# Accepted Manuscript


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PII: S0893-6080(17)30201-0
DOI: http://dx.doi.org/10.1016/j.neunet.2017.08.011
Reference: NN 3810

To appear in: *Neural Networks*

Received date : 10 February 2017
Revised date : 16 June 2017
Accepted date : 22 August 2017

Please cite this article as: Lou, X., Swamy, M.N.S., A new approach to optimal control of conductance-based spiking neurons. *Neural Networks* (2017), http://dx.doi.org/10.1016/j.neunet.2017.08.011

# A New Approach to Optimal Control of Conductance-based Spiking Neurons*

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September 6, 2017

**Abstract**

This paper presents an algorithm for solving the minimum-energy optimal control problem of conductance-based spiking neurons. The basic procedure is (1) to construct a conductance-based spiking neuron oscillator as an affine nonlinear system, (2) to formulate the optimal control problem of the affine nonlinear system as a boundary value problem based on the Pontryagin's maximum principle, and (3) to solve the boundary value problem using the homotopy perturbation method. The construction of the minimum-energy optimal control in the framework of the homotopy perturbation technique is novel and valid for a broad class of nonlinear conductance-based neuron models. The applicability of our method in the FitzHugh-Nagumo and Hindmarsh-Rose models is validated by simulations.

**Keywords:** Spiking neurons, optimal control, homotopy perturbation method.

## 1 Introduction

Oscillatory neurons exhibit voltage spikes known as action potentials, which can be controlled by electrical stimulation [1, 2]. The ability to control spiking activities may play an important role in the treatment of many neurological diseases [3, 4]. In recent years, concerning the neuron energy efficiency [5, 6], optimal electrical stimulation in single neuron for different control performances, such as minimum energy and spike times, has received much attention. It is also applicable to the treatment of subthalamic nucleus [7]. In these and many other neurological applications, considerations of optimal electrical stimulation, especially low-power electrical stimuli, are desired, since application of high power stimuli is harmful to the biological tissues and the reduction of power consumption in a neurological implant is essential in order to reduce its size and lengthen its lifetime [8].

In the field of theoretical neuroscience and automatic control, many control methods have been used or developed for minimum-power or minimum-energy controls of spiking

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neurons [8]-[12]. For instance, in the context of phase-reduced neuron models, a constrained magnitude input current as the stimulus is applied to cause a neuron to spike with a maximal firing rate in [9]. Dasanayake and Li analyzed the optimal control that leads to targeted spiking times for neuron oscillators described by phase models and analytically derived the minimum-power current stimuli [10]. Later, they designed the charge-balanced minimum-power controls for phase models of neuron oscillators [8]. A single input control strategy was developed in [11] for the problem of desynchronizing a network of coupled neurons and solved the problem by utilizing the discrete-time dynamic programming (DTDP) method. Recently, based on the phase model characterized by the neuron's phase response curve (PRC), Nabi et al. [12] demonstrated the applicability of optimal control theory for designing minimum energy charge-balanced input waveforms for single periodically-firing in vitro neurons from brain slices of Long-Evans rats. More recently, Wilson et al. [13] developed an energy optimal control strategy to entrain heterogeneous noisy neurons and apply it to numerical models of noisy phase oscillators and to in vitro hippocampal neurons.

Although the phase-reduced models using PRC is a parsimonious and effective way to describe how a neuron responds to a stimulus, it is only valid for periodic spiking or firing neurons. Therefore, some researchers have paid much attention to the control of spiking neurons in the context of conductance-based neuron models which are more intricate than phase-reduced models [14]-[18]. An event-based energy-optimal desynchronizing control for coupled reduced Hodgkin-Huxley neurons was established in [14] by means of the Hamilton-Jacobi-Bellman approach. In [15], the tracking control with low energy input current stimuli of a reference membrane voltage in the reduced Hodgkin-Huxley neuron model was addressed to support investigations of neuron dynamics and energy efficiency. In [18], a low energy extracellular electric field input was designed for a reduced two-compartment model based on Pinsky-Rinzel model, and applied to drive the neuron to closely track a prescriptive spike train. However, it is generally impossible to solve analytically the optimal control problem of conductance-based neuron models. Recently, several approaches, such as the DTDP method [11] and Level Set Methods Toolbox (ToolboxLS) [16] using a Lax-Friedrichs scheme and Runge-Kutta time stepping scheme, have been proposed to solve *numerically* optimal control of neuron models.

Low-power electrical stimuli play an important role in reducing power consumption in a neurological implant. In this paper, we consider the minimum-energy optimal control problem in neuron models by adopting the homotopy perturbation method (HPM) [19, 20, 21], which has been applied in optimal control of neuron oscillators described by phase models [23] and is powerful for *analytically* solving nonlinear boundary value problems (BVPs). We utilize the HPM to solve Hamilton equations using iteration formulas derived from two operators corresponding to the Hamilton equations. In [24], the convergence of the HPM has been proved. Also, Biazar et al. in [24] have pointed out that a solution using this method can be considered as the sum of an infinite series which converges rapidly to accurate solutions, but the convergence rate depends on a nonlinear operator $N$. In contrast to the Gauss pseudospectral method (which is an orthogonal collocation method where the collocation points are the Legendre-Gauss points, [25]) and the successive approximation approach [26, 27], the proposed scheme allows us to derive the analytical solution to the minimum-energy optimal control problem of spiking neurons within a few iterations. Moreover, the derived optimal control law will lead to accurate enough suboptimal trajectory and less control energy.

The remainder of the paper is organized as follows. In Section 2, by using the Pontryagin's maximum principle, we obtain the minimum-energy optimal control law via solving a BVP of a general conductance-based spiking neuron model which is formulated as an affine nonlinear system. In Section 3, we briefly review the HPM principle for solving the nonlinear BVP. In Section 4, we describe how the HPM can be used to solve the minimum-energy optimal control problem of conductance-based spiking neuron models. In Section 5, the HPM is illustrated in the well-known FitzHugh-Nagumo model and the Hindmarsh-Rose model. Finally, conclusions and future work are given in Section 6.

## 2 Problem statement

In this section, we introduce the minimum-energy optimal control problem of the general spiking neurons.

