3}\left(2p_{i'h} + 2p_{h(n-1)} - p_{hi'} - p_{(n-1)h}\right) + \frac{1}{6}$, then

$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} = \frac{(n-1)}{3}[(2p_{ih} - p_{hi}) - (2p_{i'h} - p_{hi'})]$, but for reciprocal relation $p_{hi} = 1 - p_{ih}$ then $\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} = (n-1)[p_{ih} - p_{i'h}]$

$\Rightarrow \sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} = (1-n)[p_{i'h} - p_{ih}]$

Thus $\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih}$. When the left hand side of this equals to 0, which means $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i'k} \Longrightarrow 0 > p_{i'h} - p_{ih} \Longrightarrow p_{i'h} < p_{ih}$ which satisfies (9.1), (9.2) and (9.3) ■

Example 2: Suppose a decision-maker provides his assessments for one criterion on four alternatives using following reciprocal fuzzy preference relation:

$$\begin{matrix} & A_1 & A_2 & A_3 & A_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} 0.5 & 0.41 & 0.62 & 0.66 \\ 0.59 & 0.5 & 0.71 & 0.75 \\ 0.38 & 0.29 & 0.5 & 0.54 \\ 0.34 & 0.25 & 0.46 & 0.5 \end{pmatrix}} \end{matrix}$$

This preference relation is 100% consistent and yields following ranking scores using (9):

$$A_2(0.319) > A_1(0.274) > A_3(0.214) > A_4(0.193)$$

When A_4 is removed, the consistency degree is still 100%. Likewise, the ranking order is:

$$A_2(0.4) > A_1(0.34) > A_3(0.260)$$

There is no rank reversal because (9.1) is satisfied

$S_2^4 > S_1^4 \Rightarrow \sum_{k=1}^{n-1} p_{2k} - \sum_{k=1}^{n-1} p_{1k} > p_{1h} - p_{2h} \Rightarrow \{1.8 - 1.53\}0.27 > \{0.66 - 0.75\} - 0.09.$

$S_2^4 > S_3^4 \Rightarrow \sum_{k=1}^{n-1} p_{2k} - \sum_{k=1}^{n-1} p_{3k} > p_{3h} - p_{2h} \Rightarrow \{1.8 - 1.17\}0.63 > \{0.54 - 0.75\} - 0.21.$

$S_1^4 > S_3^4 \Rightarrow \sum_{k=1}^{n-1} p_{1k} - \sum_{k=1}^{n-1} p_{3k} > p_{3h} - p_{1h} \Rightarrow \{1.53 - 1.17\}0.36 > \{0.54 - 0.66\} - 0.12.$

5. *Proposed Score Aggregation Methods*

Based on these results, the only way to ensure ranking order is free of rank reversal in the preference relations is by ensuring that the preference relation(s) is perfectly consistent. However, to some extent this is hard to achieve in real world problems, especially in a group decision-making environment where there is a tradeoff between consistencies and consensus (Herrera-Viedma et al., 2007a). Thus, we need to handle rank reversal when it is not desirable by maintaining some guidelines that deal with the dependency of the data/information. Here we present three scenarios that are possible to happen to the set of alternatives during the decision process: a new alternative is introduced, an existing alternative is removed, or a new one replaces an alternative.

Note: This is only applied if the set of alternatives have been modified.

Proposition 4: The following formulation does better than the sum normalization method in avoiding rank reversal in reciprocal preference relations when a new alternative, h, is introduced:

$$\tilde{S}_i^{n+1} = \frac{2\sum_{k=1,h\notin n}^{n} p_{ik} + p_{ih} + p_{ih}^e}{(n+1)^2} \qquad \forall i \in n+1 \tag{10}$$

where p_{ih}^e is the estimated preference degree calculated by (5).

Proof:

When $P^{n+1}$is perfectly consistent, then $S_i^{n+1} = \tilde{S}_i^{n+1}$since $p_{ih} = p_{ih}^e$ thus

$\tilde{S}_i^{n+1} = \frac{2\sum_{k=1,h\notin n}^{n} p_{ik}+p_{ih}+p_{ih}^e}{(n+1)^2} = \frac{2\sum_{k=1,h\notin n}^{n}(p_{ik}+p_{ih})}{(n+1)^2} = \frac{2\sum_{k=1}^{n+1} p_{ik}}{(n+1)^2} = S_i^{n+1}$. However, when

P^n has an acceptable consistency degree but $h \in n+1$ is not, then the ranking generated by the sum normalization method might be affected by the information of h. Thus, integrating the values driven by the consistency property (5) and the provided ones for h will improve the consistency degree of P^{n+1}. The chance of rank reversal decreases as the consistency increases.

