Cooperative Control Reconfiguration in Networked Multi-Agent Systems

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ABSTRACT

Cooperative Control Reconfiguration in Networked Multi-Agent Systems

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Development of a network of autonomous cooperating vehicles has attracted significant attention during the past few years due to its broad range of applications in areas such as autonomous underwater vehicles for exploring deep sea oceans, satellite formations for space missions, and mobile robots in industrial sites where human involvement is impossible or restricted, to name a few. Motivated by the stringent specifications and requirements for depth, speed, position or attitude of the team and the possibility of having unexpected actuators and sensors faults in missions for these vehicles have led to the proposed research in this thesis on cooperative fault-tolerant control design of autonomous networked vehicles.

First, a multi-agent system under a fixed and undirected network topology and subject to actuator faults is studied. A reconfigurable control law is proposed and the so-called distributed Hamilton-Jacobi-Bellman equations for the faulty agents are derived. Then, the reconfigured controller gains are designed by solving these equations subject to the faulty agent dynamics as well as the network structural constraints to ensure that the agents can reach a consensus even in presence of a fault while simultaneously the team performance index is minimized.

Next, a multi-agent network subject to simultaneous as well as subsequent actuator faults and under directed fixed topology and subject to bounded energy disturbances is

considered. An H_∞ performance fault recovery control strategy is proposed that guarantees: the state consensus errors remain bounded, the output of the faulty system behaves exactly the same as that of the healthy system, and the specified H_∞ performance bound is guaranteed to be minimized. Towards this end, the reconfigured control law gains are selected first by employing a geometric control approach where a set of controllers guarantees that the output of the faulty agent imitates that of the healthy agent and the consensus achievement objectives are satisfied. Then, the remaining degrees of freedom in the selection of the control law gains are used to minimize the bound on a specified H_∞ performance index.

Then, control reconfiguration problem in a team subject to directed switching topology networks as well as actuator faults and their severity estimation uncertainties is considered. The consensus achievement of the faulty network is transformed into two stability problems, in which one can be solved offline while the other should be solved online and by utilizing information that each agent has received from the fault detection and identification module. Using quadratic and convex hull Lyapunov functions the control gains are designed and selected such that the team consensus achievement is guaranteed while the upper bound of the team cost performance index is minimized.

Finally, a team of non-identical agents subject to actuator faults is considered. A distributed output feedback control strategy is proposed which guarantees that agents outputs' follow the outputs of the exo-system and the agents states remains stable even when agents are subject to different actuator faults.

To my parents,

for their love and endless support , and

For my sister, Azar,

without her, I would never have come so far

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LIST OF ABBREVIATIONS AND SYMBOLS

LQR	Linear Quadratic Regulator
FDI	Fault Detection, Isolation and Identification
FTC	Fault Tolerant Control
FDIR	Fault Detection, Isolation and Reconfiguration
ARE	Algebraic Riccati Equation
HJB	Hamilton Jacobi Bellman
LMI	Linear Matrix Inequalities
BMI	Bilinear Matrix Inequalities
LDI	Linear Differential Inclusion
LTI	Linear Time Invariant
LOE	Loss of Effectiveness
AUV	Autonomous Underwater Vehicles
LF	Leader Follower
LL	Leaderless

Chapter 1

Introduction

1.1 Motivation

Coordinated behaviour in animals such as flocking of birds, shoaling and schooling fish (see Figure 1.1) brings the motivation to employ a group of vehicles like spacecraft, mobile robots, or underwater vehicles instead of a single vehicle with the aim at improving the system performance, reliability, and ultimately reducing the cost of the overall mission.

The use of a group of vehicles becomes more attractive if these agents are designed to be autonomous or unmanned, which makes them appropriate for maneuvers where human involvement is dangerous, or impossible as in deploying mobile robots for planetary surface exploration, cooperative navigation, mapping and exploration, underwater sensing and monitoring applications, rescue operations, etc. (see Figure 1.2).

In safety critical missions, the agents should have the capability to cope with unexpected external influences such as environmental changes or internal events such as actuator and sensor faults. If these unexpected events are not managed successfully, they can lead to the team instability or cause sever overall team performance degradations. For example, the crash of the NASA's DART spacecraft in 2006 was due to a fault in its position sensors [1] or the crash of the Boeing freighter in 1992 could have been avoided if its control laws were reconfigured to manage it land safely [2]. Motivated by these, the fault recovery and control reconfiguration problems for safety critical systems have been extensively studied in the literature.

Though agents cooperation and interactions enhance the team performance, control design problem in networked systems is more challenging. Limited communication channels, topology variations and connectivity preservation, collision avoidance, environmental uncertainties can be listed as some important issues that are associated with the problem of cooperative control development. These issues should be added to the challenges of the control design problem of a single agent system such as actuator/ sensor faults, actuator saturations, performance criteria, model uncertainties, etc. The problem of fault recovery and control reconfiguration in multi-agent systems is more challenging as compared to that of a single agent due to limited available communication channels and has been studied only in recent years. In the following section, the existing literature on the fault recovery/control reconfiguration and cooperative control in multi-agent systems are reviewed.

Figure 1.1: Coordinated behaviour in animals. Figures borrowed from Google images

(a) Exploration and mapping with robots, Team Michigan robots [3]. Figures borrowed from Google images.

(b) Aircrafts combat. Picture courtesy: DARPA [4]. Figures borrowed from Google images.

(c) kayaks' cooperative navigation [5]. Figures borrowed from Google images.

Figure 1.2: Applications of the cooperative control.

1.2 Literature Review

1.2.1 Fault Recovery

In any physical system there is the possibility of unexpected events such as actuator fault, sensor fault or hardware/software fault. Each of these faults has its own categories and depending on the fault type and its severity it may affect the system's characteristics, degrade its performance or cause the system instability. In critical systems such as aircraft or nuclear power plants, system instability may lead to catastrophic events and cannot be tolerated. Motivated by this, considerable research has been devoted to Fault Tolerant Control Systems (FTCS) which guarantee that the system remains stable and keeps its performance in an acceptable level even in presence of sudden faults.

Comprehensive reviews on fault tolerant control approaches have been provided in [6, 7]. Fault tolerant control (FTC) approaches can be categorized from different points of view and for each method several practical applications have been provided.

Broadly speaking, FTC approaches can be classified into two main types: passive approaches and active approaches. Each one has its own advantages and disadvantages. Jiang and Yu [8] compared these two categories and presented applications, advantages and disadvantages of each of them. In passive methods, a set of possible faults is determined and then a fixed structure controller is designed to accommodate all of the faults [9–11]. In [9], Liao *et al.* investigated actuator faults and control surface impairment in aircraft. They designed a reliable robust tracking controller by using the LMI approach. In another study, Khosrowjerdi *et al.* [10] used the mixed H_2/H_∞ approach to address simultaneous fault detection and control (SFDC) problem. They used the H_2 norm to determine the fault detection objective and the H_∞ norm to evaluate the control objectives. In [11] actuator faults in nonlinear affine systems is studied and a Lyapunov-based feedback controller is presented. In all of the above works controllers are independent of the fault detection and isolation (FDI) module and do not require any information from it. However, if a fault, not belonging to the predefined fault set occurs, these controllers cannot guarantee the system stability and the system performance may deteriorate.

In contrast to passive methods, in active approaches the FDI module information is employed to reconfigure the controllers. The controllers can be selected from the set of controller, designed offline, or can be designed online by using the FDI information.

Adaptive approaches are among the most common approaches for accommodating changes in the system dynamics [12–18]. In [12, 13], adaptive fault tolerant controller is proposed to accommodate partial loss and total failure of actuator faults in uncertain linear systems in presence of exogenous disturbances. In [14], an adaptive fuzzy control is developed to compensate for the loss of effectiveness and lock in place faults in the system with unmodeled dynamics and unknown control directions. In [15], attitude tracking of flexible spacecraft with unknown inertia parameters, external disturbances, and actuator faults is studied. Symmetric diagonal-upper (SDU) factorization is employed to handle uncertainties of the spacecraft inertia matrix and the adaptive control laws are used to ensure system stability. In [16], an adaptive backstepping controller is designed to guarantee the performance bound in an uncertain nonlinear system with faulty actuators. In [17, 18] adaptive sliding mode control is employed to accommodate actuator faults in nonlinear systems. In [17], a sliding mode surface is constructed and then an adaptive sliding mode controller is designed to derive state trajectories onto the sliding mode surface and estimate the severity of actuator faults in the Markov jump nonlinear systems while in [18], a third-order sliding mode surface was employed with an adaptation law to control the attitude of a spacecraft subject to different actuator faults.

Juan *et al.* [19] investigated the optimal solution for the fault tolerant controller problem in systems with delayed measurements and states. They used the solution of the Riccati equation and Sylvester equation to ensure the system performance in presence of unmeasurable actuator and/or sensor faults in the system. Stability and H_∞ performances of linear systems in presence of actuator faults are considered in [20]. The authors suggested an indirect adaptive reliable controller to guarantee the required control objectives. In [21] composite/combined state-feedback model reference adaptive controller for linear MIMO system is proposed . The authors investigated the method efficiency for longitudinal dynamics of an aerial vehicle. In [22], Zhang *et al.* considered the FTC problem in linear systems with uncertainties. Neural networks are employed to model the effects of actuator faults on the system and then an adaptive control law is developed to compensate for the fault effects on the system.

Model predictive control (MPC) is another well-known approach for compensating the systems dynamic changes [23, 24]. In [23], two Lyapunov-based model predictive controllers are presented to stabilize the nonlinear distributed process in presence of actuator faults whereas in [24], the stability problem in a tank unit is formulated as a constrained optimization problem. A recurrent neural network is used to predict the model and the control law is obtained by solving an optimization problem.

In [25], a machine learning approach is proposed for FTC in nonlinear processes. The faulty plant behaviour is modeled with online SVM (support vector machines) and then this model is used for model-based predictive control. Fuzzy logics are used in [26,27] to compensate for actuator faults. Tong *et al*. [26], consider a class of nonlinear large-scale systems with unmeasured states and actuator faults. They employ a fuzzy logic system to estimate the unmeasured states and then the backstepping technique and nonlinear FTC theory is employed to ensure boundedness of the error signals. In [27], dynamic surface control is used to to diagnose the actuator fault, then by using fuzzy logics the gains of the adoptive control law is designed to compensate for bias and LOE actuator faults.

In [28, 29], sliding mode control is used to solve the FTC problem. In [28], the integral-type sliding mode control (ISMC) approach is employed to a class of second-order nonlinear uncertain systems, while in [29] a common sliding surface is built based on a weighted sum of the different input matrices and an adaptive sliding mode controller is designed to accommodate the degradations in the actuators.

In [30], sufficient conditions for existence of optimal performance in faulty systems are defined in terms of nonlinear fault-dependent quadratic matrix inequality. Then, these equations are reduced to Riccati equations, which are used to derive the fault tolerant controller. Boskovic *et al.* considered decentralized control reconfiguration problem for high order actuators in [31] and proposed a decentralized FTC approach for a class of nonlinear systems considering first, second and third order dynamics for actuators.

Fault accommodation in networked systems is studied in [32–34]. Zhao *et al.* [32] considered the actuator fault problem for networked system with access constraints and developed a static scheduling approach to allocate resources in the network and designed the schedule-dependent Lyapunov function to address the actuator fault and access constraints simultaneously. In [33], random transfer delays in networks are modeled with Markov chains, then observer-based fault diagnosis and control scheme is developed to estimate the fault severity and to maintain stability of the faulty network. In [34], each subsystem is assumed to be low triangular, and finite-time Lyapunov stability theorem is used to design a robust state feedback control that guarantees global finite-time stability of the system.

In [35], FTC problem in polytopic uncertain systems with actuator faults was studied. The authors presented sufficient conditions for robust stability and system performance based on the concept of affine quadratic stability. In [36], FTC problem in continuous-time piecewise affine (PWA) systems was investigated and sufficient conditions for existence of virtual actuators and virtual sensors, which guarantee closed-loop stability and reference tracking in presence of actuator and sensor faults were presented with a set of LMIs. In another study fault tolerant control in discrete-time stochastic systems was considered [37]. Simandl *et al.* [37] developed the optimal solution for active fault detection and active fault tolerant control in discrete-time stochastic systems. They employed an award/punishment strategy based on the correct/wrong behaviour of the system to address the fault problem.

In [38–42], FTC problem for particular applications are considered. In [38], Yang *et al.* considered the FTC problem in an electric vehicle. They combined the linear-quadratic control approach with the control Lyapunov function technique and proposed an optimal fault tolerant controller, which guarantees both path tracking and system performance in an electric vehicle with input constraints, actuator faults, and external resistance. In [39], a robust adaptive controller was developed to address actuator fault, component fault, input constraint and disturbance problems in spacecraft attitude simultaneously. Two parameters were adapted and the robustness to actuator full failure was also investigated. Ciubotaru *et al.* [40] studied the FTC problem for Boeing 747 aircraft. They applied different methods to this aircraft and compared the effectiveness of model matching approaches (Exact, Pseudo-Inverse Method/PIM, Modified Pseudo-Inverse Method/MPIM) in the short-period mode. In [42, 43], actuator fault accommodation in spacecraft was studied. In [42], first model reference adaptive control approach is employed to design a three-axis virtual control and then using min-max optimization the control signal by redistributing the control among the remaining actuators, whereas in [43], fuzzy logic along with adaptive control approach is employed to overcome actuator faults. In [41], adaptive neural network based fault tolerant control is proposed to compensate for external disturbances as well as actuator faults. For that purpose, radial basis neural network is employed and a nonlinear observer is designed to estimate the uncertainties and then an adaptive control law is designed to ensure system stability.

1.2.2 Cooperative Control in Multi-agent Systems

Cooperative control of multi-agent systems has attracted a lot of attentions in the past few years. Generally speaking, the cooperation in a networked multi-agent system is performed through three levels, namely, high level, mid-level and low level. The high-level includes the task assignments, timing and scheduling. The mid-level, deals with the formation keeping, consensus achievement and rendezvous, flocking or swarming, containment, synchronization and output regulation. Finally, the low-level refers to the communication management, data acquisition and agents energy management. The aim of this thesis is to design a cooperative control law for the networks of autonomous agents subject to actuator faults, which is the mid-level of team cooperation.

Cooperative control strategies can be classified into two main categories centralized and distributed (decentralized) depending on the amount of information that agents send or receive. In centralized approaches each agent has access and shares its information with all the other agents. Developing and design of these approaches are quite similar to the design problem of a single agent system but they require large amount of communication, therefore they are only applicable when the number of agents is limited. On the other hand, in the distributed strategy it is assumed that each agent has only access to data of its neighboring team members. Due to the cost and complexity of agents communication, unless there are very few agents, distributed approaches are more preferred. However, the entire team performance depends on the agents communication and their cooperations. Therefore, there is always a compromise between the amount of information that agents share and the network performance.

In this research work, the objective is to ensure that the entire team reaches a consensus (or reach a common goal), therefore, our main focus is to review the recent literature on consensus based algorithms and applications of consensus control such as formation control [44–49], flocking [50, 51], rendezvous [52, 53] are briefly reviewd.

Formation Control: In formation control, the main objective is to develop a control strategy which ensures that the agents in the network achieve a predefined geometry and possibly follow a desired trajectory that is provided by a supervisor. As the posture is unchanged during the mission, entire team acts like a rigid body. Generally, formation architectures can be classified into five different architectures: (i) Multi-Input Multi-Output (MIMO) approach [44], (ii) leader-follower approach [45, 46], (iii) virtual structure approach (or virtual leader), in which the entire formation is considered as a virtual structure [47], (iv) cyclic approach [48], and (v) behavioral approach [49].

In the MIMO architecture, the entire network is considered as a single multi-input multi output system and the formation control problem can be solved by using the design approaches for single agent systems. On the other hand, in leader-follower architecture, one or several of the agents are considered as the leader(s) which define or have access to the reference trajectory and the other agents are called followers and receive the information by sharing information among themselves. The main advantage of this structure is that the design is not complicated and the desirable trajectory can be guaranteed by defining the leader trajectory.

In the virtual structure formation, to ensure that the team follow the desired trajectory, first a virtual structure is defined. Then, the relation between the states of the virtual dynamics and the real agents are obtained and finally the control laws for each agent are developed. This structure is very useful for fixed trajectory, however if the trajectory is time varying then defining the dynamics of the virtual structure and the states' relations becomes complicated.

In the cyclic architecture, the agents are connected to each other in a cyclic form and each agents control depends on the other agents control. This makes a loop and the stability analysis for the system becomes challenging. Finally, in the behavioral approach a command is developed for various objectives e.g. obstacle avoidance, or formation. Then each agent control law is obtained as a wighted average of the developed commands. Using the average command makes the design simple but it also can lead to uncommon or sometimes strange behaviour for the system.

Flocking and Rendezvous: Flocking [54] is a form of collective behavior of large number of interacting agents with a common group objective, while in rendezvous [52, 53] the aim is to reach a certain objective in a specific time. This cooperative behaviour have applications in parallel and simultaneous transportation of vehicles, delivery of payloads, performing military missions like surveillance, and reconnaissance .

Since flocking is mainly for a large number of agents, having a collision free and obstacle avoidance maneuver is an important aspect of the design. In [54], a flocking protocol is defined for a team of agents with point mass dynamics. The authors propose a graph theoretic framework for flocking in presence of obstacles. The proposed protocol is based on analyzing the Reynolds rules and enforce agents to converge to a weighted average position. The same problem is then considered in [55,56] for fixed topology and dynamic networks of agents and flocking protocols for collision free alignment in the heading of the agents are proposed.

Consensus: The main objective of consensus algorithm is to ensure that a group of agents reach an agreement upon specific quantities, therefore it can be considered as cooperative decision making algorithm. Cooperative decision making problem in a team dates back to the work of Marschak [57] in 1955 which was then followed in [58] in 1962 and [59, 60] in 1972. However, these problems received the attention by people in control theory almost three decades later [45, 61–64] for a team of first/second order simple agents.

In [61], the performance of a team of first order agents is considered. By using graph theory, matrix theory and Lyapunov stability analysis the performance of the team as well as convergence achievement in the team is investigated. The authors considered both fixed and switching topology network and obtained the relation between the maximum tolerable delay and eigenvalues of the network graph Laplacian matrix.

In [45], coordination problem in a team of single integrator agents under undirected network topology is considered. The notion of average heading is introduced and heading alignment in both leaderless and leader-follower network are studied. In [62], the results of [45] are extended to dynamic directed topology networks. It is also shown that agents can reach a consensus if the union of the network topology graph has a spanning tree.

In [63], consensus problem is extended to a team of linear second order agents. Convergence analysis for second order consensus is provided and then necessary and/or sufficient conditions for achieving consensus in the network is obtained. In [64], the results of [63] are extended to a team of spacecraft in deep space. A PD-like control law is proposed to ensure attitude alignment with either zero or non-zero final angular velocity in the team.

The works of [45, 61–64] are then extended to a team of linear agents with general dynamics, nonlinear dynamics and different control framework including optimal control, adaptive and intelligant approaches, H_∞, and sliding mode are employed to solve the consensus achievement problem.

Decentralized optimal control is among the most attractive approaches in cooperative control design [65–75]. In most of the proposed solutions the objective is to minimize the entire team performance index. This objective is achieved by either defining a unique performance index for the team or defining an individual index for each agent. Due to the dynamics of the agents the solution of this problem would be different from that of cooperative decision making in which agents do not have dynamics and each one can be considered as a state of the entire system.

In [65–68], the optimal consensus control problem is formulated as an optimization problem subject to a set of LMIs. In [65, 66], the optimal gains are obtained by developing a Riccati equation for the entire team dynamics. Then, in order to impose the network structure, the Riccati equations are transformed into a set of LMIs and consensus achievement is guaranteed as the feasibility of the set of LMIs. In [67], an observer-type consensus protocol based on relative output measurement is proposed. Consensus problem in the team is formulated as the stability of a set of low order matrices and then the conditions for stability of these matrices or consensus agreement are presented as LMIs. In [68], the dynamics of the entire network is decomposed into consensus space and orthogonal subspaces. Then, using an H_2 design strategy and LMI formulation stability and consensus achievement are guaranteed.

In [69, 70], inverse optimal control theory and partial stability are employed to develop distributed global optimal control laws. In [69], a sufficient condition on the network graph topology for existence of distributed linear protocols that solve a global optimal LQR control problem is provided. Then, a class of graphs that satisfy this condition is introduced and it is shown that the optimal solution can be global only if the performance index includes the network topology. In [70], necessary and sufficient conditions are given for solving the cooperative optimal control problems for leaderless and leader following multi-agent systems. The proposed control locate all closed-loop eigenvalues of the multi-agent system in the desired region as well as ensures consensus agreement and optimizes the global team performance.

In [71–73], the optimal control problem is formulated as a game and a Nash equilibrium is obtained. In [71, 72], the optimality of the presented control scheme is shown based on a quadratic-invariance argument while in [73], using matrix theory, the conditions are provided as a feasibility of a set of LMIs.

In [74, 75], LQR-based framework is developed to optimize the team performance. In [74], at each node, the models of node's neighbors are used to predict the behavior of neighbors and a distributed control law is proposed to obtain the optimal solution. The proposed control law is synthesized using a simple local LQR design and it is shown that the distributed control law is stabilizing and parameters of the local LQR cost function do not affect stability. In [75], two global cost functions, namely, interaction-free and interaction-related cost functions are proposed. By using interaction free cost and LQR-based approach the optimal Laplacian matrix which corresponds to the complete graph is obtained. Then, the interaction-related cost function is used to obtain an optimal scaling associated with the pre-specified symmetric Laplacian matrix.

Adaptive control is another approach for developing consensus based algorithms [76–93]. The main advantage of these approaches is that they can compensate for the small changes in model parameters, however the control gains need to be updated continuously and that increases the complexity. Adaptive approaches can be classified into Lyapunov-based and parameter-estimation based approaches. In [76–79] neural adaptive approaches are employed to solve cooperative control problem in multi-agent systems. In [76], neural network approximation is employed and a robust adaptive approach is developed to address the leaderless consensus achievement in a network of uncertain nonlinear first order multi-agent systems. In [77], the authors consider a team of Lagrangian vehicles with directed communication graph topology. The agents dynamics and the dynamics of reference system are assumed to be unknown. The control law consists of two parts: a proportional derivative (PD) control and an adaptive tuning law. The unknown dynamics are estimated locally using neural network and the Lyapunov technique is used to design the control gains. In [78], a neural adaptive control scheme is presented to address tracking problems in a heterogenous networked nonlinear systems with unknown dynamics and disturbances. The proposed control is developed based on Lyapunov technique and is applicable to first-order as well as high-order nonlinear systems. In [79], a leaderless network of high-order nonlinear systems is considered. First, a filtered output is developed and agents high-order model is transformed into first-order systems, then the system nonlinearities are approximated using neural networks. Based on the estimated dynamics, an adaptive control law is proposed to ensure that agents achieve consensus.

In [80], a Lyapunov-based adaptive approach is proposed to guarantee that a team of first-order non-identical nonlinear agents can follow the unknown leader. The results of [80], are then extended to a team of high-order agents under fixed and strongly connected topology network in [81]. In [82], an adaptive pinning-control approach is presented to ensure state synchronization in a team of nonlinear delayed systems.

In [83–85], adaptive consensus control strategies are developed using parameterization. In [83] a linearly parameterized multi-agent system with unknown identical control directions under undirected network topology is studied. Under the assumption that all agents have the same direction, the author propose a new Nussbaum type function to estimate the unknown control direction and then employ this function to design the adaptive control law for the first-order and second-order multi-agent systems. In [84], a leader-follower team of high-order integrators with nonlinear uncertainties is considered. The nonlinear dynamics of the agents and the leader control are parameterized by using basis functions and then a distributed adaptive control law is developed to stabilize the tracking error and adaptation parameters. In [85], a linear team with non-identical unknown uncertainties is studied. The unknown dynamics is parameterized and each agent identifies its disturbance as well as the leader input. Then, a fully distributed adaptive control is developed that ensures tracking in the network. Furthermore, the results are extended to the case that agents are also subject to parameter uncertainties.

In [86], a decentralized adaptive control scheme is proposed to guarantee finite time consensus in a leader-follower network where the leader's control input signal is unknown and nonlinear. In [87], an adaptive consensus coordination problem for heterogenous unknown nonlinear multi-agent systems in networks with jointly connected topologies is studied. The author introduce persistent excitation (PE) condition for regressor matrix and a decentralized algorithm is developed for each agent to estimate the unknown parameters. Then, sufficient conditions for consensus achievement and parameter error convergence are obtained. In [88], an adaptive nonlinear control law is proposed to ensure consensus in a directed topology leader-follower network when the leader is unforced. The proposed law only uses relative information and do not require any global information about the network.

In [89, 90], model reference adaptive control is employed to solve the consensus problem in the leader-follower network. In [89], a network of single input single output agents is considered. The proposed approach employs output measurements and treat the estimation errors of tuning parameters as disturbances and by employing the theory of H_∞ control, stability of the tracking errors are guaranteed. In [90], a network of agents with partly unknown parameters and subject to bounded external disturbances is considered. First by employing model reference adaptive control approach, a base control law for disturbance free environment is proposed. Then, a compensator is added to the base control to compensate for disturbances.

In [91–93], L_1 adaptive control which is based on model reference adaptive control is employed to improve the transient performance of multi-agent systems. In [91,92], a two agent network with uncertainties is considered. A local desired trajectory and extended dynamics is defined for each agent. Using adaptive L_1 control approach each agent reaches its local objectives and it is shown that agents reach a consensus if each one follows its own local trajectory. In [93], the L_1 adaptive control is employed for leaderless and leader-follower nonlinear uncertain multi-agent system. The control law is designed in two steps: first a control is developed for ideal agents without any uncertainties and then an adaptive control law to manage the effects of uncertainties is developed.

The real network of multi-agent systems is always subject to different external disturbances and uncertainties. These disturbances may affect the total team performance and behavior. Motivated by the above the team behavior and the control design problem in a team subject to disturbances has been studied in the past few years. In [94–101], linear multi-agent systems and [102, 103], nonlinear team of agents subject to disturbances are studied.

In [94,95], the H_∞ control design in a leaderless team of agents with switching topology is studied. In [94], an output feedback control for undirected networks is developed whereas in [95], state feedback control for directed network is studied. In [96, 97], a team of high-order integrators subject to L_2 disturbances is considered. An $L_2 - L_\infty$ consensus control is proposed which its gains are obtained as solution to a set of LMIs. In [96], the H_2/H_∞ consensus control is transformed to the problem of stabilizing a set of N linear systems which are developed based on the network topology and agents dynamics. In [97], rather than L_2 disturbance, the L_1 disturbances is also considered.

In [98, 104], the team performance in presence of disturbances is studied. In [98], transient performance in a team subject to disturbances is studied and sufficient conditions for existence of H_∞/H_2 control are obtained. In [104], tracking control in linear multi-agent systems in presence of environmental disturbances is formulated as a multi-player zero-sum differential graphical game. Then, a reinforcement learning algorithm is developed to select the optimal gains that guarantee synchronization while the performance remains in an acceptable level. In [100, 101], modal decomposition approach is used to design a distributed controller for a team of identical agents. A distributed control law is proposed and the gains of the control law are obtained using LMIs along with H_2 and H_∞ criteria.

In [99], output feedback problem in a team of non-introspective team with directed topology is studied. Using matrix theory and H_∞ control theory the conditions to reduce the effects of external disturbances to a small arbitrary level are developed. In [105], homogeneous parameter-dependent (HPD) Lyapunov functions and sum of squares (SOS) technique are employed to develop a consensus control for a team where the agents are subject to polytopic uncertainties, external disturbances and time-varying/uncertain network topology.

In [102, 103], consensus control in a team of nonlinear agents with parameter uncertainties is studied. In [102], an undirected network subject to time-varying communication delays is considered and using H_∞ control theory sufficient conditions for existence of state feedback control that guarantees desirable disturbance attenuation bound are derived, whereas in [103], a team under directed network topology is studied and an output feedback control is proposed to reject the environmental disturbances.

In most of the above work the network conditions are assumed to be ideal, i.e. fixed topology. In real world as agents are moving their neighbour set may change. Generally, two main class of switching network topologies are considered in the literature: (1) the case that networks switch among a finite set of networks that each of them is either connected or has a directed spanning tree, and (2) the case that at each time the network is allowed to be disconnected but the union of graphs on the time interval is connected or has a directed spanning tree which is called jointly connected.

Consensus problem in switching topology networks of single-order agents has been studied in [45, 62, 106, 107]. In [45], Jadbabaie *et al.* considered a leaderless as well as leader-follower team under undirected topology and discussed the convergence and consensus properties in continuous-time (CT) as well as discrete-time (DT) agents. It was shown that the network reaches a consensus if the union of topology graphs is connected during the time interval. In [106], common Lyapunov function criterion is employed to investigate convergence analysis of average consensus in a leaderless network of CT/DT agents under directed, strongly connected and balanced network. In [107], a set-valued Lyapunov scheme is used to derive necessary and/or sufficient conditions for consensus achievement in networks of DT agents with unidirectional time-dependent communication links. In [62], Ren *et al.* relaxed the previous assumptions on the network topology and showed that the team can reach an agreement if the union of the collection of topology graphs during some time intervals has a spanning tree.

In [108–112], second-order consensus problem in switching topology networks is studied. In [108], a leaderless network with jointly connected interconnection and in [109], both leaderless and leader-follower network are studied. Network topologies are assumed to switche between a finite set of topologies with a directed spanning tree and sufficient conditions for consensus achievement are obtained by using Lyapunov functions. In [110–112], consensus achievement in switching topology networks and in presence of communication delays is considered and sufficient conditions for average consensus are obtained as a set of LMI. In [110, 111], an undirected jointly-connected network whereas in [112], a weakly connected and a balanced network is considered.

In [113–117], a team of general LTI systems with switching topology is studied. In [113], consensus problem in team of high order integrator agents with undirected switching topology is studied. In [114, 115], sufficient conditions for consensus in a team of leaderless and leader-follower LTI marginally stable agents with jointly connected network topology are studied. In [114], continuous-time dynamic agents and in [115], discrete-time dynamic agents are assumed. In [116], a team of leader-follower agents with unforced leader under a network that switches between a set of networks having directed spanning tree is considered. Using Lyapunov functions and M matrix theory sufficient conditions for consensus achievement are obtained.

In [118–121], a team of nonlinear agents with switching topology is considered. In [118], a distributed control law is proposed to guarantee tracking in a team of Euler-Lagrange agents where the network topology switches between a set of undirected connected networks and in [119], the results of [118] are extended to the case that agents are subject to environmental disturbances and parameter uncertainty. H_∞ optimal control law is proposed to address state synchronization and trajectory tracking. In [120], the problem in a team of nonlinear agents in networks with switching topology and bounded delay is studied and the sufficient conditions for consensus are obtained using multiple Lyapunov functions. In [121], a team of Lipschitz-Type agents are considered and a distributed control law is proposed to solve the consensus tracking in the network.

Any multi-agent system consists of a group of physical systems. Even if all the agents were developed in a same manufacture, the values of their parameters may change during time depending on the work conditions. Therefore, when a team of agents are employed to perform a single task it is very possible that agents have different parameters. Furthermore, in some application cooperation among different vehicles may be necessary such as cooperation among underwater vehicles and a ship. Motivated by that, in recent years the leader follower and leaderless consensus in a team of identical agents is extended to output regulation and output synchronization in a team of non identical (heterogenous) agents [122–137].

In [122–126], internal model principals are employed to solve output regulation problem. In [122], under the assumption that the network topology graph is cycle free distributed state feedback as well as dynamic output feedback control protocols were proposed to ensure that outputs of LTI dynamics agents with additive uncertainties track outputs of the external system whereas in [123, 124], the cycle free assumption is relaxed. However, in [123] it is assumed that state measurements are available and in [124], it is assumed that the upper bound of uncertainties norm is known priori. In [125, 126], the problem for a team of agents with time-varying uncertainties are considered and robust dynamic output feedback control laws are provided to regulate outputs of uncertain agents. In [126], a team of single input single output agents is considered and then in [125] the problem is generalized to a team of general LTI systems.

In [127, 128] output synchronization problem in time varying network topology is considered. In [127], a team of LTI agents is considered and sufficient conditions for output synchronization are derived and then based on the provided conditions, a distributed control law is proposed that only employs relative measurements. In [128], a team of linear parameter-varying (LPV) agents is considered and the proposed control law employs agents individual measurements to ensure outputs are synchronized.

In [129, 130], a high gain approach is proposed to solve output regulation problem in a heterogenous team with switching topology. A local objective and a group objective are defined for each agent. Then, an observer based control law is proposed to stabilize the tracking error for each agent which is determined based on local and global objectives. In [131, 132], a necessary and sufficient condition for synchronizing outputs of a team of LTI agents is provided and it is shown that a control can be designed to synchronize the outputs if and only if their dynamics intersect. Then, a state feedback control law is proposed that asymptotically stabilizes the tracking error between outputs of general LTI systems and external system. In [138] the results of [131, 132] are extended to the synchronization problem and a reference system is virtually constructed.

In [133–137], matrix theory and graph theory are used to address output regulation problem in non-introspective agents and without exchanging control states. In [133, 134], an observer based decentralized control is proposed for right-invertible agents with fixed topology. In [135], the previous work is extended to a network of right invertible agents with time varying topology. In [136, 137], the results of [135] are extended to the network that agents are also subject to environmental disturbances and a low-and-high gain approach is used to address the output regulation in heterogenous system with time-varying topology.

1.2.3 Fault Recovery in Multi-Agent Systems

The problem of fault recovery and control reconfiguration in multi-agent systems is more challenging as compared to that of a single agent as discussed in Section 1.2.1 and has been studied only in recent years. Cooperative fault recovery problem differs from cooperative control problem in multi-agent systems (discussed in Section 1.2.2), in the sense that cooperative control approaches (except in adaptive approaches) are mainly designed offline and in these approaches control gains are fixed or pre-specified. However, in cooperative fault recovery the control gains are developed and designed online and by using the local information of agent dynamics as well as the information provided by the Fault Detection, Isolation and Identification (FDI) modules. That is, agents update their control laws by using local information and once a fault occurs, to ensure that the entire team objectives are satisfied. The recent literature in this area are discussed in the following.

In [139,140], control reconfiguration in leader-follower network under fixed topology is studied. In [139] an adaptive mechanism is proposed to adjust the agent control gains once an actuator fault occur in a team of multi-agent system under undirected topology, parameter uncertainties and external disturbances. In [140], adaptive control approach is employed to tackle output tracking in high-order nonlinear multi-agent systems subject to LOE and additive time-varying faults. In [141–148], the control reconfiguration problem in networks with fixed graph topology is considered. In [141, 142], a high-level supervisor is proposed to address the formation flight problem in networks subject to loss of effectiveness faults (LOE). In [143–146] adaptive control strategies are employed to ensure that agents reach a consensus even when agents are subject to actuator faults that do not cause rank deficiency in the input channel matrix. In [149], two re-coordination strategies for formation keeping motion in a team of wheeled mobile robots subject to actuator faults is presented. In the first approach, the faulty agent is removed from the motion and then using Hungarian algorithm the task is assigned among the remaining agents whereas in the second one, the formation is preserved with performance degradation.

In [143–145], leaderless multi-agent systems are considered. In [145], a second-order nonlinear leaderless team is studied and actuator faults are considered as uncertainties. The norm bounding approach is used to transform the uncertainties into a virtual scaler. Then, using this scaler and Lyapunov functions the stability of the tracking error is guaranteed. In [143], neural networks are employed to estimate the actuator fault severities as well as dynamics uncertainties. Then, by using graph theory and Lyapunov function an adaptive law is proposed to ensure that the team of nonlinear second-order agents reach a consensus even when agents are subject to actuator faults. In [144] a team of agents with general LTI dynamics is considered and online adaptive laws are developed to compensate the uncertainties and faults.

In [95, 150, 151], control reconfiguration problem in switched topology networks is studied. In [150], a reconfigurable control strategy is proposed to guarantee state synchronization in a team of Euler-Lagrange systems whereas in [95, 151] a team of LTI systems subject to the LOE actuator faults is studied and a two level, i.e. the agent-level and the team-level reconfiguration strategies are presented.

In [118, 119, 152, 153] control reconfiguration in Euler-Lagrange networked systems is studied. In [152], an adaptive law is proposed to accommodate the changes in the system parameter due to faults in actuator faults and the reconfiguration is performed without the need for FDI module. On the other hand, in [118, 119, 153] the information provided by FDI module are employed to reconfigure the control laws and ensure that the team can cooperate even in presence of actuator faults.

1.3 General Problem Statement and Thesis Outline

The main objective of this thesis is to design control reconfiguration strategies for a team of autonomous vehicles. A team of linear multi-agent systems subject to three types of actuator faults, namely loss of effectiveness, outage and stuck is considered. The main concern in the development of the proposed reconfiguration strategies are as follows:

1) The proposed strategy should guarantee that agents can reach a consensus even when they are subject to different actuator faults.
2) Each agent can only communicate with few other agents (its nearest neighbors) and is unaware of the the total network behaviour therefore, the reconfiguration strategy should only employ the local information and without any external intervention.

3) The faults can cause rank deficiency in the system control channel matrix, therefore the structure of the system can change.

4) The team performance index should remain in an acceptable level.

In Chapters 3 to 6 of this thesis, the above concerns and problems are addressed. In the following a summary of these chapters are given:

- In Chapter 3, a distributed reconfigurable control law for a team of multi-agent systems is provided. On-line distributed control reconfiguration strategies are developed that employ only nearest neighbors information to guarantee the team consensus while minimizing a local cost performance index. Towards this end, the distributed Hamilton-Jacobi-Bellman equations for the faulty agent are derived and then reconfigured controllers are designed by solving these equations subject to the faulty agent dynamics and network structure constraints to ensure fault accommodation of the entire team. In the next three chapter, the above problem is extended to non-ideal conditions.

- In Chapter 4, we extend the problem of Chapter 3 to a network with directed communication topology and subject to disturbances. Furthermore, the outputs of the faulty agent are guaranteed to follow those of the healthy system. A distributed control reconfiguration strategy is proposed such that in presence of actuator faults the state consensus errors remain bounded and output of the faulty system behaves exactly the same as that of the healthy system. Furthermore, the specified H_∞

performance bound is guaranteed to be minimized in presence of bounded energy disturbances. The gains of the reconfigured control laws are selected first by employing a geometric control approach where a set of controllers guarantees that the output of the faulty agent imitates that of the healthy agent and the consensus achievement objectives are satisfied. Next, the remaining degrees of freedom in the selection of the control law gains are used to minimize the bound on a specified H_∞ performance index.

- In Chapter 5, the results of Chapter 3 are extended to a network with directed switching topology communication. The proposed control strategy ensures that the agents reach a consensus and an upper bound for the team performance index is guaranteed when the agents are subject to actuator faults and there are uncertainties and unreliabilities in the fault severity estimation. By using quadratic and convex hull (composite) Lyapunov functions, two strategies are provided to design and select the gains of the proposed distributed control laws such that the team objectives are guaranteed.

- In Chapter 6, the results of Chapter 3 are extended to a team of non-identical agents subject to limited measurement. A distributed output feedback control strategy is proposed which guarantees that agents output follow the outputs of the exo-system and the agents states remain stable even when agents are subject to actuator faults. Towards that end, the output regulation problem is transformed into two stability problems and Lyapunov functions and Gershgorin theorem are employed to obtain conditions to solve these two problems.

1.4 Thesis Contributions

In this thesis, control reconfiguration problem in multi-agent systems is tackled. The main contributions of this research are as follows:

(1) In this work we consider three types of actuator faults, namely loss of effectiveness, outage and stuck. The outage and stuck faults can cause rank deficiency which change the model structure unlike the work [139, 140, 143–146], where actuator faults is considered as parameter uncertainties which do not change the structure.

(2) The proposed control laws are fully distributed, that is the reconfigured control gains are designed by only employing the local information unlike [141, 142], which a high level supervisor is assumed to intervene and redesign the control gains once agents become faulty.

(3) A cost based reconfiguration strategy based on Hamilton Jacobi equations is presented to ensure consensus in a team with a fixed topology network and minimizing the local cost performance index. In the current literature only consensus problem is studied and the cost performance index is not considered into account.

(3) The notation of auxiliary agents are introduced and then employed to address H_∞ performance output following in the team subject to simultaneous actuator faults. The H_∞ performance consensus problem has been studied in the current literature, however output following in the team of agents subject to actuator faults has not been addressed.

(4) A distributed cost guaranteed reconfiguration strategy for switching topology networks is proposed to minimize the upper bound of the team performance index while the team consensus is ensured.

(5) A reconfigurable control law is proposed to ensure output regulation in a team of non-identical agents by using outputs measurements unlike the current literature which mostly consider control reconfiguration in a team of identical agents.

Chapter 2

Background

2.1 Graph Theory

The communication network among N agents can be denoted by a graph. A directed graph $\mathcal{G} = (V, E)$ consists of a nonempty finite set of vertices $V = \{v_1, v_2, ..., v_N\}$ and a finite set of arcs $E \subset V \times V$. The i-th vertex represents the i-th agent and the directed edge from i to j is denoted as the ordered pair $(i, j) \in E$, which implies that agent j receives information from agent i. In this case the vertex i is called the parent vertex and the vertex j is called the child vertex [154].

A graph is called bidirectional if for a pair of nodes and edge (v_i, v_j), there exists an edge (v_j, v_i), otherwise it is called directed graph or digraph. A directed path from agent

i to agent j is a sequence of arcs in E, $(v_i, v_1), (v_1 v_2), \ldots, (v_k, v_j)$. A path is a cycle if $v_i = v_j$. A directed tree is a digraph, where every vertex has exactly one tail except for one special vertex without any tail. In the other words, a connected subgraph of $\mathcal{G}$ which contains no cycle is a tree. A directed graph $\mathcal{G}$ has a directed spanning tree if there exists a node r (a root) such that all other nodes can be linked to r via a directed path. If there exists a path between any pair of distinct vertices the digraph is called to be strongly connected, whereas for undirected graph, it is called to be connected.

Neighbours of each agent i in the network are denoted by $\mathcal{N}_i = \{j|(j,i) \in E\}$. We also define the receiver neighbour set, N_i^r, which represents the set of agents that receive data from agent i by $\mathcal{N}_i^r = \{j|i \in \mathcal{N}_j\}$. The adjacency matrix of the graph $\mathcal{G}$ is given also by $G = [g_{ij}] \in \mathbb{R}^{N \times N}$ where $g_{ij} = 1$ if $j \in \mathcal{N}_i$, otherwise $g_{ij} = 0$. For undirected graph, we have $g_{ij} = g_{ji}$. The Laplacian matrix for the graph $\mathcal{G}$ is defined as $L = D - G$, where $D = \text{diag}\{d_i^{in}\}$ and $d_i^{in} = \sum_{j=1}^{N} g_{ij}$.

2.2 Consensus Achievement in Multi-agent Systems

The main objective of the consensus problem is to ensure that all the agents reach a common agreed value that is collectively decided by the agents interactions in the leaderless (LL) network topology or specified by the leader in the leader-follower (LF) network architecture. Below, we provide the mathematical representation of the agents corresponding to these two architectures.

Leaderless (LL) Network: Let the network consist of N agents where each agent is governed by the following dynamical model

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t),\ i = 1, \ldots, N, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ denote the i-th agent state and control input, respectively. The i-th agent consensus error signal is now defined according to

$$e_i(t) = \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) = (l_i \otimes I_n) x^{LL}(t), \tag{2.2}$$

where $x^{LL^{\mathrm{T}}}(t) = [x_1^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$.

Leader-Follower (LF) Network: In the leader-follower network architecture the team consists of N followers where each agent dynamics is governed by equation (2.1) and the leader whose dynamics is given by

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t), \tag{2.3}$$

where $x_0(t) \in \mathbb{R}^n$ and $u_0(t) \in \mathbb{R}^m$ represent the leader state and control input, respectively. In the LF architecture some of the followers that are designated as pinned agents are communicating with the leader and receive data from it directly. Since the leader does not receive any feedback from the followers, the leader control law can be selected as $u_0(t) = K_0 x_0(t) + r(t)$, where K_0 is the state feedback gain and $r(t)$ represents the desired trajectory of the leader. The consensus error signal for each follower is now defined as

$$\begin{aligned} e_i(t) &= g_{i0}(x_i(t) - x_0(t)) + \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) \\ &= (l_i \otimes I_n) x^{LF}(t),\ i = 1, \ldots, N \end{aligned} \tag{2.4}$$

where $x^{LF}(t) = [x_0^{\mathrm{T}}(t), x_1^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $g_{i0} = 1$ if the agent i is a pinned agent and is zero otherwise.

For the ease of notation we will use the notation $x(t)$ to address either $x^{LF}(t)$ or $x^{LL}(t)$. We are now in a position to formally introduce the objective of the consensus achievement problem.

Definition 2.1. *The team of N agents will reach a consensus if $e_i(t) \to 0$ as $t \to \infty$ for $i = 1, \dots, N$, implying that for the leaderless network architecture $x_i(t) \to x_j(t)$ as $t \to \infty$ for $i, j = 1, \dots, N$ and for the leader-follower network architecture $x_i(t) \to x_j(t) \to x_0(t)$ as $t \to \infty$ for $i, j = 1, \dots, N$.*

2.3 Hamilton Jacobi Bellman (HJB) and LQR Problem

Hamilton Jacobi Bellman (HJB) is a useful tool to find an optimal control input specially for nonlinear systems. Consider the system dynamics as

$$\dot{x}(t) = f(t, x, u),\ x(0) = x_0$$

and the corresponding cost function to be minimized as

$$J(t, x) = \int_0^\infty h(t, x, u)\mathrm{d}t,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $h : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $J : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$. The following optimization problem

$$\min J(t, x)\ \ s.t.\ \dot{x}(t) = f(t, x, u),\ x(0) = x_0,$$

has a solution if the following HJB equations have a solution [155, 156]

$$\mathcal{H}(t, x, u) = h(t, x, u) + J_x^*(t, x)^{\mathrm{T}} f(t, x, u), \tag{2.5}$$

$$u^*(t) = \arg\min_u \mathcal{H}(t, x, u), \tag{2.6}$$

$$\mathcal{H}^*(t, x, u) + \dot{J}^*(t, x) = 0, \tag{2.7}$$

where $J^*(t,x)$ is the optimal value of $J(t,x)$, $J^*_x(t,x) = \frac{\partial J^*(t,x)}{\partial x}$, $\dot{J}^*(t,x) = \frac{\mathrm{d}\, J^*(t,x)}{\mathrm{d}\, t}$ and $\mathcal{H}^*(t,x,u)$ is the optimum value of $\mathcal{H} : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. If $f = f(x,u)$ and $h = h(x,u)$, i.e. they are time-invariant, then $J^* = J^*(x)$ and $\dot{J}^* = 0$ and (2.7) reduces to $\mathcal{H}^*(x,u) = 0$.

If the plant is linear time-invariant

$$\dot{x}(t) = Ax(t) + Bu(t),$$

and the performance index is considered to be quadratic

$$J(x) = \int_0^\infty (x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t))\mathrm{d}t$$

the corresponding HJB equations will reduce to

$$\mathcal{H}(x,u) = x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t) + J_x^{*\mathrm{T}}(x)(Ax(t) + Bu(t)), \tag{2.8}$$

$$u_i^*(t) = \arg\min_u \mathcal{H}(x,u) = -R^{-1}B^{\mathrm{T}}J_x^*(t,x), \tag{2.9}$$

$$\mathcal{H}^*(x,u) = 0, \tag{2.10}$$

where $R \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ are positive definite matrices. By substituting $u(t)$ from (2.9) into (2.10), one can find the optimal control signal as $u(t) = -R^{-1}B^{\mathrm{T}}Px(t)$, where P is the solution of the following Algebraic Riccati Equation (ARE)

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q = 0. \tag{2.11}$$

The above problem is called LQR problem and can be formulated as an optimization problem subject to a set of Linear Matrix Inequalities (LMI)s. This formulation is very useful if the optimal controller with special structure is as of interest.

Lemma 2.1. *[157] The optimization problem:*

$$\begin{aligned} &\max \operatorname{trace}(P) \\ &s.t. \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} \geq 0, \\ &P > 0 \end{aligned} \tag{2.12}$$

has a solution if and only if the ARE equation (2.11) has a solution and for $R > 0$ and $Q \geq 0$, the unique optimal solution to this problem is the maximal solution of the ARE (2.11).

2.4 Actuator Fault Types

Consider the following linear time-invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.13}$$

$$y(t) = Cx(t), \tag{2.14}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are system state and input signals, A, B and C are system dynamic matrices. If the input of the system (2.13) becomes faulty then the fault is called actuator fault. Actuator faults are generally classified into three types: [158]

(1) Loss of Effectiveness (LOE): The loss of effectiveness fault is characterized by lowering the actuator gain with respect to its nominal value. Therefore, the dynamics of the i-th faulty agent can be modelled as

$$\dot{x}_i^f(t) = Ax_i^f(t) + B^f u_i^f(t), \tag{2.15}$$

where $B^f = B\Gamma_i$, $u_i^f(t) = u_i(t)$, $\Gamma_i = \text{diag}\{\Gamma_i^l\}$, for $l = 1, \dots, m$, Γ_i^l represents the effectiveness of the l-th channel of the i-th agent, $0 < \Gamma_i^l < 1$ if the l-th actuator is faulty, and $\Gamma_i^l = 1$ if it is healthy.

Let us denote the fault severity estimate and the l-th faulty actuator estimation error by $\hat{\Gamma}_i^l$ and ξ_i^l, respectively, i.e. $\Gamma_i^l = \hat{\Gamma}_i^l + \xi_i^l$. Moreover, consider the binary parameter f_i^l as the fault indicator for the l-th actuator of the i-th agent in which $f_i^l = 1$ if the actuator is faulty and $f_i^l = 0$ if the actuator is healthy. Therefore, we have

$$B^f = B\hat{\Gamma}_i + Bf_i\xi_i, \tag{2.16}$$

where $\hat{\Gamma}_i = \text{diag}\{\hat{\Gamma}_i^l\}$, $\xi_i = \text{diag}\{\xi_i^l\}$, and $f_i = \text{diag}\{f_i^l\}$.

(2) Outage (Float): The float fault occurs when the actuator output "floats" with zero value and does not contribute to the control authority. The dynamics of the i-th agent with an outage fault in its k-th actuator can be represented as

$$\dot{x}_i(t) = Ax_i(t) + B^{fk}u_i(t),\ i = 1, \dots, N, \tag{2.17}$$

where $B^{fk} = \begin{bmatrix} b^1, b^2, \dots, b^{k-1}, 0, b^{k+1}, \dots, b^m \end{bmatrix}$ and b^k is the k-th column of the matrix B.

(3) Stuck (HOF/LIP): In stuck fault the actuator "freezes" at a certain value and does not respond to subsequent commands. This fault is also called HOF (Hard Over Fault) or LIP (Lock in Place Fault). The dynamics of the i-th agent under this fault type can be modelled as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i^f(t),\ i = 1, \dots, N, \tag{2.18}$$

where $u_i^f(t) = \begin{bmatrix} u_i^1(t), \dots, u_i^{k-1}(t), \underline{u}_i^k, u_i^{k+1}(t), \dots, u_i^m(t) \end{bmatrix}^{\mathrm{T}}$, and $\underline{u}_i^k = u_i^k(t_f)$

denotes the value of the stuck command.

2.5 Geometric Control Approach

The geometric control approach deals with subspaces and properties of systems are expressed in terms of linear spaces [159, 160]. Consider the system (2.13). Let $\mathcal{X}$ be a vector space and $A : \mathcal{X} \to \mathcal{X}$ a linear map. A subspace $\mathcal{J} \subseteq \mathcal{X}$ is said to be $A-invariant$ if

$$A\mathcal{X} \subseteq \mathcal{J}.$$

Given the linear map $A : \mathcal{X} \to \mathcal{X}$ and a subspace $\mathcal{B} \subseteq \mathcal{X}$ a subspace $\mathcal{V} \subseteq \mathcal{X}$ is $(A, \mathcal{B}) - controlled\ invariant$ if the following holds

$$A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}.$$

For any $(A, \mathcal{B}) - controlled\ invariant$ subspace $\mathcal{V}$ there exists a matrix F (called a $friend$ of $\mathcal{V}$) such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

The two useful subspaces in geometric control theory are maximum $(A, \mathcal{B}){-}controlled\ invariant$ and $(A, \mathcal{C}){-}conditioned\ invariant$ containing $\mathcal{B}$. The maximum $(A, \mathcal{B}){-}controlled\ invariant$ subspace contained in Im$\{C\}$ is obtained by the recursive algorithm

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{C} \\ \mathcal{V}_i &= \mathcal{C} \bigcap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}),\ i = 2, 3, \ldots \end{aligned}$$

The minimum $(A, \mathcal{C}) - conditioned\ invariant$ containing $\mathcal{B}$ can be obtained through the recursive algorithm:

$$\begin{aligned}\mathcal{S}_1 &= \mathcal{B}\\ \mathcal{S}_i &= \mathcal{B}\bigcap A^{-1}(\mathcal{S}_{i-1}+\mathcal{C}),\ i=2,3,\dots\end{aligned}$$

2.6 Switching Systems

Let $\sigma(t):[0,\infty)\to\{1,\dots,q\}$ denote the switching signal and suppose that it represents the network graph switches. The switching sequence $0<t^1<\cdots<t^{l_k}<t^{l_k+1}<\dots$ with $l_k=1,2,\dots$ is assumed to be continuous from right, that is for $t^{l_k}\le t<t^{l_k+1}$, $G(t)=G^\sigma=G^k$, $G^k\in\{G^1\dots,G^q\}$, where $G^k=[g_{ij}^k]$ and g_{ij}^k represents the arc between the i-th and the j-th agents in this interval. In this case, the Laplacian matrix will be denoted by $L(t)=L^\sigma$, where for $t^{l_k}\le t<t^{l_k+1}$, $L^\sigma=L^k, L^k\in\{L^1,\dots,L^q\}$. The Laplacian matrix is partitioned into $L^\sigma=\begin{bmatrix}L_{11}^\sigma & L_{12}^\sigma\\ L_{21}^\sigma & L_{22}^\sigma\end{bmatrix}$ and $L^k=\begin{bmatrix}L_{11}^k & L_{12}^k\\ L_{21}^k & L_{22}^k\end{bmatrix}$, where $L_{11}^k=0$, $L_{12}^k=0$, L_{21}^k is an $N\times 1$ vector and represents the leader's links to the followers, and L_{22}^k is an $N\times N$ matrix that specifies the followers' connections.

2.7 Output Regulation

The dynamics of the i-th agent in the network is considered as

$$\dot{x}_i(t)=A_ix_i(t)+B_iu_i(t),\ i=1,\dots,N, \tag{2.19}$$

$$y_i(t)=C_ix_i(t), \tag{2.20}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{p_i}$ and $y_i \in \mathbb{R}^m$ denote the i-th agent state, control and outputs. The exo-system dynamics is given by

$$\dot{\omega}(t) = S\omega(t), \tag{2.21}$$

$$v(t) = R\omega(t), \tag{2.22}$$

where $\omega \in \mathbb{R}^r$ and $v \in \mathbb{R}^m$. This exo-system can also be considered as a (virtual) leader for the team. It is assumed that only a set of agents has access to the exo-system and its measurements. The i-th agent regulation error is defined as

$$\begin{aligned} e_i(t) &= g_{i0}(y_i(t) - v(t)) + \sum_{j \in \mathcal{N}_i} (y_i(t) - y_j(t)) \\ &= (l_i \otimes I_m)Cx(t) - g_{i0}R\omega(t), \end{aligned} \tag{2.23}$$

where $g_{i0} = 1$ if the i-th agent has access to the exo-system, otherwise $g_{i0} = 0$, l_i is the i-th row of L, $C =$ blkdiag$\{C_i\}$ and $x(t) =$ col$\{x_i(t)\}$. Agents outputs are called regulated if for $i = 1, \ldots, N$, $e_i(t) \to 0$ as $t \to \infty$ for $i = 1, \ldots, N$.

