SELECTIONS AND THEIR ABSOLUTELY CONTINUOUS INVARIANT MEASURES

ABRAHAM BOYARSKY, PAWEŁ GÓRA, AND ZHENYANG LI

ABSTRACT. Let $I = [0, 1]$ and let P be a partition of I into a finite number of intervals. Let τ_1, $\tau_2; I \to I$ be two piecewise expanding maps on P. Let $G \subset I \times I$ be the region between the boundaries of the graphs of τ_1 and τ_2. Any map $\tau : I \to I$ that takes values in G is called a selection of the multivalued map defined by G. There are many results devoted to the study of the existence of selections with specified topological properties. However, there are no results concerning the existence of selection with measure-theoretic properties. In this paper we prove the existence of selections which have absolutely continuous invariant measures (acim). By our assumptions we know that τ_1 and τ_2 possess acims preserving the distribution functions $F^{(1)}$ and $F^{(2)}$. The main result shows that for any convex combination F of $F^{(1)}$ and $F^{(2)}$ we can find a map η with values between the graphs of τ_1 and τ_2 (that is, a selection) such that F is the η-invariant distribution function. Examples are presented. We also study the relationship of the dynamics of our multivalued maps to random maps.

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada

E-mails: abraham.boyarsky@concordia.ca, pawel.gora@concordia.ca, zhenyangemail@gmail.com.

1. INTRODUCTION

Multivalued maps have application in economics [3], modeling, and rigorous numerics [8] and in dynamical systems [1, 2]. The objective of this note is to study multivalued maps whose graphs are defined by single valued maps τ_1 and τ_2, which are in the class $\mathcal{T}$ of piecewise expanding, piecewise C^2 maps from I into I. We refer to τ_1, τ_2 as the lower and upper boundaries of the graph $G \subset I \times I$. Since τ_1 and τ_2 are in $\mathcal{T}$, they possess acims with probability density functions (pdf), f_1 and f_2. Any map $\tau : I \to I$ that takes values in G is called a selection of the multivalued map defined by G. There has been much research devoted to the study of the existence of selections with specified topological properties. However, the existence of selections with acims has not been studied. In this paper we prove the existence of such selections.

Motivating examples are presented in Section 2. The first example shows that if the class of transformations is restricted only to the graphs of the lower and upper

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boundary maps τ_1 and τ_2, that is, G consists only of the graphs of these two maps, then there is no transformation that has pdf equal to a convex combination of f_1 and f_2. In the second example, we construct a selection with desired properties in the case where the upper and lower boundary maps are piecewise linear.

In Section 3 we describe the construction of selections with acims when the boundary maps of G are tent like. In Section 4 we present the main result. We assume that the lower and upper boundary maps are in $\mathcal{T}$ and have invariant distribution functions $F^{(1)}$ and $F^{(2)}$. If, for $0<\lambda<1$, the convex combination $F=\lambda F^{(1)}+(1-\lambda)F^{(2)}$ is a homeomorphism of the unit interval, then there exists a piecewise monotonic selection η, $\tau_1\le\eta\le\tau_2$, preserving the distribution function F.

In Section 5 we present an approach to finding selections based on conjugation: if τ_1 is piecewise linear and the τ_2 is conjugated to τ_1 then, for any convex combination f of f_1 and f_2 we can find a map τ with values between the graphs of τ_1 and τ_2 such that f is the invariant pdf associated with τ. In fact, τ is also a conjugacy of τ_1. In Section 6 we study the relationship between the dynamics of multivalued maps and random maps. In particular, we consider a multivalued map consisting of two graphs, and show that in general the statistical long term behaviour of an arbitrary selection of the multivalued map cannot be achieved by a position dependent random map based on the maps defining the multivalued map. A number of positive examples are also presented.

2. Motivating examples

Let us consider a multivalued map T with lower boundary map τ_1 and upper boundary map τ_2 as in Figure 1. If τ_1 preserves a density f_1 and τ_2 preserves a density f_2, then we ask whether for any convex combination $f=\lambda\cdot f_1+(1-\lambda)\cdot f_2$, $0<\lambda<1$, we can find a selection of T which preserves the density f. We present a counterexample showing that if $T=\{\tau_1,\tau_2\}$ (T is two-valued), then it may be impossible.

Example 1.

Let
$$\tau_1(x)=\begin{cases}\frac{4}{3}x, & 0\le x<\frac{3}{8};\\ 4x-1, & \frac{3}{8}\le x<\frac{1}{2};\\ -4x+3, & \frac{1}{2}\le x<\frac{5}{8};\\ -\frac{4}{3}x+\frac{4}{3}, & \frac{5}{8}\le x\le 1,\end{cases}$$
and
$$\tau_2(x)=\begin{cases}3x, & 0\le x<\frac{1}{6};\\ \frac{3}{2}x+\frac{1}{4}, & \frac{1}{6}\le x<\frac{1}{2};\\ -\frac{3}{2}x+\frac{7}{4}, & \frac{1}{2}\le x<\frac{5}{6};\\ -3x+3, & \frac{5}{6}\le x\le 1.\end{cases}$$
The invariant densities are $f_1=\frac{3}{2}\chi_{[0,1/2]}+\frac{1}{2}\chi_{[1/2,1]}$ and $f_2=\frac{2}{3}\chi_{[0,1/2]}+\frac{4}{3}\chi_{[1/2,1]}$, correspondingly. Thus, the Lebesgue measure density is a convex combination of f_1 and f_2, $1=\frac{2}{5}\cdot f_1+\frac{3}{5}\cdot f_2$. In order for a two branch map τ to leave Lebesgue measure invariant, it is necessary to satisfy
$$|\tau_1'^{-1}(x)|+|\tau_2'^{-1}(x)|=1,$$

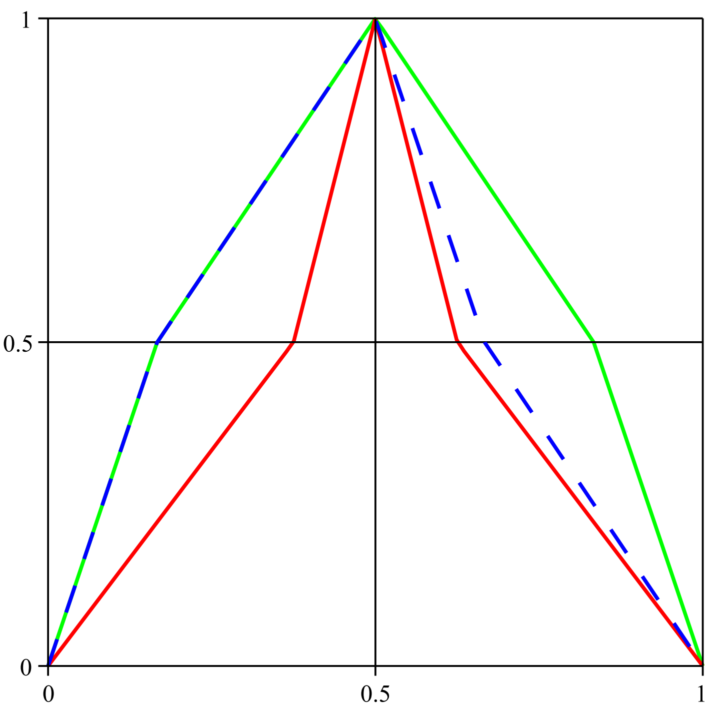

FIGURE 1. Two valued map of Example 1

at the preimages ${\tau_1}^{-1}(x)$ and ${\tau_2}^{-1}(x)$ of every point x, which is impossible.

Example 2.