For $S_i^{n+1} > S_{i\prime}^{n+1}$ then

$\frac{2}{(n-1)^2}\sum_{k=1}^{n+1} p_{ik} > \frac{2}{(n-1)^2}\sum_{k=1}^{n} p_{i\prime k} \Longrightarrow \sum_{k=1,h\notin n}^{n} p_{ik} + p_{ih} > \sum_{k=1,h\notin n}^{n} p_{i\prime k} + p_{i\prime h}$,

since $\sum_{k=1}^{n+1} p_{ik} = \sum_{k=1,h\notin n}^{n} p_{ik} + p_{ih}$ then

$\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > p_{i'h} - p_{ih}$. Thus $S_i^n > S_{i\prime}^n$ and $S_i^{n+1} > S_{i\prime}^{n+1}$ only if $\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > p_{i'h} - p_{ih}$.

However, with $\tilde{S}_i^{n+1} > \tilde{S}_{i\prime}^{n+1}$, after eliminating the constants in both sides we get,

$\Longrightarrow \sum_{k=1,h\notin n}^{n} p_{ik} + \frac{p_{ih}+p_{ih}^e}{2} > \sum_{k=1,h\notin n}^{n} p_{i'k} + \frac{p_{i'h}+p_{i'h}^e}{2} \Longrightarrow 2\sum_{k=1,h\notin n}^{n} p_{ik} - 2\sum_{k=1,h\notin n}^{n} p_{i'k} >$ $p_{i'h} + p_{i'h}^e - p_{ih} - p_{ih}^e$, but $p_{ih}^e = \frac{1}{(n+1-2)}\sum_{\substack{k=1\\k\neq i\neq h}}^{n+1}(p_{ik} + p_{kh} - 0.5)$ then

$\Longrightarrow 2\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) > p_{i'h} - p_{ih} - \frac{1}{(n-1)}\sum_{\substack{k=1\\k\neq i\neq h}}^{n+1}(p_{ik} + p_{kh} - 0.5) +$

$\frac{1}{(n-1)}\sum_{\substack{k=1\\k\neq i\prime\neq h}}^{n+1}(p_{i\prime k} + p_{kh} - 0.5)$,

$\Rightarrow 2\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) >$

$p_{i'h} - p_{ih} - \frac{1}{(n-1)} \sum_{\substack{k=1 \\ k\neq i\neq h}}^{n+1} (p_{ik} + p_{kh}) + \frac{1}{(n-1)} \sum_{\substack{k=1 \\ k\neq i'\neq h}}^{n+1} (p_{i'k} + p_{kh}),$

$\Rightarrow 2\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) > p_{i'h} - p_{ih} + \frac{1}{(n-1)} \sum_{\substack{k=1 \\ k\neq i\neq h}}^{n+1} (p_{i'k} - p_{ik}),$

but $\sum_{\substack{k=1 \\ k\neq i\neq h}}^{n+1} p_{ik} = \sum_{k=1,h\notin n}^{n} p_{ik} - 0.5$, thus

$\Rightarrow \frac{2n-1}{n-1}\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) > (p_{i'h} - p_{ih}) \Rightarrow \sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} >$
$\frac{n-1}{2n-1}(p_{i'h} - p_{ih}),$

When generating the ranking scores for P^{n+1} with sum normalization there is no rank reversal only if

$\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > p_{i'h} - p_{ih},$

but with (10) there is no rank reversal only if

$\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > \frac{n-1}{2n-1}(p_{i'h} - p_{ih})$, so clearly (10) has a higher possibility to avoid rank reversal than sum normalization. In addition, (10) ensures maintaining the sum of the scores of the alternatives at 1, $\sum_{i=1}^{n+1} \tilde{S}_i = 1$ ■

Proposition 5: The following formulation does better than the sum normalization method in avoiding rank reversal in reciprocal preference relations when an alternative h is replaced by a new alternative h':

$$\tilde{S}_i^{n\prime} = \frac{2\sum_{k=1,h'\notin n-1}^{n-1} p_{ik} + p_{ih'} + p_{ih'}^{e}}{n^2} \quad \forall i \in n \tag{11}$$

Proof:

Similar to the proof of the previous proposition.