2.8 Notations

For a vector $x = [x_1, \ldots, x_n]^{\mathrm{T}}$ we define L^1, L^2 (Euclidean norm) and L^∞ norm as $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$, $\|x\|_\infty = \max(|x_1|, \ldots, |x_n|)$. The signal $x(t)$ is also represented as $x(t) =$ col$\{x_i(t)\}$. The function sgn$\{x(t)\}$ is defined as

$$\text{sgn}\{x(t)\} = \Big[\text{sgn}\{x_1(t)\}, \ldots, \text{sgn}\{x_n(t)\}\Big]^{\mathrm{T}},$$

$$\text{sgn}\{x_i(t)\} = \begin{cases} 0 & x_i(t) = 0 \\ \frac{x_i(t)}{|x_i(t)|} & x_i(t) \neq 0 \end{cases}. \tag{2.24}$$

For the vector x the notation diag$\{x\}$ denotes a diagonal matrix that has diagonal entries x_i's. The notations I_n, 1_n and $0_{n\times m}$ denote an identity matrix of dimension $n\times n$, a unity $n\times 1$ vector with all its entries as one, and a zero matrix of dimension $n\times m$, respectively. For a matrix $X\in\mathbb{R}^{n\times n}$, the notation $X>0$ ($X\leq 0$) or $X<0$ ($X\leq 0$) implies that X is a positive definite (positive semi-definite) or a negative definite (negative semi-definite) matrix. Moreover, $\|X\|$ denotes the Euclidean norm of X. For a matrix $A\in\mathbb{R}^{m\times n}$, its 2-norm is defined by

$$\|A\|_2=\left\{\sup\frac{\|Ax\|_2}{\|x\|_2}:x\in\mathbb{R}^n,\ x\neq 0\right\}.$$

The term X^{-L} (X^{-R}) denotes the generalized left (right) inverse of the matrix X. The terms $\lambda_i(X)$, $\lambda_{\text{min}}(X)$ and $\lambda_{\text{max}}(X)$ denote the i-th eigenvalue, the smallest, and the largest eigenvalues of the matrix X, respectively. For the matrix X, $\sigma_i(X)$, $\sigma_{\text{min}}(X)$, $\sigma_{max}(X)$, denote the i-th singular value, the minimum singular value, and the largest singular value of X. The notations Im$\{X\}$ and Ker$\{X\}$ denote the image and the kernel of X. Furthermore, for the matrix X, Sym$(X)=X+X^{\text{T}}$ and for the matrix $X>0$, $\mathcal{E}(X)$ denotes the ellipsoid $\mathcal{E}(X)=\{x\in\mathbb{R}^n:x^{\text{T}}(t)Xx(t)\leq 1\}$. For matrices X_i, $i=1,\ldots,m$ the notation diag$\{X_i\}$ represent the block diagonal matrix that its diagonal entries are X_is and the rest of entries are zero. The solution to Ric(A,B,R,Q), for $Q>0$, denotes the positive definite solution $P>0$ to the following Riccati equation

$$A^{\text{T}}P+PA-PB^{\text{T}}R^{-1}BP+Q=0. \tag{2.25}$$

Fact 2.1. *For vectors $x_1,\ldots,x_N\in\mathbb{R}^n$, we always have*

$$2\sum_{i=1,i\neq j}^{N}\sum_{j=1}^{N}x_i^Tx_j\leq\sum_{i=1}^{N}x_i^Tx_i,\ \textit{and}\ (\sum_{i=1}^{N}x_i)^T(\sum_{i=1}^{N}x_i)\leq 2(\sum_{i=1}^{N}x_i^Tx_i).$$

Fact 2.2. *For any two matrices X and Y and a positive scaler α we have*

$$X^TY + Y^TX \leq \alpha X^TX + \alpha^{-1}Y^TY.$$

$$-(X^TY + Y^TX) \leq \alpha X^TX + \alpha^{-1}Y^TY.$$

Theorem 2.1. *[161] Consider the system*

$$\dot{x}(t) = Ax(t) + f(x(t), t), \tag{2.26}$$

where A is Hurwitz stable and $x(t) \in \mathbb{R}^n$ is the state vector. The system (2.26) is stable if

$$\frac{\|f(x(t), t)\|_2}{\|x(t)\|_2} < \frac{1}{\sigma_{max}(P)},$$

for all $x(t) \in \mathbb{R}^n$ and $t > 0$, where P is the solution to

$$PA + A^TP + 2I = 0.$$

Chapter 3

Cost Based Control Reconfiguration for Consensus Achievement

This chapter tackles the development of distributed control reconfiguration and fault accommodation strategies for consensus achievement in multi-agent systems in the presence of faulty agents whose actuators are unable to produce their nominal control efforts. A faulty agent can adversely affect and prevent the team from reaching an agreement and lead to the mission catastrophic performance degradations. To ensure that the faulty team pursues its consensus objectives, in this chapter on-line distributed control reconfiguration strategies are developed that employ only nearest neighbors information to guarantee the team consensus while minimizing a local cost performance index. First, distributed Hamilton-Jacobi-Bellman equations for the faulty agent are derived. Then, reconfigured controllers are designed by solving these equations subject to the faulty agent dynamics and network structure constraints to ensure fault accommodation of the entire team. The proposed reconfigurable controllers are applied to a network of autonomous underwater vehicles subject to actuator faults to demonstrate and illustrate the effectiveness and capabilities of our proposed fault recovery control strategies.

The remainder of this chapter is organized as follows. In Section 3.1, problem formulation is provided. In Section 3.2 the distributed control reconfiguration methodologies corresponding to various actuator faults are developed. In Section 3.3, the capabilities and effectiveness of our proposed controllers for accommodating actuator faults in a network of autonomous underwater vehicles (AUVs) subject to both the leader-follower as well as the leaderless network architecture are validated through extensive simulations and finally, Section 3.4 concludes the chapter. A summary of the following is presented in [147, 162].

3.1 Problem Formulation

Consider a healthy leaderless team of N agents given by (2.1) or a healthy leader-follower team consisting of a leader given by (2.3) and N agents given by (2.1). The healthy team consensus error is obtained by concatenating the agents consensus errors so that the dynamics of the team consensus errors for the leader-follower and the leaderless network architectures can be expressed as follows

$$\dot{e}(t) \quad = \quad \mathcal{A}e(t) + \mathcal{B}u(t) + L_{21} \otimes Bu_0(t), \quad \text{(LF network)}, \tag{3.1}$$

$$\dot{e}(t) \;=\; \mathcal{A}e(t) + \mathcal{B}u(t), \qquad\qquad \text{(LL network)}, \tag{3.2}$$

where $\mathcal{A} = I_N \otimes A$, $u(t) = [u_1^{\mathrm{T}}(t), u_2^{\mathrm{T}}(t), .., u_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $e(t) = [e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), .., e_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\mathcal{B} = \begin{cases} L_{22} \otimes B & \text{LF network} \\ L \otimes B & \text{LL network} \end{cases}$, L is the network graph Laplacian matrix, and L_{22} is obtained by partitioning the Laplacian matrix of the leader-follower network as $L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$, with $L_{11} = \sum_{i=1}^{N} g_{0i}$, $L_{12} = \begin{bmatrix} -g_{01}, \ldots, -g_{0N} \end{bmatrix}$, $L_{21} = \begin{bmatrix} -g_{10}, \ldots, -g_{N0} \end{bmatrix}^{\mathrm{T}}$, and $L_{22} = \begin{bmatrix} d_1 & -g_{12} & \ldots & -g_{1N} \\ -g_{21} & d_2 & \ldots & -g_{2N} \\ \vdots & \ldots & \ldots & \vdots \\ -g_{N1} & -g_{N2} & \ldots & d_N \end{bmatrix}$.

Associated with the team, the cost function of the healthy team is defined as

$$J(t_0) = \sum_{i=1}^{N} \mathsf{J}_i(t_0) = \frac{1}{2}\int_{t_0}^{\infty} (e^{\mathrm{T}}(t)\mathcal{Q}e(t) + u^{\mathrm{T}}(t)\mathcal{R}u(t))\mathrm{d}t, \tag{3.3}$$

where $J(t_0) \equiv J(e(t_0), u(.), t_0)$, $\mathsf{J}_i(t_0) \equiv \mathsf{J}_i(e_i(t_0), u_i(.), t_0) = \frac{1}{2}\int_{t_0}^{\infty}(e_i{}^{\mathrm{T}}(t)Qe_i(t) + u_i^{\mathrm{T}}(t)Ru_i(t))\mathrm{d}t$, $\mathcal{Q} = I_N \otimes Q$, $\mathcal{R} = I_N \otimes R$, Q and R are positive definite matrices. Therefore the design objective is as follows:

Healthy team objective: The overall goal is to design the control law $u(t)$ in a distributed manner so that one can ensure the agents reach a consensus and the team cost $J(t_0)$ is minimized. The following assumption is now made explicit.

Assumption 3.1. *The pair (A, B) is controllable and the pair $(A, Q^{1/2})$ is observable.*

We now assume that at time $t = t_f$ a fault occurs in the i-th agent which is now subject to either a LOE, an outage or a stuck fault. According to (2.15)-(2.18) the dynamics of this agent can be equivalently written as

$$\dot{x}_i(t) = Ax_i(t) + B_i^f u_i^r(t) + B_i^s \underline{u}_i^s, \tag{3.4}$$

where $u_i^r(t) = u(t) = [u_i^1(t), \ldots, u_i^m(t)]^{\mathrm{T}}$, $B_i^f = B\Gamma_i$ and $B_i^s = 0$ if the agent is subject to a LOE fault. Moreover, $u_i^r(t) = [u_i^1(t), \ldots, u_i^{k-1}(t), u_i^{k+1}(t), u_i^m(t)]^{\mathrm{T}}$, $B_i^f = \underline{B}^{fk} = [b^1 \ldots, b^{k-1}, b^{k+1}, b^m]$ and $B_i^s = 0$ if the agent is subject to an outage fault. Finally, $u_i^r(t) = [u_i^1(t), \ldots, u_i^{k-1}(t), u_i^{k+1}(t), u_i^m(t)]^{\mathrm{T}}$, $B_i^f = \underline{B}^{fk} = [b^1 \ldots, b^{k-1}, b^{k+1}, b^m]$ and $B_i^s = b^k$ if the agent is subject to a stuck fault where b^k is the k-th column of the matrix B. Associated with the faulty agent the local or agent-based cost performance index for the i-th agent is defined according to

$$J_i(t_f) = \mathsf{J}_i(t_f) + \sum_{j\in\mathcal{N}_i} \mathsf{J}_j(t_f), \tag{3.5}$$

where $J_i(t_f) \equiv J_i(e_i(t_f), e_j(t_f), u_i(.), u_j(.), j \in \mathcal{N}_i, t_f)$ and

$$\mathsf{J}_i(t_f) \equiv \mathsf{J}_i(e_i(t_f), u_i(.), t_f) = \frac{1}{2}\int_{t_f}^{\infty} (e_i^{\mathrm{T}}(t)Qe_i(t) + u_i^{r^{\mathrm{T}}}(t)Ru_i^r(t))\mathrm{d}t, \tag{3.6}$$

denotes the individual cost of the i-th faulty agent, $e_i(t)$ represents the consensus error of the i-th agent and Q and R are positive definite matrices. The design objective corresponding to the faulty agent/team is now defined as follows.

Faulty team/agent objective: Design $u_i^r(t)$ such that the team reaches a consensus and the performance index $J_i(t_f)$ is minimized by using local information.

Assumption 3.2. *The network communication topology is fixed and the communication links are bi-directional.*

Assumption 3.3. *The bounds on the estimated fault severities that are provided by the FDI module are known a priori, i.e.* $-\xi_{iM}^k \leq \xi_i^k \leq \xi_{iM}^k$*, where* ξ_{iM}^k*'s for* $k = 1, \ldots, m$

are known non-negative constants.

3.2 Proposed Methodology

In this section, the proposed distributed control reconfiguration strategies for two general case scenarios that are described below is presented:

- Scenario I: the mission starts at $t = t_0$, where all agents are healthy. Then, at $t = t_f$ the first fault occurs and the i-th agent becomes faulty. Multiple concurrent faults are also subsequently allowed in the i-th agent. In this case, all the neighbors of the i-th faulty agent are assumed to be healthy. The requirements of the control law to be designed are **(a)** on-line selection of the control gains, **(b)** only employing the information that the i-th agent FDI module has generated, and **(c)** employing the faulty agent and its immediate neighbors topological information that are denoted by l_i and l_j for $j \in \mathcal{N}_i$, respectively, where l_i denotes the i-th row of the network graph Laplacian matrix. In this scenario, agents share their information set including their measurements locally. Once the fault is detected, isolated, and the control is reconfigured, the fault detection, isolation, identification and reconfiguration (FDIR) information are also shared.

- Scenario II: a subsequent fault now occurs but in the j-th agent. In this case, at least one of the neighbors of the faulty agent e.g. the i-th agent is faulty with already reconfigured control and the rest of the neighbors are healthy. This control strategy is designed on-line and the reconfiguration strategy employs the FDIR information of the faulty neighbors in addition to the information that are required in the Scenario I above. In this case, the information set that the neighboring agents share among themselves includes their measurements as well as their FDIR information.

Note that the reconfiguration strategies employ two kinds of information sets, namely: 1) information that are provided for each agent off-line and before the mission is initiated, such as the agent's dynamic matrices, and the topological location of an agent and its neighbors, and 2) information that are shared on-line and among the neighboring agents during the mission, such as measurements and the FDIR information.

In the following, first we consider the healthy team and the healthy team control law is presented. Then, the control reconfiguration strategies for a team of N agents having a faulty agent that is subject to various actuator faults, as in Scenario I, are developed. Subsequently, the extension of these results to the team with subsequent and concurrent faults, as in Scenario II, is presented.

Consider a healthy team defined in Section 3.1. Let us select the control protocol for the i-th healthy agent in LL and LF network as follows

$$u_i(t) = K\sum_{j\in\mathcal{N}_i}(x_i(t)-x_j(t)) = Ke_i(t), \tag{3.7}$$

$$\begin{aligned} u_i(t) &= K(g_{i0}(x_i(t)-x_0(t)) + \sum_{j\in\mathcal{N}_i}(x_i(t)-x_j(t)) + k_{i0}u_0(t) & (3.8)\\ &= Ke_i(t) + k_{i0}u_0(t), & (3.9)\end{aligned}$$

where k_{i0}'s are solutions to $L_{22}K_0 + L_{21} = 0$, where $K_0 = \text{col}\{k_{i0}\}$. This is justified by the observation that from equations (3.1) and (3.9), we have

$$\dot{e}(t) \;\; = \;\; (\mathcal{A} + L_{22} \otimes BK)e(t) + ((L_{22}K_0 + L_{21}) \otimes B)u_0(t).$$

If k_{i0}'s are obtained such that $L_{22}K_0 + L_{21} = 0$, the second term in the above expression will vanish. Therefore, one only needs to design the gain matrix K. For the sake of notational simplicity, the same notation for the gains in both networks are employed. Consequently, and without loss of generality, the team dynamics can be expressed as

$$\dot{e}(t) = \mathcal{A}e(t) + \mathcal{B}u(t). \tag{3.10}$$

Therefore, the problem of designing a cost-based control strategy for guaranteeing consensus achievement in the healthy team is equivalent to the design of a control law that stabilizes the system (3.10) asymptotically and minimizes the team performance index (3.3).

In the next theorem the control design problem for the healthy team is presented.

Theorem 3.1. *Consider a leaderless network of N agents whose dynamics are given by equation (2.1) and a leader-follower network of N followers whose dynamics are given by equation (2.1) and the leader whose dynamics is given by equation (2.3). Moreover, let Assumptions 3.2 and 3.1 hold. The control law $u_i(t) \equiv u_i^h(t) = K^h e_i(t) = -R^{-1}B^T P^h e_i(t)$ solves the consensus problem:*

(a) in the healthy leaderless network architecture, where P^h is the solution to the follow-

ing optimization problem

$$\max \quad \text{trace } P^h \tag{3.11}$$

$$s.t. \quad \begin{bmatrix} \mathcal{P}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^T\mathcal{P} + \tilde{\mathcal{Q}} & \mathcal{P}\tilde{\mathcal{B}} \\ \tilde{\mathcal{B}}^{\mathrm{T}}\mathcal{P} & \mathcal{R} \end{bmatrix} \geq 0, \mathcal{P} = \Lambda_1^{-1} \otimes P^h, P^h > 0, \tag{3.12}$$

(b) in the healthy leader-follower network architecture, where P^h is the solution to the following optimization problem

$$\max \quad \text{trace } P^h \tag{3.13}$$

$$s.t. \quad \begin{bmatrix} \mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} + \mathcal{Q} & \mathcal{P}\mathcal{B} \\ \mathcal{B}^{\mathrm{T}}\mathcal{P} & \mathcal{R} \end{bmatrix} \geq 0, \mathcal{P} = L_{22}^{-1} \otimes P^h,\ P^h > 0, \tag{3.14}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined as in equations (3.1) and (3.2), $\mathcal{Q}$ and $\mathcal{R}$ are defined as in equation (3.3), $\tilde{\mathcal{B}} = \Lambda_1 \otimes B$, $\Lambda_1 =$ diag$\{\lambda_j\}$, λ_j's for $j = 2, \ldots, N$ denote the non-zero eigenvalues of the network graph Laplacian matrix, $\tilde{\mathcal{A}} = I_{N-1} \otimes A$, and $\tilde{\mathcal{Q}} = I_{N-1} \otimes Q$. Furthermore, under the above control protocol the minimum of the team performance index (3.3) is bounded by $e^T(t_0)\mathcal{P}e(t_0)$.

Before presenting the proof of the theorem the following lemma is provided which will be used explicitly in the proof of Theorem 3.1.

Lemma 3.1. *Provided that the network of multi-agent systems has a spanning tree and the pair (A, B) is controllable, then the pair $(\mathcal{A}, \mathcal{B})$ is controllable in the leader-follower network architecture and has n uncontrollable modes in the leaderless network architecture.*

Proof. Let $\mathcal{C}$ and $\mathbb{C}$ denote the controllability matrices for the pair $(\mathcal{A}, \mathcal{B})$ and (A, B), respectively. Therefore,

$$\begin{aligned}\mathcal{C} &= \begin{bmatrix}\mathcal{B} & \mathcal{A}\mathcal{B} & \dots & \mathcal{A}^{n(N+1)-1}\mathcal{B}\end{bmatrix} \\ &= \begin{bmatrix}L_\alpha \otimes B & L_\alpha \otimes AB & \dots & L_\alpha \otimes A^{n(N+1)-1}B\end{bmatrix} \\ &= (I \otimes C_1)(\mathbb{C}_\alpha \otimes L_\alpha)(I \otimes C_2),\end{aligned}$$

where $\mathbb{C}_\alpha = \begin{bmatrix}B & AB & \dots & A^{n(N+1)-1}B\end{bmatrix}$, $L_\alpha = L$ for the leaderless (LL) network architecture and $L_\alpha = L_{22}$ for the leader-follower (LF) network architecture, and C_1 and C_2 denote the permutation matrices. Since C_1 and C_2 are full rank matrices and $A^i = \sum_{j=0}^{n-1} \gamma_j A^j$ for $i \geq n$, we have

$$\text{rank}\{\mathcal{C}\} = \text{rank}\{L_\alpha\}\text{rank}\{\mathbb{C}\}. \tag{3.15}$$

If the network graph has a spanning tree, its Laplacian matrix has a simple zero eigenvalue [163], so that rank$\{L\} = N - 1$, and from Lemma 3.1 in [147], the matrix L_{22} is positive definite, which implies that rank$\{L_{22}\} = N$. Now, if the pair (A, B) is controllable, then rank$\{\mathbb{C}\} = n$, therefore, in the LL network architecture, we have rank$\{\mathcal{C}\} = n(N - 1)$, and the pair $(\mathcal{A}, \mathcal{B})$ has n uncontrollable modes. In the LF network architecture we have rank$\{\mathcal{C}\} = nN$ and the pair $(\mathcal{A}, \mathcal{B})$ is controllable. This completes the proof of the lemma. □

Proof. From Lemma 3.1, the pair $(\mathcal{A}, \mathcal{B})$ in the LL network has n uncontrollable modes. Therefore, to asymptotically stabilize the team consensus errors in this architecture, first the uncontrollable modes are separated from the controllable modes by using a similarity transformation and then a control law is designed to stabilize the controllable modes. Let the transformation $T = M \otimes I_n$ be applied to the system (3.2), so that one gets

$$\dot{\tilde{e}}(t) = \mathcal{A}_T \tilde{e}(t) + \mathcal{B}_T u(t), \tag{3.16}$$

where $\mathcal{A}_T = T\mathcal{A}T^{-1} = \begin{bmatrix} \tilde{\mathcal{A}} & 0 \\ 0 & A \end{bmatrix}$, $\mathcal{B}_T = T\mathcal{B} = \begin{bmatrix} \tilde{\mathcal{B}}_1 \\ 0 \end{bmatrix}$, $L = M^{\mathrm{T}}\Lambda M$, $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, $\tilde{e}(t) = Te(t) = \begin{bmatrix} \tilde{e}_1(t) \\ \tilde{e}_2(t) \end{bmatrix}$, $\tilde{\mathcal{A}} = I_{N-1} \otimes A$, $\tilde{\mathcal{B}}_1 = \Lambda_1 M_1 \otimes B$, Λ_1 is a matrix whose diagonal elements are non-zero eigenvalues of the Laplacian matrix, $M_1 \in \mathbb{R}^{N-1\times N}$, $M_2 \in \mathbb{R}^{1\times N}$, $\tilde{e}_2(t) \in \mathbb{R}^n$, and the pair $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}_1)$ is controllable. The structures of the matrices $\mathcal{A}_T$ and $\mathcal{B}_T$ reveal that the system (3.16) can be decomposed into two independent subsystems as follows:

$$S_1 : \dot{\tilde{e}}_1(t) = \tilde{\mathcal{A}}\tilde{e}_1(t) + \tilde{\mathcal{B}}_1 u(t),\ S_2 : \dot{\tilde{e}}_2(t) = A\tilde{e}_2(t), \tag{3.17}$$

and the team consensus error signal $e(t)$ can also be written as

$$e(t) = (M_1^{\mathrm{T}} \otimes I_n)\tilde{e}_1(t) + (M_2^{\mathrm{T}} \otimes I_n)\tilde{e}_2(t). \tag{3.18}$$

The second term in the right-hand side of equation (3.18) can be expressed as

$$(M_2^{\mathrm{T}} \otimes I_n)\tilde{e}_2(t) = (M_2^{\mathrm{T}} \otimes \exp(At))(M_2 L \otimes I_n)x(t_0), \tag{3.19}$$

where $x(t_0) = \begin{bmatrix} x_1^{\mathrm{T}}(t_0), x_2^{\mathrm{T}}(t_0), \ldots, x_N^{\mathrm{T}}(t_0) \end{bmatrix}^{\mathrm{T}}$. Given that $M_2 L = 0$, we have $(M_2^{\mathrm{T}} \otimes I_n)\tilde{e}_2(t) = 0$, which implies that $e(t)$ converges to zero (or the team reaches a consensus) if $\tilde{e}_1(t)$ is asymptotically stable. Since the pair $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}_1)$ is controllable, there exists a control law $u(t)$ that stabilizes the system S_1. Therefore, the control design problem for the LL network is reduced to that of stabilizing S_1. However, this control should also minimize the team performance index. For this purpose, in the following the effects of $\tilde{e}_1(t)$, $\tilde{e}_2(t)$ and $u(t)$ on the team performance index are investigated.

From the control law (3.7) we have $u(t) = \mathcal{K}e(t)$, where $\mathcal{K} = I_N \otimes K$, which can be expressed as

$$\begin{aligned} u(t) &= (I_N \otimes K)(M_1^{\mathrm{T}} \otimes I_n)\tilde{e}_1(t) + M_2^{\mathrm{T}} \otimes I_n)\tilde{e}_2(t)) \\ &= (M_1^{\mathrm{T}} \otimes K)\tilde{e}_1(t) + (M_2^{\mathrm{T}} \otimes K)\tilde{e}_2(t). \end{aligned} \tag{3.20}$$

Let $u^1(t) = (M_1^{\mathrm{T}} \otimes K)\tilde{e}_1(t)$ and $u^2(t) = (M_2^{\mathrm{T}} \otimes K)\tilde{e}_2(t)$. Considering that M_1 and M_2 are orthogonal matrices, ${u^1}^{\mathrm{T}}(t)u^2(t) = {u^2}^{\mathrm{T}}(t)u^1(t) = 0$ and the team performance index can be expressed as

$$\begin{aligned} J(t_0) &= \mathbb{J}_1(t_0) + \mathbb{J}_2(t_0) \\ &= \frac{1}{2}\int_{t_0}^{\infty}(\tilde{e}_1^{\mathrm{T}}(t)\tilde{\mathcal{Q}}\tilde{e}_1(t) + {u^1}^{\mathrm{T}}(t)\mathcal{R}u^1(t))\mathrm{d}t + \\ &\quad \frac{1}{2}\int_{t_0}^{\infty}(\tilde{e}_2^{\mathrm{T}}(t)Q\tilde{e}_2(t) + {u^2}^{\mathrm{T}}(t)\mathcal{R}u^2(t))\mathrm{d}t, \end{aligned} \tag{3.21}$$

where $J(t_0) \equiv J(\tilde{e}_1(t_0), u^1(.), \tilde{e}_2(t_0), u^2(.), t_0)$, $\mathbb{J}_1(t_0) \equiv \mathbb{J}_1(\tilde{e}_1(t_0), u^1(.), t_0)$ and $\mathbb{J}_2(t_0) \equiv \mathbb{J}_2(\tilde{e}_2(t_0), u^2(.), t_0)$. Therefore, from equations (3.17) and (3.21) the team consensus error dynamics and the performance index for the LL network architecture can be decomposed into two subsystems, namely S_1 and S_2 with the performance indices $\mathbb{J}_1(t_0)$ and $\mathbb{J}_2(t_0)$, respectively, that are governed by

$$\begin{aligned} S_1 : \dot{\tilde{e}}_1(t) &= \tilde{\mathcal{A}}\tilde{e}_1(t) + \tilde{\mathcal{B}}_1 u_1(t), \\ \mathbb{J}_1(t_0) &= \frac{1}{2}\int_{t_0}^{\infty}(\tilde{e}_1^{\mathrm{T}}(t)\tilde{\mathcal{Q}}\tilde{e}_1(t) + {u^1}^{\mathrm{T}}(t)\mathcal{R}u^1(t))\mathrm{d}t \\ &= \frac{1}{2}\int_{t_0}^{\infty}\tilde{e}_1^{\mathrm{T}}(t)I_{N-1} \otimes (Q + K^{\mathrm{T}}RK)\tilde{e}_1(t)\mathrm{d}t, \end{aligned} \tag{3.22}$$

$$\begin{aligned} S_2 : \dot{\tilde{e}}_2(t) &= A\tilde{e}_2(t), \\ \mathbb{J}_2(t_0) &= \frac{1}{2}\int_{t_0}^{\infty}(\tilde{e}_2^{\mathrm{T}}(t)Q\tilde{e}_2(t) + {u^2}^{\mathrm{T}}(t)\mathcal{R}u^2(t))\mathrm{d}t \\ &= \frac{1}{2}\int_{t_0}^{\infty}\tilde{e}_2^{\mathrm{T}}(t)(Q + K^{\mathrm{T}}RK)\tilde{e}_2(t)\mathrm{d}t. \end{aligned} \tag{3.23}$$

The dynamics above for S_1 and S_2 reveal that the control gain K has no effect on $\tilde{e}_2(t)$, implying that $J(t_0)$ is minimized if $\mathbb{J}_1(t_0)$ is minimized. Now since

$$\begin{aligned} \tilde{\mathcal{B}}_1 u^1(t) &= (\Lambda_1 M_1 \otimes B)(M_1^{\mathrm{T}} \otimes K)\tilde{e}_1(t) \\ &= (\Lambda_1 \otimes B)(I_{N-1} \otimes K)\tilde{e}_1(t), \end{aligned}$$

the system S_1 is equivalent to

$$S : \dot{\tilde{e}}_1(t) = \tilde{\mathcal{A}}\tilde{e}_1(t) + \tilde{\mathcal{B}}\tilde{u}(t). \tag{3.24}$$

This implies that stability of the system S_1 follows from the stability of the system S, where $\tilde{\mathcal{B}} = \Lambda_1 \otimes B$, $\tilde{u}(t) = \tilde{K}\tilde{e}_1(t)$, and $\tilde{K} = I_{N-1} \otimes K$. Therefore, if the control gain K is obtained such that the control law $\tilde{u}(t)$ stabilizes S and minimizes $\mathbb{J}_1(t_0)$, then the control law $u(t)$ stabilizes the system (3.2) and minimizes the team performance index $J(t_0)$.

Considering the above discussion, the optimal control design problem for the LL network can be transformed into that of a structured LQR problem. Note that the LF control design problem has similar specifications, therefore it can be transformed into a structured LQR problem. It is known that the control law $u(t) = \mathcal{K}e(t)$ asymptotically stabilizes the system given by equation (3.10) and minimizes the team performance index (3.3) if the Riccati equation

$$\mathcal{A}^{\mathrm{T}}\mathcal{P} + \mathcal{P}\mathcal{A} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P} + \mathcal{Q} = 0 \tag{3.25}$$

has a positive definite solution for $\mathcal{P}$. Moreover, the control law $\tilde{u}(t) = \tilde{\mathcal{K}}\tilde{e}_1(t)$ asymptotically stabilizes the system given by equation (3.24) and minimizes the performance index $\mathbb{J}_1(t_0)$ if the Riccati equation

$$\tilde{\mathcal{A}}^{\mathrm{T}}\mathcal{P} + \mathcal{P}\tilde{\mathcal{A}} - \mathcal{P}\tilde{\mathcal{B}}\mathcal{R}^{-1}\tilde{\mathcal{B}}^{\mathrm{T}}\mathcal{P} + \tilde{\mathcal{Q}} = 0 \tag{3.26}$$

has a positive definite solution, where $\tilde{\mathcal{K}} = -\mathcal{R}^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P}$, $\tilde{\mathcal{Q}} = I_{N-1} \otimes Q$, $\tilde{\mathcal{A}} = I_{N-1} \otimes A$

and $\tilde{\mathcal{B}} = \Lambda_1 \otimes B$. Equations (3.25) and (3.26) have positive definite solutions if the pairs $(\mathcal{A}, \mathcal{B})$ and $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ are controllable and the pairs $(\mathcal{A}, \mathcal{Q}^{\frac{1}{2}})$ and $(\tilde{\mathcal{A}}, \tilde{\mathcal{Q}}^{\frac{1}{2}})$ are observable. It is straightforward to conclude that the pairs $(\mathcal{A}, \mathcal{Q}^{\frac{1}{2}})$ and $(\tilde{\mathcal{A}}, \tilde{\mathcal{Q}}^{\frac{1}{2}})$ are observable if and only if the pair $(A, Q^{\frac{1}{2}})$ is observable. If the network has a spanning tree, then Λ_1 is a positive definite matrix so that the controllability of (A, B) implies that $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is also controllable. On the other hand, from Lemma 3.1, if (A, B) is controllable and the network has a spanning tree then the pair $(\mathcal{A}, \mathcal{B})$ is also controllable. If the above conditions hold, the control law $u(t) = \mathcal{K}e(t)$ asymptotically stabilizes the system given by equation (3.10) and minimizes the performance index J and the control law $\tilde{u}(t) = \tilde{\mathcal{K}}\tilde{e}_1(t)$ asymptotically stabilizes the system given by equation (3.24) and minimizes the performance index $\mathbb{J}_1(t_0)$. These imply that the control law $u(t)$ stabilizes the system (3.2) and minimizes the team performance index $J(t_0)$.

The only remaining condition that is needed is to ensure that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are block diagonal matrices. Towards this end, an intermediate step needs to be performed as described below. From [157], it follows that Riccati equations (3.25) and (3.26) have solutions if and only if the following maximization problems have solutions, namely

$$\max\ \mathcal{P}\ s.t.\ \mathcal{A}^{\mathrm{T}}\mathcal{P} + \mathcal{P}\mathcal{A} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P} + \mathcal{Q} \geq 0, \tag{3.27}$$

$$\max\ \mathcal{P}\ s.t.\ \tilde{\mathcal{A}}^{\mathrm{T}}\mathcal{P} + \mathcal{P}\tilde{\mathcal{A}} - \mathcal{P}\tilde{\mathcal{B}}\mathcal{R}^{-1}\tilde{\mathcal{B}}^{\mathrm{T}}\mathcal{P} + \tilde{\mathcal{Q}} \geq 0. \tag{3.28}$$

Consequently, the solution to the equation (3.27) is enforced to be $\mathcal{P} = L_{22}^{-1} \otimes P^h$ and the solution to the equation (3.28) is enforced to be $\mathcal{P} = \Lambda_1^{-1} \otimes P$, so that $\mathcal{K} = -I_N \otimes R^{-1}B^{\mathrm{T}}P^h$, which has the required and desired structure. Note that both constraints in the above optimization problems can be written as LMI's by using the Schur complement.

Therefore, the above formulation transforms the problem of solving the Riccati equations (3.25) and (3.26) to that of solving convex optimization problems subject to the LMI constraints. Since the structure of the constraints are also convex, by incorporating them into the above optimization problems, they still remain convex. It should be noted that from the results in [164] the matrix $\mathcal{P}$ can be approximated as a Kronecker product of two matrices, so that such structures for $\mathcal{P}$ can also be considered. Finally, it can be shown readily from [157] that the minimum value of the performance index (3.3) is bounded by $e^{\mathrm{T}}(t_0)\mathcal{P}e(t_0)$. $\square$

Remark 3.1. *In the above theorem, the optimization problems (3.11) and (3.13) require the network Laplacian information. However, it should be noted that since these problems are solved off-line before a mission commences, the agents only require local measurements and control is implemented in a distributed manner.*

Before our control recovery strategies are presented we need to state our main assumption regarding the nature of the faults below.

Assumption 3.4. *The following properties are assumed to hold:*

(a) Each agent is equipped with a fault detection, isolation and identification (FDI) module, which detects the presence of a fault, isolates the fault (that is the fault location), determines the type of a fault and estimates the fault magnitude/severity in cases of LOE and stuck faults.

(b) Faults are permanent and there exists a sufficient time between the occurrence of any two subsequent faults in neighboring agents to allow for the agents to reconfigure their control laws.

(c) Following the injection of a fault, the agent remains controllable, implying that the outage and the stuck faults cannot occur simultaneously in ALL the actuators of an agent, while simultaneous LOE faults are permitted.

Regarding the above assumption, we would like to point out the following: Firstly, Assumptions 3.4(a) and 3.4(c) stating that the agents are equipped with FDI module and remain controllable after the fault occurrence, respectively, are quite common in the literature and necessary for performing an on-line fault accommodation and design of active control reconfiguration strategies. Secondly, the required time between subsequent faults in neighboring agents is sufficiently small but larger than the time that is required to compute the reconfigured controller gains. Furthermore, the probability that two agents are injected with faults at exactly the same time is quite low in practice. However, in case that neighboring agents become simultaneously faulty, one can still reconfigure the agents by first allowing agents having high severity faults to be reconfigured, followed by lower severity faults to be reconfigured next, so that the simultaneous fault situation would not impose any adverse restrictions on our proposed methodology in real life applications. Finally, the problem of development of an FDI module that detects, isolates and estimates the actuator faults in multi-agent systems has already been investigated in the literature as in e.g. [165–170].

3.2.1 Control Reconfiguration Subject to the Loss of Effectiveness Fault

Assume that the i-th agent becomes faulty and its actuators effectiveness are subsequently reduced. The dynamics of the consensus error for this agent can be expressed as

$$\dot{e}_i(t) = Ae_i(t) + d_i B\Gamma_i u_i(t) - \sum_{j\in\mathcal{N}_i} Bu_j(t), \tag{3.29}$$

and the error dynamics of the nearest neighbors agents can be expressed as

$$\dot{e}_l(t) = Ae_l(t) + d_l Bu_l(t) - B\Gamma_i u_i(t) - \sum_{j\in\mathcal{N}_l, j\neq i} Bu_j(t), l \in \mathcal{N}_i \tag{3.30}$$

and finally for the remaining agents that are not in the nearest neighborhood set of the faulty agent their error dynamics can be expressed as

$$\dot{e}_k(t) = Ae_k(t) + d_k Bu_k(t) - \sum_{j\in\mathcal{N}_k} Bu_j(t),\ k \in \mathcal{N}_i^-. \tag{3.31}$$

Equations (3.29)-(3.31) indicate that any actuator fault affects the dynamics of the consensus error of the faulty agent as well as its nearest neighbors. Therefore, it follows that the faulty team can preserve its consensus if the control law of the faulty agent is reconfigured such that the consensus error of the faulty agent and its nearest neighbors remain asymptotically stable.

The reconfigured control law for the faulty agent should also minimize the cost of reaching consensus for the entire faulty team. Equation (3.29) states that the consensus error of the faulty agent depends on the information that is received from the immediate neighbors of the faulty agent. Due to this characteristics and coupling, the standard optimal control formulation cannot be solved in a straightforward manner by conventional methods and instead one needs to use Hamilton Jacobi Bellman (HJB) equations to deal with this situation as described below.

Let $J_i^*(t) \equiv J_i^*(e_i(t), e_j(t), j \in \mathcal{N}_i, t)$ denote the optimal value of the i-th agent cost function $J_i(t)$, i.e. $J_i^*(t) = \min_{u_i(t)} J_i(t)$. Then, the Hamiltonian equation for the system (3.29) and the performance index (3.5) can be expressed as

$$\begin{aligned}\mathcal{H}_i(t) &= (J_i^{e_i*}(t))^{\mathrm{T}}(Ae_i(t) + d_i B\Gamma_i u_i(t) - \sum_{j\in\mathcal{N}_i} Bu_j(t)) \\ &+ \frac{1}{2}\big(e_i^{\mathrm{T}}(t)Qe_i(t) + u_i^{\mathrm{T}}(t)Ru_i(t) \\ &+ \sum_{j\in\mathcal{N}_i} e_j^{\mathrm{T}}(t)Qe_j(t) + u_j^{\mathrm{T}}(t)Ru_j(t)\big), \qquad (3.32)\end{aligned}$$

where

$J_i^{e_i*}(t) \equiv J_i^{e_i*}(e_i(t), e_j(t), j \in \mathcal{N}_i, t) = \frac{\partial J_i^*(t)}{\partial e_i(t)}$ and $\mathcal{H}_i(t) \equiv \mathcal{H}_i(e_i(t), e_j(t), u_i(t), u_j(t), j \in \mathcal{N}_i)$. Therefore, the optimal control law for the i-th agent can be obtained as

$$u_i^*(t) = \arg\min_{u_i(t)} \mathcal{H}_i(t) = -d_i R^{-1}\Gamma_i B^{\mathrm{T}} J_i^{e_i*}(t). \qquad (3.33)$$

By substituting $u_j(t) \equiv u_j^h(t), j \in \mathcal{N}_i$ from the results of Theorem 3.1 for the healthy agent j, and also the value of $u_i^*(t)$ from equation (3.33) into equation (3.32), the optimal Hamiltonian is now obtained as

$$\begin{aligned}\mathcal{H}_i^*(t) &= \frac{1}{2}e_i^{\mathrm{T}}(t)Qe_i(t) - \frac{1}{2}d_i^2(J_i^{e_i*}(t))^{\mathrm{T}}F_i J_i^{e_i*}(t) \\ &\quad +(J_i^{e_i*}(t))^{\mathrm{T}}Ae_i(t) + \sum_{j\in\mathcal{N}_i}(J_i^{e_i*}(t))^{\mathrm{T}}FP^h e_j(t) \\ &\quad +\sum_{j\in\mathcal{N}_i}\big(\frac{1}{2}e_j^{\mathrm{T}}(t)(Q + P^h F P^h)e_j(t)\big), \qquad (3.34)\end{aligned}$$

where $F_i = B\Gamma_i R^{-1}\Gamma_i B^{\mathrm{T}}$ and $F = BR^{-1}B^{\mathrm{T}}$. Given that we are dealing with time-invariant multi-agent systems with the performance index as defined in equation (3.3),

$\mathcal{H}_i^*(t)$ satisfies $\mathcal{H}_i^*(t) = 0$, $t > 0$ which implies that

$$\begin{aligned} &\sum_{j\in\mathcal{N}_i} \big(\frac{1}{2}e_j^{\mathrm{T}}(t)(Q + P^h F P^h)e_j(t) + (J_i^{e_i*}(t))^{\mathrm{T}} F P^h e_j(t)\big) \\ + \quad & \frac{1}{2}e_i^{\mathrm{T}}(t)Qe_i(t) - \frac{1}{2}d_i^2(J_i^{e_i*}(t))^{\mathrm{T}} F_i J_i^{e_i*}(t) + (J_i^{e_i*}(t))^{\mathrm{T}} A e_i(t) \\ & = 0. \end{aligned} \tag{3.35}$$

The above represents a partial differential equation in terms of $J_i^*(t)$ which is not stright-forward to solve for a given set of general parameters. However, since $J_i(t)$ in equation (3.5) is quadratic with respect to $e_i(t)$, a possible solution can be suggested and expressed as

$$J_i^*(t) = \frac{1}{2}(e_i^{\mathrm{T}}(t)P_i e_i(t) + \sum_{j\in\mathcal{N}_i} e_j^{\mathrm{T}}(t)P^h e_j(t)). \tag{3.36}$$

By substituting $e_i(t)$ and $e_j(t)$ from equation (2.2) and given the fact that $J_i^{e_i*}(t) = P_i e_i(t)$, equation (3.35) becomes $x^{\mathrm{T}}\Psi_i x(t) = 0$ where

$$\begin{aligned} \Psi_i \;=\; & \sum_{j\in\mathcal{N}_i} \big(\mathcal{L}_{jj} \otimes (Q + P^h F P^h) + \mathcal{L}_{ij} \otimes P_i F P^h + \mathcal{L}_{ji} \otimes \\ & P^h F P_i\big) + \mathcal{L}_{ii} \otimes \big(A^{\mathrm{T}} P_i + P_i A - d_i^2 P_i F_i P_i + Q\big), \end{aligned} \tag{3.37}$$

where $\mathcal{L}_{ij} = l_i^{\mathrm{T}} l_j$. Note that $x(t) = 0$ is a trivial solution to $x^{\mathrm{T}}\Psi_i x(t) = 0$, although this equality also holds if there exists a unique positive definite matrix P_i such that $\Psi_i = 0$, which can be expressed as

$$\begin{aligned} &\sum_{j\in\mathcal{N}_i} (\mathcal{Q}_j + \mathcal{P}^h \bar{\mathcal{F}}_{jj} \mathcal{P}^h + \mathcal{P}_i \bar{\mathcal{F}}_{ij} \mathcal{P}^h + \mathcal{P}^h \bar{\mathcal{F}}_{ji} \mathcal{P}_i) \\ + \quad & \mathcal{A}_i^{\mathrm{T}} \mathcal{P}_i + \mathcal{P}_i \mathcal{A}_i - d_i^2 \mathcal{P}_i \mathcal{F}_{ii} \mathcal{P}_i + \mathcal{Q}_i = 0, \end{aligned} \tag{3.38}$$

where $\mathcal{A}_i = \mathcal{L}_{ii} \otimes A$, $\mathcal{P}^h = I_N \otimes P^h$, $\mathcal{P}_i = I_N \otimes P_i$, $\mathcal{Q}_i = \mathcal{L}_{ii} \otimes Q$, $\bar{\mathcal{F}}_{ij} = \mathcal{L}_{ij} \otimes F$, $\mathcal{F}_{ii} = \mathcal{L}_{ii} \otimes F_i$, F_i and F are defined as in equation (3.34). The following lemma now summarizes the above derivations and result.

Lemma 3.2. *Consider a team of N multi-agent systems where the control law for the* *healthy agents is designed according to Theorem 3.1. Suppose that the i-th agent is subject to LOE faults. Given an exact knowledge of the fault severity Γ_i, the control law*

$$u_i^*(t) = -d_i R^{-1} \Gamma_i B^T P_i e_i(t),$$

minimizes the performance index (3.5), where P_i is the solution to the equation (3.38).

Proof. Follows directly from the derivations preceding the lemma. □

The matrix $\mathcal{P}^h$ in equation (3.38) has already been designed according to Theorem 3.1, and therefore the matrix $\mathcal{P}_i$ is the only unknown matrix in equation (3.38). Furthermore, equation (3.38) is a Riccati equation in terms of $\mathcal{P}_i$, which is a convex function and its unique solution can be readily obtained. However, from practical considerations the assumption of perfect information generated by the FDI module may not be realistic. In other words, the FDI module information may contain uncertainties and inaccuracies as stated in Assumption 3.3. Therefore, we are concerned with a situation that is in contrast to the exact knowledge assumption that is made in Lemma 3.2.

To remedy the unavailability of an accurate information on Γ_i, the reconfigured controller is now proposed as

$$u_i^r(t) = K_i^r e_i(t),$$

where $K_i^r = -d_i R^{-1} \hat{\Gamma}_i B^{\mathrm{T}} P_i$ and $\hat{\Gamma}_i$ denotes the estimate of Γ_i that is assumed to be generated by the FDI module and $\mathcal{P}_i$ is obtained as the solution to the following equation

$$\begin{aligned} &\mathcal{A}_i^{\mathrm{T}} \mathcal{P}_i + \mathcal{P}_i \mathcal{A}_i - d_i^2 \mathcal{P}_i \tilde{\mathcal{F}}_{ii} \mathcal{P}_i + \mathcal{Q}_i + \sum_{j \in \mathcal{N}_i} (\mathcal{Q}_j + \mathcal{P}^h \bar{\mathcal{F}}_{jj} \mathcal{P}^h \\ &+ \mathcal{P}_i \bar{\mathcal{F}}_{ij} \mathcal{P}^h + \mathcal{P}^h \bar{\mathcal{F}}_{ji} \mathcal{P}_i) = 0, \end{aligned} \tag{3.39}$$

where $\tilde{\mathcal{F}}_{ii} = \mathcal{L}_{ii} \otimes \tilde{F}_i$, $\tilde{F}_i = B(\hat{\Gamma}_i - 2f_i\xi_{iM})R^{-1}\hat{\Gamma}_i B^{\mathrm{T}}$, $\bar{\mathcal{F}}_{ij}$ is defined as in equation (3.38), ξ_{iM} and f_i are defined as in equation (2.16). Note that solvability of equation (3.39) depends on the matrix $\mathcal{P}^h$ and the agent's fault severity as well as the agents dynamics, the network structure and the value of the design matrices Q and R. In fact if we denote $\mathbb{A}_i = \mathcal{A}_i + \sum_{j\in\mathcal{N}_i} \bar{\mathcal{F}}_{ij}\mathcal{P}^h$, $\mathbb{B}_i = \tilde{\mathcal{F}}_{ii}^{1/2}$ and $\mathbb{Q}_i = \mathcal{Q}_i + \sum_{j\in\mathcal{N}_i}(\mathcal{Q}_j + \mathcal{P}^h\bar{\mathcal{F}}_{jj}\mathcal{P}^h)$, then equation (3.39) has a unique positive definite solution for $\mathcal{P}_i$ if and only if the pair $(\mathbb{A}_i, \mathbb{B}_i)$ is controllable and the pair $(\mathbb{A}_i, \mathbb{Q}_i^{\frac{1}{2}})$ is observable. Therefore, the following assumption is now required.

Assumption 3.5. *The pair $(\mathbb{A}_i, \mathbb{B}_i)$ is controllable and the pair $(\mathbb{A}_i, \mathbb{Q}_i^{\frac{1}{2}})$ is observable.*

We are now in a position to state our main result of this subsection.

Theorem 3.2. *Consider a team of N multi-agent systems where the control law for the healthy agent is designed according to Theorem 3.1. Suppose that at $t = t_f$ the i-th agent is subjected to LOE faults. Provided that the reconfigured control law for this agent is set to*

$$u_i^r(t) = K_i^r e_i(t), \tag{3.40}$$

guarantees that the team reaches a consensus and the value of the local cost performance index (3.5) is given by $J_i^*(t_f) = e_i^T(t_f)P_i e_i(t_f) + J_{i\xi}(t_f)$; where $K_i^r = -d_i R^{-1}\hat{\Gamma}_i B^T P_i$, and P_i is the solution to

$$\begin{aligned} \max \quad & \text{trace } P_i \\ s.t. \quad & \begin{bmatrix} \Phi_i^1 & d_i\mathcal{P}_i\tilde{\mathcal{F}}_{ii}^{\frac{1}{2}} \\ d_i\tilde{\mathcal{F}}_{ii}^{\frac{1}{2}}\mathcal{P}_i & I \end{bmatrix} \geq 0, \begin{bmatrix} \Phi_i^2 & \mathcal{P}_i\bar{\mathcal{E}}_{ii}^{\frac{1}{2}} \\ \bar{\mathcal{E}}_{ii}^{\frac{1}{2}}\mathcal{P}_i & -d_i * I \end{bmatrix} \leq 0, \\ & \mathcal{P}_i > 0, \end{aligned} \tag{3.41}$$

where $\mathcal{P}_i = I_N \otimes P_i$, $\Phi_i^1 = \mathcal{A}_i^T \mathcal{P}_i + \mathcal{P}_i \mathcal{A}_i + \mathcal{Q}_i + \mathcal{P}_i \sum_{j\in\mathcal{N}_i} \bar{\mathcal{F}}_{ij} \mathcal{P}^h + \sum_{j\in\mathcal{N}_i} \mathcal{P}^h \bar{\mathcal{F}}_{ji} \mathcal{P}_i + \sum_{j\in\mathcal{N}_i} (\mathcal{Q}_j + \mathcal{P}^h \bar{\mathcal{F}}_{jj} \mathcal{P}^h)$, $l \in \mathcal{N}_i$, $\Phi_i^2 = d_i(\mathcal{P}_i \tilde{\mathcal{E}}_{il} \mathcal{P}^h + \mathcal{P}^h \tilde{\mathcal{E}}_{il}^T \mathcal{P}_i) + \mathcal{P}^h (d_i \bar{\mathcal{E}}_{il} - \bar{\mathcal{F}}_{il} - \bar{\mathcal{F}}_{il}^T) \mathcal{P}^h$ *for* $l \in \mathcal{N}_i$, $\tilde{\mathcal{F}}_{ii} = \mathcal{L}_{ii} \otimes \tilde{F}_i$, $\tilde{F}_i = B(\hat{\Gamma}_i - 2 f_i \xi_{iM}) R^{-1} \hat{\Gamma}_i B^T$, $\mathcal{A}_i$ *and* $\mathcal{Q}_i$ *are defined as in equation (3.38),* $\bar{\mathcal{F}}_{ij} = \mathcal{L}_{ij} \otimes F$, $F = BR^{-1}B^T$, $\tilde{\mathcal{E}}_{il} = \mathcal{L}_{il} \otimes \tilde{E}_i$, $\tilde{E}_i = B\hat{\Gamma}_i R^{-1} \hat{\Gamma}_i B^T$, $\bar{\mathcal{E}}_{il} = \mathcal{L}_{ll} \otimes \bar{E}_i$, $\bar{E}_i = B f_i \xi_{iM} R^{-1} \hat{\Gamma}_i B^T$, $J_{i\xi}(t_f) = -2d_i^2 \int_{t_f}^{\infty} e_i^T(t) P_i B f_i(\xi_{iM} + \xi_i) R^{-1} \hat{\Gamma}_i B^T P_i e_i(t) dt$, $\hat{\Gamma}_i$ *denotes the estimated value of the fault severities,* f_i *and* ξ_{iM} *are defined in equation (2.16),* $\mathcal{P}^h = I_N \otimes P^h$, *and* P^h *is obtained from Theorem 3.1.*

Proof. The objective of the reconfigured control design is to asymptotically stabilize the faulty agent consensus error dynamics while ensuring that the nearest neighboring agents' errors also remain asymptotically stable. Consider $V_i(t) = e_i^{\mathrm{T}}(t) P_i e_i(t)$ as a Lyapunov function candidate for the faulty system (3.29). The time derivative of $V_i(t)$ along the trajectories of the system (3.29) becomes

$$\begin{aligned}\dot{V}_i(t) &= (Ae_i(t) + d_i B\Gamma_i u_i(t) - \sum_{j\in\mathcal{N}_i} Bu_j(t))^{\mathrm{T}} P_i e_i(t) \\ &\quad + e_i^{\mathrm{T}}(t) P_i (Ae_i(t) + d_i B\Gamma_i u_i(t) - \sum_{j\in\mathcal{N}_i} Bu_j(t)).\end{aligned}$$

By substituting $u_i(t) \equiv u_i^r(t)$, $e_i(t)$ and $u_j(t),\ j \in \mathcal{N}_i$ from equations (3.40), (2.2) and Theorem 3.1, respectively, into the above equation yields

$$\begin{aligned}\dot{V}_i(t) &= x^{\mathrm{T}}(t)(\mathcal{L}_{ii} \otimes (A^{\mathrm{T}} P_i + P_i A - 2d_i^2 P_i E_i P_i) \\ &\quad + \sum_{j\in\mathcal{N}_i} (\mathcal{L}_{ij} \otimes P_i F P^h + \mathcal{L}_{ji} \otimes P^h F P_i)) x(t) \\ &= x^{\mathrm{T}}(t)(\mathcal{A}_i^{\mathrm{T}} \mathcal{P}_i + \mathcal{P}_i \mathcal{A}_i - 2d_i^2 \mathcal{P}_i (\mathcal{L}_{ii} \otimes E_i) \mathcal{P}_i \\ &\quad + \sum_{j\in\mathcal{N}_i} (\mathcal{P}_i \bar{\mathcal{F}}_{ij} \mathcal{P}^h + \mathcal{P}^h \bar{\mathcal{F}}_{ji} \mathcal{P}_i)) x(t), \qquad (3.42)\end{aligned}$$

where $E_i = B\hat{\Gamma}_i R^{-1} \Gamma_i B^{\mathrm{T}}$, $\bar{\mathcal{F}}_{ij} = \mathcal{L}_{ij} \otimes F$ and $F = BR^{-1}B^{\mathrm{T}}$. If the matrix P_i represents the positive definite solution to equation (3.39) then $\dot{V}_i(t)$ can be written as

$$\begin{aligned}\dot{V}_i(t) &= -x^{\mathsf{T}}(t)\big(d_i^2\mathcal{P}_i(\mathcal{L}_{ii}\otimes(2E_i-\tilde{F}_i))\mathcal{P}_i+\mathcal{Q}_i \\ &\quad +\sum_{j\in\mathcal{N}_i}(\mathcal{Q}_j+\mathcal{P}^h\bar{\mathcal{F}}_{jj}\mathcal{P}^h\big)x(t).\end{aligned}$$

This can equivalently be expressed as

$$\begin{aligned}\dot{V}_i(t) &= -e_i^{\mathsf{T}}(t)(Q+d_i^2P_iB\hat{\Gamma}_iR^{-1}(\hat{\Gamma}_i+2f_i(\xi_{iM}+\xi_i)B^{\mathsf{T}}P_i) \\ &\quad e_i(t)-\sum_{j\in\mathcal{N}_i}e_j^{\mathsf{T}}(t)(Q+P^hFP^h)e_j(t)<0,\end{aligned} \tag{3.43}$$

which implies that $e_i(t)$ is asymptotically stable.