The dynamics of a conductance-based spiking neuron oscillator can be described by

$$\begin{cases} \dot{x}(t) &= F(x(t)) + Bu(t), \\ x(t_0) &= x_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state which usually consists of the membrane potential, gating variable, recovery variable and adaptation variable; $x_0 \in \mathbb{R}^n$ is the initial state; $u(t) \in \mathbb{R}$ is the external input injected to the neuron; $B \in \mathbb{R}^n$ is a constant vector, which is typically taken as $B = [1, 0, \cdots]^\top$ when the external input acts on the membrane potential.

The system described by (1) is quite general as it includes many spiking neuron models such as the Hodgkin-Huxley model, the Hindmarsh-Rose (HR) model, the FitzHugh-Nagumo (FHN) model and so on. In particular, the FHN model in a dimensionless form [28] can be written in the form of (1) with $B = [1, 0]^\top$ and

$$F(x) = \begin{bmatrix} (v(v+a)(1-v) - w)/\delta \\ v - 0.5w \end{bmatrix} \tag{2}$$

where $x = [v, w]^\top$ is the state vector of the system, $v$ is the voltage potential of the neuron membrane and $w$ is the inactivation of the sodium channels. The parameters $a, \delta$ will be specified in Section V. In practice, when applying stimuli in a DBS setting for treatment of Parkinson's disease, it is clinically desirable to reduce the amount of energy needed per stimulation. Motivated by this, the present work is directed towards developing analysis tools to study the optimal control of spiking neuron models when controlling neurons to achieve certain desired behavior or driving the neurons to their phaseless sets. For instance, in clinical therapy for epilepsy and Parkinson's disease through desynchronization, they can be set as the unstable phaseless set.

Considering the neuron system (1), the objective is to find the minimum energy input $u^*(t)$ which would drive the states of the neuron to the phaseless set in some prespecified length of time $[t_0, t_f]$. This is achieved by minimizing an objective function

$$J = \frac{1}{2}\int_{t_0}^{t_f} [u(t)]^2 \mathrm{d}t, \tag{3}$$

with $x(t_f) = x_f$ where $x_f$ is the desired terminal state. Extending the optimal control concepts to the neuron systems is motivated by control applications where a high-power stimulus is harmful to the biological tissues and low-power electrical stimulus plays an important role in reducing the power consumption in a neurological implant.

Consider the Hamiltonian for system (1) as:

$$\mathcal{H}(x, \lambda) \;=\; \tfrac{1}{2}[u(t)]^2 + \lambda^\top(t)(F(x(t)) + Bu(t)), \tag{4}$$

where $\lambda$ is the co-state variable vector.

According to the Pontryagin's maximum principle, the optimality conditions are obtained by the following nonlinear BVP with differential equations

$$\begin{cases} \dot{x}(t) &= -BB^\top\lambda(t) + F(x(t)), \\ \dot{\lambda}(t) &= -\left(\frac{\partial F(x(t))}{\partial x(t)}\right)^\top \lambda(t), \\ x(t_0) &= x_0, \\ x(t_f) &= x_f, \end{cases} \tag{5}$$

and the optimal control law is given by

$$u^*(t) = -B^\top\lambda(t),\; t \in [t_0, t_f]. \tag{6}$$

**Remark 2.1.** *Note that Eq.(5) contains a nonlinear two-point BVP, which is in general hard to be solved analytically except for a few simple cases. The solution of the BVP problem (5) poses numerical challenges. Although the DTDP method [11], Level Set Methods Toolbox (ToolboxLS) [16] and the MATLAB* `bvp4c()` *routine [29] are applicable to numerically solve this two-point BVP, it is hard to carry out further system analysis using the derived optimal control law based on the nonanalytic solutions. In the next section, we introduce the HPM to overcome this difficulty.*

# 3 Principle of HPM

In this section, we briefly review the basic idea of He's HPM [20]. Consider the following nonlinear differential equation:

$$L(v(r)) + N(v(r)) = 0, \quad r \in \Omega, \tag{7}$$

with the boundary condition

$$\mathcal{B}(v, \frac{\partial v}{\partial n}) = 0 \quad r \in \Gamma, \tag{8}$$

where $\Omega$ is the definition domain of $v$, $L$ is a linear operator, $N$ is a nonlinear operator, $\Gamma$ is the boundary of domain $\Omega$, $\mathcal{B}$ is a boundary operator, and $\frac{\partial}{\partial n}$ denotes a differential along the normal direction drawn outwards from $\Omega$.

Now we can construct a homotopy for (7) as follows:

$$\begin{cases} H(\tilde{v},p) &= L(\tilde{v}) - L(v_{\text{ini}}) \\ & \quad +p\big(L(v_{\text{ini}}) + N(\tilde{v})\big) = 0, \\ p \in [0,1], & r \in \Omega, \end{cases}$$

where $p \in [0,1]$ is an embedding parameter called homotopy parameter, and $v_{\text{ini}}$ is an initial guess approximation for the solution of (7), which satisfies the boundary condition in (8). Obviously, when $p = 0$ and $p = 1$ the following holds:

$$\begin{cases} H(\tilde{v},0) &= L(\tilde{v}) - L(v_{\text{ini}}) = 0, \\ H(\tilde{v},1) &= L(\tilde{v}) + N(\tilde{v}) = 0. \end{cases} \tag{9}$$

Assume that the solution of (9) is a power series in $p$:

$$\tilde{v} = \tilde{v}^{(0)} + p\tilde{v}^{(1)} + p^2\tilde{v}^{(2)} + \cdots \tag{10}$$

or $\tilde{v} = \sum\limits_{k=0}^{\infty} p^k \tilde{v}^{(k)}$.

Setting $p = 1$, we derive the approximate solution of (7)

$$v = \lim_{p\to 1} \tilde{v} = \tilde{v}^{(0)} + \tilde{v}^{(1)} + \tilde{v}^{(2)} + \cdots . \tag{11}$$

The convergence of series (11) has been discussed in [22, 19].

