Sources of Mature Students' Difficulties in Solving Different Types of Word Problems in Mathematics

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Abstract

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Maria-Josée Bran Lopez

There are many different types of research done on algebra learning. In particular, word problems have been used to analyze students' thought process and to identify difficulties in algebraic thinking. In this thesis, we show the importance of quantitative reasoning in problem solving. We gave 14 mature students, who were re-taking an introductory course on algebra, four word problems of different types to solve: a connected problem, a disconnected problem, a problem with contradictory data and a problem where students were asked to assess the correctness of a fictional solution. In selecting these types of problems we have drawn on the research of Sylvine Schmidt and Nadine Bednarz on the difficulties of passing from arithmetic to algebra in mathematical problem solving. We present the students' solutions and a detailed analysis of these solutions, seeking to identify the sources of the difficulty these students had in producing correct solutions. We sought these sources in the defects of quantitative reasoning, arithmetic mistakes, and algebraic mistakes. The attention to quantitative reasoning was inspired by the research of Pat Thompson and Stacey Brown. Defects of quantitative reasoning appeared to be an important reason why the students massively failed to solve the problems correctly, more important than their lack of technical algebraic skills. Therefore, teaching procedures and algebraic technical skills is not enough for students to develop problem solving skills. There should be a focus on developing students' quantitative reasoning. Students need to have a good understanding of relations between quantities. Defects of quantitative reasoning create obstacles that prevent mature students from successfully solving any type of word problem.

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1 INTRODUCTION AND RATIONALE

The sense of meaningfulness comes with the ability of 'seeing' abstract ideas hidden behind the symbols. (Sfard & Linchevski, 1994, p. 224)

It is not a secret that many students struggle with algebra. Booker, Windsor (2010) agree that "for many students, the development of algebra in high school has often marked the end of enjoyment in mathematics and the onset of a feeling of mathematical inadequacy" (Booker, Windsor, 2010, p.412). This has motivated many researchers to study 'school algebra' in different ways and to try to develop new approaches to make learning algebra meaningful. One question to ask is *what makes algebra learning so difficult?* In order to answer this question, we need to understand students' reasoning.

Although there are many studies on how to improve algebra education, in this thesis, we are mainly interested in students' approaches to solve word problems. More specifically, we are concerned about the sources of difficulty that mature students (21 years or older) returning to university have to face and overcome when they are required to retake algebra. Mature students have complex backgrounds, and when asked to retake an algebra course, they bring all the misconceptions and obstacles they have developed in their previous studies. As instructors and researchers, it is important to identify those obstacles that block students from learning a new way of thinking in and about algebra. While there are multiple ways to analyze the difficulties that come with algebra learning, we are not interested in the transition from arithmetic to algebra, nor the complexity of algebra notation. We believe that quantitative reasoning (Thompson, 1993; Thompson & Saldanha, 2003; Brown, 2012) is the key to solving any word problem, even with arithmetic thinking.

1.1 RESEARCH QUESTION

In this study, we try to identify and understand the sources of the difficulties that students are facing when solving four types of word problems in an elementary algebra class: a connected word problem, a disconnected word problem (Schmidt & Bednarz, 2002), a problem with contradictory data, and a problem of finding the flaw in a fictional solution of a word problem. We seek the sources of the difficulties in:

- the type of the problem the student has to solve;
- defects of quantitative reasoning;
- arithmetic mistakes;
- algebraic mistakes, and
- epistemological obstacles (Sierpinska, 1990) related to algebra.

1.2 METHOD

We recruited 14 participants from an elementary level algebra course for mature students offered in a large, urban, North American university. We selected four word problems, each of a different type, for the participants to solve individually. Some of the participants were then interviewed on their solutions. The solutions were then analyzed, with a focus on identifying the sources of difficulty listed in the previous question. Specific manifestations of these sources of difficulty were identified and coded with easy to remember short names. Simple counting of frequencies was used to decide on the importance of a source of difficulty.

1.3 RESULTS

We conclude that all the mentioned sources of difficulty have a role in students' difficulties, but the most important part seems to be played by important defects of quantitative reasoning, some of which are related with specific epistemological obstacles related to algebra.

1.4 OVERVIEW OF THE THESIS

The thesis is composed of 6 chapters.

The first chapter is the present introduction. In the second chapter, we review selected literature on the nature of algebra, on the obstacles that were encountered and overcome in its historical development, on approaches to teaching algebra, and the difficulties in learning the subject.

In Chapter 3, we discuss the conceptual framework used for this study. In particular, we identify different types of word problems, different types of reasoning, and obstacles related to quantitative reasoning, arithmetic skills and algebraic skills.

In Chapter 4, the methodology, the research procedures, and the research instrument are presented.

Chapter 5 contains the results of our analysis of the data, and the conclusions that could be drawn from them regarding our research question.

Finally, Chapter 6 contains the summary and conclusion of this study. We also make some recommendations for teaching algebra to mature students and for future research.

2 REVIEW OF LITERATURE ON ALGEBRA TEACHING AND LEARNING

In this literature review, we will outline what other studies suggest about algebra teaching, students' difficulties and different types of word problems. However, it is important to first define algebra.

2.1 WHAT IS ALGEBRA?

Culturally, algebra has always been linked to variables. Usiskin (1988) mentions that "algebra starts as the art of manipulating sums, products, and powers of numbers… [and] school algebra has to do with the understanding of letters" (p.7). But using letters in solving a problem is not enough to make the solution algebraic. In algebra, letters are used in expressions that represent relations between known and unknown quantities and the manipulation of these expressions according to certain stable rules produces information (e.g., about the values of the unknowns) that were not obvious at the start. This property is referred to as "operational symbolism": in algebra, letters are not just shorthand for objects, they are part of an operational symbolism. It is the first characteristic of algebra in a definition found in the works of the historian Michael Mahoney. According to this author, three elements need to be present in algebraic thinking:

1. Operational Symbolism.
2. The preoccupation with mathematical relations rather than with mathematical objects, which relations determine the structures constituting the subject-matter of modern algebra. The algebraic mode of thinking is based, then, on relational rather than on predicate logic.
3. Freedom from any ontological questions and commitments and, connected with this, abstractness rather than intuitiveness.

(a quote from Mahoney, in Charbonneau, 1996, p. 15)

Usiskin identifies four conceptions of algebra in teaching: algebra as generalized arithmetic, as a means to solve certain problems, as the study of relationships among quantities, and as the study of structures. However, according to Sfard (1995), algebra is related to "any kind of mathematical endeavor concerned with generalized computational processes, whatever the tools used to convey this generality" (p.18). In other words, algebra is more than just symbols. It is also a way of thinking. Drijvers, Goddijn, & Kindt (2010) suggest that there is no exact definition of what algebra is in general. Some definitions are based on the historical context. The word *algebra* comes from the Arabic word *al-jabr* which Al-Khwarizmi, the author of *Hisab al-jabr w'al-muqabala,* defined as eliminating subtractions. Other definitions are associated with abstract algebra (Drijvers, Goddijn & Kindt, 2010). However, in our context, "algebra at school is strongly associated with verbs such as solve, manipulate, generalize, formalize, structure and abstract" (Drijvers, Goddijn & Kindt, 2010, p.8).

2.2 WHY TEACH ALGEBRA?

Consequently, there are multiple views on *why* to teach algebra. For our society, it is assumed that every student should learn algebra. Drijvers, Goddijn & Kindt (2010) mention how algebra is not only taught to students for computational skills, but also for "the development of strategic problem solving and reasoning skills, symbol sense and flexibility, rather than [just] procedural fluency" (p.5). Nevertheless, there is still work to be done in algebra teaching. Brown (2012), Doorman & Drijvers (2010), Sajka (2003), and Schmittau & Morris (2004) agree that algebra teaching has to be improved. "In the future, there will be a greater need for "flexible analytical reasoning skills, rather than for procedural skills. Consequently, algebra education should change its goals; it should focus on new epistemologies and aim at new types of understanding" (Drijvers, Goddijn & Kindt, 2010, p.5). Sajka (2003) argues that it is important to focus on the student's process to find a solution rather than on the ability to solve a problem. Schmittau & Morris (2004) mention the importance of using problems "that require [students] to go beyond prior methods, or challenge them to look at prior methods in altogether new ways, in order to attain a complete theoretical understanding of concepts" (p.62). Since algebra is more than just its notation, all the articles of this review suggest a shift of priorities from a focus on the correctness of solutions to a focus on students' reasoning.

2.3 WHY IS LEARNING ALGEBRA DIFFICULT AND WHAT TO DO ABOUT IT?

Several studies have been done on students' difficulties related to algebra. Drijvers, Goddijn & Kindt (2010) mention three main obstacles: the general abstraction of algebra, generalization and overgeneralization, and the variable as a process and as an object. In addition, Sajka (2003) observes that many students have difficulties understanding the task they are given. Such difficulties are created by either the intrinsic ambiguities of mathematical notation, the students' own misinterpretations, or "the restricted context in which some symbols occur in teaching and a limited choice of mathematical tasks at school" (Sajka, 2003, p.229). On the other hand, Schmidt & Bednarz (2002) mention the difficult transition from arithmetic to algebra, and how the students do not see the linkage between arithmetic and algebra. The major difficulties in this transition are: the "fundamental changes involving the very nature of the type of reasoning to be employed; […] the different relationships involving symbolic writing; and […] the kind of control performed in each of the two areas of knowledge" (Schmidt & Bednarz, 2002, p.269).

Beyond the obstacles that algebra learning has to overcome, several studies focus on the actual teaching of algebra in order to help students to have a deeper understanding of the mathematical concepts. There are multiple approaches to algebra mentioned in these studies. One of these approaches is from Bednarz Kieran & Lee (1996), which involves generalization, problem solving, and modeling and functions. The approach of Usiskin (1988), and Drijvers, Goddijn & Kindt (2010) involves defining algebra by analyzing all its different components and the role of variables. Moreover, Usiskin's (1988) main concern is to know "the extent to which students should be required to be able to do various manipulative skills" (p.8). On the other hand, Doorman & Drijvers (2010), and Sajka (2003) suggest that a functional approach can provide opportunities for algebraic activity. "The functional view is connected to the patterns and formulas and restriction stands. Even if algebraic expressions and formulas are important ways to represent functions, the function perspective is different because of its dynamic dependency perspective and its representational tools" (Doorman & Drijvers, 2010, p.126). Sajka (2003) suggests that using functions, instead of standard procedures, might help teachers to identity students, even the ones with good grades, who lack a complete understanding of the concepts. Conversely, Schmidt & Bednarz (2002), and Schmittau & Morris (2004) focus on the types of problems used in class. Schmidt & Bednarz (2002) mention 4 types of problems: connected problems, disconnected problems, problems with a contradiction, and problems that require the analysis of an incorrect solution. Their idea is that these types of problems, especially the connected and disconnected problems, can help teachers to identify students with arithmetical reasoning or algebraic reasoning. Schmittau & Morris (2004) use the Davydov's curriculum to provide children early algebra experiences in order to develop theoretical thinking and prepare them to give meaning to algebraic concepts. Finally, Brown (2012) suggests in her project that students can experience, even in an arithmetic context, three forms of early algebraic thinking: relational thinking, functional thinking, and advanced mathematical thinking. Brown's idea is that no matter the type of problem students are asked to do, there is always a way to turn their task into an opportunity for them to generalize, analyze, explain their reasoning, and to learn with understanding.

Another view on the reasons why learning algebra is difficult is the historical - epistemological perspective: if geometry and arithmetic were developed already in the Antiquity but it has taken mathematicians many centuries to develop algebra as we know it today, then there must have been some important conceptual ("epistemological") obstacles that these mathematicians had to overcome. We deal with these obstacles in section 2.7.

2.4 FOCUS ON ALGEBRAIC THINKING RATHER THAN ON CORRECT APPLICATION OF ALGEBRAIC PROCEDURES

Many researchers believe that focusing on algebraic thinking is the key to help students move away from applying procedures to understanding the concepts. Windsor (2010) defines algebraic thinking as "a perspective that values, enriches and improves the thinking required to understand algebraic concepts" (p.665). In addition, it "is a crucial and fundamental element of mathematical thinking and reasoning" (Windsor, 2010, p.665). Norton & Windsor (2012) mention that "algebraic thinking is the activity of doing, thinking and talking about mathematics from a generalized and relational perspective", and that facilitates solving more complex problems. There are many benefits to focusing on algebraic thinking; Booker & Windsor (2010) suggest that it can help students have a flexible mind to interpret problems, give them a better understanding of generalization, and allow them to see the meaningful use of symbolism. Moreover, "the benefits of developing students' algebraic thinking can offer students a more meaningful conceptualization of algebra beyond the mechanics and procedures often associated with algebra" (Booker & Windsor, 2010, p. 419).

2.5 FOCUS ON QUANTITATIVE REASONING

Other research has been done on expanding the focus of computational skills. Instead of changing the curriculum goals, researchers have thought to provide teachers with opportunities to "support [students'] work towards understanding and explaining their own and others' approaches to arithmetic tasks" (Brown, 2012, p.28). Brown wants teachers to be able to distinguish numeric reasoning from quantitative reasoning. She uses the definition of quantitative reasoning from (Thompson, 1993):

> Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships…. A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities. In that regard, quantitative reasoning bears a strong resemblance to the kind of reasoning customarily emphasized in algebra instruction. (Thompson, 1993, p. 165)

For Brown, quantitative reasoning doesn't necessarily imply using variables, but it does involve relational thinking (Brown, 2012, p. 21). An example is given in Table 1.

	Response	Analysis
Child A	"... cause 3 and 4 makes 7 and 2 and 5 makes 7. So, it's true because they're both 7" (p.21)	Child A's answer is only based on computed quantities.
Child B	"So ... umm, they're the same because if you take 1 from the 3 and add it to the 4 it makes 5" (p.21)	Child B goes beyond the computation. There is an understanding of arithmetic properties. Also, Child B focuses on transforming one expression into the other, which is algebraic thinking.

Table 1: Relational vs. Numeric Thinking: Two responses to the question: ***3 + 4 = 2 + 5 True or False?*** *(Brown, 2012)*

The goal is for students to focus on the relationships and to engage in quantitative reasoning when solving a problem. This prepares them to later learn algebra and use letters as part of operational symbolism rather than shorthand.

2.6 WORD PROBLEMS AS BOTH A DIAGNOSTIC AND DIDACTIC TOOL

Many researchers use word problems to analyze students' solutions and observe algebraic thinking. If the goal is to analyze students' reasoning and understanding of the concepts, then there is a need for a greater attention on the environment they learn in. There is agreement that teachers have a big influence on students and it is important to observe their view on arithmetic and algebra, and the types of problems they give to students. As a result, Schmidt & Bednarz's goal (2002) is to identify word problems that would help teachers get an insight into students' types of reasoning. “There is a distinction between algebraic and arithmetical types of reasoning” (Schmidt & Bednarz, 2002, p.69). Bednarz and Janvier’s (1996), as mentioned in Schmidt & Bednarz (2002), describe two types of problems (see Table 2).

Connected Problem	Disconnected Problem
A problem where “a relationship can be easily established between two known quantities, thus leading to the possibility of arithmetical reasoning (from the known quantities to the unknown quantity at the end of the process)” (Bednarz & Janvier, 1996, p. 123)	A problem where “no direct bridging can be established between the known quantities” (Bednarz & Janvier, 1996, p. 123)

Table 2: Connected and Disconnected Problems

<u>Example of a Connected Problem</u> with a diagram that explains the connection between known quantities and unknown quantities (Schmidt & Bednarz, 2002, p. 85) is given in Figure 1.

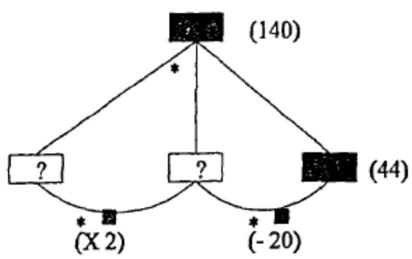

Figure 5: ‘The Biology Class’ (connected problem).
There are 140 students in the biology class. Some students are in the laboratory, others are watching a film, and still others are doing research in the library. There are twice as many students in the film room as in the laboratory, and there are 20 fewer students in the library than in the film room. If there are 44 students in the library, how many are in the laboratory and how many are in the film room?
Two filled-in spaces in a row (known quantities) facilitate using an arithmetical approach

Figure 1: Example of Connected Problem

<u>Example of a Disconnected Problem</u> with a diagram that explains the connection between known quantities and unknown quantities (Schmidt & Bednarz, 2002, p. 84) is given in Figure 2.