Again, consider the two maps of Example 1. If we allow at least one of the branches of the map τ to be between the maps τ_1 and τ_2, then we can achieve the invariance of Lebesgue measure. For example, the map given below preserves Lebesgue measure. Its graph is shown in Figure 1 using dashed lines.

$$\tau(x) = \begin{cases} \tau_2(x), & 0 \le x < \frac{1}{2}; \\ -3x + \frac{5}{2}, & \frac{1}{2} \le x < \frac{4}{6}; \\ -\frac{3}{2}x + \frac{3}{2}, & \frac{4}{6} \le x \le 1. \end{cases}$$

3. SELECTIONS FOR SPECIAL CASE OF TENT-LIKE MAPS

We assume that both maps τ_1, τ_2 are increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$ and have values 0 at 0 and 1, 1 at 1/2. We do not assume that the lower map τ_1 is conjugated to the upper map τ_2. Let us assume that τ_1 preserves measure μ_1 and τ_2 preserves measure μ_2, not necessarily absolutely continuous. Let $F^{(1)}, F^{(2)}$ be the distribution functions of measures μ_1, μ_2, respectively ($F^i(x) = \mu_i([0, x])$, $i = 1, 2$). Let $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$, $0 < \lambda < 1$, and let F be the distribution function of μ:

$$F(x) = \mu([0, x]).$$

We are looking for a map η satisfying $\tau_1 \le \eta \le \tau_2$ that preserves the distribution function F (or equivalently measure μ).

We introduce the function $s : [0, 1/2] \to [1/2, 1]$, which relates the branches η_1, η_2 of η. Let

$$\eta_2(x) = \eta_1(s^{-1}(x)). \tag{1}$$

The Frobenius-Perron operator of η is given by

$$(P_\eta F)(x) = F(\eta_1^{-1}(x)) + 1 - F(\eta_2^{-1}(x)).$$

Thus the fixed point of this operator is given by

$$F(x) = F(\eta_1^{-1}(x)) + 1 - F(\eta_2^{-1}(x)),$$

or

$$F(\eta_1(z)) = F(z) + 1 - F(s(z)),$$

or

$$F(s(z)) = 1 + F(z) - F(\eta_1(z)),$$

which allows us to find s once η_1 is given:

$$s(z) = F^{-1}(1 + F(z) - F(\eta_1(z))). \tag{2}$$

Thus, once we construct η_1 satisfying $\tau_{1,1} \le \eta_1 \le \tau_{2,1}$ we obtain η_2 and have to check if it satisfies the required inequalities. We will show that η_1 can be chosen in such a way that the graph of η_2 is between the graphs of τ_1 and τ_2.

Let us assume

$$\tau_{2,1}^{-1} \le \eta_1^{-1} \le \tau_{1,1}^{-1}, \tag{3}$$

which is equivalent to $\tau_{1,1} \le \eta_1 \le \tau_{2,1}$, and

$$F = \lambda F^{(1)} + (1-\lambda) F^{(2)}. \tag{4}$$

The fixed points of the Frobenius-Perron operators for τ_1 and τ_2 yield

$$F^{(1)}(x) = F^{(1)}(\tau_{1,1}^{-1}(x)) + 1 - F^{(1)}(\tau_{1,2}^{-1}(x)),$$
$$F^{(2)}(x) = F^{(2)}(\tau_{2,1}^{-1}(x)) + 1 - F^{(2)}(\tau_{2,2}^{-1}(x)),$$

or

$$\begin{aligned} F^{(1)}(\tau_{1,2}^{-1}(x)) &= F^{(1)}(\tau_{1,1}^{-1}(x)) + 1 - F^{(1)}(x), \\ F^{(2)}(\tau_{2,2}^{-1}(x)) &= F^{(2)}(\tau_{2,1}^{-1}(x)) + 1 - F^{(2)}(x), \\ F(\eta_2^{-1}(x)) &= F(\eta_1^{-1}(x)) + 1 - F(x), \end{aligned} \tag{5}$$

where we have also included the foregoing fixed point equation for η. We want to show that

$$\tau_{1,2}^{-1}(x) \le \eta_2^{-1}(x) \le \tau_{2,2}^{-1}(x),$$

or, equivalently, that

$$F(\tau_{1,2}^{-1}(x)) \le F(\eta_2^{-1}(x)) \le F(\tau_{2,2}^{-1}(x)). \tag{6}$$

First, we will show that it is possible to choose η_1 in such a way that

$$F(\tau_{1,2}^{-1}(x)) \le F(\eta_2^{-1}(x)). \tag{7}$$

Using (4) and (5) we obtain the following inequalities, all of which are equivalent to (7).

$$\lambda F^{(1)}(\tau_{1,2}^{-1}(x)) + (1-\lambda) F^{(2)}(\tau_{1,2}^{-1}(x)) \le F(\eta_1^{-1}(x)) + 1 - F(x), \tag{8}$$

$$\begin{aligned}&\lambda\left[F^{(1)}(\tau_{1,1}^{-1}(x))+1-F^{(1)}(x)\right]+(1-\lambda)F^{(2)}(\tau_{1,2}^{-1}(x))\\&\quad\le\lambda F^{(1)}(\eta_1^{-1}(x))+(1-\lambda)F^{(2)}(\eta_1^{-1}(x))+1-\lambda F^{(1)}(x)-(1-\lambda)F^{(2)}(x),\end{aligned}\tag{9}$$

$$\begin{aligned}&\lambda F^{(1)}(\tau_{1,1}^{-1}(x))+(1-\lambda)F^{(2)}(\tau_{1,2}^{-1}(x))\\&\quad\le\lambda F^{(1)}(\eta_1^{-1}(x))+(1-\lambda)F^{(2)}(\eta_1^{-1}(x))-(1-\lambda)[F^{(2)}(x)-1].\end{aligned}\tag{10}$$

Using (5), we obtain

$$\begin{aligned}&\lambda F^{(1)}(\tau_{1,1}^{-1}(x))+(1-\lambda)F^{(2)}(\tau_{1,2}^{-1}(x))\\&\quad\le\lambda F^{(1)}(\eta_1^{-1}(x))+(1-\lambda)F^{(2)}(\eta_1^{-1}(x))-(1-\lambda)\left[F^{(2)}(\tau_{2,1}^{-1}(x))-F^{(2)}(\tau_{2,2}^{-1}(x))\right],\end{aligned}\tag{11}$$

or,

$$\begin{aligned}&\lambda\left[F^{(1)}(\tau_{1,1}^{-1}(x))-F^{(1)}(\eta_1^{-1}(x))\right]+(1-\lambda)\left[F^{(2)}(\tau_{2,1}^{-1}(x))-F^{(2)}(\eta_1^{-1}(x))\right]\\&\quad\le(1-\lambda)\left[F^{(2)}(\tau_{2,2}^{-1}(x))-F^{(2)}(\tau_{1,2}^{-1}(x))\right],\end{aligned}\tag{12}$$

which is also equivalent to (7). Since $\tau_{2,2}^{-1}\ge\tau_{1,2}^{-1}$ the right hand side of (12) is positive, independent of the choice of η_1. Since $\tau_{2,1}^{-1}\le\eta_1^{-1}\le\tau_{1,1}^{-1}$, the first term on the left hand side is positive (and zero for $\eta_1=\tau_{1,1}$) and the second term is negative. This shows that there is an interval $I_1(x)$ touching $\tau_{1,1}(x)$ such that if we choose $\eta_1(x)$ in this interval, then the inequality (12) and thus (7) will be satisfied.