Next, substituting $\tilde{v}$ from (10) into $N(\tilde{v})$ and then expanding $N$ in a power series with respect to the parameter $p$ together with the claim in [30]

$$\frac{\partial^n}{\partial p^n} N(\tilde{v})_{p=0} = \frac{\partial^n}{\partial p^n} N\Big(\sum_{k=0}^{n} p^k \tilde{v}^{(k)}\Big)_{p=0},$$

we have

$$\begin{aligned} N(\tilde{v}) &= N(\tilde{v})\big|_{p=0} + \tfrac{\partial N(\tilde{v})}{\partial p}\big|_{p=0} p + \cdots \\ &= N(\tilde{v}^{(0)}) + \big(\tfrac{\partial N(\tilde{v})}{\partial \tilde{v}}\tfrac{\partial \tilde{v}}{\partial p}\big)\big|_{p=0} p + \cdots \\ &= N(\tilde{v}^{(0)}) + \tfrac{\partial N(\tilde{v})}{\partial \tilde{v}}\big|_{\tilde{v}=\tilde{v}^{(0)}} \tilde{v}^{(1)} p + \cdots . \end{aligned} \tag{12}$$

Then, we construct a new homotopy for (7) as follows:

$$\begin{aligned} H(\tilde{v},p) &= L(\tilde{v}) - L(v_{\text{ini}}) \\ & \quad +p\big(L(v_{\text{ini}}) + N(\tilde{v}^{(0)})\big) \\ & \quad +p^2\big(\tfrac{\partial N(\tilde{v})}{\partial \tilde{v}}\big|_{\tilde{v}=\tilde{v}^{(0)}} \tilde{v}^{(1)}\big) \\ & \quad + \cdots = 0. \end{aligned} \tag{13}$$

By equating the coefficients of the same powers of $p$ in (13), we get

$$\begin{cases} p^0: & L(\tilde{v}^{(0)}) - L(v_{\text{ini}}) = 0 \\ p^1: & L(\tilde{v}^{(1)}) + L(v_{\text{ini}}) + N(\tilde{v}^{(0)}) = 0 \\ p^2: & L(\tilde{v}^{(2)}) + \frac{\partial}{\partial p} N\Big(\sum\limits_{k=0}^{1} p^k \tilde{v}^{(k)}\Big) = 0 \\ & \vdots \\ p^j: & L(\tilde{v}^{(j)}) + \frac{1}{j-1!}\frac{\partial^{j-1}}{\partial p^{j-1}} N\Big(\sum\limits_{k=0}^{j-1} p^k \tilde{v}^{(k)}\Big) = 0 \end{cases} . \tag{14}$$

Then, the $M$th order approximate solution is

$$v^{(M)} = \sum_{i=0}^{M} \tilde{v}^{(i)}.$$

**Remark 3.1.** *There are two important points that should be noted regarding the work in [20] and the present work. First, the HPM in [20] was introduced for nonlinear differential equations and applied to solve an ordinary differential equation. There was no explicit solution for solving linear or nonlinear optimal control problems, especially for some spiking neuron models which may be singular. Second, it is not easy to directly apply the HPM to the optimal control of spiking neurons. One of the main contributions in the present work is the formulation of the minimum-energy optimal control problem into a framework where the HPM could be applied. Typically, to solve optimal control problems of spiking neurons, direct methods [11, 16] that convert the problem into a nonlinear programming one using the discretization or parameterization techniques are adopted. Such methods generally lead to numerical solutions or approximate solutions, while new methods to obtain analytical solutions of optimal control for spiking neurons are rare; these have motivated the study of using the HPM.*

# 4 Minimum-Energy Optimal Control Via HPM

In this section, we apply the HPM for solving the minimum-energy optimal control problem. In order to perform this scheme, let us define two operators as follows:

$$F_1(x(t), \lambda(t)) \triangleq \dot{x}(t) + BB^\top \lambda(t) - F(x(t)), \tag{15}$$

$$F_2(x(t), \lambda(t)) \triangleq \dot{\lambda}(t) + \left( \frac{\partial F(x(t))}{\partial x(t)} \right)^\top \lambda(t). \tag{16}$$

By the BVP (5), it follows that

$$F_s(x(t), \lambda(t)) = 0, \quad s = 1, 2. \tag{17}$$

**Theorem 4.1.** *Consider the minimum-energy optimal control problem of conductance-based spiking neuron system in (1) with the performance index in (3). Using the HPM, the optimal trajectory and the optimal control law can be determined as follows:*

$$\begin{cases} x^*(t) &= \sum_{i=0}^{\infty} \tilde{x}^{(i)}(t), \\ u^*(t) &= -B^\top \sum_{i=0}^{\infty} \tilde{\lambda}^{(i)}(t), \end{cases} \tag{18}$$

*where $t \in [t_0, t_f]$, $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ for $i \geq 0$ are obtained by solving a sequence of linear time-invariant two point BVPs.*

*Proof.* To solve system (17) by HPM, we construct the following homotopy:

$$F_s(x(t),\lambda(t)) = L_s(x(t),\lambda(t)) + N_s(x(t),\lambda(t)) \tag{19}$$

where $s = 1, 2$,

$$\begin{cases} L_1(x(t),\lambda(t)) & \triangleq & \dot{x}(t) + BB^\top\lambda(t) \\ L_2(x(t),\lambda(t)) & \triangleq & \dot{\lambda}(t) \end{cases} \tag{20a}$$

$$\begin{cases} N_1(x(t),\lambda(t)) & \triangleq & -F(x(t)) \\ N_2(x(t),\lambda(t)) & \triangleq & \left(\frac{\partial F(x(t))}{\partial x(t)}\right)^\top \lambda(t). \end{cases} \tag{20b}$$

Now construct two homotopies $\tilde{x}(t,p) : [t_0,t_f] \times [0,1] \to \mathbb{R}^n$ and $\tilde{\lambda}(t,p) : [t_0,t_f] \times [0,1] \to \mathbb{R}^n$ for (19) satisfying

$$\begin{aligned} &L_s[(\tilde{x}(t,p),\tilde{\lambda}(t,p)) - (x_{\text{ini}}(t),\lambda_{\text{ini}}(t))] \\ &+ pL_s(x_{\text{ini}}(t),\lambda_{\text{ini}}(t)) + pN_s(\tilde{x}(t,p),\tilde{\lambda}(t,p)) = 0 \end{aligned} \tag{21}$$

where $s = 1, 2$, $p \in [0,1]$ is an embedding parameter which is called homotopy parameter. The initial approximations $x_{\text{ini}}(t)$ and $\lambda_{\text{ini}}(t)$ for the solution of (5) are chosen as the solution of the following equations

$$\begin{cases} L_1(x_{\text{ini}}(t),\lambda_{\text{ini}}(t)) = 0, \\ L_2(x_{\text{ini}}(t),\lambda_{\text{ini}}(t)) = 0, \\ x_{\text{ini}}(t_0) = x_0, \quad x_{\text{ini}}(t_f) = x_f. \end{cases} \tag{22}$$