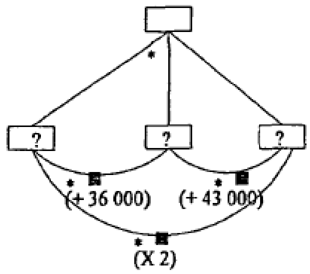

Figure 4: 'Arsène Ponton' (disconnected problem)
Arsène Ponton leaves his fortune to his three nieces, Marie, Chantal, and Sophie. He gives Marie twice as much money as Chantal, Sophie $36,000 more than Chantal, and, finally, Marie $43,000 more than Sophie. How much money do Marie, Chantal, and Sophie actually receive?
No bridge can be established between the two known quantities

Figure 2: Example of Disconnected Problem

A solution was considered 'arithmetical' if "the participant was found to have used a synthetic type of solution, or consistently used known numbers to perform a series of operations that he or she considered necessary" (Schmidt & Bednarz, 2002, p. 70). And a solution was considered 'algebraic' if "the participant adopted an analytical approach, wherein his or her solution was centered on an unknown number that was temporarily replaced by some notational figure (a letter or a word)" (Schmidt & Bednarz, 2002, p. 70). Based on their research, arithmetical-type students were able to solve connected problems but had a difficult time solving disconnected problems. On the other hand, algebraic-type students had no difficulty solving both types of problems.

Schmidt & Bednarz (2002) also used two other types of problems: a word problem with erroneous relationships between quantities, and a word problem where the task is to analyze an incorrect solution. The goal of giving the word problem with the erroneous relationships was to 'break' the numeric progression and observe if students noticed a contradiction in their solution. Unlike the arithmetic-type student, the algebraic-type student showed a grasp of the relationships and opted for an overall analysis. As for the problem with the wrong solution, it made possible for the researchers “to bring out how [students] controlled an algebraic treatment which requires detachment from quantities” (Schmidt & Bednarz, 2002, p. 72).

There is a clear gap between arithmetic reasoning and algebraic thinking, and the problems used by Schmidt & Bednarz (2002) can help teachers identity the type of reasoning of each student based on their procedures. Some questions arise after their research such as:

> Do students still prefer the trial-and-error method, or do they see the utility of generalization?
> Are they able to detach themselves from the context?
> Do they see algebra as a powerful tool to represent relationships between quantities? (Schmidt & Bednarz, 2002)

Without any doubt, it is important to know the kind of relationship that students have with algebra.

Schmittau & Morris (2004) focus on problems that have not been “broken down into steps for the children” (p.62) and where no hints were given. Their objective is “the development of the ability to think theoretically, which then enables […] an understanding of mathematics concepts at their most abstract and generalized level” (Schmittau & Morris, 2004, p.61). Similarly to Brown (2012), these authors agree with the importance of relationships between quantities, and that “cognitive development occurs when one is confronted with a problem for which previous methods of solution are inadequate” (Schmittau & Morris, 2004, p.62). Because their approach is based on Davydov’s curriculum, their goal is also to improve students reasoning and help them go beyond numeric reasoning.

2.7 EPISTEMOLOGICAL OBSTACLES RELATED TO ALGEBRA

According to the historians Bashmakova & Smirnova (2000), the historical evolution of algebra went through four stages:

- Numerical algebra of ancient Babylonia

- Geometric algebra of classical antiquity (5th- 1st century BCE)
- The rise of literal algebra (1st CE - end of 16th century)
- Creation of the theory of algebraic equations (17th-18th century)
- Formation of the foundations of modern algebra (1830s - 1930s)

The second stage mentioned above goes counter the belief in mathematics education that algebra is a kind of extension of arithmetic – a generalized arithmetic. Although it is true that it is part of the evolution of arithmetic, algebra is also closely related to geometry.

> Geometric analysis, as well as the theory of proportions, played an important role in the development of algebra in the Renaissance. Until Viète's algebraic revolution at the end of the 16th century, geometry was a means to prove algebraic rules, and, likewise, algebra was a means to solve some geometrical problems. (Charbonneau, 1996, p.15)

But "Viète's revolution" required that mathematicians detach their thinking from the geometric meanings of quantities. These meanings were limiting the development of an abstract theory of algebra, with its own language (the "operational symbolism" mentioned in section 2.1) and laws independent from the "ontology" of geometric meanings, which were becoming an "obstacle."

2.7.1 The notion of epistemological obstacle

Throughout history, we see a constant change in reasoning and methods used. Those changes were triggered when mathematicians were becoming aware of obstacles – limitations in their ways of thinking – when they wanted to solve new problems. In order to understand the difficulties related to algebraic thinking and learning, it is useful to identify the obstacles mathematicians in the past had to overcome to develop our modern algebra. Those historical obstacles are called "epistemological obstacles", as opposed to "cognitive obstacles" that are caused by the limitations of the human brain, and to "didactic obstacles" that result from the way mathematics is taught in school (Brousseau, 1997). Some epistemological obstacles related to mathematics survive in the common culture although they are overcome in research mathematicians' thinking. For example, when we say, in ordinary language, that something has a "limit", we exclude the possibility of this something being infinite; but in mathematics, some infinite sequences have limits. The belief that "limit" and "infinite" are contradictory is an epistemological obstacle that mathematicians overcame when developing Calculus, but this obstacle is still present in students today. But not all epistemological obstacles survive; some are totally forgotten, some are replaced by opposite beliefs and habits of thinking.

According to Sierpinska (1990; 1994), the notions of understanding and overcoming epistemological obstacles are closely linked. Understanding is viewed in a positive way since it "looks forward to the new ways of knowing" (Sierpinska, 1990, p. 28). On the other hand, epistemological obstacles are often seen as a negative aspect of learning since it focuses on what is "wrong, insufficient, in our ways of knowing" (Sierpinska, 1990, p. 28). Either point of view indicates that when we realize that our knowledge is not enough or that our methods are incorrect, we are facing an obstacle. Overcoming those obstacles lead to a better understanding of mathematical concepts and we start to think in a different way.

An epistemological obstacle indicates a way of knowing that is valid but in a limited area. Back in ancient Babylonia, mathematicians did not only lack knowledge; they had a different way of thinking. Until they faced an obstacle and reviewed their mathematical system, they didn't see the need to develop new concepts.

It is important to note that "all our understanding is based on our previous beliefs, prejudgments, preconceptions, convictions, unconscious schemes of thought" (Sierpinska, 1990, p. 28). They are the material for epistemological obstacles. Thus, there is no way of escaping them in learning something new.

2.7.2 Epistemological obstacles in the historical development of algebra

In this review we will mention only two obstacles related to algebra. We will call them: Quantitative obstacle, and Ontological obstacle. Their identification was inspired by Charbonneau's account of the historical development of algebra (Charbonneau, 1996).

2.7.2.1 Quantitative Obstacle (QO) – The measure of a quantity is not abstracted from the quantity as an object

Algebra was "based on the measure of geometrical magnitudes and relations between these measures" (Charbonneau, 1996, p. 16). Geometry was very present in ancient Greek mathematics. Charbonneau (1996) suggests that the reason geometry was used by algebraists was to "demonstrate the accuracy of rules otherwise given as numerical algorithms" (p. 26) and because "geometry was one way to represent general reasoning without involving specific magnitudes" (p. 26). Drawings were used to solve problems. However, this implied that numbers had geometrical meaning. Letters were used to represent lines, which had a certain shape and length. Whether it was the shape that a Proposition referred to or the length depended on the context in which the word "line" was used. Length as a measure expressed by a number was not abstracted from the geometric object "line."

> "When a new magnitude [came] from an operation on two magnitudes, the new magnitude [had] a meaning only in relation with those from which it [came]" (Charbonneau, 1996, p. 19).

Letters or symbols did not have any meaning on their own; operations such as addition and multiplication still carried the original reference of those symbols to geometrical objects. As an example, what, today, we call the product of two numbers, would be called a "rectangle", referring to both the shape and the area of a rectangle made from two segments with numbers as lengths (the Greeks did not have a special symbol for multiplication; they used words to speak about operations).

An example of the functioning of this obstacle can be gleaned from Euclid's *Elements*, Proposition II.14:

To construct a square equal to a given rectilinear figure.

In our modern notation, we can rewrite this proposition as $x^2 = A$, where x represents the length of the side of the square and A is the area of the rectilinear figure; for us, this is an equality of two numbers. We can solve the problem by simply taking the square root of the number A. But finding the length of the side was not sufficient for Euclid. It is a figure he was looking for: a square. The length of its side was just one aspect of this figure. His problem was to construct this figure, using a straightedge and a compass. He needed to construct the side of a square with the same area as the area of a given rectangle. If Euclid had the modern algebraic notation, he would represent the problem in the Proposition II.14 not as $x^2 = A$ but as $xx = ab$ or, more likely, as the proportion $\frac{x}{a} = \frac{b}{x}$ where all the variables represent lengths of segments, a and b are assumed already constructed, and x remains to be constructed. The geometric reference of the variables is never ignored.

From the perspective of this obstacle, since any problem had to be represented with a figure, any relation or equation with dimensions higher than 3 made no sense. Also expressions such as $x^2 + x$ did not make sense because it did not make sense to add a square to its side – the result was not a known geometric figure.

Dividing an area by a length – taking their ratio – was also inconceivable in Greek geometry. It is inconceivable because of the restriction expressed in Definition V.3 – "A ratio is a sort of relation in respect of size between two magnitudes of the same kind." So ratios could only be taken between quantities of the same kind: squares to squares, circles to circles, rectilinear figures to rectilinear figures, etc. When using proportions, the length of a segment could only be compared to another length, not an area. In Euclid's geometry, the ratio of the circumference of a circle (a curved line) to its diameter (a straight line) is inconceivable. Similarly, the ratio of the area of a circle to the area of the square built on its diameter is inconceivable. This excludes the number π from Euclid's geometry. The idea of constancy of the ratio of the area of a circle to the square of its diameter is expressed, in Euclid, in a roundabout way, in Proposition XII.3: "Circles are to one another as the squares on their diameters." This keeps magnitudes of the same kind together in the ratios equated in the proportion.

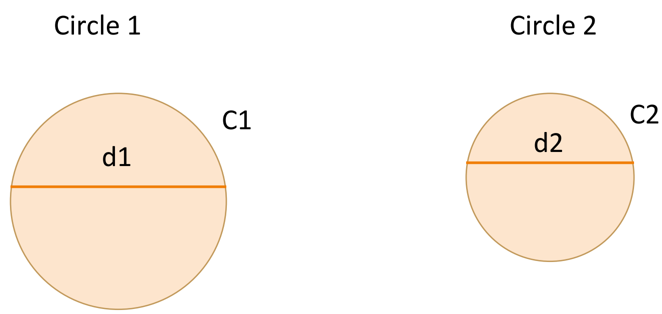

Figure 3: Comparison of two circles

This way of thinking was a serious obstacle to the development of Calculus because it hindered the notion of velocity and, generally, rate of change, which is the basis of the concept of derivative. If one wanted to remain faithful to Euclid's notion of ratio and respected the rule of not mixing quantities of different kinds in a single ratio, then, rather than speaking directly about two bodies moving with the same velocity (the ratio of distance to time), one would have to say that the ratio of the distances the bodies covered was the same as the ratio of the times they covered them.

2.7.2.2 Ontological Obstacle (OO) – The reference of variables to quantities they represent is carried along in solving a problem

This obstacle hinders the development of an operational symbolism, since not all operations on the symbols can be readily interpreted as actions on the objects represented by the variables.

Although geometry helped the development of some algebraic properties, there were always ontological restrictions that stopped more concepts to be explored. In this geometric context, mathematicians had a constant need to make connections between the operations they performed and their geometric reference. Even though ancient Greek symbolism was very different from ours, they had a complete notation, and they focused on the relations between quantities. Nevertheless, the level of abstractness was still low because of the ontological attachment they gave to objects. On the other hand, however, the obstacle of ontology sustained the development of quantitative reasoning, which, as many mathematics educators realize today, is crucial in supporting students' ability to solve more complex word problems and their transition from arithmetic to algebra.

3 CONCEPTUAL FRAMEWORK

3.1 INTRODUCTION

According to previous studies and results, algebraic thinking is an important element of teaching algebra. From using functions to solving word problems, all the articles reviewed in the previous chapter pointed towards analyzing students' reasoning. As we explained in the introduction, this research was motivated by an interest in understanding how students solve word problems in order to find new ways of helping them. In analyzing students' solutions, we used several concepts to identify and name different aspects of their thinking. These concepts were drawn from different research works and not from a single theory. It is therefore "a conceptual framework" (Eisenhart, 1991) that we are using, rather than a theoretical framework.

In our study, we used the types of word problems described by Schmidt & Bednarz (2002), and, in fact, their examples of these types of problems as our research instrument. In our analysis of participants' solutions we used the same authors' notions of algebraic and arithmetic solutions as well as the types of letter use identified in (Küchemann, 1981), and the idea of quantitative reasoning mentioned by Brown (Brown, 2012).

Brown (2012) emphasizes the importance of quantitative reasoning in word problems, and Schmidt & Bednarz (2002) focus on algebraic thinking using disconnected problems. The close link between both studies is the focus on relations between quantities. Solving a word problem, whether arithmetic or algebraic, requires an emphasis on the relationships given in the problem. This is why a well-developed quantitative reasoning is essential for solving algebraic problems. Defects of quantitative reasoning, on the other hand, can be associated with the epistemological obstacles related to algebra.

The four types of problems mentioned in Schmidt & Bednarz (2002) can be used to indicate algebraic thinking. More specifically, disconnected problems can be used to encourage algebraic thinking, since arithmetical methods are not useful in solving these problems. On the other hand, Brown's (2012) project focuses on the reasoning behind the procedures used by students.

3.2 TYPES OF WORD PROBLEMS

Based on the research done by Bednarz & Janvier (1996) and Schmidt & Bednarz (2002), we chose four types of word problems to reveal participant's difficulties in problem solving.

3.2.1 Type 1: Connected Problem

Schmidt & Bednarz (2002) described a connected problem as an arithmetic problem. They suggest that "a relationship can be easily established between [the] known quantities, thus leading to the possibility of arithmetical reasoning (from the known quantities to the unknown quantity at the end of the process)" (Bednarz & Janvier, 1996, p. 123).

3.2.2 Type 2: Disconnected Problem

A disconnected problem is more related to algebra (Schmidt & Bednarz, 2002). They suggest in this type of problem "no direct bridging can be established between the known quantities" (Bednarz & Janvier, 1996, p. 123).

3.2.3 Type 3: Problem with a contradiction

This type of problem is like any word problem given to students who are learning algebra. However, an erroneous relationship is added between the different objects. In this case, the calculations should reveal that there is a contradiction in the given relations.

3.2.4 Type 4: Analysis of a problem with an incorrect solution

For this type of problem, the students are given a typical word problem, but this time, the solution of an imaginary student is also given. They are asked to analyze the solution and determine whether the solution is correct or incorrect, and to justify their answer. “Students' analysis of this problem [makes] it possible to bring out how they controlled an algebraic treatment which requires detachment from quantities” (Schmidt & Bednarz, 2002, p.72).

Each word problem was chosen specifically to identify the thought processes of the participants. As mentioned previously, connected problems and disconnected problems not only show the type of approach students are more inclined to use, but also reveal how well students are at expressing relations given in the problem.

The word problem with a contradiction is a problem that contains many relations. Since it contains a contradiction, it allows us to identify students who focus on the given relations and make sure their final answer satisfies all the relations given in the problem. A problem with an incorrect solution is a type of problem the students are not used to solve in test or assignments. Analyzing a solution allows them to choose their own approach. It reveals the importance they give to the relationships described in the problem and in the solution. Both of these problems are more centered on quantitative reasoning than algebraic thinking as such. Each of these word problems is unique. Students have to face problems that take them away from a memorized method and encourage them to understand the problem and focus on relations.