Now, we will show the second part of (6), i.e., that it is possible to choose η_1 in such a way that

$$F(\eta_2^{-1}(x))\le F(\tau_{2,2}^{-1}(x)).\tag{13}$$

As above, we manipulate (13) to obtain

$$\begin{aligned}&(1-\lambda)\left[F^{(2)}(\eta_1^{-1}(x))-F^{(2)}(\tau_{2,1}^{-1}(x))\right]+\lambda\left[F^{(1)}(\eta_1^{-1}(x))-F^{(1)}(\tau_{1,1}^{-1}(x))\right]\\&\quad\le\lambda\left[F^{(1)}(\tau_{2,2}^{-1}(x))-F^{(1)}(\tau_{1,2}^{-1}(x))\right].\end{aligned}\tag{14}$$

Since $\tau_{2,2}^{-1}\ge\tau_{1,2}^{-1}$ the right hand side is positive, independent of the choice of η_1. Since $\tau_{2,1}^{-1}\le\eta_1^{-1}\le\tau_{1,1}^{-1}$, the first term on the left hand side is positive (and zero for $\eta_1=\tau_{2,1}$) and the second term is negative. This shows that there is an interval $I_2(x)$ touching $\tau_{2,1}(x)$ such that if we choose $\eta_1(x)$ in this interval, the inequality (14) and thus (13) will be satisfied.

Now, we will show that there exists an η_1 satisfying inequalities (3), (12) and (14). Note that the left hand size of (12) is equal to minus the left hand side of (14), while both right hand sides are positive. We choose η_1 such that the left hand side of (12) (and also of (14)) is 0. Then, both inequalities are satisfied. This is possible, at least if we assume that $F^{(1)}$, $F^{(2)}$ are continuous.

We now solve for η_1 as follows:

$$\lambda F^{(1)}(\eta_1^{-1}(x))+(1-\lambda)F^{(2)}(\eta_1^{-1}(x))=(1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x))+\lambda F^{(1)}(\tau_{1,1}^{-1}(x)),$$

which implies that

$$F(\eta_1^{-1}(x))=(1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x))+\lambda F^{(1)}(\tau_{1,1}^{-1}(x)),$$

or,

$$\eta_1^{-1}(x)=F^{-1}\left((1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x))+\lambda F^{(1)}(\tau_{1,1}^{-1}(x))\right),\tag{15}$$

assuming F is strictly increasing.

Now we show that $\eta_1(x)$ defined by equation (15) satisfies assumption (3), i.e., the graph of η_1 is located between the graphs of $\tau_{1,1}$ and $\tau_{2,1}$. First,

$$\tau_{2,1}^{-1}(x) \le \eta_1^{-1}(x)$$

is equivalent to

$$\tau_{2,1}^{-1}(x) \le F^{-1}\left((1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(x))\right),$$

thus,

$$F(\tau_{2,1}^{-1}(x)) \le (1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(x)),$$

or

$$\lambda F^{(1)}(\tau_{2,1}^{-1}(x)) + (1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x)) \le (1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(x)),$$

or

$$F^{(1)}(\tau_{2,1}^{-1}(x)) \le F^{(1)}(\tau_{1,1}^{-1}(x)),$$

which is true since $\tau_{2,1}(x) \ge \tau_{1,1}(x)$ and both are increasing. On the other hand,

$$\eta_1^{-1}(x) \le \tau_{1,1}^{-1}(x)$$

is equivalent to

$$F^{-1}\left((1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(x))\right) \le \tau_{1,1}^{-1}(x),$$

or

$$(1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(x)) \le F(\tau_{1,1}^{-1}(x)),$$

or

$$(1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(x)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(x)) \le \lambda F^{(1)}(\tau_{1,1}^{-1}(x)) + (1-\lambda)F^{(2)}(\tau_{1,1}^{-1}(x)),$$

or

$$F^{(2)}(\tau_{2,1}^{-1}(x)) \le F^{(2)}(\tau_{1,1}^{-1}(x)),$$

which is true by the same reason as in the first case.

Note that $\eta_1(x)$, defined by equation (15), is increasing and continuous since all the functions defining $\eta_1(x)$ are increasing and continuous. Also,

$$\eta_1^{-1}(0) = F^{-1}\left((1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(0)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(0))\right) = F^{-1}(0) = 0,$$

$$\eta_1^{-1}(1) = F^{-1}\left((1-\lambda)F^{(2)}(\tau_{2,1}^{-1}(1)) + \lambda F^{(1)}(\tau_{1,1}^{-1}(1))\right) = F^{-1}\left(F(1/2)\right) = 1/2.$$

Actually, for the tent like map we are considering, it is not necessary to assume that the maximum is achieved at $1/2$, it can be any point in $(0,1)$.

4. Main result

Our main result is the following theorem.

Theorem 1. *Let us consider a multivalued map from the unit interval into itself whose lower and upper boundary maps τ_1 and τ_2 are piecewise monotonic, their invariant distribution functions $F^{(1)}$ and $F^{(2)}$ are continuous, and for any $0 < \lambda < 1$, the convex combination $F = \lambda F^{(1)} + (1-\lambda)F^{(2)}$ is a homeomorphism of the unit interval. Then, there exists a piecewise monotonic selection η, $\tau_1 \le \eta \le \tau_2$, preserving the distribution function F.*

Proof. We assume that the partition points for τ_1 and τ_2 are the same: $a_0 = 0 < a_1 < a_2 < \cdots < a_m = 1$. Let $I_j = [a_{j-1}, a_j]$, $j = 1, 2, 3, \ldots, m$. On each interval I_j, $\tau_{1,j} := \tau_1|_{I_j}$ and $\tau_{2,j} := \tau_2|_{I_j}$ share the same monotonicity, where we understand $\tau_{1,j}$ and $\tau_{1,j}$ as the natural extensions of branches of τ_1 and τ_2, respectively.

For any interval $[a, b] \subseteq [0, 1]$, given a monotone continuous function $h : [a, b] \to [0, 1]$ (not necessarily onto), we define its extended inverse as follows. Let

$$h^{\max} = \max \{h(x) | x \in [a, b]\},$$

and

$$h^{\min} = \min \{h(x) | x \in [a, b]\}.$$

If h is increasing, then its extended inverse is defined as

$$\overline{h^{-1}}(x) = \begin{cases} a\ , \text{ for } \ x \in [0, h^{\min}]; \\ h^{-1}(x)\ , \text{ for } \ x \in [h^{\min}, h^{\max}]; \\ b\ , \text{ for } \ x \in [h^{\max}, 1]. \end{cases}$$

If h is decreasing, then its extended inverse is defined as

$$\overline{h^{-1}}(x) = \begin{cases} b\ , \text{ for } \ x \in [0, h^{\min}]; \\ h^{-1}(x)\ , \text{ for } \ x \in [h^{\min}, h^{\max}]; \\ a\ , \text{ for } \ x \in [h^{\max}, 1]. \end{cases}$$

We define the extended inverse of each branch of η by

$$\overline{\eta_j^{-1}}(x) = F^{-1}\left(\lambda F^{(1)}(\overline{\tau_{1,j}^{-1}}(x)) + (1-\lambda)F^{(2)}(\overline{\tau_{2,j}^{-1}}(x))\right), \tag{16}$$

where $j = 1, 2, 3, \ldots, m$. The function η defined in this way, after the vertical segments are removed, has the same number of branches as τ_1 and τ_2, and each branch of it has the same monotonicity as the corresponding branches of the boundary maps.

First, we show that the graph of η is located between the graphs of τ_1 and τ_2. For some $j \in \{1, 2, \ldots, m\}$, we will show this for the case when $\tau_{1,j}$ and $\tau_{2,j}$ are increasing. The proof for the case when $\tau_{1,j}$ and $\tau_{2,j}$ are decreasing is similar. We need to show:

$$\overline{\tau_{2,j}^{-1}}(x) \le \overline{\eta_j^{-1}}(x) \le \overline{\tau_{1,j}^{-1}}(x),$$

which is equivalent to

$$F\left(\overline{\tau_{2,j}^{-1}}(x)\right) \le F\left(\overline{\eta_j^{-1}}(x)\right) \le F\left(\overline{\tau_{1,j}^{-1}}(x)\right),$$

or, using (16),

$$\begin{aligned} \lambda F^{(1)}(\overline{\tau_{2,j}^{-1}}(x)) \quad + \quad & (1-\lambda)F^{(2)}(\overline{\tau_{2,j}^{-1}}(x)) \\ & \le \lambda F^{(1)}(\overline{\tau_{1,j}^{-1}}(x)) + (1-\lambda)F^{(2)}(\overline{\tau_{2,j}^{-1}}(x)) \\ & \le \lambda F^{(1)}(\overline{\tau_{1,j}^{-1}}(x)) + (1-\lambda)F^{(2)}(\overline{\tau_{1,j}^{-1}}(x)), \end{aligned}$$

which is true, because $\overline{\tau_{2,j}^{-1}}(x) \le \overline{\tau_{1,j}^{-1}}(x)$ since $\tau_{1,j}$ and $\tau_{2,j}$ are increasing.