Next, to investigate the stability properties of the consensus error signals corresponding to the nearest neighbors of the faulty agent, consider $V_l(t)=e_l^{\mathsf{T}}(t)P^he_l(t)$ for $l\in\mathcal{N}_i$. Note that $V_l(t)$ denotes a Lyapunov function candidate for the system (3.30), where P^h is obtained from Theorem 3.1. The time derivative of $V_l(t)$ along the trajectories of the system (3.30) becomes

$$\begin{aligned}\dot{V}_l(t) &= \Big(Ae_l(t)+d_lBu_l(t)-\sum_{j\in\mathcal{N}_l,j\neq i}Bu_j(t)\Big)^{\mathsf{T}}P^he_l(t) \\ &\quad +e_l^{\mathsf{T}}(t)P^h(Ae_l(t)+d_lBu_l(t)-\sum_{j\in\mathcal{N}_l,j\neq i}Bu_j(t)) \\ &\quad -(u_i^{\mathsf{T}}(t)\Gamma_iB^{\mathsf{T}}P^he_l(t)+e_l^{\mathsf{T}}(t)P^hB\Gamma_iu_i(t)).\end{aligned}$$

We need to ensure that $\dot{V}_l(t)$ remains negative definite subsequent to the invocation of the reconfiguration control law (3.40). Let us denote $\dot{V}_l(t)$ corresponding to the reconfigured system and the healthy system by $\dot{V}_l^r(t)$ and $\dot{V}_l^h(t)$, respectively. Since $\dot{V}_l^h(t)<0$, it is sufficient to ensure that $\dot{V}_l^r(t)\leq\dot{V}_l^h(t)$, which is achieved if the following inequality holds:

$$u_i^{r\mathrm{T}}(t)\Gamma_i B^{\mathrm{T}} P^h e_l(t) + e_l^{\mathrm{T}}(t) P^h B \Gamma_i u_i^r(t) \geq$$

$$u_i^{h\mathrm{T}}(t) B^{\mathrm{T}} P^h e_l(t) + e_l^{\mathrm{T}}(t) P^h B u_i^h(t),\ l \in \mathcal{N}_i, \tag{3.44}$$

where the control laws $u_i^h(t)$ and $u_i^r(t)$ are specified according to Theorem 3.1 and equation (3.40), respectively. By substituting $u_i^h(t)$, $u_i^r(t)$, $e_i(t)$ and $e_l(t)$ into the inequality (3.44), it follows that this inequality holds if

$$d_i(\mathcal{P}_i \mathcal{E}_{il} \mathcal{P}^h + \mathcal{P}^h \mathcal{E}_{il}^{\mathrm{T}} \mathcal{P}_i) - (\mathcal{P}^h \bar{\mathcal{F}}_{il} \mathcal{P}^h + \mathcal{P}^h \bar{\mathcal{F}}_{il}^{\mathrm{T}} \mathcal{P}^h) \leq 0,\ l \in \mathcal{N}_i \tag{3.45}$$

where $\mathcal{E}_{il} = \mathcal{L}_{il} \otimes E_i$. The inequality (3.45) cannot be verified due to the unavailability of Γ_i in E_i. Let us denote the left-hand side of (3.45) by T and decompose E_i into $E_i = \tilde{E}_i + B\hat{\Gamma}_i R^{-1} \xi_i B^{\mathrm{T}}$, where $\tilde{E}_i = B\hat{\Gamma}_i R^{-1} \hat{\Gamma}_i B^{\mathrm{T}}$. Therefore, T can be written as

$$\begin{aligned} T &= d_i(\mathcal{P}_i \tilde{\mathcal{E}}_{il} \mathcal{P}^h + \mathcal{P}^h \tilde{\mathcal{E}}_{il}^{\mathrm{T}} \mathcal{P}_i + \mathcal{P}_i \hat{\mathcal{E}}_{il} \mathcal{P}^h + \mathcal{P}^h \hat{\mathcal{E}}_{il}^{\mathrm{T}} \mathcal{P}_i) \\ &\quad -(\mathcal{P}^h \bar{\mathcal{F}}_{il} \mathcal{P}^h + \mathcal{P}^h \bar{\mathcal{F}}_{il}^{\mathrm{T}} \mathcal{P}^h) \end{aligned} \tag{3.46}$$

where $\hat{\mathcal{E}}_{il} = \mathcal{L}_{il} \otimes \hat{E}_i$, $\tilde{\mathcal{E}}_{il} = \mathcal{L}_{il} \otimes \tilde{E}_i$ and $\hat{E}_i = B\hat{\Gamma}_i R^{-1} f_i \xi_i B^{\mathrm{T}}$. Let

$$\mathcal{P}_i \hat{\mathcal{E}}_{il} \mathcal{P}^h = \mathcal{L}_{il} \otimes P_i B \xi_i R^{-1} \hat{\Gamma}_i B^{\mathrm{T}} P^h.$$

Given that $\xi_i = \mathrm{diag}\{|\xi_i^k| \mathrm{sgn}\{x_i^k\}\}$, by using the fact that $X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \leq X^{\mathrm{T}}X + Y^{\mathrm{T}}Y$ for $X^{\mathrm{T}} = l_i^{\mathrm{T}} \otimes P_i B \mathrm{diag}\{|\xi_i^k|^{\frac{1}{2}} \mathrm{sgn}\{x_i^k\}\} R^{-\frac{1}{2}} \hat{\Gamma}_i^{\frac{1}{2}}$ and $Y = l_l \otimes \mathrm{diag}\{|\xi_i^k|^{\frac{1}{2}} \mathrm{sgn}\{x_i^k\}\}$ $R^{-\frac{1}{2}} \hat{\Gamma}_i^{\frac{1}{2}} B^{\mathrm{T}} P^h$, the expression (3.46) is reduced to

$$\begin{aligned} T &\leq d_i(\mathcal{P}_i \tilde{\mathcal{E}}_{il} \mathcal{P}^h + \mathcal{P}^h \tilde{\mathcal{E}}_{il}^{\mathrm{T}} \mathcal{P}_i + \mathcal{P}_i(\mathcal{L}_{ii} \otimes P_i B |\xi_i| R^{-1} \hat{\Gamma}_i B^{\mathrm{T}}) \\ &\quad \mathcal{P}_i + \mathcal{P}^h(\mathcal{L}_{ll} \otimes P_i B |\xi_i| R^{-1} \hat{\Gamma}_i B^{\mathrm{T}}) \mathcal{P}^h) - \mathcal{P}^h(\bar{\mathcal{F}}_{il} + \bar{\mathcal{F}}_{il}^{\mathrm{T}}) \mathcal{P}^h \\ &\leq d_i(\mathcal{P}_i \tilde{\mathcal{E}}_{il} \mathcal{P}^h + \mathcal{P}^h \tilde{\mathcal{E}}_{il}^{\mathrm{T}} \mathcal{P}_i + \mathcal{P}_i \bar{\mathcal{E}}_{ii} \mathcal{P}_i + \mathcal{P}^h \bar{\mathcal{E}}_{il} \mathcal{P}^h) \\ &\quad -\mathcal{P}^h(\bar{\mathcal{F}}_{il} + \bar{\mathcal{F}}_{il}^{\mathrm{T}}) \mathcal{P}^h \end{aligned}$$

where $|\xi_i| = \mathrm{diag}\{|\xi_i^k|\}$, $\bar{\mathcal{E}}_{il} = \mathcal{L}_{ll} \otimes \bar{E}_i$ and $\bar{E}_i = B f_i \xi_{iM} R^{-1} \hat{\Gamma}_i B^{\mathrm{T}}$. This implies that if

$\mathcal{P}_i$ is found such that

$$d_i(\mathcal{P}_i\tilde{\mathcal{E}}_{il}\mathcal{P}^h + \mathcal{P}^h\tilde{\mathcal{E}}_{il}^{\mathrm{T}}\mathcal{P}_i + \mathcal{P}_i\bar{\mathcal{E}}_{ii}\mathcal{P}_i)$$

$$+\mathcal{P}^h(d_i\bar{\mathcal{E}}_{il} - \bar{\mathcal{F}}_{il} - \bar{\mathcal{F}}_{il}^{\mathrm{T}})\mathcal{P}^h \leq 0, \tag{3.47}$$

then the inequality (3.45), and consequently the inequality (3.44) will also be satisfied.

The above observations on the properties of $e_i(t)$ and $e_l(t),\ l \in \mathcal{N}_i$ indicate that the proposed reconfigurable controller guarantees the asymptotic stability of the consensus errors if the matrix $\mathcal{P}_i$ satisfies the equation (3.39) as well as the inequality (3.47). Given that equation (3.39) is a Riccati equation in terms of $\mathcal{P}_i$, it has a solution if and only if the following maximization problem is feasible [157]:

$$\max \qquad \mathrm{trace}\, \mathcal{P}_i \tag{3.48}$$

$$s.t.\ \mathcal{A}_i^{\mathrm{T}}\mathcal{P}_i + \mathcal{P}_i\mathcal{A}_i - d_i^2\mathcal{P}_i\tilde{\mathcal{F}}_{ii}\mathcal{P}_i + \mathcal{Q}_i + \sum_{j\in\mathcal{N}_i}(\mathcal{Q}_j$$

$$+\mathcal{P}^h\bar{\mathcal{F}}_{jj}\mathcal{P}^h + \mathcal{P}_i\bar{\mathcal{F}}_{ij}\mathcal{P}^h + \mathcal{P}^h\bar{\mathcal{F}}_{ji}\mathcal{P}_i) \geq 0, \tag{3.49}$$

which implies that the problem of solving the Riccati equation (3.39) can be formulated as a maximization problem subject to the constraint (3.49). Using the Schur complement, the constraint (3.49) and the inequality (3.47) can be written according to the following LMIs, namely

$$\begin{bmatrix} \Phi_i^1 & d_i\mathcal{P}_i\tilde{\mathcal{F}}_{ii}^{\frac{1}{2}} \\ d_i\tilde{\mathcal{F}}_{ii}^{\frac{1}{2}}\mathcal{P}_i & I \end{bmatrix} \geq 0 \ \text{ and } \ \begin{bmatrix} \Phi_i^2 & \mathcal{P}_i\bar{\mathcal{E}}_{ii}^{\frac{1}{2}} \\ \bar{\mathcal{E}}_{ii}^{\frac{1}{2}}\mathcal{P}_i & -d_i * I \end{bmatrix} \leq 0,$$

where Φ_i^1 and Φ_i^2 are defined as in the problem (3.41). The above inequalities can be imposed as extra constraints on the maximization problem (3.48).

Now, given that $u_i(t) = -d_iR^{-1}\hat{\Gamma}_iB^{\mathrm{T}}P_ie_i(t)$ and $u_j(t) = -R^{-1}B^{\mathrm{T}}P^he_j(t)$, by integrating both sides of (3.43) from t_f to ∞ one obtains

$$\int_{t_f}^{\infty} \dot{V}_i(t)\mathrm{d}t = -\int_{t_f}^{\infty} \big(e_i^{\mathrm{T}}(t)Qe_i(t) + u_i^{\mathrm{T}}(t)Ru_i(t) + \sum_{j\in\mathcal{N}_i}(e_j^{\mathrm{T}}(t)Q_je_j(t) + u_j^{\mathrm{T}}(t)Ru_j(t))\big)\mathrm{d}t$$
$$-2d_i^2\int_{t_f}^{\infty} e_i^{\mathrm{T}}(t)P_iB\hat{\Gamma}_iR^{-1}f_i(\xi_{iM}+\xi_i)B^{\mathrm{T}}P_i\mathrm{d}t. \qquad (3.50)$$

Therefore,

$$J_i^*(t_f) = \mathsf{J}_i(t_f) + \sum_{j\in\mathcal{N}_i}\mathsf{J}_j(t_f) = e_i^{\mathrm{T}}(t_f)P_ie_i(t_f) + J_{i\xi}(t_f),$$

where $J_{i\xi}(t_f) = -2d_i^2\int_{t_f}^{\infty} e_i^{\mathrm{T}}(t)P_iB\hat{\Gamma}_iR^{-1}f_i(\xi_{iM}+\xi_i)B^{\mathrm{T}}P_i\mathrm{d}t$. Since $e_i(t)$ is asymptotically stable, $J_{i\xi}(t_f)$ will be bounded and this completes the proof. □

3.2.2 Control Recovery Subject to the Stuck and the Outage Faults

Provided that the k-th actuator of the i-th agent freezes at a particular constant value of $\underline{u}_i^k$ corresponding to the stuck fault, the dynamics of the i-th faulty agent is then governed by

$$\dot{e}_i(t) = Ae_i(t) + d_iBu_i^f(t) - \sum_{j\in\mathcal{N}_i}Bu_j(t), \qquad (3.51)$$

and the dynamics of the nearest neighbor agents of the faulty agent is

$$\dot{e}_l(t) = Ae_l(t) + d_lBu_l(t) - Bu_i^f(t) - \sum_{j\in\mathcal{N}_l, j\neq i}Bu_j(t),\ l\in\mathcal{N}_i, \qquad (3.52)$$

where $u_i^f(t) = \Big[u_i^1(t),\ldots,u_i^{k-1}(t),\underline{u}_i^k,u_i^{k+1}(t),\ldots,u_i^m(t)\Big]^{\mathrm{T}}$. The dynamics of the remaining healthy agents that are not communicating with the i-th faulty agent are governed by equation (3.31). Consider the system (3.4). In order to design a reconfigured control law $u_i^r(t)$ that accommodates the effects of these two faults three scenarios are considered, namely:

Scenario (i) $\underline{u}_i^k = 0$,

Scenario (ii) $\underline{u}_i^k \neq 0$, however the FDI module provides the exact value of $\underline{u}_i^k$ and $b^k \subset \mathrm{Im}\{\underline{B}^{fk}\}$. This scenario is now designated as the **Condition (a)** in the remainder, and

Scenario (iii) $\underline{u}_i^k \neq 0$ however either the FDI module estimation information is inaccurate, i.e. $\underline{u}_i^k = \hat{\underline{u}}_i^k + \xi_i^k$ with $|\xi_i^k| \leq \xi_{iM}^k$ or $b^k \not\subset \mathrm{Im}\{\underline{B}^{fk}\}$.

Below the specific control laws for each scenario is developed and presented.

Scenario (i): This case refers to the outage fault and the structure of the reconfigured control law is now considered as $u_i^r(t) = K_i^r e_i(t)$. By following along the same steps as those provided for the LOE fault in Theorem 3.2, a similar optimization problem can be obtained. The solution to this problem can be used to compute the control gains K_i^r. The optimization problem and design of the control gains are provided in Theorem 3.3.

Scenario (ii): For this case, the reconfigured control law is selected as $u_i^r(t) = K_i^r e_i(t) + \underline{u}_i^r$ where K_i^r is designed as in the Scenario (i) above. If $b^k \subset \mathrm{Im}\{\underline{B}^{fk}\}$ and the exact value of $\underline{u}_i^k$ is available then there exists $\underline{u}_i^r$ such that

$$\sum_{j=1, j\neq k}^{m} b^j \underline{u}_i^{jr} = -b^k \underline{u}_i^k. \tag{3.53}$$

That is the effects of the stuck fault can be fully compensated for, where $\underline{u}_i^{jr}$ is the j-th element of $\underline{u}_i^r$ and b^j is the j-th column of the matrix B. However, in order to have a finite performance index, a modified cost performance index is now defined for the i-th faulty agent as

$$J_i^M(t_f) = \mathsf{J}_i^M(t_f) + \sum_{j\in\mathcal{N}_i} \mathsf{J}_j(t_f), \tag{3.54}$$

where $J_i^M(t_f) \equiv J_i^M(e_i(t_f), \tilde{u}_i(.), e_j(t_f), u_j(.), j \in \mathcal{N}_i, t_f)$, $\mathsf{J}_i^M(t_f) \equiv \mathsf{J}_i^M(e_i(t_f), \tilde{u}_i(.), t_f) = \int_{t_f}^{\infty}(e_i^{\mathrm{T}}(t)Qe_i(t) + \tilde{u}_i^{\mathrm{T}}(t)\bar{R}\tilde{u}_i(t))\mathrm{d}t$, with J_j defined as in equation (3.6), $\tilde{u}_i(t) = u_i^r(t) - u_i^s$, $u_i^s = \lim_{t\to\infty} u_i^r(t)$, and $\bar{R}$ is obtained by removing the k-th column and row of the matrix R. In both the above two cases, as shown in Theorem 3.3 subsequently, one can guarantee that the consensus error converges to zero asymptotically.

Scenario (iii): If Condition (a) does not hold and there does not exist a vector $\underline{u}_i^r$ such that equation (3.53) holds, the error due to the stuck fault cannot be fully accommodated for. Therefore, one has to accept a degraded performance objective and design the reconfigured control law that guarantees only <u>boundedness</u> of the consensus error. In this scenario, the control law is selected as $u_i^r(t) = K_i^r e_i(t) + \underline{u}_i^r$, where K_i^r is designed as in Scenario (i) and $\underline{u}_i^r$ is selected as the solution to:

$$\min_{\underline{u}_i^r} \sum_{j=1, j\neq k}^{m} |b^j \underline{u}_i^{jr} + b^k \underline{\hat{u}}_i^k|, \tag{3.55}$$

where $\underline{\hat{u}}_i^k$ is the estimated value of the stuck fault generated by the FDI module. This control law stabilizes the consensus error of the faulty agent and the consensus error bound will be $e_i^s = \lim_{t\to\infty} e_i(t) = A_{cl}\Pi_i^k$, where $\Pi_i^k = \sum_{j=1,j\neq k}^m b^j \underline{u}_i^{jr} + b^k \underline{u}_i^k$ and A_{cl} denotes the dynamics of the faulty agent consensus error and is the i-th block of the block-matrix $\mathcal{A}_{cl}^{-1}.(l_i^{\mathrm{T}} \otimes I_n)$ where $\mathcal{A}_{cl} = -(\mathcal{A} + (L_{22} \otimes I_n)\mathcal{B}^f\mathcal{K})$, $\mathcal{B}^f = \mathrm{diag}\{B, \ldots, \underline{B}^{fk}, \ldots, B\}$, $\mathcal{K} = \mathrm{diag}\{K, \ldots, K_i^r, \ldots, K\}$.

The next theorem summarizes the above results formally.

Theorem 3.3. *Consider a team of N multi-agent systems where the control law for the healthy agent is governed according to Theorem 3.1.*

(A) *Suppose at $t = t_f$ the k-th actuator of the i-th agent is subject to the outage fault. The distributed reconfigured control law $u_i^r(t) = K_i^r e_i(t)$ guarantees the team consensus and the local cost performance index (3.5) is given by $J_i^*(t_f) = e_i^T(t_f)P_i e_i(t_f)$ if the following optimization problem has a positive definite solution to P_i, namely*

$$
\begin{aligned}
\max \quad & \text{trace } P_i \\
s.t. \quad & \begin{bmatrix} \Phi_i & \mathcal{P}_i\mathcal{F}_{ii}^{\frac{1}{2}} \\ \mathcal{F}_{ii}^{\frac{1}{2}}\mathcal{P}_i & I \end{bmatrix} \geq 0 \\
& d_i(\mathcal{P}_i\mathcal{F}_{il}\mathcal{P}^h + \mathcal{P}^h\mathcal{F}_{il}^T\mathcal{P}_i) \leq \mathcal{P}^h\bar{\mathcal{F}}_{il}\mathcal{P}^h + \mathcal{P}^h\bar{\mathcal{F}}_{il}^T\mathcal{P}^h,\ l \in \mathcal{N}_i, \\
& \mathcal{P}_i = I_N \otimes P_i > 0,
\end{aligned}
\tag{3.56}
$$

where $J_i^*(t_f) \equiv J_i^*(e_i(t_f), t_f)$, $K_i^r = -d_i\bar{R}^{-1}\underline{B}^{fk^T}P_i$, $\Phi_i = \mathcal{A}_i^T\mathcal{P}_i + \mathcal{P}_i\mathcal{A}_i + \mathcal{Q}_i + \mathcal{P}_i\sum_{j\in\mathcal{N}_i}\bar{\mathcal{F}}_{ij}\mathcal{P}^h + \sum_{j\in\mathcal{N}_i}\mathcal{P}^h\bar{\mathcal{F}}_{ji}\mathcal{P}_i + \sum_{j\in\mathcal{N}_i}\mathcal{Q}_j + \mathcal{P}^h\bar{\mathcal{F}}_{jj}\mathcal{P}^h$, $\mathcal{P}^h$ is obtained from Theorem 3.1, $\bar{\mathcal{F}}_{ij}$, $\mathcal{A}_i$ and $\mathcal{Q}_i$ are defined as in equation (3.38), $\mathcal{F}_{il} = \mathcal{L}_{il} \otimes F_i$, $\mathcal{F}_{ii} = \mathcal{L}_{ii} \otimes F_i$, and $F_i = d_i^2\underline{B}^{fk}\bar{R}^{-1}\underline{B}^{fk^T}$.

(B) *Suppose at $t = t_f$ the k-th actuator of the i-th agent freezes at the specified value $\underline{u}_i^k$ and let the reconfigured control law be selected as $u_i^r(t) = K_i^r e_i(t) + \underline{u}_i^r$. It can then be shown that:*

(B-1) *The control $u_i^r(t)$ guarantees that the team reaches a consensus and the local cost performance index (3.54) is given by $J_i^{M*}(t_f) = e_i^T(t_f)P_i e(t_f)$, if Condition (a) holds; where $J_i^{M*}(t_f) \equiv J_i^{M*}(e_i(t_f), t_f)$, $K_i^r = -d_i\bar{R}^{-1}\underline{B}^{fk^T}\mathcal{P}_i$, $\mathcal{P}_i$ is the solution to the maximization problem (3.56), and $\underline{u}_i^r$ is a solution to equation (3.53).*

(B-2) *The control $u_i^r(t)$ guarantees that consensus error of the faulty agent remains bounded and the local cost performance index (3.54) is given by $J_i^{M*}(t_f) = e_i^T(t_f)P_ie_i(t_f) + J_{id}(t_f)$; where $J_i^{M*}(t_f) \equiv J_i^{M*}(e_i(t_f), t_f)$, $K_i^r = -d_i\bar{R}^{-1}\underline{B}^{fk^T}\mathcal{P}_i$, $\mathcal{P}_i$ is the solution to the maximization problem (3.56), $\underline{u}_i^r$ and Π_i^k are defined as in the problem (3.55), $J_{id}(t_f) \equiv J_{id}(e(t_f), t_f) = 2\Pi_i^k P_i\bar{e}_i - e_i^{sT}(t_f)P_ie_i^s$ and $\bar{e}_i = \int_{t_f}^{\infty} e_i(t)dt$.*

Proof. Consider $V_i(t) = e_i^{\mathrm{T}}(t)P_ie_i(t)$ as a Lyapunov function candidate for the system (3.51) and let us replace $B\Gamma_iu_i(t)$ by $B_i^fu_i^r(t) + B_i^s\underline{u}_i^s$. By following along the same lines as provided in the proof of Theorem 3.2, the above results can be shown and concluded. These details are not included due to space limitations. □

3.2.3 Control Recovery Subject to Subsequent Concurrent Faults

In the previous two subsections, it was assumed that the faulty agent is subject to only one type of fault and only associated with the LOE fault type one can have multiple faults in different actuator channels and all the neighbors of the faulty agent are healthy. In this subsection, our previous results are extended to the cases where the faulty agents are subject to concurrent multiple fault types and when subsequent faults do occur in faulty agents. Towards this end, and without loss of generality, let us assume that the i-th agent with m actuators becomes faulty, such that the first m_{1i} actuators are subject to the LOE fault, the actuators $m_{1i} + 1$ to m_{2i} are subject to the outage fault, the actuators $m_{2i} + 1$ to m_{3i} are subject to the stuck fault, and finally the actuators $m_{3i} + 1$ to m are healthy.

In this case, the consensus error dynamics of the i-th agent becomes

$$\dot{e}_i(t) = Ae_i(t) + d_i B_i^f u_i^f(t) - \sum_{j\in\mathcal{N}_i} Bu_j(t), \tag{3.57}$$

where $B_i^f = \underline{B}_i\Gamma_i$, $\underline{B}_i = \begin{bmatrix}\underline{B}_{i1} & \underline{B}_{i3} & \underline{B}_{i4}\end{bmatrix}$, $\underline{B}_{i1} = \begin{bmatrix}b^1 & \dots & b^{m_{1i}}\end{bmatrix}$, $\underline{B}_{i2} = \begin{bmatrix}b^{m_{1i}+1} & \dots & b^{m_{2i}}\end{bmatrix}$, $\underline{B}_{i3} = \begin{bmatrix}b^{m_{2i}+1} & \dots & b^{m_{3i}}\end{bmatrix}$, $\underline{B}_{i4} = \begin{bmatrix}b^{m_3+1} & \dots & b^{m}\end{bmatrix}$, $u_i^f(t) = \begin{bmatrix}u_i^{loe}(t)^{\mathrm{T}} & u_i^s(t_f)^{\mathrm{T}} & u_i^h(t)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, $u_i^{loe}(t) = \begin{bmatrix}u_i^1(t) & \dots & u_i^{m_{1i}}(t)\end{bmatrix}^{\mathrm{T}}$, $u_i^s(t_f) = \begin{bmatrix}u_i^{m_{2i}+1}(t_f) & \dots & u_i^{m_{3i}}(t_f)\end{bmatrix}^{\mathrm{T}}$, $u_i^h(t) = \begin{bmatrix}u_i^{m_{3i}+1}(t) & \dots & u_i^{m}(t)\end{bmatrix}^{\mathrm{T}}$ and $\Gamma_i = \mathrm{diag}\{\Gamma_i^1, \dots, \Gamma_i^{m_{1i}}, \underbrace{1, \dots, 1}_{m-m_{2i}}\}$. Comparing equation (3.57) with equations (3.29) and (3.51), and considering the results that are stated in Theorems 3.2 and 3.3, justifies that the reconfigured control law can be selected as

$$u_i^r(t) = K_i^r e_i(t) + \underline{u}_i^r,$$

where $K_i^r = -d_i R^{-1}\hat{\Gamma}_i \bar{B}_i^{f^{\mathrm{T}}} P_i$, P_i is the solution to the maximization problem (3.41) for $\tilde{F}_i = \underline{B}_i^f(\hat{\Gamma}_i - 2f_i\xi_{iM})R^{-1}\hat{\Gamma}_i\underline{B}_i^f$, $\bar{B}_i^{f^{\mathrm{T}}} = \begin{bmatrix}\underline{B}_{i1} & 0_{n\times(m_{3i}-m_{1i}+1)} & \underline{B}_{i4}\end{bmatrix}$, $\underline{B}_i^f = \begin{bmatrix}\underline{B}_{i1} & \underline{B}_{i4}\end{bmatrix}$, and the remaining variables are the same as those defined in Theorem 3.2. Furthermore, $\underline{u}_i^r$ can be obtained as the solution to

$$\min_{\underline{u}_i^r} \sum_{k=1}^{m_{1i}} |b^k\hat{\Gamma}_i^k\underline{u}_i^{kr} + \sum_{k=m_{3i}+1}^{m} b^k\underline{u}_i^{kr} + \sum_{k=m_{2i}+1}^{m_{3i}} b^k\hat{\underline{u}}_i^k|.$$

Let us now consider that one of the i-th faulty agent neighbors, say agent l, becomes also faulty. The dynamics of the l-th faulty agent can be expressed as

$$\dot{e}_l(t) = Ae_l(t) + d_l B_l^f u_l^f(t) - B_i^f u_i^r(t) - \sum_{j\in\mathcal{N}_l, l\neq i} Bu_j(t) - \Pi_i, \tag{3.58}$$

where B_l^f, $u_l^f(t)$ are as defined in (3.57) for $l = i$ and $\Pi_i = \sum_{k=1}^{m_{1i}} b^k\Gamma_l^k\underline{u}_i^{kr} + \sum_{k=m_{3i}+1}^{m} b^k\underline{u}_i^{kr} + \sum_{k=m_{3i}+1}^{m} b^k\underline{u}_i^{kr} + \sum_{k=m_{2i}+1}^{m_{3i}} b^k\underline{u}_i^k$.

Consider the dynamics of the consensus errors that are governed by equations (3.57) and (3.58) and the results that are obtained in Theorems 3.2 and 3.3. It can be shown that the reconfigured control law for the l-th agent can be designed in the same manner as that of the i-th agent by utilizing the matrices P_i and F_i for the i-th faulty neighbor of the l-th agent alternatively as opposed to P^h and $\bar{F}$ that are given in Theorems 3.2 and 3.3. Note that if the stuck faults are as specified in Scenario (iii) then $\Pi_i \neq 0$ and the reconfigured control law can only guarantee the boundedness of the consensus error. These details are omitted due to space limitations.

3.3 Simulation Results

In this section, our proposed control recovery approaches are applied and evaluated to a leader-follower as well as a leaderless network of Autonomous Underwater Vehicles (AUVs). Towards this end, two AUVs networks are considered and the team behavior and its performance under several scenarios are investigated. Specifically, we consider cases when **(a)** the agents are healthy, **(b)** when the agents are subject to the LOE, the outage and the stuck actuator faults, and **(c)** when there are delays in generating the FDI information. The team consists of Sentry Autonomous Underwater Vehicles (AUVs). Sentry, made by the Woods Hole Oceanographic Institution [171], is a fully autonomous underwater vehicle which is capable of surveying to the depth of 6000 m. The linearized model of the sentry is of six degrees of freedom, but it is commonly decomposed into four non-interacting subsystems, namely: the speed subsystem (u), the steering subsystem (v, r, ψ), the diving subsystem (ω, q, z, θ) and the roll subsystem (ϕ).

3.3.1 Leader-follower (LF) Network of AUVs

In the first part of simulations, our mission is to have all the agents maneuver at a given depth. Therefore, we only consider the diving subsystem. The diving subsystem of the i-th agent is governed according to equation (2.1) with

$$A = \begin{bmatrix} -0.51 & 0.56 & 0 & 0.001 \\ 0.22 & 0.35 & 0 & -0.01 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0.143 & 0.118 \\ -0.126 & 0.156 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

, $x_i(t) = \begin{bmatrix} \omega_i(t), q_i(t), z_i(t), \theta_i(t) \end{bmatrix}^{\mathrm{T}}$, $u_i(t) = \begin{bmatrix} \delta_i^b(t), \delta_i^s(t) \end{bmatrix}^{\mathrm{T}}$, where $\omega_i(t)$, $q_i(t)$, $z_i(t)$, $\theta_i(t)$, $\delta_i^b(t)$ and $\delta_i^s(t)$ denote the heave speed, pitch rate, depth, pitch, bone and stern plane deflections, respectively.

First, let us consider the network communication graph as shown in Figure 3.1. It is assumed that the agent 0 designates the team leader and its desired depth trajectory represents a trapezoid as shown in Figures 3.2-3.4, and where agents 1 to 6 designate the followers.

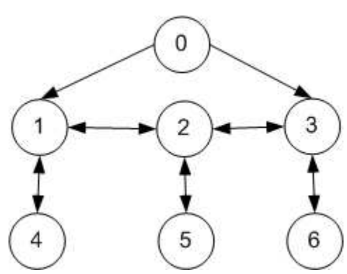

Figure 3.1: The topology of the leader-follower (LF) network of AUVs.

The initial state of the vehicles is considered as $x_0^0 = \begin{bmatrix}3, 0, 500, 2\end{bmatrix}^{\mathrm{T}}$, $x_0^1 = \begin{bmatrix}2, 0, 450, -1\end{bmatrix}^{\mathrm{T}}$, $x_0^2 = \begin{bmatrix}1, 0, 480, -3\end{bmatrix}^{\mathrm{T}}$, $x_0^3 = \begin{bmatrix}3, 0, 550, 1\end{bmatrix}^{\mathrm{T}}$, $x_0^4 = \begin{bmatrix}4, 0, 600, 4\end{bmatrix}^{\mathrm{T}}$, $x_0^5 = \begin{bmatrix}2, 0, 510, -1\end{bmatrix}^{\mathrm{T}}$, and $x_0^6 = \begin{bmatrix}2.5, 0, 500, -5\end{bmatrix}^{\mathrm{T}}$. Suppose the mission starts at $t = 0\ s$ and is terminated at the time $t = 1000\ s$, and its objective is to have all the six followers follow the leader depth trajectory. The following Scenarios A-E are considered for conducting our simulation case studies where the summary of the agents and the team performance indices are provided in Table 3.1.

Scenario A: In this case the performance and behavior of the healthy team is investigated. Therefore, the agents control laws are designed and specified according to Theorem 3.1. The results are shown in Figure 3.2.

In Scenarios B, C and D below the behavior and performance of the team subject to the LOE fault, the outage fault and the stuck fault are studied, respectively. Towards this end, during $0 \leq t < 200\ s$ the agents and their control laws are considered to be the same as those governed by the Scenario A. At time $t = 200\ s$ the faults are injected and after a delay of $\Delta\ s$ the control reconfiguration for the faulty agent is initiated. The details corresponding to each scenario are provided below:

Scenario B: The LOE fault is injected to the agent 5 where the effectiveness of the first and the second actuators are reduced to 50% and 30% of their nominal values respectively, and the fault severity estimation uncertainty is considered to be 10% and Δ's are selected as $\Delta = 250\ s$, $400\ s$, $500\ s$ corresponding to consecutively larger delays in invoking the recovery strategy.

<u>**Scenario C**</u>: The outage fault is injected to the second actuator of the agent 1 and Δ's are selected as $\Delta = 20\ s,\ 40\ s$, and $60\ s$.

<u>**Scenario D:**</u> The stuck fault is injected to the second actuator of the agent 3. The stuck fault is considered as $u_3^2 = 1$ and Δ's are selected as $\Delta = 10\ s,\ 25\ s$, and $35\ s$.

<u>**Scenario E:**</u> In this scenario the effects of the concurrent outage and LOE faults are considered. For this purpose, at $t_1 = 200\ s$ the outage fault is injected to the second actuator of the agent 4 and at $t_2 = 500\ s$, a 50% loss of effectiveness is injected to the second actuator of the agent 3. The initiations of the recovery strategy delay for the first and the second faults are considered as $\Delta_1 = 50\ s,\ 100\ s$, and $150\ s$ and $\Delta_2 = 100\ s,\ 200\ s$, and $300\ s$, respectively.

<u>**Scenario F**</u>: In this scenario <u>simultaneous faults</u> are considered. For this purpose at $t = 200\ s$ an outage fault is injected to the second actuator of the agent 1 and a 50% loss of effectiveness is injected to the second actuator of the agent 4. The agent 1 is reconfigured immediately and the agent 4 is reconfigured after one second of delay.

The agents and the team performance indices for the above scenarios are given in Table 3.1, but due to space limitations, only the depth trajectories for certain scenarios are shown in Figures 3.2, 3.3 and 3.4.

Finally, in order to evaluate and compare the performance capabilities of our proposed approach, we also consider a control reconfiguration design strategy as implemented by a High-Level Supervisor (HLS), along the lines that are proposed in [142]. Specifically, the scenarios A-E are repeated by using this control strategy that is based on an LQR control reconfiguration approach. The supervisor receives the FDI information from all the agents and redesigns the agents control laws according to an LQR approach and by considering the network structure.

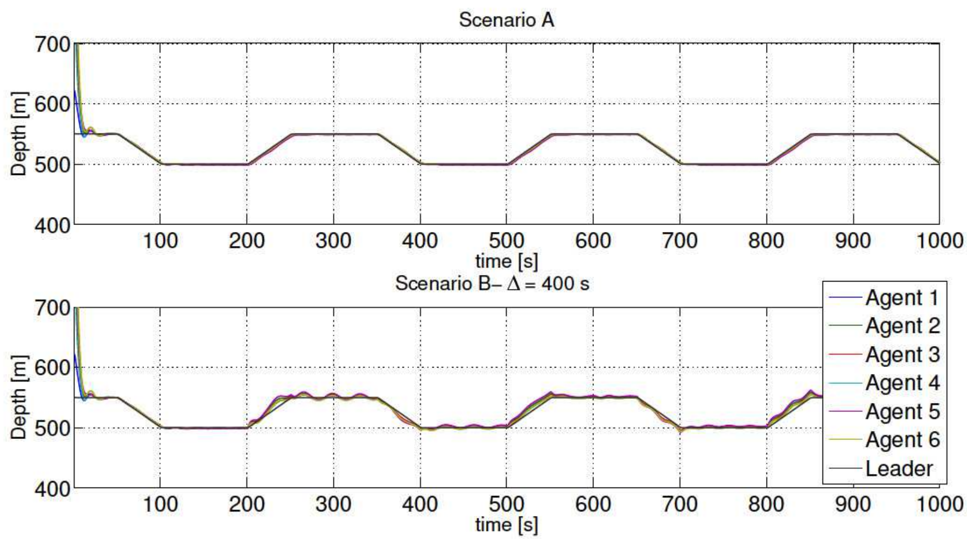

Figure 3.2: The depth trajectories of the agents corresponding to the Scenarios A and B.

The following is a summary of our general observations and conclusions that are derived based on the above results:

- In presence of the LOE fault the team remains stable. However, the fault deteriorates the team performance and by reconfiguring the faulty agent control law the team performance is shown to be improved.
- In presence of the outage fault in one agent the entire team becomes unstable. Fig-

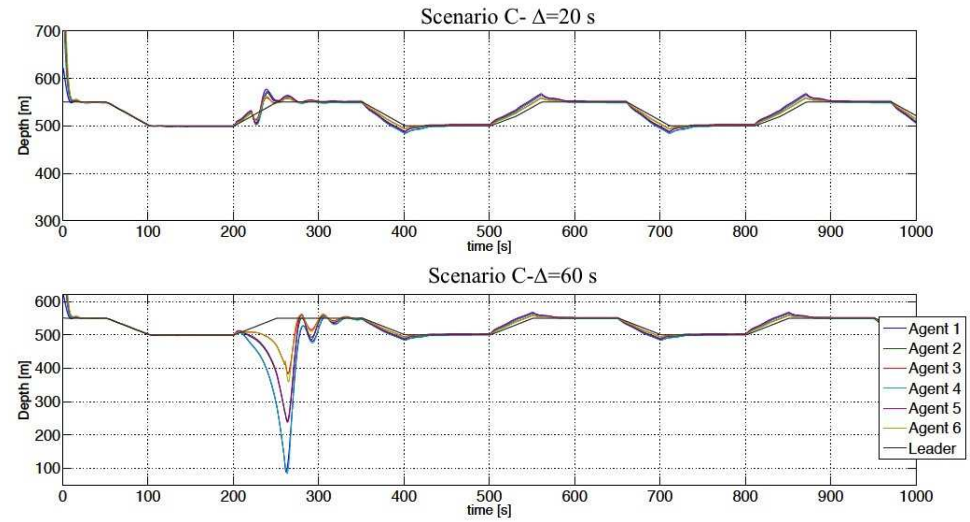

Figure 3.3: The depth trajectories of the agents corresponding to the Scenario C.

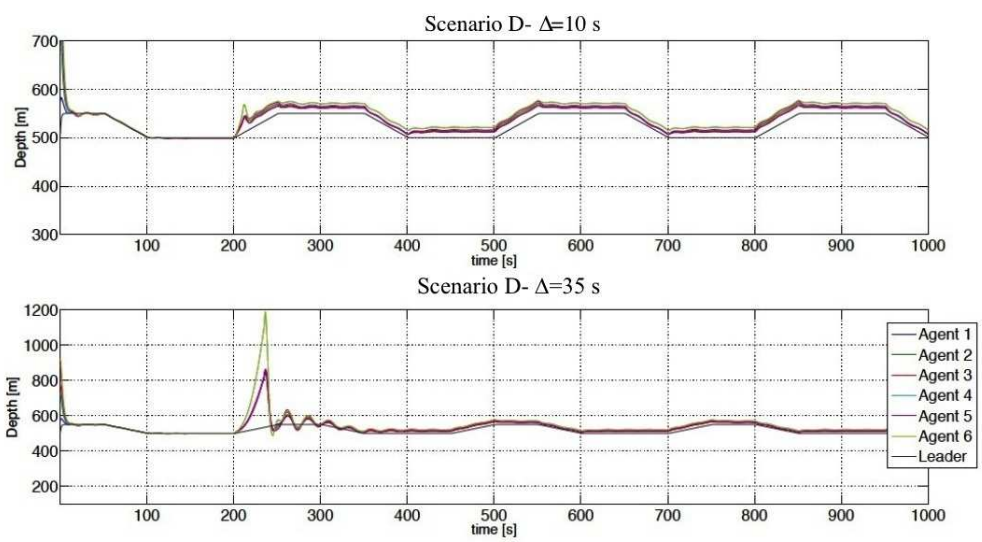

Figure 3.4: The depth trajectories of the agents corresponding to the Scenario D.

Scenarios and Delays in Recovery Invocation	J_1	J_2	J_3	J_4	J_5	J_6	J	J/J_A	J/J_A with LQR based by using HLS [142]
A	4.29e+06	4.81e+05	2.41e+06	1.93e+06	7.75e+04	2.38e+05	9.42e+06	1	1
B; $\Delta = 250$	4.31e+06	5.08e+05	2.44e+06	1.94e+06	2.60e+05	2.38e+05	9.69e+06	1.028	1.022
B; $\Delta = 400$	4.31e+06	5.15e+05	2.44e+06	1.95e+06	2.65e+05	2.41e+05	9.72e+06	1.030	1.027
B; $\Delta = 500$	4.32e+06	5.19e+05	2.44e+06	1.95e+06	2.73e+05	2.41e+05	9.74e+06	1.034	1.032
C; $\Delta = 20$	1.94e+07	5.13e+05	2.43e+06	1.94e+06	8.94e+04	2.36e+05	2.47e+07	2.62	2.48
C; $\Delta = 40$	2.60e+07	5.19e+05	2.43e+06	1.95e+06	9.76e+04	2.40e+05	3.12e+07	3.31	2.94
C; $\Delta = 60$	6.58e+07	5.54e+05	2.45e+06	2.01e+06	1.60e+05	2.69e+05	7.12e+07	7.55	4.87
D; $\Delta = 10$	4.23e+06	5.09e+05	3.10e+06	4.53e+06	1.15e+05	2.55e+05	1.29e+07	1.36	1.182
D; $\Delta = 25$	4.43e+06	5.11e+05	3.15e+06	5.05e+06	1.19e+05	2.57e+05	1.35e+07	1.43	1.23
D; $\Delta = 35$	4.59e+06	4.54e+05	4.70e+06	2.47e+07	2.19e+05	2.95e+05	3.51e+07	3.72	2.97
E; $\Delta_1 = 50, \Delta_2 = 100$	4.33e+06	4.93e+05	2.45e+06	3.82e+08	8.81e+04	2.38e+05	3.90e+08	40.5	36.23
E; $\Delta_1 = 100, \Delta_2 = 200$	4.59e+06	5.73e+05	2.49e+06	4.34e+08	2.08e+05	2.97e+05	4.42e+08	45.89	40.85
E; $\Delta_1 = 150, \Delta_2 = 300$	1.28e+07	3.02e+06	3.62e+06	2.14e+09	3.91e+06	2.17e+06	2.17e+09	225.5	158.14
F; $\Delta_1 = 0, \Delta_2 = 1$	1.45e+07	6.04e+05	2.50e+06	1.11e+07	1.66e+05	4.67e+05	3.04e+07	3.23	3.02

Table 3.1: The performance indices corresponding to the leader-follower (LF) network (note Δ denotes the delay in invoking the control recovery strategy, J denotes the team performance index (with possibly faulty agents) and J_A denotes the healthy team performance index).

ure 3.3 confirms that the proposed reconfiguration control law strategy can accommodate this fault successfully. Furthermore, Figure 3.3 shows that in this scenario the team performance is quite sensitive to the FDI and control recovery delay and the mission will fail completely if the fault is not recovered before $60\ s$. Therefore, $\Delta = 60$ s can be considered as the largest tolerable FDI delay. Note that no simulations were conducted with larger delays.

- The team behavior that is subject to the stuck fault is similar to that of the outage fault. This implies that this fault can also make the team unstable. The reconfigured control law ensures that the team that is subject to the stuck fault in one agent, such as the agent 3, has a stable consensus error. Since in this case Condition (a) does not hold as stated in Theorem 3.3-B2, it follows from Figure 3.4 that the consensus errors remain only bounded and do not converge to zero.

- The index J/J_A represents the ratio of the team performance index (J) corresponding to each scenario with respect to the healthy team performance index (J_A). This index illustrates how different fault scenarios and the FDI and control recovery delays affect the team performance. For example, the delay of $10\ s$ in Scenario C increases the team performance by 153%, whereas the same delay for the Scenario D increases the performance by only 36%.

- Comparing the results of the team that employs our distributed control reconfiguration strategy with the team that employs the LQR-based control recovery designed by the HLS [142], it can be concluded (as shown in the last two columns of Table 3.1) that when the team is subject to LOE fault or stuck/outage with small

FDI delays both approaches result in a similar cost performance values, whereas when the team is subject to outage or stuck fault and large FDI delays the LQR-based scheme performance is better. In fact, the team that employs the LQR-based scheme has a faster response and deviations of the agents state from the leader state are compensated in a shorter time period and the overall team performance was improved. Regarding this observation we should point out that: first, the design procedure in LQR-based approach is centralized and requires the team information while our approach is distributed and uses only the local information. Second, these scenarios (stuck and outage) correspond to unstable systems where the deviations are large, so that the fault can be detected quickly or the FDI and control recovery delays can be kept small.

3.3.2 Leaderless (LL) Network of AUVs

In the second set of simulation studies, let us assume that the agent 0 is removed from the network. This implies that one now has a leaderless (LL) network consisting of six AUVs and the resulting communication graph is the same as that of the followers communication graph shown in Figure 3.1. The agents initial conditions are selected to be the same as that of the initial conditions of the agents 1 to 6 in the LF network and similar fault scenarios are now considered, except that in this case study the fault injection time is considered to be at $t = 10\ s$ and various FDI and control recovery delays are considered. Figures 3.5 and 3.6 depict the simulated leaderless team which show that the team remains stable in presence of various actuator fault scenarios (this is achieved by reconfiguring the control laws where the final consensus state becomes close to that of the healthy team state). The ratio of the team performance index corresponding to each scenario with respect to the healthy team performance index are also given in the last column of Table 3.2. These ratios show how the reconfigured control improves the performance of the faulty team.

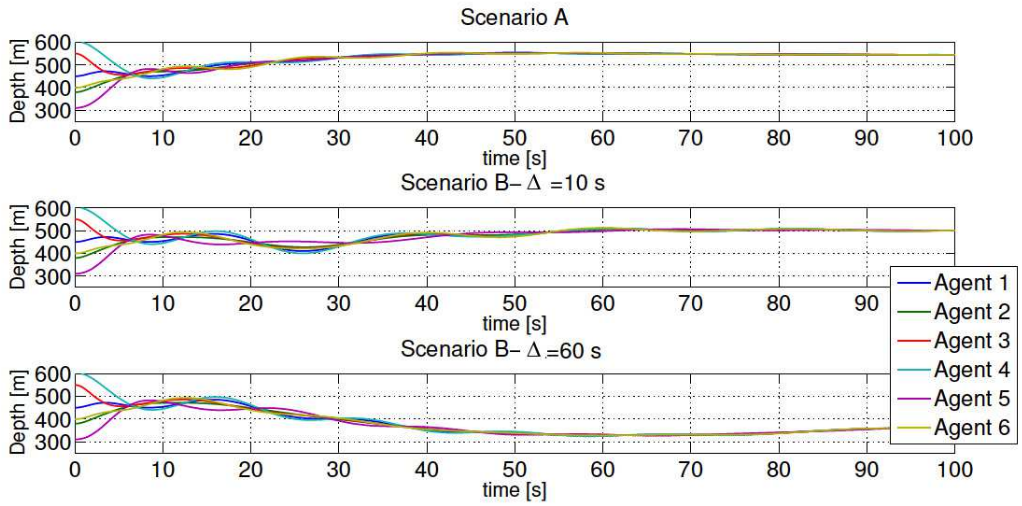

Figure 3.5: The depth trajectories of the agents corresponding to the Scenarios A and B.

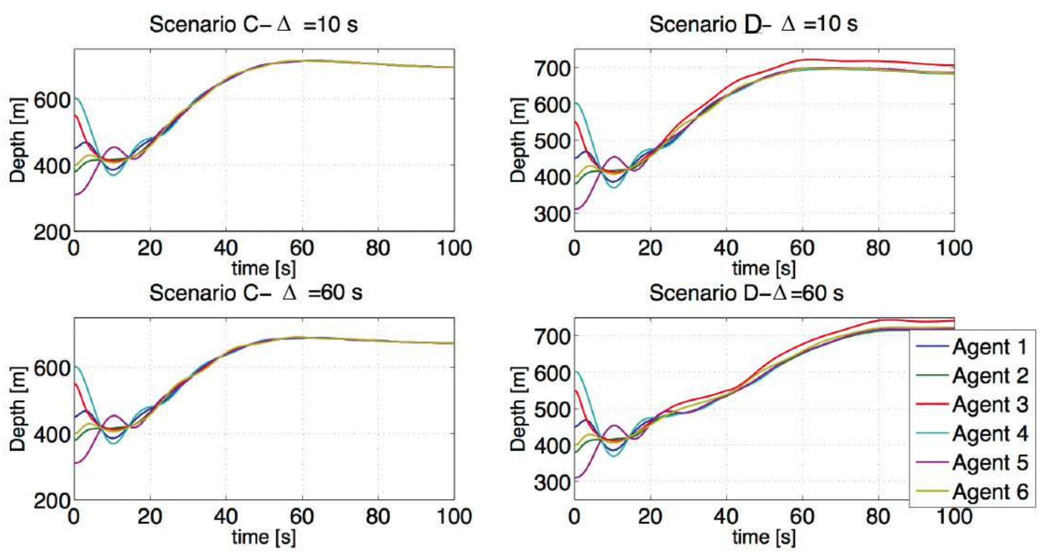

Figure 3.6: The depth trajectories of the agents corresponding to the Scenarios C and D.

Scenarios and Delays in Recovery Invocation	J_1	J_2	J_3	J_4	J_5	J_6	J	J/J_A
A	7.14e+05	2.78e+06	9.87e+06	2.71e+06	1.35e+06	2.41e+06	1.98e+07	1
B; $\Delta = 10$	7.88e+05	3.00e+06	1.06e+07	3.14e+06	1.93e+06	2.52e+06	2.19e+07	1.106
B; $\Delta = 30$	7.80e+05	3.02e+06	1.06e+07	3.14e+06	1.96e+06	2.54e+06	2.21e+07	1.115
B; $\Delta = 60$	8.32e+05	3.03e+06	1.07e+07	3.19e+06	2.10e+06	2.54e+06	2.24e+07	1.131
C; $\Delta = 10$	7.41e+05	2.99e+06	1.06e+07	3.09e+06	1.58e+06	2.27e+06	2.13e+07	1.075
C; $\Delta = 30$	8.24e+05	3.00e+06	1.06e+07	3.26e+06	1.73e+06	2.52e+06	2.19e+07	1.106
C; $\Delta = 60$	8.27e+05	3.00e+06	1.06e+07	3.26e+05	1.73e+06	2.53e+04	2.23e+07	1.126
D; $\Delta = 10$	7.38e+05	3.00e+06	1.07e+07	3.16e+06	1.60e+06	2.40e+06	2.16e+07	1.091
D; $\Delta = 30$	1.45e+06	3.65e+07	1.17e+07	4.08e+06	2.40e+06	3.11e+06	2.64e+07	1.333
D; $\Delta = 60$	1.97e+06	4.15e+06	1.23e+07	4.64e+06	2.89e+06	3.59e+06	2.95e+07	1.490

Table 3.2: The performance indices corresponding to the leaderless (LL) network (note Δ denotes the delay in invoking the control recovery strategy, J denotes the team performance index (with possibly faulty agents) and J_A denotes the healthy team performance index).

3.3.3 Comparative Evaluation with an Alternative Method

In order to compare our results with an alternative methodology in the literature, the decentralized control approach proposed in [142] is applied to the LL network of AUVs in which the 4-th agent is subject to the LOE fault as stated in Scenario B. The consensus errors corresponding to this approach as well as our approach are shown in Figure 3.7. It can be seen that in both approaches the errors converge to zero, however the convergence of the errors in the decentralized approach of [142] is less oscillatory than our approach. However, in the reconfigured control law in [142] all agents FDI information and measurements should be sent to a high-level supervisor which designs the reconfigured control whereas in our approach only the nearest neighbor agents share their FDI information and measurements and local information are taking into account in the reconfigured control design.

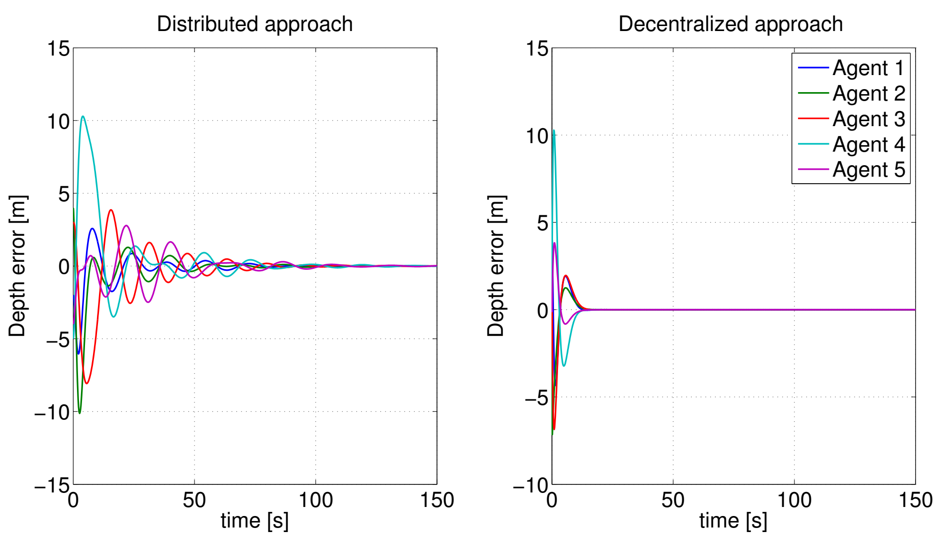

Figure 3.7: The depth trajectories errors corresponding to the agents 1 to 5 by comparing our proposed scheme with that of the decentralized scheme.

3.4 CONCLUSIONS

In this chapter, consensus achievement in multi-agent systems in presence of actuator faults is investigated and distributed reconfigurable/recovery control strategies are developed to accommodate for the concurrent and simultaneous actuator faults in the team. Recovery control strategies are developed by solving local Hamilton-Jacobi-Belman equations to asymptotically stabilize the team consensus errors and to minimize the agent-based cost performance indices. The proposed control recovery approaches are designed for networks having leaderless (LL) and leader-follower (LF) topological architectures and are applied to a team of autonomous underwater vehicles (AUVs). The team behavior and the agent-level as well as the team-level performance indices subject to various fault scenarios are studied. Simulation results confirm the effectiveness of our proposed reconfiguration/recovery strategies in accommodating actuator faults in multi-agent systems.

Chapter 4

H_∞ based Fault accommodation

In this chapter, an H_∞ performance fault recovery control problem for a team of multi-agent systems that is subject to actuator faults is studied. The main objective here is to design a distributed control reconfiguration strategy for a team with faulty agents such that: (a) the state consensus errors remain bounded, and the output of the faulty system behaves exactly the same as that of the healthy system in disturbance free environment and (b) the specified H_∞ performance bound is guaranteed to be minimized in presence of bounded energy disturbances. The gains of the reconfigured control laws are selected first by employing a geometric approach where a set of controllers guarantees that the output of the faulty agent imitates that of the healthy agent and the consensus achievement objectives are satisfied. Next, the remaining degrees of freedom in selection of the control law gains are used to minimize the bound on a specified H_∞ performance index.

The effects of uncertainties and imperfections in the FDI module decision in correctly estimating the fault severity as well as delays in invoking the reconfigured control laws are investigated and a bound on the maximum tolerable estimation uncertainties are obtained. The proposed distributed and cooperative control recovery approach is applied to a team of five autonomous underwater vehicles to demonstrate its capabilities and effectiveness in accomplishing the overall team requirements subject to various actuator faults, delays in invoking the recovery control, fault estimation and isolation imperfections and unreliabilities under various control recovery scenarios.

The remainder of this chapter is organized as follows. In Section 4.1, the problem formulation are provided. Section 4.2 presents the distributed H_∞ performance control reconfiguration methodology developed corresponding to actuator faults is presented. In Section 4.3, proposed control reconfiguration methodology is applied to a network of autonomous underwater vehicles (AUVs) subject to actuator faults and ocean current disturbances are validated through extensive simulations. Finally, Section 4.4 concludes the chapter. A summary of the following is presented in [148, 172, 173].

4.1 Problem Formulation

In this work, our main goal and objective is to design a state feedback reconfigurable or recovery control strategy in a directed network of multi-agent systems that seeks consensus in presence of three types of actuator faults and environmental disturbances. Suppose the i-th agent becomes faulty and its first m_o actuators are subject to the outage fault,

m_o+1 to m_s actuators are subject to the stuck fault, while the remaining $m-m_s$ actuators are either subject to the LOE fault or are healthy. Using equations (2.15)-(2.18) the model of i-th faulty agent that is subject to three types of actuator faults can be expressed as

$$\begin{aligned}&\dot{x}_i^f(t) = Ax_i^f(t) + B_i^f u_i^f(t) + B_\omega \omega_i(t),\ x_i^f(t_f) = x_i(t_f),\ t \geq t_f, \\ &y_i^f(t) = Cx_i^f(t),\end{aligned} \tag{4.1}$$

where $B_i^f = \begin{bmatrix} B_i^o & B_i^s & B_i^r \end{bmatrix}$, $B_i^o = \begin{bmatrix} b^1, \dots, b^{m_o} \end{bmatrix}$, $B_i^s = \begin{bmatrix} b^{m_o+1}, \dots, b^{m_s} \end{bmatrix}$, $B_i^r = \begin{bmatrix} b^{m_s+1}, \dots, b^m \end{bmatrix} \Gamma_i$, $\Gamma_i = \mathrm{diag}\{\Gamma_i^k\}$, $k = m_s+1, \dots, m$, Γ_i^k denotes the k-th actuator effectiveness and fault severity factor, $u_i^f(t) = \begin{bmatrix} 0_{1\times m_o} & (\underline{u}_i^s)^{\mathrm{T}} & (u_i^r(t))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\underline{u}_i^s = \begin{bmatrix} u_i^{m_o+1}(t_f), \dots, u_i^{m_s}(t_f) \end{bmatrix}^{\mathrm{T}}$, $u_i^r(t) = \begin{bmatrix} u_i^{m_s+1}(t), \dots, u_i^m(t) \end{bmatrix}^{\mathrm{T}}$.

Considering the structure of the control law $u_i^f(t)$ and the matrix B_i^f, it follows that only the actuators m_s+1 to m are available to be reconfigured. Therefore, to proceed with our proposed control recovery strategy the model (4.1) is rewritten as follows

$$\begin{aligned}&\dot{x}_i^f(t) = Ax_i^f(t) + B_i^r u_i^r(t) + B_i^s \underline{u}_i^s + B_\omega \omega_i(t),\ x_i^f(t_f) = x_i(t_f),\ t \geq t_f, \\ &y_i^f(t) = Cx_i^f(t).\end{aligned} \tag{4.2}$$

The main objective of the control reconfiguration or control recovery is to design and select $u_i^r(t)$ such that the state consensus errors remain bounded and $y_i^f(t) = y_i^h(t)$, for $t \geq t_f$, when $\omega_i(t) \equiv 0$, $i = 0, \dots, N$, and the environmental disturbances are attenuated for $\omega_i(t) \neq 0$, where $y_i^h(t) = y_i(t)$, $i = 1, \dots, N$, and $y_i(t)$ is defined as in equation (2.1).

To develop our proposed reconfiguration control laws, a **virtual auxiliary system** associated with each agent is introduced as follows

$$\dot{x}_i^a(t) = Ax_i^a(t) + Bu_i^a(t),\ x_i^a(t_0) = x_{i0}^a,\ i = 1, \dots, N, \tag{4.3}$$
$$y_i^a(t) = Cx_i^a(t),$$

where $x_i^a(t) \in \mathbb{R}^n$, $u_i^a(t) \in \mathbb{R}^m$ and $y_i^a(t) \in \mathbb{R}^q$ denote the state of the auxiliary system corresponding to the i-th agent, its control and output signals, respectively. Furthermore, the disagreement error for each auxiliary system is also defined as

$$e_i^a(t) = \sum_{j \in \mathcal{N}_i} (x_i^a(t) - x_j^a(t)) + g_{i0}(x_i^a(t) - x_0(t)). \tag{4.4}$$

The auxiliary system that is defined in (4.3) is “virtual" and is not subject to actuator faults or disturbances, and hence it can be used as the reference model for designing the reconfigured control laws of the actual system (2.1) once it is subjected to actuator faults.