Proposition 6: The following formulation prevents rank reversal from occurring in reciprocal preference relations when an alternative, h, is removed:

$$\tilde{S}_i^{n-1} = \frac{2(\sum_{k=1}^{n-1} p_{ik} + p_{ih})}{n^2 - 2\sum_{k=1}^{n} p_{hk}} = \frac{2\sum_{k=1}^{n} p_{ik}}{n^2 - 2\sum_{k=1}^{n} p_{hk}} \quad \forall i \neq h \tag{12}$$

Proof:

$S_i = \frac{2}{n^2}\sum_{k=1}^{n} p_{ik} \Rightarrow n^2 S_i = 2\sum_{k=1}^{n} p_{ik}$, when $S_i^n > S_{i'}^n$ then

$n^2 S_i > n^2 S_{i'}$ Thus if we divide both sides by any constant greater than zero, the inequality will not be affected. Therefore, we divide both sides by $n^2 - 2\sum_{k=1}^{n} p_{hk}$ since $2\sum_{k=1}^{n} p_{hk}$ is always less than n^2, where n is the number of alternatives of the original problem. We get:

$$\frac{n^2 S_i}{n^2 - 2\sum_{k=1}^{n} p_{hk}} > \frac{n^2 S_{i'}}{n^2 - 2\sum_{k=1}^{n} p_{hk}} \Longrightarrow \frac{2\sum_{k=1}^{n} p_{ik}}{n^2 - 2\sum_{k=1}^{n} p_{hk}} > \frac{2\sum_{k=1}^{n} p_{ik}}{n^2 - 2\sum_{k=1}^{n} p_{hk}}$$

$\Rightarrow \tilde{S}_i^{n-1} > \tilde{S}_{i'}^{n-1} \;\; \forall i \neq h$, and this formulation also ensures maintaining the sum of the scores of the alternatives at 1, $\sum_{i=1}^{n-1} \tilde{S}_i = 1$ ■

6. *Numerical Example*

Suppose that four decision-makers provide their assessments (by fuzzy preference relations) on four alternatives as follows:

$$P^1 = \begin{pmatrix} 0.50 & 0.38 & 0.20 & 0.28 \\ 0.62 & 0.50 & 0.32 & 0.40 \\ 0.80 & 0.68 & 0.50 & 0.58 \\ 0.72 & 0.6 & 0.42 & 0.50 \end{pmatrix} \qquad P^2 = \begin{pmatrix} 0.50 & 0.38 & 0.25 & 0.33 \\ 0.62 & 0.50 & 0.37 & 0.45 \\ 0.75 & 0.63 & 0.50 & 0.58 \\ 0.67 & 0.55 & 0.42 & 0.50 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.50 & 0.75 & 0.55 & 0.41 \\ 0.25 & 0.50 & 0.30 & 0.16 \\ 0.45 & 0.70 & 0.50 & 0.36 \\ 0.59 & 0.84 & 0.64 & 0.50 \end{pmatrix} \qquad P^4 = \begin{pmatrix} 0.50 & 0.40 & 0.30 & 0.60 \\ 0.60 & 0.50 & 0.40 & 0.70 \\ 0.70 & 0.60 & 0.50 & 0.80 \\ 0.40 & 0.30 & 0.20 & 0.50 \end{pmatrix}$$

After several rounds of discussion, they reach an acceptable level of consensus, which results in the following collective preference relation, which has been aggregated by a weighted averaging operator assuming equal weights for decision-makers:

$$P^c = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.48 & 0.26 & 0.41 \\ 0.52 & 0.5 & 0.35 & 0.55 \\ 0.74 & 0.65 & 0.5 & 0.69 \\ 0.59 & 0.45 & 0.31 & 0.5 \end{pmatrix} \end{array}$$

This preference relation is 95% consistent. If we calculate the ranking score by the sum normalization method (9), then we get the following ranking order:

$$A_3(0.323) \succ A_2(0.24) \succ A_4(0.231) \succ A_1(0.206).$$

A. Adding a new alternative

Now consider that the decision-makers introduce a new alternative A_5. Going through the consensus process, they end up with the following collective preference relation:

$$P^c = \begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & 0.5 & 0.48 & 0.26 & 0.41 & 0.52 \\ A_2 & 0.52 & 0.5 & 0.35 & 0.55 & 0.92 \\ A_3 & 0.74 & 0.65 & 0.5 & 0.69 & 0.25 \\ A_4 & 0.59 & 0.45 & 0.31 & 0.5 & 0.55 \\ A_5 & 0.48 & 0.08 & 0.75 & 0.45 & 0.5 \end{matrix}$$

The consistency degree of this preference relation has dropped to 78.5% and the new ranking order by (9) is:

$$A_2(0.227) \succ A_3(0.226) \succ A_4(0.192) \succ A_5(0.181) \succ A_1(0.174)$$

Notice that by introducing A_5, which is a non-dominant alternative, the ranking order for the first two alternatives has reversed. This is because the collective preference relation is not perfectly consistent and thus, there is no guarantee that (9.1) and (9.3) are satisfied.