Now, for any $x \in [0, 1]$, using the previous notation, we have

$$\overline{\eta^{-1}}([0, x]) = \bigcup_{j=1}^{m} [\overline{\eta_j^{-1}}^{l}(x), \overline{\eta_j^{-1}}^{r}(x)],$$

$$\overline{\tau_1^{-1}}([0,x]) = \bigcup_{j=1}^{m} [\overline{\tau_{1,j}^{-1}}^l(x), \overline{\tau_{1,j}^{-1}}^r(x)],$$

$$\overline{\tau_2^{-1}}([0,x]) = \bigcup_{j=1}^{m} [\overline{\tau_{2,j}^{-1}}^l(x), \overline{\tau_{2,j}^{-1}}^r(x)],$$

where the bars over inverses of maps η^{-1}, τ_1^{-1} and τ_2^{-1} imply that the extended inverses are used for each branch. Note that all the three maps have the same monotonicity for each corresponding branch. Moreover, for some x, the intervals appearing on the right hand side of the above preimages may only contain one point. For example, if $x \in [0, \tau_{1,j}^{\min}]$, where $j \in \{1, 2, \dots, m\}$, then $\overline{\tau_{1,j}^{-1}}^l(x) = \overline{\tau_{1,j}^{-1}}^r(x) = a_{j-1}$ when $\tau_{1,j}$ is increasing.

For the maps τ_1 and τ_2, the Frobenius-Perron equation implies

$$F^i(x) = \sum_{j=1}^{m} \left[F^i\left(\overline{\tau_{i,j}^{-1}}^r(x)\right) - F^i\left(\overline{\tau_{i,j}^{-1}}^l(x)\right)\right],$$

$i = 1, 2$.

Using (16) and the fact that η, τ_1 and τ_2 have the same monotonicity on each interval I_j, $j \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned}
F\left(\overline{\eta_j^{-1}}^r(x)\right) - F\left(\overline{\eta_j^{-1}}^l(x)\right) &= \lambda F^{(1)}\left(\overline{\tau_{1,j}^{-1}}^r(x)\right) + (1-\lambda) F^{(2)}\left(\overline{\tau_{2,j}^{-1}}^r(x)\right) \\
&\quad - \left[\lambda F^{(1)}\left(\overline{\tau_{1,j}^{-1}}^l(x)\right) + (1-\lambda) F^{(2)}\left(\overline{\tau_{2,j}^{-1}}^l(x)\right)\right] \\
&= \lambda \left[F^{(1)}\left(\overline{\tau_{1,j}^{-1}}^r(x)\right) - F^{(1)}\left(\overline{\tau_{1,j}^{-1}}^l(x)\right)\right] \\
&\quad + (1-\lambda)\left[F^{(2)}\left(\overline{\tau_{1,j}^{-1}}^r(x)\right) - F^{(2)}\left(\overline{\tau_{1,j}^{-1}}^l(x)\right)\right].
\end{aligned}$$

Thus, denoting the measure corresponding to F by μ, we have

$$\begin{aligned}
\mu\left(\eta^{-1}([0,x])\right) &= \sum_{j=1}^{m} F\left(\overline{\eta_j^{-1}}^r(x)\right) - F\left(\overline{\eta_j^{-1}}^l(x)\right) \\
&= \lambda \sum_{j=1}^{m} \left[F^{(1)}\left(\overline{\tau_{1,j}^{-1}}^r(x)\right) - F^{(1)}\left(\overline{\tau_{1,j}^{-1}}^l(x)\right)\right] \\
&\quad + (1-\lambda) \sum_{j=1}^{m} \left[F^{(2)}\left(\overline{\tau_{1,j}^{-1}}^r(x)\right) - F^{(2)}\left(\overline{\tau_{1,j}^{-1}}^l(x)\right)\right] \\
&= \lambda F^{(1)}(x) + (1-\lambda) F^{(2)}(x) = F(x),
\end{aligned}$$

which implies that the map η defined in (16) preserves F. This completes the proof. $\square$

We construct an example as follows. Let φ_1 and φ_2 be homeomorphisms of $[0,1]$ onto itself defined by

$$\varphi_1(x) = \begin{cases} 2x^2, & \text{for } 0 \le x < 1/2; \\ 1 - 2(1-x)^2, & \text{for } 1/2 \le x \le 1, \end{cases}$$

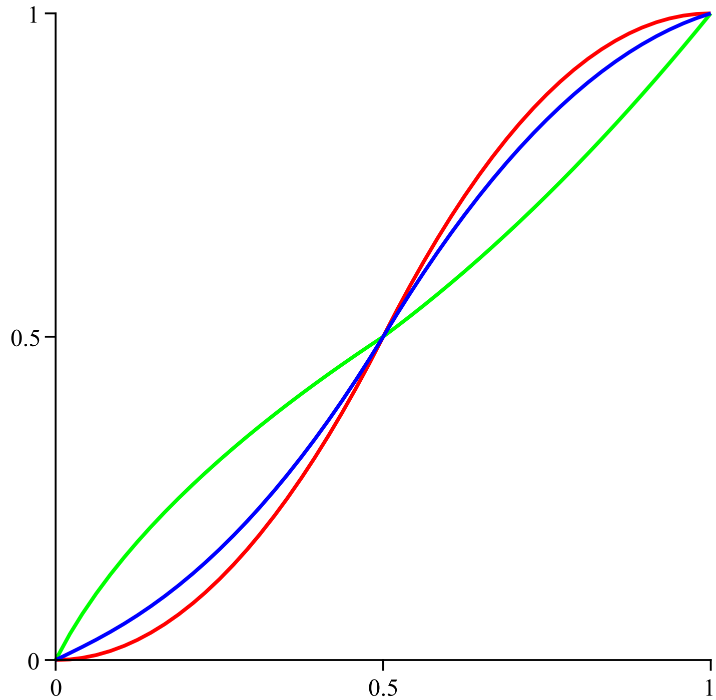

FIGURE 2. Invariant distribution functions $F^{(1)}$, $F^{(2)}$ and F.

$$\varphi_2(x) = \begin{cases} -\frac{1}{4} + \frac{1}{4}\sqrt{1+16x}, & \text{for } 0 \le x < 1/2; \\ \frac{1}{2}\left(x^2 + \frac{1}{2}(x+1)\right), & \text{for } 1/2 \le x \le 1. \end{cases}$$

Define the maps $\tau_1 = \varphi_1^{-1} \circ S \circ \varphi_1$ and $\tau_2 = \varphi_2^{-1} \circ S \circ \varphi_2$, where S is the tent map. The graphs of τ_1 and τ_2 are shown in Figure 3. The invariant distribution function for τ_1 is $F^{(1)} = \varphi_1$, and the invariant distribution function for τ_2 is $F^{(2)} = \varphi_2$ (Corollary 1). Let $\lambda = 3/4$ and $F = \lambda F^{(1)} + (1-\lambda)F^{(2)}$. The distribution functions are shown in Figure 2. In Figure 3 we show the selection η constructed using formula (16).

5. Another method of partially solving the problem

In this section we generally assume that lower and upper boundary maps τ_1 and τ_2 are conjugated and use this conjugation to construct a selection. The following result is well known.