The H_∞ performance index corresponding to the i-th healthy agent (2.1) and the i-th faulty agent (4.2) is now defined according to

$$J_i = \int_{t_0}^{\infty} \big((x_i(t) - x_0(t))^{\mathrm{T}}(x_i(t) - x_0(t)) - \gamma^2(\omega_i^{\mathrm{T}}(t)\omega_i(t) + \omega_0^{\mathrm{T}}(t)\omega_0(t))\big)\mathrm{d}t, \tag{4.5}$$

$$J_i^f = \int_{t_f}^{\infty} (\xi_i^{f^{\mathrm{T}}}(t)\xi_i^f(t) - \gamma_f^2\omega_i^{\mathrm{T}}(t)\omega_i(t))\mathrm{d}t, \tag{4.6}$$

where $\xi_i^f(t) = x_i^f(t) - x_i^a(t)$, and γ and γ_f represent the disturbance attenuation bounds. Based on the above definitions, the team performance index is now defined by $J = \sum_{i=1}^N J_i$. Under the control laws $u_i(t)$, $i = 1, \dots, N$, the H_∞ performance index bound for the healthy team is attenuated if $J = \sum_{i=1}^N J_i \leq 0$, $\forall\ \omega_i \in \mathcal{L}_2$. Furthermore, the H_∞ performance index for the i-th faulty agent is attenuated if $J_i^f \leq 0$, $\forall\ \omega_i \in \mathcal{L}_2$, $i = 0, \dots, N$. It should be noted that the performance indices (4.5) and (4.6) are not and cannot be calculated directly as the disturbance is unknown and the aim of the proposed approach is to minimize the performance indices without directly calculating them.

We are now in a position to formally state the problem that we consider in this work.

Definition 4.1. *(a) The state consensus H_∞ performance control problem for the healthy team is solved if in absence of disturbances, the agents follow the leader states and consensus errors converge to zero asymptotically, and in presence of disturbances, the prescribed H_∞ performance bound for the healthy team is attenuated, i.e. $J = \sum_{i=1}^{N} J_i \leq 0$.*
(b) The H_∞ performance control reconfiguration problem with stability is solved if in absence of disturbances the state consensus errors remain bounded while the output of the faulty agent behaves the same as those of the healthy system outputs, and in presence of disturbances the disturbance attenuation bound is minimized and $J_i^f \leq 0$.

We also assume the following assumptions hold in this chapter.

Assumption 4.1. *(a) The network graph is directed and has a spanning tree, and (b) The leader control input is bounded and the upper bound is known.*

Assumption 4.2. *(a) The agents are stabilizable and remain stabilizable even after the fault occurrence.*
(b) Each agent is equipped with a local FDI module which detects with possible delays and correctly isolates the fault in the agent and also estimates the severity of the fault with possible errors in the case of the LOE or stuck faults.

Regarding the above assumptions the following clarifications are in order. First, the

Assumptions 4.1-(a) and 4.2-(a) are quite common for consensus achievement and fault recovery control design problems, respectively. Second, it is quite necessary that in most practical applications one considers a leader whose states are ensured to be bounded. Moreover, in practical scenarios the actuators are quite well understood and described and their maximum deliverable control effort and bound they can tolerate are readily available and known. Therefore Assumptions 4.1-(b) is also not restrictive. Furthermore, in Subsection 4.2.2 we analyze the system behavior for situations where either Assumption 4.2-(c) does not hold or the estimated fault severities by the FDI module are not accurate. We obtain the maximum uncertainty bound that our proposed approaches can tolerate. However, as stated in Assumption 4.2-(b), we require the correct actuator location as well as the type of the fault for guaranteeing that our proposed reconfigured control laws will yield the desired design specifications and requirements.

As far as Assumption 2-(c) is concerned, it should be noted that this assumption is indeed quite realistic for the following observations and justifications. The transient time that any cooperative or consensus-based controller takes to settle down and the overall team objectives are satisfied is among one of the design consideration and specification for the controller selection. In most practical consensus achievement scenarios dealing with a healthy team, the transient time associated with the agent response is ensured to be settled down in a very small fraction of the entire mission time, and in most cases the healthy transient time takes a few seconds to minutes to die out. Therefore, it is quite realistic and indeed practical that during this very short and initial operation of the system, the agents are assumed to be fault free. In other words, we will not initiate the mission with agents that are faulty from the outset. It is highly unlikely that during the very first few moments after the initiation of the mission a fault occurs in the agents. Based on the above explanations and observations Assumption 2-(c) is meaningful and quite realistic.

4.2 Proposed Methodology

In this section, our proposed reconfigurable control law is introduced and developed. Since each agent only shares its information with its nearest neighbors, the reconfiguration control strategy also employs the same information as well as the agent's FDI module information.

Consider the dynamics of the i-th faulty agent as given by (4.2). As defined above $\xi_i^f(t) = x_i^f(t) - x_i^a(t)$, with $x_i^f(t)$ denoting the i-th faulty agent state and $x_i^a(t)$ defined in (4.3). Let $z_i(t) = C\xi_i^f(t)$ denote the deviation of the output of the faulty agent from its associated auxiliary agent output. Then, the dynamics associated with $\xi_i^f(t)$ can be obtained as

$$\begin{aligned} \dot{\xi}_i^f(t) &= A\xi_i^f(t) + B_i^r u_i^r(t) + B_i^s \underline{u}_i^s - Bu_i^a(t) + B_\omega \omega_i(t),\ t \geq t_f, \\ z_i(t) &= C\xi_i^f(t). \end{aligned} \tag{4.7}$$

Moreover, the faulty agent consensus error is defined as

$$e_i^f(t) = \sum_{j\in\mathcal{N}_i}(x_i^f(t) - x_j(t)) + g_{i0}(x_i^f(t) - x_0(t)). \tag{4.8}$$

Lemma 4.1. *The faulty agent consensus error (4.8) is stable if $e_i^a(t)$ and $\xi_i(t) = x_i(t) - x_i^a(t)$ are asymptotically stable and $\xi_i^f(t)$ is stabilized.*

Proof. From the auxiliary error dynamics (4.7), one can express the state consensus error dynamics for the i-th faulty agent, i.e. $e_i^f(t)$ according to

$$\begin{aligned} e_i^f(t) &= \sum_{j\in\mathcal{N}_i}(x_i^f(t)-x_j(t))+g_{i0}(x_i^f(t)-x_0(t)) \\ &= e_i^a(t)+(d_i+g_{i0})\xi_i^f(t)-\sum_{j\in\mathcal{N}_i}\xi_j(t). \end{aligned}$$

Therefore if the control law $u_i^r(t)$ can be reconfigured such that $\xi_i^f(t)$ is stabilized then it follows that $e_i^f(t)$ will be stable. This completes the proof of the lemma. $\square$

The above lemma shows that stability of the faulty agent's consensus error can be guaranteed by reconfiguring the control law $u_i^r(t)$ such that $\xi_i^f(t)$ is stable. This implies that one can transform the control reconfiguration problem to a stabilization problem, hence in the next two subsections we consider the problem of stabilizing $\xi_i^f(t)$. However, as seen from (4.7), the dynamics of ξ_i^f depends on the control of the healthy agents. Therefore, before presenting our proposed control reconfiguration strategy, the control law for the healthy team (where it is assumed without loss of any generality that all the agents are healthy) is presented below.

In this work, the following general control law structure is utilized,

$$u_i(t)=K_{1i}\xi_i(t)+K_{2i}e_i^a(t)+c_{i0}\text{sgn}(Ke_i^a(t)), \tag{4.9}$$

which is the generalization of the one developed in [174] and is given by

$$u_i(t)=c_1Ke_i(t)+c_2\text{sgn}(Ke_i(t)), \tag{4.10}$$

where $\xi_i(t)=x_i(t)-x_i^a(t)$, and $e_i^a(t)$ is given by (4.4) and

$$e_i(t) = g_{i0}(x_i(t)-x_0(t))+\sum_{j\in\mathcal{N}_i}(x_i(t)-x_j(t)), \tag{4.11}$$

where $g_{i0}=1$ if the agent i is a pinned agent or is directly communicating with the leader and is zero otherwise. The followings comments summarize the main characteristics of the control law (4.9):

(1) In the control law (4.9) an agent employs and communicates only the auxiliary states $x_i^a(t)$ that are unaffected by both disturbances and faults. In contrast in standard consensus control schemes such as (4.10) the actual states $x_i(t)$ are employed and communicated from the nearest neighbor agents. Hence, the utilization of (4.9) avoids the propagation of the adverse effects of the disturbances and faults through out the team of multi-agent systems. This along with the degrees of freedom in designing the control recovery laws allow us to manage the i-th faulty agent by only reconfiguring the control law of the faulty agent, and moreover it also provides us with the capability to recover simultaneous faults in multiple agents.

(2) The gain K_{1i} is designed such that the states of the i-th agent follow the states of its associated auxiliary agent, while the gain K_{2i} is designed such that the states of the auxiliary agents reach a consensus and follow the leader state.

(3) Each agent receives only the auxiliary agents states in its nearest neighbor set as opposed to their actual states that is conventionally required in standard multi-agent consensus approaches.

(4) The control law (4.9) is shown subsequently to solve the consensus problem in a directed network topology that is subject to environmental disturbances, whereas the control law (4.10) solves the consensus problem in disturbance free environment and where the network topology is assumed to be undirected. The procedure for selecting and designing the gains of the control law (4.9) is provided in Theorem 4.1. Moreover, the structure of the proposed control law of this agent are provided in Figures 4.1 and 4.2.

Theorem 4.1. *The control law $u_i(t) = u_{it}(t) + u_{ic}(t)$ solves the H_∞ performance state consensus problem in a team of N follower agents whose dynamics are given by (2.1) and the leader dynamics that is given by (2.3), if $u_{it}(t)$ and $u_{ic}(t)$ are selected as follows:*

$$u_{it}(t) = K_{1i}\xi_i(t)$$

$$u_{ic}(t) = u_i^a(t) = K_{2i}e_i^a(t) + K_{i0}(t),$$

where $e_i^a(t)$ is defined as in (4.4), $K_{1i} = c_1 K$, $K_{2i} = c_{2i}K$, $K_{i0}(t) = c_{i0}sgn(Ke_i^a(t))$, $sgn\{.\}$ is defined as in (2.24), $K = -B^T P$, $c_1 = \frac{c_3}{2}$, and finally the positive definite matrix P is the solution to

$$A^T P + PA - c_3 PBB^T P + 2\gamma^{-2}c_4^{-1}PB_\omega B_\omega^T P + d_0^* I < 0,$$

and c_{2i} and c_3 are solutions to

$$C_2 L_{22}^T + L_{22}C_2 > c_3 I,\ c_3 > 0,\ C_2 = diag\{c_{2i}\} > 0,$$

where d_0^* denotes the number of pinned agents, γ^2 is the desired disturbance attenuation bound, $c_4^{-1} = \max\{1, N^{-1}\lambda_{min}^{-1}(L_{22}^T L_{22})\}$, and c_{i0}'s are the solutions to the inequalities

$$d_0^* u_{0M} - d_i c_{i0} + \sum_{j\in\mathcal{N}_i} c_{j0} < 0,\ c_{i0} > 0,\ i, j = 1, \ldots, N,$$

where u_{0M} denotes the upper bound of the leader control signal, i.e., $\|u_0(t)\|_\infty \le u_{0M}$ for all $t \ge t_0$.

Proof. The team reaches a consensus if $x_i(t) \to x_j(t) \to x_0(t)$. This goal is also achieved if agents' controls are designed such that $x_i(t) \to x_i^a(t)$ ($\xi_i(t) \to 0$) and

$x_i^a(t) \to x_0(t)$ ($e_i^a(t) \to 0$) for $i = 1, \ldots, N$. This implies that the consensus achievement problem can be re-stated as the problem of asymptotically stabilizing $\xi_i(t)$ and $e_i^a(t)$ simultaneously.

In the following, first we discuss the stability criterion and disturbances attenuation for $e_i^a(t)$ and $\xi_i(t)$ in Parts A and B, respectively and then in Part C, we derive the conditions that satisfy the requirements for both Parts A and B that in fact solve the H_∞ performance state consensus.

Part A: From (4.3) and (4.4), the dynamics of $e^a(t) = \text{col}\{e_i^a(t)\}$ can be obtained as

$$\dot{e}^a(t) = \mathcal{A}e^a(t) + \mathcal{B}u^a(t) + \mathcal{B}_0 u_0(t) + \mathcal{B}_\omega \omega_0(t), \tag{4.12}$$

where $u^a(t) = \text{col}\{u_i^a(t)\}$, $\mathcal{A} = I_N \otimes A$, $\mathcal{B} = L_{22} \otimes B$, $\mathcal{B}_0 = L_{21} \otimes B$, $\mathcal{B}_\omega = L_{21} \otimes B_\omega$. Let us select $u_i^a(t)$ as $u_i^a(t) = K_{2i}e_i^a(t) + c_{i0}\text{sgn}(Ke_i^a(t))$, then the system (4.12) becomes

$$\begin{aligned} \dot{e}^a(t) &= (\mathcal{A} + L_{22}C_2 \otimes BK)e^a(t) + (L_{22}C_0 \otimes B)\text{sgn}((I \otimes K)e^a(t)) + \mathcal{B}_0 u_0(t) \\ &\quad + \mathcal{B}_\omega \omega_0(t), \end{aligned} \tag{4.13}$$

where $C_2 = \text{diag}\{c_{2i}\}$ and $C_0 = \text{diag}\{c_{i0}\}$. Since the sgn function is discontinuous, in order to conduct the stability analysis of the system (4.13), it is replaced with its differential inclusion (for more details refer to [175, 176]) representation as follows

$$\begin{aligned} \dot{e}^a(t) &\in^{a.e.} \mathcal{K}[(\mathcal{A} + L_{22}C_2 \otimes BK)e^a(t) + (L_{22}C_0 \otimes B)\text{sgn}((I \otimes K)e^a(t)) \\ &\quad + \mathcal{B}_0 u_0(t) + \mathcal{B}_\omega \omega_0(t)], \end{aligned} \tag{4.14}$$

where the operator $\mathcal{K}[.]$ is defined as in [175, 176] to investigate its Filipov solutions. Now, we require to define the Lyapunov function candidate $V(e^a(t))$ to study the stability properties of the error dynamics system. For this purpose, let us select $V(e^a(t)) =$

$e^{a^{\mathrm{T}}}(t)\mathcal{P}e^a(t)$, as a Lyapunov function candidate for the system (4.14), where $\mathcal{P} = I_N \otimes P$. Also, let $K = -B^{\mathrm{T}}P$, so that the set-valued derivative of $V(e^a(t))$ along the trajectories of the system (4.14) is given by

$$\begin{aligned}\dot{\tilde{V}}(e^a(t)) &= \mathcal{K}[e^{a^{\mathrm{T}}}(t)\big(I_N \otimes (A^{\mathrm{T}}P + PA) - (C_2L_{22}^{\mathrm{T}} + L_{22}C_2) \otimes PBB^{\mathrm{T}}P\big)e^a(t) \\ &\quad +2e^{a^{\mathrm{T}}}(t)(I \otimes PB)(L_{21} \otimes I)u_0(t) - 2e^{a^{\mathrm{T}}}(t)(I \otimes PB)(L_{22}C_0 \otimes I) \\ &\quad \mathrm{sgn}((I \otimes B^{\mathrm{T}}P)e^a(t)) + 2\omega_0^{\mathrm{T}}(t)\mathcal{B}_\omega^{\mathrm{T}}\mathcal{P}e^a(t)].\end{aligned} \tag{4.15}$$

Let $T_1(t) = e^{a^{\mathrm{T}}}(t)(I \otimes PB)(L_{21} \otimes I)u_0(t)$, $T_2(t) = e^{a^{\mathrm{T}}}(t)(I \otimes PB)(L_{22}C_0 \otimes I)\mathrm{sgn}((I \otimes B^{\mathrm{T}}P)e^a(t))$, $\bar{e}_i(t) = B^{\mathrm{T}}Pe_i^a(t)$ and $\bar{e}(t) = \mathrm{col}\{\bar{e}_i(t)\}$. Since $T_1(t)$ is a scaler, $T_1(t) \leq \|T_1(t)\|_1$, and one has

$$T_1(t) \leq \|T_1(t)\|_1 \leq \|(L_{21}^{\mathrm{T}} \otimes I)(I \otimes B^{\mathrm{T}}P)e^a(t)\|_1\|u_0(t)\|_\infty. \tag{4.16}$$

Then

$$\begin{aligned}T_1(t) &\leq \|(L_{21}^{\mathrm{T}} \otimes I)\|_\infty\|(I \otimes B^{\mathrm{T}}P)e^a(t)\|_1\|u_0(t)\|_\infty \\ &\leq d_0^*\|(I \otimes B^{\mathrm{T}}P)e^a(t)\|_1 u_{0M} = d_0^* u_{0M}\sum_{i=1}^{N}\sum_{k=1}^{m}|\bar{e}_i^k(t)|,\end{aligned} \tag{4.17}$$

where $\bar{e}_i^k(t)$ is the k-th element of $\bar{e}_i(t) = \Big[\bar{e}_i^1(t), \ldots, \bar{e}_i^m(t)\Big]^{\mathrm{T}}$ and we use the fact that $\|L_{21}\|_\infty = 1$. On the other hand, $T_2(t)$ can be written as

$$T_2(t) = \bar{e}^{\mathrm{T}}(t)(L_{22}C_0 \otimes I)\mathrm{sgn}(\bar{e}(t)) = \sum_{i=1}^{N} T_{2i}(t), \tag{4.18}$$

where

$$\begin{aligned}T_{2i}(t) &= \bar{e}_i^{\mathrm{T}}(t)(d_ic_{i0}\mathrm{sgn}\{\bar{e}_i(t)\} - \sum_{j\in\mathcal{N}_i} c_{j0}\mathrm{sgn}\{\bar{e}_j(t)\}) \\ &= \sum_{k=1}^{m}\bar{e}_i^k(t)\big(d_ic_{i0}\mathrm{sgn}\{\bar{e}_i^k(t)\} - \sum_{j\in\mathcal{N}_i} c_{j0}\mathrm{sgn}\{\bar{e}_j^k(t)\}\big).\end{aligned}$$

Let $T_{2i}^k(t) = \bar{e}_i^k(t)\big(d_ic_{i0}\mathrm{sgn}\{\bar{e}_i^k(t)\} - \sum_{j\in\mathcal{N}_i} c_{j0}\mathrm{sgn}\{\bar{e}_j^k(t)\}\big)$, then three cases can be considered depending on the value of $\bar{e}_i^k(t)$ as follows:

i) $\bar{e}_i^k(t) = 0$, then $T_{2i}^k(t) = 0$.

ii) $\bar{e}_i^k(t) > 0$, then $\text{sgn}\{\bar{e}_i^k(t)\} = 1$. Since $c_{j0} > 0$ and $\text{sgn}\{\bar{e}_j^k(t)\} \in \{-1, 0, 1\}$, it follows that

$$d_i c_{i0} - \sum_{j \in \mathcal{N}_i} c_{j0} \leq d_i c_{i0} \text{sgn}\{\bar{e}_i^k(t)\} - \sum_{j \in \mathcal{N}_i} c_{j0} \text{sgn}\{\bar{e}_j^k(t)\} \leq d_i c_{i0} + \sum_{j \in \mathcal{N}_i} c_{j0},$$

and if c_{i0}, $i = 1, \ldots, N$ are designed such that $d_i c_{i0} - \sum_{j \in \mathcal{N}_i} c_{j0} > 0$, then

$$|\bar{e}_i^k(t)|(d_i c_{i0} - \sum_{j \in \mathcal{N}_i} c_{j0}) \leq T_{2i}^k(t) \leq |\bar{e}_i^k(t)|(d_i c_{i0} + \sum_{j \in \mathcal{N}_i} c_{j0}). \tag{4.19}$$

iii) $\bar{e}_i^k(t) < 0$, then $\text{sgn}\{\bar{e}_i^k(t)\} = -1$ and $e_i^k(t) = -|e_i^k(t)|$. Therefore,

$$-d_i c_{i0} - \sum_{j \in \mathcal{N}_i} c_{j0} \leq d_i c_{i0} \text{sgn}\{\bar{e}_i^k(t)\} - \sum_{j \in \mathcal{N}_i} c_{j0} \text{sgn}\{\bar{e}_j^k(t)\} \leq -d_i c_{i0} + \sum_{j \in \mathcal{N}_i} c_{j0}.$$

Again if c_{i0}, $i = 1, \ldots, N$ are designed such that, $d_i c_{i0} - \sum_{j \in \mathcal{N}_i} c_{j0} > 0$, then

$$|\bar{e}_i^k(t)|(d_i c_{i0} - \sum_{j \in \mathcal{N}_i} c_{j0}) \leq T_{2i}^k(t) \leq |\bar{e}_i^k(t)|(d_i c_{i0} + \sum_{j \in \mathcal{N}_i} c_{j0}). \tag{4.20}$$

Let $T_3(t) = T_1(t) - T_2(t)$. From the inequalities (4.17)-(4.20) it follows that

$$\begin{aligned} T_3(t) &\leq d_0^* u_{0M} \sum_{i=1}^{N} \sum_{k=1}^{m} |\bar{e}_i^k(t)| - \sum_{i=1}^{N} \sum_{k=1}^{m} |\bar{e}_i^k(t)|(d_i c_{i0} - \sum_{j \in \mathcal{N}_i} c_{j0}) \\ &= \sum_{i=1}^{N} \sum_{k=1}^{m} |\bar{e}_i^k(t)| \big(d_0^* u_{0M} - d_i c_{i0} + \sum_{j \in \mathcal{N}_i} c_{j0}\big). \end{aligned} \tag{4.21}$$

Suppose that c_{2i}s and c_3 are obtained such that

$$C_2 L_{22}^{\mathrm{T}} + L_{22} C_2 > c_3 I, \; c_3 > 0, \; C_2 = \text{diag}\{c_{2i}\} > 0. \tag{4.22}$$

Now by using the Fact 2.2 for the last term in the right-hand side of (4.15) with $X = (L_{21} \otimes I_m)\omega_0(t)$, $Y = (I_N \otimes B_\omega^{\mathrm{T}} P)e^a(t)$ and $\alpha = \frac{\gamma^2}{2} c_4$, and also the inequalities (4.21) and (4.22), the expression (4.15) can be replaced with the following inequality

$$\begin{array}{rcl}\dot{\tilde{V}}(e^a(t)) & \leq & \mathcal{K}[e^{a^{\mathrm{T}}}(t)\big(I_N \otimes (A^{\mathrm{T}}P + PA - c_3 PBB^{\mathrm{T}}P)\big)e^a(t) \\ & & +2\sum_{i=1}^{N}\sum_{k=1}^{m}|\bar{e}_i^k|\big(d_0^* u_{0M} - d_i c_{i0} + \sum_{j\in\mathcal{N}_i} c_{j0}\big) + \frac{\gamma^2}{2}c_4\omega_0^{\mathrm{T}}(t)(L_{21}^{\mathrm{T}}L_{21}\otimes I_m)\omega_0(t) \\ & & +2\gamma^{-2}c_4^{-1}e^{a^{\mathrm{T}}}(t)(I\otimes PB_\omega B_\omega^{\mathrm{T}}P)e^a(t)].\end{array}$$

Since now the right hand side of the above inequality is continuous, the operator $\mathcal{K}[.]$ can be removed. Let $d_0^* = L_{21}^{\mathrm{T}}L_{21}$ and add $d_0^* e^{a^{\mathrm{T}}}(t)e^a(t)$ to both sides of the above inequality then it follows that

$$\dot{\tilde{V}}(e^a(t)) - \frac{\gamma^2}{2}d_0^* c_4\omega_0^{\mathrm{T}}(t)\omega_0(t) + d_0^* e^{a^{\mathrm{T}}}(t)e^a(t) \leq g(e^a(t)), \tag{4.23}$$

where

$$\begin{array}{rcl}g(e^a(t)) & = & e^{a^{\mathrm{T}}}(t)\big(I_N\otimes(A^{\mathrm{T}}P + PA - c_3PBB^{\mathrm{T}}P + d_0^* I + 2\gamma^{-2}c_4^{-1}PB_\omega B_\omega^{\mathrm{T}}P)\big)e^a(t) \\ & & +2\sum_{i=1}^{N}\sum_{k=1}^{m}|\bar{e}_i^k(t)|\big(d_0^* u_{0M} - d_i c_{i0} + \sum_{j\in\mathcal{N}_i} c_{j0}\big).\end{array}$$

From [175], we require $g(e^a(t))$ to be negative definite, which will be achieved if P is obtained such that

$$A^{\mathrm{T}}P + PA - c_3PBB^{\mathrm{T}}P + 2\gamma^{-2}c_4^{-1}PB_\omega B_\omega^{\mathrm{T}}P + d_0^* I < 0, \tag{4.24}$$

and c_{i0} are selected such that

$$d_0^* u_{0M} - d_i c_{i0} + \sum_{j\in\mathcal{N}_i} c_{j0} < 0, i = 1, \ldots, N. \tag{4.25}$$

Therefore, if c_{i0}, $i = 1, \ldots, N$ and P are selected as the solutions to (4.25) and (4.24), the function $g(.)$ will be negative definite and for $\omega_0(t) \equiv 0$, it follows that $\dot{\tilde{V}}(e^a(t)) < 0$, or equivalently the consensus errors are asymptotically stable.

Now, if the initial conditions are set to zero and the disturbance is the only input to the agents, then by integrating the left-hand side of (4.23) one gets

$$\int_{t_0}^{\infty}(e^{a^{\mathrm{T}}}(t)e^a(t)-\frac{\gamma^2}{2}c_4\omega_0^{\mathrm{T}}(t)\omega_0(t))\mathrm{d}t<0. \tag{4.26}$$

Given that $e^a(t)=(L_{22}\otimes I_n)\xi^a(t)$, $\xi^a(t)=x^a(t)-1_N\otimes x_0(t)$ and $x^a(t)=\mathrm{col}\{x_i^a(t)\}$, it follows that

$$\lambda_{\mathrm{m}}\xi^{a^{\mathrm{T}}}(t)\xi^a(t)\leq e^{a^{\mathrm{T}}}(t)e^a(t)\leq\lambda_{\mathrm{M}}\xi^{a^{\mathrm{T}}}(t)\xi^a(t), \tag{4.27}$$

where $\lambda_{\mathrm{m}}=\lambda_{\min}(L_{22}^{\mathrm{T}}L_{22})$ and $\lambda_{\mathrm{M}}=\lambda_{\max}(L_{22}^{\mathrm{T}}L_{22})$. Hence, from the inequalities (4.26) and (4.27) it follows that

$$\int_{t_0}^{\infty}\lambda_{\mathrm{m}}\xi^{a^{\mathrm{T}}}(t)\xi^a(t)\mathrm{d}t-\int_{t_0}^{\infty}\frac{\gamma^2}{2}c_4\omega_0^{\mathrm{T}}(t)\omega_0(t)\mathrm{d}t<0,$$

and by selecting $c_4=N\lambda_{\mathrm{m}}$ one gets

$$\frac{\int_{t_0}^{\infty}\xi^{a^{\mathrm{T}}}(t)\xi^a(t)\mathrm{d}t}{\int_{t_0}^{\infty}\omega_0^{\mathrm{T}}(t)\omega_0(t)\mathrm{d}t}<\frac{N}{2}\gamma^2. \tag{4.28}$$

<u>Part B</u>: Under our proposed control law the dynamics of the i-th auxiliary agent tracking error, $\xi_i(t)$, can be expressed as

$$\dot{\xi}_i(t)=(A+c_1BK)\xi_i(t)+B_\omega\omega_i(t). \tag{4.29}$$

Consider $V_i(\xi_i(t))=\xi_i^{\mathrm{T}}(t)P\xi_i(t)$ as a Lyapunov function candidate for the system (4.29) and select $K=-B^{\mathrm{T}}P$. It then follows that

$$\dot{V}_i(\xi_i(t)) = \xi_i^{\mathrm{T}}(t)(A^{\mathrm{T}}P+PA-2c_1PBB^{\mathrm{T}}P)\xi_i(t)+2\xi_i^{\mathrm{T}}(t)PB_\omega\omega_i(t),$$

and by following along the same steps as in Part A, the above equality can be written as

$$\begin{aligned}\dot{V}_i(\xi_i(t))-\frac{\gamma^2}{2}\omega_i^{\mathrm{T}}(t)\omega_i(t)+\xi_i^{\mathrm{T}}(t)\xi_i(t)&\leq\xi_i^{\mathrm{T}}(t)(A^{\mathrm{T}}P+PA-2c_1PBB^{\mathrm{T}}P\\&+2\gamma^{-2}PB_\omega B_\omega^{\mathrm{T}}P+I)\xi_i(t).\end{aligned}$$

Now if $P>0$ is obtained such that

$$A^{\mathrm{T}}P+PA-2c_1PBB^{\mathrm{T}}P+2\gamma^{-2}PB_\omega B_\omega^{\mathrm{T}}P+I<0, \tag{4.30}$$

then $\dot{V}_i(\xi_i(t))-\frac{\gamma^2}{2}\omega_i^{\mathrm{T}}(t)\omega_i(t)+\xi_i^{\mathrm{T}}(t)\xi_i(t)<0$. This implies that for $\omega_i(t)\equiv 0$, we have

$\dot{V}_i(\xi_i(t)) < 0$, and for $\omega_i(t) \neq 0$, one gets

$$\frac{\int_{t_0}^{\infty} \xi_i^{\mathrm{T}}(t)\xi_i(t)\mathrm{d}t}{\int_{t_0}^{\infty} \omega_i^{\mathrm{T}}(t)\omega_i(t)\mathrm{d}t} \leq \frac{\gamma^2}{2}. \tag{4.31}$$

Part C: In order to obtain the positive definite matrix P that satisfies the inequalities (4.24) and (4.30) and also guarantees the disturbance bound attenuation, let us set c_1 and c_4 as $c_1 = \frac{c_3}{2}$ and $c_4^{-1} = \max\{1, N^{-1}\lambda_{\mathrm{m}}^{-1}\}$, respectively. Given that $d_0^* \geq 1$, it can be observed that if P satisfies

$$A^{\mathrm{T}}P + PA - c_3PBB^{\mathrm{T}}P + 2\gamma^{-2}c_4^{-1}PB_\omega B_\omega^{\mathrm{T}}P + d_0^*I < 0, \tag{4.32}$$

then inequalities (4.24) and (4.30) will both hold, where c_3 is the solution to (4.22). On the other hand

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}(x_i(t) - x_0(t))^{\mathrm{T}}(x_i(t) - x_0(t))\mathrm{d}t \\
&= \frac{1}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}(\xi_i(t) + \xi_i^a(t))^{\mathrm{T}}(\xi_i(t) - \xi_i^a(t))\mathrm{d}t = \\
&\frac{1}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}(\xi_i^{\mathrm{T}}(t)\xi_i(t) + \xi_i^{a\mathrm{T}}(t)\xi_i^a(t) + 2\xi_i^{\mathrm{T}}(t)\xi_i^a(t))\mathrm{d}t \leq \\
&\frac{1}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}(\xi_i^{\mathrm{T}}(t)\xi_i(t) + \xi_i^{a\mathrm{T}}(t)\xi_i^a(t) + \xi_i^{\mathrm{T}}(t)\xi_i(t) + \xi_i^a(t)\xi_i^{a\mathrm{T}}(t))\mathrm{d}t = \\
&\sum_{i=1}^{N}\int_{t_0}^{\infty}(\xi_i^{\mathrm{T}}(t)\xi_i(t) + \xi_i^{a\mathrm{T}}(t)\xi_i^a(t))\mathrm{d}t
\end{aligned}$$

Now from equations (4.28) one has

$$\begin{aligned}
\int_{t_0}^{\infty}\xi^{a\mathrm{T}}(t)\xi^a(t)\mathrm{d}t &= \sum_{i=1}^{N}\int_{t_0}^{\infty}\xi^{a\mathrm{T}}_{\;i}(t)\xi_i^a(t)\mathrm{d}t \leq \frac{N}{2}\gamma^2\int_{t_0}^{\infty}\omega_0^{\mathrm{T}}(t)\omega_0(t)\mathrm{d}t \\
&= \frac{\gamma^2}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}\omega_0^{\mathrm{T}}(t)\omega_0(t)\mathrm{d}t,
\end{aligned}$$

and by using (4.31)

$$\sum_{i=1}^{N}\int_{t_0}^{\infty}\xi_i^{\mathrm{T}}(t)\xi_i(t)\mathrm{d}t \leq \frac{\gamma^2}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}\omega_i^{\mathrm{T}}(t)\omega_i(t)\mathrm{d}t,$$

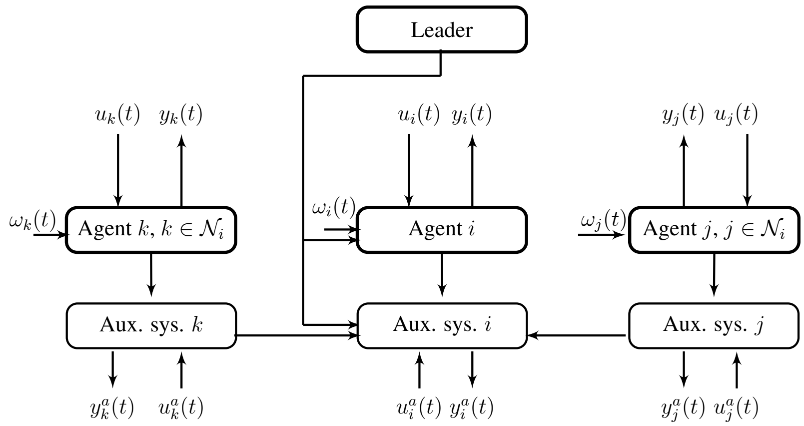

Figure 4.1: The schematic of the i-th pinned agent and its nearest neighbor agents j and k, which are not pinned.

then it follows that

$$\frac{1}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}(x_i(t)-x_0(t))^{\mathrm{T}}(x_i(t)-x_0(t))\mathrm{d}t \le \frac{\gamma^2}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}(\omega_i^{\mathrm{T}}(t)\omega_i(t)+\omega_0^{\mathrm{T}}(t)\omega_0(t))\mathrm{d}t.$$

Therefore, the team H_∞ performance upper bound can be expressed as

$$\begin{aligned}&\frac{\sum_{i=1}^{N}\int_{t_0}^{\infty}(x_i(t)-x_0(t))^{\mathrm{T}}(x_i(t)-x_0(t))\mathrm{d}t}{\sum_{i=1}^{N}\int_{t_0}^{\infty}(\omega_i^{\mathrm{T}}(t)\omega_i(t)+\omega_0^{\mathrm{T}}(t)\omega_0(t))\mathrm{d}t}\\ \le\ &\gamma^2.\end{aligned}$$

The above inequality implies that $J \le 0$, or equivalently the healthy team H_∞ performance criterion holds. This along with the properties of the stability of $e_i^a(t)$ and $\xi_i(t)$, as stated in Parts A and B, imply that our proposed control law solves the H_∞ performance state consensus problem for the healthy team. $\square$

4.2.1 H_∞ Performance Control Reconfiguration

Consider the representation of an agent subject to presence of faults be specified as in Subsection 4.1, and given by the equation (4.2) or equivalently by the transformed model

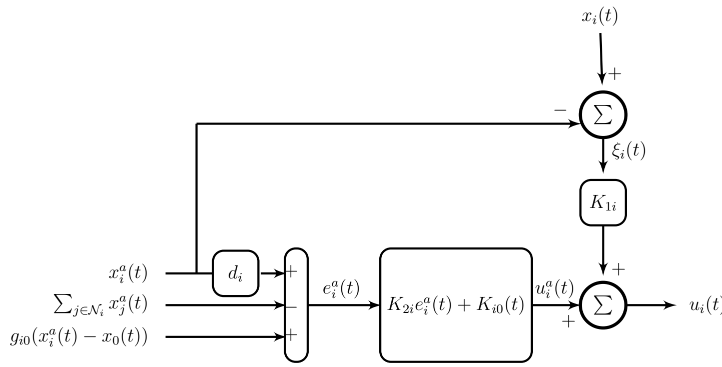

Figure 4.2: The i-th agent cooperative control structure and its associated auxiliary system control laws, where $\xi_i(t) = x_i(t) - x_i^a(t)$ and $e_i^a(t)$ is defined in (4.4).

(4.7). Our proposed reconfigured control law for the i-th faulty agent is now given by

$$u_i^r(t) = K_{1i}^r \xi_i^f(t) + K_{2i}^r u_i^a(t) + \underline{u}_i^C, \tag{4.33}$$

where K_{1i}^r, K_{2i}^r are control gains and $\underline{u}_i^C$ is the control command to be designed later. Therefore the dynamics of the closed-loop faulty agent (4.7) becomes

$$\begin{aligned} \dot{\xi}_i^f(t) &= (A + B_i^r K_{1i}^r)\xi_i^f(t) + (B_i^r K_{2i}^r - B)u_i^a(t) + B_i^s \underline{u}_i^s + B_i^r \underline{u}_i^C \\ &\quad + B_\omega \omega_i(t), \\ z_i(t) &= C\xi_i^f(t). \end{aligned} \tag{4.34}$$

As per Definition 4.1, the H_∞ control reconfiguration objectives can now be stated as that of selecting the gains K_{1i}^r and K_{2i}^r and the control command $\underline{u}_i^r$ such that (a) $\xi_i^f(t)$ is stable, (b) $z_i(t) \equiv 0$ (that is, $y_i^f(t) = y_i^a(t)$) for $\omega_i(t) \equiv 0$, $t \geq t_f$, and (c) $\frac{\int_{t_f}^{\infty} \xi_i^{f^{\mathrm{T}}}(t)\xi_i^f(t)\mathrm{d}t}{\int_{t_f}^{\infty} \omega_i^{\mathrm{T}}(t)\omega_i(t)\mathrm{d}t} \leq \gamma_f^2$ for $\omega_i(t) \neq 0$. In order to pursue the reconfiguration strategy we required the following assumption, we later discuss how deviation of this assumption affect the results.

Assumption 4.3. *Under the fault scenario, there still enough actuator redundancy to* *compensate for the fault, i.e.*

$$B_i^s \underline{u}_i^s \subset \textit{Im}\{B_i^r\}, \tag{4.35}$$

then there exists a control signal $\underline{u}_i^C$ such that

$$B_i^s \underline{u}_i^s + B_i^r \underline{u}_i^C = 0. \tag{4.36}$$

Subject to the above condition, equation (4.34) now becomes

$$\begin{aligned} \dot{\xi}_i^f(t) &= (A + B_i^r K_{1i}^r)\xi_i^f(t) + (B_i^r K_{2i}^r - B)u_i^a(t) + B_\omega \omega_i(t), \\ z_i(t) &= C\xi_i^f(t). \end{aligned} \tag{4.37}$$

Let us temporarily assume that $\omega_i(t) \equiv 0$, then

$$z_i(t) = Ce^{(A+B_i^r K_{1i}^r)(t-t_f)}\xi_i^f(t_f) + \int_{t_f}^{t} Ce^{(A+B_i^r K_{1i}^r)(t-s)}(B_i^r K_{2i}^r - B)u_i^a(s)\mathrm{d}s. \tag{4.38}$$

From (4.38), to ensure that the outputs of the faulty agent do not deviate after fault, both terms should be zero or negligible. The first term will be negligible if the agents reach a consensus before fault occurrence i.e. $\xi_i^f(t_f) \simeq 0$ or if K_{1i}^r is designed such that $e^{(A+B_i^r K_{1i}^r)(t-t_f)}$ damps very fast. This can be achieved easily if $\lambda_{max}\{A + B_i^r K_{1i}^r\}$ is small enough. On the other hand, according to Theorem 4.1, $u_i^a(t) = u_{ic}(t) = K_{2i}e_i^a(t) + K_{i0}(t)$. Given that the control gains are designed such that $e_i^a(t)$ is asymptotically stable and $K_{i0}(t)$ is bounded, $u_i^a(t)$ also remains bounded. Considering that $u_i^a(t)$ does not depend on the dynamics of $\xi_i^f(t)$, it can be treated as a disturbance to the system (4.37). Consequently, the problems of *(i)* enforcing $z_i(t) \equiv 0$ (for $t \geq t_f$, $\omega_j(t) \equiv 0$, $j = 0, \ldots, N$ and any $u_i^a(t)$), and *(ii)* stabilizing $\xi_i^f(t)$, is similar to that of the disturbance decoupling problem with stability (DDPS), as studied in [177].

The geometric approach that is based on the theory of subspaces [160] is the most popular method for solving the DDPS problem. Towards this end, we first introduce the required subspaces as follows: $\mathcal{B}_i^r = \text{Im}\{B_i^r\}$, $\mathcal{C} = \text{Ker}\{C\}$, $\mathcal{V}^*$ and $\mathcal{V}_g^*$ denote the maximal (A, B_i^r) controlled invariant subspace that is contained in $\mathcal{C}$, and the maximal internally stable (A, B_i^r) controlled invariant subspace that is contained in $\mathcal{C}$, respectively.

Following the procedure in [160], if K_{2i}^r and K_{1i}^r are selected such that

$$\text{Im}\{B_i^r K_{2i}^r - B\} \subset \mathcal{V}^*, (A + B_i^r K_{1i}^r)\mathcal{V}^* \subset \mathcal{V}^* \tag{4.39}$$

then the second term in (4.38) will also vanish. On the other hand, if K_{2i}^r and K_{1i}^r are selected such that

$$\text{Im}\{B_i^r K_{2i}^r - B\} \subset \mathcal{V}_g^*, (A + B_i^r K_{1i}^r)\mathcal{V}_g^* \subset \mathcal{V}_g^*, \tag{4.40}$$

then the second term in (4.38) will also vanish and $\xi_i^f(t)$ will be stable due to the stability of the subspace $\mathcal{V}_g^*$. Unfortunately, there is no systematic approach to explicitly obtain $\mathcal{V}_g^*$, implying that $\mathcal{V}_g^*$ cannot be computed and employed directly for obtaining K_{2i}^r that satisfies the condition (4.40). Therefore, we are required to transform the condition (4.40) into a verifiable one. Once such a controller is obtained, one can then ensure that $z_i(t) \equiv 0$ and $\xi_i^f(t)$ will remain stable.

Given that $\mathcal{V}^*$ is (A, B_i^r) controlled invariant, there exists a matrix K_{1i}^r, a friend of $\mathcal{V}^*$, [160] such that $A_c\mathcal{V}^* \subset \mathcal{V}^*$, where $A_c = A + B_i^r K_{1i}^r$. Now, by invoking the Theorem 3.2.1 of [160], for a matrix A_c and its associated $\mathcal{V}^*$, there always exists a nonsingular transformation T such that

$$\bar{A}_c = T^{-1} A_c T = \begin{bmatrix} \bar{A}_c^1 & \bar{A}_c^2 \\ 0 & \bar{A}_c^3 \end{bmatrix}, \tag{4.41}$$

where $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, $\mathrm{Im}\{T_1\} = \mathcal{V}^*$ and T_2 is any matrix that renders T nonsingular. By substituting $A_c = A + B_i^r K_{1i}^r$ into (4.41), it follows that

$$\bar{A}_c = \bar{A} + \bar{B}_i^r \bar{K}_{1i}^r, \tag{4.42}$$

where $\bar{A} = T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$, $\bar{B}_i^r = T^{-1}B_i^r = \begin{bmatrix} \bar{B}_{i1}^r \\ \bar{B}_{i2}^r \end{bmatrix}$ and $\bar{K}_{1i}^r = K_{1i}^r T$. Now, if $\bar{K}_{1i}^r$ is partitioned as $\bar{K}_{1i}^r = \begin{bmatrix} \bar{K}_{1i}^{r1} & \bar{K}_{1i}^{r2} \end{bmatrix}$, from (4.41) and (4.42) it can be concluded that there exists $\bar{K}_{1i}^{r1}$ such that

$$\bar{A}_{21} + \bar{B}_{i2}^r \bar{K}_{1i}^{r1} = 0.$$

Furthermore, under the transformation T, the system (4.37) can be re-written as

$$\begin{aligned} \dot{\bar{\xi}}_i^f(t) &= \bar{A}_c \bar{\xi}_i^f(t) + \bar{E}_i u_i^a(t) + \bar{B}_\omega \omega_i(t), \\ z_i(t) &= \bar{C} \bar{\xi}_i^f(t), \end{aligned} \tag{4.43}$$

where $\bar{\xi}_i^f(t) = T^{-1}\xi_i^f(t)$, $\bar{A}_c = \begin{bmatrix} \bar{A}_c^1 & \bar{A}_c^2 \\ 0 & \bar{A}_c^3 \end{bmatrix}$, $\bar{A}_c^1 = \bar{A}_{11} + \bar{B}_{i1}^r \bar{K}_{1i}^{r1}$, $\bar{A}_c^2 = \bar{A}_{12} + \bar{B}_{i1}^r \bar{K}_{1i}^{r2}$, $\bar{A}_c^3 = \bar{A}_{22} + \bar{B}_{i2}^r \bar{K}_{1i}^{r2}$, $\bar{E}_i = \bar{B}_i^r K_{2i}^r - \bar{B} = \begin{bmatrix} \bar{B}_{i1}^r K_{2i}^r - \bar{B}_1 \\ \bar{B}_{i2}^r K_{2i}^r - \bar{B}_2 \end{bmatrix}$, $\bar{B} = T^{-1}B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}$, $\bar{C} = CT = \begin{bmatrix} 0 & \bar{C}_2 \end{bmatrix}$, $\bar{B}_\omega = T^{-1}B_\omega$. We are now in a position to state the main result of this subsection.

Theorem 4.2. *Consider a team that consists of a leader that is governed by (2.3) and N follower agents that are governed by (2.1), and their control laws are designed and specified according to Theorem 4.1. Suppose at time $t = t_f$ the i-th agent becomes faulty* *and its dynamics is now governed by (4.2) where Assumption 4.2 also hold. The control law (4.33) solves the H_∞ performance control reconfiguration problem with stability where the H_∞ upper bound is given by $\gamma_f^2 = \alpha^{-1}\lambda_{min}^{-1}\{(TT^T)^{-1}\}$ if $\underline{u}_i^C$ is obtained as a solution to (4.36), $K_{1i}^r = \begin{bmatrix} Y_1X_1^{-1} & Y_2X_2^{-1} \end{bmatrix} T^{-1}$, and K_{2i}^r is the solution to*

$$\bar{B}_{i2}^r K_{2i}^r - \bar{B}_2 = 0, \tag{4.44}$$

where T is defined in (4.41), X_i and Y_i's, $i = 1, 2$ are solutions to

$$\textit{max } \alpha \textit{ s.t.} \begin{bmatrix} \Theta & X \\ X & -I \end{bmatrix} < 0,\ X = \textit{diag}\{X_1, X_2\} > 0,\ \bar{A}_{21}X_1 + \bar{B}_{i2}^r Y_1 = 0, \tag{4.45}$$

where $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix}$, $\Theta_1 = X_1\bar{A}_{11}^T + Y_1^T\bar{B}_{i1}^{r^T} + \bar{A}_{11}X_1 + \bar{B}_{i1}^r Y_1 + \alpha\bar{B}_\omega^1\bar{B}_\omega^{1^T}$, $\Theta_2 = \bar{A}_{12}X_2 + \bar{B}_{i1}^r Y_2 + \alpha\bar{B}_\omega^1\bar{B}_\omega^{2^T}$, $\Theta_3 = X_2\bar{A}_{22}^T + Y_2^T\bar{B}_{i2}^{r^T} + \bar{A}_{22}X_2 + \bar{B}_2Y_2 + \alpha\bar{B}_\omega^2\bar{B}_\omega^{2^T}$, $\bar{A}_{11}$, $\bar{A}_{21}$, $\bar{A}_{12}$, $\bar{B}_{i1}$ and $\bar{B}_{i2}$ are defined as in (4.42) and $\bar{B}_\omega$ and $\bar{B}_2$ are defined as in (4.43).

Proof. Consider the system (4.43). Given that the two inputs $\omega_i(t)$ and $u_i^a(t)$ are bounded and independent from each other, one can investigate their effects separately. Therefore, the proof is provided in three parts, namely: in Part A we assume that $\omega_i(t) \equiv 0$ and the set of all control gains that guarantee $z_i(t) = 0$ and stabilize $\xi_i^f(t)$ are obtained. Next, in Part B we assume that the disturbance is the only input to the agent and obtain the gains that minimize the H_∞ performance index and guarantee stability as well. Finally, in Part C, the control gains that satisfy both Parts A and B are obtained.

Part A: Let $\omega_i(t) \equiv 0$ so that we have

$$\begin{aligned} \dot{\bar{\xi}}_i^f(t) &= \bar{A}_c \bar{\xi}_i^f(t) + \bar{E}_i u_i^a(t), \\ z_i(t) &= \bar{C} \bar{\xi}_i^f(t). \end{aligned}$$

Since $\bar{A}_c$ is an upper-triangular matrix, the matrix $e^{\bar{A}_c t}$ is also upper-triangular and can be written as $e^{\bar{A}_c t} = \begin{bmatrix} e^{\bar{A}_c^1 t} & F_2(t) \\ 0 & e^{\bar{A}_c^3 t} \end{bmatrix}$, where $F_2(t) = \int_{t_0}^{t} e^{\bar{A}_c^1 (t-s)} \bar{A}_c^2 e^{\bar{A}_c^3 s} \mathrm{d}s$. Under Assumption 4.2-(c), $\xi_i^f(t_f) = 0$ and $z_i(t)$ can be written as

$$z_i(t) = \int_{t_f}^{t} \bar{C}_2 e^{\bar{A}_c^3 (t-s)} (\bar{B}_{i2}^r K_{2i}^r - \bar{B}_2) u_i^a(s) \mathrm{d}s. \tag{4.46}$$

If K_{2i}^r is obtained such that

$$\bar{B}_{i2}^r K_{2i}^r - \bar{B}_2 = 0,$$

then $z_i(t) \equiv 0$, which implies that the above condition is equivalent to (4.39). Moreover, if $\bar{K}_{1i}^{r1}$ and $\bar{K}_{1i}^{r2}$ are selected such that $\bar{A}_{11} + \bar{B}_{i1}^r \bar{K}_{1i}^{r1}$ and $\bar{A}_{22} + \bar{B}_{i2}^r \bar{K}_{1i}^{r2}$ are Hurwitz, then $\bar{A}_c$ will also be Hurwitz. Given that $u_i^a(t)$ is bounded and $\bar{A}_c$ is Hurwitz, then $\bar{\xi}_i(t)$ will also be bounded. Therefore, condition (4.40) is equivalent to obtaining the matrices $\bar{K}_{1i}^{r1}$, $\bar{K}_{1i}^{r2}$ and K_{2i}^r such that

$$\bar{A}_{21} + \bar{B}_{i2}^r \bar{K}_{1i}^{r1} = 0, \tag{4.47}$$

$$\bar{A}_{11} + \bar{B}_{i1}^r \bar{K}_{1i}^{r1} \text{ is Hurwitz}, \tag{4.48}$$

$$\bar{A}_{22} + \bar{B}_{i2}^r \bar{K}_{1i}^{r2} \text{ is Hurwitz}, \tag{4.49}$$

$$\bar{B}_{i2}^r K_{2i}^r - \bar{B}_2 = 0. \tag{4.50}$$

Part B: Let the agents be only affected by the disturbances, then we obtain

$$\begin{aligned} \dot{\bar{\xi}}_i^f(t) &= \bar{A}_c \bar{\xi}_i^f(t) + \bar{B}_\omega \omega_i(t), \\ z_i(t) &= \bar{C} \bar{\xi}_i^f(t). \end{aligned} \tag{4.51}$$

Consider a Lyapunov function candidate $V_i^f(\bar{\xi}_i^f(t)) = \bar{\xi}_i^{f^{\mathrm{T}}}(t)P\bar{\xi}_i^f(t)$, where $P = \mathrm{diag}\{P_1, P_2\} > 0$. The time derivative of $V_i^f(t)$ along the trajectories of the system (4.51) is given by

$$\dot{V}_i^f(t) = \bar{\xi}_i^{f^{\mathrm{T}}}(t)(\bar{A}_c^{\mathrm{T}}P + P\bar{A}_c)\bar{\xi}_i^f(t) + 2\bar{\xi}_i^{f^{\mathrm{T}}}(t)P\bar{B}_\omega\omega_i(t).$$

By applying Fact 2.2 to the second term in the right hand side of the above equation with $X^{\mathrm{T}} = \bar{\xi}_i^{f^{\mathrm{T}}}(t)P\bar{B}_\omega$, $Y = \omega_i(t)$ and $\alpha = \gamma^{-2}$, and adding $\bar{\xi}_i^{\mathrm{T}}(t)\bar{\xi}_i(t)$ to both sides one gets

$$\dot{V}_i^f(t) - \gamma^2\omega_i^{\mathrm{T}}(t)\omega_i(t) + \bar{\xi}_i^{f^{\mathrm{T}}}(t)\bar{\xi}_i^f(t) \leq \bar{\xi}_i^{f^{\mathrm{T}}}(t)\Lambda\bar{\xi}_i^f(t), \tag{4.52}$$

where

$$\Lambda = \begin{bmatrix} \bar{A}_c^{1^{\mathrm{T}}}P_1 + P_1\bar{A}_c^1 & P_1\bar{A}_c^2 \\ \bar{A}_c^{2^{\mathrm{T}}}P_1 & \bar{A}_c^{3^{\mathrm{T}}}P_2 + P_2\bar{A}_c^3 \end{bmatrix} + \gamma^{-2}\begin{bmatrix} P_1\bar{B}_\omega^1\bar{B}_\omega^{1^{\mathrm{T}}}P_1 & P_1\bar{B}_\omega^1\bar{B}_\omega^{2^{\mathrm{T}}}P_2 \\ P_2\bar{B}_\omega^2\bar{B}_\omega^{1^{\mathrm{T}}}P_1 & P_2\bar{B}_\omega^2\bar{B}_\omega^{2^{\mathrm{T}}}P_2 \end{bmatrix} + I,$$

and $\bar{B}_\omega^1$ and $\bar{B}_\omega^2$ are such that $\bar{B}_\omega = \begin{bmatrix} \bar{B}_\omega^1 \\ \bar{B}_\omega^2 \end{bmatrix}$. If the matrices P_1 and P_2 are obtained such that

$$\Lambda = \begin{bmatrix} \bar{A}_c^{1^{\mathrm{T}}}P_1 + P_1\bar{A}_c^1 & P_1\bar{A}_c^2 \\ \bar{A}_c^{2^{\mathrm{T}}}P_1 & \bar{A}_c^{3^{\mathrm{T}}}P_2 + P_2\bar{A}_c^3 \end{bmatrix} + \gamma^{-2}\begin{bmatrix} P_1\bar{B}_\omega^1\bar{B}_\omega^{1^{\mathrm{T}}}P_1 & P_1\bar{B}_\omega^1\bar{B}_\omega^{2^{\mathrm{T}}}P_2 \\ P_2\bar{B}_\omega^2\bar{B}_\omega^{1^{\mathrm{T}}}P_1 & P_2\bar{B}_\omega^2\bar{B}_\omega^{2^{\mathrm{T}}}P_2 \end{bmatrix} + I < 0, \tag{4.53}$$

then the right hand side of (4.52) will be negative definite and we have

$$\dot{V}_i^f(t) - \gamma^2\omega_i^{\mathrm{T}}(t)\omega_i(t) + \bar{\xi}_i^{f^{\mathrm{T}}}(t)\bar{\xi}_i^f(t) < 0.$$

Consequently, by integrating both sides of the above inequality, one gets

$$\frac{\int_{t_f}^{\infty}\bar{\xi}_i^{f^{\mathrm{T}}}(t)\bar{\xi}_i^f(t)\mathrm{d}t}{\int_{t_f}^{\infty}\omega_i^{\mathrm{T}}(t)\omega_i(t)\mathrm{d}t} < \gamma^2.$$

Now, given that $\bar{\xi}_i^{f^{\mathrm{T}}}(t) = T^{-1}\xi_i^f(t)$, the H_∞ performance bound for $\xi_i^f(t)$ can be obtained as

$$\frac{\int_{t_0}^{\infty}\xi_i^{\mathrm{T}}(t)\xi_i(t)\mathrm{d}t}{\int_{t_0}^{\infty}\omega_i^{\mathrm{T}}(t)\omega_i(t)\mathrm{d}t} \leq \gamma^2\lambda_{\min}^{-1}(T^{-\mathrm{T}}T^{-1}) = \gamma_f^2.$$

Part C: From Parts A and B, it follows that $\bar{K}_{1i}^{r1}$ should satisfy (4.47) and (4.48), $\bar{K}_{1i}^{r2}$ should satisfy (4.49) and K_{2i}^r should satisfy (4.50), while the inequality (4.53) should also hold. Note that if there exist matrices P_1 and P_2 such that (4.53) holds then $\bar{A}_c$ will be Hurwitz. This implies that if the inequality (4.53) holds then (4.48) and (4.49) will hold. Therefore, the problem is reduced to solving the equality (4.50) for K_{2i}^r and solving (4.47) and (4.53) simultaneously for $\bar{K}_{1i}^{r1}$ and $\bar{K}_{1i}^{r2}$. Equation (4.50) is linear with respect to K_{2i}^r and can be solved easily, whereas considering the structure of $\bar{A}_c^i$ for $i = 1, 2, 3$, the inequality (4.53) is nonlinear with respect to P_1, P_2 and γ. However, by multiplying both sides by P^{-1} and using the known change of variables $X = \text{diag}\{X_1, X_2\}$, $X_1 = P_1^{-1}$, $X_2 = P_2^{-1}$, $Y_1 = \bar{K}_{i1}^{r1} P_1^{-1}$, $Y_2 = \bar{K}_{i1}^{r2} P_2^{-1}$, $\alpha = \gamma^{-2}$ and using the Schur complement, the inequality (4.53) can be transformed into the following LMI condition:

$$\begin{bmatrix} \Theta & X \\ X & -I \end{bmatrix} < 0, \tag{4.54}$$

where $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^{\mathrm{T}} & \Theta_3 \end{bmatrix}$, $\Theta_1 = X_1 \bar{A}_{11}^{\mathrm{T}} + Y_1^{\mathrm{T}} \bar{B}_{i1}^{r^{\mathrm{T}}} + \bar{A}_{11} X_1 + \bar{B}_{i1}^r Y_1 + \alpha \bar{B}_\omega^1 \bar{B}_\omega^{1^{\mathrm{T}}}$, $\Theta_2 = \bar{A}_{12} X_2 + \bar{B}_{i1}^r Y_2 + \alpha \bar{B}_\omega^1 \bar{B}_\omega^{2^{\mathrm{T}}}$ $\Theta_3 = X_2 \bar{A}_{22}^{\mathrm{T}} + Y_2^{\mathrm{T}} \bar{B}_{i2}^{r^{\mathrm{T}}} + \bar{A}_{22} X_2 + \bar{B}_2 Y_2 + \alpha \bar{B}_\omega^2 \bar{B}_\omega^{2^{\mathrm{T}}}$. Therefore, the control gains $\bar{K}_{1i}^{r1}$ and $\bar{K}_{1i}^{r2}$ satisfy the requirements of Parts A and B if the solutions to the inequality (4.54) also satisfy (4.47). These requirements can be achieved provided that the gains are obtained as solutions to the following optimization problem, namely

$$\max \alpha \text{ s.t. } \quad \begin{bmatrix} \Theta & X \\ X & -I \end{bmatrix} < 0,\ X > 0,\ \bar{A}_{21} X_1 + \bar{B}_{i2}^r Y_1 = 0.$$

Subject to the above conditions the upper bound for the H_∞ performance index and the reconfigured control gain K_{1i}^r are now specified according to $\gamma^2 = \alpha^{-1} \lambda_{\min}^{-1}\{(TT^{\mathrm{T}})^{-1}\}$ and $K_{1i}^r = \begin{bmatrix} Y_1 X_1^{-1} & Y_2 X_2^{-1} \end{bmatrix} T^{-1}$, and this completes the proof of the theorem. $\square$

The following algorithm summarizes the required steps that one needs to follow for designing the reconfigured control law gains.