Expanding $\tilde{x}(t,p)$ and $\tilde{\lambda}(t,p)$ as the Maclaurin series in an small embedding parameter $p$ yields

$$\begin{cases} \tilde{x}(t,p) = \tilde{x}^{(0)}(t) + p\tilde{x}^{(1)}(t) + p^2\tilde{x}^{(2)}(t) + \cdots \\ \tilde{\lambda}(t,p) = \tilde{\lambda}^{(0)}(t) + p\tilde{\lambda}^{(1)}(t) + p^2\tilde{\lambda}^{(2)}(t) + \cdots \end{cases} \tag{23}$$

where $\tilde{x}^{(n)}(t) = \frac{1}{n!}\frac{\partial^n \tilde{x}(t,p)}{\partial p^n}\Big|_{p=0}$ and $\tilde{\lambda}^{(n)}(t) = \frac{1}{n!}\frac{\partial^n \tilde{\lambda}(t,p)}{\partial p^n}\Big|_{p=0}$.

Substituting (22) and (23) into (21), and rearranging terms with respect the order of $p$, we can obtain $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ for $i \geq 0$ by solving the following sequence of linear time-invariant two point BVPs in a recursive manner

$$p^0 : \begin{cases} L_1(\tilde{x}^{(0)},\tilde{\lambda}^{(0)}) - L_1(x_{\text{ini}}(t),\lambda_{\text{ini}}(t)) = 0, \\ L_1(\tilde{x}^{(0)},\tilde{\lambda}^{(0)}) - L_2(x_{\text{ini}}(t),\lambda_{\text{ini}}(t)) = 0, \\ \tilde{x}^{(0)}(t_0) = x_0, \quad \tilde{x}^{(0)}(t_f) = x_f, \end{cases} \tag{24a}$$

$$p^1 : \begin{cases} \begin{bmatrix} L_1(\tilde{x}^{(1)},\tilde{\lambda}^{(1)}) \\ L_2(\tilde{x}^{(1)},\tilde{\lambda}^{(1)}) \end{bmatrix} + \begin{bmatrix} L_1(x_{\text{ini}}(t),\lambda_{\text{ini}}(t)) \\ L_2(x_{\text{ini}}(t),\lambda_{\text{ini}}(t)) \end{bmatrix} \\ \qquad + \begin{bmatrix} N_1(X_0,D_0) \\ N_2(X_0,D_0) \end{bmatrix} = 0, \\ \tilde{x}^{(1)}(t_0) = 0, \quad \tilde{x}^{(1)}(t_f) = 0, \end{cases} \tag{24b}$$

$$p^2:\begin{cases}\begin{bmatrix} L_1(\tilde{x}^{(2)},\tilde{\lambda}^{(2)}) \\ L_2(\tilde{x}^{(2)},\tilde{\lambda}^{(2)})\end{bmatrix}+\begin{bmatrix}\frac{\partial}{\partial p}N_1(X_1,D_1)\big|_{p=0} \\ \frac{\partial}{\partial p}N_2(X_1,D_1)\big|_{p=0}\end{bmatrix}=0, \\ \tilde{x}^{(2)}(t_0)=0, \qquad \tilde{x}^{(2)}(t_f)=0,\end{cases} \tag{24c}$$

$$\vdots$$

$$p^j:\begin{cases}\begin{bmatrix} L_1(\tilde{x}^{(j)},\tilde{\lambda}^{(j)}) \\ L_2(\tilde{x}^{(j)},\tilde{\lambda}^{(j)})\end{bmatrix}+\begin{bmatrix}\frac{1}{j-1!}N_{p1} \\ \frac{1}{j-1!}N_{p2}\end{bmatrix}=0, \\ \tilde{x}^{(j)}(t_0)=0, \qquad \tilde{x}^{(j)}(t_f)=0,\end{cases} \tag{24d}$$

where $N_{p1}=\frac{\partial^{j-1}}{\partial p^{j-1}}N_1(X_{j-1},D_{j-1})\big|_{p=0}$, $N_{p2}=\frac{\partial^{j-1}}{\partial p^{j-1}}N_2(X_{j-1},D_{j-1})\big|_{p=0}$, $X_k=\sum_{i=0}^{k}p^i\tilde{x}^{(i)}$, $D_k=\sum_{i=0}^{k}p^i\tilde{\lambda}^{(i)}$, $k=0,1,\cdots,j-1$, $j=1,2,\cdots$.

After obtaining $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ for $i\geq 0$, by setting $p=1$ the solution of the nonlinear BVP in (5) can be expressed as

$$\begin{cases} x^*(t)=\tilde{x}(t,1)=\sum_{i=0}^{\infty}\tilde{x}^{(i)}(t), \\ \lambda^*(t)=\tilde{\lambda}(t,1)=\sum_{i=0}^{\infty}\tilde{\lambda}^{(i)}(t). \end{cases} \tag{25}$$

Furthermore, using (6) yields

$$\begin{cases} x^*(t)=\sum_{i=0}^{\infty}\tilde{x}^{(i)}(t), \\ u^*(t)=-B^\top\lambda^*(t)=-B^\top\sum_{i=0}^{\infty}\tilde{\lambda}^{(i)}(t). \end{cases} \tag{26}$$

This completes the proof. □

Theorem 4.1 presents the optimal trajectory and the optimal control law, but it is impractical to use it as it contains infinite series. Therefore, to apply the HPM in practice, we propose the following efficient algorithm for solving the minimum-energy optimal control problem. In this algorithm, the $M$th order suboptimal trajectory-control pair is obtained by replacing $\infty$ with a finite positive integer $M$ in (18), i.e., the $M$th order approximate solution is

$$\begin{cases} x^{(M)}(t) = \sum_{i=0}^{M}\tilde{x}^{(i)}(t), \\ \lambda^{(M)}(t) = \sum_{i=0}^{M}\tilde{\lambda}^{(i)}(t). \end{cases} \tag{27}$$

*Practical Algorithm.*

Step 1. Obtain $x_{\text{ini}}(t)$ and $\lambda_{\text{ini}}(t)$ from the linear time-invariant BVP in (22). Set $\tilde{x}^{(0)}(t)=x_{\text{ini}}(t)$, $\tilde{\lambda}^{(0)}(t)=\lambda_{\text{ini}}(t)$, and $i=1$.