3.3 Concepts used to characterize participants’ solutions

In order to characterize each participant’s solution, we classified their use of letters according to the types identified in (Küchemann, 1981), decided if their solution was arithmetic, algebraic or neither based mainly on (Schmidt & Bednarz, 2002), and sought to identify the defects in their quantitative reasoning (Thompson, 1993; Brown, 2012; Thompson & Saldanha, 2003), and in their arithmetic and algebraic skills. We used the notion of epistemological obstacle (Sierpinska A. , 1990) to explain some of those defects in our discussion of the results of our analyses.

3.3.1 Ways of understanding letters in mathematics

To a certain extent, we took into account the way the solver used letter symbols when classifying whether a solution was arithmetic or algebraic. Küchemann (1981) identified 6 types of letter use, of which only the last three treat letters as part of an operational symbolism (Charbonneau, 1996).

> **Letter evaluated** – This category applies to responses where the letter is assigned a numerical value from the outset.
>
> **Letter not used** – Here the children ignore the letter, or at best acknowledge its existence but without giving it a meaning.
>
> **Letter used as an object** – The letter is regarded as a shorthand for an object or as an object in its own right.
>
> **Letter used as a specific unknown** – Children regard the letter as a specific but unknown number, and can operate upon it directly.
>
> **Letter used as a generalized number** – The letter is seen as representing, or at least as being able to take, several values rather than just on.
>
> **Letter used as a variable** – The letter is used a representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.
>
> (Küchemann, 1981, p. 104)

3.3.2 Quantitative Reasoning vs. Numerical Reasoning

In order to identify the types of reasoning of the participants, it is essential to differentiate quantitative reasoning from numerical reasoning. According to Thompson (1993) "a prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities" (p.165). On the other hand, numerical reasoning focuses mainly on the numbers given in a problem. One example of these types of reasoning is given by Thompson (1993) when he mentions that quantitative difference and numerical difference are not synonyms. Numerical difference refers to the numerical result of subtraction. Quantitative difference of two quantities is their comparison by “the amount by which one quantity exceeds the other” (Thompson, 1993, 166).

Another example is given by the distinction between numerical equations and quantity equations. Thompson & Saldanha (2003) gave the following problem to students:

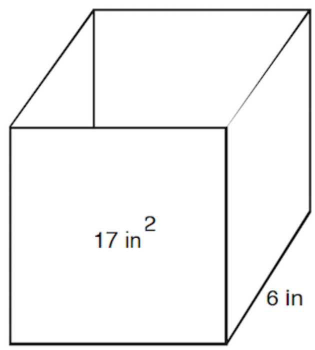

Figure 4: What is the volume of this box?

One student asked for more information to find the volume. He needed the measure of the other two sides to calculate the volume of the box. When asked if he could use 17 in^2 to find the answer, he responded: “No. It’s just the area of that face” (Thompson & Saldanha, 2003, p. 18). This student saw the volume formula as a numerical formula because the numbers “had no relation to evaluating quantities’ magnitudes” (Thompson & Saldanha, 2003, p.18).

Another student saw the problem as partly done for him. He mentioned that the last step was to multiply 17 by 6. He knew that he did not need all the dimensions to solve the problem. This student saw the volume formula as a quantity formula. “To him, [the formula of the volume] was: V = [LW][D], where [LW] produced an area, and [LW][D] produced the volume” (Thompson & Saldanha, 2003, p.18). Clearly, “quantity equations suggest a quantity’s construction” (Thompson & Saldanha, 2003, p.17).

Thus, Quantitative reasoning requires an analysis in terms of relationships and quantities independently from the numerical values (Thompson, 1993).

3.3.3 Arithmetic vs. Algebraic reasoning in problem solving

A further distinction to establish for this study is the important difference between arithmetic and algebraic reasoning. Charbonneau (1996) mentions characteristics of algebra in order to evaluate an algebraic way of thinking. He clarifies that algebra is not just an extension of arithmetic; it is a way of manipulate relations. Moreover, algebra is not only a question of symbolism. Although symbolism is central to algebra and it is used as a language, it is also used to name "something that has no name" (Charbonneau, 1996, p. 35). Symbolism on its own has no meaning; it is used to solve problems. "The power of Viète's algebra comes from the fact that operations on letters are defined operationally but not semantically" (Charbonneau, 1996, p.35). More importantly, algebra is about analysis. "The core of analysis is the hypothesis, that is, the assumption that the problem is solved […] it imposes the development of a certain way of representing the unknown magnitudes that are considered given by hypothesis" (Charbonneau, 1996, p.36).

Similarly, Schmidt & Bednarz (2002) define both types of reasoning. "Arithmetic proceeds synthetically, from the known to the unknown" (Schmidt & Bednarz, 2002, p.69). A procedure would be considered arithmetical if the solution consistently used only the known values to be able to perform operations. Conversely, algebraic reasoning "adopts an analytical method, which proceeds *from the unknown to the known*" (Schmidt & Bednarz, 2002, p.69). In this case, a procedure would be considered algebraic if the solution focused on the unknown value that would temporarily be replaced by a letter or a symbol in order to manipulate it in equations.

Both Charbonneau (1996) and Schmidt & Bednarz (2002) agree that arithmetic reasoning focuses on the known values and algebraic reasoning works with both the known and the unknown values from the start.

3.3.4 Epistemological obstacles

Epistemological obstacles should not be seen as having only negative effects on understanding. Those obstacles did lead to new discoveries. It is true that mathematicians, such as Descartes, or Newton, saw the need to overcome those obstacles that led to more abstract concepts, to algebraic thinking, to analytic geometry and calculus. Without algebra, Newton would not have seen the relationship between tangent and quadrature problems and there would be no Fundamental Theorem of Calculus. But these obstacles were exactly the "shoulders of giants"[1] on which they stood.

In Chapter 2, we identified two epistemological obstacles related to algebra: the Quantitative obstacle and the Ontological obstacle. In the Discussion section of the results of our analysis, they will be linked with some of the defects of quantitative reasoning that we discovered in the participants.

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[1] https://en.wikiquote.org/wiki/Isaac_Newton

4 METHODOLOGY

4.1 SOURCES OF DATA

The main motivation of this study was to help mature students who return to school and are required to re-take an elementary algebra class to satisfy the prerequisites for the academic program of their choice. In order to help them, it was necessary to first understand the sources of their difficulties. This thesis gives an account of this first, diagnostic phase of the process, with data obtained from a total of 14 students, 13 of whom were taking an elementary algebra course at the time of the research, and one who took the course in the previous year. The participants volunteered to this study. They were asked to solve 4 word problems, and their solutions were the data in this research. They had to work individually and had one hour to do it. This was followed by interviewing students on their solutions. We recorded the interviews and used it when we needed some clarifications from the solutions in order to classify them. Since we wanted to observe the problem solving behavior of the participants in their role as mature students, we chose to conduct this study as close as possible to the school environment. This is why the interviews were not individual, but in groups, in the format of tutorial discussions, in which the majority of the participants participated. There were two group discussions, and one interview was conducted individually, with the participant who took the algebra course in the previous year. The discussion was started by the researcher's question: "So how did you solve the problems?" The researcher let the participants speak but did not ask leading questions or evaluate the interventions as correct or not. These interventions were taken into account as data in the research. Later, the participants received feedback from the researcher on their solutions, but this feedback and its impact on the participants' problem solving skills are not part of this study.

4.2 DATA COLLECTION INSTRUMENT

All the participants were given 4 word problems to solve individually.

Each of the four problems was of a different type. The types of problems were based on the research done by Schmidt & Bednarz (2002). Question 4 was adapted from this article and the others were constructed by the researcher. Our goal was to not only observe arithmetic and algebraic behaviors in all the questions, but also to analyze quantitative reasoning, especially with Question 3 and Question 4.

4.2.1 Question 1 (Connected Problem)

Question 1 was formulated as follows:

> Lisa has an hourly salary of $17.50. If, last month, after 13% tax deduction, her salary was $548.10, how many hours did she work?

Using the schema from Schmidt & Bednarz (2002), the diagram in

Figure 5 illustrates the data and relations given in the problem.

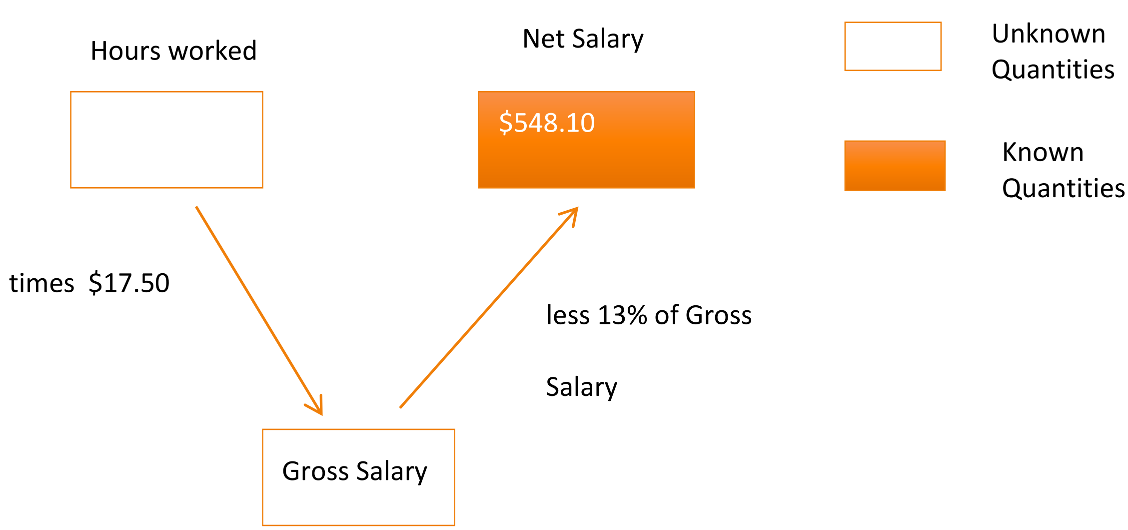

Figure 5: Connected Diagram of Question 1

This word problem can be solved using an arithmetic or an algebraic approach (see Table 3), but it was chosen because it is a connected problem. In other words, using an arithmetic approach is enough because the unknown quantity (the number of hours) can be obtained by a chain of arithmetic operations on known quantities from a known quantity. To see this, one needs, however, to "invert the relational arrows" in the diagram. One needs to translate the information "Gross salary reduced by 13% is \$548.10" into the equivalent one, "(100-13)% of \$548.10 is the Gross salary." And the information "\$17.50 per hour times the number of hours is the Gross salary" must be translated into "the Gross salary divided by hourly salary is the number of hours worked." Therefore the problem is not a straightforward connected problem, and the fact that it involves percents – a difficult concept – makes it even more challenging.

Arithmetic Solution	Algebraic Solution
$100 - 13 = 87$ So, \$548.10 represents 87% of the gross salary. $\frac{548.10}{0.87} = 630$ \$630 is the gross salary. $\frac{630}{17.50} = 36$ So, Lisa worked 36 hours. Checking answer: $17.50\ \$\ per\ hour \times 36\ hours = 630\ \$$ $630\ \$ \times 0.13 = 81.9\ \$$ $630\ \$ - 81.9\ \$ = 548.10\ \$$	Let a represent the number of hours Lisa worked last month. $17.50a - (17.50a \times 0.13) = 548.10$ $17.50a - 2.275a = 548.10$ $15.225a = 548.10$ $a = \frac{548.10}{15.225}$ $a = 36$ So, Lisa worked 36 hours. Checking answer (the same as in Arithmetic solution)

Table 3: Arithmetic and Algebraic solutions of Question 1

4.2.2 Question 2 (Disconnected Problem)

Question 2 was formulated as follows:

Marvin is 9 years and 3 months older than his youngest sister Mary, who is 10 times younger than her mother Miriam. In two years, Marvin and Mary's ages together will be half their mother's age. What are Miriam, Marvin, and Mary's ages today?

Based on Schmidt & Bednarz (2002), the following diagram illustrates the relationships and data given in the problem.

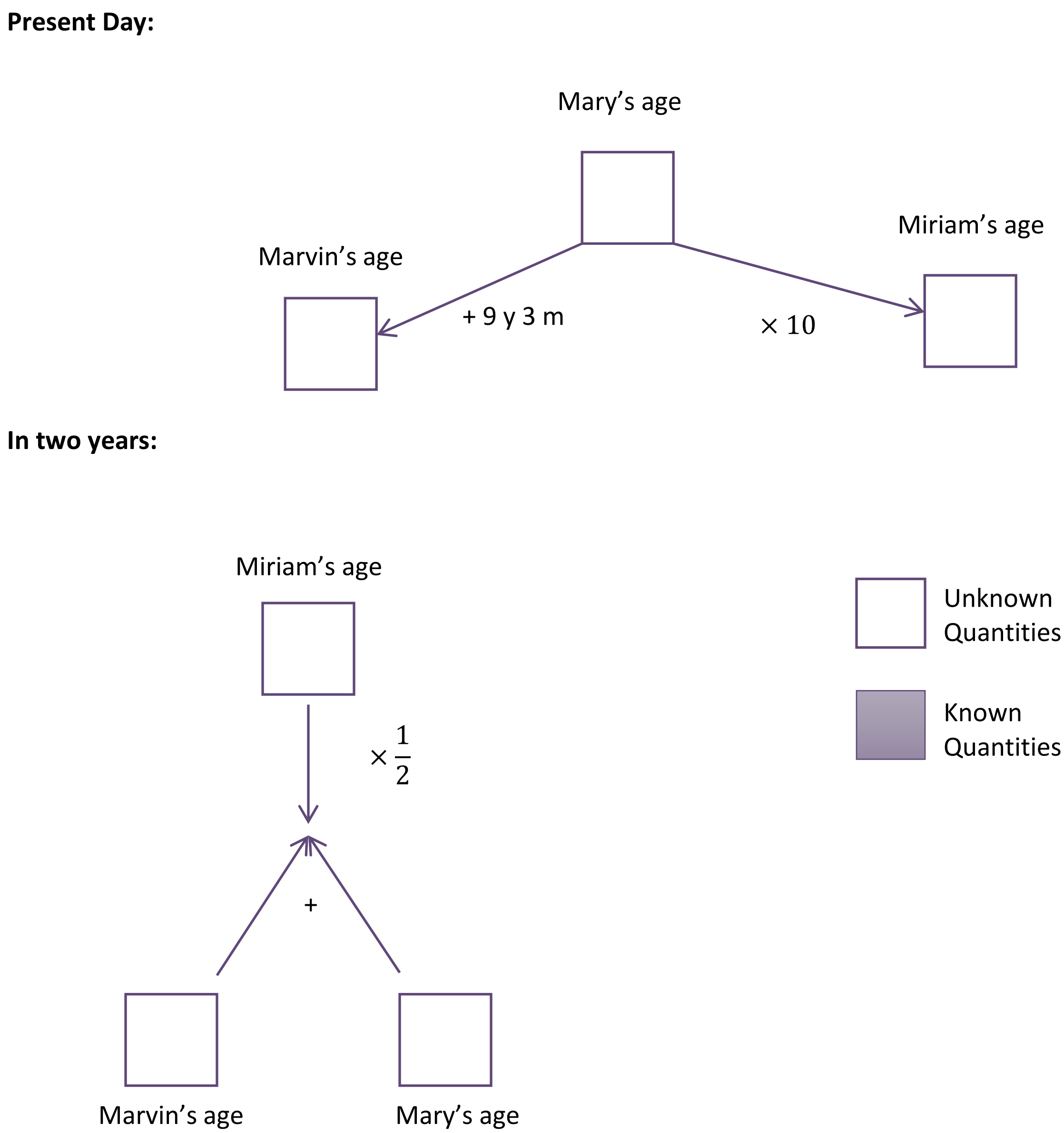

Figure 6: Disconnected Diagram of Question 2

This word problem was chosen because it is a disconnected problem. In other words, using an arithmetic approach will not be enough. It would be very difficult to solve this problem without treating at least one unknown as known, representing it by a letter or a line segment and representing the relations in form of equations, because all we are given are relations between three unknown quantities. This forces the reader to look at the relations, which requires quantitative reasoning. Also, this problem can be considered as complex since the reader is required to keep in mind multiple relationships in order to understand and solve the problem (Thompson, 1993). An algebraic solution using three unknowns is shown in

Table 4.