Proposition 1. *Let τ_1 and τ_2 be interval $[0,1]$ maps preserving densities f_1 and f_2, correspondingly, and conjugated by a diffeomorphism (or at least absolutely continuous homeomorphism) h:*

$$\tau_2 = h^{-1} \circ \tau_1 \circ h.$$

Then,

$$f_2 = (f_1 \circ h) \cdot |h'|.$$

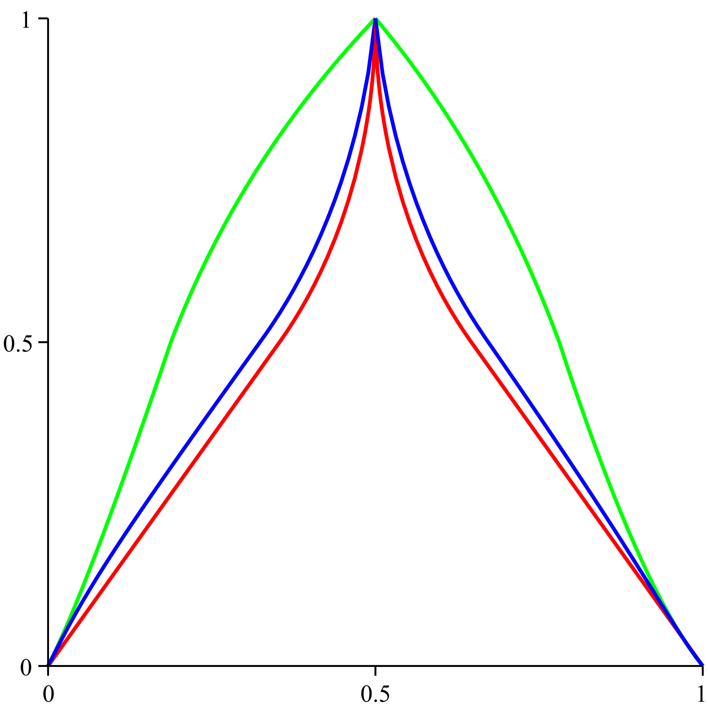

FIGURE 3. Transformations τ_1, τ_2 and η.

Corollary 1. *If τ_1 is the tent map, then*

$$f_2 = |h'|,$$

or equivalently

$$h(x) = \pm \int_0^x f_2(t)dt.$$

Proposition 2. *Let τ_1 be a piecewise linear Markov map of $[0,1]$ onto itself preserving density f_1. (This means that there is a partition $\mathcal{P}$ such that $\mathcal{P} = \{I_i\}_{i=1}^n$ and $\tau_1(I_j)$ is a union of consecutive elements of $\mathcal{P}$ for any $1 \le j \le n$. Then, f_1 is piecewise constant $f_1 = \sum_{i=1}^n c_i \chi_{I_i}$ [4].) Let τ_2 be a map conjugated to τ_1 by a diffeomorphism (or at least an absolutely continuous homeomorphism) h preserving the partition $\mathcal{P}$,*

$$\tau_2 = h^{-1} \circ \tau_1 \circ h.$$

Then, τ_2 preserves the density

$$f_2 = |h'| \cdot \sum_{i=1}^n c_i \chi_{I_i},$$

and

$$|h'| = f_2 \cdot \sum_{i=1}^n \frac{1}{c_i} \chi_{I_i}.$$

Proof. We have

$$f_2 = (f_1 \circ h) \cdot |h'| = \sum_{i=1}^{n} c_i \chi_{I_i} \circ h \cdot |h'| = |h'| \cdot \sum_{i=1}^{n} c_i \chi_{I_i}.$$

□

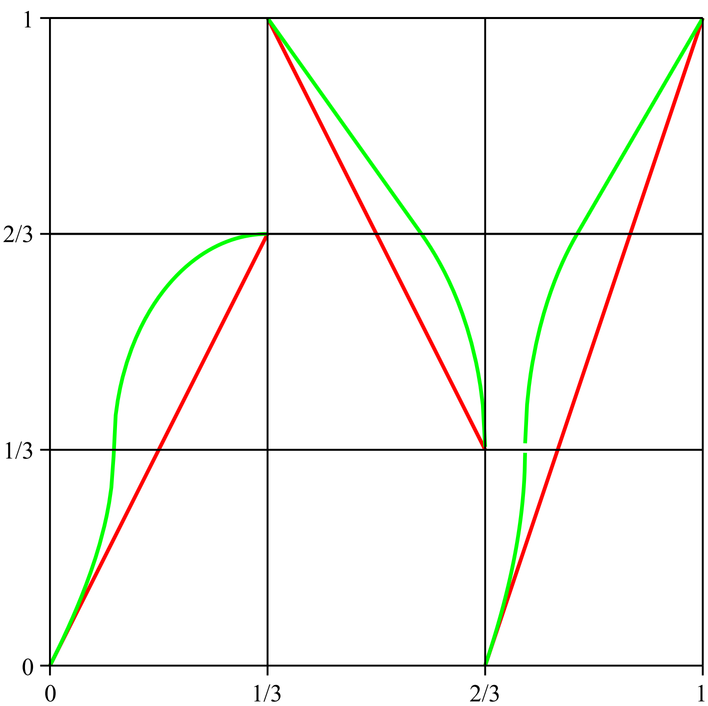

FIGURE 4. Piecewise linear Markov map τ_1 and the conjugated map τ_2.

Let us now consider a more general multivalued map T with lower boundary map τ_1 and upper boundary map τ_2 as in Figure 4. The map T is typically infinitely valued: $T(x) = [\tau_1(x), \tau_2(x)]$, $x \in [0,1]$. If τ_1 preserves a density f_1 and τ_2 preserves a density f_2, then we ask if, for any convex combination $f = \lambda \cdot f_1 + (1-\lambda) \cdot f_2$, $0 < \lambda < 1$, we can find a selection of T which preserves the density f. We give conditions under which this holds.

Theorem 2. *Let τ_1 be a piecewise linear Markov map (on partition $\mathcal{P}$) of $[0,1]$ onto itself preserving the density $f_1 = \sum_{i=1}^{n} c_i \chi_{I_i}$. Let τ_2 be a map conjugated to τ_1 by an increasing absolutely continuous homeomorphism h preserving the partition $\mathcal{P}$, that is,*

$$\tau_2 = h^{-1} \circ \tau_1 \circ h.$$

Let the density f_2 be τ_2 invariant. Then, for any convex combination $f = \lambda \cdot f_1 + (1-\lambda) \cdot f_2$, $0 < \lambda < 1$, we can find a selection of T which preserves the density f.

Proof. By Proposition 2 we have $|h'| = f_2 \cdot \sum_{i=1}^n \frac{1}{c_i}\chi_{I_i}$. Assuming that h is increasing

$$h(x) = \int_0^x f_2(t) \cdot \sum_{i=1}^n \frac{1}{c_i}\chi_{I_i}(t)dt.$$

Using Proposition 2 again, if we define the following conjugation

$$\begin{aligned} g(x) &= \int_0^x f(t) \cdot \textstyle\sum_{i=1}^n \frac{1}{c_i}\chi_{I_i}(t)dt = \int_0^x \left(\lambda \cdot f_1(t) + (1-\lambda)\cdot f_2(t)\right) \cdot \sum_{i=1}^n \frac{1}{c_i}\chi_{I_i}(t)dt \\ &= \int_0^x \left(\lambda \cdot 1 + (1-\lambda) \cdot \left(f_2(t) \cdot \textstyle\sum_{i=1}^n \frac{1}{c_i}\chi_{I_i}(t)\right)\right) dt = \lambda \cdot x + (1-\lambda)\cdot h(x), \end{aligned}$$

then $\tau = g^{-1} \circ \tau_1 \circ g$ preserves the density f. Note, g is also increasing.