Step 2. Compute the $i$th order terms $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ from the sequence of linear time-invariant BVP. Set $M=i$ and calculate $x^{(M)}(t)$ and $\lambda^{(M)}(t)$ from (27), and then derive $u^{(M)}(t)$ from (18).

Step 3. Calculate $J^{(M)}$ given by

$$J^{(M)} = \frac{1}{2}\int_{t_0}^{t_f} [u^{(M)}(t)]^2 \mathrm{d}t.$$

Step 4. If the following inequality

$$\left|\frac{J^{(M)} - J^{(M-1)}}{J^{(M)}}\right| < \varepsilon$$

holds for a given small enough constant $\varepsilon > 0$ (which represents the tolerance error bound), go to step 5; else replace $i$ by $i + 1$ and go to Step 2.

Step 5. Stop the algorithm; obtain $x^{(M)}(t)$ and $u^{(M)}(t)$ which are accurate enough.

**Remark 4.2.** *From the expressions in* (18)*, we know that the optimal trajectory and the optimal control law are provided analytically with their dependence only on time, which gives the opportunity for further studies. The HPM is a combination of homotopy in topology and classic perturbation techniques. In contrast to other approximate methods, it avoids solving directly the nonlinear BVP and provides a convenient way to obtain the analytic or approximate solutions for the optimal control problems of spiking neurons. However, one limitation of using the HPM is that it may be complicated to compute the derivatives of the nonlinear operator* $N_i(X_i, D_i)$ *in* (24)*, which may affect the convergence rate though it is tolerable as illustrated in the simulations later. It remains an open problem in our further work.*

Denote $\xi(t) = [x^\top(t)\ \lambda^\top(t)]^\top$ and

$$W(\xi) = \begin{bmatrix} N_1(x(t), \lambda(t)) - BB^\top\lambda(t) \\ N_2(x(t), \lambda(t)) \end{bmatrix}.$$

Next, the Banach's fixed point theorem is applied for convergence study of the series (27).

**Theorem 4.3.** *(Convergence) For the BVP* (5) *with the homotopy* (19)*, suppose* $W : \mathbb{R}^n \to \mathbb{R}^n$ *is a contractive nonlinear mapping, i.e.,*

$$\|W(\xi) - W(\xi^*)\| \leq \gamma\|\xi - \xi^*\|, \gamma \in (0, 1) \tag{28}$$

*for all* $\xi, \xi^* \in \mathbb{R}^n$*. Then the sequence* $\{\xi_m\}_{m=1}^{\infty}$ *with* $\xi_m = W(\xi_{m-1})$ *generated by the HPM may be regarded as satisfy* $\lim\limits_{m\to\infty} \xi_m = \xi^*$*.*

*Proof.* Since $W : \mathbb{R}^n \to \mathbb{R}^n$ is a contractive nonlinear mapping, then, according to Banach's fixed point theorem, $W$ has a unique fixed point $\xi^*$ that is $W(\xi^*) = \xi^*$. By induction, for $n = 1$ we have

$$\|\xi_1 - \xi^*\| = \|W(\xi_0) - W(\xi^*)\| \leq \gamma\|\xi_0 - \xi^*\|.$$

Now assume that $\|\xi_{m-1} - \xi^*\| \leq \gamma^{m-1}\|\xi_0 - \xi^*\|$, then, using the induction hypothesis and (28), it follows that

$$\begin{aligned}\|\xi_m - \xi^*\| &= \|W(\xi_{m-1}) - W(\xi^*)\| \\ &\leq \gamma\|\xi_{m-1} - \xi^*\| \\ &\leq \gamma^m\|\xi_0 - \xi^*\|. \end{aligned} \tag{29}$$

Therefore, by the fact $\lim_{m\to\infty} \gamma^m = 0$, one can obtain that $\lim_{m\to\infty} \|\xi_m - \xi^*\| = 0$, i.e., $\lim_{m\to\infty} \xi_m = \xi^*$. This completes the proof. □

Theorem 4.3 provides a sufficient condition for convergence of the homotopy sequence, but the condition seems difficult to be verified since $W(\xi)$ contains the state iteration solutions for all $t \geq 0$. Finding new easily-verified condition remains an open problem in our future work.

**Remark 4.4.** *Since the external input or the input current stimulus $u(t)$ acts only on the membrane potential, it is possible to apply the proposed algorithm in practice. For instance, the algorithm may be performed on neurons of Long-Evans rats, specifically* in vitro *pyramidal neurons in the CA1 region of rat hippocampus, for which a pulse control method [3] and a novel approach for designing minimum-energy input waveforms [12] have been experimentally implemented. However, the challenge for illustrating these type of experimental applications is not only in modelling such a biological neuron in a special region of a tested animal, but also in determining the uncertain model parameters and solving technical issues like physical implementation of the optimal control law and the computational requirements. Therefore, in our case, we only provide numerical examples in the next section to demonstrate the efficiency of the proposed algorithm.*

# 5 Application: two neuron models

This section presents the results from simulation of the proposed HPM-based minimum-energy optimal control design applied to two biological spiking (neuron) models.

## 5.1 FHN model

As is well-known, the Hodgkin-Huxley model is very important in describing the transmission of an action potential through a cell membrane [31]. However, due to the large number of variables, the phase space dynamics of the equation is hard to be visualized. The FHN model was introduced as a simplified model of the cell membrane [32], and experimentally verified by Nagumo et al [33] using electrical circuits. Here, we consider the FHN model [28, 16] described in a dimensionless form

$$\left\{\begin{array}{lll} \delta\dot{v} &=& v(v+0.6)(1-v) - w + I, \\ \dot{w} &=& v - 0.5w \end{array}\right. \tag{30}$$

where $v$ is the voltage potential of the neuron membrane, $w$ is the inactivation of the sodium channels, $I$ represents the input current, $\delta$ acts as the singular perturbation

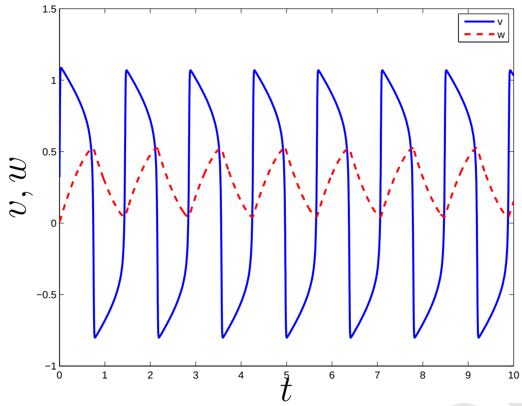



Figure 1: The spiking oscillation of the FHN model with initial values $v(0) = 0.32$, $w(0) = 0.01$.

parameter and $v$ evolves on a much faster time scale than $w$. When $\delta = 0.01$ and the input current value $I = 0.2$, the neural system exhibits periodic spiking dynamics (see, Figure 1).