<u>**Algorithm for Design of the Fault Reconfiguration Controller Gains**</u>:

1) Obtain the maximal (A, B_i^r) controlled invariant subspace, $\mathcal{V}^*$, either by using the iterative algorithm that is proposed in [160] or by using the Geometric Approach Toolbox [178] (available online). Set T_1 such that $\mathcal{V}^* = \text{Im}\{T_1\}$ and select T_2 such that $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ is a nonsingular matrix.

2) Obtain $\bar{A}_{11}$, $\bar{A}_{21}$, $\bar{A}_{12}$, $\bar{B}_{i1}$ and $\bar{B}_{i2}$ as in (4.42) and $\bar{B}_\omega$ and $\bar{B}_2$ as in (4.43).

3) Solve the optimization problem (4.45) for X_1, X_2, Y_1 and Y_2.

4) Set K_{1i}^r as $K_{1i}^r = \begin{bmatrix} Y_1 X_1^{-1} & Y_2 X_2^{-1} \end{bmatrix} T^{-1}$.

5) Solve equation (4.44) for K_{2i}^r.

6) Solve equation (4.36) for $\underline{u}_i^r$.

7) Set $u_i^r(t) = K_{1i}^r \xi_i^f(t) + K_{2i}^r u_i^a(t) + \underline{u}_i^r$.

8) Set $u_i^f(t) = \begin{bmatrix} 0_{1\times m_o} & (\underline{u}_i^s)^{\mathrm{T}} & (u_i^r(t))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$.

In view of the Theorem 4.2 and the above Algorithm the following results can be obtained immediately.

Corollary 4.1 (Presence of only the LOE fault)**.** *Suppose the actuators are either healthy or subject to the LOE fault. In this case,* B_i^r *in (4.42) is given by* $B_i^r = B\Gamma_i$*, where*

$\Gamma_i = diag\{\Gamma_i^k\}$, $k = 1, \ldots, m$. *Furthermore, the faulty control law* $u_i^f(t)$, *and the reconfigured control law,* $u_i^r(t)$, *for the* i*-th faulty agent are designed according to*

$$u_i^f(t) = u_i^r(t)$$

$$u_i^r(t) = K_{1i}^r \xi_i^f(t) + K_{2i}^r u_i^a(t),$$

where the control gains K_{1i}^r *and* K_{2i}^r *are designed according to the Steps 4 and 5 of the above algorithm.*

Corollary 4.2 (Presence of only the outage fault)**.** *Suppose the actuators* 1 *to* m_o *are subject to the outage fault and the remaining actuators are healthy. In this case,* B_i^r *in (4.42) is given by* $B_i^r = \begin{bmatrix} b^{m_o+1} & \ldots & b^m \end{bmatrix}$. *Furthermore, the faulty control law* $u_i^f(t)$, *and the reconfigured control law,* $u_i^r(t)$, *for the* i*-th faulty agent are designed according to*

$$u_i^f(t) = \begin{bmatrix} 0_{1\times m_o} & (u_i^r(t))^T \end{bmatrix}^T,$$
$$u_i^r(t) = K_{1i}^r \xi_i^f(t) + K_{2i}^r u_i^a(t),$$

where the control gains K_{1i}^r *and* K_{2i}^r *are designed according to the Steps 4 and 5 of the above algorithm.*

Corollary 4.3 (Presence of only the stuck fault)**.** *Suppose the actuators* 1 *to* m_s *are subject to the stuck and the remaining actuators are healthy. In this case,* B_i^r *in (4.42) is given by* $B_i^r = \begin{bmatrix} b^{m_s+1} & \ldots & b^m \end{bmatrix}$. *Furthermore, the faulty control law* $u_i^f(t)$, *and the reconfigured control law,* $u_i^r(t)$, *for the* i*-th faulty agent are designed according to*

$$u_i^f(t) = \begin{bmatrix} (\underline{u}_i^s)^T & (u_i^r(t))^T \end{bmatrix}^T,$$
$$u_i^r(t) = K_{1i}^r \xi_i^f(t) + K_{2i}^r u_i^a(t) + \underline{u}_i^r,$$

where the control gains K_{1i}^r *and* K_{2i}^r *are designed according to the Steps 4 and 5 and the* *control command $\underline{u}_i^r$ is obtained according to the Step 6 of the above algorithm.*

Similar results corresponding to the combination of any two of the considered three types of faults can also be developed. These straightforward results that follow from Theorem 4.2 and the Corollaries 4.1-4.3 are not included here for brevity.

4.2.2 The Existence of Solutions and Analysis

In previous subsection, a cooperative control strategy to ensure consensus achievement and control reconfiguration subject to actuator faults and environmental disturbances was proposed and conditions under which the objectives are guaranteed were provided. In the following, we discuss the properties of solutions if certain required conditions are not satisfied. We consider five cases that are designated as I to V below.

Case I: If the Assumption 4.2-(c) does not hold, i.e., the fault occurs during the transient period, then $\xi_i^f(t_f) \neq 0$, and the first term in (4.38) will be non-zero. However, since K_{1i}^r is designed such that $A+B_i^r K_{1i}^r$ is Hurwitz this term will vanish asymptotically. Note that the delay in receiving the information from the FDI module and activating the control reconfiguration will also result in $\xi_i^f(t_f) \neq 0$, and causes a similar effect. **Case II**: If $\bar{B}_2 \not\subset \text{Im}\{\bar{B}_{i2}^r\}$, then (4.44) does not have a solution. In this case, we may obtain K_{2i}^r as a solution to

$$\min_{K_{2i}^r} \text{trace}\{B_{i2}^r K_{2i}^r - \bar{B}_2\}.$$

Corresponding to this choice of K_{2i}^r, the second term of (4.38) will remain non-zero and we have $z_i(t) \neq 0$ but bounded. However, if $\bar{K}_{1i}^r$ is designed according to Part B in the proof of Theorem 4.2, one can still guarantee boundedness of the state consensus errors.

<u>Case III</u>: Suppose the estimated value of the stuck fault command, that is $\underline{u}_i^s$, is not accurate, namely $\underline{u}_i^s = \hat{\underline{u}}_i^s + \epsilon_i$, where $\hat{\underline{u}}_i^s$ and ϵ_i denote the estimated stuck command and its error. Equation (4.36) can then be expressed as $(B_i^s \hat{\underline{u}}_i^s + B_i^r u_i^C) + B_i^s \epsilon_i = 0$. Since ϵ_i is unknown, therefore to obtain u_i^C we instead use the following optimization problem, namely

$$\min_{u_i^C} \text{ trace}\{B_i^s \hat{\underline{u}}_i^s + B_i^r u_i^C\}.$$

Let $\eta_i = (B_i^s \hat{\underline{u}}_i^s + B_i^r u_i^C) + B_i^s \epsilon_i$. Consider the control law (4.33) as designed in Theorem 4.2. It follows that for $\omega_i(t) \equiv 0$, equation (4.37) becomes $\dot{\xi}_i^f(t) = A_c \xi_i^f(t) + (B_i^r K_{2i}^r - B) u_i^a(t) + \eta_i$, where $A_c = A + B_i^r K_{1i}^r$. Under Assumption 4.2, and for K_{2i}^r as a solution to (4.44), it follows that $z_i(t) = C A_c^{-1}(e^{A_c(t-t_f)} - I)\eta_i$. Given that A_c is Hurwitz, the above equation implies that after a transient period the error between the output of the faulty agent and its associated auxiliary system, or equivalently the output tracking error reaches a constant steady state value, i.e. $\lim_{t\to\infty} z_i(t) = -C A_c^{-1} \eta_i$. Consequently, under this scenario one can still observe that the state consensus errors remain bounded.

<u>Case IV</u>: Let the estimated actuator loss of effectiveness factor or severity be subject to uncertainties, i.e. $\Gamma_i^k = \hat{\Gamma}_i^k + \epsilon_i^k$, where $\hat{\Gamma}_i^k$ is the estimate of the fault severity that is provided by the FDI module, and ϵ_i^k is an unknown estimation error uncertainty. Consider equation (4.42). Since $\Gamma_i \neq \hat{\Gamma}_i$ we have $\bar{B}_i^r = \hat{\bar{B}}_i^r + \bar{B}_i^{r\epsilon}$, where $\hat{\bar{B}}_i^r = T^{-1} B_i^{fr} \hat{\Gamma}_i$, $\bar{B}_i^{r\epsilon} = T^{-1} B_i^{fr} \Upsilon_i$, $\Upsilon_i = \text{diag}\{\epsilon_i^k\}$, $k = m_s + 1, \ldots, m$ and $B_i^{fr} = \begin{bmatrix} b^{m_s+1} \ldots, B^m \end{bmatrix}$.

In order to analyze the impact of these uncertainties on our previous results, we need to investigate both the matching condition, namely equation (4.44), and the stability of the tracking error $\xi_i^f(t)$.

Since Υ_i is unknown, one cannot determine the gain K_{2i}^r such that (4.44) holds. This implies that unlike $z_i(t)$ (given in Part A of the proof), one cannot ensure $z_i(t) \equiv 0$. On the other hand, $\bar{A}_c$ in (4.43) should be replaced by $\tilde{A}_c = \bar{A}_c + \bar{A}_c^\epsilon = \bar{A} + \hat{\bar{B}}_i^r \bar{K}_{1i}^r + \bar{B}_i^{r\epsilon} \bar{K}_{1i}^r$. Following along the same steps as those utilized in Subsection 4.2.1 for now $\bar{A}_c = \bar{A} + \hat{\bar{B}}_i^r \bar{K}_{1i}^r$, one can obtain the control gain $\bar{K}_{1i}^r$ that makes $\bar{A}_c$ Hurwitz. Hence, for $\omega_i(t) \equiv 0$, equation (4.43) can be written as $\dot{\bar{\xi}}_i^f(t) = \big(\bar{A}_c + \bar{A}_c^\epsilon\big)\xi_i^f(t) + \bar{E}_i u_i^a(t)$. In order to analyze the robustness we can use the results of Theorem 2.1, however we need to rewrite the above equation as below

$$\dot{\bar{\xi}}_i^f(t) = \bar{A}_c \xi_i^f(t) + \bar{E}_i u_i^a(t) + f(\xi_i^f(t)),$$

where $f(\xi_i^f(t)) \triangleq \bar{A}_c^\epsilon \xi_i^f(t)$. Given that $\bar{A}_c^\epsilon = \sum_{l=m_s+1}^{m} \epsilon_i^l b^l k^l$, we get

$$\|\bar{A}_c^\epsilon\|_2 \leq \sum_{l=m_s+1}^{m} |\epsilon_i^l| \|b^l k^l\|_2,$$

where b^l and k^l denote the $l - m_s$-th column of B_i^{fr} and the $l - m_s$-th row of $\bar{K}_{1i}^r$, respectively. Now, by using Theorem 2.1, if there exist $\bar{\epsilon}_{\text{imax}}^l > 0$, $l = 1, \ldots, m - m_s$ such that

$$\sum_{l=m_s+1}^{m} \bar{\epsilon}_{\text{imax}}^l \|b^l k^l\|_2 \leq \frac{1}{\sigma_{min}(P)}, \tag{4.55}$$

and $|\epsilon_i^l| \leq \bar{\epsilon}_{\text{imax}}^l$, then the matrix $\tilde{A}_c$ remains Hurwitz, where P is a positive definite matrix solution to $P\bar{A}_c + \bar{A}_c^{\mathrm{T}} P = -2I$. This along with the boundedness of $u_i^a(t)$ implies that $\bar{\xi}_i^f(t)$ will also remain bounded.

Case V: Suppose that the fault *is recovered after a delay* of Δ s, i.e. $t_r = t_f + \Delta$, where t_f and t_r denote the time that the fault occurs and the time that the control reconfiguration is invoked. During the time $t_f \le t \le t_r$, the tracking dynamics of the i-th agent, i.e., $\xi_i^f(t)$, becomes

$$\dot{\xi}_i^f(t) = (A + c_1 B_i^f K)\xi_i^f(t) + (B_i^f - B)u_i^a(t), \ t_f \le t < t_r.$$

Therefore, one gets $x_i^f(t) = \exp((A + c_1 B_i^f K)(t - t_f))(x_i(t_f) - x_i^a(t_f)) + \int_{t_f}^{t} \exp((A + c_1 B_i^f K)(t - s))u_i^a(s)\mathrm{d}s + x_i^a(t)$. If the fault causes $A + c_1 B_i^f K$ to become non-Hurwitz, then $x_i^f(t)$ will grow exponentially. Now, let x_i^M denote the maximum allowable upper bound on the agent's state (this can be specified for example based on the maximum speed of the moving agent or the maximum depth for surveying under the water), then invoking the reconfigured control law cannot be delayed beyond Δs, where the maximum delay in invoking the reconfigured controller is denoted by Δ and can be obtained by solving the following equation $x_i^M = x_i^a(t_f + \Delta) + \exp((A + c_1 B_i^f K)\Delta)(x_i(t_f) - x_i^a(t_f)) + \int_{t_f}^{t_f+\Delta} \exp((A + c_1 B_i^f K)(t_f + \Delta - s))u_i^a(s)\mathrm{d}s$. This implies that if the fault is not recovered before $t = t_f + \Delta$ s, the faulty agent may no longer be recoverable to satisfy the overall mission requirements and specifications at all times.

4.3 Simulation Results

In this section, our proposed control recovery approach is applied to a network of Autonomous Underwater Vehicles (AUVs). The team behavior is studied under several scenarios, namely when the agents are healthy and also when the agents are subject to simultaneous LOE, outage and stuck actuator faults, uncertainties in the FDI module information and delays in invoking the control reconfiguration. The team is considered to consist of five Sentry Autonomous Underwater Vehicles (AUVs). Sentry, made by the Woods Hole Oceanographic Institution [171], is a fully autonomous underwater vehicle that is capable of surveying to the depth of 6000 m and is efficient for forward motions.

The nonlinear six degrees of freedom equations of motion in the body-fixed frame in the horizontal plane is given by [179]:

$$\begin{aligned}&\mathbb{M}\dot{\nu} + \mathbb{C}(\nu)\nu + \mathbb{D}(\nu, \phi_f)\nu + g(\eta) = b(\phi, h),\\&\dot{\eta} = \mathbb{J}(\eta)\nu,\end{aligned}$$

where $\mathbb{M}$, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{J}$ denote the inertia matrix, the moment/forces matrix, the damping matrix and the transformational matrix, respectively. The terms $g(\eta)$ and $b(\phi_f, h)$ denote the hydrostatic restoring forces and the truster input, respectively, and are given by

$$b(\phi_f, h) = \begin{bmatrix} (h_{fp} + h_{fs})\cos\phi_{ff} + (h_{ap} + h_{as})\cos\phi_{af} \\ 0 \\ (h_{fp} + h_{fs})\sin\phi_{ff} + (h_{ap} + h_{as})\sin\phi_{af} \\ b_t(h_{fp} - h_{fs})\sin\phi_{ff} + b_t(h_{ap} - h_{as})\sin\phi_{af} \\ -a_{ff}(h_{fp} + h_{fs})\sin\phi_{ff} - a_{af}(h_{ap} + h_{as})\sin\phi_{af} \\ b_t(h_{fp} - h_{fs})\cos\phi_{ff} + b_t(h_{ap} - h_{as})\cos\phi_{af} \end{bmatrix},$$

$g(\eta) = \begin{bmatrix} 0 & 0 & 0 & z_{BG}\cos\theta\sin\phi W & z_{BG}\sin\theta W & 0 \end{bmatrix}^{\mathrm{T}}$, where $\phi_f = [\phi_{ff}\ \phi_{af}]^{\mathrm{T}}$ and $h = [h_{fp}\ h_{fs}\ h_{ap}\ h_{as}]^{\mathrm{T}}$ denote the foil angles and the truster inputs, respectively. The term $\eta = [\eta_1^{\mathrm{T}}\ \eta_2^{\mathrm{T}}]^{\mathrm{T}}$, where $\eta_1 = [x\ y\ z]^{\mathrm{T}}$ denotes the inertial position and $\eta_2 = [\phi\ \theta\ \psi]^{\mathrm{T}}$ denotes the inertial orientation. Also, $\nu = [\nu_1^{\mathrm{T}}\ \nu_2^{\mathrm{T}}]^{\mathrm{T}}$, with $\nu_1 = [\bar{u}\ v\ w]^{\mathrm{T}}$ denotes the body-fixed linear velocity and $\nu_2 = [p\ q\ r]^{\mathrm{T}}$ denotes the angular velocity. Finally, z_{BG}

and W denote the vertical distance between the center of the buoyancy and the center of the mass and the vehicle weight, respectively.

For the Sentry vehicle, the horizontal position is controlled indirectly through the heading subsystem, i.e. v, r, ψ, and surge speed subsystem, i.e. $\bar{u}$. Therefore, for control purposes the states x and y are ignored. Moreover, under the assumptions that (a) the truster and foil angles do not affect each other, (b) the pitch and the pitch rate, i.e. θ and q are sufficiently small, and (c) the foil angles are sufficiently small, then the states p, ϕ are also ignored for control design and are considered passive [171]. Therefore, for the control design in the near horizontal maneuver under the operating point $\nu_1^o = [u^o \; 0 \; w^o]^{\mathrm{T}}$, $\nu_2^o = 0_{3\times1}$ and $\eta^o = 0_{6\times1}$, the linear model of the Sentry AUV is reduced to the following subsystems:

$$
\begin{bmatrix} \dot{\bar{u}}(t) \\ \dot{v}(t) \\ \dot{r}(t) \\ \dot{\psi}(t) \end{bmatrix} = \begin{bmatrix} a_{11}u^o & 0 & 0 & 0 \\ 0 & a_{22}u^o & a_{26}u^o & 0 \\ 0 & a_{62}u^o & a_{66}u^o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}(t) \\ v(t) \\ r(t) \\ \psi(t) \end{bmatrix}
$$

$$
+ \begin{bmatrix} m_{11} & m_{11} & m_{11} & m_{11} \\ m_{26}b_h & -m_{26}b_h & m_{26}b_h & -m_{26}b_h \\ m_{66}b_h & -m_{66}b_h & m_{66}b_h & -m_{66}b_h \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{fp}(t) \\ h_{fs}(t) \\ h_{ap}(t) \\ h_{as}(t) \end{bmatrix},
$$

$$
\begin{bmatrix} \dot{w}(t) \\ \dot{q}(t) \\ \dot{z}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} a_{33}u^o & a_{35}u^o & 0 & -m_{m35}z_{GB}W \\ a_{53u^o} & a_{55}u^o & 0 & -m_{55}z_{GB}W \\ 1 & 0 & 0 & u^o \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ q(t) \\ z(t) \\ \theta(t) \end{bmatrix}
$$

$$
+ \begin{bmatrix} \alpha_h^{11} & \alpha_h^{12} \\ \alpha_h^{21} & \alpha_h^{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{ff}(t) \\ \phi_{af}(t) \end{bmatrix},
$$

where $\alpha_h^{11} = \frac{\beta_h}{1+\beta_h}h_{31} + (u^o)^2 f_{31}$, $\alpha_h^{12} = \frac{1}{1+\beta_h}h_{32} + (u^o)^2 f_{32}$, $\alpha_h^{21} = \frac{\beta_h}{1+\beta_h}h_{51} + (u^o)^2 f_{51}$, and $\alpha_h^{22} = \frac{1}{1+\beta_h}h_{52} + (u^o)^2 f_{52}$. The detail relationships between the above parameters and the system parameters are provided in [171].

For underwater vehicles, the ocean current is considered as a disturbance to the system, i.e. $\omega(t) = V_c(t)$, where $V_c(t)$ denotes the ocean current. In [180], the ocean current is modeled by a first order Gauss-Markov Process as governed by $\dot{V}_c(t) + \mu V_c(t) = v(t)$, where $\mu \geq 0$ and $v(t)$ is a Gaussian white noise. For $\mu = 0$, the model becomes a random walk, i.e. $\dot{V}_c(t) = v(t)$. Therefore, the disturbance signal that is applied to the i-th agent is expressed as $\omega_i(t) = V_{ci}(t) = \int_{t_0}^{t} v_i(t)\mathrm{d}t + V_{ci}(t_0)$.

In conducting our simulations we only consider the speed-heading subsystems, i.e. $\bar{u}, v, r, \psi$. To obtain a linear model, the forward (surge) speed u^o is set to $u^o = 1$ and all the parameters are considered to be the same as those in [171, 173]. The numerical values of the triple (A, B, C) for the i-th agent is governed by

$$A = \begin{bmatrix} -0.0401 & 0 & 0 & 0 \\ 0 & -0.709 & -0.648 & 0 \\ 0 & -1.770 & 1.414 & 0 \\ 0 & 0 & 1.00 & 0 \end{bmatrix}, B = 1.0e-03 \begin{bmatrix} 0.44 & 0.44 & 0.44 & 0.44 \\ 0.06 & -0.06 & 0.06 & -0.06 \\ 0.49 & -0.49 & 0.49 & -0.49 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_\omega = \begin{bmatrix} 0.023 & 0.017 & 0.03 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, x_i(t) = \begin{bmatrix} \bar{u}_i(t) & v_i(t) & r_i(t) & \psi_i(t) \end{bmatrix}^{\mathrm{T}},$$

$$u_i(t) = \begin{bmatrix} h_{ifp}(t) & h_{ifs}(t) & h_{iap}(t) & h_{ias}(t) \end{bmatrix}^{\mathrm{T}}.$$

The network topology considered is as shown in Figure 4.3. The leader control law

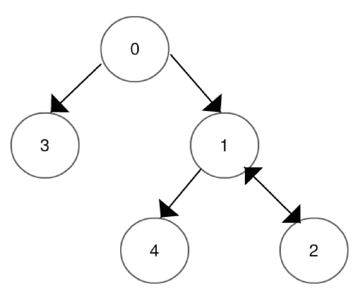

Figure 4.3: The topology of the leader-follower network of given AUVs.

is selected as $u_0(t) = K_0 x_0(t) + F_0 r(t)$, and the desired leader speed $r(t) = \bar{u}_{\text{desired}}(t)$ is defined according to Figure 4.4. The objective of the team cooperative control is to ensure that all the agents follow the leader output (surge speed) trajectory, while their yaw angle, sway and yaw rate remain bounded. The acceptable errors between the desired trajectory and the actual trajectories are considered to be less than 10% in the steady state. The following scenarios are now considered:

Scenario 1: Faulty team without control reconfiguration: In this scenario, it is assumed

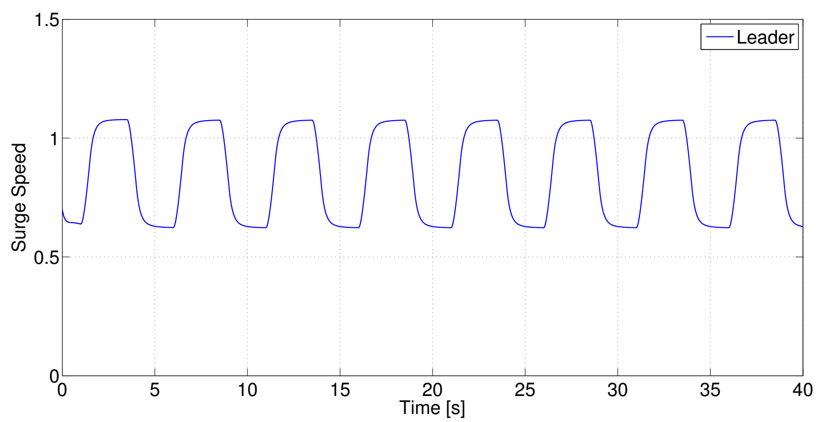

Figure 4.4: The desired leader surge speed trajectory.

that no control reconfiguration is invoked after the occurrence of the faults. The specifics for the mission considered are as follows where the followers state trajectories are depicted in Figure 4.5.

A) All the agents are healthy and the agent control law is designed according to Theorem 4.1 and using YALMIP toolbox for MATLAB.

B) At time $t = t_f = 25\ s$, the agents 1 and 2 become faulty. Agent 1 loses its second actuator i.e. $h^f_{1fs}(t) \equiv 0$, $t \geq 25$. Agent 2 loses 30% of its first actuator and its second actuator gets stuck at $\underline{u}^s_2 = 1$, i.e. $h^f_{2fp}(t) = 0.7 h_{2fp}(t)$ and $h^f_{2fs}(t) \equiv 1$ for $t \geq 25\ s$.

Figure 4.5 clearly shows that if a reconfiguration control strategy is not invoked, the agents become unstable and their states grow exponentially unbounded. Therefore, it is necessary to reconfigure the agent's control law after the occurrence of this fault.

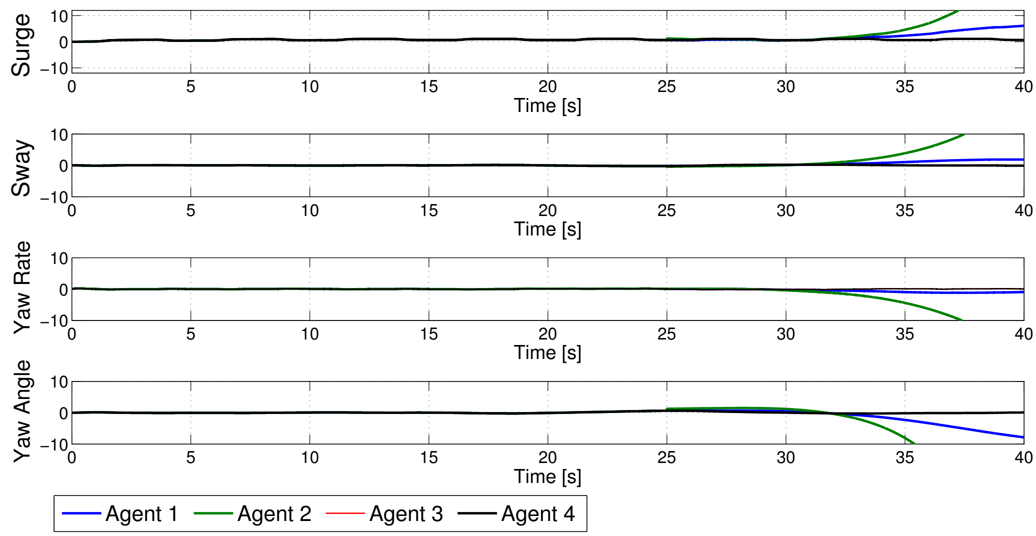

Figure 4.5: The followers trajectories corresponding to the Scenario 1.

Scenario 2: Control reconfiguration subject to delays in invoking the reconfigured control law: Unlike the previous scenario, in this scenario control reconfiguration laws are invoked to the faulty agents. However, it is assumed that there are delays in the time that the FDI module communicates this information to the faulty agents and the agents reconfigured controls are invoked. The specifics for the execution of the mission are as follows where the followers state trajectories are depicted in Figure 4.6.

A) All the agents are healthy and the agent control law is similar to the Scenario 1.

B) At time $t = t_f = 25\ s$, the agents 1 and 2 become faulty. The fault scenario that is considered is the same as that of Step **B)** in Scenario 1.

C) The control laws for both faulty agents are reconfigured according to Theorem 4.2 at

$t = t_r = 30\ s$.

Figure 4.6, depicts that by invoking the reconfigured control laws one can now stabilize all the agents. The delay in invoking the control reconfiguration causes a transient period in which the agent states diverge and will not follow the leader (refer to discussion in Subsection 4.2.2, **Case V**). However, after the transients have died out, the agent reach a consensus with the leader state.

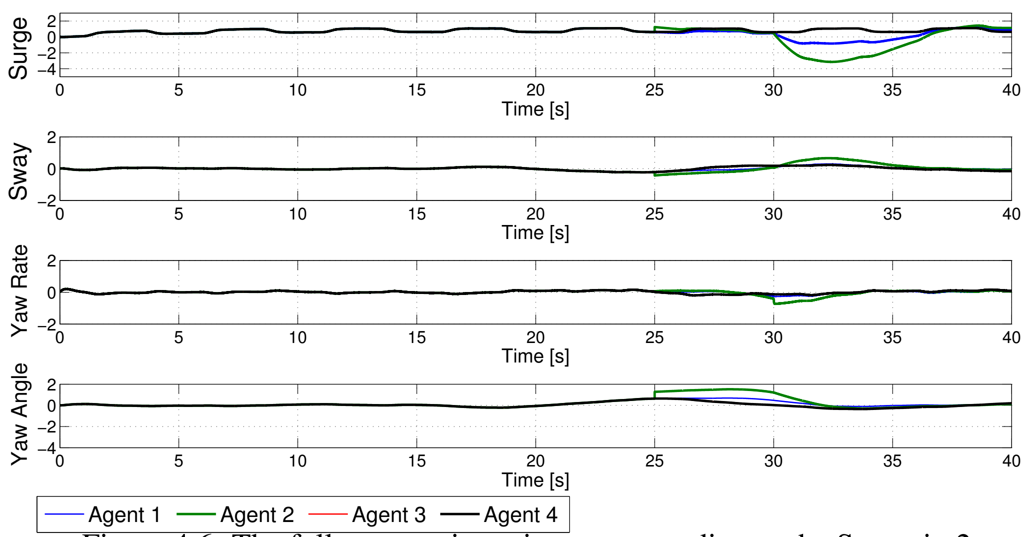

Figure 4.6: The followers trajectories corresponding to the Scenario 2.

Scenario 3: Control reconfiguration subject to fault estimation uncertainties: In this scenario, we consider a similar fault scenario as in the previous scenarios. However, it is assumed that the estimated fault severities are subject to unreliabilities, errors and uncertainties. Using the inequality (4.55) the upper bound on uncertainties is obtained as $\bar{\epsilon}^1_{2max} = 0.146$, implying that the reconfigured control law stabilizes the errors provided that it is designed based on $0.554 \leq \hat{\Gamma}^1_2 \leq 0.846$. To investigate how accurate this range is, various levels of uncertainties and mismatches are considered and it is observed that the control gains that are designed for $\hat{\Gamma}_2^1 < 0.86$ stabilize the errors whereas for $\hat{\Gamma}_2^1 \geq 0.87$ the state consensus errors become unstable. This indicates that the bound provided by the inequality (4.55) provides an acceptable approximation to the maximum allowable fault severities estimation errors and uncertainties. The agents state simulation responses correspond to $\hat{\Gamma}_2^1 = 0.6$ and $\underline{\hat{u}}_i^s = 0.9$, and are depicted in Figure 4.7. Figure 4.7 shows

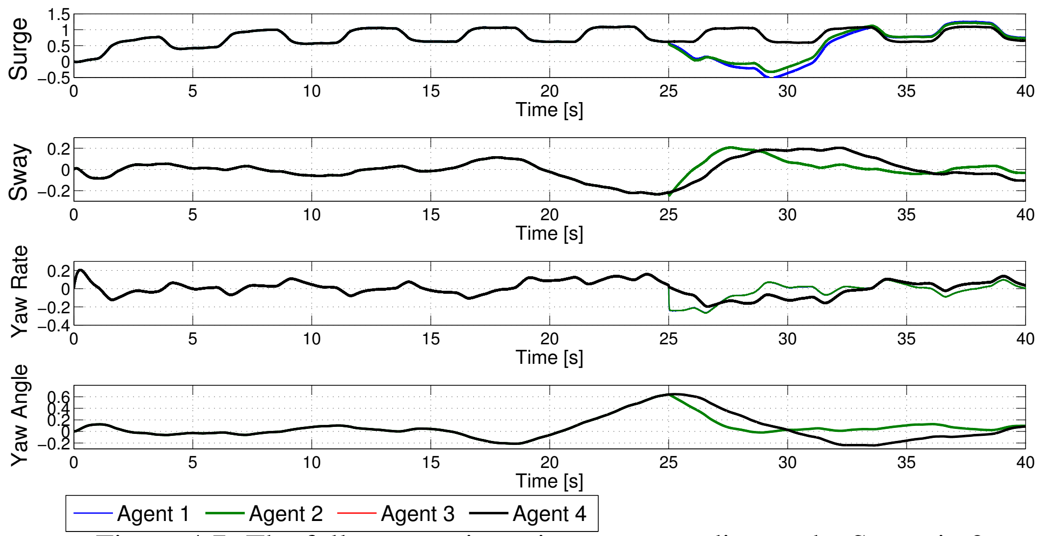

Figure 4.7: The followers trajectories corresponding to the Scenario 3.

that by invoking the reconfigured control law, the agent states will no longer diverge and the recovery control strategy stabilizes the agent states. In fact, in this scenario the agents do follow the changes in the leader speed trajectory, although the error between the faulty agent speed trajectory and the leader speed trajectory will not vanish but converges asymptotically to a small constant value.

4.4 Conclusions

In this work, a cooperative and distributed reconfigurable control law strategy is developed and designed to control and reconfigure faulty agents from three types of actuator faults, namely loss of effectiveness, outage, and stuck faults that guarantee boundedness of the state consensus errors for a network of multi-agent systems. It is shown that the proposed control strategy can ensure an H_∞ performance bound attenuation for the team agents when they are subjected to environmental disturbances and actuator faults. Our proposed reconfigured control law ensures that the output of the faulty agent matches that of the healthy agent in absence of disturbances. Moreover, the control law also guarantees that the state consensus errors remain bounded. Furthermore, in presence of environmental disturbances the H_∞ disturbance attenuation bound is ensured to be minimized. The effectiveness of our proposed cooperative control and reconfigurable approaches are evaluated by applying them to a network of autonomous underwater vehicles. Extensive simulation case studies are also considered to demonstrate the capabilities and advantages of our proposed strategies subject to FDI module uncertainties, erroneous decisions, and imperfections.

Chapter 5

Control Reconfiguration in Switched Topology networks

In this chapter, distributed control reconfiguration strategies for directed switching topology networked multi-agent systems are developed and investigated. The proposed control strategies are invoked when the agents are subject to actuator faults and while the available fault detection and isolation (FDI) modules provide inaccurate and unreliable information on the estimation of faults severities. Our proposed strategies will ensure that the agents reach a consensus while an upper bound on the team performance index is obtained and minimized. Three types of actuator faults are considered, namely: the loss of effectiveness fault, the outage fault, and the stuck fault. By utilizing quadratic and convex hull (composite) Lyapunov functions, two cooperative and distributed recovery strategies are designed and provided to select the gains of the proposed control laws such that the team objectives are guaranteed. Our proposed reconfigurable control laws are applied to a team of autonomous underwater vehicles (AUVs) under directed switching topologies and subject to simultaneous actuator faults. Simulation results demonstrate the effectiveness of our proposed distributed reconfiguration control laws in compensating for the effects of sudden actuator faults and subject to fault diagnosis module uncertainties and unreliabilities.

5.1 Problem Formulation

Consider a multi-agent network having a leader that is governed by (2.3) and N follower agents whose dynamics are given by (2.1). Without loss of any generality, assume that the agents 1 to N_f become faulty at the time $t = t_f$, where for the i-th faulty agent the first m_{oi} actuators are subject to the outage fault, the actuators $m_{oi}+1$ to m_{si} are subject to the stuck fault, and the remaining $m - m_{si}$ actuators are either subject to the LOE fault or are healthy. Using equations (2.15)-(2.18), the model of i-th faulty agent can be expressed as

$$\dot{x}_i^f(t) = Ax_i^f(t) + B_i^f u_i^f(t),\ i = 1, \ldots, N_f, \tag{5.1}$$

where $B_i^f = [B_i^o\ B_i^s\ B_i^r]$, $B_i^o = [b^1, \ldots, b^{m_{oi}}]$, $B_i^s = [b^{m_o+1}, \ldots, b^{m_s}]$, $B_i^r = \bar{B}_i^r \Gamma_i = \hat{B}_i^r + B_{i\xi}$, $\bar{B}_i^r = \begin{bmatrix} b^{m_{si}+1}, \ldots, b^m \end{bmatrix}$, $\Gamma_i =$diag$\{\Gamma_i^k\}$, $k = m_{si}+1, \ldots, m$, Γ_i^k denotes the k-th actuator effectiveness and the fault severity factor, $\hat{B}_i^r = \bar{B}_i^r \hat{\Gamma}_i$, $B_{i\xi} = \bar{B}_i^r f_i \xi_i$, $\hat{\Gamma}_i =$ diag$\{\hat{\Gamma}_i^k\}$, $f_i =$ diag$\{f_i^k\}$, $\xi_i =$ diag$\{\xi_i^k\}$, Γ_i^k, $\hat{\Gamma}_i^k$, ξ_i^k and f_i^k are defined as in (2.16),

$u_i^f(t) = \begin{bmatrix} 0_{1\times m_{oi}} & (\underline{u}_i^s)^{\mathrm{T}} & (u_i^r(t))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\underline{u}_i^s = \hat{\underline{u}}_i^s + \underline{u}_i^{\xi}$, $\underline{u}_i^s = \begin{bmatrix} u_i^{m_{oi}+1}(t_f), \dots, u_i^{m_{si}}(t_f) \end{bmatrix}^{\mathrm{T}}$, $\hat{\underline{u}}_i^s = \begin{bmatrix} \hat{\underline{u}}_i^{m_{oi}+1}, \dots, \hat{\underline{u}}_i^{m_{si}} \end{bmatrix}^{\mathrm{T}}$, $\underline{u}_i^{\xi} = \begin{bmatrix} \underline{u}_i^{m_{oi}+1\xi}, \dots, \underline{u}_i^{m_{si}\xi} \end{bmatrix}^{\mathrm{T}}$, and $u_i^r(t) = \begin{bmatrix} u_i^{m_{si}+1}(t), \dots, u_i^m(t) \end{bmatrix}^{\mathrm{T}}$.

Considering the structure of the matrix B_i^f and the control law $u_i^f(t)$, it follows that for the i-th faulty agent only the actuators $b^{m_{s_i}+1}$ to b^m are available for actuation. Therefore, to proceed with our proposed reconfiguration control strategy that will *manage* all the *three* types of actuator faults the faulty system (5.1) is rewritten as follows

$$\dot{x}_i^f(t) = Ax_i^f(t) + B_i^r u_i^r(t) + B_i^s \underline{u}_i^s, i = 1, \dots, N_f. \tag{5.2}$$

Let us now define associated with the i-th agent the following performance index

$$J_i = \frac{1}{2}\int_{t_f}^{T} e_i^{\mathrm{T}}(t) Q e_i(t) \mathrm{d}t, \tag{5.3}$$

where $e_i(t)$ is defined as in (2.4), T is a finite time, and Q is a positive definite matrix with appropriate dimension.

Control Reconfiguration Objective: Design the control laws $u_i^h(t)$ and $u_i^r(t)$ such that:

(i) the agents reach a consensus, i.e. $e_i(t) \to 0$ as $t \to \infty$, and *(ii)* the team performance J_f upper bound is minimized, where

$$J_f = \sum_{i=1}^{N} J_i = \frac{1}{2}\int_{t_f}^{T} e^{\mathrm{T}}(t) \mathcal{Q} e(t) \mathrm{d}t,$$

$e(t) = \mathrm{col}\{e_i(t)\}$ and $\mathcal{Q} = I_N \otimes Q$ ($\otimes$ denotes the Kronecker product).

In the remainder of this work the following assumptions are considered to hold.

Assumption 5.1. *(a) The switching signal has finite number of discontinuities at each bounded time and there is a small number number τ such that $t^{l_k+1} - t^{l_k} \geq \tau > 0$ and*

the network topology switches changes among a finite set of communication graphs that have directed spanning trees.
(b) The leader control input is designed such that its states and control signals remain bounded.

Assumption 5.2. *(a) The agents are stabilizable and remain stabilizable even after the fault occurrence.*
(b) Each agent is equipped with a local FDI module that correctly detects and isolates the faulty actuator but it can only estimate the severity of the LOE and stuck faults with uncertainties, unreliabilities, and inaccuracies.

5.2 Proposed Methodology

Consider a multi-agent team that consists of N healthy followers that are governed by (2.1) and a leader that is given by (2.3). The overall dynamics of the team consensus error can be expressed as

$$\dot{e}(t) = \mathcal{A}e(t) + \mathcal{B}^{\sigma}u(t) + \mathcal{B}_0^{\sigma}u_0(t), \tag{5.4}$$

where $e(t) = \text{col}\{e_i(t)\}$, $e_i(t)$ is defined as in (2.4), $u(t) = \text{col}\{u_i^h(t)\}$, $\mathcal{A} = I_N \otimes A$, $\mathcal{B}^{\sigma} = L_{22}^{\sigma} \otimes B$ and $\mathcal{B}_0^{\sigma} = L_{21}^{\sigma} \otimes B$.

Our goal is to design $u(t)$ in (5.4) that is distributed in its implementation, implying that it uses only local information and measurements. Moreover, it is reconfigurable (the control gains can be redesigned) by using local network information, i.e. the agent dynamics as well as the FDI module information once a fault has occurred and detected.

Since each agent only communicates with its nearest neighbors, once the network topology switches each agent notices its new neighboring set, and is unaware of the entire network graph topology at that instant. Therefore, we either require a high-level supervisor (a team-level supervisory control law) that monitors the entire network and can decide on the appropriate control laws associated with each network topology or that we should employ a fixed fault-tolerant (reconfigurable) control law that renders the team stability corresponding to all the possible topologies.

In this work, we aim at designing a fully distributed and an active fault-tolerant control law that guarantees consensus subject to any possible topology and that can be reconfigured based on the availability of only local information. Towards this end, we introduce a local virtual system that is associated with the i-th agent as follows:

$$\dot{\bar{x}}_i(t) = A\bar{x}_i(t) + Bu_i^g(t). \tag{5.5}$$

The dynamics of this virtual system does not change during the mission whether the real agents are faulty or healthy.

If $u_i^g(t)$ is designed such that $\bar{x}_i(t) \to \bar{x}_j(t) \to x_0(t)$ and $u_i(t)$ is designed such that $x_i^*(t) \to \bar{x}_i(t)$, where $* = \{h, f\}$, then we can ensure that $x_i^*(t) \to x_0(t)$. The main motivation for employing $\bar{x}_i(t)$ to guarantee team consensus ($x_i^*(t) \to x_0(t)$) is that $\bar{x}_i(t)$ is not subject to a fault, therefore it can be used to guarantee consensus even when the agents are faulty. For this purpose, let us also define $\eta_i^h(t) = x_i^h(t) - \bar{x}_i(t)$, $\eta_i^f(t) = \begin{bmatrix} (\eta_{1i}^f(t))^{\mathrm{T}} & (\eta_{2i}^f(t))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\eta_{1i}^f(t) = x_i^f(t) - \bar{x}_i(t)$, $\eta_{2i}^f(t) = F_i \int_{t_f}^{t} \eta_{1i}^f(\tau)\mathrm{d}\tau$, $\bar{e}_i(t) = \sum_{j\in\mathcal{N}_i(t)}(\bar{x}_i(t) - \bar{x}_j(t)) + g_{0i}(t)(\bar{x}_i(t) - x_0(t))$, $\bar{e}_{iI}(t) = F_i \int_{t_f}^{t} \bar{e}_i(\tau)\mathrm{d}\tau$, and where F_i is a design matrix to be specified subsequently.

Consequently, each agent behavior is determined according to the following systems

$$\dot{\eta}_i^h(t) = A\eta_i^h(t) + B(u_i^h(t) - u_i^g(t)), i = N_f + 1, \ldots, N, \tag{5.6}$$

$$\dot{\eta}_i^f(t) = A_i^a \eta_i^f(t) + B_i^{af} u_i^f(t) - B^a u_i^g(t) = A_i^a \eta_i^f(t) + B_i^{ar} u_i^r(t) + B_i^{as} \underline{u}_i^s - B^a u_i^g(t),$$
$$i = 1, \ldots, N_f, \tag{5.7}$$

$$\dot{\bar{e}}_i(t) = A\bar{e}_i(t) + d_i(t)Bu_i^g(t) - \sum_{j\in\mathcal{N}_i(t)} Bu_j^g(t) - g_{i0}(t)Bu_0(t), i = 1, \ldots, N, \tag{5.8}$$

$$\dot{\bar{e}}_i^a(t) = A_i^a \bar{e}_i^a(t) + d_i(t)B^a u_i^g(t) - \sum_{j\in\mathcal{N}_i(t)} B^a u_j^g(t) - g_{i0}(t)B^a u_0(t), i = 1, \ldots, N, \tag{5.9}$$

where $\bar{e}_i^a(t) = [\bar{e}_i^{\mathrm{T}}(t)\ \bar{e}_{iI}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $A_i^a = \begin{bmatrix} A & 0 \\ F_i & 0 \end{bmatrix}$, $B_i^{a*} = \begin{bmatrix} B_i^* \\ 0 \end{bmatrix}$, $* = \{r, f, s\}$, and $B^a = \begin{bmatrix} B \\ 0 \end{bmatrix}$.

Let $u_i^g(t)$ be selected as

$$u_i^g(t) = K_{2i}\bar{e}_i^a(t) = K_{21i}\bar{e}_i(t) + K_{22i}\bar{e}_{iI}(t), \tag{5.10}$$

where $K_{2i} = [K_{21i}\ K_{22i}]$ and the control gains K_{21i} and K_{22i} should be selected such that $\bar{e}_i(t)$ is asymptotically stable. Before presenting the procedure for designing the control gains, first the distributed control laws for both the healthy and the faulty agents as well as the sufficient conditions for consensus achievement of the faulty team are presented in Lemma 5.1. In fact, the following lemma transforms the consensus achievement problem for the healthy and faulty teams into equivalent stability problems.

Lemma 5.1. *Let the control gains K_{21i} and K_{22i} be designed such that $\bar{e}_i(t)$ is asymp-*

totically stable. Then, it follows that

(a) The healthy control law

$$u_i^h(t) \;=\; u_i^{lh}(t) + u_i^g(t) = K_{1i}^h \eta_i^h(t) + u_i^g(t), \tag{5.11}$$

ensures that $\eta_i^h(t)$ *is asymptotically stable and* $x_i^h(t) \to x_0(t)$ *as* $t \to \infty$ *if* K_{1i}^h *is designed such that* $A + BK_{1i}^h$ *is Hurwitz, where* $u_i^g(t)$ *is defined according to (5.10).*

(b) The reconfigured control law

$$u_i^r(t) \;=\; u_i^{lf}(t) + G_i u_i^g(t) = K_{1i}^f \eta_i^f(t) + G_i u_i^g(t), \tag{5.12}$$

guarantees that $\eta_{1i}^f(t)$ *is asymptotically stable,* $\eta_{2i}^f(t)$ *is bounded and* $x_i^f(t) \to x_0(t)$ *as* $t \to \infty$ *if* K_{1i}^f *is designed such that* $A_i^a + B_i^{ar} K_{1i}^f$ *is Hurwitz, where* $A_i^a = \begin{bmatrix} A & 0 \\ F_i & 0 \end{bmatrix}$, $B_i^{ar} = \begin{bmatrix} B_i^r \\ 0 \end{bmatrix}$, F_i *is selected such that it is full rank, and* $G_i \in \mathbb{R}^{(m-m_{si})\times m}$ *is a gain matrix to be specified.*

The results of the following lemma is employed to prove the results of Lemmas 5.1-5.6.

Lemma 5.2. *Consider the system*

$$\dot{x}(t) = A_c x(t) + f(t), x(t_0) = x_0, \tag{5.13}$$

where $A_c = \begin{bmatrix} A_{11} & A_{12} \\ F & 0 \end{bmatrix}$ *is Hurwitz,* $f(t) = [f_1^T, 0]^T$ *is bounded and has a finite limit as* $t \to \infty$*, i.e.* $\|f(t)\| \leq f_M$ *and* $\lim_{t\to\infty} f(t) = F_0$*,* $F_0 \in \mathbb{R}^n$ $x(t) = [x_1^T(t), x_2^T(t)]^T$*, and* F *is a full rank matrix. Then, it follows that* $x(t)$ *is bounded,* $\|x(t)\| \leq$ *exp*$\{\alpha\} \max\{\|x_0\|, \frac{f_M}{\beta}\}$, *where* $\alpha = \mathit{log}\{\frac{\sigma_{\max}\{S\}}{\sigma_{\min}\{S\}}\}$, $\beta = -\max_i \mathit{real}\{\lambda_i\{A\}\}$, S *is such that* $A_c = SJ_AS^{-1}$ *and* J_A *is the Jordan decomposition of* A_c*. Moreover,* $x_1(t) \to 0$ *as* $t \to \infty$ *and* $x_2(t)$ *remains bounded for all time.*

Proof. The solution of the system (5.13) is obtained as

$$x(t) = \exp(A_c(t-t_0))x_0 + \int_{t_0}^{t} \exp(A_c(t-\tau))f(\tau)\mathrm{d}\tau.$$

Let us express the Jordan canonical decomposition of A_c as $A = SJ_AS^{-1}$. Hence, given that $\exp(A_ct) = S\exp(J_At)S^{-1}$, [1] we have $\|\exp(At)\| \leq \exp(\alpha - \beta t)$, where $\alpha = \log\{\frac{\sigma_{\max}\{S\}}{\sigma_{\min}\{S\}}\}$ and $\beta = -\max_i \mathrm{real}\{\lambda_i\{A_c\}\}$. Therefore,

$$\begin{aligned}\|x(t)\| &\leq \exp(\alpha - \beta(t-t_0))\|x_0\| + \int_{t_0}^{t} \exp(\alpha - \beta(t-\tau))\|f(\tau)\|\mathrm{d}\tau, \\ &\leq \exp(\alpha - \beta(t-t_0))\|x_0\| + \frac{f_M}{\beta}(\exp(\alpha) - \exp(\alpha - \beta(t-t_0)) = \Phi(t).\end{aligned}$$

It follows that $\Phi(t_0) = \exp\{\alpha\}\|x_0\|$ and $\lim_{t\to\infty}\Phi(t) = \exp\{\alpha\}\frac{f_M}{\beta}$, Moreover, based on the definition of $\Phi(t)$ it can be seen that $\Phi(t)$ is monotonically decreasing if $f_M - \beta\|x_0\| < 0$, and is monotonically increasing if $f_M - \beta\|x_0\| > 0$. Therefore, $\|x(t)\| \leq \exp\{\alpha\}\max\{\|x_0\|, \frac{f_M}{\beta}\}$, that is $x(t)$ is bounded. Now if $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$, it follows that $x_1(t)$ and $x_2(t) = F\int_{t_0}^{t} x_1(\tau)\mathrm{d}\tau$ remain bounded, so that $\int_{t_0}^{t} x_1(\tau)\mathrm{d}\tau$ is bounded. Since $f(t)$ is bounded, $\dot{x}_1(t)$ is also bounded. Therefore, by using the Barbalat's lemma since $\int_{t_0}^{t} x_1(\tau)\mathrm{d}\tau$ and $\dot{x}_1(t)$ are bounded, it follows that $x_1(t) \to 0$, or it is asymptotically stable. □

[1] Note that $\|\exp(A_ct)\| \leq \|S\|\|\exp(J_At)\|\|S^{-1}\|$, $\|S\| = \sigma_{\max}\{S\}$, $\|S^{-1}\| = \sigma_{\max}\{S^{-1}\}$. Now, given that $\sigma_i\{A_c\}\sigma_n\{B\} \leq \sigma_i\{A_cB\}$, (i is correct) $i = 1, \ldots, n$ for $A_c = S^{-1}$, $B = S$, $i = 1$ we have $\sigma_1\{S^{-1}\} = \sigma_{\max}\{S^{-1}\} \leq \frac{1}{\sigma_n\{S\}} = \sigma_{\min}^{-1}\{S\}$. Therefore, $\alpha = \log\frac{\sigma_{\max}\{S\}}{\sigma_{\min}\{S\}}$.

Proof. If $\bar{e}_i(t)$ is asymptotically stable then one obtains $\bar{x}_i(t) \to x_0(t)$ as $t \to \infty$.

(a) Under the control law (5.11), the dynamics of $\eta_i^h(t)$ is given by

$$\dot{\eta}_i^h(t) = (A + BK_{1i}^h)\eta_i^h(t). \tag{5.14}$$

The matrix $A + BK_{1i}^h$ is Hurwitz, and hence, $\eta_i^h(t)$ is asymptotically stable, and $x_i^h(t) \to \bar{x}_i(t)$ as $t \to \infty$. Since $\bar{x}_i(t) \to x_0(t)$, it follows that $x_i^h(t) \to x_0(t)$, as $t \to \infty$.

(b) Under the control law (5.12), the dynamics of $\eta_i^f(t)$ can be expressed as

$$\dot{\eta}_i^f(t) \;\; = \;\; (A_i^a + B_i^{ar}K_{1i}^f)\eta_i^f(t) + B_i^{as}\underline{u}_i^s + (B_i^{ar}G_i - B^a)u_i^g(t),$$

where $B_i^{ar} = [(B_i^r)^{\mathrm{T}}\ 0]^{\mathrm{T}}$ and $B_i^{as} = \begin{bmatrix}(B_i^s)^{\mathrm{T}}\ 0\end{bmatrix}^{\mathrm{T}}$. If $\bar{e}_i(t)$ is asymptotically stable, then $\bar{e}_{iI}(t)$ is bounded and from (5.10), it follows that $u_i^g(t)$ is also bounded. Since $B_i^{as}\underline{u}_i^s$ is also bounded, if $A_i^a + B_i^{ar}K_{1i}^f$ is Hurwitz, then by using Lemma 5.2, with $f(t) = B_i^{as}\underline{u}_i^s + (B_i^{ar}G_i - B^a)u_i^g(t)$, $\eta_{1i}^f(t)$ is asymptotically stable and $\eta_{2i}^f(t)$ is stable, implying that $x_i^f(t) \to \bar{x}_i(t)$ as $t \to \infty$. This along with $\bar{x}_i(t) \to x_0(t)$ leads to $x_i^f(t) \to x_0(t)$ as $t \to \infty$. □

Considering the control laws (5.11) and (5.12), one can observe that the term $u_i^g(t)$ is common in both the healthy agent and the faulty agent control laws. Therefore, the objectives of the faulty team may be guaranteed by reconfiguring $u_i^{lh}(t)$ and designing the control law $u_i^{lf}(t)$ and the gain G_i appropriately. The above result is not constructive in the sense that it does not provide a procedure for selecting the control gains.

In the following, our design strategy will be provided and discussed in detail. However in order to select the gains in an optimal control framework, it is also required to express the team cost performance index in terms of $\bar{e}(t) = \text{col}\{\bar{e}_i(t)\}$, $\bar{e}^a(t) = \text{col}\{\bar{e}_i^a(t)\}$, $\bar{e}_i^a(t) = \begin{bmatrix} \bar{e}_i^{\mathrm{T}}(t) & \bar{e}_{iI}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$ and $\eta^f(t)$.