We will prove that $\tau_1 \le \tau \le \tau_2$. Consider $x \in I_i \in \mathcal{P}$. Let $\beta = 1-\lambda$. The function τ_1 is piecewise linear on I_i, so $\tau_1(\lambda x + \beta y) = \lambda\tau_1(x) + \beta\tau_1(y)$, $x, y \in I_i$. First, we will prove that $\tau_1 \le \tau$ or equivalently $g \circ \tau_1 \le \tau_1 \circ g$. For $x \in I_i$ we have

$$g(\tau_1(x)) = \lambda\tau_1(x) + \beta h(\tau_1(x)) \le \lambda\tau_1(x) + \beta\tau_1(h(x)) = \tau_1(g(x)).$$

We used the inequality $h \circ \tau_1 \le \tau_1 \circ h$ which is equivalent to $\tau_1 \le h^{-1} \circ \tau_1 \circ h = \tau_2$.

Now, we prove that $\tau \le \tau_2$ or that $g^{-1} \circ \tau_1 \circ g \le \tau_2$ or, equivalently, that $\tau_1 \circ g \le g \circ \tau_2$. Again, we consider $x \in I_i$:

$$\tau_1(g(x)) = \tau_1(\lambda x + \beta h(x)) = \lambda\tau_1(x) + \beta\tau_1(h(x)).$$

We also have

$$g(\tau_2(x)) = \lambda\tau_2(x) + \beta h(\tau_2(x)) = \lambda\tau_2(x) + \beta h(h^{-1}(\tau_1(h(x)))) = \lambda\tau_2 + \beta\tau_1(h(x)).$$

Since $\tau_1 \le \tau_2$ the proof is complete. □

Example 3.

In this example we show existence of a selection τ in a situation when the lower boundary map is not onto. Let us consider the tent map

$$\tau_2(x) = 1 - 2|x - 1/2|,$$

and

$$\tau_1(x) = \begin{cases} 4x^2 & \text{, for } 0 \le x < 1/4, \\ 2x - 1/4 & \text{, for } 1/4 \le x < 1/2, \\ -2x + 7/4 & \text{, for } 1/2 \le x < 3/4, \\ 4(1-x)^2 & \text{, for } 3/4 \le x \le 1, \end{cases}$$

shown in Figure 5.

The invariant densities are $f_2 = 1$ for τ_2 and $f_1 = 2\chi_{[1/4,3/4]}$ for τ_2. For any $0 < \lambda < 1$ their convex combination is

$$f = \lambda f_1 + (1-\lambda)f_2 = (1-\lambda)\chi_{[0,1/4]\cup[3/4,1]} + (1+\lambda)\chi_{[1/4,3/4]}.$$

We are looking for the selection τ satisfying $\tau_1 \le \tau \le \tau_2$ and preserving f. We must have $\tau(0) = \tau(1) = 0$ since both τ_1 and τ_2 satisfy these conditions. We also must have $\tau(1/2) = 1$ since f is supported on the whole $[0,1]$. We will look for a symmetric map τ.

For $x \in [0, 1/4]$, we have

$$(1-\lambda) = \frac{(1-\lambda)}{\left|\tau'(\tau_{(1)}^{-1}(x))\right|} + \frac{(1-\lambda)}{\left|\tau'(\tau_{(2)}^{-1}(x))\right|},$$

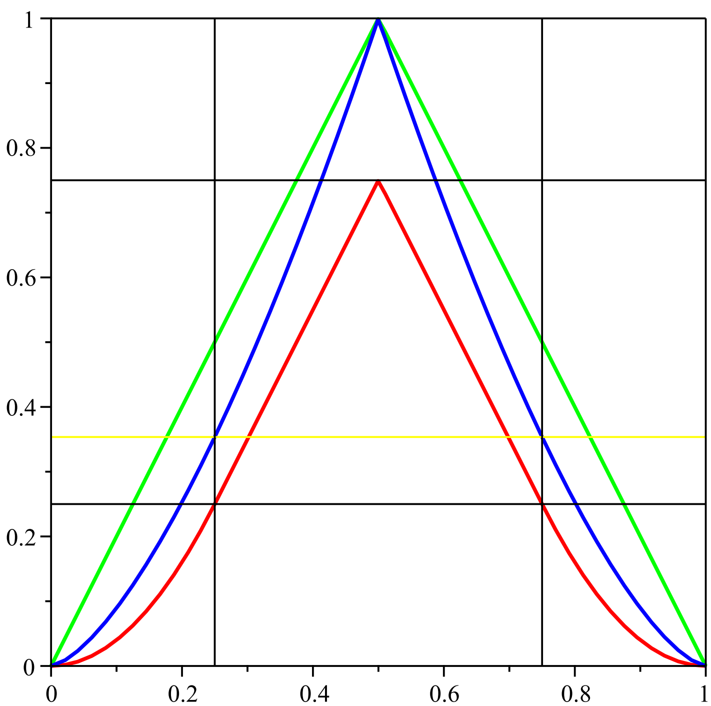

FIGURE 5. Maps τ_1, τ_2 and map τ we are looking for.

or

$$1 = \frac{1}{\left|\tau'(\tau_{(1)}^{-1}(x))\right|} + \frac{1}{\left|\tau'(\tau_{(2)}^{-1}(x))\right|}.$$

Thus, by symmetry of τ:

$$|\tau'(x)| = 2\ ,\ \text{for}\ \ x \in [0, \tau_{(1)}^{-1}(1/4)] \cup [1 - \tau_{(1)}^{-1}(1/4), 1].$$

For $x \in [1/4, \tau(1/4)]$, we have

$$(1+\lambda) = \frac{(1-\lambda)}{\left|\tau'(\tau_{(1)}^{-1}(x))\right|} + \frac{(1-\lambda)}{\left|\tau'(\tau_{(2)}^{-1}(x))\right|},$$

which, by symmetry of τ, implies

$$|\tau'(x)| = 2(1-\lambda)/(1+\lambda)\ ,\ \text{for}\ \ x \in [\tau_{(1)}^{-1}(1/4), 1/4] \cup [3/4, 1 - \tau_{(1)}^{-1}(1/4)].$$

For $x \in [\tau(1/4), 3/4]$, we have

$$(1+\lambda) = \frac{(1+\lambda)}{\left|\tau'(\tau_{(1)}^{-1}(x))\right|} + \frac{(1+\lambda)}{\left|\tau'(\tau_{(2)}^{-1}(x))\right|},$$

which, by symmetry of τ, implies

$$|\tau'(x)| = 2\ ,\ \text{for}\ \ x \in [1/4, \tau_{(1)}^{-1}(3/4)] \cup [1 - \tau_{(1)}^{-1}(3/4), 3/4].$$

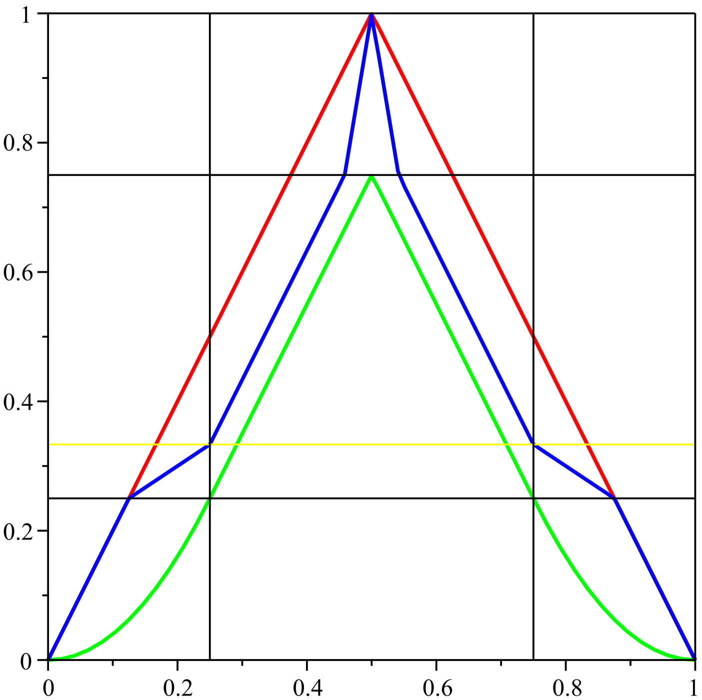

Figure 6. Map τ for $\lambda = 1/2$.