However, if we apply(10), which relies on improving the consistency of the added alternative, we get the following ranking order:

$$A_3(0.251) \succ A_2(0.207) \succ A_4(0.19) \succ A_5(0.181) \succ A_1(0.171)$$

This ranking order is similar to the original problem except that alternative A_5 has been placed in its right ranking position among the alternatives.

B. Replacing an alternative

Now, let us consider that alternative A_2 has been replaced by $A_{2'}$, in the original problem. The collective preference relation is 81% consistent for the collective preference relation below:

$$P^c = \begin{matrix} & A_1 & A_2, & A_3 & A_4 \\ A_1 & 0.5 & 0.55 & 0.26 & 0.41 \\ A_2, & 0.45 & 0.5 & 0.6 & 0.45 \\ A_3 & 0.74 & 0.4 & 0.5 & 0.69 \\ A_4 & 0.59 & 0.55 & 0.31 & 0.5 \end{matrix}$$

The following are the ranking orders obtained by the sum normalization method (9) and the proposed formula(11), respectively:

Obtained by (9): $A_3(0.291) \succ A_{2'}(0.25) \succ A_4(0.244) \succ A_1(0.215)$

Obtained by (11): $A_3(0.314) \succ A_{2'}(0.25) \succ A_4(0.236) \succ A_1(0.2)$

Note that both methods generate the same ranking order but with different score values.

C. Removing an alternative

Consider Example 1 again,

$$\begin{matrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0.5 & 0.55 & 0.62 & 0.65 \\ A_2 & 0.45 & 0.5 & 0.7 & 0.75 \\ A_3 & 0.38 & 0.3 & 0.5 & 0.85 \\ A_4 & 0.35 & 0.25 & 0.15 & 0.5 \end{matrix}$$

Where the preference relation is 82% consistent and has the following ranking order, by (9):

$$A_2(0.3) > A_1(0.29) > A_3(0.254) > A_4(0.156)$$

We saw that when alternative A_4 is removed, the consistency degree increases but the ranking order has reversed between the first and the second:

$$A_1(0.371) > A_2(0.367) > A_3(0.262)$$

However, if we apply (12) we get the following ranking order:

$$A_2(0.356) > A_1(0.344) > A_3(0.3)$$

which is consistent with the ranking order of the original problem.

7. Conclusions

In this paper, we have proved that consistency of the data/information is the main cause of rank reversal in preference relation. Also, we have shown that when the preference relations are perfectly consistent then neither a weighted averaging aggregation operator nor an arithmetic mean aggregation operator could cause rank reversal. This is also true for the score aggregation operator, particularly, the sum normalization method. However, when the preference relation is inconsistent, which is usually the case in real life decision problems, then the score aggregation operator could generate rank reversal when the set of alternatives is modified. Thus, we proposed modified score aggregation operators that could be used when a change in the set of alternatives is done. The proposed score aggregation operators integrate the consistency element to reduce the chances of rank reversal. We show that the proposed operators perform better than the sum normalization method in avoiding rank reversal when a change happens in the set of alternatives.

This work was based on additive consistency in reciprocal preference relation. As future work, we would like to extend this work to preference relation based on multiplicative consistency. Multiplicative consistency is as important as additive consistency. Moreover, we would like to investigate the possibility of establishing a score aggregation method that has the ability to handle rank reversal in general cases.

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Solution	Author(s)
Max normalization method	Belton and Gear (1983)
Exclude/remove near or similar copies of the alternatives	Saaty (1987)
Geometric mean aggregation method	Lootsma (1993); Sheu (2004); Ishizaka and Labib (2011); Barzilai and Golany (1994); Wang and Triantaphyllou (2008); Barzilai and Lootsma (1997)
Find the intervals of all rank reversals	Farkas et al. (2004)
Graphical approach	Rodriguez et al. (2013)

Research Highlights

1. The paper deals with the subject of rank reversal in decision making.

2. Reciprocal preference relations based on additive consistency are studied.

3. Inconsistency of information and score aggregation are found as main causes.

4. Two score aggregation methods are proposed.

5. Numerical application shows better results than other tested methods.