Based on the given definitions of $\eta_i^f(t)$ and $\eta_i^h(t)$ one has $x_i^f(t) = [I\ 0]\eta_i^f(t) + \bar{x}_i(t)$ and $x_i^h(t) = \eta_i^h(t) + \bar{x}_i(t)$, then we can define $x(t)$ as follows:

$$\begin{aligned} x(t) &= [(x_1^f(t))^{\mathrm{T}}, \ldots, (x_{N_f}^f(t))^{\mathrm{T}}, (x_{N_f+1}^h(t))^{\mathrm{T}}, \ldots, (x_N^h(t))^{\mathrm{T}}] \\ &= [[I\ 0]\eta_1^{f^{\mathrm{T}}}(t), \ldots, [I\ 0]\eta_{N_f}^{f^{\mathrm{T}}}(t), \eta_{N_f+1}^{h^{\mathrm{T}}}(t), \ldots, \eta_N^{h^{\mathrm{T}}}(t)]^{\mathrm{T}} + \bar{x}(t) = S\eta^f(t) + \bar{x}(t), \end{aligned}$$

where $\bar{x}(t) = \text{col}\{\bar{x}_i(t)\}$, $S = \text{diag}\{I_{N_f} \otimes \begin{bmatrix} I_n & 0_{n\times n} \end{bmatrix}, I_{(N-N_f)} \otimes I_n\}$, $\eta^f(t) = \text{col}\{\eta_i^*(t)\}$, $* = f$ for $i = 1, \ldots, N_f$ and $* = h$ for $i = N_f+1, \ldots, N$. Therefore, $e(t)$ can be written as

$$\begin{aligned} e(t) &= (L_{22}^\sigma \otimes I_n)x(t) + (L_{21}^\sigma \otimes I_n)x_0(t) \\ &= (L_{22}^\sigma \otimes I_n)(S\eta^f(t) + \bar{x}(t)) + (L_{21}^\sigma \otimes I_n)x_0(t) \\ &= (L_{22}^\sigma \otimes I_n)S\eta^f(t) + \big((L_{22}^\sigma \otimes I_n)\bar{x}(t) + (L_{21}^\sigma \otimes I_n)x_0(t)\big) \\ &= (L_{22}^\sigma \otimes I_n)S\eta^f(t) + \bar{e}(t). \end{aligned}$$

Based on the above representation for $e(t)$, the team cost can be expressed as

$$\begin{aligned} J_f &= \frac{1}{2}\int_{t_f}^{T} (\bar{e}(t) + (L_{22}^\sigma \otimes I_n)S\eta^f(t))^{\mathrm{T}}\mathcal{Q} \\ &\quad (\bar{e}(t) + (L_{22}^\sigma \otimes I_n)S\eta^f(t))\mathrm{d}t. \end{aligned} \tag{5.15}$$

Now using Fact 2.1 we have

$$\begin{aligned} &\frac{1}{2}(\bar{e}(t) + (L_{22}^\sigma \otimes I_n)S\eta^f(t))^{\mathrm{T}}\mathcal{Q}(\bar{e}(t) + (L_{22}^\sigma \otimes I_n)S\eta^f(t)) \leq \\ &\bar{e}^{\mathrm{T}}(t)\mathcal{Q}\bar{e}(t) + \eta^f(t)^{\mathrm{T}}S^{\mathrm{T}}(L_{22}^\sigma \otimes I_n)^{\mathrm{T}}\mathcal{Q}(L_{22}^\sigma \otimes I_n)S\eta^f(t) = \\ &\bar{e}^{\mathrm{T}}(t)\mathcal{Q}\bar{e}(t) + \eta^f(t)^{\mathrm{T}}S^{\mathrm{T}}((L_{22}^\sigma)^{\mathrm{T}}L_{22}^\sigma \otimes Q)S\eta^f(t), \end{aligned}$$

and therefore,

$$J_f \quad \leq \quad \int_{t_f}^{T} \bar{e}^{\mathrm{T}}(t)\mathcal{Q}\bar{e}(t) + \eta^f(t)^{\mathrm{T}} S^{\mathrm{T}}((L_{22}^{\sigma})^{\mathrm{T}} L_{22}^{\sigma} \otimes Q) S\eta^f(t)\mathrm{d}t. \tag{5.16}$$

The cost J_f is a function of $\bar{e}(t)$ and $\eta^f(t)$. Under the control law (5.10), the dynamics of $\bar{e}^a(t)$ can be written as

$$\begin{aligned} \dot{\bar{e}}^a(t) &= (I_N \otimes A_i^a)e^a + (L_{22}^{\sigma} \otimes B^a)u^g(t) + \bar{\mathcal{B}}_0^{a\sigma} u_0(t) \\ &= \bar{\mathcal{A}}_2^{\sigma} \bar{e}^a(t) + \bar{\mathcal{B}}_0^{a\sigma} u_0(t), \end{aligned} \tag{5.17}$$

where $\bar{\mathcal{A}}_2^{\sigma} = \mathcal{A}^a + \bar{\mathcal{B}}^{a\sigma} K_2$, $\mathcal{A}^a = I_N \otimes A_i^a$, $\bar{\mathcal{B}}^{a\sigma} = L_{22}^{\sigma} \otimes B^a$, $K_2 = \mathrm{diag}\{K_{2i}\}$, and $\bar{\mathcal{B}}_0^{a\sigma} = L_{21}^{\sigma} \otimes B^a$. However, under the fault scenario that is defined in Section 5.1 and the reconfigured control law (5.12) the dynamics of $\eta^f(t)$ can be expressed as follows

$$\dot{\eta}^f(t) = \mathcal{A}_1^f \eta^f(t) + (\mathcal{B}^{ar} G - \mathcal{B}^a)u^g(t) + \mathcal{B}^s \underline{u}^s, \tag{5.18}$$

where $\mathcal{A}_1^f = \mathrm{diag}\{\mathcal{A}_{1i}^*\}$, $* = f$ for $i = 1, \ldots, N_f$ and $* = h$ for $i = N_f + 1, \ldots, N$, $\mathcal{A}_{1i}^f = A_i^a + B_i^{ar} K_{1i}^f$ and $\mathcal{A}_{1i}^h = A + BK_{1i}^h$, $\mathcal{B}^s = \mathrm{col}\{(\mathrm{diag}\{B_i^{as}\}), 0_{n(N-N_f)\times\sum_{i=1}^{N_f} m_{si}}\}$, $\mathcal{B}^{ar} = \mathrm{diag}\{\mathrm{diag}\{B_i^{ar}\}, I_{N-N_f} \otimes B\}$, $\mathcal{B}^a = \mathrm{diag}\{\mathrm{diag}\{B^a\}, I_{N-N_f} \otimes B\}$, $B_i^* = \begin{bmatrix} B_i^* \\ 0 \end{bmatrix}$, $* = s, ar, a$, $G = \mathrm{diag}\{\mathrm{diag}\{G_i\}, I_{m(N-N_f)}\}$, $u^g(t) = \mathrm{col}\{u_i^g(t)\}$, $u_i^g(t)$ is defined as in (5.10), $\eta^f(t) = \eta_i^f(t)$ and $\underline{u}^s = \mathrm{col}\{\underline{u}_i^s\}$.

If one employs the approach that is proposed in [181, 182], the problem of minimizing J_f subject to (5.17) and (5.18) can be relaxed to a time-varying optimization problem which can be solved by only using computationally intensive numerical algorithms. Therefore, instead of minimizing the cost function J_f, we proceed to minimize its upper bound. As stated in Subsection 2.6 for $t^{l_k} \leq t < t^{l_k+1}$, $L_{22}^{\sigma} = L_{22}^{k} \in \{L_{22}^{1} \ldots, L_{22}^{g}\}$. On the other hand, we have $(L_{22}^k)^{\mathrm{T}} L_{22}^k \leq \lambda_{\max}\{(L_{22}^k)^{\mathrm{T}} L_{22}^k\} I_N$, $k = 1, \ldots, q$. [2] Therefore,

$$
\begin{aligned}
(L_{22}^{\sigma})^{\mathrm{T}} L_{22}^{\sigma} \otimes Q &\leq (\lambda_M I_N) \otimes Q, \\
S^{\mathrm{T}}\big((L_{22}^{\sigma})^{\mathrm{T}} L_{22}^{\sigma} \otimes Q\big) S &\leq S^{\mathrm{T}}(\lambda_M I \otimes Q) S \\
&= \lambda_M \begin{bmatrix} I_{N_f} \otimes \bar{I}^{\mathrm{T}} \bar{I} & 0 \\ 0 & I_{(N-N_f)} \end{bmatrix} \otimes Q \\
&\leq \lambda_M \mathcal{Q}^f, \qquad (5.19)
\end{aligned}
$$

where $\bar{I} = \begin{bmatrix} I & 0_{n\times n} \end{bmatrix}$, $\lambda_M = \max\{\lambda_{\max}\{(L_{22}^k)^{\mathrm{T}} L_{22}^k\}\}$, $k = 1, \ldots, q$ and $\mathcal{Q}^f = \mathrm{diag}\{I_{N_f} \otimes \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, I_{N-N_f} \otimes Q\}$.

Consequently, by using the above inequalities the upper bound of J_f can be obtained as follows

$$J_f \leq \mathbb{J}_1 + \mathbb{J}_{2f}, \qquad (5.20)$$

with

$$\mathbb{J}_1 = \int_{t_f}^{T} \bar{e}^{\mathrm{T}}(t) \mathcal{Q} \bar{e}(t) \mathrm{d}t, \text{ and } \mathbb{J}_{2f} = \bar{\lambda} \int_{t_f}^{T} (\eta^f(t))^{\mathrm{T}} \mathcal{Q}^f \eta^f(t) \mathrm{d}t, \qquad (5.21)$$

where $\bar{\lambda} = \max\{1, \lambda_M\}$. Note that in the performance index (5.20), the term $\mathbb{J}_{2f}$ only depends on $\eta^f(t)$ whereas $\mathbb{J}_1$ depends on $\bar{e}(t)$.

Therefore, the problem of minimizing the upper bound of J_f such that the agents reach a consensus can be reduced to stating the following two problems

$$
\begin{aligned}
\min_{K_2} \sup \mathbb{J}_1 &= \min_{K_2} \sup \int_{t_f}^{T} \bar{e}^{\mathrm{T}}(t) \mathcal{Q} \bar{e}(t) \mathrm{d}t \qquad (5.22) \\
s.t. \qquad & \dot{\bar{e}}^a(t) = \bar{\mathcal{A}}_2^{\sigma} \bar{e}^a(t) + \bar{\mathcal{B}}_0^{a\sigma} u_0(t), \qquad (5.23)
\end{aligned}
$$

[2]Note that $\mathbb{L}^k = (L_{22}^k)^{\mathrm{T}} L_{22}^k$, $k = 1, \ldots, q$ are positive semi-definite matrices. Let $\lambda_1, \ldots, \lambda_N$ denote all the eigenvalues of $\mathbb{L}^k$. Then eigenvalues of $\lambda_{\max} I - \mathbb{L}^k$ are $\lambda_{\max} - \lambda_1, \ldots, \lambda_{\max} - \lambda_N$. Consequently, $\min_i(\lambda_{\max} - \lambda_i) = \lambda_{\max} - \lambda_{\min} \geq 0$, hence $\lambda_{\max} I - \mathbb{L}^k \geq 0$.

and

$$\begin{aligned}\min_{K_1^f}\sup \mathbb{J}_{2f} &= \bar{\lambda}\min_{K_1^f}\sup\int_{t_f}^{T}(\eta^f(t))^{\mathrm{T}}\mathcal{Q}^f\eta^f(t)\mathrm{d}t \\ s.t. \qquad & \dot{\eta}^f(t) = \mathcal{A}_1^f\eta^f(t) + (\mathcal{B}^{ar}G - \mathcal{B}^a)u^g(t) + \mathcal{B}^s\underline{u}^s,\end{aligned} \tag{5.24}$$

where $\bar{\mathcal{A}}_2^\sigma = \mathcal{A}^a + \bar{\mathcal{B}}^{a\sigma}K_2$, $\mathcal{A}_1^f = \mathrm{diag}\{\mathcal{A}_{1i}^*\}$, $\mathcal{A}_{1i}^* = A_i^a + B_i^{a*}K_{1i}^*$, $K_1^f = \mathrm{diag}\{K_{11}^f, \ldots, K_{1N_f}^f, K_{1(N_f+1)}^h, \ldots, K_{1N}^h\}$, $K_2 = \mathrm{diag}\{[K_{21i}\, K_{22i}]\}$, $\bar{e}(t) = \mathrm{col}\{\bar{e}_i(t)\}$, $\bar{e}^a(t) = \mathrm{col}\{[\bar{e}_i^{\mathrm{T}}\, \bar{e}_{iI}^{\mathrm{T}}]^{\mathrm{T}}\}$, where for $i = 1, \ldots, N_f$, we have $B_i^{a*} = B_i^{ar}$ and $K_{1i}^* = K_{1i}^f$, whereas $B_i^{a*} = B_i^a$ and $K_{1i}^* = K_{1i}^h$ for $i = N_f + 1, \ldots, N$, $\eta^f(t)$, $\mathcal{B}^{ar}$, G, $\mathcal{B}$, $\mathcal{B}^s$, $\underline{u}^s$ are defined as in (5.18) and $u_0(t)$ is defined as in (2.3).

Note that based on its defined index $\mathbb{J}_1$ is a function of $\bar{e}(t)$ and not $\bar{e}^a(t)$. However, to ensure the system stability we require the stability of all the states, therefore the dynamical model (5.23) is included in the problem (5.22) to be solved subject to the dynamics of $\bar{e}^a(t)$ and not $\bar{e}(t)$. In the following, first our proposed design for solving (5.22) is given, and then we will discuss the solution of (5.24).

Due to our original condition that the network topology switches at times $t = t^{l_k}$, $l_k = 1, 2, \ldots$, the right-hand side of system (5.23) is not a continuous function of time. In fact, since $\sigma : [t_f, \infty) \to \{1, \ldots, q\}$, the system (5.23) is equivalent to the following linear differential inclusion (LDI) system [183]

$$\dot{\bar{e}}^a(t) \in^{a.e.} \bar{\mathcal{A}}_2\bar{e}^a(t) + \bar{\mathcal{B}}_0^a u_0(t), \tag{5.25}$$

where $\bar{\mathcal{A}}_2 \in \{\bar{\mathcal{A}}_2^1 \ldots, \bar{\mathcal{A}}_2^q\}$, $\bar{\mathcal{B}}_0^a \in \{\bar{\mathcal{B}}_0^{a1}, \ldots, \bar{\mathcal{B}}_0^{aq}\}$, $\bar{\mathcal{A}}_2^k = \mathcal{A}^a + \bar{\mathcal{B}}^{ak}K_2$, $\mathcal{A}^a = I_N \otimes A_i^a$, $K_2 = \mathrm{diag}\{K_{2i}\}$, $\bar{\mathcal{B}}^{ak} = L_{22}^k \otimes B^a$, $\bar{\mathcal{B}}_0^{ak} = L_{21}^k \otimes B^a$, L_{22}^k and L_{21}^k are defined as in Subsection 2.6 and $k = 1, \ldots, q$, and "a.e." stands for almost everywhere.

The above notation implies that the derivative of $\bar{e}^a(t)$ satisfies (5.25) almost everywhere, which is due to the discontinuity of the right-hand side of (5.25) at times t^{l_k}, $l_k = 1, 2, \ldots$, where the network topology switches. In order to solve the problem (5.22) we also introduce the "relaxed" version of (5.25) below

$$\dot{\bar{e}}^a(t) \in^{a.e} \text{co}(\bar{\mathcal{A}}_2^k \bar{e}^a(t) + \bar{\mathcal{B}}_0^{ak} u_0(t)), k = 1, \ldots, q, \tag{5.26}$$

where "co" stands for the convex hull. Then by employing the governing dynamics (5.26) one can use the generalized derivative to perform stability analysis and also to obtain the upper bound of $\mathbb{J}_1$. However, we should note that (i) the set of trajectories of the system (5.23) is dense into the set of trajectories of the relaxed system (5.26) [184], (ii) the right-hand side of (5.26) is one-sided Lipschitz, therefore it has a unique solution [185], and (iii) the system (5.23) is stable if and only if the system (5.26) is stable [186]. Therefore, the relaxation above is not restrictive, and consequently the following problem follows

$$\begin{aligned} \min_{K_2} \sup \mathbb{J}_1 \quad &= \quad \min_{K_2} \sup \int_{t_f}^{T} \bar{e}^{\mathrm{T}}(t) \mathcal{Q} \bar{e}(t) \mathrm{d}t \\ s.t. \qquad & \dot{\bar{e}}^a(t) \in^{a.e} \text{co}(\bar{\mathcal{A}}_2^k \bar{e}^a(t) + \bar{\mathcal{B}}_0^{ak} u_0(t)), \bar{\mathcal{A}}_2^k = \mathcal{A}^a + \bar{\mathcal{B}}^{ak} K_2, \end{aligned} \tag{5.27}$$

In [187], the convex hull Lyapunov functions are proposed for analyzing saturated systems and linear differential inclusion (LDI) systems and it is shown that they can improve the analyzed performance when compared to the use of quadratic Lyapunov functions. In order to solve the problem (5.27), we develop and compare *two* strategies, namely one by using quadratic Lyapunov functions that are common for linear systems and the second by utilizing convex hull Lyapunov functions. The results obtained are subsequently provided in Lemma 5.3 and Lemma 5.4, respectively. Although, the derivation and analysis provided in Lemma 5.4 are more involved, nevertheless our simulation results illustrate and show that despite a higher complexity one can achieve an improved performance index as compared to the results obtained in Lemma 5.3.

Lemma 5.3. *Consider the problem specified by (5.27). If* K_2 *is selected as* $K_2 = \mathit{diag}\{[K_{21i}\ K_{22i}]\}$*, with* $K_{21i} = -c_i B^T P_1^a$*, and* $K_{22i} = -c_i B^T P_2^a$*,* $i = 1, \ldots, N$*, then it follows that* $\bar{e}_i(t)$ *is asymptotically stable and* $\bar{e}^a(t)$ *is bounded. Moreover, the performance index* $\mathbb{J}_1$ *is upper bounded by*

$$\mathbb{J}_1^{ub-quad} = V_q(\bar{e}^a(t_f)) + \Delta^{quad},$$

where P_1^a *and* P_2^a *are associated with the symmetric block matrix* $P^a = \begin{bmatrix} P_1^a & P_2^a \\ P_2^a & P_3^a \end{bmatrix}$*, where* P^a *is the solution to Ric*$(A_i^a, B^a, \frac{1}{2\bar{c}}I, Q^a)$*,* $Q^a = \mathit{diag}\{Q, \epsilon_2 Q_2\}$*,* $\epsilon_2 > 0$ *is an arbitrary small number,* Q_2 *is a positive definite matrix with appropriate dimension and* $\bar{c}$ *and* c_i *are solutions to*

$$L_{22}^k C + C(L_{22}^k)^T \geq 2\bar{c} I_N > 0,\ C = \mathit{diag}\{c_i\} > 0, k = 1, \ldots, q, \tag{5.28}$$

and $\Delta^{quad} = 2\sum_{i=1}^N \int_{t_f}^T g_{i0}(t)(\bar{e}_i^a(t))^T P^a B^a u_0(t)dt$*, and*

$$\Delta^{quad} \leq (T - t_f)\lambda_{\min}\{Q^a\} G_q \bar{e}^a_{\max},$$

so that the upper bound of $\bar{e}^a(t)$ *is given by* $\bar{e}^a_{\max} = \max\{G_q, (\frac{\lambda_{\max}\{P^a\}}{\lambda_{\min}\{P^a\}})^{\frac{1}{2}} \|\bar{e}^a(t_f)\|\}$*, where* $G_q = \frac{2\lambda_{\max}\{P^a\} b_m u_{0M}}{\lambda_{\min}\{Q^a\}}$*,* $b_m = \max_k \|\mathcal{B}_0^{ak}\|$*,* u_{0M} *is specified such that* $\|u_0(t)\| \leq u_{0M}$ *and* $V_q(\bar{e}^a(t)) = (\bar{e}^a(t))^T (I_N \otimes P^a)\bar{e}^a(t)$*.*

Proof. Let $V_q(\bar{e}^a(t)) = (\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)\bar{e}^a(t)$ be a Lyapunov function candidate for the system (5.26). Therefore, from [176] it follows that its derivative along the trajectories of this system is given by

$$\frac{dV_q(t)}{dt} \in^{a.e.} \dot{\tilde{V}}_q(t),$$

where $\dot{\bar{V}}_q(t)$ is the set value derivative of $V_q(t)$ and

$$\dot{\bar{V}}_q(t) = \{\nabla V_q(\bar{e}^a(t)).v, v \in \text{co}(\bar{\mathcal{A}}_2^k \bar{e}^a(t) + \bar{\mathcal{B}}_0^{ak} u_0(t))\}. \tag{5.29}$$

Therefore,

$$\begin{aligned}
\frac{dV_q(t)}{dt} \in^{a.e.} \dot{\bar{V}}_q(t) &= \nabla V_q^{\mathrm{T}}(\bar{e}^a(t))(\bar{\mathcal{A}}_2^k \bar{e}^a(t) + \mathcal{B}_0^{ak} u_0(t)) \\
&= 2(\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)(\bar{\mathcal{A}}_2^k \bar{e}^a(t) + \mathcal{B}_0^{ak} u_0(t)) \\
&= (\bar{e}^a(t))^{\mathrm{T}} \Lambda^k \bar{e}^a(t) + 2(\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)\mathcal{B}_0^{ak} u_0(t),
\end{aligned} \tag{5.30}$$

where $\Lambda^k = \text{Sym}\big((I_N \otimes P^a)\bar{\mathcal{A}}_2^k\big)$, $k = 1, \ldots, q$. The term Λ^k can be expressed as

$$\begin{aligned}
\Lambda^k &= (\bar{\mathcal{A}}_2^k)^{\mathrm{T}}(I_N \otimes P^a) + (I_N \otimes P^a)\bar{\mathcal{A}}_2^k \\
&= (\mathcal{A}^a + \mathcal{B}^{ak} K_2)^{\mathrm{T}}(I_N \otimes P^a) + (I_N \otimes P^a)(\mathcal{A}^a + \mathcal{B}^{ak} K_2) \\
&= (I_N \otimes A_i^a + (L_{22}^k \otimes B^a)K_2)^{\mathrm{T}}(I_N \otimes P^a) + (I_N \otimes P^a)(I_N \otimes A_i^a \\
&\quad +(L_{22}^k \otimes B^a)K_2) \\
&= I_N \otimes (A^{a^{\mathrm{T}}} P^a + P^a A_i^a) + K_2^{\mathrm{T}}(L_{22}^k \otimes B^a)^{\mathrm{T}}(I_N \otimes P^a) \\
&\quad +(I_N \otimes P^a)(L_{22}^k \otimes B^a)K_2.
\end{aligned}$$

Let us select K_{2i} as $K_{2i} = \begin{bmatrix} K_{21i} & K_{22i} \end{bmatrix} = -c_i B^{a^{\mathrm{T}}} P^a = -c_i [B^{\mathrm{T}} \; 0] \begin{bmatrix} P_1^a & P_2^a \\ P_2^a & P_3^a \end{bmatrix} = -c_i B^{\mathrm{T}} [P_1^a \; P_2^a]$ then

$$\begin{aligned}
\Lambda^k &= I_N \otimes (A^{a^{\mathrm{T}}} P^a + P^a A_i^a) - (C \otimes B^{a^{\mathrm{T}}} P^a)^{\mathrm{T}}(L_{22}^k \otimes B^a)^{\mathrm{T}}(I_N \otimes P^a) - \\
&\quad (I_N \otimes P^a)(L_{22}^k \otimes B^a)(C \otimes B^{a^{\mathrm{T}}} P^a) \\
&= I_N \otimes (A^{a^{\mathrm{T}}} P^a + P^a A_i^a) - (C L_{22}^{k^{\mathrm{T}}} \otimes P^a B^a B^{a^{\mathrm{T}}} P^a) - (L_{22}^k C \otimes P^a B^a B^{a^{\mathrm{T}}} P^a) \\
&= I_N \otimes (A^{a^{\mathrm{T}}} P^a + P^a A_i^a) - (L_{22}^k C + C L_{22}^{k^{\mathrm{T}}}) \otimes P^a B^a B^{a^{\mathrm{T}}} P^a,
\end{aligned}$$

where $C = \text{diag}\{c_i\}$. If C satisfies the inequality (5.28), then

$$\Lambda^k \leq I_N \otimes (A^{a^{\mathrm{T}}} P^a + P^a A_i^a - 2\bar{c} P^a B^a (B^a)^{\mathrm{T}} P^a).$$

Now, if $P^a > 0$ is the solution to Ric$(A_i^a, B^a, \frac{1}{2\bar{c}}I, Q^a)$, $Q^a =$ diag$\{Q, \epsilon_2 Q_2\}$, $\epsilon_2 > 0$, or equivalently the following minimization problem

$$\min \text{ Trace}\{P^a\} s.t. \ (A_i^a)^{\mathrm{T}} P^a + P^a A_i^a - 2\bar{c} P^a B^a (B^a)^{\mathrm{T}} P^a + Q^a \leq 0,$$

then

$$\Lambda^k \ \leq \ -(I_N \otimes Q^a) < 0. \tag{5.31}$$

From (5.30) and (5.31) one obtains

$$\dot{\bar{V}}_q(\bar{e}^a(t)) \ \leq \ -(\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes Q^a)\bar{e}^a(t) + 2(\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)\mathcal{B}_0^{ak} u_0(t) \tag{5.32}$$

Let $b_m = \max_k \|\mathcal{B}_0^{ak}\|$ and u_{0M} be such that $\|u_0(t)\| \leq u_{0M}$, then the inequality (5.32) is reduced to

$$\begin{aligned} \dot{\bar{V}}_q(\bar{e}^a(t)) \ &\leq \ -\lambda_{\min}\{Q^a\}\|\bar{e}^a(t)\|^2 + 2\|\bar{e}^a(t)\|\lambda_{\max}\{P^a\} b_m u_{0M} \\ &= \ -\|\bar{e}^a(t)\|(\lambda_{\min}\{Q^a\}\|\bar{e}^a(t)\| - 2\lambda_{\max}\{P^a\} b_m u_{0M}) \end{aligned}$$

Also let $W(\bar{e}^a(t)) \equiv -\|\bar{e}^a(t)\|(\lambda_{\min}\{Q^a\}\|\bar{e}^a(t)\| - 2\lambda_{\max}\{P^a\} b_m u_{0M})$, then for $\|\bar{e}^a(t)\| > G_q = \frac{2\lambda_{\max}\{P^a\} b_m u_{0M}}{\lambda_{\min}\{Q^a\}}$, we have

$$\dot{\bar{V}}_q(\bar{e}^a(t)) \leq W(\bar{e}^a(t)) < 0,$$

that is for $\|\bar{e}^a(t)\| > G_q$ and for all switching topologies, we obtain $W(\bar{e}^a(t)) < 0$. Therefore, $\bar{e}^a(t)$ converges to $\|\bar{e}^a(t)\| \leq G_q$ and remains in

$$\mathcal{R} = \{\bar{e}^a(t) | \|\bar{e}^a(t)\| \leq G_q\}. \tag{5.33}$$

This implies that $\bar{e}^a(t)$ remains bounded, so that $\bar{e}(t)$ and $\bar{e}_I(t)$ are also bounded. To show that $\bar{e}(t) \to 0$ as $t \to \infty$, let $g(t) \triangleq \int_{t_f}^{t} \bar{e}(s)\mathrm{d}s = F^{-g_i}\bar{e}_I(t)$ and $\dot{g}(t) \triangleq \bar{e}(t)$. Since $u_0(t)$ is bounded from (5.26), the term $\ddot{g}(t) \triangleq \dot{\bar{e}}(t)$ is also bounded. Therefore, by invoking Lemma 1 in [188], since $g(t)$ and $\ddot{g}(t)$ are bounded it can be concluded that

$\dot{g}(t) = \bar{e}(t) \to 0$ as $t \to \infty$ or $\bar{e}(t)$ is asymptotically stable.

In order to obtain the upper bound of the performance index, we first obtain the upper bound of $\|\bar{e}^a(t)\|$ as follows. Below two possible cases are considered:

(i) $\|\bar{e}^a(t_f)\| > G_q$: In this case $\dot{V}_q(t_f) < 0$, moreover, for $t \geq t_f$ and $\|\bar{e}^a(t)\| > G_q$ we have $\dot{V}_q(t) < 0$, which implies that $V_q(t) \leq V_q(t_f)$. Note that $V_q(t)$ is continuous and $\dot{V}_q(t)$ may be discontinuous at the switching instants but since $W(\bar{e}^a(t))$ is continuous and negative definite, $V_q(t)$ is decreasing. Since $V_q(t_f) \leq \lambda_{\max}\{P^a\}\|\bar{e}^a(t_f)\|^2$ and $\lambda_{\min}\{P_i^a\}\|\bar{e}^a(t)\|^2 \leq V_q(t)$, one obtains $\|\bar{e}^a(t)\| \leq (\frac{\lambda_{\max}\{P^a\}}{\lambda_{\min}\{P^a\}})^{\frac{1}{2}}\|\bar{e}^a(t_f)\|$. Once $\bar{e}^a(t)$ reaches the boundary of $\mathcal{R}$ it remains inside it there after. In fact since outside of $\mathcal{R}$, $\dot{V}_q(t) < 0$, the trajectories of $\bar{e}^a(t)$ cannot leave the boundary of $\mathcal{R}$, and therefore it remains inside it. Therefore, for $t \geq t_f$, $\|\bar{e}^a(t)\| \leq (\frac{\lambda_{\max}\{P^a\}}{\lambda_{\min}\{P^a\}})^{\frac{1}{2}}\|\bar{e}^a(t_f)\|$.

(ii) $\|\bar{e}^a(t_f)\| \leq G_q$: The solution remains inside $\mathcal{R}$ (using the analysis that was stated above it cannot leave the boundary of $\mathcal{R}$). Therefore, from (i) and (ii) for $t \geq t_f$, we obtain $\|\bar{e}^a(t)\| \leq \max\{G_q, (\frac{\lambda_{\max}\{P^a\}}{\lambda_{\min}\{P^a\}})^{\frac{1}{2}}\|\bar{e}^a(t_f)\|\} = \bar{e}^a_{\max}$.

Now, let the right-hand side of the inequality (5.32) be expressed as follows

$$\begin{aligned}
\dot{V}_q(\bar{e}^a(t)) &\leq -(\bar{e}(t))^{\mathrm{T}}(I_N \otimes Q)\bar{e}(t) - \epsilon_2(\bar{e}_I(t))^{\mathrm{T}}(I_N \otimes Q_2)\bar{e}_I(t) + \\
&\quad 2(\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)\mathcal{B}_0^{ak}u_0(t) \\
&\leq -(\bar{e}(t))^{\mathrm{T}}(I_N \otimes Q)\bar{e}(t) + 2(\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)\mathcal{B}_0^{ak}u_0(t).
\end{aligned}$$

Then, by integrating both sides of the above inequality from t_f to T it follows that

$$V_q(\bar{e}^a(T)) - V_q(\bar{e}^a(t_f)) \leq -\int_{t_f}^{T} \bar{e}^{\mathrm{T}}(t)(I_N \otimes Q)\bar{e}(t)\mathrm{d}t$$
$$+2\int_{t_f}^{T} (\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)\mathcal{B}_0^{ak}u_0(t)\mathrm{d}t$$

Therefore,

$$\mathbb{J}_1 \leq V_q(\bar{e}^a(t_f)) - V_q(\bar{e}^a(T)) + \Delta^{quad} \leq V_q(\bar{e}^a(t_f)) + \Delta^{quad} = \mathbb{J}_1^{ub-quad},$$

where

$$\begin{aligned} \Delta^{quad} &= 2\int_{t_f}^{T} (\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)\mathcal{B}_0^{ak}u_0(t)\mathrm{d}t \\ &= 2\int_{t_f}^{T} (\bar{e}^a(t))^{\mathrm{T}}(I_N \otimes P^a)(L_{21}(t) \otimes B^a)^a u_0(t)\mathrm{d}t. \end{aligned}$$

Now since $\|\bar{e}^a(t)\| \leq \bar{e}^a_{\max}$ and $G_q = \frac{2\lambda_{\max}\{P^a\}b_m u_{0M}}{\lambda_{\min}\{Q^a\}}$, we have

$$\Delta^{quad} \leq (T - t_f)\lambda_{\min}\{Q^a\}G_q\bar{e}^a_{\max},$$

and this completes the proof of Lemma 5.3. □

The main advantage of developing Lemma 5.3 is that the control gains are obtained as solutions to convex optimization problems, therefore they can be readily solved. In order to improve the achievable team performance bound, the convex hull (composite) Lyapunov functions are employed next to design and select the control gains. However, we first require to introduce certain preliminaries that are stated below.

Let P^ls, $l = 1, \ldots, M$ denote positive definite matrices and define

$$\Theta^M = \left\{\theta = [\theta^1, \ldots, \theta^M] \in \mathbb{R}^M : \sum_{l=1}^{M} \theta^l = 1, \theta^l \geq 0\right\}. \tag{5.34}$$

Associated with P^ls and θ^ls, the following convex hull function [189] can be defined

$$V_c(\bar{e}^a(t)) = \min_{\theta\in\Theta^M}\left\{(\bar{e}^a(t))^{\mathrm{T}}\big(\sum_{l=1}^{M}\theta^l P^l\big)^{-1}\bar{e}^a(t)\right\}, \tag{5.35}$$

where $M > 1$ is an arbitrary integer. The first level set of $V_c(\bar{e}^a(t))$ can now be expressed as

$$L_{V_c} := \left\{\sum_{l=1}^{M}\theta^l\bar{e}^{al}(t) : \bar{e}^{al}(t) \in \mathcal{E}((P^l)^{-1}), \theta^l \in \Theta^M\right\},$$

which is in fact the convex hull of the level sets of $\mathcal{E}((P^l)^{-1}),\ l = 1,\ldots,M$ (refer to the notation Subsection 2.8).

Let us denote the extreme points [187], of L_{V_c} by ∂L_{V_c} and the set of extreme points of L_{V_c} by E. Note that a point on the boundary is an extreme point if it cannot be expressed as a combination of other points, otherwise it is called an ordinary point. Consequently, E can be expressed as

$$E = \bigcup_{l=1}^{M} E^l = \bigcup_{l=1}^{M}\Big\{\partial L_{V_c}\bigcap\partial\mathcal{E}((P^l)^{-1})\Big\}. \tag{5.36}$$

We are now in a position to state our next result.

Lemma 5.4. *Consider the problem specified by (5.27). If for $\gamma > 0$ there exist a block diagonal matrix K_2, positive definite matrices P^l and non-negative numbers λ_{jl}^k, $l = 1,\ldots,M$, $k = 1,\ldots,q$ such that*

$$\mathit{Sym}\{\mathcal{A}^a P^l + \mathcal{B}^{ak}K_2P^l\} + 2\gamma P^l\bar{\mathcal{Q}}P^l \leq \sum_{j=1}^{M}\lambda_{jl}^k(P^j - P^l), \tag{5.37}$$

then it follows that $\bar{e}_i(t)$ is asymptotically stable while $\bar{e}^a(t)$ is bounded and the performance index $\mathbb{J}_1$ is upper bounded by

$$\mathbb{J}_1^{ub-con} = \gamma^{-1}\big(V_c(\bar{e}^a(t_f)) + \Delta^{con}\big), \tag{5.38}$$

where

$$\Delta^{con}=2\int_{t_f}^{T}(\bar{e}^a(t))^T(P(\theta^*))^{-1}(L_{21}(t)\otimes B^a)u_0(t)dt\le(T-t_f)\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}G_c\bar{e}^a_{\max},$$

$P(\theta^*)=\sum_{l=1}^{M}\theta^{l*}P^l$, $\theta^*=$ *arg* $\min_{\theta\in\Theta^M}(\bar{e}^a(t))^T\big(\sum_{l=1}^{M}\theta^lP^l\big)^{-1}\bar{e}^a(t)$, $V_c(\bar{e}^a(t))$ *is defined as in (5.35),* $G_c=\frac{2u_{0M}b_m\tilde{p}}{\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}}$, $\bar{e}^a_{\max}=\max\{G_c,(\frac{(\min_l\underline{p}^l)^{-1}}{(\max_l\bar{p}^l)^{-1}})^{\frac{1}{2}}\|\bar{e}^a(t_f)\|\}$, $b_m=\max_k\|\mathcal{B}_0^{ak}\|$, $\tilde{p}=(\min_l\underline{p}^l)^{-1}$, $\underline{p}^l=\lambda_{\min}\{P^l\}$, $\bar{p}^l=\lambda_{\max}\{P^l\}$, u_{0M} *is the upper bound of* $u_0(t)$, *i.e.* $\|u_0(t)\|\le u_{0M}$, $\bar{\mathcal{Q}}=I_N\otimes(diag\{Q,\epsilon_2Q_2\})$, $\epsilon_2>0$ *and* $\alpha>0$ *are small numbers,* Q_2 *is a positive definite matrix with appropriate dimension and* Θ^M *is defined as in (5.34).*

Before stating the proof of Lemma 5.4, we require the following lemma from [187].

Lemma 5.5. *[187] For a given* $\bar{e}^a(t)\in\mathbb{R}^{2nN}$, *suppose* $\theta^*\in\Theta$ *is the optimal* θ, *i.e. it is a solution to*

$$\theta^*=arg\min_{\theta\in\Theta^M}(\bar{e}^a(t))^T\big(\sum_{l=1}^{M}\theta^lP^l\big)^{-1}\bar{e}^a(t).$$

Without loss of generality assume that $\theta^{l*}>0$, $l=1,\ldots,M_0$ *and* $\theta^{l*}=0$, $l=M_0+1,\ldots,M$. *Denote*

$$P(\theta^*)=\sum_{l=1}^{M_0}\theta^{l*}P^l,\ \bar{e}^{al}(t)=P^lP(\theta^*)^{-1}\bar{e}^a(t),$$

then $V_c(\bar{e}^a(t))=V_c(\bar{e}^{al}(t))=(\bar{e}^{al}(t))^T(P^l)^{-1}\bar{e}^{al}(t)$. *Moreover,* $\nabla V_c(\bar{e}^a(t))=\nabla V_c(\bar{e}^{al}(t))=2(P^l)^{-1}\bar{e}^{al}(t)$, $\bar{e}^a(t)=\sum_{l=1}^{M_0}\theta^l\bar{e}^{al}(t)$, $\bar{e}^{al}(t)\in\mathcal{E}((P^l)^{-1})$, $l=1,\ldots,M_0$, $\bar{e}^{al}(t)=$ *col*$\{\bar{e}_i^{al}(t)\}$ *and* $\bar{e}_i^{al}(t)=\begin{bmatrix}\bar{e}_i^l(t) & \bar{e}_{iI}^l(t)\end{bmatrix}$.

Proof of Lemma 5.4. Consider

$$V_c(\bar{e}^a(t))=\min_{\theta\in\Theta^M}\left\{(\bar{e}^a(t))^{\mathrm{T}}\left(\sum_{j=1}^{M}\theta^jP^j\right)^{-1}\bar{e}^a(t)\right\},$$

as a value function (Lyapunov function candidate) for the system (5.26). Recall that each $\bar{e}^a(t) \in \partial L_{V_c}$ is either an extreme point or an ordinary point. If $\bar{e}^a(t)$ is an extreme point, i.e. $\bar{e}^{al}(t) \in E^l$, then $\bar{e}^a(t)$ cannot be written as a convex combination of other points and $\bar{e}^a(t) \in \mathcal{E}((P^l)^{-1}),$ for some l. On the other hand, if $\bar{e}^a(t)$ is an ordinary point, then it can be written as a convex combination of extreme points as follows

$$\bar{e}^a(t) = \sum_{l=1}^{M_0} \theta^l \bar{e}^{al}(t),\ \bar{e}^{al}(t) \in \mathcal{E}((P^l)^{-1}),$$

where $M_0 \geq 1$ is an arbitrary integer. Therefore, in the following first we investigate the extreme points, $\bar{e}^a(t) \in E^l$, and then the results are extended to the ordinary points.

Let $\bar{e}^a(t) \in E^l$, then

$$\begin{aligned} \frac{dV_c(\bar{e}^a(t))}{dt} \in^{a.e.} \dot{\bar{V}}_c(\bar{e}^a(t)) &= \nabla V_c^{\mathrm{T}}\big(\bar{e}^a(t)\big)\big(\bar{\mathcal{A}}_2^k \bar{e}^{al}(t) + \mathcal{B}_0^{ak} u_0(t)\big) \\ &= \nabla V_c^{\mathrm{T}}(\bar{e}^a(t))\big((\mathcal{A}^a + \mathcal{B}^{ak} K_2)\bar{e}^{al}(t) + \mathcal{B}_0^{ak} u_0(t)\big), \end{aligned} \tag{5.39}$$

where $\dot{\bar{V}}_c(\bar{e}^a(t))$ is the set value derivative as defined in (5.29), $k = 1, \ldots, q$ and $l = 1, \ldots, M$. Recall from Lemma 5.5 that $\nabla V_c(\bar{e}^a(t)) = \nabla V_c(\bar{e}^{al}(t))$ therefore

$$\begin{aligned} \dot{\bar{V}}_c(\bar{e}^a(t)) &= 2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\big(\mathcal{A}^a + \mathcal{B}^{ak} K_2)\bar{e}^{al}(t) + \mathcal{B}_0^{ak} u_0(t)\big) \\ &= (\bar{e}^{al}(t))^{\mathrm{T}}\mathrm{Sym}\big((P^l)^{-1}\mathcal{A}^a + (P^l)^{-1}\mathcal{B}^{ak} K_2\big)\bar{e}^{al}(t) \\ &\quad +2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak} u_0(t). \end{aligned}$$

Let $\Lambda^k = \mathrm{Sym}\big((P^l)^{-1}\mathcal{A}^a + (P^l)^{-1}\mathcal{B}^{ak} K_2\big)$ and suppose for $\gamma > 0$, $\lambda_{jl}^k \geq 0$, $P^l > 0$ and K_2 exist such that

$$\Lambda^k + 2\gamma\bar{\mathcal{Q}} \leq \sum_{j=1}^{M} \lambda_{jl}^k (P^l)^{-1}(P^j - P^l)(P^l)^{-1}, \tag{5.40}$$

where $\bar{\mathcal{Q}} = I_N \otimes (\mathrm{diag}\{Q, \epsilon_2 Q_2\})$, $\epsilon_2 > 0$ and $Q_2 > 0$. According to [187], for each

$\bar{e}^{al}(t) \in E^l$, we have

$$(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}(P^j - P^l)(P^l)^{-1}\bar{e}^{al}(t) \leq 0, \; j = 1, \ldots, M. \tag{5.41}$$

Therefore, from (5.40) and (5.41) it follows that

$$\begin{aligned}
\dot{V}_c(\bar{e}^a(t)) &\leq -2(\bar{e}^{al}(t))^{\mathrm{T}}(\gamma\bar{\mathcal{Q}} + \sum_{j=1}^{M}\lambda_{jl}^k(P^l)^{-1}(P^j - P^l)(P^l)^{-1})\bar{e}^{al}(t) \\
&\quad +2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak}u_0(t) \\
&\leq -(\bar{e}^{al}(t))^{\mathrm{T}}(\gamma\bar{\mathcal{Q}} + \sum_{j=1}^{M}\lambda_{jl}^k(P^l)^{-1}(P^j - P^l)(P^l)^{-1})\bar{e}^{al}(t) \\
&\quad +2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak}u_0(t) \\
&\leq -\gamma(\bar{e}^{al}(t))^{\mathrm{T}}\bar{\mathcal{Q}}\bar{e}^{al}(t) + 2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak}u_0(t) \\
&\leq -\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}\|\bar{e}^{al}(t)\|^2 + 2\|\bar{e}^{al}(t)\|u_{0M}\underline{p}_l^{-1}b_m = W_c(\bar{e}^a(t)),
\end{aligned} \tag{5.42}$$

where $b_m = \max_k \|\mathcal{B}_0^{ak}\|$, $\underline{p}_l = \lambda_{\min}\{P^l\}$ and u_{0M} is such that $\|u_0(t)\| \leq u_{0M}$. The above inequality implies that for $\|\bar{e}^{al}(t)\| > \frac{2u_{0M}\underline{p}_l^{-1}b_m}{\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}}$ and all network topologies, $W_c(\bar{e}^{al}(t)) \geq \max_k \dot{V}_c(t)$ negative definite, so that $\bar{e}^{al}(t)$ converges to

$$\mathcal{R}_1 = \{\bar{e}^a(t) | \|\bar{e}^{al}(t)\| \leq \frac{2u_{0M}\underline{p}_l^{-1}b_m}{\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}}\}$$

and remains inside it. Let $\tilde{p} = \max_l(\underline{p}^l)^{-1}$, then the largest region that includes all $\bar{e}^{al}(t)$ is

$$\mathcal{R} = \{\bar{e}^a(t) | \|\bar{e}^a(t)\| \leq G_c = \frac{2u_{0M}b_m\tilde{p}}{\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}}$$

that is $\|\bar{e}^{al}(t)\| \leq G_c$. This implies that $\bar{e}^l(t) = [(\bar{e}^l(t))^{\mathrm{T}} \; (\bar{e}_I^l(t))^{\mathrm{T}}]^{\mathrm{T}}$ remains bounded, hence $\bar{e}^l(t)$ and $\bar{e}_I^l(t)$ also remain bounded. We now need to show that $\bar{e}^l(t)$ is asymptotically stable. Since $u_0(t)$ is bounded, then from the dynamic equation (5.26) for $\bar{e}^a(t) \equiv \bar{e}^{al}(t)$, it follows that $\dot{\bar{e}}^{al}(t)$ is bound. Let $g(t) \triangleq \int_{t_f}^{t}\bar{e}^l(s)\mathrm{d}s = F^{-gi}\bar{e}_I^l(t)$, $\dot{g}(t) \triangleq \bar{e}^l(t)$ and $\ddot{g}(t) \triangleq \dot{\bar{e}}^l(t)$. Since $g(t)$ and $\ddot{g}(t)$ are bounded, according to Lemma 1 in [188] $\dot{g}(t) \triangleq \bar{e}^l(t) \to 0$ as $t \to \infty$ or $\bar{e}^l(t)$ is asymptotically stable.

Now, suppose $\bar{e}^a(t) \in \partial L_{V_c}$, therefore it can be written as $\bar{e}^a(t) = \sum_{l=1}^{M_0} \theta^l \bar{e}^{al}(t)$. By using Lemma 5.5 and the previous condition for $\bar{e}^a(t)$, $\dot{\bar{V}}_c(\bar{e}^a(t))$ can be written as

$$\begin{aligned}
\frac{dV_c(\bar{e}^a(t))}{dt} \in^{a.e.} \dot{\bar{V}}_c &= 2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}(\mathcal{A}^a + \mathcal{B}^{ak}K_2)\sum_{l=1}^{M_0}\theta^l \bar{e}^{al}(t) \\
&\quad +2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak}u_0(t) \\
&= \sum_{l=1}^{M_0}\theta^l 2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}(\mathcal{A}^a + \mathcal{B}^{ak}K_2)\bar{e}^{al}(t) \\
&\quad +2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak}u_0(t) \\
&= \sum_{l=1}^{M_0}\theta^l(\bar{e}^{al}(t))^{\mathrm{T}}\Lambda^k\bar{e}^{al}(t) + 2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak}u_0(t) \quad (5.43)
\end{aligned}$$

If P^l and K_2 satisfy (5.40), we have the following inequality

$$\dot{\bar{V}}_c \leq -2\gamma\sum_{l=1}^{M_0}\theta^l(\bar{e}^{al}(t))^{\mathrm{T}}\bar{\mathcal{Q}}\bar{e}^{al}(t) + 2(\bar{e}^{al}(t))^{\mathrm{T}}(P^l)^{-1}\mathcal{B}_0^{ak}u_0(t). \quad (5.44)$$

Inequality (5.44) is in terms of $\bar{e}^{al}(t)$, however to obtain $\dot{V}_c(\bar{e}^a(t))$ we used the gradient of V_c at $\bar{e}^a(t)$, therefore we require to express the right-hand side of (5.44) in terms of $\bar{e}^a(t)$. Given that $0 < \theta^l \leq 1$, $\sum_{l=1}^{M_0} \theta^l = 1$ and $\bar{e}^{al}(t) = P^l P(\theta^*)^{-1} \bar{e}^a(t)$ one gets

$$\begin{aligned}
\sum_{l=1}^{M_0} \theta^l \dot{\bar{V}}_c(\bar{e}^a(t)) &\leq -2\gamma \sum_{l=1}^{M_0} \theta^l \sum_{l=1}^{M_0} \theta^l (\bar{e}^{al}(t))^{\mathrm{T}} \bar{\mathcal{Q}} \bar{e}^{al}(t) \\
&\quad +2 \sum_{l=1}^{M_0} \theta^l (\bar{e}^a(t))^{\mathrm{T}} (P(\theta^*))^{-1} \mathcal{B}_0^{ak} u_0(t) \\
&= -2\gamma \sum_{l=1}^{M_0} \big((\theta^1)^2 (\bar{e}^{a1}(t))^{\mathrm{T}} \bar{\mathcal{Q}} \bar{e}^{a1}(t) + \cdots + (\theta^{M_0})^2 (\bar{e}^{aM_0}(t))^{\mathrm{T}} \\
&\quad \bar{\mathcal{Q}} \bar{e}^{aM_0}(t)\big) + 2(\bar{e}^a(t))^{\mathrm{T}} (P(\theta^*))^{-1} \mathcal{B}_0^{ak} u_0(t) \\
&= -2M_0\gamma\big((\theta^1 \bar{e}^{a1}(t))^{\mathrm{T}} \bar{\mathcal{Q}} (\theta^1 \bar{e}^{a1}(t)) + \cdots + (\theta^{M_0} \bar{e}^{aM_0}(t))^{\mathrm{T}} \\
&\quad \bar{\mathcal{Q}} (\theta^{M_0} \bar{e}^{aM_0}(t))\big) + 2(\bar{e}^a(t))^{\mathrm{T}} (P(\theta^*))^{-1} \mathcal{B}_0^{ak} u_0(t) \\
&\leq -2\gamma\big((\theta^1 \bar{e}^{a1}(t))^{\mathrm{T}} \bar{\mathcal{Q}} (\theta^1 \bar{e}^{a1}(t)) + \cdots + (\theta^{M_0} \bar{e}^{aM_0}(t))^{\mathrm{T}} \\
&\quad \bar{\mathcal{Q}} (\theta^{M_0} \bar{e}^{aM_0}(t))\big) + 2(\bar{e}^a(t))^{\mathrm{T}} (P(\theta^*))^{-1} \mathcal{B}_0^{ak} u_0(t) \\
&\leq -\gamma(\sum_{l=1}^{M_0} \theta^l \bar{e}^{al}(t))^{\mathrm{T}} \bar{\mathcal{Q}} \sum_{l=1}^{M_0} \theta^l \bar{e}^{al}(t) \\
&\quad +2(\bar{e}^a(t))^{\mathrm{T}} (P(\theta^*))^{-1} \mathcal{B}_0^{ak} u_0(t),
\end{aligned}$$

where in the last inequality we invoked Fact 2.1. Therefore,

$$\dot{\bar{V}}_c(t) \quad \leq \quad -\gamma\big(\bar{e}^a(t))^{\mathrm{T}} \bar{\mathcal{Q}} \bar{e}^a(t)\big) + 2(\bar{e}^a(t))^{\mathrm{T}} (P(\theta^*))^{-1} \mathcal{B}_0^{ak} u_0(t). \tag{5.45}$$

Similar to the inequality (5.42), it can be seen that for $\|\bar{e}^a(t)\| > \frac{2b_m \|u_0(t)\| \|P(\theta^*)^{-1}\|}{\gamma \lambda_{\min}\{\bar{\mathcal{Q}}\}}$ and all network topologies, $\max \dot{\bar{V}}_c(t)$ is negative. Now, based on the definition of $P(\theta^*)$ one has

$$P(\theta^*) = \sum_{l=1}^{M_0} \theta^{l*} P^l \geq \sum_{l=1}^{M_0} \theta^{l*} (\underline{p}^l) I \geq \sum_{l=1}^{M_0} \theta^{l*} (\min_l \underline{p}^l) I = (\min_l \underline{p}^l) I$$

and

$$P(\theta^*) = \sum_{l=1}^{M_0} \theta^{l*} P^l \leq \sum_{l=1}^{M_0} \theta^{l*} (\bar{p}^l) I \leq \sum_{l=1}^{M_0} \theta^{l*} (\max_l \bar{p}^l) I = (\max_l \bar{p}^l) I,$$

where $\underline{p}^l = \lambda_{\min}\{P^l\}$ and $\bar{p}^l = \lambda_{\max}\{P^l\}$. Therefore, $(\max_l \bar{p}^l)^{-1} I \leq P(\theta^*)^{-1} \leq$

$(\min_l \underline{p}^l)^{-1} I = \tilde{p} I$. That is, $\bar{e}^a(t)$ converges to

$$\mathcal{R} = \{\bar{e}^a(t) | \|\bar{e}^a(t)\| \leq G_c\},$$

and remains inside $\mathcal{R}$. This implies that $\bar{e}^a(t)$, $\bar{e}(t)$ and $\bar{e}_I(t)$ remain bounded. Moreover, based on (5.26), it follows that $\dot{\bar{e}}^a(t)$ is bounded. Hence, since $g(t) \triangleq \bar{e}_I(t) = F^{-gi} \int_{t_f}^{t} \bar{e}(s) \mathrm{d}s$, and $\ddot{g}(t) \triangleq \dot{\bar{e}}_I(t)$ are bounded according to Lemma 1 in [188], it follows that $\dot{g}(t) \triangleq \bar{e}(t) \to 0$ as $t \to \infty$, that is $\bar{e}(t)$ is asymptotically stable.

To obtain the upper bound of $\mathbb{J}_1$, similar to the proof of Lemma 5.3, first we obtain the upper bound of $\|\bar{e}^a(t)\|$. Two cases are possible:

(i) $\|\bar{e}^a(t_f)\| > G_c$: For $t \geq t_f$ and $\|\bar{e}^a(t)\| > G_c$, we have $\dot{V}_c(t) < 0$ therefore $V_c(t) \leq V_c(t_f)$. Using the upper and lower bounds of $P(\theta^*)$ and the definition of $V_c(\bar{e}^a(t))$, one obtains the following inequality

$$(\max_l \bar{p}^l)^{-1} \|\bar{e}^a(t)\|^2 \leq V_c(\bar{e}^a(t)) \leq \|\bar{e}^a(t)\|^2 (\min_l \underline{p}^l)^{-1}.$$

Therefore, it follows that

$$(\max_l \bar{p}^l)^{-1} \|\bar{e}^a(t)\|^2 \leq V_c(\bar{e}^a(t)) \leq V_c(\bar{e}^a(t_f)) \leq \|\bar{e}^a(t_f)\|^2 (\min_l \underline{p}^l)^{-1},$$

which implies that $\|\bar{e}_i^a(t)\| \leq (\frac{(\min_l \underline{p}^l)^{-1}}{(\max_l \bar{p}^l)^{-1}})^{\frac{1}{2}} \|\bar{e}^a(t_f)\|$. Once a trajectory reaches the boundary of $\mathcal{R} = \{\bar{e}^a(t) | \|\bar{e}^a(t)\| \leq G_c\}$, it remains inside $\mathcal{R}$. In other words, since outside of $\mathcal{R}$, $V_c(t)$ is decreasing, when a trajectory reaches the boundary this region it cannot leave it and remains on it or enters inside it. That is, we will have $\|\bar{e}^a(t)\| \leq (\frac{(\min_l \underline{p}^l)^{-1}}{(\max_l \bar{p}^l)^{-1}})^{\frac{1}{2}} \|\bar{e}^a(t_f)\|$.

(ii) $\|e^a(t_f)\| \leq G_c$: The solution remains inside $\mathcal{R}$.

Therefore, from (i) and (ii) for $t \geq t_f$, it follows that

$$\|\bar{e}^a(t)\| \leq \max\{G_c, (\frac{(\min_l \underline{p}^l)^{-1}}{(\max_l \bar{p}^l)^{-1}})^{\frac{1}{2}} \|\bar{e}^a(t_f)\|\} = \bar{e}^a_{\max}$$

.

Now given that

$$-(\bar{e}^a(t))^{\mathrm{T}}\bar{\mathcal{Q}}\bar{e}^a(t) = -(\bar{e}(t))^{\mathrm{T}}\mathcal{Q}\bar{e}(t) - \epsilon_2(\bar{e}_I(t))^{\mathrm{T}}(I_N \otimes Q_2)\bar{e}_I(t) \leq -(\bar{e}(t))^{\mathrm{T}}\mathcal{Q}\bar{e}(t)$$

the inequality (5.45) can be reduced to

$$\dot{V}_c(t) \quad \leq \quad -\gamma\big(\bar{e}^{\mathrm{T}}(t)\mathcal{Q}\bar{e}(t)\big) + 2(\bar{e}^a(t))^{\mathrm{T}}(P(\theta^*))^{-1}\mathcal{B}_0^{ak}u_0(t).$$

Therefore, by integrating both sides of the above inequality from t_f to T it follows that

$$V_c(\bar{e}^a(T)) - V_c(\bar{e}^a(t_f)) \leq -\alpha\mathbb{J}_1 + \Delta^{con}$$

where $\Delta^{con} = 2\int_{t_f}^{T}(\bar{e}^a(t))^{\mathrm{T}}(P(\theta^*))^{-1}(L_{21}(t) \otimes B^a)u_0(t)\mathrm{d}t$. Therefore,

$$\mathbb{J}_1 \leq \gamma^{-1}(V_c(\bar{e}^a(t_f)) - V_c(\bar{e}^a(T)) + \Delta^{con}) \leq \gamma^{-1}(V_c(\bar{e}^a(t_f)) + \Delta^{con}) = \mathbb{J}_1^{ub-con}. \tag{5.46}$$

Furthermore, since $\|\bar{e}^a(t)\| \leq \bar{e}^a_{\max}$ and $G_c = \frac{2u_{0M}b_m\tilde{p}}{\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}}$, it follows that

$$\Delta^{con} \leq (T - t_f)\gamma\lambda_{\min}\{\bar{\mathcal{Q}}\}G_c\bar{e}^a_{\max}.$$

Finally, by multiplying the inequality (5.40) from both sides by P^l one gets

$$\mathrm{Sym}\{\mathcal{A}^aP^l + \mathcal{B}^{ak}K_2P^l\} + 2\gamma P^l\bar{\mathcal{Q}}P^l \leq \sum_{j=1}^{M}\lambda^k_{jl}(P^j - P^l) < 0,$$

which is the BMI condition that is stated in the lemma. $\square$

The inequality (5.37) is a bilinear matrix inequality (BMI) for $M > 1$ and for $M = 1$ it can be transformed into a convex LMI condition. Solving the BMI in a general form is still an open research problem, but numerous approaches have been proposed in the literature to tackle this problem. These approaches can be classified into two main categories, namely (i) global approaches, and (ii) local approaches. Branch-and-bound approaches [190–192] and sum-of-squares approach [193] are examples of global methods.