For $x \in [3/4, 1]$, we have

$$(1-\lambda) = \frac{(1+\lambda)}{\left|\tau'(\tau_{(1)}^{-1}(x))\right|} + \frac{(1+\lambda)}{\left|\tau'(\tau_{(2)}^{-1}(x))\right|},$$

which, by symmetry of τ, implies

$$|\tau'(x)| = 2(1+\lambda)/(1-\lambda) \ , \ \text{for} \ \ x \in [\tau_{(1)}^{-1}(3/4), 1-\tau_{(1)}^{-1}(3/4)].$$

In Figures 6 and 7 we present graphs of τ for $\lambda = 1/2$ and $\lambda = 1/10$. The slopes are $2, 2/3, 2, 6$ for the first and $2, 18/11, 2, 22/9$ for the second.

6. Multivalued Maps and Random Maps

We define a random map to be a finite collection of maps as follows: let $R = (\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K)$, where τ_k are maps of an interval, p_k are position dependent probabilities which are assumed to be measurable, $p_k(x) \geq 0$ for $k = 1, 2, ..., K$ and $\sum_{k=1}^{K} p_k(x) = 1$. At each step, the random map R moves the point x to $\tau_k(x)$ with probability $p_k(x)$. For fixed $\{\tau_1, \tau_2, \ldots, \tau_K\}$, R can have different invariant probability density functions, depending on the choice of the (weight) functions $\{p_1, p_2, \ldots, p_K\}$. Let f_k be an invariant density of τ_k, $k = 1, \ldots, K$. It is shown in [5, 7] that for any positive constants a_k, $k = 1, \ldots, K$, satisfying $\sum_{k=1}^{K} a_k = 1$, there exists a system of weight probability functions $p_1, \ldots, p_K$ such

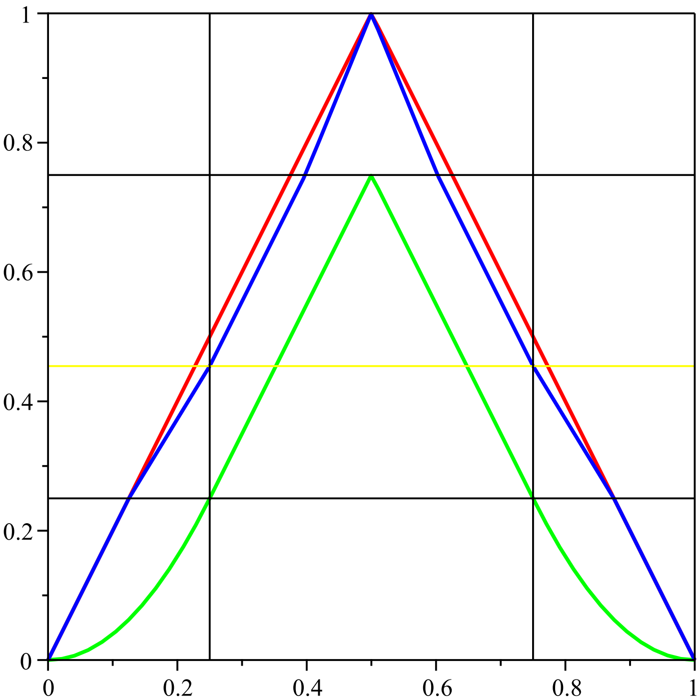

FIGURE 7. Map τ for $\lambda = 1/10$.

that the density $f = a_1 f_1 + \cdots + a_K f_K$ is invariant under the random map $R = \{\tau_1, \ldots, \tau_K; p_1, \ldots, p_K\}$, where

$$p_k = \frac{a_k f_k}{a_1 f_1 + \cdots + a_K f_K} \quad , \quad k = 1, 2, \ldots, K,$$

(It is assumed that $0/0 = 0$.)

Let us now consider a multivalued map consisting of a lower boundary map τ_1 and an upper boundary map τ_2, with density functions f_1 and f_2, respectively. Let f be any convex combination of f_1 and f_2. Then, by the foregoing result, we can construct a position dependent random map on the graphs of τ_1 and τ_2 whose unique pdf is f.

A related problem is to consider a piecewise expanding selection having density function f; can we find a probability function $p(x)$ such that the resulting random map $R = (\tau_1, \tau_2; p, 1-p)$ has f as its density function?

In general this problem does not have a positive solution (see Example 4 below). However, in many cases the solution can be found. A simple example of this situation can be shown from τ_1 and τ_2 of Example 1. Let us consider the triangle map, τ, whose graph fits in between the graphs of τ_1 and τ_2. It can be shown that $R = (\tau_1, \tau_2; 0.75, 0.25)$ has Lebesgue measure as its invariant measure.

Another, more general result in this direction can be established by considering τ_1 to be a piecewise linear Markov map where τ_2 is conjugated to τ_1 by $g(x) = \lambda x + (1-\lambda)h(x)$, where h conjugates the upper map τ_2 to the lower map τ_1. Then τ

has pdf f which is a convex combination of f_1 and f_2. Hence by the main result of [7], we know that there exists a position dependent random map $R = (\tau_1, \tau_2; p, 1-p)$ which has f as its pdf.

Example 4.

We consider the semi-Markov ([6]) piecewise linear maps

$$\tau_1(x) = \begin{cases} \frac{4}{3}x \ , & \text{for } \ 0 \le x < \frac{3}{20}; \\ 16x - \frac{11}{5} \ , & \text{for } \ \frac{3}{20} \le x < \frac{1}{5}; \\ 5x \ (\mathrm{mod})\ 1 \ , & \text{for } \ \frac{1}{5} \le x \le 1, \end{cases} \qquad \tau_2(x) = \begin{cases} 16x \ , & \text{for } \ 0 \le x < \frac{1}{20}; \\ \frac{4}{3}x + \frac{11}{15} \ , & \text{for } \ \frac{1}{20} \le x < \frac{1}{5}; \\ 5x \ (\mathrm{mod})\ 1 \ , & \text{for } \ \frac{1}{5} \le x \le 1, \end{cases}$$

whose graphs are shown in Figure 8. For the selection τ we choose the map $\tau(x) = 5x \pmod 1$ preserving Lebesgue measure.

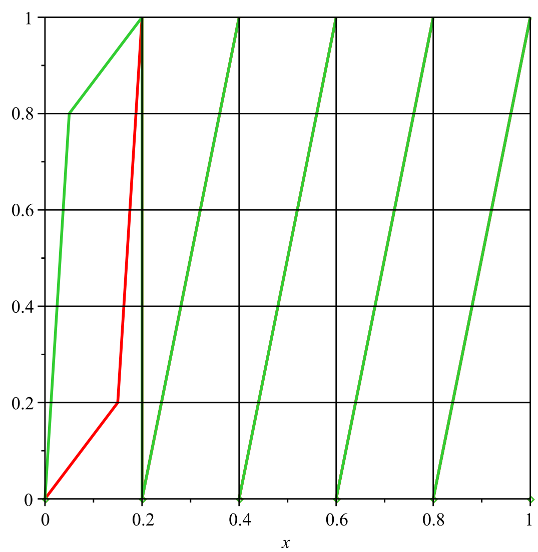

FIGURE 8. Boundary maps in the counterexample

We will show that there is no solution, i.e., there is no position dependent random map based on τ_1, τ_2 that preserves Lebesgue measure.