An algebraic solution of Question 2
Let x represent Marvin's age (in years) Let y represent Mary's age (in years) Let z represent Miriam's age (in years) Note: 9 years and 3 months equals to 9.25 years (1) $x = y + 9.25$ (2) $10y = z$ (3) $(x + 2) + (y + 2) = \frac{z+2}{2}$ Substitute x and z in (3): $$[(y + 9.25) + 2] + (y + 2) = \frac{(10y + 2)}{2}$$ $$2y + 13.25 = 5y + 1$$ $$12.25 = 3y$$ $$\frac{12.25}{3} = y$$ $$\frac{12 + 0.25}{3} = y$$ $$4 + \frac{1}{12} = y$$ Note: $\frac{1}{12}$ represents 1 month. So, **Mary is 4 years and 1 month old**. Then, Marvin's age is 9 years and 3 months plus 4 years and 1 month old, meaning that **Marvin is 13 years and 4 months old**. And so, Miriam's age is 10 times 4 years and 1 month. So **Miriam is 40 years and 10 months old**.

Table 4: Algebraic Solution of Question 2

4.2.3 Question 3 (Problem with contradictory data)

Question 3 was formulated as follows:

> A coffee shop charges \$13.7 for 2 hot chocolates and 2 pieces of cheesecake. Three hot chocolates and one piece of cheesecake cost \$11.05, and 2 pieces of cheesecake and one hot chocolate cost \$12.6. What is the cost of one piece of hot chocolate and one hot chocolate in this coffee shop?

This question appears to be a typical word problem. However, it has contradictory data, and no numerical answer can be found. This word problem can be solved with an arithmetic and algebraic approaches (see Table 5).

Arithmetic Solution	Algebraic Solution
\$13.7 is the cost of 2 hot chocolates (HC) and 2 pieces of cheesecake (PC). So, half of \$13.7 or \$6.85 is the cost of 1 HC and 1 piece of PC.  \$6.85 We know \$11.05 is the cost of 3 HC and 1 PC,  \$11.05 But 3 HC and 1 PC is the same as 1 HC, 1 PC, and 2 HC. Since we already know the cost of 1 HC and 1 PC, we can calculate the cost of 2 HC: \$11.05 – \$6.85 = \$4.2 So, \$4,2 is the cost of 2 HC. Thus, 1 HC costs \$2.1. Therefore 1 PC costs: \$6.85 – \$2.1 = \$4.75 Checking the answer: $2(\$2.1) + 2(\$4.75) = \$13.7$ – Correct $3(\$2.1) + (\$4.75) = \$11.05$ – Correct $(\$2.1) + 2(\$4.75) = \$11.6 \neq 12.6$ – Incorrect There is a contradiction. So, no solution.	Let a represent the number of dollars that one hot chocolate costs, and let b represent the number of dollars that one piece of cheesecake costs. (1) $13.7 = 2a + 2b$ (2) $11.05 = 3a + b$ (3) $12.6 = a + 2b$ Isolate a in (3): $a = 12.6 - 2b$ Substitute a in (1): $13.7 = 2(12.6 - 2b) + 2b$ $13.7 = 25.2 - 4b + 2b$ $2b = 11.50$ $b = 5.75$ Back at (3): $a = 12.6 - 2(5.75) = 1.10$ So, one hot chocolate costs \$1.10 and one piece of cheesecake costs \$5.75. Checking the answer: If $a = 1.10$ and $b = 5.75$, then ○ (1) $13.7 = 2(1.10) + 2(5.75)$ - Correct (2) $11.05 = 3(1.10) + (5.75)$ - Incorrect (3) $12.6 = (1.10) + 2(5.75)$ - Correct There is a contradiction in the data. So, no solution.

Table 5: Arithmetic and Algebraic Solutions of Question 3

4.2.4 Question 4 (Analyze a fictional solution of a given word problem)

Question 4 was formulated as follows:

> Jean solves the problem: "Brigitte goes to the store. She buys the same number of books and records. The books cost \$2 each and the records \$6 each. She spends \$40 in all. How many books and records did she buy?"
>
> Jean answers the problem as follows:
>
> $2x + 6y = 40$
>
> Since $x = y$, I can write:
>
> $2x + 6x = 40$
>
> $8x = 40$
>
> The last equation shows that 8 books cost \$40 so one book costs \$5.
>
> Questions:
>
> 1. Is this solution correct? Justify your answer.
>
> 2. Does the last equation indeed show that 1 book cost \$5?

In this question, students are asked to analyze an incorrect solution. This question is quite different from the other questions. Students are not used to having this type of word problem, and so we are interested to observe their thought processes. There are different ways of responding to the problem. Two are presented in Table 6.

Solution 1	Solution 2
1. Is this solution correct? Justify your answer. The solution is incorrect. Although the algebra in the solution is correct, the concluding statement is incorrect. From the equations used by Jean, x has to represent the number of books bought and y has to represent the number of records bought. In the concluding statement, Jean interprets x and y as prices, which is incorrect. 2. Does the last equation indeed show that 1 book cost $5? No, because the text says that one book costs $2. The last equation shows that Jean bought 5 books.	1. Is this solution correct? Justify your answer. No it is not correct. We can view this problem in terms of proportions. Since Brigitte bought the same amount of books and records, we can interpret it as she bought a certain amount of pairs of books and records. If one pair costs $8, how many pairs did Brigitte buy to pay $40 in total? So, 5 pairs cost $40. Thus, she bought 5 books and 5 records. 2. Does the last equation indeed show that 1 book cost $5? No, because the text says that one book costs $2. The last equation shows that Jean bought 5 books.

Table 6: Possible acceptable responses to Question 4

4.3 METHOD OF ANALYSIS OF THE DATA

For each question, we grouped solutions by similar approaches and reasoning after analyzing each solution according to the following characteristics:

Question 1, Question 2 and Question 3:

- Answer, correct or not, checked or not
- Type of solution (algebraic or arithmetic)
- Type of letter use
- Defects of quantitative reasoning
- Defects of arithmetic skills
- Defects of algebraic skills

Question 4:

- Flaw discovered or not
- Type of letter use
- Defects of quantitative reasoning
- Defects of arithmetic skills
- Defects of algebraic skills

5 RESULTS

The presentation of the results will start by a detailed description and analysis of the participants' responses to the four word problems they were asked to solve. This will be followed by a summary of the observations, suggesting the possible sources of the participants' difficulties in mathematical problem solving.

5.1 DECISIONS REGARDING THE ANALYSIS OF PARTICIPANT'S RESPONSES

While participants had different backgrounds, they were all mature students (at least 21 years old) that had to re-learn algebra by taking a high school level algebra course at the university in order to be admitted into the academic programs of their choice. In the following analysis, we do not differentiate the different groups of students that were part of the research. We consider them as participants that had to solve four word problems and were asked then to discuss their solutions. The main focus was not to see if the participants could get the right answer, but to analyze their thought processes. Thus, we will analyze all the solutions of each word problem and describe each participant's process by the type of approach (arithmetic or algebraic), and defects observed mainly in quantitative reasoning, and arithmetic and algebraic skills. Participants who took part in the session of solving Problems 1 and 2, have been numbered S1, S2,…, S12. Participants S11 and S12 did not come to the session of solving questions 3 and 4; instead, two new participants joined the session; they have been labeled S13 and S14. There were 12 participants in each session.

5.2 TYPES OF OBSERVED DEFECTS IN PARTICIPANT'S SOLUTIONS

Looking for similarities among participants' responses in view of grouping or classifying them somehow, we came to identify specific types of defects, which we then grouped into larger categories. Since we will be using these categories in describing participants' responses, we list and describe them below. For easier reference, we code the categories with brief names whose meaning, we hope, will be easy to remember.

5.2.1 Defects of quantitative reasoning

- **Measure = Object**: Measure of an object is not distinguished from the object (This defect can be regarded as a symptom of the Geometrical obstacle or the Obstacle of Ontology)
- **Quantity = Number**: Quantities are not distinguished from abstract numbers; no attention is paid to the units in which quantities are expressed (This defect can be attributed to an obstacle opposite to the Geometric obstacle and the Obstacle of Ontology since the numerical value of the measure of a quantity is quickly abstracted from the quantity with little or no relation with the quantity; we call it the Numerical obstacle).
- **Bad Quantitative Grammar**: Quantitative statements are formulated incorrectly (e.g., are incomplete or contain contradictions).
- **Quantitative Negligence**: Not all conditions on the quantities in the problem are taken into account; not paying attention to details of expressions regarding relations between quantities.
- **Nonsense Manipulation**: Operations on equations do not make sense in terms of the meaning of variables as quantities.
- **Additive Conception of Percent:** If the expression $a\%$, where a is a number, is treated as an abstract number (and not as a multiplicative relation between two quantities) then a statement such as "Lisa's salary after 13% tax deduction was 548.1 \$" could be written as $Gross\ salary - 13\% = 548.1$ \$. So, performing a formal operation on this equation, $Gross\ salary = 548.1\ \$ + 13\%$. Most participants, at this point, returned to the correct multiplicative and relational conception of percent and recalled that a percent is always a percent of something. For most, this something was the given net salary, so they calculated

that $ross\ salary\ = 548.1\$ + 0.13 \times 548.1\ \$ = 619.353\ \$$. The behavior described thus far was coded as suffering from the defect of *Additive Conception of Percent*. There were a few participants who, after writing that $Gross\ salary\ = 548.1\ \$ + 13\%$ represented 13% as 0.13 and obtained that Lisa's gross salary was 548.23$. These participants' solutions were coded as presenting both the *Additive Conception of Percent* and the *Quantity = Number* defect, since they treated $a\%$ as an abstract number.

- **Reasoning unnecessarily complicated**

5.2.2 Defects of arithmetic skills

- **Poor number sense**: e.g., multiplying a value by a number and then dividing the result by the same number and expecting a different value; subtracting a bigger number from a smaller number (both positive) and obtaining a positive number, etc.
- **Computational Negligence**: e.g., copying the output from a calculator incorrectly; not paying attention to the position of a digit in a decimal representation of a number, etc.

5.2.3 Defects of algebraic skills

- Any mistake and misconception related to **algebraic notation and manipulation**: e.g., distributive law ignored, incorrect use of the equal sign, etc.

Note: *Letter used as an object*, mentioned in Chapter 3, is considered as a defect of algebraic skills. However, if will be mentioned separately in the analysis of each solution under the heading "Letter use", since some uses will be correct.

5.3 PRESENTATION OF PARTICIPANTS’ RESPONSES

5.3.1 Question 1

We recall the statement of the problem:

> Lisa has an hourly salary of $17.50. If, last month, after 13% tax deduction, her salary was $548.10, how many hours did she work?

The correct answer is 36 hours, and the different ways of solving the problem have been discussed in section 4.2.

We have grouped the participants’ solutions according to their correctness and type of reasoning used. We obtained seven groups of solutions.

5.3.1.1 The answer and reasoning correct

Three participants’ solutions fell into this category: S1, S2 and S3.

We start by presenting their solutions in the form of typewritten transcripts. The symbol “//” is used to represent a new paragraph (or line) in the written solution. The symbol “*” is used to represent the multiplication signs, × or · , in the original solutions.

S1’s solution

A. 17.50/h // 0.13 tax --> 15.225/h // 15.225x = 548.1 // x= 548.1/15.225 // x=36 // Lisa worked 36 hours // B. 17.50 * 3 = 52.50 // 52.50 x 0.13=6.825 // 52.50 - 6.825 = 45.675 // 17.50 *0.13 = 2.275 // 17.50-2.225=15.225 * 3 = 45.675 // The total salary is equal to the hourly rate times the number of hours worked. I wasn't sure if the tax deduction were applied on the hourly rate or the total salary but discovered that it wouldn't make a difference by arbitrarily choosing 3 hours of work to test. see B. After this discovery, equation A was used to solve the problem with basic algebra. After applying the 13% deduction.

S2’s solution

17.50 x 36 h = 630 // 630 * 13 % = 548.10

Note regarding S2’s solution: It seems that this solution is like checking a final answer. Even though, taken literally, this solution has contradictions and mistakes, the participant got the right answer.

Since 36 is not an easy number to guess, we agreed to given him the benefit of the doubt and conclude that his quantitative reasoning was correct. We assumed that he found the gross salary using a calculator and then divided that amount by the hourly salary to obtain the number of hours. We assumed that he reasoned as follows:

$100 - 13 = 87$

$So, 87\%\ of\ the\ gross\ salary\ equals\ to$ $548.10

$548.10 \div 0.87 = 630$ --- $630 is the gross salary

$630 \div 17.50 = 36$

S3's solution

[Summary only; the actual solution is very long; it's transcription is given below] 13% of $100 is $13.00, leaving $87.00 // 0.13÷87=0.00149425 // 0.00149425 · 548.10 · 100=81.899 // 548.10+81.90=630 // 630 ÷17.50=36 hours

Full transcript of S3's solution

Thought Process:

1. Employee gets $17.50 per hour.

2. Last month, received $548.10, after enduring a 13% tax d.

3. Need to find out how much salary she earned without the tax, so I need to reverse the tax.

4. Then, I just need to divide the before-tax amount by a divisor of the hourly amount in order to achieve the number of hours.

5. If 13% tax is deducted from an easy-to-figure-out sample $100, then $13.00 is taken off, leaving $87.00.

6. $87 \div 0.13 = wrong$ 669.23; should have done $0.13 \div 87 = 0.00149425$. Trying to find out what number, when multiplied to 87, will return the pre-tax amount. If $87 \times 0.00149425 = 13$, then that is the amount of tax paid when given the net amount of $87 and the rate of 13%.

7. Therefore, 0.00149425×548.10 should give me the amount of paid on the original amount.

8. Test: $0.00149425 \cdot 548.10 \cdot 100 = 81.899$

9. So, $548.10 + 81.90 = 630$. Then $630 \div 17.50 = 36$ hours worked.

10. Therefore, the after-tax amount needs to be the dividend and the percentile rate of tax needs to be the divisor and the quotient would be the tax paid.

Note: S3's solution is very complicated but after analyzing his solution, we believe that the operations he does can be explained as follows:

$$\frac{0.13}{87} \times 548.10 \times 100 = \frac{0.13}{0.87} \times 548.10 = 0.13 \times \frac{548.10}{0.87} = 0.13\, X$$

Since $548.10 = X - 0.13X$ then $548.10 = 0.87X$ and so $X = \frac{548.10}{0.87} = 630$.

The number of hours is calculated from: $\frac{630}{17.50} = 36$

Characteristics of the solutions in this group are presented in Table 7.

Characteristics of the solution	**S1**	**S2**	**S3**
Answer	36 hours	36 hours	36 hours
Checks solution	Yes	Yes	No
Type of solution: arithmetic, algebraic	Partly algebraic, partly arithmetic	Arithmetic	Arithmetic
Type of letter use	Letter used as a specific unknown	Letter not used	Letter not used
Defects of quantitative reasoning	None observed	**Bad quantitative grammar** (e.g., 630 × 13 % = 548.10 implies that 13 % of 630$+630$= 548.10$)	Reasoning correct but unnecessarily complicated
Defects of arithmetic skills	None observed	None observed	None observed
Defects of algebraic skills	Unsure of distributivity law; Incorrect use of the equal sign – chain writing of operations (17.50-2.225=15.225 * 3 = 45.675)	None observed	None observed

Table 7. Question 1 - Solutions of the type: answer and reasoning correct

5.3.1.2 Answer correct but reasoning based on additive conception of percent

Only one participant's solution belongs to this category, S4.

S4's solution:

17.50x // 17.50 + x =548.10 // 548.10 (13%) = 71.25 // 620 -- Gross pay // x - y = Net Salary // x - .13y = 548.10 // x = 548.10 + .13y // 620/17.50 = 36 hours // Ans: Lisa worked 36 hrs // 17.50 @ 36 hrs: $630 // 630 less 13% tax deduction: 81.90 Tax deduction // 630 - 81.90 = 548.10 NET PAY

Characteristics of this solution are presented in Table 8.