They provide a global solution, but are computationally intensive and not polynomial in time. Therefore, they are mostly applicable and effective for low dimensional problems with few variables. Given the possible large dimensionality of our problem (the inequality (5.37)) these approaches are deemed not to be appropriate here.

On the other hand, local approaches are computationally less demanding and are effective for practical applications, but do not guarantee that they yield a global solution. These approaches are mainly based on linearization or the idea of decomposing the variables into two sets. The linearization based approaches like path following approach [194], or the decomposition-based approaches e.g. $D-K$ iteration [195], $V-K$ iteration [196] and coordinate descent [197,198] are well-known local approaches. Since decomposition approaches do not perform well when the number of variables is medium or large, in this work we employ the path following approach and simulation results show that the approach is suited well for addressing our problem. The following algorithm, which is based on the path following provides the required steps to design and select the gain K_2.

Algorithm 5.1. • *Initialization:*

- *Select an integer* $M \geq 2$*, small positive numbers* δ_1*,* δ_2*,* $\delta_1 \gg \delta_2$*,* $\gamma > 0$ *and* $t_{\max}$ *as the maximum number of iterations.*
- *Set* $t = 1$*,* $\lambda_{jk}^{l1} = 0$*,* $P^{l1} = P^0$*,* $l = 1, \ldots, M$ *and change the variables as* $Y = K_2 P^l = K_2 P^0$ *and solve the following convex optimization problem:*

$$Sym\{\mathcal{A}^a P^0 + \mathcal{B}^{ak} K_2^1 Y\} + 2\gamma P^0 \bar{\mathcal{Q}} P^0 \leq 0$$

- *Set* $K_2^1 = Y(P^0)^{-1}$ *and* $P^{l1} = P^0$

- *At the t-th iteration: for $t \leq t_{\max}$,*

  1) *Solve the following LMI for dP^l, dK_2 and $d\lambda^l_{jk}$*

$$
\begin{aligned}
&\mathit{Sym}\{\mathcal{A}^a(P^{lt} + dP^l) + \mathcal{B}^{ak}(K_2^t(P^{lt} + dP^l) \\
&+ dK_2 P^{lt})\} + 2\gamma\big((P^{lt} + dP^l)\bar{\mathcal{Q}}P^{lt} + P^{lt}\bar{\mathcal{Q}}dP^l\big) \leq \\
&\sum_{j=1}^{M}\big((\lambda^{kt}_{jl} + d\lambda^k_{jl})(P^{jt} - P^{lt}) + \lambda^{kt}_{jl}(dP^j - dP^l)\big), \\
&P^{lt} + dP^l > 0, \lambda^{kt}_{jl} + d\lambda^k_{jl} \geq 0, \|dP^l\| \leq \delta_1\|P^{lt}\|, \\
&\|dK_2\| \leq \delta_1\|K_2^t\|, dK_2 = \mathit{diag}\{dK_{2i}\}
\end{aligned}
$$

  2) *If $\|dP^l\| \leq \delta_2$ and $\|dK_2\| \leq \delta_2$ exit the algorithm, otherwise go to the next step.*

  3) *Set $P^{l(t+1)} = P^{lt} + dP^l$, $K_2^{t+1} = K_2^t + dK_2$ and $\lambda^{k(t+1)}_{jl} = \lambda^{kt}_{jl} + d\lambda^k_{jl}$, $t = t + 1$ and go to step 1) as long as $t \leq t_{\max}$.*

Now, let us return to the problem (5.24). The problem (5.24) is an optimization one subject to a time-invariant dynamical system. The main objective and goal is to design the gain K_1^f such that $\eta^f_{1i}(t)$ and $\eta^h_i(t)$ are asymptotically stable and $\eta^f_{i2}(t)$ remains bounded while the upper bound of $\mathbb{J}_{2f}$ is minimized. This problem is quite similar to standard optimal control design problems, however to design the control gains associated with the faulty agents, one requires to consider the fact that the exact fault severities are not available.

To remedy the above concern, the gains will be designed to achieve robustness with respect to the FDI module uncertainties and unreliabilities. Note that if K_2 is designed according to Lemma 5.3 or Lemma 5.4, then $\bar{e}(t)$ is asymptotically stable and $u^g(t)$ will be bounded. Given that the dynamics of $\bar{e}(t)$ does not depend on the dynamics of $\eta^f(t)$ and $\underline{u}^s$ is constant, $u^g(t)$ and $\underline{u}^s$ can be considered as disturbances for this system. In the next lemma a design strategy for selecting the control gains that solve this problem is provided.

Lemma 5.6. *The control gains matrices $K_{1i}^h = -B^T P^h$ and $K_{1i}^f = -(\hat{B}_i^r)^T P_i^r$ guarantee that $\eta_i^h(t)$ and $\eta_{1i}^f(t)$ are asymptotically stable and $\eta_{2i}^f(t)$ remains bounded while the cost index $\mathbb{J}_{2f}$ is upper bounded (ub) by*

$$\mathbb{J}_{2f}^{ub} \;=\; \sum_{i=1}^{N_f} V_i^f(t_f) + \Delta^f + \sum_{i=N_f+1}^{N} (\eta_i^h(t_f))^T P^h \eta_i^h(t_f),$$

where $V_i^f(t) = (\eta_i^f(t))^T P_i^r \eta_i^f(t)$, P^h is the solution to Ric$(A, B, \frac{1}{2}I, \bar{\lambda}Q)$, P_i^r is the solution to Ric$(A_i^a, \bar{B}_i^{ar}, R^f, \bar{\lambda}Q^f)$, $\bar{B}_i^{ar} = [(\bar{B}_i^r)^T \; 0_{m-m_{si}\times n}]^T$, $Q^f = diag\{Q, \epsilon_1 Q_2\}$, $R^f = \frac{1}{2}\hat{\Gamma}_i^{-1}(\hat{\Gamma}_i - f_i\xi_{iM})^{-1}$, $\bar{\lambda} = \max\{1, \lambda_M\}$, $\epsilon_1 > 0$ is an arbitrary small number, Q_2 is positive definite matrix with appropriate dimension, G_i is the solution to $\min_{G_i} \hat{B}_i^{ar} G_i - B^a$ for $i = 1, \ldots, N_f$, $\hat{B}_i^{ar} = [(\hat{B}_i^r)^T, 0_{(m-m_{si}\times n)}]^T$, $\Delta^f = \sum_{i=1}^{N_f}(\Delta_{1i}^f + \Delta_{2i}^f)$, with $\Delta_{1i}^f = -2\int_{t_f}^T (\eta_i^f(t))^T P_i^r \bar{B}_i^{ar} f_i \hat{\Gamma}_i(\xi_i + \xi_{iM})(\bar{B}_i^{ar})^T P_i^r \eta_i^f(t) dt$, and $\Delta_{2i}^f = 2\int_{t_f}^T (\eta_i^f(t))^T P_i^r \big(B_i^{as}\underline{u}_i^s + (B_i^{ar}G_i - B^a)u_i^g(t)\big)dt$,

$-4(T-t_f)(\eta_{iM}^f)^2 \lambda_{\max}\{P_i^r\}^2 \|\bar{B}_i^{ar}\|^2 \|\hat{\Gamma}_i\| \|\xi_{iM}\| \leq \Delta_{1i}^f \leq 0$,

$\Delta_{2i}^f \leq 2(T-t_f)\eta_{iM}^f \lambda_{\max}\{P_i^r\}\Big(\|\bar{B}_i^{as}\|(\|\hat{\underline{u}}_i^s\| + \|\underline{u}_i^\xi\|) + \big(\|\bar{B}_i^a\|(\|\hat{\Gamma}_i\| + \|f_i\xi_{iM}\|)\|G_i\| + \|B^a\|\big)\|K_{2i}\|\bar{e}_{\max}^a\Big)$, $\eta_{iM}^f = exp\{\alpha\}\max\{\|\eta_i^f(t_f)\|, \frac{\|\hat{\underline{u}}_i^s\| + \|\underline{u}_i^M\|}{\beta}\}$, $\alpha = log\frac{\sigma_{\max}\{S\}}{\sigma_{\min}\{S\}}$, $\beta = -\max real\{\lambda_j\{A_i^a + B_i^{ar}K_{1i}^f\}\}$, the matrix S is such that $S(A_i^a + B_i^{ar}K_{1i}^f)S^{-1} = J_{A_{cl}}$, $J_{A_{cl}}$ is the Jordan transformation of $A_i^a + B_i^{ar}K_{1i}^f$, $\hat{\underline{u}}_i^s$ is the estimated stuck command, and $\underline{u}_i^M$ is the upper bound of command estimation errors.

Proof. Our main objective is to select K^h_{1i} and K^f_{1i} such that $\eta^h_i(t)$ and $\eta^f_{1i}(t)$ are asymptotically stabilized, which ensures that $\eta^f_{2i}(t)$ is bounded to obtain the upper bound of $\mathbb{J}_{2f}$. We require to select K^f_{1i} online and by using only local information. Recall that $\mathcal{Q}^f = \text{diag}\{I_{n_f} \otimes \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, I_{N-N_f} \otimes Q\}$, that is $\mathcal{Q}^f$ is a block diagonal matrix. Therefore, one can partition $\mathbb{J}_{2f}$ as $\mathbb{J}_{2f} = J_{21} + J_{22}$, where $J_{21} = \sum_{i=1}^{N_f} J_{21i}$, $J_{22} = \sum_{i=N_f+1}^{N} J_{22i}$, $J_{21i} = \int_{t_f}^{T} \bar{\lambda}(\eta^f_{1i}(t))^{\mathrm{T}} Q \eta^f_{1i}(t)\mathrm{d}t$ and $J_{22i} = \int_{t_f}^{T} \bar{\lambda}(\eta^h_i(t))^{\mathrm{T}} Q \eta^h_i(t)\mathrm{d}t$. Given that $\mathcal{A}^f_1$ is also a block diagonal matrix, the problem (5.24) can be equivalently solved by tackling a set of N problems associated with each agent, that is N problems that can be solved locally. Specifically, for the faulty agents

$$\begin{aligned} \min_{K^f_{1i}} \sup J_{21i} &= \min_{K^f_{1i}} \sup \int_{t_f}^{T} \bar{\lambda}(\eta^f_{1i}(t))^{\mathrm{T}} Q \eta^f_{1i}(t)\mathrm{d}t,\ i = 1, \ldots, N_f, \quad (5.47) \\ & s.t.\ \dot{\eta}^f_i(t) = (A^a_i + B^{ar}_i K^f_{1i})\eta^f_i(t) + (B^{ar}_i G_i - B^a_i)u^g_i(t) \\ & +B^{as}_i \underline{u}^s_i, \quad (5.48) \end{aligned}$$

and for the healthy system we will have

$$\begin{aligned} \min_{K^h_{1i}} \sup J_{22i} &= \min_{K^h_{1i}} \sup \int_{t_f}^{T} \bar{\lambda}(\eta^h_i(t))^{\mathrm{T}} Q \eta^h_i(t)\mathrm{d}t, i = N_f + 1, \ldots, N, \quad (5.49) \\ & s.t.\ \dot{\eta}^h_i(t) = (A + BK^h_{1i})\eta^h_i(t). \quad (5.50) \end{aligned}$$

For $i = N_f + 1, \ldots, N$, in the problem (5.49)-(5.50), let $K^h_{1i} = -B^{\mathrm{T}} P^h$, where P^h is the unique solution to Ric$(A, B, \bar{\lambda}Q, \frac{1}{2}I)$. In this case

$$\Lambda \triangleq (A + BK^h_{1i})^{\mathrm{T}} P^h + P^h(A + BK^h_{1i}) = -Q < 0,$$

which implies that $A + BK^h_{1i}$ is Hurwitz and $\eta^h_i(t)$ is asymptotically stable. Furthermore,

$$\begin{aligned} J_{22i} &= -\int_{t_f}^{T} (\eta^h_i(t))^{\mathrm{T}} \big((A + BK^h_{1i})^{\mathrm{T}} P^h + P^h(A + BK^h_{1i})\big)\eta^h_i(t)\mathrm{d}t = -\int_{t_f}^{T} \frac{d\varpi^h_i(t)}{dt}\mathrm{d}t \\ &= \varpi^h_i(t_f) - \varpi^h_i(T), \end{aligned}$$

where $\varpi_i^h(t) = \left(\eta_i^h(t)\right)^{\mathrm{T}} P^h \eta_i^h(t)$. Therefore, $J_{22i} \leq \varpi_i^f(t_f) = \left(\eta_i^h(t_f)\right)^{\mathrm{T}} P^h \eta_i^h(t_f) = J_{22i}^{ub}$, and it follows that

$$J_{22}^{ub} = \sum_{i=N_f+1}^{N} J_{22i}^{ub} = \sum_{i=N_f+1}^{N} \left(\eta_i^h(t_f)\right)^{\mathrm{T}} P^h \eta_i^h(t_f).$$

It should be noted that solving the Riccati equation $\mathrm{Ric}(A, B, \bar{\lambda}Q, \frac{1}{2}I)$ is equivalent to solving the following minimization problem [157]

$$\min \left(\eta_i^h(t_f)\right)^{\mathrm{T}} P^h \eta_i^h(t_f) \; s.t. \; A^{\mathrm{T}} P_i^h + P_i^h A - 2PBB^{\mathrm{T}}P + \bar{\lambda}Q \leq 0, \tag{5.51}$$

which is the solution to $\mathrm{Ric}(A, B, \bar{\lambda}Q, \frac{1}{2}I)$ minimizing J_{12i}^{ub}.

Subsequently, for $i = 1, \ldots, N_f$, we require to design K_{1i}^f such that $\eta_{1i}^f(t)$ becomes asymptotically stable and obtain the upper bound of (or the guaranteed cost for) J_{11i}. Let G_i be the solution to

$$\min_{G_i} \hat{B}_i^{ar} G_i - B^a,$$

where $\hat{B}_i^{ar} = [(\hat{B}_i^r)^{\mathrm{T}}, 0_{(m-m_{si}\times n)}]^{\mathrm{T}}$. Then, the design objectives are guaranteed by following through two steps: (i) the conditions that make $A_i^a + B_i^{ar} K_{1i}^f$ Hurwitz are obtained and then by using Lemma 5.2 it can be concluded that $\eta_{1i}^f(t)$ is asymptotically stable and (ii) by using the results of part (i) the upper bound (ub) of J_{11i} is obtained.

Step i: Consider the system (5.48). The matrix $A_i^a + B_i^{ar} K_{1i}^f$ is Hurwitz if there exists P_i^r such that

$$\Lambda = (A_i^a + B_i^{ar} K_{1i}^f)^{\mathrm{T}} P_i^r + P_i^r (A_i^a + B_i^{ar} K_{1i}^f) < 0.$$

Towards this end, let us select K_{1i}^f as $K_{1i}^f = -(\hat{B}_i^{ar})^{\mathrm{T}} P_i^r$, where P_i^r is the solution to the $\mathrm{Ric}(A_i^a, \bar{B}_i^{ar}, \bar{\lambda}Q^f, \frac{1}{2}\hat{\Gamma}_i^{-1}(\hat{\Gamma}_i - f_i\xi_{iM})^{-1})$, $Q^f = \mathrm{diag}\{Q, \epsilon_1 Q_2\}$, ϵ_1 is an arbitrary small positive number, and Q_2 is a positive definite matrix with an appropriate dimension. Then, one obtains

$$
\begin{aligned}
\Lambda &= \mathrm{Sym}\big((A_i^a - B_i^{ar}(\hat{B}_i^{ar})^{\mathrm{T}}P_i^r)^{\mathrm{T}}P_i^r\big) \\
&= \mathrm{Sym}(P_i^r A_i^a) - 2P_i^r\hat{B}_i^{ar}(B_i^{ar})^{\mathrm{T}}P_i^r \\
&= \mathrm{Sym}(P_i^r A_i^a) - 2P_i^r\hat{B}_i^{ar}(\hat{\Gamma}_i + f_i\xi_i)(\bar{B}_i^{ar})^{\mathrm{T}}P_i^r \\
&= \mathrm{Sym}(P_i^r A_i^a) - 2P_i^r\bar{B}_i^{ar}\hat{\Gamma}(\hat{\Gamma}_i + f_i\xi_i)(\bar{B}_i^{ar})^{\mathrm{T}}P_i^r \\
&= -\big(\bar{\lambda}Q^f + P_i^r\bar{B}_i^{ar}\hat{\Gamma}_i\big(2(\hat{\Gamma}_i + f_i\xi_i) - 2(\hat{\Gamma}_i - f_i\xi_{iM})\big)(\bar{B}_i^{ar})^{\mathrm{T}}P_i^r\big) \\
&= -\big(\bar{\lambda}Q^f + 2P_i^r\bar{B}_i^{ar}\hat{\Gamma}_i f_i(\xi_i + \xi_{iM})(\bar{B}_i^{ar})^{\mathrm{T}}P_i^r\big) < 0.
\end{aligned}
$$

The above inequality implies that under the proposed control law, $A_i^a + B_i^{ar}K_{1i}^f$ is Hurwitz. Let

$$
f(t) \equiv (B_i^{ar}G_i - B_i^a)u_i^g(t) + B_i^{as}\underline{u}_i^s = (B_i^{ar}G_i - B_i^a)(K_{2i}\bar{e}_i^a(t)) + B_i^{as}\underline{u}_i^s.
$$

Given that $\underline{u}_i^s$ and $\bar{e}_i^a(t)$ are bounded (based on the results of Lemmas 5.3 and 5.4) and by using Lemma 5.2, with $A_c \equiv A_i^a + B_i^{ar}K_{1i}^f = \begin{bmatrix} A & 0 \\ F & 0 \end{bmatrix} + \begin{bmatrix} B_i^a \\ 0 \end{bmatrix} K_{1i}^f$, it follows that the system (5.48) is stable or $\eta_{1i}^f(t)$ is asymptotically stable and $\eta_{2i}^f(t)$ is bounded.

Step ii: To obtain the upper bound of J_{21i}, let us express it as follows

$$
\begin{aligned}
J_{21i} &= \int_{t_f}^{T} \bar{\lambda}(\eta_{1i}^f(t))^{\mathrm{T}}Q\eta_{1i}^f(t)\mathrm{d}t \\
&\leq \int_{t_f}^{T} \bar{\lambda}\big((\eta_{1i}^f(t))^{\mathrm{T}}Q\eta_{1i}^f(t) + \epsilon_1(\eta_{2i}^f(t))^{\mathrm{T}}Q_2\eta_{2i}^f(t)\big)\mathrm{d}t \\
&= \int_{t_f}^{T} \bar{\lambda}(\eta_i^f(t))^{\mathrm{T}}Q^f\eta_i^f(t)\mathrm{d}t
\end{aligned}
$$

Given that $K_{1i}^f = -(\hat{B}_i^{ar})^{\mathrm{T}} P_i^r$ and

$$\begin{aligned}
-\bar{\lambda} Q^f &= P_i^r A_i^a + (A_i^a)^{\mathrm{T}} P_i^r - 2 P_i^r \bar{B}_i^{ar} \hat{\Gamma}_i (\hat{\Gamma}_i - f_i \xi_{iM}) (\bar{B}_i^{ar})^{\mathrm{T}} P_i^r \\
&= P_i^r A_i^a + (A_i^a)^{\mathrm{T}} P_i^r - 2 P_i^r \bar{B}_i^{ar} \hat{\Gamma}_i (\hat{\Gamma}_i + f_i \xi_i) (\bar{B}_i^{ar})^{\mathrm{T}} P_i^r \\
&\quad + 2 P_i^r \bar{B}_i^{ar} \hat{\Gamma}_i f_i (\xi_i + \xi_{iM}) (\bar{B}_i^{ar})^{\mathrm{T}} P_i^r
\end{aligned}$$

one obtains

$$\begin{aligned}
J_{21i} &= -\int_{t_f}^{T} (\eta_i^f(t))^{\mathrm{T}} \big((A_i^a + B_i^{ar} K_{1i}^f)^{\mathrm{T}} P_i^r + P_i^r (A_i^a + B_i^{ar} K_{1i}^f) \\
&\quad - 2 P_i^r \bar{B}_i^{ar} \hat{\Gamma}_i f_i (\xi_i + \xi_{iM}) (\bar{B}_i^{ar})^{\mathrm{T}} P_i^r \big) \eta_i^f(t) \mathrm{d}t.
\end{aligned}$$

Let $\varpi_i^f(t) \triangleq (\eta_i^f(t))^{\mathrm{T}} P_i^r \eta_i^f(t)$ then

$$\begin{aligned}
J_{21i} &\leq -\int_{t_f}^{T} \frac{d\varpi_i^f(t)}{dt} \mathrm{d}t + \int_{t_f}^{T} 2(\eta_i^f(t))^{\mathrm{T}} P_i^r (B_i^{as} \underline{u}_i^s + (B_i^{ar} G_i - B^a) u_i^g(t)) \mathrm{d}t \\
&\quad - \int_{t_f}^{T} (\eta_i^f(t))^{\mathrm{T}} (2 P_i^r \bar{B}_i^{ar} \hat{\Gamma}_i f_i (\xi_i + \xi_{iM}) (\bar{B}_i^{ar})^{\mathrm{T}} P_i^r \big) \eta_i^f(t) \mathrm{d}t \\
&= \varpi_i^f(t_f) - \varpi_i^f(T) + \int_{t_f}^{T} 2(\eta_i^f(t))^{\mathrm{T}} P_i^r (B_i^{as} \underline{u}_i^s + (B_i^{ar} G_i - B^a) K_{2i} \bar{e}^a(t)) \mathrm{d}t \\
&\quad - \int_{t_f}^{T} (\eta_i^f(t))^{\mathrm{T}} (2 P_i^r \bar{B}_i^{ar} \hat{\Gamma}_i f_i (\xi_i + \xi_{iM}) (\bar{B}_i^{ar})^{\mathrm{T}} P_i^r \big) \eta_i^f(t) \mathrm{d}t
\end{aligned}$$

Therefore,

$$J_{21i} \leq \varpi_i^f(t_f) - \varpi_i^f(T) + \Delta_{1i}^f + \Delta_{2i}^f, \tag{5.52}$$

where

$$\begin{aligned}
\Delta_{1i}^f &= -2 \int_{t_f}^{T} \eta_i^{f^{\mathrm{T}}}(t) P_i^r \bar{B}_i^{ar} f_i \hat{\Gamma}_i (\xi_i + \xi_{iM}) (\bar{B}_i^{ar})^{\mathrm{T}} P_i^r \eta_i^f(t) \mathrm{d}t \\
\Delta_{2i}^f &= 2 \int_{t_f}^{T} \eta_i^{f^{\mathrm{T}}}(t) P_i^r \big(B_i^{as} \underline{u}_i^s + (B_i^{ar} G_i - B^a) K_{2i} \bar{e}^a(t) \big) \mathrm{d}t.
\end{aligned}$$

To obtain the range of values of Δ_{1i}^f and Δ_{2i}^f in (5.52), let us express the Jordan canonical decomposition of $A_{cli} \triangleq A_i^a + B_i^{ar} K_{1i}^f$ as $A_{cli} = S J_{A_{cl}} S^{-1}$. Then, since

$\|\underline{u}_i^s\| \leq \|\hat{\underline{u}}_i^s\| + \|\underline{u}_i^\xi\| \leq \|\hat{\underline{u}}_i^s\| + \|\underline{u}_i^M\|$ from Lemma 5.2, we have

$$\|\eta_i^f(t)\| \leq \eta_{iM}^f = \exp\{\alpha\} \max\{\|\eta_i^f(t_f)\|, \frac{\|\hat{\underline{u}}_i^s\| + \|\underline{u}_i^M\|}{\beta}\},$$

where $\alpha = \log\frac{\sigma_{\max}\{S\}}{\sigma_{\min}\{S\}}$ and $\beta = -\max \mathrm{real}\{\lambda_j\{A_{cli}\}\}$. Considering that $\xi_i \leq \xi_{iM}$, $P_i^r \leq \lambda_{\max}\{P_i^r\}I$ we have

$$\eta_i^{f^{\mathrm{T}}}(t)P_i^r\bar{B}_i^{ar}f_i\hat{\Gamma}_i(\xi_i + \xi_{iM})(\bar{B}_i^{ar})^{\mathrm{T}}P_i^r\eta_i^f(t) \leq 2(\eta_{iM}^f)^2\lambda_{\max}\{P_i^r\}^2\|\bar{B}_i^{ar}\|^2\|\hat{\Gamma}_i\|\|\xi_{iM}\|.$$

Therefore, one obtains

$$-4(T - t_f)(\eta_{iM}^f)^2\lambda_{\max}\{P_i^r\}^2\|\bar{B}_i^{ar}\|^2\|\hat{\Gamma}_i\|\|\xi_{iM}\| \leq \Delta_{1i}^f \leq 0.$$

On the other hand, since $\|(B_i^{ar}G_i - B^a)K_{2i}\bar{e}^a(t)\| \leq \big(\|\bar{B}_i^a\|(\|\hat{\Gamma}_i\| + \|f_i\xi_{iM}\|)\|G_i\| + \|B^a\|\big)\|K_{2i}\|\bar{e}_{\max}^a$ we have $\Delta_{2i}^f \leq 2(T - t_f)\eta_{iM}^f\lambda_{\max}\{P_i^r\}\Big(\|\bar{B}_i^{as}\|(\|\hat{\underline{u}}_i^s\| + \|\underline{u}_i^\xi\|) + \big(\|\bar{B}_i^a\|(\|\hat{\Gamma}_i\| + \|f_i\xi_{iM}\|)\|G_i\| + \|B^a\|\big)\|K_{2i}\|\bar{e}_{\max}^a\Big)$.

Now, given that $\eta_{1i}^f(t)$ is asymptotically stable, and $\eta_i^f(t)$ is bounded, $\varpi_i^f(T) \geq 0$ is bounded and by using (5.52) it follows that

$$J_{21i} \leq \varpi_i^f(t_f) - \varpi_i^f(T) + \Delta_{1i}^f + \Delta_{2i}^f \leq \varpi_i^f(t_f) + \Delta_{1i}^f + \Delta_{2i}^f.$$

Therefore, $J_{21} = \sum_{i=1}^{N_f} J_{21i}$ is upper bounded by

$$J_{21}^{ub} = \sum_{i=1}^{N_f} \varpi_i^f(t_f) + \sum_{i=1}^{N_f}(\Delta_{1i}^f + \Delta_{2i}^f), \tag{5.53}$$

Finally, the upper bound of $\mathbb{J}_{2f}$ is obtained as follows

$$\begin{aligned}\mathbb{J}_{2f}^{ub} &= J_{21}^{ub} + J_{22}^{ub} = \sum_{i=1}^{N_f} \varpi_i^f(t_f) + \Delta^f \\ &\quad + \sum_{i=N_f+1}^{N} (\eta_i^h(t_f))^{\mathrm{T}}P^h\eta_i^h(t_f),\end{aligned}$$

where $\Delta^f = \sum_{i=1}^{N_f}(\Delta_{1i}^f + \Delta_{2i}^f)$, and this completes the proof of Lemma 5.6. □

Remark 5.1. *It should be noted that in absence of an LOE fault or if one can ensure that the information provided by the FDI module are exact, then* $\Delta_{1i}^f = 0$.

Now we are in the position to state our main results.

Theorem 5.1. *The control laws (5.11) and (5.12) solve the control recovery problem in a faulty multi-agent system described in Subsection 5.1, if K_{2i} is designed as in Lemma 5.3 and K_{1i}^h and K_{1i}^f are designed according to Lemma 5.6. Moreover, under these control laws the team cost performance index is upper bounded by*

$$J_f^{ub-quad} = \mathbb{J}_1^{ub-quad} + \mathbb{J}_{2f}^{ub}, \tag{5.54}$$

where $\mathbb{J}_1^{ub-quad}$ is specified in Lemma 5.3 and $\mathbb{J}_{2f}^{ub}$ is specified in Lemma 5.6.

Proof. Follows from the results derived in Lemmas 5.3 and 5.6. □

Theorem 5.2. *The control laws (5.11) and (5.12) solve the fault recovery problem in a faulty multi-agent system as described in Subsection 5.1 and the team performance index cost is upper bounded by*

$$J_f^{ub-con} = \mathbb{J}_1^{up-con} + \mathbb{J}_{2f}^{ub}, \tag{5.55}$$

if $K_{2i} = [K_{21i}\ K_{22i}]$ is the solution to the Algorithm 5.1, where $\mathbb{J}_1^{ub-con}$ is given as in (5.38), $\mathbb{J}_{2f}^{ub}$, K_{1i}^f and K_{1i}^h are defined as in Lemma 5.6 .

Proof. Follows from the results derived in Lemmas 5.4, 5.6 and the previous discussions. □

Based on the results obtained in Theorems 5.1 and 5.2, the following corollary for a team of healthy agents is concluded immediately.

Corollary 5.1. *Suppose all the agents are healthy and the cooperative task starts at $t = t_0$. The control law*

$$u_i(t) = K_{1i}^h \eta_i^h + K_{21i}\bar{e}_i(t) + K_{22i}\bar{e}_{iI}(t),$$

solves the consensus problem in a healthy team with a directed switching topology network consisting of a leader given by (2.3) and N followers given by (2.1) if K_{1i}^h is designed according to Lemma 5.6 and K_{21i} and K_{22i} are designed according to Lemma 5.3 or Algorithm 5.1. Moreover, under this control law the team cost performance index is upper bounded by

$$J_h^{quad} = \sum_{i=1}^{N} (\eta_i^h(t_0))^T P^h \eta_i^h(t_0) + \mathbb{J}_1^{ub-quad},$$
$$J_h^{con} = \sum_{i=1}^{N} (\eta_i^h(t_0))^T P^h \eta_i^h(t_0) + \mathbb{J}_1^{ub-con},$$

where all the gains and parameters are defined as in Theorems 5.1 and 5.2.

5.3 Simulation Results and Case Studies

In this section, our proposed methodologies are applied to and illustrated for a team of multi-agents that is composed of seven Sentry Autonomous Underwater Vehicles (AUVs) [171]. Sentry, made by the Woods Hole Oceanographic Institution, is a fully autonomous underwater vehicle and is capable of surveying to the depth of 6000 m.

The linearized model of the Sentry is of six degrees of freedom, but it is commonly decomposed into four non-interacting subsystems, namely (i) the speed subsystem, (ii) the steering subsystem, (iii) the diving subsystem, and (iv) the roll subsystem. In conducting simulation scenarios, our interest is to enforce all the agents follow the leader's depth trajectory, therefore we only consider the diving subsystem.

Assume that the near horizontal speed is $1.5\ m/s$ so that by using the parameters of the Sentry AUV presented in [171], the diving subsystem for the i-th agent can be governed according to (2.1) with

$$A = \begin{bmatrix} -0.61 & 0.68 & 0 & 0.001 \\ 0.27 & -0.43 & 0 & -0.01 \\ 1.00 & 0 & 0 & 1.00 \\ 0 & 1.00 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.205 & 0.17 \\ -0.18 & 0.22 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where $x_i(t) = \begin{bmatrix} \omega_i(t), q_i(t), z_i(t), \theta_i(t) \end{bmatrix}^{\mathrm{T}}$, $u_i(t) = \begin{bmatrix} \delta_i^b(t), \delta_i^s(t) \end{bmatrix}^{\mathrm{T}}$, with $\omega_i(t)$, $q_i(t)$, $z_i(t)$, $\theta_i(t)$, $\delta_i^b(t)$ and $\delta_i^s(t)$ denoting the heave speed, pitch rate, depth, pitch, bone and the stern plane deflections, respectively. Furthermore, it is assumed that the agents are subject to additive disturbances. The disturbance input channel matrix is considered as $B_\omega = [0.0402, 0.0311, 0, 0]^{\mathrm{T}}$ and the disturbance signal is considered as a random walk [180].

Let agent 0 designate the leader and the agents 1 to 6 designate the follower agents. The communication network switches among three topologies that are denoted by G^1, G^2 and G^3 and are given in Figure 5.1. The topologies are assumed to switch according to $G^1 \rightarrow G^3 \rightarrow G^2 \rightarrow G^3 \rightarrow G^1$ every 100 sec. Suppose the mission starts at $t = 0$ sec. and is terminated at the time $t = T = 1000$ sec. The following scenarios are considered for conducting our simulation case studies.

Scenario A - Healthy Team: In this scenario, all the agents are considered to be

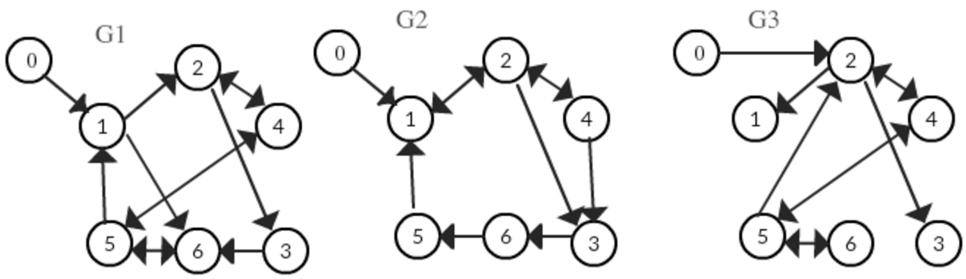

Figure 5.1: The three switching network graph topologies for one leader and six followers.

healthy and the control laws are designed according to the Corollary 5.1. The control gain K_{1i}^h is designed according to Lemma 5.6, whereas to design K_{21i} and K_{22i} we consider two sets of gains that are obtained according to Lemma 5.3 and Algorithm 5.1. As stated in [189], generally by increasing M the performance of the team is improved, however this is at the cost of adding more variables and constraints. Moreover, in many cases by having $M = 2$ one can achieve an acceptable performance improvement as compared to the case when $M = 1$, where the constraints become LMIs instead of BMIs. Using the gains that are designed based on the Algorithm 5.1 with $M = 2$ and $M = 3$ the team performance index becomes $J_h^{con} = 1.4319e + 08$ and $J_h^{con} = 1.183e + 08$, respectively. On the other hand, by employing the control gains that are designed according to Lemma 5.3, the team performance index becomes $J_h^{quad} = 5.8625e + 08$. On the other words, by using the convex hull function (results of Lemma 5.3) the team performance index is significantly improved. Figure 5.2 depicts the agents depth trajectory corresponding to this scenario.

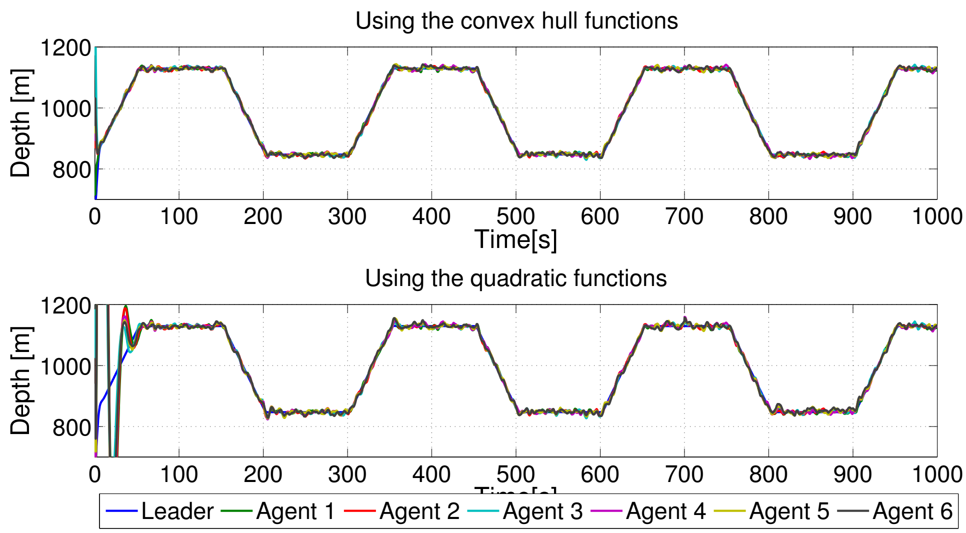

Figure 5.2: The agents depth trajectories corresponding to Scenario A.

In Scenarios B, C and D below the behavior and performance of the team subject to the LOE fault, the outage fault, and the stuck fault are studied, respectively. Towards this end, during $0 \leq t < 400$ $sec.$ the agents and their control laws are considered to be the same as those governed by the Scenario A. At time $t = 400$ $sec.$ the faults are injected and after a delay of Δ $sec.$ the control reconfigurations for the faulty agent according to Theorems 5.1 and 5.2 are initiated. The delay times Δ's are selected as $\Delta = 10$ $sec.$, 40 $sec.$, and $80 sec.$ corresponding to consecutive delays in formally invoking the recovery strategy. The details corresponding to each scenario are provided below:

<u>**Scenario B**</u>: The LOE fault is injected in the agent 2 where the loss of effectiveness of the first and second actuators are 70% and 50% of their nominal values, respectively, and moreover the outage fault is injected in the second actuator of the agent 4. It is assumed that the upper bound of the fault severity estimation uncertainty is taken as 10%.

Scenario C: The stuck fault is injected in the second actuator of the agent 6 where the stuck command is considered as $u_3^2 = 1$. Moreover, the the outage fault is injected in the first actuator of the agent 3.

Scenario D: The LOE fault is injected in the agent 3 where the loss of effectiveness of the first and second actuators are reduced to 40% and 30% of their nominal values, respectively. Moreover, the outage fault is injected in the second actuator of the agent 4 and a stuck fault is injected in the second actuator of the agent 6 where the stuck command is considered as $u_3^2 = 1$.

The team performance indices for the above scenarios are given in Table 5.1, but due to space limitations, only the depth trajectories for certain scenarios are depicted in Figures 5.3, 5.4 and 5.5.

Scenarios and Delays in Recovery Invocation	J^{con}	J^{quad}	$\frac{J^{quad}-J^{con}}{J^{con}} * 100$
A	1.4319e+08	5.8625e+08	309.42
B; $\Delta = 10$	2.85290e+08	8.5069e+08	66.7445
B; $\Delta = 40$	4.6336e+08	1.2193e+09	61.997
B; $\Delta = 80$	2.8363e+09	5.3597e+09	47.081
C; $\Delta = 10$	1.0823e+09	2.5618e+09	57.7549
C; $\Delta = 40$	8.9964e+10	1.737e+11	48.226
C; $\Delta = 80$	4.6880e+11	8.5470e+11	45.149
D; $\Delta = 10$	9.6067e+08	1.8192e+09	47.193
D; $\Delta = 40$	3.8690e+10	7.2626e+10	47.727
D; $\Delta = 80$	1.6346e+11	3.0603e+11	46.589

Table 5.1: The indices corresponding to the network performance index under the scenarios A-D, where Δ denotes the delay in formally invoking the control recovery strategy.

Figures 5.3, 5.4 and 5.5 demonstrate that when the agents become faulty, the team

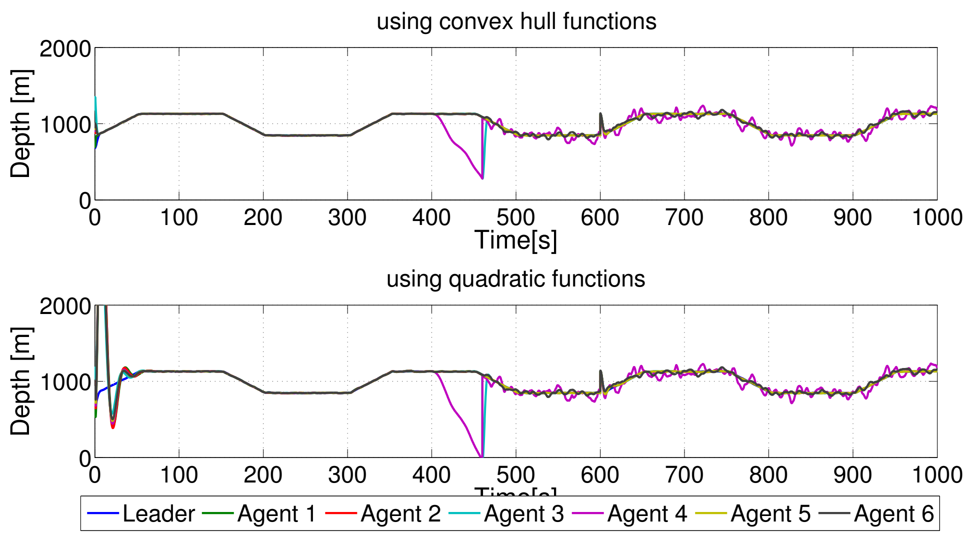

Figure 5.3: The agents depth and pitch angle trajectories corresponding to the Scenario B-$\Delta = 40$ s.

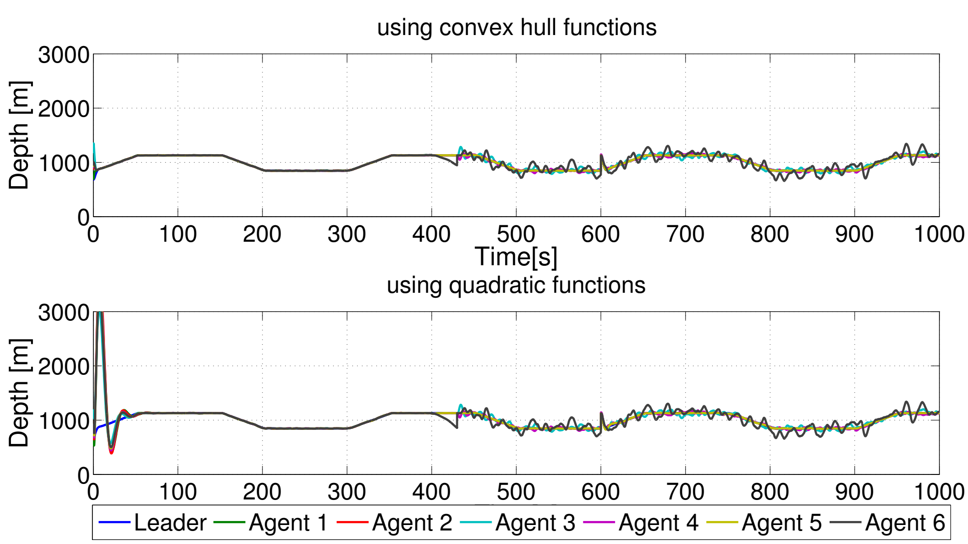

Figure 5.4: The agents depth and pitch angle trajectories corresponding to the Scenario C-$\Delta = 10$ s.

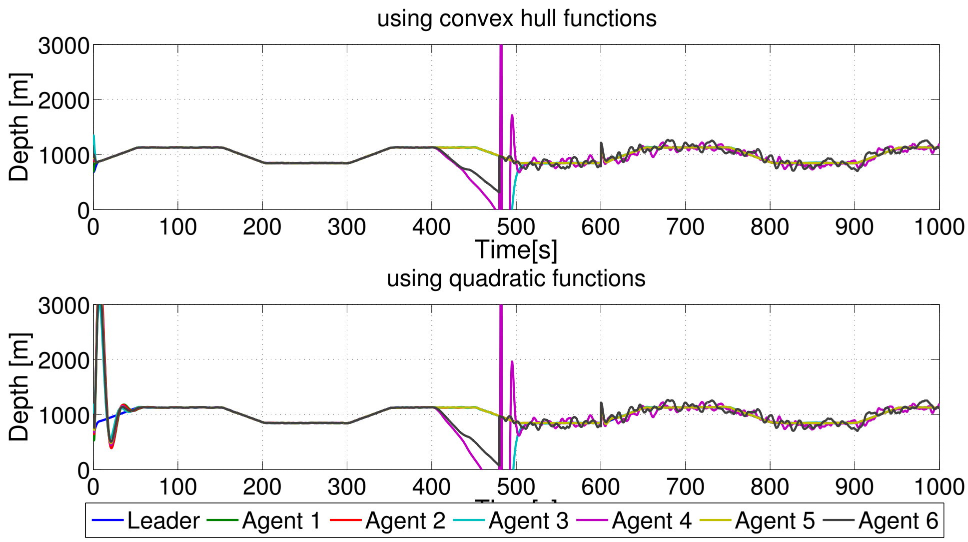

Figure 5.5: The agents depth and pitch angle trajectories corresponding to the Scenario D-$\Delta = 80$ s.

cannot reach a consensus with the control gains that are designed based on the healthy agents and the team becomes *unstable*. However, once the reconfigured control gains are initiated and invoked the team again becomes stable and can then reach a consensus. Moreover, Table 5.1 indicates that by employing convex hull function for selecting the control gains instead of the quadratic function the overall team performance can be significantly improved. Note that when the delay in invoking the reconfigured control gains increases the followers' states deviation from the leader's states grows and the team performance index increases as well. In other words, delays in invoking the reconfigured control will deteriorate the overall team performance.

5.4 Conclusions

In this work, the fault recovery control problem of networked multi-agent systems under directed switching topology is investigated. Reconfigurable control laws are proposed that accommodates the loss of effectiveness, the outage and the stuck actuator faults as well as uncertainties, unreliabilities, and inaccuracies in the FDI module estimations information. Two strategies have been proposed to design and select the gains of the proposed control laws which employ only local available information to ensure that the team simultaneously reaches a consensus in presence of actuator faults and minimize the upper bound of the team performance index. The proposed methodology is applied to a team of Sentry autonomous underwater vehicles and the team behavior and performance under various fault scenarios are investigated and compared. Simulation results demonstrate and illustrate the effectiveness of our proposed reconfigurable control approaches in recovering different actuator faults in the network.

Chapter 6

Distributed Cooperative Output Regulation Control Reconfiguration Design

In this chapter, cooperative dynamic output control regulation problem in a heterogeneous network of multi-agent systems subject to actuator faults is investigated. A distributed dynamic output feedback control strategy is proposed which guarantees that the agents output follow the outputs of an exo-system while the agents states remain bounded even when the agents are subject to various types of actuator faults. It should be noted that only a very *limited* set of agents have access to the exo-system and its measurements. Three types of faults are considered, namely, the loss of effectiveness (LOE), the outage and the stuck faults. The reconfigured control law is designed online and only employs local measurements as well as information that are provided by the fault detection, isolation and identification (FDII) module locally to redesign the gains of the proposed control laws. Furthermore, an upper bound on the team performance index under our proposed control laws is obtained. Our proposed approach is applied and implemented to a team of autonomous underwater vehicles (AUVs) where its effectiveness and capabilities are validated through simulation case studies.

6.1 Problem Formulation

Consider a network of N agents that is governed by the dynamics (2.19) as well as the exo-system dynamics (2.21). Without loss of any generality, it is assumed that all the agents are faulty at $t = t_f$, where for the i-th faulty agent its first m_{oi} actuators are subject to the outage fault, the actuators $m_{oi} + 1$ to m_{si} are subject to a stuck fault, and the remaining $m_i - m_{si}$ actuators are either subject to the LOE fault or are healthy. Using equations (2.15)-(2.18), the model of the i-th faulty agent can be expressed as

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i^f u_i^f(t), \\ y_i(t) &= C_i x_i(t), \end{aligned} \tag{6.1}$$

where $B_i^f = [B_i^o \ B_i^s \ B_i^r]$, $B_i^o = [b^1, \ldots, b^{m_{oi}}]$, $B_i^s = [b^{m_{oi}+1}, \ldots, b^{m_{si}}]$, $B_i^r = \bar{B}_i^r \Gamma_i$, $\bar{B}_i^r = \begin{bmatrix} b^{m_{si}+1}, \ldots, b^{m} \end{bmatrix}$, $\Gamma_i = \text{diag}\{\Gamma_i^l\}$, Γ_i^l is the severity of the l-th actuator of the i-th agent, $u_i^f(t) = \begin{bmatrix} 0_{1\times m_{oi}} & (\underline{u}_i^s)^{\mathrm{T}} & (u_i(t))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\underline{u}_i^s = \begin{bmatrix} \underline{u}_i^{m_{oi}+1}, \ldots, \underline{u}_i^{m_{si}} \end{bmatrix}^{\mathrm{T}}$, and $u_i(t) = \begin{bmatrix} u_i^{m_{si}+1}(t), \ldots, u_i^{m_i}(t) \end{bmatrix}^{\mathrm{T}}$.

Considering the structure of the control law $u_i^f(t)$ and the matrix B_i^f, it follows that only the actuators $m_{si}+1$ to m_i are available for use in the control reconfiguration. Therefore, to proceed with our proposed control recovery strategy the model (6.1) is now rewritten as follows

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i^r u_i(t) + B_i^s \underline{u}_i^s, \\ y_i(t) &= C_i x_i(t). \end{aligned} \tag{6.2}$$

Output Control Reconfiguration Objective: The main objective of the reconfigurable control law is to develop and design $u_i(t)$ for the faulty multi-agent network that is governed by (6.2) such that the output regulation errors $e_i(t)$ for $i = 1, \ldots, N$ remain stable.

In this work, the following assumptions are assumed to hold.

Assumption 6.1. *The matrix S is anti-Hurwitz and the pair (S, R) is observable.*

Assumption 6.2. *The triple (A_i, B_i^r, C_i) is controllable and observable.*

Assumption 6.3. *The matrix $\begin{bmatrix} A_i - \lambda_k I & B_i^r \\ C_i & 0 \end{bmatrix}$ is full rank for $\lambda_k \in \sigma(S)$.*

Assumption 6.4. *The network communication topology has a directed spanning tree.*

The following lemma will be used to derive our results in the subsequent sections.

Lemma 6.1. *[199] For the matrices A, B, C, E and F with appropriate dimensions, the matrix equation*

$$XS = AX + BU + E \textit{ and } 0 = CX + F$$

have a solution (X, U) for any E and F, if and only if for all $\lambda_k \in \sigma(S)$, $\begin{bmatrix} A - \lambda_k I & B \\ C & 0 \end{bmatrix}$ *is full rank.*

Lemma 6.2. *[200] Let M be a matrix with negative diagonal elements and non-negative off-diagonal elements. If $x(t, t_f, x_0) \in \mathbb{R}^n$ and $y(t, t_f, y_0) \in \mathbb{R}^n$ are solutions to*

$$\dot{x}(t) \preceq Mx(t), x(t_f) = x_0$$

$$\dot{y}(t) = My(t), y(t_f) = y_0,$$

and $x_0 = y_0$, then $x(t, t_f, x_0) \preceq y(t, t_f, y_0)$ for all $t \in [t_f, \infty)$, where $\preceq$ implies that $\dot{x}_i(t) \leq \sum_{j=1}^{n} M(i,j)x_j(t)$, $i = 1, \ldots, n$.

6.2 Proposed Methodology

In this section, our proposed dynamic output control reconfiguration scheme is developed. Given that each agent only communicates with its nearest neighboring agents and only limited number of agents have access to the measurements of the exo-system, the control strategy should be distributed. In order to solve the problem first the necessary conditions for stabilizing output errors are obtained and the team cost performance index is obtained in terms of the agents states and control states. Second, by introducing some auxiliary variables the conditions are transformed and decomposed into two sets and the team cost performance is also expressed as the summation of two costs which are associated with the new variables. Finally, it is shown that by solving the yielded problems one can guarantee that the agents outputs follow the outputs of the exo-system while the upper bound of the team cost performance index is obtained.

Consider a team of multi-agent systems that consists of N agents as specified in Section 6.1 and the exo-system that is governed by (2.21). Let us select the i-th agent *dynamic output control* strategy as

$$\dot{x}_{ci}(t) = A_{ci}x_{ci}(t) + \sum_{j\in\mathcal{N}_i} A_{cij}x_{cj}(t) + B_{ci}e_i(t), \tag{6.3}$$

$$u_i(t) = C_{ci}x_{ci}(t) + \underline{u}_i^r, \tag{6.4}$$

where A_{ci}, A_{cij}, B_{ci} and C_{ci} denote the controller gains and $\underline{u}_i^r$ denotes the control command to be selected, $x_{ci}(t) \in \mathbb{R}^{\rho_i}$, denotes the control state and $e_i(t)$ is defined as in (2.23).

Let us now select the control command $\underline{u}_i^r$ as the solution to

$$B_i^s\underline{u}_i^s + B_i^r\underline{u}_i^r = 0. \tag{6.5}$$

Remark 6.1. *Equation (6.5) has a solution if there exists sufficient control effort in the faulty agent such that*

$$B_i^s\underline{u}_i^s \subset \mathit{Im}\{B_i^r\}. \tag{6.6}$$

The closed-loop dynamics of the agents and dynamic control laws can now be obtained as

$$\dot{x}^a(t) = \mathcal{A}x^a(t) + \mathcal{B}\omega(t), \tag{6.7}$$

$$e(t) = \bar{C}x^a(t) + \bar{E}\omega(t), \tag{6.8}$$

where $\mathcal{A} = \begin{bmatrix} A & BC_c \\ B_c\tilde{C} & A_c \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ B_c\bar{E} \end{bmatrix}$, $\bar{C} = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}$, $\bar{E} = -\begin{bmatrix} g_{10} \\ \vdots \\ g_{N0} \end{bmatrix} \otimes \mathcal{R}$, $\tilde{C} = \mathbb{L}C$, $\mathbb{L} = (L + L_\beta) \otimes I_m$, $L_\beta = \text{diag}\{g_{i0}\}$, $A = \text{blkdiag}\{A_i\}$, $B = \text{blkdiag}\{B_i^r\}$, $C = \text{blkdiag}\{C_i\}$, $A_c = \begin{bmatrix} A_{c1} & \dots & A_{c1N} \\ \vdots & \dots & \vdots \\ A_{cN1} & \dots & A_{cN} \end{bmatrix}$, $A_{cij} = 0$ if $j \notin \mathcal{N}_i$, $B_c = \text{blkdiag}\{B_{ci}\}$, $C_c = \text{blkdiag}\{C_{ci}\}$, $x^a(t) = \text{col}\{x(t), x_c(t)\}$, $x(t) = \text{col}\{x_i(t)\}$, $x_c(t) = \text{col}\{x_{ci}(t)\}$, $e(t) = \text{col}\{e_i(t)\}$, and $i = 1, \dots, N$. We are now in a position to state our first result.

Lemma 6.3. *Consider a faulty network that consists of N agents given by (6.2) and the exo-system dynamics given by (2.21). Suppose that Assumptions (6.1)-(6.4) hold. The dynamic output control strategy (6.3)-(6.4) guarantees that the output regulation errors given by (2.23) are asymptotically stable if the controller gains A_{ci}, A_{cij}, B_{ci} and C_{ci} are designed such that the matrix $\mathcal{A}$ is Hurwitz and there exist $\Pi = col\{\Pi_i\}$ and $\Theta = col\{\Theta_i\}$ such that the following equations have a solution*

$$\begin{bmatrix} \Pi \\ \Theta \end{bmatrix} S - \mathcal{A} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} + \mathcal{B} = 0 \textit{ and } \bar{E} - \bar{C} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} = 0.$$

Proof. Let us first define the auxiliary states $\tilde{x}(t) = x^a(t) + \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} \omega(t)$, so that from equations (6.7) and (6.8), we have

$$\dot{\tilde{x}}(t) = \mathcal{A}\tilde{x}(t) + (\begin{bmatrix} \Pi \\ \Theta \end{bmatrix} S - \mathcal{A} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} + \mathcal{B})\omega(t), \tag{6.9}$$

$$e(t) = \bar{C}\tilde{x}(t) + (\bar{E} - \bar{C} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix})\omega(t). \tag{6.10}$$

If the matrices A_{ci}, B_{ci} and C_{ci} are designed such that $\mathcal{A}$ is Hurwitz, then under Assumption 6.1 for any $\mathcal{B}$ there exist Π and Θ that solve the following Sylvester equation

$$\begin{bmatrix} \Pi \\ \Theta \end{bmatrix} S - \mathcal{A} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} + \mathcal{B} = 0, \tag{6.11}$$

so that we will have

$$\dot{\tilde{x}}(t) = \mathcal{A}\tilde{x}(t). \tag{6.12}$$

Given that $\mathcal{A}$ is now Hurwitz, the first term in (6.10) goes to zero asymptotically and for steady state response of $e(t)$ one gets

$$e(t) \to (\bar{E} - \bar{C} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix})\omega(t) \text{ as } t \to \infty.$$

Therefore, if the condition stated in the lemma holds and

$$\bar{E} - \bar{C} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} = 0, \tag{6.13}$$

then the output regulation errors are asymptotically stable and $e_i(t) \to 0$ as $t \to \infty$, and this completes the proof of the lemma. $\square$

Let us now assume that the gains are selected as specified in Lemma 6.3 and the matrix $\mathcal{A}$ is Hurwitz. According to the proof of Lemma 6.3, when $\mathcal{A}$ is Hurwitz, $\lim_{t\to\infty} \tilde{x}(t) =$

0 so that the steady state response of $x^a(t)$ can be expressed as

$$x^a(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \to - \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} \omega(t) \text{ as } t \to \infty$$

that is $x(t) \to -\Pi\omega(t)$ and $x_c(t) \to -\Theta\omega(t)$ as $t \to \infty$. Therefore, the steady state values of output and control signals can be obtained as $y_i^{ss}(t) = C_i\Pi_i\omega(t)$ and $u_i^{ss}(t) = -C_{ci}\Theta_i\omega(t) + \underline{u}_i^r$.