For neurons that are firing in synchronism, an efficient way to desynchronize the population is to drive the state of the neuron model to its phaseless set, a point at which its phase is undefined and is extremely sensitive to background noise [16]. For the FHN model (30), the origin is the phaseless set contained by the basin of attraction of the periodic orbit [16], and thus our objective is to construct a minimum energy control law to reach an arbitrarily small neighborhood of the origin. This is achieved by minimizing an objective function

$$J[I(t)] = \frac{1}{2}\int_0^T [I(t)]^2 \mathrm{d}t. \tag{31}$$

The Hamiltonian in this case is given by

$$\mathcal{H}(v, w, \alpha, \beta) = \tfrac{1}{2}[I(t)]^2 + \alpha\dot{v} + \beta\dot{w}, \tag{32}$$

where $\alpha$ and $\beta$ are the co-state variables.

According to the Pontryagin's maximum principle, the optimality conditions are obtained as follows

$$\begin{cases} \delta\dot{v} &= \frac{\partial\mathcal{H}}{\partial\alpha} = v(v+0.6)(1-v) - w + I, \\ \dot{w} &= \frac{\partial\mathcal{H}}{\partial\beta} = v - 0.5w, \\ \dot{\alpha} &= -\frac{\partial\mathcal{H}}{\partial v} = -\alpha(0.8v - 3v^2 + 0.6)/\delta - \beta, \\ \dot{\beta} &= -\frac{\partial\mathcal{H}}{\partial w} = \alpha/\delta + 0.5\beta. \end{cases} \tag{33}$$

The boundary conditions for $v$ and $w$ are

$$\begin{cases} v(0) &= v_0, \\ w(0) &= w_0, \\ v(T) &= v_f, \\ w(T) &= w_f. \end{cases} \tag{34}$$

and the boundary conditions for $\alpha$ and $\beta$ are free. The optimal control law is given by:

$$\frac{\partial \mathcal{H}}{\partial I} = 0 \Rightarrow I^*(t) = -\alpha(t)/\delta.$$

Though the neuron system is amenable to standard ordinary differential equation initial value problem solvers, it is numerically challenging to derive the solution of the BVP problem (33)-(34). To solve this problem using the method in Section III, we begin by implicitly defining a homotopy map

$$\begin{aligned} & H(v, w, \alpha, \beta, p) \\ = \; & L(v, w, \alpha, \beta) - L(v_{\text{ini}}, w_{\text{ini}}, \alpha_{\text{ini}}, \beta_{\text{ini}}) \\ & + p\big(L(v_{\text{ini}}, w_{\text{ini}}, \alpha_{\text{ini}}, \beta_{\text{ini}}) + N(v, w, \alpha, \beta)\big) \\ = \; & 0. \end{aligned} \tag{35}$$

Using (33), we can define the linear and nonlinear parts of the system dynamics by

$$L(v, w, \alpha, \beta) = \left[\delta \dot{v} + \alpha/\delta,\ \dot{w},\ \dot{\alpha},\ \dot{\beta}\right]^\top,$$

$$N(v, w, \alpha, \beta) = \begin{bmatrix} v(v+0.6)(v-1) + w \\ -(v - 0.5w) \\ \alpha(0.8v - 3v^2 + 0.6)/\delta + \beta \\ -\alpha/\delta - 0.5\beta \end{bmatrix}.$$

Then, we can expand the variables $v, w, \alpha, \beta$ using power series expansions in the embedding parameter $p$ and follow the procedure in *Practical Algorithm* to obtain the approximate solution

$$\begin{cases} v^*(t) &= \sum_{i=0}^{\infty} \tilde{v}^{(i)}(t), \\ w^*(t) &= \sum_{i=0}^{\infty} \tilde{w}^{(i)}(t), \\ \alpha^*(t) &= \sum_{i=0}^{\infty} \tilde{\alpha}^{(i)}(t), \\ \beta^*(t) &= \sum_{i=0}^{\infty} \tilde{\beta}^{(i)}(t), \\ I^*(t) &= -\alpha^*(t)/\delta. \end{cases} \tag{36}$$

Before performing the simulation, consider the initial value $v_0 = 1, w_0 = -0.1$, the objective being to control the states of the neuron system to the phaseless point $v_f = 0, w_f = 0$ within $T = 0.2$ seconds, while maintaining the performance given by (31). For simplicity, take $\delta = 1$ to eliminate input and voltage scaling. Let the initial approximations of co-state variables $\alpha$ and $\beta$ be denoted by $\alpha_0$ and $\beta_0$, respectively. In our implementation, we perform the simulation in Matlab to numerically solve the optimal control problem. The differential equations in (24) for each order of $p$ are computed using symbolic manipulation, and are then evolved in time to obtain the successive approximations. Following the procedure given in *Practical Algorithm*, we can solve the presented sequence of the BVP in (33)-(34) with initial values $v_0, w_0, \alpha_0$ and $\beta_0$ as follows

$$\left\{\begin{array}{lll} \tilde{v}^{(0)}(t) = v_0, & \tilde{v}^{(1)}(t) = (0.1 - \alpha_0)t, & \cdots \\ \tilde{w}^{(0)}(t) = w_0, & \tilde{w}^{(1)}(t) = \frac{21}{20}t, & \cdots \\ \tilde{\alpha}^{(0)}(t) = \alpha_0, & \tilde{\alpha}^{(1)}(t) = (1.6\alpha_0 - \beta_0)t, & \cdots \\ \tilde{\beta}^{(0)}(t) = \beta_0, & \tilde{\beta}^{(1)}(t) = (\alpha_0 + 0.5\beta_0)t, & \cdots \end{array}\right. \tag{37}$$

To obtain an accurate enough suboptimal trajectory, we set the tolerance error bound $\varepsilon = 0.01$. In this case, convergence is achieved after four iterations, i.e. $\left|\frac{J^{(3)}-J^{(2)}}{J^{(3)}}\right| = 0.0014 < 0.01$ and a minimum of $J^{(3)} = 11.1503$ is obtained.