Characteristics of the solution	S4
Answer	36 hours
Checks solution	Yes
Type of solution: arithmetic, algebraic	Arithmetic
Letter used as…	Letter evaluated
Defects of quantitative reasoning	Additive conception of percent **Bad quantitative grammar** (does not use operation signs consistently: 17.50x and 17.50 + x =548.10; 548.10 (13%) = 71.25)
Defects of arithmetic skills	None observed
Defects of algebraic skills	Interchanges the meaning of the letters in an equation Incorrect use of the equals sign (= used for rough approximations: 620/17.50 = 36 hours)

Table 8. Question 1 - Solution of type: Answer correct but reasoning incorrect

5.3.1.3 The answer is incorrect but the reasoning is almost correct

This category is also represented by a single participant's solution, S11.

S11's solution:

548.1 * 0.13 = 71.25 // 548.1 - 71.25 = 426.84 // 17.50 x = 478.84 // x = 27.36 hours // Mary worked 27.36 hours

Characteristics of S11's solution are presented in Table 9.

Characteristics of the solution	S11
Answer	27.36 hours
Checks solution	No
Type of solution: arithmetic, algebraic	Partly arithmetic, partly algebraic
Type of letter use	Letter used as a specific unknown
Defects of quantitative reasoning	**Quantitative negligence**: not paying attention to details of expressions regarding relations between quantities ("after tax reduction" misread as "before tax deduction")
Defects of arithmetic skills	None observed
Defects of algebraic skills	None observed

Table 9. Question 1 - Solution of the type: Answer incorrect but reasoning correct

5.3.1.4 The answer is incorrect and reasoning incorrect, based on additive conception of percent

Five participants' solutions fell into this category: S5, S6, S7, S8, and S9. We present transcripts of their solutions below.

S5's solution:

548.1 * 0.13 = 71.253 // = 71.25 + 548.1 = 619.353 $ (before taxes) // 619.353 ÷ 17.50 = 35.3916 hrs // = 35.39 hrs worked // Rounded = 35.4 hrs

S6's solution:

$ 71.253 deducted // $ 548.10 + 71.35 = $ 619.35÷ 17.50 /hr = 35.39 hrs

S7's solution:

548.10 $ * 13% = 71.53 $ // 548.10 $ + 71.53 $ = 619.53 $ // 619.53 : 17.50 = 35.3916 hours

S8's solution:

hours = x // 13% = 0.13 // total salary = y // y = 548.10 + 548.10 * 0.13 // x = y / 15.50 // y = 548.10 + 71.253 = 619.353 // x = 619.353 / 15.5 = 35.39 // She worked ~ 35 h per week or if it was a 31 day month => ~ 1085 hours the last month

S9's solution:

17.50 (h) - 0.13 = 548.1 // 17.50 (h) = 548.1 + 0.13 // 17.50 (h) = 548.23 // h = 548.23 / 17.50 // h = 31.32

These solutions are characterized in Table 10.

Characteristics of the solution	S5, S6	S7	S8	S9
Answer	35.39 hours	35.3916 hours	~1085 hours	31.32
Checks solution	No	No	No	No
Type of solution: arithmetic, algebraic	Arithmetic	Arithmetic	Arithmetic	Algebraic
Type of letter use	Letter not used	Letter not used	Letter evaluated	Letter used as a specific unknown
Defects of quantitative reasoning	Additive Conception of Percent **Bad Quantitative Grammar** (quantities equated with abstract numbers)	Additive conception of percent **Bad quantitative grammar** (divides a pure number by dollars and obtains hours)	Additive conception of percent **Quantitative negligence** (monthly salary taken as weekly salary and then weekly number of hours taken as daily number of hours) **Quantity = Number** (treating percents as abstract numbers: 13% = 0.13)	Additive conception of percent
Defects of arithmetic skills	None observed	Computational negligence (548.1 x 0.13 = 71.53 instead of 71.253)	None observed	None observed
Defects of algebraic skills	Incorrect use of the equal sign – chain writing of operations; e.g., $\$\,548.10 + 71.35 = \$\,619.35 \div 17.50\,/hr = 35.39\,hrs$	None observed	None observed	None observed

Table 10. Question 1 - Solutions of the type: Answer incorrect and reasoning incorrect, based on the Additive Conception of Percent

5.3.1.5 *The answer is incorrect and based on a wild guess*

One of the solutions looked as a very rough copy and it was very difficult to find a reason for the answer given.

S10's solution

$17.5/hrs$ // $1\ month$ \$548.1 // Tax 13% // $It's\ x + 13\%y = 548.1$ // $40 \cdot 17.5 = \$700$ // $\frac{x}{700} \cdot \frac{13}{100} = \$91$ // She worked 40 hrs

S10's solution is characterized in Table 11.

Characteristics of the solution	S10
Answer	40 hours
Checks solution	No
Type of solution: arithmetic, algebraic	Arithmetic
Letter used as…	Letter used as an object
Defects of quantitative reasoning	**Quantitative negligence** (x + 13%y = 548.1; "after tax deduction" is read as "before tax deduction") **Bad quantitative grammar**: Equating pure numbers with quantities (40 · 17.5=\$700)
Defects of arithmetic skills	None observed
Defects of algebraic skills	None observed

Table 11. Question 1 - Solution of the type: answer incorrect and based on a wild guess

5.3.1.6 *The answer is incorrect and complete misunderstanding of percents and lack of number sense*

The last category also contains only one solution; that of S12.

Here is the transcript of this solution.

S12's solution:

$Hourly\ Salary$ \$17.50 // $After\ tax$ 13% // $= \$548.10$ // $547.97 - 548.1 = 0.13$ // $548.1 \times 0.13 = 71.25$ // $71.25 \div 0.13 = 548.1$ // $548.1 \div 17.50$ // $\Rightarrow$ $31.32\ hours$

S12's solution is characterized in Table 12.

Characteristics of the solution	S12
Answer	31.32 hours
Checks solution	No
Type of solution: arithmetic, algebraic	Arithmetic
Letter used as…	Letter not used
Defects of quantitative reasoning	**Quantity = Number**: Understanding of percent as abstract numbers
Defects of arithmetic skills	Number sense lacking (multiplies by a number then divides the result by the same number; subtracts a bigger number from a smaller one, both positive, and obtains a positive number, incorrect even in its absolute value)
Defects of algebraic skills	None observed

Table 12. Question 1 - Solution of type: Misunderstanding of percents and lack of number sense

5.3.1.7 Summary of analysis of Question 1

Categories of solutions

- The answer and the reasoning correct: 3 out of 12 solutions
- The answer correct and reasoning incorrect based on additive conception of percent: 1 out of 12
- The answer incorrect but reasoning almost correct: 1 out of 12
- The answer is incorrect and reasoning based on additive conception of percent: 5 out of 12
- Other incorrect answers: 2 out of 12

Characteristics of the solutions

<u>Checks solution</u>: 3 out of 12

<u>Types of solutions</u>:

- Purely arithmetic solution: 9 out of 12
- Purely algebraic solution: 1 out 12
- Partly arithmetic and partly algebraic solution: 2 out 12

Types of letter use:

- Letter not used: 6 out 12
- Letter evaluated: 2 out 12
- Letter used as an object: 1 out 12
- Letter used as a specific unknown: 3 out of 12

Defects of quantitative reasoning

- Bad quantitative grammar: 7 out of 12 people displayed this defect

  Examples:

  - Inconsistent use of operation signs
  - Adding abstract numbers to quantities
  - Dividing a pure number by dollars and obtaining hours
  - Equating pure numbers with quantities
  - Additive conception of percent: 6 out of 12 people
- Quantitative Negligence: 3 out of 12
  - Example: “after tax deduction” is read as “before tax deduction”
- Quantity = Number: 2 out of 12
  - Example: a% is an abstract number that can be added to any other number or quantity (symptom of numerical obstacle)
- The reasoning is correct but unnecessarily complicated: 1 out 12

Defects of arithmetic skills: 2 out of 12

Examples:

- Computational negligence: 1 out of 12
- Poor number sense: 1 out of 12

Defects of algebraic skills: observed in 4 out 12 solutions

There were 6 instances of defect of algebraic skills. The defects were of three types:

1. Incorrect use of the equal sign: 4 instances out of the 6

Examples:

- "=" used for rough approximations: $620/17.50 = 36\ hours$
- Chain writing of operations leading to incorrect use of the equals sign:
    - $548.1 * 0.13 = 71.253 = 71.25 + 548.1 = 619.353\ \$$
    - $\$\,548.10 + 71.35 = \$\,619.35 : 17.50\ /hr = 35.39\ hrs$
    - $17.50 - 2.225 = 15.225 \times 3 = 45.675$
- Interchanging the meaning of the letters in an equation: 1 instance out of 6
- Distributivity of multiplication with respect with addition not internalized: 1 instance out of 6

5.3.2 Question 2

We recall the text of Question 2:

> Marvin is 9 years and 3 months older than his youngest sister Mary, who is 10 times younger than her mother Miriam. In two years, Marvin and Mary's ages together will be half their mother's age. What are Miriam, Marvin, and Mary's ages today?

The correct answer is: Mary is 4 years and 1 month; Marvin is 13 years and 4 months; Miriam is 44 years and 10 months. The solution of the problem has been discussed in section 4.2.

For this question, none of the participants involved in this study were able to find the correct answer. Some only represented the given relations – these we put in one group. A second group contains those who attempted to also find the ages of the characters in the problem. Within these two groups, we identified subgroups characterized by specific types of defects of quantitative reasoning.

5.3.2.1 Given relations among ages represented; no further solution attempted

In this group, some participants only copied the assumptions almost in the same form as in the question (2 solutions were of this type); others tried to represent those assumptions in the form of equations (4 solutions).

5.3.2.1.1 Given assumptions copied only

This was the case of S2's and S5's and solutions, which are transcribed below:

S2's solution

Marvin is 9 y and 3 m // youngest Mary // Miriam // Marvin // Mary

S5's solution

Marvin 9 yrs 3 mths // Mary 10 × younger than mother //

→ 9 ¼ · 10

Characteristics of these solutions are given in Table 13.

Characteristics of the solution	S2, S5
Answer	No answer
Checks solution	No
Type of solution: arithmetic, algebraic	Neither
Type of letter use	Letter not used
Defects of quantitative reasoning	Quantitative negligence (not all conditions taken into account)
Defects of arithmetic skills	None observed
Defects of algebraic skills	None observed

Table 13. Question 2 - Solutions of type: Given assumptions copied only

5.3.2.1.2 Given relations among ages represented by equations

In this group, the solutions go beyond just copying the assumptions and show attempts to represent the given relations among ages of the three people. In only one solution (S1), the quantitative grammar is acceptable and no quantitative negligence can be observed – all conditions are taken into account. Other solutions (3) all present some quantitative defect. Some use bad quantitative grammar but take into account all the conditions (S11, S12); in one solution, both bad grammar and quantitative negligence can be observed (S7).

We start by presenting transcripts of the solutions in this group.

S1's solution:

9y 3mnth // 9y · 12m = 108m + 3 months // Marvin // Mary 10 x younger // Miriam // $In\ two\ years\ (24\ months)$ // $Marvin\ age = v\ in\ months$ // $Mary\ age = Y\ in\ months$ // $Miri\square m\ age = r\ in\ months$ // $v = 111\ months + Y$ // $Y = r/10$ // $Y + v + 24 = \frac{r}{2}$ // First I converted everything to months. Then assign each age a variable. Then did not know how to progress and was defeated.

Note: S1 consciously uses letters as specific unknowns – he says he assigns each age a variable, and represents the relations by equations – but his solution cannot be classified as arithmetic or algebraic, because he does not process these equations.

S11's solution:

Mary $= x + 9/3$ // Miriam $= x - 10x$ // M and M $= 2x = 1/2$ // $x + 9 + x - 10 + 2x = 1/2$

S12's solution:

Marvin 9 years 3 months older // Mary 10 times younger than Miriam // 2 years = Marvin + Mary's ages Together // Marvin: $x + 9^{3}$ // Mary: $2x = ½$

S7's solution:

Marvin: a // Mary: b // Miriam: c // a = 111 + b, ["months" on top of 111] // b = 10c

The solutions in this group are presented in Table 14.

Characteristics of the solution	S1	S11, S12	S7
Answer	No answer	No answer	No answer
Checks solution	No	No	No
Type of solution: arithmetic, algebraic	Neither	Neither	Neither
Type of letter use	Letter used as a specific unknown	Letter used as an object	Letter used as an object
Defects of quantitative reasoning	None observed	**Bad quantitative grammar** (Equations do not represent quantitative relations adequately 2x = ½: "2x" is Mary's age in 2 years; not converting quantities into the same unit 9/3 is 9 years and 3 months – S11); (Equations do not represent quantitative relations adequately 2x = ½: "2x" is Mary's age in 2 years; not converting quantities into the same unit 9 3 is 9 years and 3 months – S12)	**Bad quantitative grammar**: Equations do not always represent relations assumed in the problem (the "Students-and-Professors" mistake, where the multiplicative relationship between two quantities is reversed (Clement, 1982)
Defects of arithmetic skills	None observed	None observed	None observed
Defects of algebraic skills	Unable to solve system of equations with 3 unknowns Incorrect use of the equal sign – chain writing of operations (9y * 12m = 108m + 3 months)	None observed	None observed

Table 14. Question 2 - Solutions of type: Given relations represented by equations, but no attempt at finding the ages

5.3.2.2 An attempt to find the ages is made

Six solutions fell into this category. All suffered from bad quantitative grammar, but three presented no quantitative negligence (S8, S9 and S10); this negligence was observed in the remaining three (S3, S4, and S6).

5.3.2.2.1 Solutions with bad quantitative grammar but no quantitative negligence

First, we present transcripts of the solutions of S8, S9 and S10 which all belong to this category.

S8's solution:

Mary $= x$ // Marvin $= x + 9.25$ // Miriam $= x \cdot 10 = y$ // $2(x + (x + 9.25)) = 1/2\, y$ // $2(2x + 9.25) = 1/2 \cdot 10x$ // $2(2x + 9.25) = 5x$ // $4x + 18.5 = 5x$ // $4x - 5x = -18.5$ // $-x = -18.5$ // $x = 18.5$

S9's solution:

Mrvn = x // Mry = y // Miriam = z // $x = 3$ // $9.3 + y = x$ // $10\, y = z$ // [*]$x + 2 + y + 2 = z/2$ // [**] $10\, y = z$ // [multiply equation * by -10] // [***] $-10x - 20 - 10y - 20 = -5$ // [10 y from eq.** and -10y from eq. *** cancel out, giving] $-10x - 20 - 20 = -4$ // -10 $x - 40 = -4$ // $-10\, x = 40 + 4$ // $x = 44/(-10)$ // $x = 4.4$

S10's solution:

Marvin + 9.3 = 111 months // Mary x10 younger than Miriam = 120 months // Marvin age -> x // Mary age -> y // Miriam age --> z // +9.3x + (2X 10 y) = // 12 +3 =15 // 14 + 5 = 19 // If Marvin is 12, then Mary is 3, which means Mary will be 38 // Marvin -> 12 // Mary 3 //Miriam 36

Characteristics of solutions in this category are presented in Table 15.

Characteristics of the solution	**S8**	**S9**	**S10**
Answer	x = 18.5	[Marvin's age] x = 4.4	Marvin: 12, Mary: 3, Miriam: 36
Checks solution	No	No	No
Type of solution: arithmetic, algebraic	Algebraic	Algebraic	Arithmetic
Type of letter use	Letter used as a specific unknown	Letter used as a specific unknown	Letter used as an object
Defects of quantitative reasoning	**Bad quantitative grammar**: Equations do not represent quantitative relations adequately (The equation 2(x + (x + 9.25)) = 1/2 y is intended to represent what would happen in 2 years - multiplies by 2 instead of adding 2)	**Bad quantitative grammar**: Equations do not represent quantitative relations adequately (x+2+y+2 = z/2) **Quantity = Number**: not converting quantities into the same unit (9.3 is 9 years and 3 months)	**Bad quantitative grammar:** Equations do not represent quantitative relations adequately +9.3x + (2X 10 y) =; mixed units months and years: used both 111 months and 9.3 years **Quantity = Number**: not converting quantities into the same unit 9.3 is 9 years and 3 months;
Defects of arithmetic skills	None observed	**Computational negligence**: Ignores contradictions in the expressions she is writing (x = 4.4 and x= 44/(-10))	None observed
Defects of algebraic skills	None observed	Ignores some terms when adding two equations together	Incorrect use of the equal sign and the addition sign(+9.3x + (2X 10 y) =)

Table 15. Question 2 – Solutions of type: Attempt at finding the ages is made, using bad quantitative grammar but no quantitative negligence

5.3.2.2.2 Solutions with bad quantitative grammar and quantitative negligence

The solutions of S3, S4 and S6 presented these two defects.