Let us consider $p_1(x)$, non-constant, on $[0, 1/5]$ (the values on $[1/5, 1]$ are not important). Let $\phi_1 = \tau_1^{-1}$ on $[0, 1/5]$, $\psi_1 = \tau_2^{-1}$ on $[0, 1/5]$, $\phi_i = \tau_1^{-1} = \tau_2^{-1}$ on $[(i-1)/5, i/5]$, $i = 2, 3, 4, 5$. The Frobenius-Perron operator of R is

(17)

$$\begin{aligned} (P_R f)(x) = {} & \frac{3}{4} p_1(\phi_1(x)) f(\phi_1(x)) \chi_{[0,1/5]} + \frac{1}{16} \left(1 - p_1(\psi_1(x))\right) f(\psi_1(x)) \chi_{[0,1/5]} \\ & + \frac{1}{16} p_1(\phi_1(x)) f(\phi_1(x)) \chi_{[1/5,4/5]} + \frac{1}{16} \left(1 - p_1(\psi_1(x))\right) f(\psi_1(x)) \chi_{[1/5,4/5]} \\ & + \frac{1}{16} p_1(\phi_1(x)) f(\phi_1(x)) \chi_{[4/5,1]} + \frac{3}{4} \left(1 - p_1(\psi_1(x))\right)) f(\psi_1(x)) \chi_{[4/5,1]} \\ & + \frac{1}{5} \left(f(\phi_2(x)) + f(\phi_3(x)) + f(\phi_4(x)) + f(\phi_5(x)) \right). \end{aligned}$$

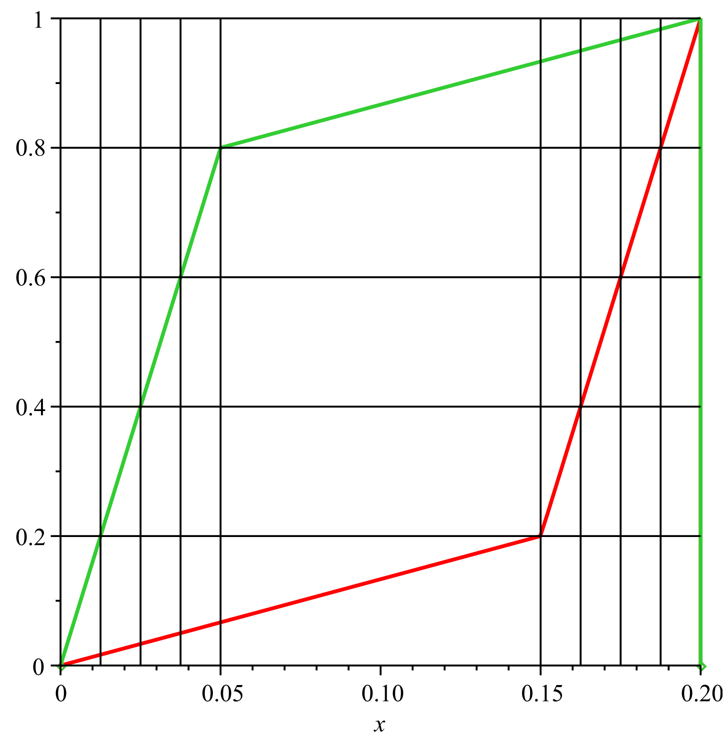

FIGURE 9. Maps τ_1, τ_2 on interval $[0,1/5]$ (not to scale).

If we assume that $f=1$ is preserved by P_R, then equation (17) reduces to

$$\begin{aligned}\frac{1}{5} &= \frac{3}{4}p_1(\phi_1(x))\chi_{[0,1/5]} + \frac{1}{16}\left(1-p_1(\psi_1(x))\right)\chi_{[0,1/5]}\\ &+ \frac{1}{16}p_1(\phi_1(x))\chi_{[1/5,4/5]} + \frac{1}{16}\left(1-p_1(\psi_1(x))\right)\chi_{[1/5,4/5]}\\ &+ \frac{1}{16}p_1(\phi_1(x))\chi_{[4/5,1]} + \frac{3}{4}\left(1-p_1(\psi_1(x))\right))\chi_{[4/5,1]}.\end{aligned} \tag{18}$$

We introduce a map $\tau_{21}:[0,1/5]\to[0,1/5]$, $\tau_{21}=\tau_2^{-1}\circ\tau_1$, (see Fig. 10) defined by

$$\tau_{21}(y)=\begin{cases}\frac{1}{12}x\ , & \text{for } \ 0\le x<\frac{3}{20};\\ x-\frac{11}{80}\ , & \text{for } \ \frac{3}{20}\le x<\frac{15}{80};\\ 12x-\frac{11}{5}\ , & \text{for } \ \frac{15}{80}\le x\le\frac{1}{5}.\end{cases}$$

We assume that the solution p_1 exists and is a probability, i.e., its values are between 0 and 1. In particular it is defined on the interval $[1/80,4/80]$ and on interval $[12/80,15/80]$. Let us consider equation (18) for $x\in[1/5,4/5]$. We have

$$\frac{1}{5}=\frac{1}{16}p_1(\phi_1(x))+\frac{1}{16}\left(1-p_1(\psi_1(x))\right), \tag{19}$$

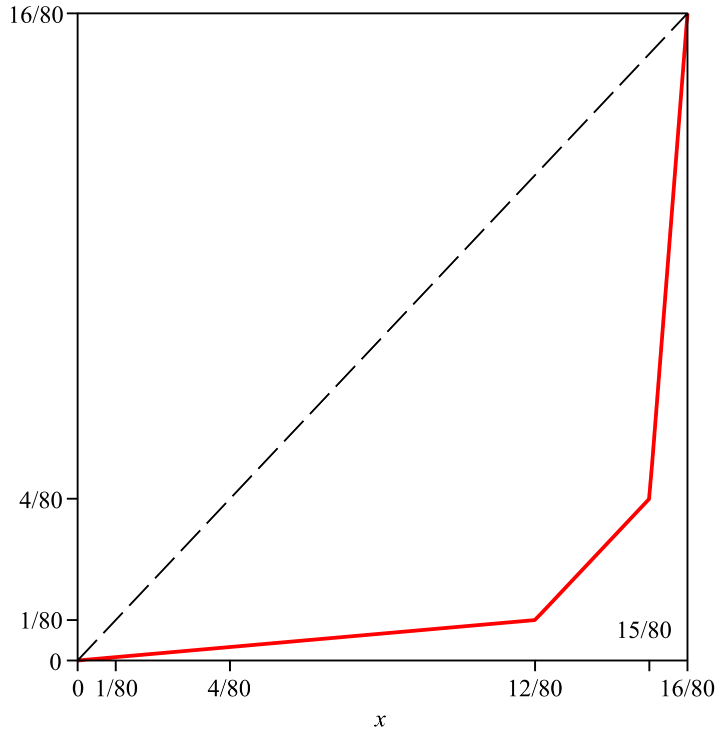

FIGURE 10. The map τ_{21} on $[0, 1/5]$.

or, substituting $x = \phi_1^{-1}(y)$, $y \in [12/80, 15/80]$,

$$\frac{1}{5} = \frac{1}{16} p_1(y) + \frac{1}{16}\left(1 - p_1(\psi_1(\phi_1^{-1}(y)))\right). \tag{20}$$

Using the equality $\tau_{21} = \psi_1 \circ \phi_1^{-1}$, this can be rewritten as

$$p_1(y) = \frac{11}{5} + p_1(\tau_{21}(y)).$$

Note that $\tau_{21}([12/80, 15/80]) = [1/80, 4/80]$. Whatever are the values of p_1 on $[1/80, 4/80]$, this implies that the values on $[12/80, 15/80]$ are strictly larger than 1. This contradicts the assumptions on p_1.

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(A. Boyarsky) Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada
E-mail address, A. Boyarsky: `abraham.boyarsky@concordia.ca`

(P. Góra) Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada
E-mail address, P. Góra: `pawel.gora@concordia.ca`

(Z. Li) Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada
E-mail address, Z. Li: `zhenyangemail@gmail.com`