By considering the final state condition, we have

$$\sum_{i=0}^{M} \tilde{v}^{(i)}(t) = v_f, \sum_{i=0}^{M} \tilde{w}^{(i)}(t) = w_f, \tag{38}$$

yielding $\alpha_0 = 5.5591$ and $\beta_0 = 5.4788$. The associated optimal control is

$$\begin{array}{rcl} I^*(t) & \approx & -\alpha^*(t)/\delta \\ & = & -109.816t^3 + 80.3195t^2 \\ & & -3.4157t - 5.5591. \end{array} \tag{39}$$

In order to verify whether the derived control achieves the objectives or not, it is applied to the neuron system to obtain controlled state trajectories. We compare the results of the HPM with the state trajectories and the optimal control signals obtained using the Gauss pseudospectral method (GPM) [25] and the successive approximation approach (SAA) [27]. As an orthogonal collocation method, GPM is an efficient optimization approach based on approximating the state and control trajectories using interpolating polynomials and has been used extensively throughout the world both in academia and industry [34, 35]. It is employed to verify the effectiveness of HPM though it only gives numerical solutions. SAA was first introduced in [26], and then was improved to design optimal controllers for a class of nonlinear systems with a quadratic performance index in [27]. By using SAA in [27], one can obtain an approximate semi-analytic optimal control law, but an exact analytic one is unavailable due to the existence of a nonlinear compensation term.

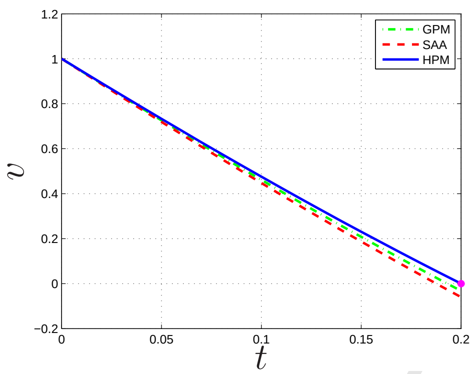
Figures 2 and 3 provide simulation results for minimum-energy optimal control of the FHN model for these three methods. Figure 2 shows the state trajectories of the solutions. It is seen that compared to the SAA, both the HPM and the GPM can achieve the phaseless point better. Figure 3 shows the optimal control law of the solutions. It can be seen that among the three methods, the HPM needs the least energy. This is quantatively confirmed by the computational results given in Table 1, where the iteration length, the computational time (CT) and the objective function value $J$ for each of the methods are given. The error between the real and the desired terminal states, defined as $E = \sqrt{(v(T) - v_f)^2 + (w(T) - w_f)^2}$, is also listed in Table 1. Obviously, the benefit of utilizing the HPM is clear. It requires the least computational time with least control energy and provides a good accuracy on the desired state.

Note that the DTDP method [11] is also a powerful optimization method for solving the problem of optimal control of spiking neurons numerically. However, we do not compare quantitatively our algorithm with the DTDP method, since in the latter method

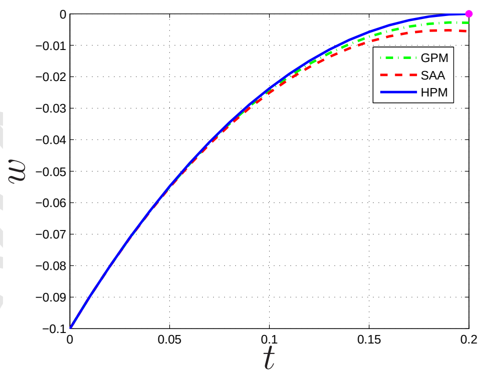
Table 1: Comparing computational efficiency

| Method | Number of Iterations | CT (s) | $J$ | $E$ |
|---|---|---|---|---|
| HPM | 3 | 1.38 | 11.1503 | 0.0140 |
| GPM | 40 | 4.78 | 12.1128 | 0.0324 |
| SAA | 7 | 2.23 | 12.7246 | 0.0602 |

one solves the problem in a discrete state space through discretization and the derived optimal control signals are discrete. Note that by reducing the mesh size, the number of states grows quickly, which is restricted by available computational power. Most importantly, the HPM has a significant advantage that it provides an analytical approximate solution of the optimal control law as a function of the time variable.




(a) $t$ vs $v(t)$




(b) $t$ vs $w(t)$

Figure 2: The state solutions under different methods

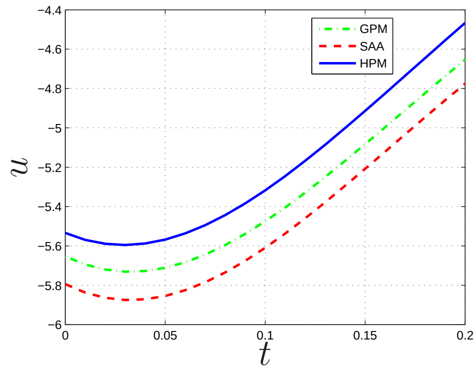



Figure 3: The optimal control solutions under different methods

## 5.2 HR model

In this subsection, we briefly investigate the minimum-energy optimal control of the HR equations of 1984 (see, [36, Section 1.4]).

The HR equations for the 1984 model are given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f(x,y) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} I(t) \tag{40}$$

where $f(x,y) := \begin{bmatrix} y - x^3 + 3x^2 \\ 1 - 5x^2 - y \end{bmatrix}$, $x$ represents the membrane potential, $y$ is an internal, or recovery, variable, and $I(t)$ is the external, applied, or clamping current at time $t$.

Similar to the FHN model (30), we aim to minimize the objective function (31) and the Hamiltonian is thus given by

$$\mathcal{H}(x,y,\alpha,\beta) \quad = \quad \tfrac{1}{2}[I(t)]^2 + \alpha\dot{x} + \beta\dot{y}, \tag{41}$$

where $\alpha$ and $\beta$ are the co-state variables.