S3's solution:

The key to figure here is the mother, but information on the mother is lacking. So, I need to check for the children. // Marvin: 11 y. 3 mths; Mary: 2 years; Miriam: 20 // Let x1 = marvin's age // x2 = Mary's age // x3 = Miriam's age // x1 = x2 + 10 // x -x = 10 // 0x = 10? // x2 = x3 ÷ 10 // x2 = x3 + 10 // x2 - x3 = 10 // 0x = 10-0 // x = 0 // x3 = x2 *10 // x3 = x2/10 // x3/10 = x2/10

S4's solution:

Marvin is 111 months older than Mary (9)(12) + 3 = 111 // Marvin's age in 2 years = 24 months. 111 + 24 = 135 months // mary is 10 times younger than her mom // 135 - 10x = 24 // 135 - 24 = 10x // 111 = 10x // 11.1 = x // Mary's age // Miriam's age presently = // Marvin's age presently = 9 yr 3m // Mary's age presently = // In two years = Marvin age: // Mary's age:

S6's solution:

x Marvin // y Mary // z Miriam // x + y + 2 (yrs) = 1/2 (mother's age) // Marvin 11 yrs + 3 months

Characteristics of these solutions are in Table 16.

Characteristics of the solution	S3	S4	S6
Answer	Marvin: 11 y. 3 mths; Mary: 2 years; Miriam: 20	Marvin's age: 111 months; Miriam's age: 11.1	Marvin is 11 yrs + 3 months
Checks solution	No	No	No
Type of solution: arithmetic, algebraic	Partly arithmetic and partly algebraic	Algebraic	Arithmetic
Type of letter use	Letter used as a specific unknown	Letter used as a specific unknown	Letter used as an object
Defects of quantitative reasoning	**Bad quantitative grammar**: Inability to write quantitative statements correctly (tried 3 different equations: $x_1 = x_2 + 10, x_2 = x_3 \div 10$ and $x_3 = x_2 \cdot 10$.) **Quantitative negligence**: does not take into account all the conditions on the quantities in the problem (only uses the relation between Mary and Miriam) **Nonsense manipulation**: operations performed on the equations do not correspond to the relations between quantities that the equation is supposed to represent ($x_1 = x_2 + 10$ is transformed into $x - x = 10$ which implies that $x_1 - x_2 = 0$)	**Bad quantitative grammar:** equations do not represent relations assumed in the problem (135 - 10x = 24 does not consider Miriam's age being twice of the sum of her children's age in two years; not converting quantities into the same unit) **Quantitative negligence** (wrote that Marvin is 9 years and 3 months old, instead of that much older than his sister)	**Bad quantitative grammar** (Equations do not represent quantitative relations adequately: x + y + 2 (yrs) = 1/2 (mother's age)) **Quantitative negligence:** does not take into account all the conditions on the quantities in the problem (does not mention the relation between Marvin's age and Mary's age in the equation x + y + 2 (yrs) = 1/2 (mother's age)) **Quantity = Number:** not minding the unit (converting 9 years and 3 months into 9.3)
Defects of arithmetic skills	None observed	None observed	None observed
Defects of algebraic skills	Incorrect use of algebraic rules: Different letters in an equation represent the same number if they are written with the same letter but different indices ($x_2 - x_3$ becomes $0x$; $x_3 = x_2 \cdot 10$ becomes $x_3 = \frac{x_2}{10}$)	None observed	None observed

Table 16. Question 2- Solutions of type: Attempt to find the ages, but bad quantitative grammar and quantitative negligence

5.3.2.3 Summary of analysis of Question 2

Categories of solutions

- Given relations among ages represented only; no attempt at finding the ages: 6 out of 12 solutions
    - Assumptions copied only: 2 out of the 6 solutions
    - Relations represented by equations: 4 out of the 6 solutions
        - No defects of quantitative reasoning: 1 out of the 4
        - Bad quantitative grammar and no quantitative negligence: 2 out of the 4
        - Bad quantitative grammar and quantitative negligence: 1 out of the 4

- An attempt to find the ages is made: 6 out of 12 solutions
    - Bad quantitative grammar and no quantitative negligence: 3 out of the 6 solutions
    - Bad quantitative grammar and quantitative negligence: 3 out of the 6 solutions

Characteristics of the solutions

<u>Checks solution</u>: 0 out of 12

<u>Types of solutions:</u>

- Purely arithmetic solutions: 2 out of 12
- Purely algebraic solutions: 3 out of 12
- Mixed arithmetic-algebraic solution: 1 out of 12
- Not arithmetic and not algebraic: 6 out 12

<u>Types of letter use:</u>

- Letter not used: 2 out of 12
- Letter used as an object: 5 out of 12

- Letter used as a specific unknown: 5 out of 12

<u>Defects of quantitative reasoning</u>

Most solutions presented defects of quantitative reasoning; only one solution was almost free from them.

- Bad quantitative grammar: 6 out of 12

  Examples:

  - Equations do not always represent relations assumed in the problem: ages in two years are represented by multiplication by 2
    - $2x = ½$
    - $2(x + (x + 9.25)) = 1/2\, y$
    - $x + y + 2\ (yrs) = 1/2\ (mother's\ age)$
- Quantity = Number: 3 out of 12
  - Not converting quantities into the same unit
    - 9 3 is 9 years and 3 months
    - 9.3 is 9 years and 3 months
    - 9/3 is 9 years and 3 months
- Quantitative Negligence: 6 out of 12
  - Not all conditions taken into account
  - Misreading the conditions
- Nonsense manipulation: 1 out of 12

  Example:

  - Operations performed on the equations do not correspond to the relations between quantities that the equation is supposed to represent

Defects of arithmetic skills

There was 1 instance of defect of arithmetic skills. The defect was of that of **computational negligence**: a contradiction in the expressions written $\left(x = 4.4 \; and \; x = \frac{44}{-10}\right)$ was ignored.

Defects of algebraic skills: observed in 4 out of 12 solutions

Examples:

- Incorrect use of equal sign and signs of operations:
    - Chain writing of operations leading to incorrect use of the equals sign: 9y * 12m = 108m + 3 months
    - $+9.3x + (2X\ 10\ y) =$
- Treating variables represented by a letter with a subscript as if they represented equal numbers

5.3.3 Question 3

The text of Question 3 was:

> A coffee shop charges $13.7 for 2 hot chocolates and 2 pieces of cheesecake. Three hot chocolates and one piece of cheesecake cost $11.05, and 2 pieces of cheesecake and one hot chocolate cost $12.6. What is the cost of one piece of hot chocolate and one hot chocolate in this coffee shop?

Correct Answer: No solution because the given data is contradictory. The first two conditions imply the costs $4.75 for the cheesecake and $2.10 for hot chocolate, and this contradicts the third condition.

The contradiction was discovered in only 2 out the 12 solutions (S1 and S4). Among the remaining 10 solutions, one suffered mainly from defects of algebraic skills; the other nine – mainly from defects of quantitative reasoning. In four solutions (S8, S9, S10 and S14), costs were calculated from two equations representing some of the conditions given in the problem, but these values were not tested for satisfying the remaining condition. In these solutions, the quantitative grammar was generally good, but they suffered from the quantitative negligence defect. In the other five solutions (S2, S3, S5, S6, and S13), the defect consisted in not distinguishing between a cup of hot chocolate (a piece of cheesecake) and the cost of a cup of hot chocolate (piece of cheesecake) – what we labeled, the Measure = Objet defect.

5.3.3.1 Contradiction discovered

As mentioned, S1 and S4 discovered the contradiction and thus solved the problem correctly.

Here are the transcripts of these solutions.

S1's solution:

$A. Cheese = x \quad choco = y$ // $2x + 2y = 13.7$ // $2x + 2y = 13.7$ // $2 * \quad x + 3y = 11.05$ // $-2 * \quad 2x + y = 12.6$ // $2x + 2x - 4x + 2y + 6y - 2y = 13.7 + 22.1 - 25.2$ // $6y = 10.6$ // $y = 1.76$ // $2x + 2(1.76) = 13.7$ // $2x + 3.52 = 13.7$ // $2x = 13.7 - 3.52$ // $2x = 10.8$ // $x = 5.09$ //

$B.$ $2x + x + 2x + 2y + 3y + y = 13.7 + 11.05 + 12.6$ // $5x + 6y = 37.35$ // $x = (37.35 - 6y)/5$ // $5((37.35 - 6y)/5) + 6y = 37.35$ // $(186.75 - 30y)/5 + 6y = 37.35$ // $37.35 - 6y + 6y = 37.35$ //

$C. 2x + x + 2y + 3y = 13.7 + 11.05$ // $3x + 5y = 24.75$ // $2x + y = 12.6 \quad * -5$ // $3x - 10x + 5y - 5y = 24.75 - 63$ // $-7x = -38.25$ // $x = 5.46$

These algebraic equations need to be combined to solve the problem. However, a variable needs to be isolated. I multiplied the =11.05 equation by 2 and the =12.06 equation by -2 to remove the x variable and solve for y. After solving for y, I plugged that number into the equation of =13.70 and solved for x. However, this does not produce the correct answer. I tried two more things and failed miserably. In B and C, the values I get do not work for every total price value.

S4's solution:

Let x represent the cost of hot chocolate // let y represent the cost of cheesecake // 2x + 2y = 13.70 // x + 2y = 12.60 // 3x + 1y = 11.05 // 2x + 2y = 13.70 // x + 2y = 12.60 // 2x + 2y = 13.70 // (-1)-x - 2y = -12.60 // x = 1.10 hot chocolate // 2(1.10) + 2y = 13.70 // 2.20 + 2y = 13.70 // 2y = 11.50 // y =5.75 // x + 2(5.75) = 12.60 // x+11.50 = 12.60 // x= 1.10 // x= $1.10 hot chocolate // y = $5.75 cheesecake // (a) 2(5.75) +2(1.10) = 11.50 +2.20=13.70 // (c) 2(5.75)+1.10 = 22.50 + 1.10 = 12.60 // Ans: Cheesecake cost $5.75 // Hot chocolate cost $1.10

Note: S4 had not discovered the contradiction in the written solution above, but, in the interview, without prompting, the participant realized that not all the conditions have been checked and performed the check. The contradiction was discovered.

The solutions of S1 and S4 are characterized in Table 17.

Characteristics of the solution	**S1, S4**
Answer	No solution exists
Checks solution	Yes
Type of solution: arithmetic, algebraic	Algebraic
Type of letter use	Letter used as a specific unknown
Defects of quantitative reasoning	None observed
Defects of arithmetic skills	None observed
Defects of algebraic skills	None observed

Table 17. Question 3 - Solutions of type: Contradiction discovered

5.3.3.2 Contradiction not discovered

5.3.3.2.1 Concrete solution found from two equations but failure to check against a third

In this group of solutions, we observed quantitative negligence but the quantitative grammar was good.

Four solutions represent this category: S8, S9, S10 and S14. Their transcripts follow.

S8's solution:

a = cheesecake // b = hot chocolate // 1. 2b + 2a = 13.7$ //2. a + 3b = 11.05$ // 3. 2a + b = 12.6$ // a = 11.05 - 3b // 1. 2b + 2(11.05 - 3b) = 13.70$ // 2b + 22.10 - 6b = 13.70$ // -4b = -8.4$ // b = 2.10$ // 2. a + 3b = 11.05 // a = 11.05 - 3*2.10$ // a = 11.05 - 6.3 = 4.75$ // Cheesecake = $4.75 // Hot chocolate =$ 2.10

S9's solution:

13.7 = 2HC + 2CC // 11.5 = 3HC + 1CC // 12.6 = 1HC + 2CC // [Multiply 2nd eq by -2] -23 = -6HC - 2CC // [Add the last equation to the 3rd] -10.4 = -5HC // HC = 2.08 //

[plug the value of HC into the 2nd eq.] 11.5 = 2.08 (3) + 1CC // 11.5 = 6.24 + 1CC // 11.5 - 6.24 = 1CC // 5.26 = CC

S10's solution:

$13.7/ 2 HC & 2 cheesecake // $11.05 / 3 HC & 1 cheesecake // $12.6 / 1 HC & 2 cheesecake // 13.7/2 => $6.85 for one cheesecake and hot chocolate

S14's solution:

Price of hot chocolate = x // Price of one piece of cheesecake = y // 2x + 2 y= 13.7 // 3x + y = 11.05 // x + 2 y= 12.6 // y = 11.05 - 3x // x + 2(11.05 - 3x) = 12.6 // x + 23 - 6x = 12.6 // -5x = -10.4 // x = 2.08 // 2(2.08) + 2y = 13.7 // 4.16 + 2y = 13.7 // 2y = 9.54 // y = 4.77 // Price of hot chocolate = $2.08 // Price of cheesecake = $4.77

The above solutions are characterized in Table 18.

Characteristics of the solution	**S8, S14**	**S9**	**S10**
Answer	[concrete prices given in dollars]	[concrete prices given as abstract numbers]	[concrete prices given in dollars]
Checks solution	No	No	No
Type of solution: arithmetic, algebraic	Algebraic	Algebraic	Arithmetic
Type of letter use	Letter used as a specific unknown	Letter used as a specific unknown	Letter not used
Defects of quantitative reasoning	**Quantitative negligence** – not all conditions taken into account	**Quantitative negligence** – not all conditions taken into account **Quantity = Number** (price given as abstract number)	**Quantitative negligence** – not all conditions taken into account
Defects of arithmetic skills	None observed	None observed	None observed
Defects of algebraic skills	Unable to solve system of equations with 3 unknowns Incorrect use of the equal sign – chain writing of operations (9y * 12m = 108m + 3 months)	None observed	None observed

Table 18. Question 3 - Solutions of type: Contradiction not discovered; not all conditions taken into account but quantitative grammar generally good

5.3.3.2.2 Measure of the object is not distinguished from the object

Five solutions (S2, S3, S5, S6, and S13) were characterized by treating “hot chocolate” and “cheesecake” as the unknowns in the problem and using the symbols of addition and equality as shorthand to represent the relations in the problem. Here are transcripts of these solutions.

S2’s solution:

coffee shop 13.70 for 2 h.ch + 2 p cheese // hot chocolate = x // cheesecake = y // 2x + 2y = 13.7 // 3x + 1y = 11.05 // x + 2y = 12.60

S3’s solution:

Let x = hot chocolate; let y = cheesecake // so: 2x + 2y = $13.70 // 3x + y = $11.05 // y + 2x = $12.60 // 2x + 2y = 13.70

S5’s solution:

Let x = 2 hot chocolates // let y = 2 pieces of cheesecake // 1. 2x + 2y = 13.70$ // 1y + 3x = 11.05$ // 2y + 1x = 12.60$ // Here's where I get lost (plugging in the equations)

S6’s solution:

2 hot chocolate + 2 pieces cheesecake = $13.70 // 3 hot chocolate + 1 pieces cheesecake = $11.05 // 1 hot chocolate + 2 pieces cheesecake = $12.60 // How much 1 hot c + 1 cheesecake? = 2 // 2.65 // 1.10 --- cost of 1 hot chocolate // x + y = 2

S13’s solution

hot chocolate = x // cheesecake = y // 13.07 = 2x + 2y // 2. 11.05 = 3x + 1y // 12.06 = 1x + 2y // 1. 13.07 = 2x + 2y // 2x = -13.07 + 2y // x =(-13.07 + 2y)/ 2

These solutions have been characterized in Table 19.

Characteristics of the solution	**S2, S3, S5**	**S6**	**S13**
Answer	[No numerical answer]	[concrete price of hot chocolate given as an abstract number]	[no numerical answer]
Checks solution	[Not applicable – there is nothing to check]	No	[Not applicable]
Type of solution: arithmetic, algebraic	Neither	Arithmetic	Algebraic
Type of letter use	Letter used as an object	Letter used as an object	Letter used as a specific unknown
Defects of quantitative reasoning	**Measure = Object** (cost identified with hot chocolate or cheesecake) **Quantity = Number** (price given as abstract number)	**Measure = Object** (cost identified with hot chocolate or cheesecake) **Quantity = Number** (price given as abstract number)	**Measure = Object** (cost identified with hot chocolate or cheesecake) **Quantity = Number** (price given as abstract number)
Defects of arithmetic skills	None observed	None observed	**Computational negligence** ($13.07 = 2x + 2y$; takes 13.07 to mean the same as 13.7)
Defects of algebraic skills	Algebraic language not used as an operational symbolism – therefore the attempt at solving the problem is not algebraic	Algebraic language not used as an operational symbolism – therefore the attempt at solving the problem is not algebraic	Fails to change the sign when moving a term to the other side of the equation

Table 19. Question 3 - Solution of type: Contradiction not discovered; cost not distinguished from the objects having that cost.