Now associated with the i-th faulty agent we define quadratic cost performance index as follows

$$J_i^f = \int_{t_f}^{\infty} \big((y_i(t) - y_i^{ss}(t))^{\mathrm{T}} Q_i (y_i(t) - y_i^{ss}(t)) + \tilde{u}_i^{\mathrm{T}}(t) R_i \tilde{u}_i(t)\big)\mathrm{d}t, \tag{6.14}$$

where $\tilde{u}_i(t) = u_i(t) - u_i^{ss}(t)$, $u_i^{ss}(t)$ and $y_i^{ss}(t)$ denote the steady state values of $u_i(t)$ and $y_i(t)$ i.e. $u_i^{ss}(t) = C_{ci}\Theta_i\omega(t) + \underline{u}_i^r$, $y_i^{ss}(t) = C_i\Pi_i\omega(t)$, Q_i and R_i are positive definite matrices with appropriate dimensions. The above cost function represents the cost of the i-th agent transient time in regulating its output. In the following the upper bound of the team cost which is defined as $J = \sum_{i=1}^{N} J_i^f$ under the proposed control law will be obtained.

Let $u^{ss}(t) = \mathrm{col}\{u_i^{ss}(t)\}$, therefore from the definition of $u_i^{ss}(t)$ one has $u^{ss}(t) = C_c x_c(t) + \underline{u}^r = -C_c\Theta\omega(t) + \underline{u}^r$ and it follows that

$$\begin{aligned}
\tilde{u}(t) &= u(t) - u^{ss}(t) = C_c x_c(t) + \underline{u}^r - u^{ss}(t) = \begin{bmatrix} 0 & C_c \end{bmatrix} x^a(t) + C_c\Theta\omega(t) \\
&= \begin{bmatrix} 0 & C_c \end{bmatrix} x^a(t) + \begin{bmatrix} 0 & C_c \end{bmatrix} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} \omega(t) = \begin{bmatrix} 0 & C_c \end{bmatrix} (x^a(t) + \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} \omega(t)) \\
&= \begin{bmatrix} 0 & C_c \end{bmatrix} \tilde{x}(t),
\end{aligned} \tag{6.15}$$

where $\tilde{u}(t) = \mathrm{col}\{\tilde{u}_i(t)\}$.

The overall team performance index following the occurrence of a fault can now be expressed as

$$
\begin{aligned}
J &= \sum_{i=1}^{N} \int_{t_f}^{\infty} \big((y_i(t) - y_i^{ss}(t))^{\mathrm{T}} Q_i (y_i(t) - y_i^{ss}(t)) \\
&\quad + (u_i(t) - u_i^{ss}(t))^{\mathrm{T}} R_i (u_i(t) - u_i^{ss}(t))\big)\mathrm{d}t, \\
&= \int_{t_f}^{\infty} (y(t) - y^{ss}(t))^{\mathrm{T}} Q (y(t) - y^{ss}(t)) + (u(t) - u^{ss}(t))^{\mathrm{T}} R (u(t) - u^{ss}(t))\big)\mathrm{d}t \\
&= \int_{t_f}^{\infty} (y(t) - y^{ss}(t))^{\mathrm{T}} Q (y(t) - y^{ss}(t)) + \tilde{u}^{\mathrm{T}}(t) R \tilde{u}(t)\big)\mathrm{d}t,
\end{aligned}
$$

where $y(t) = \mathrm{col}\{y_i(t)\}$, $y^{ss}(t) = \mathrm{col}\{y_i^{ss}(t)\}$, $R = \mathrm{diag}\{R_i\}$, and $Q = \mathrm{diag}\{Q_i\}$. Given that $\lim_{t\to\infty} x(t) + \Pi\omega(t) = 0$, and based on the definition of $y_i^{ss}(t)$, we now have

$$
y(t) - y^{ss}(t) = \begin{bmatrix} C & 0 \end{bmatrix} (x^a(t) + \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} \omega(t)) = \begin{bmatrix} C & 0 \end{bmatrix} \tilde{x}(t).
$$

Consequently, by considering (6.15), the overall team performance cost can be obtained as

$$
\begin{aligned}
J &= \int_{t_f}^{\infty} \tilde{x}^{\mathrm{T}}(t)(\begin{bmatrix} C & 0 \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} C & 0 \end{bmatrix})\tilde{x}(t) + \tilde{x}^{\mathrm{T}}(t)(\begin{bmatrix} 0 & C_c \end{bmatrix}^{\mathrm{T}} R \begin{bmatrix} 0 & C_c \end{bmatrix})\tilde{x}(t) \\
&= \int_{t_f}^{\infty} \tilde{x}^{\mathrm{T}}(t) \begin{bmatrix} C^{\mathrm{T}} Q C & 0 \\ 0 & C_c^{\mathrm{T}} R C_c \end{bmatrix} \tilde{x}(t)\mathrm{d}t.
\end{aligned}
$$

Therefore, the design problem can now be stated as that of selecting the matrices A_{ci}, A_{cij}, B_{ci}, and C_{ci} such that the outputs regulation errors are stabilized or equivalently the following conditions are satisfied simultaneously

$$\mathcal{A} \text{ is Hurwitz,} \tag{6.16}$$

$$\begin{bmatrix} \Pi \\ \Theta \end{bmatrix} S - \mathcal{A} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} + \mathcal{B} = 0, \tag{6.17}$$

$$\bar{E} = \bar{C} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix}, \tag{6.18}$$

and also determining the upper bound of the overall team performance index which is denoted by J^{ub}, $J \leq J^{ub}$.

In order to select the controller gains that satisfy the above requirements simultaneously we need certain preliminary developments as follows: first in Lemma 6.4 the conditions under which equations (6.17) and (6.18) simultaneously can have solutions are investigated. Then, some transformations are introduces and employed to transform the stability problem into the problem of designing the gains to ensure that the matrices are Hurwitz.

Lemma 6.4. *Under Assumption 6.3, equation (6.17) and (6.18) have a solution for* Π *and* Θ *simultaneously if* A_c *includes the eigenvalues of the matrix* S *as given by (2.21).*

Proof. Equations (6.17) and (6.18) can be expressed as follows

$$\Pi S - A\Pi - BC_c\Theta = 0, \tag{6.19}$$

$$\Theta S - B_c\tilde{C}\Pi - A_c\Theta + B_c\bar{E} = 0, \tag{6.20}$$

$$\bar{E} = \bar{C} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix}. \tag{6.21}$$

By substituting (6.21) into (6.20) for $\bar{E}$ it follows that

$$\Theta S - B_c(\tilde{C}\Pi - \bar{E}) - A_c\Theta = \Theta S - B_c(\tilde{C}\Pi - \begin{bmatrix} \tilde{\mathcal{C}} & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix}) - A_c\Theta = \Theta S - A_c\Theta = 0,$$

which along with equation (6.19) can be written as

$$\begin{bmatrix} \Pi \\ \Theta \end{bmatrix} S - \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} = 0. \tag{6.22}$$

Equation (6.22) is a homogeneous Sylvester equation and has a non-zero solution only if the matrix $\begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}$ includes the eigenvalues of S. Given that the eigenvalues of A depend on the agents' dynamics which cannot be changed, equation (6.22) can have a solution only if the matrix A_c is designed such that it includes the eigenvalues of S, and this completes the proof of the lemma. □

Remark 6.2. *As shown subsequently, the values of* Π *and* Θ *do not affect the control gains and only determine the upper bound of the overall team cost index that is denoted by* J.

We are now in a position to introduce a procedure for selecting the controller gains. Let us partition $x_{ci}(t)$ as $x_{ci}(t) = [x_{ci}^{1^{\mathrm{T}}}(t)\ x_{ci}^{2^{\mathrm{T}}}(t)]^{\mathrm{T}}$, where $x_{ci}^1(t) \in \mathbb{R}^{\rho_i - r}$ and $x_{ci}^2(t) \in \mathbb{R}^r$, r is the dimension of $\omega(t)$. Then, the dynamics (6.3) can be expressed as

$$\begin{bmatrix} \dot{x}_{ci}^1(t) \\ \dot{x}_{ci}^2(t) \end{bmatrix} = \begin{bmatrix} A_{ci}^{11} & A_{ci}^{12} \\ A_{ci}^{21} & A_{ci}^{22} \end{bmatrix} \begin{bmatrix} x_{ci}^1(t) \\ x_{ci}^2(t) \end{bmatrix} + \sum_{j \in \mathcal{N}_i} \begin{bmatrix} A_{cij}^{11} & A_{cij}^{12} \\ A_{cij}^{21} & A_{cij}^{22} \end{bmatrix} \begin{bmatrix} x_{cj}^1(t) \\ x_{cj}^2(t) \end{bmatrix} + \begin{bmatrix} B_{ci}^1 \\ B_{ci}^2 \end{bmatrix} e_i(t).$$

Let select A_{ci}^{21} and A_{cij}^{21} as $A_{ci}^{21} = 0$, $A_{cij}^{21} = 0$ and define

$$X^a(t) = T_1 x^a(t) = [x^{\mathrm{T}}(t)\ x_c^{1^{\mathrm{T}}}(t)\ x_c^{2^{\mathrm{T}}}(t)]^{\mathrm{T}},$$

where $x_c^1(t) = \mathrm{col}\{x_{ci}^1(t)\}$, $x_c^2(t) = \mathrm{col}\{x_{ci}^2(t)\}$ and T_1 is the nonsingular transformation to rearrange $x^a(t)$ to $X^a(t)$, then the dynamics of $X^a(t)$ is

$$\dot{X}^a(t) = \tilde{\mathcal{A}} X^a(t) + \begin{bmatrix} 0 \\ B_c^1 \bar{E} \\ B_c^2 \bar{E} \end{bmatrix} \omega(t),$$

where $\tilde{\mathcal{A}} = \begin{bmatrix} A & BC_c^1 & BC_c^2 \\ B_{c1}\tilde{C} & \multicolumn{2}{c}{\multirow{2}{*}{$\bar{\mathrm{A}}_c$}} \\ B_c^2\tilde{C} & & \end{bmatrix}$, $\bar{A}_c = \begin{bmatrix} A_c^{11} & A_c^{12} \\ 0 & A_c^{22} \end{bmatrix}$, $B_c = \begin{bmatrix} B_c^1 \\ B_c^2 \end{bmatrix}$, $A_c^{22} =$ blkdiag$\{A_{ci}^{22}\}$, $B_c^1 =$ blkdiag$\{B_{ci}^1\}$ and $B_c^2 =$ blkdiag$\{B_{ci}^2\}$. Suppose that the pairs (A_{ci}^{22}, B_{ci}^2), $i = 1, \ldots, N$ are selected such that they are controllable and A_{ci} includes the eigenvalues of S, C_c^1 and C_c^2 are selected such that their dimensions are compatible with the dimensions of $x_c^1(t)$ and $x_c^2(t)$, respectively. Now, let $\tilde{X}(t) = X^a(t) + \begin{bmatrix} \Pi \\ \Theta_1 \\ \Theta_2 \end{bmatrix} \omega(t)$, then it can be shown that

$$\dot{\tilde{X}}(t) = \tilde{\mathcal{A}} \tilde{X}(t) \tag{6.23}$$

Consequently, by applying the linear transformation $T_2 = \begin{bmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{bmatrix}$ to the system $\dot{\tilde{X}}(t) = \tilde{\mathcal{A}} \tilde{X}(t)$, the transformed system becomes

$$\dot{\tilde{x}}_T(t) = \mathcal{A}_T \tilde{x}_T(t),$$

where $\tilde{x}_T(t) = \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t), \tilde{x}_{2T}^{\mathrm{T}}(t), \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}} = T_2 \tilde{X}(t)$ and

$$\mathcal{A}_T = \begin{bmatrix} A + BC_c^1 & BC_c^1 & BC_c^2 \\ A_c^{11} - (A + BC_c^1 - B_c^1 \tilde{C}) & A_c^{11} - BC_c^1 & A_c^{12} - BC_c^2 \\ B_c^2 \tilde{C} & 0 & A_c^{22} \end{bmatrix}. \tag{6.24}$$

Let us now select $A_c^{11} = A + BC_c^1 - B_c^1 \tilde{C}$, $A_c^{12} = BC_c^2$, and let $\bar{x}(t) = [\tilde{x}_{1T}^{\mathrm{T}}(t), \tilde{x}_{3T}^{\mathrm{T}}(t), \tilde{x}_{2T}^{\mathrm{T}}(t)]^{\mathrm{T}}$. It now follows that

$$\dot{\bar{x}}(t) = \bar{\mathcal{A}} \bar{x}(t) = \begin{bmatrix} \bar{\mathcal{A}}_1 & \bar{\mathcal{A}}_2 \\ 0 & \bar{\mathcal{A}}_3 \end{bmatrix} \bar{x}(t), \tag{6.25}$$

where $\bar{\mathcal{A}}_1 = \begin{bmatrix} A + BC_c^1 & BC_c^2 \\ B_c^2 \tilde{C} & A_c^{22} \end{bmatrix}$, $\bar{\mathcal{A}}_2 = \begin{bmatrix} BC_c^1 \\ 0 \end{bmatrix}$, and $\bar{\mathcal{A}}_3 = A - B_c^1 \tilde{C}$.

The design strategy for each of the above matrices will be discussed below. However, we first require to express the team performance index in terms of the new state $\bar{x}(t)$. Towards this end, we have

$$\begin{aligned} u(t) - u^{ss}(t) &= \begin{bmatrix} 0 & C_c \end{bmatrix} \tilde{x}(t) = \begin{bmatrix} 0 & C_c^1 & C_c^2 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{x}_T(t) \\ &= \begin{bmatrix} C_c^1 & C_c^2 & C_c^1 \end{bmatrix} \bar{x}(t), \\ y(t) - y^{ss}(t) &= \begin{bmatrix} C & 0 \end{bmatrix} \tilde{x}(t) = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{x}_T(t) = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \tilde{x}_T(t) \\ &= \begin{bmatrix} C & 0 & 0 \end{bmatrix} \bar{x}(t), \end{aligned}$$

where $\bar{x}(t)$ is as given in (6.25). Therefore,

$$
\begin{aligned}
J &= \int_{t_f}^{\infty} \big((y(t)-y^{ss}(t))^{\mathrm{T}}Q(y(t)-y^{ss}(t)) + (u(t)-u^{ss}(t))^{\mathrm{T}}R(u(t)-u^{ss}(t))\big)\mathrm{d}t \\
&= \int_{t_f}^{\infty} \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t) & \tilde{x}_{2T}^{\mathrm{T}}(t) \end{bmatrix} \bar{\mathcal{Q}} \begin{bmatrix} \tilde{x}_{1T}(t) \\ \tilde{x}_{3T}(t) \\ \tilde{x}_{2T}(t) \end{bmatrix} \mathrm{d}t \\
&= \int_{t_f}^{\infty} \big(\begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix} \bar{\mathcal{Q}}_1 \begin{bmatrix} \tilde{x}_{1T}(t) \\ \tilde{x}_{3T}(t) \end{bmatrix} + 2 \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix} \bar{\mathcal{Q}}_2 \tilde{x}_{2T}(t) \\
&\quad + \tilde{x}_{2T}^{\mathrm{T}}(t)\bar{\mathcal{Q}}_3\tilde{x}_{2T}(t)\big)\mathrm{d}t,
\end{aligned}
$$

where $\bar{\mathcal{Q}} = \begin{bmatrix} \bar{\mathcal{Q}}_1 & \bar{\mathcal{Q}}_2 \\ \bar{\mathcal{Q}}_2^{\mathrm{T}} & \bar{\mathcal{Q}}_3 \end{bmatrix} = \begin{bmatrix} C^{\mathrm{T}}QC + C_c^{\mathrm{T}}RC_c & C_c^{\mathrm{T}}RC_c^1 \\ (C_c^1)^{\mathrm{T}}RC_c & (C_c^1)^{\mathrm{T}}RC_c^1 \end{bmatrix}$. Now, by using Fact 2.2 with $\alpha = 1$ one gets

$$
\begin{aligned}
J &\leq \int_{t_f}^{\infty} \big(\begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix} \bar{\mathcal{Q}}_1 \begin{bmatrix} \tilde{x}_{1T}(t) \\ \tilde{x}_{3T}(t) \end{bmatrix} + \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix} C_c^{\mathrm{T}}RC_c \begin{bmatrix} \tilde{x}_{1T} \\ \tilde{x}_{3T}(t) \end{bmatrix} \\
&\quad + \tilde{x}_{2T}^{\mathrm{T}}(t)(C_c^1)^{\mathrm{T}}RC_c^1\tilde{x}_{2T}(t) + \tilde{x}_{2T}^{\mathrm{T}}(t)\bar{\mathcal{Q}}_3\tilde{x}_{2T}(t)\big)\mathrm{d}t \\
&= \int_{t_f}^{\infty} \big(\begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix} (C^{\mathrm{T}}QC + 2C_c^{\mathrm{T}}RC_c) \begin{bmatrix} \tilde{x}_{1T}(t) \\ \tilde{x}_{3T}(t) \end{bmatrix} \\
&\quad + 2\tilde{x}_{2T}^{\mathrm{T}}(t)(C_c^1)^{\mathrm{T}}RC_c^1\tilde{x}_{2T}(t)\big)\mathrm{d}t.
\end{aligned}
$$

Given that the spectrum of $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_T$ are the same and $\bar{\mathcal{A}}$ is a block triangular matrix, if $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_3$ are Hurwitz, then $\bar{\mathcal{A}}$ as well as $\bar{\mathcal{A}}_T$ and $\mathcal{A}$ are Hurwitz. Now, considering the above representation for $\bar{\mathcal{A}}_T$, the overall team cost J, can be expressed and partitioned as follows:

$$\begin{aligned}
J &= \mathbb{J}_1 + \mathbb{J}_2 \\
&= \int_{t_f}^{\infty} \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix} (Q_2 + 2C_c^{\mathrm{T}} R C_c) \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t) \\ \tilde{x}_{3T}^{\mathrm{T}}(t) \end{bmatrix} \mathrm{d}t \\
&\quad +2\int_{t_f}^{\infty} \tilde{x}_{2T}^{\mathrm{T}}(t)(C_c^1)^{\mathrm{T}} R C_c^1 \tilde{x}_{2T}(t)\mathrm{d}t, \\
s.t. \quad & \begin{bmatrix} \dot{\tilde{x}}_{1T}(t) \\ \dot{\tilde{x}}_{3T}(t) \\ \dot{\tilde{x}}_{2T}(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{A}}_1 & \bar{\mathcal{A}}_2 \\ 0 & \bar{\mathcal{A}}_3 \end{bmatrix} \begin{bmatrix} \tilde{x}_{1T}(t) \\ \tilde{x}_{3T}(t) \\ \tilde{x}_{2T}(t) \end{bmatrix}
\end{aligned}$$

where $\mathbb{J}_1$ denotes the first cost term and $\mathbb{J}_2$ denotes the second cost term and where $Q_2 = \mathrm{diag}\{Q_{2i}\}$ and $Q_{2i} = C_i^{\mathrm{T}} Q_i C_i$.

The above representation has interesting features that can be stated as follows:

1) $\mathbb{J}_1$ is a function of only the states $\tilde{x}_{1T}(t)$ and $\tilde{x}_{3T}(t)$ while $\mathbb{J}_2$ is a function of only $\tilde{x}_{2T}(t)$,

2) If $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_3$ are Hurwitz, the entire multi-agent team will also be stable, and

3) Q_2 and $(C_c^1)^{\mathrm{T}} R C_c^1$ are block diagonal matrices.

Motivated by the above, one now reformulate our problem as obtaining J_1^{ub} and J_2^{ub} such that $\mathbb{J}_1 \leq J_1^{ub}$ and $\mathbb{J}_2 \leq J_2^{ub}$ and also designing the control gains such that $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_3$ become Hurwitz. Specifically, we are only interested in distributed reconfigurable control design strategies that will render the matrices $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_3$ Hurwitz. In other words, the agents should design their controller gains by using only local information when an actuator fault has occured and is detected by the FDI module while leading to changes in their dynamics.

In fact, if for $t < t_f$, the matrices $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_3$ are Hurwitz and a fault occurs at $t = t_f$, then for $t \geq t_f$, $\bar{\mathcal{A}}_1$ may become anti-Hurwitz, so that the control gains should be reconfigured online by incorporating the information that are provided by the FDI module to ensure that matrices $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_3$ remain Hurwitz.

Theorems 6.1 and 6.2 below discuss the conditions under which $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_3$ remain Hurwitz following the detection, isolation and identification of the actuator faults.

Theorem 6.1. *By invoking the controller gains* $C_{ci} = \begin{bmatrix} C_{ci}^1 & C_{ci}^2 \end{bmatrix} = -R_i^{-1}(B_i^a)^T P_i$*,* $i = 1, \dots, N$*, the matrix* $\bar{\mathcal{A}}_1$ *is Hurwitz if if* A_{ci}^{22}*,* B_{ci}^{22} *are selected such that* $(A_{ci}^{22}, B_{ci}^{22})$ *is controllable and* A_{ci}^{22} *includes the eigenvalues of* S *and for* $q_i > 0$ *and* $\beta > 1$ *there exist* $P_i > 0$*,* $i = 1, \dots, N$ *as solutions to*

$$\max \ \textit{Trace}\{P_i\} \quad s.t. \quad \begin{bmatrix} A_{ia}^T P_i + P_i A_{ia} + (Q_{2i} + q_i I) & P_i B_i^a \\ (B_i^a)^T P_i & R_i \end{bmatrix} \geq 0, \ \beta I \leq P_i \leq \alpha_i I, \tag{6.26}$$

where $A_i^a = \begin{bmatrix} A_i & 0 \\ d_i B_{ci}^2 C_i & A_{ci}^{22} \end{bmatrix}$, $B_i^a = \begin{bmatrix} B_i^r \\ 0 \end{bmatrix}$, $\alpha_i \leq \beta \frac{(\lambda_{min}\{Q_{2i}\}+q_i)\gamma_i}{\sum_{j\in\mathcal{N}_i} g\|B_{ci}^{2a} C_j^a\|^2}$ *and* $\gamma_i \ll d_i^{-1}\|B_i^a R_i^{-1}(B_i^a)^T\|$.

Proof. The control gain C_{ci}s should be designed such that $\bar{\mathcal{A}}_1$ is Hurwitz. This is guaranteed in two steps: first the gains are designed such that the following system

$$\dot{\tilde{y}}(t) = \mathcal{A}_1 \tilde{y}(t) \tag{6.27}$$

is asymptotically stable, where $\tilde{y}(t) = \text{col}\{\tilde{y}_i(t)\}$,

$$\mathcal{A}_1 = \begin{bmatrix} A_1^a + B_1^a C_{c1} & -g_{12} B_{c1}^{2a} C_2^a & \dots & -g_{1N} B_{c1}^{2a} C_N^a \\ -g_{21} B_{c2}^{2a} C_1^a & A_2^a + B_2^a C_{c2} & \dots & -g_{2N} B_{c2}^{2a} C_N^a \\ \vdots & \ddots & \vdots & \\ -g_{N1} B_{cN}^{2a} C_1^a & -g_{N2} B_{cN}^{2a} C_2^a & \dots & A_N^a + B_N^a C_{cN} \end{bmatrix}, \tag{6.28}$$

$A_i^a = \begin{bmatrix} A_i & 0 \\ d_i B_{ci}^2 C_i & A_{ci}^{22} \end{bmatrix}$, $B_i^a = \begin{bmatrix} B_i^r \\ 0 \end{bmatrix}$, $B_{ci}^{2a} = \begin{bmatrix} 0 \\ B_{ci}^2 \end{bmatrix}$ and $C_i^a = \begin{bmatrix} C_i & 0 \end{bmatrix}$. Second, since the spectrum of the matrix $\mathcal{A}_1$ is the same as that of the spectrum of $\bar{\mathcal{A}}_1$, if one has negative eigenvalues the other will be the same. On the other words, if (6.27) is asymptotically stable then $\mathcal{A}_1$ and so that $\bar{\mathcal{A}}_1$ are Hurwitz.

Let $v_i(\tilde{y}_i(t)) = \tilde{y}_i^{\mathrm{T}}(t) P_i \tilde{y}_i(t)$, $P_i > 0$, then

$$\begin{aligned} \dot{v}_i(\tilde{y}_i(t)) &= \tilde{y}_i^{\mathrm{T}}(t)\big((A_i^a + B_i^a C_{ci})^{\mathrm{T}} P_i + P_i (A_i^a + B_i^a C_{ci})\big)\tilde{y}_i(t) \\ &\quad - \sum_{j\in\mathcal{N}_i} \big(\tilde{y}_j^{\mathrm{T}}(t)(B_{ci}^{2a} C_j^a)^{\mathrm{T}} P_i \tilde{y}_i(t) + \tilde{y}_i^{\mathrm{T}}(t) P_i (B_{ci}^{2a} C_j^a)\tilde{y}_j(t)\big). \end{aligned}$$

By applying Fact 2.2 to the last two terms of the right-hand side of the above equality yields

$$\begin{aligned} \dot{v}_i(\tilde{y}_i(t)) &\leq \tilde{y}_i^{\mathrm{T}}(t)\big((A_i^a + B_i^a C_{ci})^{\mathrm{T}} P_i + P_i (A_i^a + B_i^a C_{ci})\big)\tilde{y}_i(t) \\ &\quad + \sum_{j\in\mathcal{N}_i} \big(\gamma_i^{-1} \tilde{y}_j^{\mathrm{T}}(t)(B_{ci}^{2a} C_j^a)^{\mathrm{T}} (B_{ci}^{2a} C_j^a)\tilde{y}_j(t) + \gamma_i \tilde{y}_i^{\mathrm{T}}(t) P_i P_i \tilde{y}_i(t)\big). \end{aligned}$$

Let $C_{ci} = -R_i^{-1}(B_i^a)^{\mathrm{T}} P_i$, where P_i is the solution to

$$\max \text{Trac}\{P_i\}\ s.t. (A_i^a)^{\mathrm{T}} P_i + P_i A_i^a - P_i B_i^a R_i^{-1} (B_i^a)^{\mathrm{T}} P_i + Q_{2i} + q_i I \leq 0, \tag{6.29}$$

[1] then it follows that

$$\begin{aligned}\dot{v}_i(\tilde{y}_i(t)) &\leq -\tilde{y}_i^{\mathrm{T}}(t)\big(P_i(B_i^a R_i^{-1}(B_i^a)^{\mathrm{T}} - d_i\gamma_i)P_i + Q_{2i} + q_i I\big)\tilde{y}_i(t) \\ &\quad +\gamma_i^{-1}\sum_{j\in\mathcal{N}_i}\|B_{ci}^{2a}C_j^a\|_2^2\tilde{y}_j^{\mathrm{T}}(t)\tilde{y}_j(t).\end{aligned}$$

If γ_i is selected such that $d_i\gamma_i \ll \|B_i^a R_i^{-1}(B_i^a)^{\mathrm{T}}\|_2^2$, and P_i the solution to (6.29) also satisfies $\beta I \leq P_i \leq \alpha_i I$, with $\alpha_i, \beta > 0$ then one has

$$\dot{v}_i(\tilde{y}_i(t)) \leq -\tilde{y}_i^{\mathrm{T}}(t)\big(P_i B_i^a R_i^{-1}(B_i^a)^{\mathrm{T}} P_i + Q_i + q_i I\big)\tilde{y}_i(t) + \gamma_i^{-1}\sum_{j\in\mathcal{N}_i}\|B_{ci}^{2a}C_j^a\|_2^2\tilde{y}_j^{\mathrm{T}}(t)\tilde{y}_j(t).$$

Consequently, it follows that

$$\beta\tilde{y}_i^{\mathrm{T}}(t)\tilde{y}_i(t) \leq v_i(\tilde{y}_i(t)) = \tilde{y}_i^{\mathrm{T}}(t)P_i\tilde{y}_i(t) \leq \alpha_i\tilde{y}_i^{\mathrm{T}}(t)\tilde{y}_i(t),$$

hence

$$\frac{v_i(\tilde{y}_i(t))}{\alpha_i} \leq \tilde{y}_i^{\mathrm{T}}(t)\tilde{y}_i(t) \leq \frac{v_i(\tilde{y}_i(t))}{\beta}.$$

Subsequently,

$$\dot{v}_i(\tilde{y}_i(t)) \leq -\frac{\lambda_{min}\{Q_{2i}\} + q_i}{\alpha_i}v_i(\tilde{y}_i(t)) + \frac{\gamma_i^{-1}}{\beta}\sum_{j\in\mathcal{N}_i}\|B_{ci}^{2a}C_j^a\|_2^2 v_j(\tilde{y}_j(t)),$$

and one will have

$$\dot{v}(\tilde{y}(t)) \preceq M v(\tilde{y}(t)),$$

where $v(\tilde{y}(t)) = \mathrm{col}\{v_i(\tilde{y}_i(t))\}$ and

$$M = \begin{bmatrix} -\frac{\lambda_{min}\{Q_{21}\}+q_1}{\alpha_1} & \dots & g_{1N}\frac{\gamma_1^{-1}}{\beta}\|B_{c1}^{2a}C_N^a\|_2^2 \\ \vdots & \ddots & \vdots \\ g_{N1}\frac{\gamma_N^{-1}}{\beta}\|B_{cN}^{2a}C_1^a\|_2^2 & \dots & -\frac{\lambda_{min}\{Q_{2N}\}+q_N}{\alpha_N} \end{bmatrix}.$$

Using Lemma 6.2, for $v_0 = v(t_f) = z(t_f) = z_0$, it follows that $v(t, t_f, v_0) \preceq z(t, t_f, z_0)$,

[1] Given that $(B_{ci}^{2a}C_j^a)^{\mathrm{T}}(B_{ci}^{2a}C_j^a)$ is positive semi-definite matrix

$$(B_{ci}^{2a}C_j^a)^{\mathrm{T}}(B_{ci}^{2a}C_j^a) \leq \lambda_{max}\{(B_{ci}^{2a}C_j^a)^{\mathrm{T}}(B_{ci}^{2a}C_j^a)\}I = \|B_{ci}^{2a}C_j^a)\|_2^2 I.$$

where $z(t)$ is the solution to

$$\dot{z}(t) = Mz(t).$$

On the other hand, from the Gerschgorin circle theorem if

$$\frac{\lambda_{min}\{Q_{2i}\} + q_i}{\alpha_i} > \frac{\gamma_i^{-1}}{\beta} \sum_{j \in \mathcal{N}_i} \|B_{ci}^{2a} C_j^a\|_2^2 > 0,$$

then all the eigenvalues of M will be on the open left-half plane and $z(t)$ will be asymptotically stable. Given that $\|\tilde{y}(t)\|^2 \leq \frac{v_i(\tilde{x}_{1,3}(t))}{\beta}$ and $v_i(t, t_f, v_0) \leq z_i(t, t_f, z_0)$, it follows that $\tilde{y}(t)$ is asymptotically stable. Therefore, the system (6.27) is asymptotically stable which implies that $\mathcal{A}_1$ and so that $\bar{\mathcal{A}}_1$ are Hurwitz and this completes the proof of the theorem. □

Theorem 6.2. *Let $B_{ci}^1 = \mu_i X_{2i}^{-1} C_i^T$, $i = 1, \ldots, N$, where positive definite matrix X_{2i} is the solution to*

$$A_i^T X_{2i} + X_{2i} A_i - (C_i^T C_i + \eta_{2i} I) + \eta_{1i} X_{2i} X_{2i} \leq 0, \tag{6.30}$$

with $\eta_{1i} > 1$, $\eta_{2i} \geq 2\|(C_{ci}^1)^T R_i C_{ci}^1\|$, and μ_i, $i = 1, \ldots, N$ are obtained such that

$$\mathbb{L}^T \mu + \mu \mathbb{L} - I > 0, \ \mu = diag\{\mu_i\}, \tag{6.31}$$

then the matrix $\bar{\mathcal{A}}_3$ is Hurwitz.

Proof. Let $B_{ci}^1 = \mu_i X_{2i}^{-1} C_i^{\mathrm{T}}$. We now have

$$X_2^{-1} \bar{\mathcal{A}}_3^{\mathrm{T}} + \bar{\mathcal{A}}_3 X_2^{-1} = X_2^{-1} A^{\mathrm{T}} + A X_2^{-1} - X_2^{-1} C^{\mathrm{T}} \big((\mathbb{L}^{\mathrm{T}} \mu + \mu \mathbb{L}) \otimes I_m\big) C X_2^{-1}, \tag{6.32}$$

where $X_2^{-1} = \mathrm{col}\{X_{2i}^{-1}\}$. For X_{2i} selected as as the solution to (6.30), one has

$$X_2^{-1} \bar{\mathcal{A}}_3^{\mathrm{T}} + \bar{\mathcal{A}}_3 X_2^{-1} \leq X_2^{-1} (C^{\mathrm{T}} C + \eta_2) X_2^{-1} - \eta_1 - X_2^{-1} C^{\mathrm{T}} \big((\mathbb{L}^{\mathrm{T}} \mu + \mu \mathbb{L}) \otimes I_m\big) C X_2^{-1},$$

where $\eta_2 = \mathrm{diag}\{\eta_{2i} I\}$ and $\eta_1 = \mathrm{diag}\{\eta_{1i} I\}$. If $\mu = \mathrm{diag}\{\mu_i\}$, and μ_i is the solution to

(6.31) then

$$X_2^{-1}\bar{\mathcal{A}}_3^{\mathrm{T}} + \bar{\mathcal{A}}_3 X_2^{-1} < -X_2^{-1}\eta_2 X_2^{-1} - \eta_1 < 0, \tag{6.33}$$

which implies that $\bar{\mathcal{A}}_3$ is Hurwtiz, and this completes the proof of the Theorem. □

We are now in the position to state our main result.

Theorem 6.3. *With the application of the dynamic output feedback control laws (6.3) and (6.4) to the faulty multi-agent network, the agents outputs are regulated and the team performance index is upper bounded by*

$$J^{ub} = (\begin{bmatrix} x(t_f) + \Pi\omega(t_f) \\ \Theta\omega(t_f) \end{bmatrix})^T \begin{bmatrix} \mathcal{P}_1^1 - \mathcal{P}_3 & -\mathcal{P}_3 & \mathcal{P}_1^2 \\ -\mathcal{P}_3 & 2\mathcal{P}_3 & 0 \\ \mathcal{P}_1^2 & 0 & \mathcal{P}_1^3 \end{bmatrix} \begin{bmatrix} x(t_f) + \Pi\omega(t_f) \\ \Theta\omega(t_f) \end{bmatrix}$$

where $\underline{u}_i^r$ is the solution to (6.5) and the control gains are selected as

$A_{ci} = \begin{bmatrix} A_i + B_i^r C_{ci}^1 - d_i B_{ci}^1 C_i & B_i^r C_{ci}^2 \\ 0 & A_{ci}^{22} \end{bmatrix}$, $A_{cij} = \begin{bmatrix} B_{ci}^1 C_j & 0 \\ 0 & 0 \end{bmatrix}$, $B_{ci} = \begin{bmatrix} B_{ci}^1 \\ B_{ci}^{22} \end{bmatrix}$, A_{ci}^{22}, B_{ci}^{22}, A_{ci}^{22} *and* $C_{ci} = \begin{bmatrix} C_{ci}^1 & C_{ci}^2 \end{bmatrix}$ *are designed as in Theorem 6.1,* B_{ci}^1 *is designed as in Theorem 6.2,* Π *and* Θ *are the simultaneous solutions to (6.17) and (6.18), and* $\mathcal{P}_1 = \begin{bmatrix} \mathcal{P}_1^1 & \mathcal{P}_1^2 \\ \mathcal{P}_1^2 & \mathcal{P}_1^3 \end{bmatrix}$ *is the solution to*

$$\bar{\mathcal{A}}_1^T \mathcal{P}_1 + \mathcal{P}_1 \bar{\mathcal{A}}_1 = -(Q_2 + 2C_c^T R C_c) \tag{6.34}$$

with $\mathcal{P}_3 = diag\{X_{2i}\}$, where X_{2i} is the solution to (6.30).

Proof. The cost performance index $\mathbb{J}_1$ is bounded if $\tilde{x}_{1iT}(t)$ and $\tilde{x}_{3iT}(t)$ are asymptotically stable. If C_{ci} is designed as in Theorem 6.1, the matrix $\bar{\mathcal{A}}_1$ is Hurwitz, which implies that $\tilde{x}_{1iT}(t)$ and $\tilde{x}_{3iT}(t)$ are asymptotically stable. Therefore, there exists a $\mathcal{P}_1 > 0$

such that

$$\bar{\mathcal{A}}_1^{\mathrm{T}}\mathcal{P}_1 + \mathcal{P}_1\bar{\mathcal{A}}_1 = -(Q_2 + 2C_c^{\mathrm{T}}RC_c).$$

Hence, we have

$$\begin{aligned}
\mathbb{J}_1 &= \int_{t_f}^{\infty} \begin{bmatrix}\tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t)\end{bmatrix} (Q_2 + 2C_c^{\mathrm{T}}RC_c) \begin{bmatrix}\tilde{x}_{1T}^{\mathrm{T}}(t) \\ \tilde{x}_{3T}^{\mathrm{T}}(t)\end{bmatrix} \mathrm{d}t \\
&= -\int_{t_f}^{\infty} \begin{bmatrix}\tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t)\end{bmatrix} (\bar{\mathcal{A}}_1^{\mathrm{T}}\mathcal{P}_1 + \mathcal{P}_1\bar{\mathcal{A}}_1) \begin{bmatrix}\tilde{x}_{1T}^{\mathrm{T}}(t) \\ \tilde{x}_{3T}^{\mathrm{T}}(t)\end{bmatrix} \mathrm{d}t \\
&= -\int_{t_f}^{\infty} \frac{d\Psi_1(t)}{dt}\mathrm{d}t = \Psi_1(t_f) - \Psi_1(\infty) = \Psi_1(t_f),
\end{aligned} \tag{6.35}$$

where $\Psi_1(t) = \begin{bmatrix}\tilde{x}_{1T}^{\mathrm{T}}(t) & \tilde{x}_{3T}^{\mathrm{T}}(t)\end{bmatrix} \mathcal{P}_1 \begin{bmatrix}\tilde{x}_{1T}^{\mathrm{T}}(t) \\ \tilde{x}_{3T}^{\mathrm{T}}(t)\end{bmatrix}$.

On the other hand, $\mathbb{J}_2$ is bounded if $\tilde{x}_{2iT}(t)$ is asymptotically stable, which is guaranteed if $\bar{\mathcal{A}}_3$ is Hurwitz or B_{ci}^1s are designed as in Lemma 6.2. Recall that $\eta_{2i} \geq \|(C_{ci}^1)^{\mathrm{T}}RC_{ci}^1\|$, therefore by multiplying both sides of the inequality (6.33) by $\mathcal{P}_3 = X_2$, one gets

$$\bar{\mathcal{A}}_3^{\mathrm{T}}\mathcal{P}_3 + \mathcal{P}_3\bar{\mathcal{A}}_3 < -\eta_2 - \mathcal{P}_3\eta_1\mathcal{P}_3 \leq -(C_c^1)^{\mathrm{T}}RC_c^1. \tag{6.36}$$

Hence,

$$\begin{aligned}
\mathbb{J}_2 &= 2\int_{t_f}^{\infty} \tilde{x}_{2T}^{\mathrm{T}}(t)((C_c^1)^{\mathrm{T}}RC_c^1)\tilde{x}_{2T}(t)\mathrm{d}t \\
&\leq -2\int_{t_f}^{\infty} \tilde{x}_{2T}^{\mathrm{T}}(t)\big(\bar{\mathcal{A}}_3^{\mathrm{T}}\mathcal{P}_3 + \mathcal{P}_3\bar{\mathcal{A}}_3\big)\tilde{x}_{2T}(t)\mathrm{d}t \\
&= -2\int_{t_f}^{\infty} \frac{d\Psi_3(t)}{dt}\mathrm{d}t = \Psi_3(t_f) - \Psi_3(\infty) = \Psi_3(t_f),
\end{aligned} \tag{6.37}$$

where $\Psi_3(t) = 2\tilde{x}_{2T}^{\mathrm{T}}(t)\mathcal{P}_3\tilde{x}_{2T}(t)$. Now, from the derivations in (6.35) and (6.37) we obtain

$$
\begin{aligned}
J &= \mathbb{J}_1 + \mathbb{J}_2 \leq \bar{x}^{\mathrm{T}}(t_f) \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & 2\mathcal{P}_3 \end{bmatrix} \bar{x}(t_f) \\
&= \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t_f) & \tilde{x}_{3T}^{\mathrm{T}}(t_f) & \tilde{x}_{2T}^{\mathrm{T}}(t_f) \end{bmatrix} \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & 2\mathcal{P}_3 \end{bmatrix} \begin{bmatrix} \tilde{x}_{1T}(t_f) \\ \tilde{x}_{3T}(t_f) \\ \tilde{x}_{2T}(t_f) \end{bmatrix} \\
&= \begin{bmatrix} \tilde{x}_{1T}^{\mathrm{T}}(t_f) & \tilde{x}_{2T}^{\mathrm{T}}(t_f) & \tilde{x}_{3T}^{\mathrm{T}}(t_f) \end{bmatrix} \begin{bmatrix} \mathcal{P}_1^1 & 0 & \mathcal{P}_1^2 \\ 0 & 2\mathcal{P}_3 & 0 \\ \mathcal{P}_1^2 & 0 & \mathcal{P}_1^3 \end{bmatrix} \begin{bmatrix} \tilde{x}_{1T}(t_f) \\ \tilde{x}_{2T}(t_f) \\ \tilde{x}_{3T}(t_f) \end{bmatrix} \\
&= \tilde{x}_T^{\mathrm{T}}(t_f) \begin{bmatrix} \mathcal{P}_1^1 & 0 & \mathcal{P}_1^2 \\ 0 & 2\mathcal{P}_3 & 0 \\ \mathcal{P}_1^2 & 0 & \mathcal{P}_1^3 \end{bmatrix} \tilde{x}_T(t_f),
\end{aligned}
$$

where $\mathcal{P}_1 = \begin{bmatrix} \mathcal{P}_1^1 & \mathcal{P}_1^2 \\ \mathcal{P}_1^2 & \mathcal{P}_1^3 \end{bmatrix}$. From the definition of $\tilde{x}_T(t)$, we have $\tilde{x}_T(t_f) = \begin{bmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{X}(t_f)$ and $\tilde{X}(t_f) = \begin{bmatrix} x(t_f) \\ X_c(t_f) \end{bmatrix} + \begin{bmatrix} \Pi \\ \Theta_1 \\ \Theta_2 \end{bmatrix} \omega(t_f)$, where Π, Θ_1 and Θ_2 are obtained by solving (6.17) and (6.18), simultaneously. Since $X_c(t)$ represents the controller states, let $X_c(t_f) = 0$, so that J^{ub} can be obtained as

$$
J^{ub} = \left(\begin{bmatrix} x(t_f) + \Pi\omega(t_f) \\ \Theta_1\omega(t_f) \\ \Theta_2\omega(t_f) \end{bmatrix}\right)^{\mathrm{T}} \begin{bmatrix} \mathcal{P}_1^1 - \mathcal{P}_3 & -\mathcal{P}_3 & \mathcal{P}_1^2 \\ -\mathcal{P}_3 & 2\mathcal{P}_3 & 0 \\ \mathcal{P}_1^2 & 0 & \mathcal{P}_1^3 \end{bmatrix} \begin{bmatrix} x(t_f) + \Pi\omega(t_f) \\ \Theta_1\omega(t_f) \\ \Theta_2\omega(t_f) \end{bmatrix}, \tag{6.38}
$$

and this completes the proof of the theorem. $\square$

6.3 Simulation Case Studies and Results

In this section, our proposed control recovery approach is applied to a network of heterogenous Autonomous Underwater Vehicles (AUV)s. The team is considered to consist of six Sentry Autonomous Underwater Vehicles (AUV)s. Sentry, made by the Woods Hole Oceanographic Institution [171], is a fully autonomous underwater vehicle that is capable of surveying to the depth of 6000 m and is efficient for forward motions.

To conduct our simulation studies we consider the diving subsystem of the linearized model of Sentry. To obtain a linearized model of this subsystem, the forward (surge) speed u^o is set to $u^o = 1$ and the numerical values for the triple (A_i, B_i, C_i) for the i-th agent, where the team contains two sets of homogenous agents (so that the team is considered overall as heterogenous) are governed by

$$A_i = \begin{bmatrix} -0.509 & 0.566 & 0 & 0.001 \\ 0.225 & -0.355 & 0 & -0.0105 \\ 1.00 & 0 & 0 & 1.00 \\ 0 & 1.00 & 0 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0.146 & 0.120 \\ -0.128 & 0.160 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{for } i = 1, 3, 5$$

$$A_i = \begin{bmatrix} -0.464 & 0.442 & 0 & 0.0012 \\ 0.213 & -0.337 & 0 & -0.0134 \\ 1.00 & 0 & 0 & 1.00 \\ 0 & 1.00 & 0 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0.134 & 0.110 \\ -0.117 & 0.1460 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{for } i = 2, 4, 6$$

$$C_i = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \ i = 1, \ldots, 6,$$

where $x_i(t) = \begin{bmatrix} \omega_i(t), q_i(t), z_i(t), \theta_i(t) \end{bmatrix}^{\mathrm{T}}$, $u_i(t) = \begin{bmatrix} \delta_i^b(t), \delta_i^s(t) \end{bmatrix}^{\mathrm{T}}$, $\omega_i(t)$, $q_i(t)$, $z_i(t)$, $\theta_i(t)$, $\delta_i^b(t)$ and $\delta_i^s(t)$ denote the heave speed, pitch rate, depth, pitch, bone and the stern plane deflections, respectively.

The exo-system dynamic matrices and the network topology Laplacian matrix are given by

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \mathcal{R} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 0 & -1 & 3 \end{bmatrix}.$$

It is assumed that *only* the first agent is connected to the exo-system directly, and the following scenarios are considered to evaluate the performance of our proposed reconfigurable control laws.

Scenario A- *healthy team*: In this scenario, all the agents are considered to be healthy and their dynamics are governed as stated above. By selecting the control gains as specified in Theorem 6.3 for $B_i^r = B_i$, the agents output depth trajectories along with the exo-system output are depicted in Figure 6.1.

In Scenarios B, C and D below, (*faulty team*) the behavior and performance of the team subject to three types of actuator faults, namely the LOE fault, the outage fault, and the stuck fault are considered. Towards this end, during $0 \leq t < 50$ sec the agents and their control laws are considered to be the same as those governed by the Scenario A. At time $t = 50$ sec the faults are injected and after a delay of Δ sec the control gains are reconfigured by invoking Theorem 6.3.The details corresponding to each scenario are provided below:

Scenario B: At $t = 50$ sec the agent 2 becomes faulty and its second actuator is lost while the actuators of the agent 5 are subject to a LOE fault and the effectiveness of its first and second actuators are 60% and 70% of their nominal values, respectively. The delays in invoking the reconfigured control are considered to be $\Delta = 3, 7, 10$ sec.

Scenario C: The stuck fault is injected in the second actuator of the agent 4 where the stuck command is considered as $u_3^2 = 5$ while the same fault is also injected to the agent

5. The delays in invoking the reconfigured control are considered to be $\Delta = 6, 12, 17$ sec.

Scenario D: In this scenario, the agents 1, 4 and 6 are assumed to be faulty. An outage fault is injected to the first actuator of the agent 1 and the second actuator of the agent 6, the agent 4 is subject a LOE fault whereas its actuator effectiveness is reduced by 50%. The delays in invoking the reconfigured control are considered to be $\Delta = 2, 12, 20$ sec.

The team performance indices for the above scenarios are provided in Table 6.1, however due to space limitations, only the depth trajectories for certain scenarios are depicted in Figures 6.1-6.3.

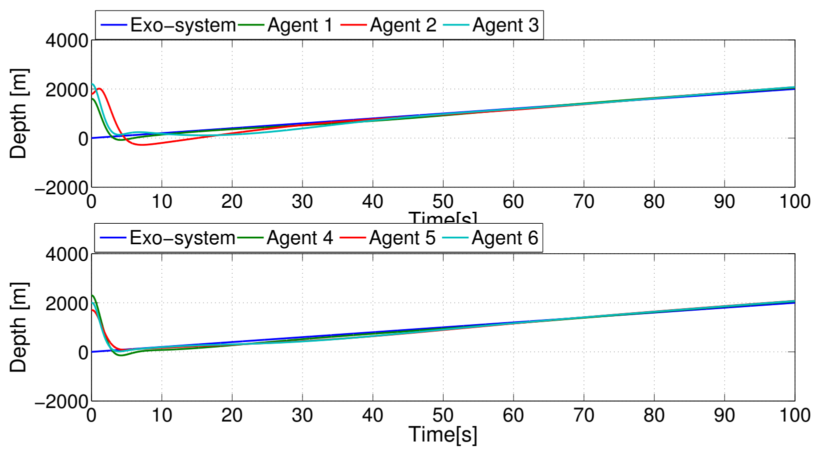

Figure 6.1: The agents depth trajectories corresponding to the Scenario A.

Figure 6.1 illustrates that when all the agents are healthy they can track the output trajectory of the exo-system with the least steady state errors. Figures 6.2 and 6.3 demonstrate that once the agents become faulty the team becomes unstable as confirmed by presence of large tracking error. Therefore, the agents no longer can follow the desired

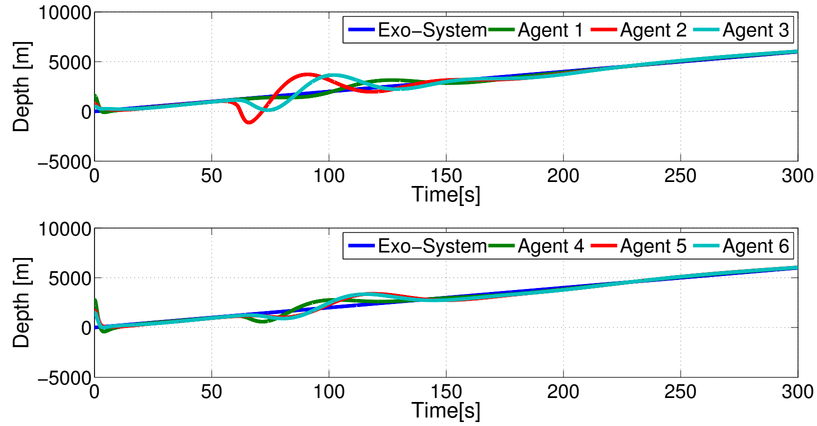

Figure 6.2: The agents depth trajectories corresponding to the Scenario B with $\Delta = 10$ sec.

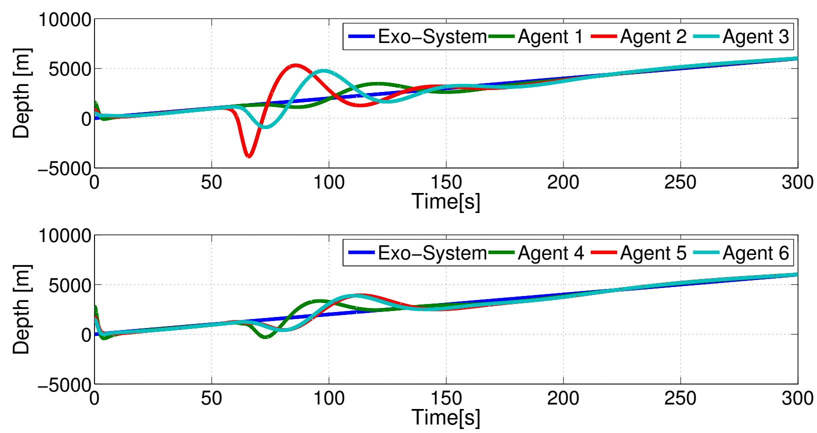

Figure 6.3: The agents depth trajectories corresponding to the Scenario C with $\Delta = 12$ sec.

Scenarios and Delays in Recovery Invocation	J
A	1.6449e+11
B; $\Delta = 3$	6.922e+11
B; $\Delta = 7$	2.678e+12
B; $\Delta = 10$	2.205e+13
C; $\Delta = 6$	14.702e+11
C; $\Delta = 12$	3.88e+13
C; $\Delta = 17$	7.838e+15
D; $\Delta = 2$	5.13e+14
D; $\Delta = 12$	5.22e+14
D; $\Delta = 20$	8.904e+15

Table 6.1: The indices corresponding to the network performance index under the Scenarios A-D, where Δ denotes the delay in invoking the reconfigured control gains.

trajectory of the exo-system. However when the reconfigured control laws are invoked the team becomes stable and subsequently the agents can manage to track the outputs of the exo-system with still quite acceptable steady state errors.

6.4 Conclusions

In this work, the distributed output control reconfigurable regulation problem in a network of heterogenous multi-agent systems subject to actuator faults is studied. Three types of actuator faults are considered and a distributed reconfigurable control strategy is proposed to ensure that the outputs of the agents follow the outputs of an exo-system. The proposed approach uses only local information consisting of output measurements as well as the fault detection and isolation information from the agents in the nearest neighbours. Furthermore, the upper bound of the team cost performance index under the proposed control law is obtained. Towards this end, the regulation problem is transformed into two stability problems which are solved in a distributed fashion. The proposed distributed control recovery approach is applied to a team of unmanned underwater vehicles and simulation results indicate the capabilities and effectiveness of the proposed approach in accommodating three different types of actuator faults simultaneously and subject to delays in invoking the control reconfiguration strategies.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this thesis the cooperative control reconfiguration problem in a team of autonomous multi-agents was considered. A team of linear time-invariant systems that is seeking consensus/output regulation while agents may be subject to three types of actuator faults was considered. The objectives were to develop fully distributed control reconfiguration strategies using only local available information to ensure that the team can cooperate among themselves and the specified team performance indices remain in an acceptable level. Towards this end, the control reconfiguration problem is transformed into stability problem and optimal control framework is employed to ensure the required team objectives.

In Chapter 3, cost-performance based control reconfiguration in a multi-agent systems under fixed, undirected network topology is considered. Distributed Hamilton-Jacobi-Bellman equations for the faulty agents are derived and reconfigured controllers are designed by solving these equations subject to the faulty agent dynamics and network structure constraints to ensure the fault the accommodation of the entire team. The proposed control scheme accommodates actuator faults in a single agent as well as concurrent actuator faults in the team and minimizes the faulty agent local cost performance index.

In Chapter 4, an H_∞ performance fault recovery control problem for a team of multi-agent systems under directed, fixed topology and subject to environmental disturbances is studied. The objectives were to design a distributed control reconfiguration strategy such that in presence of actuator fault the output of the faulty system behaves exactly the same as that of the healthy system, while the state consensus error remain bounded as well as minimizing the specified H_∞ performance bound in presence of bounded energy disturbances. The gains of the reconfigured control laws are selected first by employing a geometric control approach where a set of controllers guarantees that the output of the faulty agent imitates that of the healthy agent and the consensus achievement objectives are satisfied. Next, the remaining degrees of freedom in the selection of the control law gains are used to minimize the bound on a specified H_∞ performance index. The proposed control reconfiguration methodology accommodates single, concurrent as well as simultaneous actuators faults in the team.

In Chapter 5, control reconfiguration problem in multi-agent systems subject to directed switching topology networks and severity estimation uncertainties and unreliabilities are considered. It is assumed that each agent can observe the change in its neighbor agents and is unaware of the entire network changes. Control reconfiguration problem is transformed into two stability problems, in which one can be solved offline while the other should be solved online and based on the information each agent receives from fault detection and isolation module. Using quadratic and convex hull Lyapunov functions the control gains are developed such that the team consensus achievement is guaranteed while the upper bound of the team cost performance index is minimized. The proposed approach accommodates single, concurrent as well as simultaneous actuators faults in the team.

In chapter 6, output regulation problem in a network of non-identical vehicles is considered. Three types of actuator faults are considered and a distributed strategy is proposed to ensure output of the agents follow the outputs of an exo-system whereas the agents states remain bounded. The proposed approach uses local information including output measurements as well as the fault detection and isolation information of the agents in the agents neighbourhood. Furthermore, the upper bound of the team cost performance index under the proposed control law is obtained. First, distributed output regulation problem is transformed into two stability problems. Then using Lyapunov stability theorems and Gershgorin circle theorem theorems, these problems are solved. The proposed approach accommodates single, concurrent as well as simultaneous actuators faults in the team.

In conclusion, in this thesis a framework for developing distributed reconfiguration control strategies in multi-agent systems subject to three types of actuator faults are proposed. The proposed methodologies are applied to a team of unmanned underwater vehicles and the effectiveness of the proposed approaches in compensating for actuator faults are validated through simulating many fault scenarios.

7.2 Suggestions for Future Work

Some of the future extensions of the present research are as follows:

- Developing a reconfiguration strategy for the team that is subject to delays in sharing information.
- Extending the proposed framework for deterministic switching topologies to the stochastic switching topologies network.
- Developing an optimal, computationally effective approach to improve the overall team performance and investigate the optimality gap of using the proposed sub-optimal solutions.
- Extending the proposed methodology for heterogenous agents for the team with switching topology and in presence of environmental disturbances and uncertainties in FDI estimations.
- Extending the proposed framework for accommodating outage fault in a team of

linear time-invariant systems to a team of nonlinear agents .

- Combining the proposed result for H_∞ and cost-based control reconfiguration to ensure that both performances remain in an acceptable level.

- Investigating the transient behaviour and state constraints in development of the reconfigured control laws, as agents are subject to physical limitations in values that their states can tolerate. Also, considering the limitations of the control efforts that can be applied to each agent.

- Providing a priority decision making rule for reconfiguring faults in agents that depends on the effect of the actuator in the system stability and performance. This is due to the fact that different actuators may have different effects on the system stability and performance and reconfiguring any fault without explicitly taking into account this aspect can be inefficient to the entire team performance.

- Extending of the proposed strategies for the multi-agent systems that are subject to simultaneous actuator and sensor faults.

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