According to the Pontryagin's maximum principle, the optimality conditions are obtained as follows

$$\begin{cases} \dot{x} &= \frac{\partial\mathcal{H}}{\partial\alpha} = y - x^3 + 3x^2 + I, \\ \dot{y} &= \frac{\partial\mathcal{H}}{\partial\beta} = 1 - 5x^2 - y, \\ \dot{\alpha} &= -\frac{\partial\mathcal{H}}{\partial v} = -\alpha(6x - 3x^2) + 10\beta x, \\ \dot{\beta} &= -\frac{\partial\mathcal{H}}{\partial w} = -\alpha + \beta. \end{cases} \tag{42}$$

The boundary conditions for $x$ and $y$ are

$$\begin{cases} x(0) &= x_0, \\ y(0) &= y_0, \\ x(T) &= x_f, \\ y(T) &= y_f. \end{cases} \tag{43}$$

The boundary conditions for $\alpha$ and $\beta$ are free. The optimal control law is given by:

$$\frac{\partial\mathcal{H}}{\partial I} = 0 \Rightarrow I^* = -\alpha.$$

To solve this problem using the method in Section III, we begin by defining a homotopy map implicitly

$$\begin{aligned} & H(x,y,\alpha,\beta,p) \\ = \ & L(x,y,\alpha,\beta) - L(x_{\text{ini}}, y_{\text{ini}}, \alpha_{\text{ini}}, \beta_{\text{ini}}) \\ & + p\big(L(x_{\text{ini}}, y_{\text{ini}}, \alpha_{\text{ini}}, \beta_{\text{ini}}) + N(x,y,\alpha,\beta)\big) = 0. \end{aligned} \tag{44}$$

Now define the linear and nonlinear parts of the system dynamics by

$$L(x,y,\alpha,\beta) = \left[\dot{x} + \alpha,\ \dot{y},\ \dot{\alpha},\ \dot{\beta}\right]^\top,$$

$$N(x,y,\alpha,\beta) = \begin{bmatrix} -(y - x^3 + 3x^2) \\ -(1 - 5x^2 - y) \\ \alpha(6x - 3x^2) - 10\beta x \\ \alpha - \beta \end{bmatrix}.$$

Our objective is to control the states of the neuron system to the final value $x_f = 0, y_f = 0$ within $T = 0.3$ seconds. Following the procedure given in *Practical Algorithm* with initial values $x_0 = 1, y_0 = 2, \alpha_0$ and $\beta_0$, we can solve the presented sequence of the BVP in (42)-(43) as follows

$$\begin{cases} \tilde{x}^{(0)}(t) = x_0, & \tilde{x}^{(1)}(t) = -\big(\alpha_0 - 4\big)t, & \cdots \\ \tilde{y}^{(0)}(t) = y_0, & \tilde{y}^{(1)}(t) = -6t, & \cdots \\ \tilde{\alpha}^{(0)}(t) = \alpha_0, & \tilde{\alpha}^{(1)}(t) = (-3\alpha_0 + 10\beta_0)t, & \cdots \\ \tilde{\beta}^{(0)}(t) = \beta_0, & \tilde{\beta}^{(1)}(t) = -(\alpha_0 - \beta_0)t, & \cdots \end{cases} \tag{45}$$

After five iterations, convergence is achieved with the tolerance error bound $\varepsilon = 0.03$ and a minimum of $J^{(5)} = 44.8908$. Moreover, the initial values of the co-state variables are $\alpha_0 = -4.9495$ and $\beta_0 = 5.9367$. The associated optimal control is

$$\begin{aligned} I^*(t) &\approx -\alpha^*(t) \\ &= -19107.93t^5 + 10523.79t^4 - 789.81t^3 \\ &\quad -134.36t^2 - 75t + 4.9495. \end{aligned} \tag{46}$$

Since the nonlinear function in the HR model (40) does not satisfy $f(0,0) = 0$, the SAA is not applicable. Hence, we only compare the performance with the GPM to illustrate the effectiveness of the HPM. The error $E$ between the real and the desired terminal states is also employed and the results are given in Table 2 including the number of iterations, the computational time, the objective function $J$ and the error $E$. It is observed that using the HPM, the minimum-energy optimal control problem of the HR model is effectively solved and only five iterations can lead to highly accurate solutions. In contrast to the GPM, the HPM performs better in computational time, control energy and accuracy. It is also demonstrated from the neuron applications that the minimum stimulus of a single control from only one observable state (usually, the membrane potential) can lead to the goal of controlling the neuron models. This conclusion agrees with empirical evidence from practical biological experiments and observations [?].

Table 2: Comparing computational efficiency

| Method | Number of Iterations | CT (s) | $J$ | $E$ |
|---|---|---|---|---|
| HPM | 5 | 1.61 | 44.8908 | $2.07e-5$ |
| GPM | 58 | 4.89 | 46.1440 | 0.1339 |

## 6 Conclusions

In this paper, we have studied the minimum-energy optimal control of spiking neurons. By utilizing the homotopy perturbation method and combining the Pontryagin's maximum principle, the optimal control law has been determined in the form of a rapid convergent series with easily computable terms. Without solving a nonlinear BVP directly, the proposed method has provided an iterative optimization-free technique to solve Hamilton equations by the use of iteration formulas derived from two operators corresponding to Hamilton equations. We have applied the proposed method to the FitzHugh-Nagumo and Hindmarsh-Rose models. Comparisons of the results by using the Gauss pseudospectral method and the successive approximation approach have also been carried out. It has been shown that this homotopy perturbation method can be used with good analytical approximate results, where it is difficult to solve the optimal control problem in an explicit analytical form. We have also attempted to develop approximate solutions for the Hodgkin-Huxley model. Unfortunately, it was hard to derive an approximate analytical solution for this model. One reason may be due to the presence of highly nonlinear functions in the Hodgkin-Huxley model, restricting the possibility of rapid convergence of an approximate solution to the exact one. We intend to tackle this problem and explore the convergence conditions for this model in the future.

## Acknowledgements

This work is partially supported by National Natural Science Foundation of China (61473136), the Fundamental Research Funds for the Central Universities (JUSRP51322B) and the 111 Project (B12018). The authors would like to thank the Action Editor and the anonymous reviewers for their very valuable comments and suggestions, which have helped in improving the quality of the paper.

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