5.3.3.2.3 Solution based on nonsense manipulation of symbols

The solution of S7 did not seem to make sense, and appeared not to belong to any of the previously described categories. The cost of a piece of cheesecake was obtained as a negative number, after many algebraic manipulations which had little meaning in terms of the assumed quantitative relations.

S7's solution

13.70 = 2 hot chocolates and 2 pieces of cheesecake // 11.05 = 1 piece of cheesecake and 3 hot chocolates // 12.60 = 2 pieces of cheesecake and 1 hot chocolate // Let x = cost of hot chocolate // let y = cost of piece of cheesecake // 13.70 = 2x + 2y // 11.05 = 3x + 1y // 12.60 = 1x + 2y // 11.05 = 3x + 1y - 33.15 = -3x -3y // 22.1/-2 = -2y/-2 // -11.05 = y

We characterize S7's solution in Table 20.

Characteristics of the solution	S7
Answer	[price of cheesecake given as a negative number]
Checks solution	No
Type of solution: arithmetic, algebraic	Algebraic
Letter used as…	Letter used as a specific unknown
Defects of quantitative reasoning	**Nonsense manipulation**: Operations on equations do not make sense quantitatively ($-22.15 = -3x - 3y$ does not follow from any of the assumptions about the costs of hot chocolate and cheesecake in the problem)
Defects of arithmetic skills	Number sense lacking (11.05 – 33.15 = 22.1)
Defects of algebraic skills	Does not apply the distributivity law

Table 20. Question 3 - Solution of type: Contradiction not discovered - Nonsense manipulation of symbols

5.3.3.3 Summary of analysis of Question 3

Categories of solutions

- Contradiction discovered: 2 out of 12
- Contradiction not discovered: 10 out of 12

- Concrete solution found from two equations but failure to check against a third: 4 out of 10
- Measure of the object is not distinguished from the object: 5 out of 10
- Solution based on nonsense manipulation of symbols: 1 out of 10

Characteristics of the solutions

Checks solution: 2 out of 12

Types of solutions:

- Purely arithmetic solutions: 2 out of 12
- Purely algebraic solutions: 7 out of 12
- Neither algebraic nor arithmetic solutions: 3 out of 12

Types of letter use:

- Not used: 1 out of 12
- As an object: 4 out of 12
- As a specific unknown: 7 out of 12

Defects of quantitative reasoning

We observed defects of quantitative reasoning in 10 out of 12 solutions. These are the types of defects:

- Measure = Object: 5 out of 10 students

  Measure of the object is not distinguished from the object

  - Example: hot chocolate = x; x [on top of] 2 hot chocolate

- Nonsense manipulation: 1 out of 10 students

  Operations on equations do not make sense quantitatively

  - Example: $-22.15 = -3x - 3y$ does not follow from any of the assumptions about the costs of hot chocolate and cheesecake in the problem

- Quantity = Number: 6 out of 10 students
    - Example: Price is written as an abstract number
- Quantitative Negligence: 4 out of 10 students
    - Example: Not all conditions on the quantities taken into account in solving the problem

Defects of arithmetic skills

The following type of defects in arithmetic was observed in 2 out of 12 solutions:

- Computational negligence

Example:

- No attention paid to the place value of digits: $13.07 = 2x + 2y$; takes 13.07 to mean the same as 13.7

- Number sense lacking

Example:

- Subtracting a bigger positive number from a smaller positive number and obtaining a positive number: $11.05 - 33.15 = 22.1$

Defects of algebraic skills

The following types of defects in arithmetic were observed in 6 out of 12 student:

- Does not use algebraic language as an operational symbolism: 4 out of 6
- Does not apply the distributivity law: 1 out of 6
- Fails to change the sign when moving a term to the other side of the equation: 1 out of 6

  Example:

$$13.07 = 2x + 2y$$

$$2x = -13.07 + 2y$$

$$x = \frac{-13.07 + 2y}{2}$$

5.3.4 Question 4

We recall the text of Question 4.

> Jean solves the problem: "Brigitte goes to the store. She buys the same number of books and records. The books cost \$2 each and the records \$6 each. She spends \$40 in all. How many books and records did she buy?" Jean answers the problem as follows: 2x + 6y = 40, since x = y, I can write: 2x + 6y = 40, 8x = 40. The last equation shows that 8 books cost \$40, so one book costs \$5."
>
> Questions:
>
> 1. Is this solution correct? Justify your answer.
>
> 2. Does the last equation indeed show that 1 book cost \$5?

Note: The second question was asked in case participants looked only at the algebraic calculations and ignored the conclusion in the last sentence of Jean's solution.

Correct Answer for Question 4.1: Jean's calculations are correct. However, the conclusion is incorrect. One book does not cost \$5. Based on the problem, it costs \$2. The unknown x represents the number of books Brigitte bought and the unknown y represents the number of records Brigitte bought. Thus, Brigitte bought 5 books and 5 records.

Correct Answer for Question 4.2: No. The unknown x represents the number of books Brigitte bought and the unknown y represents the number of records Brigitte bought. Thus, the last equation shows that Brigitte bought 5 books and 5 records.

In two solutions (S1 and S8), the flaw of Jean's solution was clearly and correctly identified. One solution (S9) also identified the flaw, but it contained also a statement that raised hesitations as to the participant's clear awareness of the flaw. We decided to give this solution the benefit of the doubt and classified it in the same group as S1 and S8, as a correct solution.

Six solutions claimed that there is a flaw in Jean's solution. Four of them attributed it to irrelevant factors. According to four of these (S3, S5, S6 and S7), the flaw was in the assumption that x = y. A fourth solution (S2) appeared to suggest that the answer was numerically incorrect. The fifth solution (S14) only stated that Jean's solution is incorrect, but no justification was given.

In three solutions (S4, S10 and S13), Jean's solution was considered correct, possibly because participants ignored the conclusion and looked only at the sequence of algebraic equations.

5.3.4.1 The flaw of Jean's solution is correctly identified

Three solutions fell into this category: S1, S8 and S9. Their transcripts are below.

> **S1's solution**
>
> [Question 4.1]: The solution is correct in showing how many books and records she bought. She bought the same number of each, so, they are indeed the same variable. They both equal 5.
>
> [Question 4.2]: The last equation does not show that 1 book costs 5$. It shows she bought 5 books at 2$ each. The price for the merchandises is already indicated in the question, they are not unknown variables.
>
> **S8's solution**
>
> [Question 1.] $\overbrace{\text{\# books}}^{a} = \overbrace{\text{\# records}}^{b}$ // 2$ * a + $6 * b = 40 $ // 8 $ x = 40 $ // x = 5 # books & # records // No, it shows the number of books and # of records bought // because a # book = a # record bought // So she bought 5 books for 2 $ each & 5 records for 6$ each.
>
> [Question 2.] No
>
> **S9's solution**
>
> Books = 2$ // Records = 6$ // Total (40$) // [Question 1]: it is not correct, because x does not equal y in price but in quantity // [Question 2]: Yes, if it were correctly done, the last line shows that 1 book is 5$

The characteristics of these solutions are in

Table 21.

Characteristics of the solution	S1, S8	S9
Answer	[Jean's solution is incorrect, because 5 refers to number of books or records and not the cost]]	"it is not correct, because x does not equal y in price but in quantity"
Type of letter use	Letter used as a specific unknown	Letter used as a specific unknown
Defects of quantitative reasoning	None observed	None observed
Defects of arithmetic skills	None observed	None observed
Defects of algebraic skills	None observed	None observed

Table 21. Question 4 - Solutions of type: Flaw correctly identified

5.3.4.2 The flaw of Jean's solution is incorrectly identified

As mentioned, six solutions fell into this category.

In three of these solutions and from the interviews, the flaw was clearly attributed to the assumption that x equals y. Here are the transcripts of these solutions.

S5's solution

[Question 1.] NO → $x \neq y$

[Question 2.] x = books // y = records // therefore $x \neq y$ // $2x + 6y = 40$

S6's solution

[Question 1.] NO [because] x doesn't = y

[Question 2.]Says in the question that books cost $2,so it can' t be $5. // So 8 books would cost $16.

S7's solution

[Did not write anything. In the interview, he mentioned during the interview that the solution was incorrect because x could not equal to y.]

In one solution (S3), the flaw was seen also in the assumption that $x = y$, but not because of the

Measure = Object defect of quantitative reasoning but rather because of the Quantitative negligence defect (this participant appeared not to notice the assumption that the number of books bought was the same as the number of records bought) and Bad quantitative grammar. Here is the transcript of S3's solution.

S3's solution

The answer could be anything because she could buy 17 books and 1 records or 14 books and 2 records and so on… (e.g., 11 books and 3 records). // let x = books; let y = records // x does not necessarily = y

[Question 1]: No, it is not [correct]. Books cost 2$ each, not 5.

[Question 2]: Yes it does [show that a book costs $5], but the cost of 1 book is $2 not $5.

[Then solves the problem for himself, but ignoring the assumption that # books = # records] Let x = number of books.// Let y = number of records// So $2x + $6y= $40 // $2x = $40 - $6y // $2x /2 = ($40 - $6y)/2 // $x = $20 - $6y // Solve for y // $2x + 6y = $40 // 2($20 - $6y) + 6y = $40 // ($40 - $12y) + 6y = $40 // -$12y + $6y = $40 - $40 // -$ 6y = $0 // -6y/6 = $0/6 // y = 0

Note: The Bad Quantitative grammar appeared in the way quantities were used by S3 in his response to question 4.2. He interpreted the last equation in Jean's solution as being about quantities of dollars (so the price) and not about abstract numbers:

$$\$8x = \$40$$

He then divided both sides by 8 (not 8 dollars) and obtained:

$$\$x = \$5$$

So the result is 5 dollars, not 5 books. This could justify, for him, Jean's conclusion as being correct.

In the solution of S2, the flaw appeared to be attributed to a computational mistake.

S2's solution

[Question 1.] NO

[Question 2.] # the books $= x$ // # the records $= y$ // $2x + 6y = 40$ // $6y = 40 - 2$ // $y = \frac{38}{6}$ // $y = 6$ // $2x + 36 = 40$ // $2x = 40 - 36 =$ $x = \frac{4}{2}$ // $x = 2$

The fifth solution (S14) only stated that Jean's solution is incorrect, but no justification was given.

The other solutions in this group are characterized in Table 22.

Characteristics of the solution	S5, S6, S7	S2	S3
Answer	[Not correct because $x \neq y$]	[Not correct because of computational mistakes]	[Not correct because x is not necessary equal to y; still, the conclusion could be correct if reasoning interpreted in a certain way]
Type of letter use	Letter used as an object	Letter used as a specific unknown	Letter used as a specific unknown
Defects of quantitative reasoning	**Measure = Object** (number of objects not distinguished from the objects)	None observed	**Quantitative negligence** (misses the assumption that the number of books was equal to the number of records) **Bad quantitative grammar** (see note to S3's solution above)
Defects of arithmetic skills	None observed	Computational negligence: 38/6 = 6	None observed
Defects of algebraic skills	None observed	Incorrect processing of equations: believes that 2x + 6y = 40 implies 6y = 40 – 2	Not applying the distributivity law: $\frac{a+b}{c} = \frac{a}{c} + b$; $c(a + b) = ca + b$ Circular substitutions

Table 22. Question 4 – Solutions of type: Flaw incorrectly identified or acknowledged but not identified

5.3.4.3 Acceptance of Jean's solution as correct

Three solutions (S4, S10 and S13), claimed that Jean's solution was correct. Here are the transcripts. S4 and S13 appeared to look only at the equations in Jean's solution and ignored to conclusion in their evaluation. S4 interpreted the last equation correctly, in terms of number of books (or records), not prices.

S4's solution

[Question 1]: Yes, the solution is correct; because if 2(5) + 6(5) = 40 [then] $10 + $30 = $40;

[Question 2]: She bought 5 books and 5 records. Book costs $2.00.

S13's solution appears to only check the algebra and finding nothing wrong with it.

S13's solution

$8x = 40$ // $x = \frac{40}{8} = 5$

S10's solution is ambiguous. The answer to Question 4.1 is "no", but it is "yes" to Question 4.2. The participant appears to demonstrate why Jean's answer is incorrect numerically. He starts by writing some equations, but treats the letters as shorthand for "books" and "records" and guesses (incorrectly) that 10 books for 2 $ each and 5 records for 6$ each would total $40.

S10's solution

[Question1]: No. // 2x + 6y = 40 // x+y = ? // 5x + 5y = 40 // 2(5)x + 6(5)y = $40 // 10 books + 5 records = $40

[Question 2]: Yes, it does.

The three solutions in this group are characterized in Table 23.

Characteristics of the solution	**S4, S13**	**S10**
Answer	[Correct because algebra correct]	[Answer numerically incorrect, but conclusion correct]
Type of letter use	Letter used as a specific unknown	Letter used as an object
Defects of quantitative reasoning	None observed	**Bad Quantitative Grammar**: In "x + y = ?" the sign + means "and" in the sentence "books and records"; the equation 5x + 5y = 40 does not represent the given relations)
Defects of arithmetic skills	None observed	**Computational negligence** (10 books for $2 each and 5 records for $6 each cost $40)
Defects of algebraic skills	None observed	None observed

Table 23. Question 4 - Solutions of type: Jean's solution accepted as correct

5.3.4.4 Summary of analysis of Question 4

Categories of solutions

- The flaw of Jeans' solution correctly identified: 3 solutions out of 12
- The flaw of Jean's solution is incorrectly identified: 6 out of 12
- Acceptance of Jean's solution as correct: 3 out of 12

Characteristics of the solutions

Types of letter use:

- As a specific unknown: 7 out of 12
- As an object: 4 out of 12
- [We cannot say anything about the type of letter use]: 1 out 12

Defects of quantitative reasoning

- None observed (for lack of evidence): 7 out of 12
- **Measure = Object** observed in 3 out of 12 student
- **Quantitative Negligence** observed in 1 out of 12
- **Bad Quantitative Grammar** observed in 2 out of 12

Defects of arithmetic skills

- Computational negligence, observed in 2 students out of 12.

  Examples:

  - "10 books for $2 and 5 records for $6 costs $40"; 10 books and 5 records should cost $50, not $40.
  - $\frac{38}{6} = 6$; but $\frac{38}{6}$ does not equal 6.

Defects of algebraic skills

Observed in 2 students out of 12. The types are:

- Distributivity law not observed
- Adding the same number on both sides of the equation is not observed
- Circular substitutions

5.4 SUMMARY OF RESULTS OF ANALYSIS OF ALL QUESTIONS

In the following summary, we will combine the results discussed in section 5.3. We will compare the type of solutions from Question 1 and Question 2, compare the letter use in all the questions, and list all the defects found in the solutions.

5.4.1 Types of solutions for Question 1 and Question 2

Since Question 1 is a connected problem and Question 2 is a disconnected problem, we expected the solutions to be more often arithmetic in Question 1 and more often algebraic in Question 2. From the analysis and

Table 24, the majority of the participants indeed used an arithmetic approach for Question 1. On the other hand, for Question 2, only 2 out of 12 had an arithmetic solution. Unexpectedly, however, the disconnected problem did not produce many algebraic solutions. Half of the solutions for Question 2 could not be classified as arithmetic or algebraic. Most participants tried to use letters and equations to solve the problem, but only 6 used letters as specific unknowns. In as many as 4 of the 12 solutions, letters were viewed as objects. The number of defects prevented the solutions to be classified as arithmetic or algebraic.

	Question 1	Question 2
Purely arithmetic solutions	9	2
Purely algebraic solutions	1	3
Mix of arithmetic and algebraic approaches in a solution	2	1
Solution not arithmetic and not algebraic	0	6

Table 24. Comparison of types of solutions in Questions 1 and 2

5.4.2 Types of letter use

For Question 1, the connected problem, 6 participants out of 12 chose not to use letters, and nine, in total, did not use letters in an algebraic way. On the other hand, in Question 2, the disconnected problem, 5 participants used letters as specific unknowns. This algebraic use of letters was more present also in Questions 3 and 4, although they could be solved arithmetically, with good quantitative reasoning. (Table 25) But good quantitative reasoning was rare among the participants (see next section). Algebraic use of letters in Questions 3 and 4 turned out not to be of much help in solving these problems successfully, however there were 2 correct responses in Question 3 and 3 correct responses in Question 4.

	Question 1	Question 2	Question 3	Question 4
Letter not used	6	2	1	0
Letter evaluated	2	0	0	0
Letter used as an object	1	5	4	4
Letter as a specific unknown	3	5	7	7

Table 25. Types of letter use in Questions 1-4

5.4.3 Defects of Quantitative Reasoning

There were multiple types of defects of quantitative reasoning observed in each question.

The following defects were found in this study:

- **Bad Quantitative Grammar** was the most frequent defect. It was found in all the questions except for Question 3, and was observed at least once in 11 of the total of 14 participants in the study. It was found in 6 solutions in Question 1 (S2, S4, S5, S6, S7, S10); 9 solutions in Question 2 (S3, S4, S6, S7, S8, S9, S10, S11, S12), and 2 solutions in Question 4 (S3, and S10).

  Examples: Inconsistent use of operation signs; Adding abstract numbers to quantities; Representing relations assumed in a problem by inappropriate operations (ages in two years are represented by multiplication by 2; the Students-and-Professors mistake); using signs of operations and the equals sign as shorthand for ordinary words such as "and" or "is"; dividing by 8 both sides of an equation such as $8x = $40, where x represents a number of objects and $8x the price of these x objects, and obtaining $x = $5, as if "$" represented a variable; 630 × 13 % = 548.10 is claimed to imply that 13 % of 630$+630$= 548.10$; not converting quantities into the same units before operating on them.

- **Quantitative Negligence** was found in all the questions. It was observed at least once in 10 of the 14 participants in the study. It appeared in 3 solutions in Question 1 (S8, S10 and 11); 5 solutions in Question 2 (S2, S3, S4, S5 and S6); 4 solutions in Question 3 (S8, S9, S10 and S14) and in 1 solution in Question 4 (S3).

  Examples: Misreading the conditions on the quantities in a problem: monthly salary taken as weekly salary and then weekly number of hours taken as daily number of hours; "after tax deduction" read as "before tax deduction"; "Marvin is 9 years 3 months older than Mary" read as “Marvin is 9 years 3 months old”, etc.

- **Quantity = Number** was found in all the questions except for Question 4. It was observed in 9 of the 14 participants at least once: in 2 solutions in Question 1 (S8, S12); 3 solutions in Question 2 (S6, S9, S10); 6 solutions in Question 3 (S2, S3, S5, S6, S9, S13).

  Examples: treating percents as abstract numbers (13% = 0.13); equating 9 years 3 months with the number 9.3; giving a price by an abstract number.

- **Measure = Object** was found in Question 3 and Question 4. It was observed in 6 of the 14 participants in the study: in 5 solutions in Question 3 (S2, S3, S5, S6, S13) and 3 solutions in Question 4 (S5, S6, S7).

  Examples: not distinguishing between a cup of hot chocolate and the price of a cup of hot chocolate, a book and a price of a book, etc.

- **Additive conception of percent** was found only in Question 1. It was observed in 6 participants (S4, S5, S6, S7, S8, and S9). This defect is explained in section 5.2.1.

- **Nonsense Manipulation** was found in Question 2 and Question 3. It was observed in 2 participants: once in Question 2 (S3) and once in Question 3 (S7).

- **Unnecessarily complicated reasoning** was found only in Question 1, and was observed in only one participant (S3).

5.4.4 Defects of Arithmetic skills

Overall, participants were not lacking in arithmetic skills. The most frequent defect was

- Computational negligence

which was found in all the questions, and it was observed in 1 solution in Question 1 (S7); 1 solution in Question 2 (S9); 2 solutions in Question 3 (S7 and S13) and in 2 solutions in Question 4 (S2 and S1). Examples copying the output of 548.1 x 0.13 on a calculator as 71.53 instead of 7.253; writing x

= 4.4 and x = 44/(-10) side by side and not noticing the contradiction; taking 13.07 to be the same as 13.70; dividing 38 by 6 and obtaining 6; claiming that 10 books for $2 each and 5 records for 6$ each cost 40 together.

The more serious arithmetic skills defect of

- Poor number sense

was found only in 2 solutions: 1 solution in Question 1 (S12) and 1 solution in Question 3 (S7).

Examples: multiplying by a number then dividing the result by the same number; subtracting a bigger number from a smaller one, both positive, and obtaining a positive number.

5.4.5 Defects of Algebraic skills

We have grouped the many examples of defects of algebraic skills observed in the solutions into a few larger categories. The largest is the failure to respect the basic rules of algebraic processing of expressions, which we called "Bad algebraic grammar." The categories are presented below, with examples of their manifestation.

- **Bad algebraic grammar**: observed in 14 solutions across all questions.
  - Distributivity law – not applied: observed in 3 solutions, in Question 1 (S1), Question 3 (S7) and Question 4 (S3). In Question 4, S3 appeared to follow rules such as:

    $\frac{a+b}{c} = \frac{a}{c} + b;\ c(a+b) = ca + b$
  - Incorrect processing of equations: observed in 4 solutions, in Question 2 (S3, S9), Question 3 (S13), and Question 4 (S2).

    Examples include: ignoring some terms when adding two equations together (Q2-S9); failing to change the sign when moving a term to the other side of the equation (Q3-S13); ignoring the variable when moving a term to the other side of the equation 2x + 6y = 40 implies 6y = 40 – 2 (Q4-S2); treating different letters in an equation as representing the same number if they are written with the same letter but different indices ($x_2 - x_3$ becomes $0x$) (Q2-S3)

  - Incorrect use of the equal sign: observed in 6 solutions: in Question 1 (S1, S4, S5, S6) and Question 2 (S1, S10).

Examples: chain writing of operations (17.50-2.225=15.225 * 3 = 45.675, Q1-S1; 9y * 12m = 108m + 3 months, Q2-S1; $\$ 548.10 + 71.35 = \$ 619.35 \div 17.50\ /hr = 35.39\ hrs$ Q1-S5, S6; using "=" for rough approximations (620/17.50 = 36 hours, Q1-S4); writing nothing after the equal sign (+9.3x + (2X 10 y) =, Q2-S10).

  - Interchanging the meaning of the letters in an equation: Observed in one solution, in Question 1 (S4)

- Algebraic language not used as an operational symbolism but as shorthand: observed in 4 solutions in Question 3 (S2, S3, S5, S6)
- **Circular substitutions** were observed in one solution in Question 4 (S3): x is represented in terms of y based on an equation and then plugged back into the same equation.
- **Inability to solve a system of equations with 3 unknowns** was observed in 3 solutions: 1 in Question 2 (S1), and 2 in Question 3 (S8, S14).

5.5 DISCUSSION

When analyzing Question 1 and Question 2, we were interested in the correlation of the type of word problem and the type of approach used to solve the word problem. As mentioned previously, Bednarz & Schmidt (2002) used connected and disconnected problems to reveal students' reasoning. In our study, the same thing happened. There was a strong correlation between connected problems and arithmetic solutions. Most of the participants used an arithmetic approach and did not use any letter in their solution. On the other hand, the analysis of the disconnected problem, Question 2, revealed a decrease of arithmetic solutions and an increase in the use of letter as a specific unknown. However, only 3 out of 12 participants produced an algebraic solution.

In the analysis, the type of letter use was an indicator of the type of solution, so one could expect the relationships, *non-algebraic use of letter – arithmetic solution*, or *algebraic use of letter – algebraic solution*. (Letter not used, Letter evaluated and Letter used as an object are considered non-algebraic uses of letter). But the relationships were not as straightforward. In the connected problem, Question 1, there was a close relationship between type of letter use and type of solution: arithmetic solutions coincided with non-algebraic uses of letter (S2, S3, S4, S5, S6, S7, S8, S10, and S12). If letter was used as a specific unknown, the solution was algebraic or mixed arithmetic-algebraic (S1, S9, S11).

In Question 2 – the disconnected problem, if the letter was used in a non-algebraic way (7 solutions), the solution was arithmetic (S6, S10) or neither algebraic nor arithmetic (S2, S5, S7, S11, S12). Using letter as a specific unknown coincided with an algebraic solution in 3 solutions only (S4, S8, S9); one solution was partly algebraic (S3) and in one case (S1) – neither algebraic not arithmetic.

In Question 3, the 7 solutions which used letter as a specific unknown were exactly those classified as algebraic (S1, S4, S7, S8, S9, S13 and S14). Three of the 4 solutions which used letter as an object (shorthand for hot chocolate or cheesecake) could not be classified as arithmetic of algebraic (S2, S3, and S5). The fourth one (S6) using letter as an object was classified as arithmetic or very weak evidence and could also count as neither arithmetic nor algebraic. The single solution where the letter was not used was classified as arithmetic.

In Question 4, solutions were not classified as arithmetic or algebraic because it required an evaluation of a given solution, not a solution.

In Question 3 and Question 4, we were not interested whether the participant used an algebraic approach or not but hoped to reveal more information about their quantitative reasoning. To a certain extent, these questions did fulfill our expectation because they revealed the Measure = Object defect of quantitative reasoning, which was hidden in Questions 1 and 2. Especially Question 3 has the potential to diagnose this type of defect: it was found in 5 out of 12 solutions.

In Schmidt & Bednarz (2002) there seems to be an assumption that arithmetic approaches to problem solving as such are an obstacle to algebraic approaches. We did not make this assumption. We observed that numeric thinking is an obstacle for algebraic thinking. Following Brown's characterization of quantitative reasoning (Brown, 2012), we assumed that it is the defects of quantitative reasoning that are more likely to create obstacles to successful problem solving using any approach. One of the characteristics of quantitative reasoning highlighted by Brown is that it focuses on relations between quantities more than on the quantities themselves. So the disconnected problems, which give relations between quantities rather than the measures of the quantities, force looking at the relations and therefore require quantitative reasoning for solving. On the other hand, Bednarz and Schmidt (1997) say that disconnected problems are more likely to provoke algebraic solutions. This suggests a link between algebraic solutions and quantitative reasoning and why a well-developed quantitative reasoning could be a prerequisite for successful algebraic problem solving.

Brown (2012) mentioned in her research how it is possible to use any type of problem and make small changes that can allow students improve their quantitative reasoning and make them better prepared to develop algebraic thinking. Using that idea, we used Question 3 and Question 4, which could seem as typical word problems, but in fact are completely new to students. Question 3 reads as an ordinary word problem; however, by adding an element of contradiction, we were able to reveal the shortcomings of students' approaches to problem solving. We were also able to see not only their technical skills in solving systems of equations but also to determine if they saw the final answer as just a number or a value that had to make sense in terms of the quantities in the initial problem. As for Question 4, since this time they had to evaluate somebody else's solution, they could not just reproduce routine behaviors in writing a solution, and we were able to observe how they used letters and how much they paid attention to the quantities in the problem, the units, and the concluding statement in the fictional solution.

Brown (2012) mentioned the key to help students is to turn a word problem, whether arithmetic or algebraic, into a problem that allows students have an algebraic experience. If we compare Question 1 and Question 3, we notice that, although Question 3 is not considered as a connected problem, both questions can be solved with an arithmetic approach. However, Question 3 was modified to have more relations in the text and have contradictory data. As a result, Question 3 triggered more students to use an algebraic approach and to use letters as specific unknowns.

The research done by Schmidt & Bednarz was to clarify "the difficulties encountered in bridging arithmetic and algebra in a problem-solving context" (Schmidt & Bednarz, 2002, p. 82). On the other hand, the goal of Brown (Brown, 2012) was to provide instructors and students with opportunities to experience quantitative reasoning with any type of problem. In our study, we confirmed the connection between connected problems and arithmetic solution, and observed the importance of quantitative reasoning.

Most of the participants failed to solve correctly all the questions. Although correctness of the answers was not our primary concern in this study, the low success rate in solving the problems was disturbing for us: in Question 1, only three solutions concluded with the correct answer obtained by correct reasoning; in Question 2, no correct answer was obtained; in Question 3 – two answers were correct and there were three correct answers to Question 4. It was even more disturbing that the participants did not seem to be interested in the correctness of their answers; they rarely checked them.

One explanation of this massive failure is that the solutions presented many defects of quantitative reasoning. One or more of the defects we called *Quantitative Negligence*, *Bad Quantitative Grammar*, *Quantity = Number* or *Measure = Object* were found in all except one (S1) of the 14 participants in our study. Four participants displayed all four defects and three participants – the first three. Five participants displayed 2 of the defects. Although Question 1, compared to the other questions, could be considered the easiest one, most of the participants (9) failed to solve it correctly because they did not focus on the relationships given in the problem, ignored units, and confused quantities with abstract numbers. Most of the mistakes in this question were related to poor understanding of percents, especially – the defect we called "additive conception of percent."

Some defects of quantitative reasoning could be linked to the epistemological obstacles identified in Chapter 2. The Measure = Object defect can be seen as a manifestation of the Quantitative obstacle: the measure of an object is not abstracted from the object. This obstacle is a defect of quantitative reasoning because it indicates a strong focus on the object rather than the given relationships.

On the other hand, we realized that certain participants were facing the extreme opposite of QO: we could call it the Numerical obstacle, present in the defect of Quantity = Number: quantities in a problem are ignored; the numbers count only. Historically, the Numerical obstacle could be identified especially in the formal approaches to mathematics in the 19^{th} and 20^{th} centuries: ignoring the units by the numbers and the meanings of the numbers of units as referring to quantities in statements of application problems, and operating on the pure numbers only.

The Ontological obstacle could explain why some participants were unable to move beyond naming the quantities with letters. They could not operate algebraically on these letters because these operations did not make sense: how can one add a hot chocolate to a cheesecake?

6 CONCLUSIONS

Many studies done on algebra education are about problem solving. Our study focused on analyzing four specific types of problems and on understanding students' solutions. We gave an emphasis to their approaches and the kind of defects that could have affected their reasoning. We observed students that tried to use algebraic approaches but failed due to a lack of understanding of relations between quantities. No matter the type of problem or the context, we concluded that quantitative thinking was required. We found in this research possible reasons why mature students fail in solving word problems.

This modest study confirmed, for us, the postulate advanced by Stacey Brown (2012) that, to improve students' algebraic problem solving skills, it would be more effective to train them in quantitative reasoning than to focus on practicing techniques of solving equations, although these techniques are also very important.

We identified defects of quantitative reasoning in all four types of problems. Just like Brown (2012) mentions in her project, it does not matter what kind of problem you give to the students, what really matters is how you use the problems to help them to have an algebraic experience. Similarly, based on the results that we have obtained from this research, the question now is what are we going to do as teachers? In order to help students develop algebraic thinking, we need to be aware of these flaws and address them. For future research, the defects mentioned in this study can be used to develop new teaching methods and new exercises to help students overcome those obstacles. Teachers could create exercises specific to quantitative defects they observe in students' solutions; use word problems that would encourage students to develop a better understating on specific concepts, such as percents and the difference between measure of an object and the object itself in a mathematical context. Another possible idea for teachers is that instead of writing the complete solution of a word problem in front of the students, they could have group discussions of different ways to solve the given word problem and ask questions in order to guide them to a correct reasoning. Further investigations of students' reasoning can reveal more aspects on how to improve teaching of algebra. The aim is to provide students with a deeper understanding of mathematical concepts, and to give meaning to their work.

The participants were students who were seeking help in MATH 200. It is safe to assume that they found the course difficult. We are aware that this could have influenced the results of this study. However, as instructors and researchers, they are the students who need our attention. They need help in identifying and overcoming their obstacles. They should be given the opportunity to try new methods and practice on problems that would develop their quantitative reasoning and that would encourage algebraic thinking. Students' defects of quantitative reasoning are not a negative effect of learning, they can be used to identify the obstacles students need to overcome, and to indicate an adjustment in tests and lectures according to those defects.

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